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The contribution of Gustav R. Kirchhoff to the dynamics of tapered beams

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Gustav Kirchhoff has been credited, among many other renowned achievements, as the first scientist who tackled and solved the problem of studying the transversal vibrations of beams with variable cross-section. His contribution, which was presented in 1879 and published in the following year, is nowadays almost forgotten in the international scientific community, with the only exception of the German-speaking countries. For this reason it is rediscovered and thoroughly discussed here, with an exegetical approach. For completeness' sake an unabridged translation into English (the first one, to the best of the authors' knowledge) is provided in the appendix for the interested readers.

1 Introduction

The outspreading of the ideals of the Age of Enlightenment and the outbreak of French revolution produced in the world, among many other effects, the decline of Latin language as the *lingua franca* of scientific communication, as it has been for centuries before[1]. As a consequence the language used by G. W. Leibniz, I. Newton, D. Bernoulli, L. Euler in the world of Mechanics was suddenly superseded by national languages, so that around 1830 a French Mechanician like Augustin-Louis Cauchy (1789–1857) was publishing his researches in French, while in the meantime the Italian Gabrio Piola (1794–1850) published his in Italian (see, e.g. [10], [11], [12], [13]) and the German Friedrich W. Bessel (1784–1846) used German for his; the most noticeable exception being Carl Friedrich Gauss (1777–1855), who still used to publish in Latin up to 1832. It is important to remark that the use of different languages did not block the spreading of the research work nor prevent at all fruitful discussions between these scientists: hence, the existence in the XIX century of a multi-lingual international community of mechanicians, where no single language was prevailing on the other ones, has to be seen as a happy occurrence in the history of science. A different trend took place instead in the last 70 years, namely after the end of WWII, since English increasingly became the *de facto* standard language for scientific communication, thus bringing to a rapid fading of all other foreign languages for exchanging research results. As a consequence many important cornerstones of Mechanics were forgotten simply because they were written in a different language and English translations were not available[2].

This is precisely what has happened to the Memoir that Gustav Kirchhoff devoted to the transversal vibrations of variable-section beams, which started a fruitful research vein during the last twenty years of the XIX and in the XX century: nowadays it is, wrongly, overlooked. For precisely the purpose of reviving this important research work, it has

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[1] The consequences of the loss of a common language for science has always produced remarkable effects: for a detailed discussion of this point, see the outstanding book by L. Russo [56].

[2] See, among many others, [17], where an annotated translation of Hellinger's encyclopedic article (originally published in German) appeared. Therein Russo's theory contradicting a linear growth of knowledge and the effects of missing a *lingua franca* is also discussed.

been translated into English and it is here proposed again along with a commentary and complete analysis of the procedure which Kirchhoff followed in his way of exposing the relevant theory.

The theory of transversal vibration of uniform beams, which had been developed as a result of the researches started by Daniel Bernoulli (1700–1782) and Leonhard Euler (1707–1783), had already reached a rather complete development at Kirchhoff's time. For instance, in the framework of linear elastic behavior, experimental results had been already established, exploiting an acoustic background, by Ernst F. Chladni (1756–1827) [8] in 1830. By the year 1858 a reasonably complete understanding of the vibration modes of a uniform beam for different boundary conditions were already available since Joseph Stefan (1835–1893), who is mostly famous nowadays for the Stefan-Boltzmann law of radiation, published his paper *On the transversal vibrations of an elastic beam*, [61].

The topic of the essay is the study of transversal vibrations of a tapered (variable cross-section) cantilever beam in the vertical plane containing the beam axis (z) and one of the principal inertia axis (x) of each cross-section. First, the general case is given a solution; subsequently attention is focused on beams having the shape of linearly varying wedges and cones. In particular, Kirchhoff's aim was to provide the computation of the fundamental frequency (tone) and of the maximum deflection at the free end, under the condition that the maximum elastic strain is not exceeded anywhere along the beam; solutions were then compared to the case of a uniform beam.

The rest of the paper is organized as follows: in Section 2 a brief sketch of Kirchhoff's life and his achievements in Mechanics are outlined; then, in Section 3 a detailed analysis of Kirchhoff's procedure is presented and commented upon. Section 4 gives some details about Kirchhoff's legacy in the theory of transversal vibrations of tapered beams. In Appendix A some information about the different versions of this memoir and their availability is given, as well as some translation notes. Finally, the English translation of the unabridged essay is given in Appendix B.

2 Kirchhoff's life and contribution to Mechanics

A short *resumé* of Kirchhoff's life, giving an essential view of the most important achievements, is presented here; the interested reader can find a more detailed description in the commemorative memoir of Robert von Helmholtz (1862–1889) [67], the eldest son of Hermann Helmholtz and of his second wife Anna von Mohl (1834–1899). An English translation of this article is also available [68]. More specifical descriptions of Kirchhoff's contribution to several branches of Physics, mostly spectroscopy, can be found in [70], [19], [58], [62], [9].

2.1 Kirchhoff's life

Gustav Robert Kirchhoff was born on March 12, 1824 in Königsberg, Eastern Prussia (now Kaliningrad, Russia), the son of Friedrich Kirchhoff, a law councilor, and of Johanna Henriette Wittke.

In 1843 he entered Albertus University of Königsberg, which had been founded in 1544: Carl Gustav J. Jacobi (1804–1851), Franz E. Neumann (1798–1895) and Friedrich J. Richelot (1808–1875), of whom he married the daughter Clara in 1857, were his teachers. He graduated from the University in 1847 with researches on electrical current, (Kirchhoff's laws) extending Ohm's work; it is remarkable that before graduating he had two papers, namely [25], [26]—which he signed as *Studiosus* (i.e. Student) *Kirchhoff*—published on the highly renowned journal *Annalen der Physik*, which during years 1824–1876 was also known as *Poggendorffs Annalen*, from the name of the Editor-in-chief.

In 1848, being impossible for him to reach Paris for enjoying a research grant due to the political turmoils of that year, he joined Berlin University as a *Privatdocent* (unpaid post); in 1850 he was appointed as an adjunct professor at University of Breslau (now Wrocław, Poland); in the same year Kirchhoff published his paper *On the equilibrium and motion of an elastic disc* [27], which is a fundamental contribution to the theory of thin plates, following the pioneering works by Sophie Germain (1776–1831), Simeón-Denis Poisson (1781–1840) and Claude-Henri Navier (1785–1836); it was indeed in this paper that Kirchhoff gave, for the first time, the correct form of boundary conditions.

In 1854 he was appointed professor of Physics at the University of Heidelberg, with the support of Robert W. Bunsen (1811–1899), whom he had already met in Breslau in 1852, and Hermann Helmholtz (1821–1894). There Bunsen and Kirchhoff began to cooperate on spectroscopy and in 1860 they coauthored the first paper of a series about *Chemical analysis through spectral observations* [40]. In 1861 together they discovered caesium (Cs) and rubidium (Rb) while studying the chemical composition of the Sun via its spectral signature. For their achievements in spectroscopy they were the first recipients in 1877 of the *Davy medal* presented by the Royal Society of London.

After the death of his wife Clara in 1869, Kirchhoff was left with four children and married three years later a second time. His new wife was Luise Brömmel, a matron of the university hospital. In 1875, due to serious health problems produced by a fall on the staircase, which compelled him for a long time to move only with crutches or in a wheelchair, and made hard for him the life in a laboratory, he accepted the newly created chair of Theoretical Physics at the University of Berlin and began to write *Lectures on Mathematical Physics* in 4 volumes. Only the first of them, *Mechanics* [33], appeared during his life; the other three, were posthumously edited by Kurt Hensel (*Mathematical Optics* [41]) and by Max Planck (*Electricity and Magnetism* [42]; *Theory of Heat* [43]).

On October 29, 1879 at the Royal Prussian Academy of Sciences in Berlin he presented the paper *On the transversal vibrations of a beam of variable cross-section* [34], where the problem of flexural vibrations of non uniform beams was addressed and solved for the first time.

Fig. 1 (a) A photographic portrait of Gustav Robert Kirchhoff in his late years. Image taken from [37]. The same portrait appears in two commemorative stamps issued in 1974 on the occasion of the 150-th anniversary of Kirchhoff's birth: (b) Stamp issued by the Bundespost Berlin; (c) Stamp issued by the DDR mail.

In 1883–1884 Gustav Kirchhoff was Rector of the University of Berlin; he died in Berlin on October 17, 1887 and was buried in Alter St.-Matthäus graveyard in Berlin-Schöneberg. His grave is still standing.

A portrait of Kirchhoff in his late years, reproduced here from [37] (the same image appears also in [41]), is shown in Figure 1(a). To celebrate the 150-th anniversary of Kirchhoff's birth, a commemorative stamp was issued in 1974 by both mail services of the two then existing (before re-unification) German states, Federal Republic of Germany (BRD) and German Democratic Republic (DDR); they are shown in Figure 1(b) and Figure 1(c), respectively.

2.2 Kirchhoff's scientific contributions

According to the *Catalogue of scientific papers* edited by the Royal Society of London [59] (see vols. 1, 3, 8, 10, 16), during his life Kirchhoff authored 64 different[3] papers, 7 of them in cooperation: four with Robert Bunsen and three with Gustav Hansemann.

In his *Collected essays* [37], which were edited by himself during the last period of his life and appeared in 1882, only 38 contributions are listed; in the *Supplement* [5], which was edited by Ludwig Boltzmann after Kirchhoff's death, and was printed in 1891, 9 more contributions are reported.

In the whole scientific production of Kirchhoff, papers dealing with solid and structural mechanics form a relatively small group, but some of them played an important role in shaping and developing both disciplines of *Theory of Elasticity* and *Strength of Materials*. In their monumental work, Todhunter and Pearson [65] devoted 69 pages to Kirchhoff, presenting an account of 16 of his works (among them they reviewed [27], [28], [29], [30], [33], [41], [34], [38], [39]). The paper which is here taken into consideration has been carefully addressed by them in Art. 1302–1307 (see [65], pages 92–98).

[3] Emphasis has to be placed on the word *different* since (already) in those days it was rather common to publish the same paper more than once, eventually in abridged form, for instance in a journal and in the proceedings of some Academy of Sciences, to ensure a better spreading of the research results.

3 A detailed analysis of Kirchhoff's solution

To motivate his research on tapered beams, Kirchhoff wrote, at the beginning of the paper (see [34] or [35]) these sentences (here translated into English): "The transversal vibrations of cylindrical beams are theoretically and experimentally treated in detail; the vibrations of a beam whose cross-section is variable are not however, up to now, more closely investigated, even though, besides the mathematical interest which they deserve, they possess in this respect a practical one, too, because for a beam which oscillates with a free end, the amplitude of vibration of this end can be much larger, without exceeding the elasticity limit, when toward this end the beam is tapered, than when the cross-section is everywhere the same."

The scope of the work is also clearly defined in the following sentence: "The following considerations are referred to a beam which forms a prism or a cone with an extremely small angle, with the edge or the sharp tip at the free end."

Starting from these assumptions the analysis is carried out carefully. On the other hand, from this beginning the reader can realize how Kirchhoff's choice of words is precise and how the structure of the speech is fully developed, while preserving an admirable clear style. This is outlined, in the above mentioned commemoration by Robert von Helmholtz [67]– [68], where it is explicitly written: "The words stand as if hewn in stone, each one at its place, the logical comprehension of each duly considered; we find here condensed into a few lines what would have taken others pages to describe; only when the existing words seemed not precise enough, he uses circumlocutions and definitions, and that mostly in mathematical language."

The solution of the vibration problem for tapered beams, as first obtained by Kirchhoff, will be analyzed in detail and commented upon where necessary. Figure 2 should allow the reader to follow without difficulties the development; in particular a Cartesian reference system is adopted; the z-axis coincides with the beam axis, connecting the centroid of all cross-sections; x and y are the principal axes of inertia, and vibrations are assumed to occur in the x-z plane. The origin is located at the free end of the beam, while the opposite one is fixed i.e. clamped, so that a cantilever beam is obtained. It has to be remarked that, for the particular cases considered by Kirchhoff, the orientation of the reference system is optimal for imposing the boundary conditions, while this is no more true, in general, if tapered beams having the shape of a frustum or of an otherwise truncated solid need to be studied.

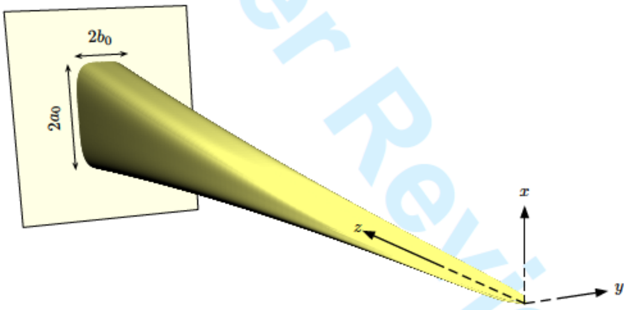

Fig. 2 Perspective sketch illustrating the general case of tapered beams analyzed by Kirchhoff: here a beam with an *hyperelliptical* cross-section and different tapers in the x-z and y-z planes is shown. The adopted Cartesian reference system is clearly marked.

After defining the area q and the second area moment k with respect to y of a generic cross-section of the beam, see, e.g., eq. (B.1), Kirchhoff introduces (denoting by ξ, μ and E transversal displacement, density, and Young's modulus, respectively) kinetic, eq. (B.2), and potential energy, eq. (B.3), and suggests that the equation of motion could be deduced by Hamilton's principle. Clearly Kirchhoff considers variational principles as a basic tool in Mechanics, following the tradition settled by Lagrange and recognizes their importance when exploring new fields in mechanics: see, e.g., [2], [7], [14].

The governing equation (B.4) is then provided, without any deduction but by taking it from Lord Rayleigh's reference [63], along with the relevant boundary conditions[4]. In particular, considering that δ is used as the symbol of variation, he outlines that for a fixed end or for a free end either shear force or deflection must vanish, as well as either bending moment or slope needs to be zero: this is shown in eq. (B.5) (for the general case) and in eq. (B.8) when variables have been separated to solve the equation of motion for studying the transversal oscillations.

3.1 The analyzed problem, general case

After presenting the equation of motion, a partial differential one, Kirchhoff proceeds to solve it by separation of variables and assuming that the tapered beam is vibrating according to the fundamental frequency: then eq. (B.6) holds, the angular

[4] It has to be remarked that the idea exploited by Lord Rayleigh to obtain the equation, which was later studied by Kirchhoff, has been used many times to get Generalized Beam Theories; among many others, see these papers: [16], [18], [22], [48], [55], [57], [64].

frequency λ being a constant and u (vibration mode) depending only on z. The resulting ordinary differential equation (ODE) to be solved is then given by eq. (B.7); then he devises a method for solving it by adopting a power series expansion.

3.1.1 Statement of the problem for the general case

For the general case (see also [65], page 93, or [69], Figure 1) it is assumed that the contour of the cross-section is described by an an implicit function χ of both coordinates x and y:

$$\chi(x, y) = 0, \tag{3.1}$$

and moreover that the following laws of variation of the cross-section are given:

$$x = z^m f_1(\vartheta), \qquad y = z^n f_2(\vartheta), \tag{3.2}$$

where m and n are, in general, real constant values (even though Kirchhoff in the presented applications considers only the case where $m, n \in \mathbb{N}$), f_1, f_2 are given functions of a variable ϑ, which is independent of z.

As an example, for drawing Figure 2 it has been assumed that the boundary of the generic cross-section is defined by this equation:

$$\chi(x, y) = \left(\frac{x}{a}\right)^4 + \left(\frac{y}{b}\right)^4 - 1 = 0, \tag{3.3}$$

where the maximum extensions a and b of the cross-section, in the x and y directions respectively, are governed by these taper rules:

$$a = a_0 \left(\frac{z}{l}\right)^{3/2}, \qquad b = b_0 \left(\frac{z}{l}\right)^{1/3}. \tag{3.4}$$

So the beam, which has been assumed to have a length $l = 160$ and a transversal cross-section defined by a fourth-order Lamé curve (a *hyperellipse*) with semi-diameters $a_0 = 10$, $b_0 = 5$, (these values are referred to the clamped end of the beam) presents different tapers in the x-z ($m = 3/2$) and in the y-z ($n = 1/3$) planes. The contour curve (3.1) of the cross-section is represented, in this particular case, by the hyperellipse eq. (3.3), whose parametric equations are simply:

$$x(\vartheta) = \pm a(\cos\vartheta)^{\frac{1}{2}}, \qquad y(\vartheta) = \pm b(\sin\vartheta)^{\frac{1}{2}}, \qquad 0 \le \vartheta < \frac{\pi}{2}. \tag{3.5}$$

Thus, taking into account the taper laws (3.4), it is possible to arrive at this explicit form of the two equations (3.2):

$$x = \pm z^m \frac{a_0}{l^m}(\cos\vartheta)^{\frac{1}{2}}; \qquad y = \pm z^n \frac{b_0}{l^n}(\sin\vartheta)^{\frac{1}{2}}, \tag{3.6}$$

implying that x/z^m and y/z^n are given functions of the section boundary, namely

$$\frac{x}{z^m} = f_1(\vartheta); \qquad \frac{y}{z^n} = f_2(\vartheta); \qquad f_1(\vartheta) = \pm\frac{a_0}{l^m}(\cos\vartheta)^{\frac{1}{2}}; \qquad f_2(\vartheta) = \pm\frac{b_0}{l^n}(\sin\vartheta)^{\frac{1}{2}}. \tag{3.7}$$

That is precisely what Kirchhoff means in an extremely concise way in his original statement and in the resulting eq. (B.9).

Next, denoting by q' and k' the values of the cross-section area and second area moment corresponding to $z = 1$, and considering that the former depends linearly on both x and y, while the latter depends cubically on x and linearly on y, Kirchhoff succeeds in providing the expressions of q and k for any cross-section, see eq. (B.10), as functions of q', k' and z alone. Accordingly, the governing ODE becomes eq. (B.11); after the required differentiations and some rearrangements are performed, it reads:

$$z^{2m}\frac{d^4u}{dz^4} + 2(3m+n)z^{2m-1}\frac{d^3u}{dz^3} + (3m+n)(3m+n-1)z^{2m-2}\frac{d^2u}{dz^2} = \alpha^2\lambda^2 u, \tag{3.8}$$

where the following short-hand notation has been introduced:

$$\alpha = \sqrt{\frac{q'\mu}{k'E}}. \tag{3.9}$$

The solution method adopted by Kirchhoff is the following: a solution (integral) of the previous ODE is sought under the form of a series expansion, by setting:

$$u = \sum_{r=0}^{\infty} A_r z^{h+r}, \tag{3.10}$$

(where, in general, $h \in \mathbb{R}$) and substituting in eq. (3.8) to obtain an identity, so that, when both sides of it are multiplied by z^{4-2m}, it results:

$$z^h \sum_{r=0}^{\infty} (g A_r z^r - \alpha^2 \lambda^2 A_r z^{r+4-2m}) = 0, \tag{3.11}$$

where the following short-hand notation has been introduced:

$$g = (h+r)(h+r-1)[(h+r-2)(h+r-3) + 2(h+r-2)(3m+n) + (3m+n)(3m+n-1)]. \tag{3.12}$$

In order to satisfy eq. (3.11) as an identity, it appears that r has to be an integer multiple of $4-2m$, say $r = s(4-2m)$, $(s = 0, 1, \dots, \infty)$ so that eq. (3.10) can be replaced by eq. (B.12); then for $s = 0$ the fourth-order algebraic equation $g = 0$, the so called *indicial equation*, has to be solved for h, as shown by eq. (B.13), providing the four roots $h_1 = 0$; $h_2 = 1$; $h_3 = 2-3m-n$; and $h_4 = 3-3m-n$. Finally, by assuming $A_0 = A$, the coefficients A_1, A_2, etc. of the power expansion (B.12) are obtained recursively by placing $s = 1$, $s = 2$, and so on (i.e. $r = 1(4-2m)$, $r = 2(4-2m), \dots$) into eq. (3.12) and then equating the coefficients of the same powers of z in eq. (3.11): the results for the first two terms are presented in eq. (B.14) and eq. (B.15). After that Kirchhoff states that the general integral of the ODE (B.11) is obtained by choosing h as one of the four roots (h_1, h_2, h_3, h_4) of the indicial equation, selecting for any h defined in this way a different value for the coefficient A and then summing up the results, the complete expression of u is obtained.

3.1.2 Properties of the solution

Kirchhoff then analyzes the solution and, without expanding further the results, makes the following clarifying statements

- The convergent series representing u proceeds by *increasing* powers of z if $m < 2$, by *decreasing* powers of z if $m > 2$;
- In the limiting case $m = 2$ the solution is obtained by the sum of the 4 values that expression (B.16) takes when h is chosen as one of the four roots h_1, h_2, h_3, h_4 of the resulting indicial equation: $g = \alpha^2\lambda^2$, where g is computed for $m = 2$, as shown by eq. (B.17), and A is given a different value for each value of h;
- In cases when two of the given values of h coincide, or when one of the factors of A_1, A_2 disappears, the given form of the general integral loses its validity. The correct solution is then obtained by a sum of power series which are partly multiplied by $\ln z$. The coefficients are then determined by the same procedure.

As a consequence, only one of the two constants governing the beam taper, namely m, which controls the cross-section variation in the plane of vibration, i.e. in the x-z plane, does actually influence the power series solution.

Remark 1.

The outlined solution method practically coincides with what nowadays is known as Frobenius' method (see for instance [23], [24], [6]), which is an improvement of a technique originally developed by Carl G. Neumann (1832–1925) for finding the solutions of Bessel's equation [51]. Ferdinand Georg Frobenius (1849–1917) [21] had already published (in 1873) his fundamental paper in a well-known journal (Journal für die reine und angewandte Mathematik = Journal for pure and applied Mathematics)[5]. Kirchhoff himself had already or would still have published some contributions (like for instance [27], [29], [31], [32] or [36]) on the same journal, but inexplicably he does not make any reference to the work of Frobenius. □

3.2 The analyzed problem, particular cases

Given the general solution, Kirchhoff studies next two particular cases, namely the linearly-varying wedge ($m = 1$ and $n = 0$) and the linearly-varying cone ($m = n = 1$), see Figure 3.

[5] It is also known as *Crelles Journal* from August Leopold Crelle (1780–1855) who founded the journal in 1826 in Berlin and was the journal's first editor until his death.

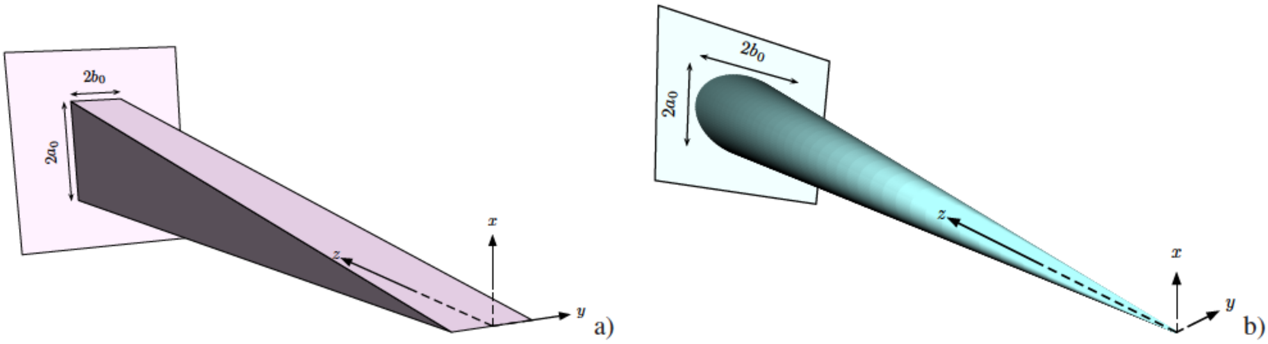

Fig. 3 Particular cases of tapered beams analyzed by Kirchhoff. (a): Rectangular cross-section and wedge-shaped tapered beam ($m = 1$, $n = 0$), i.e. linear taper in the x direction and no taper in the y direction. (b): Circular or, more generally, elliptical cross-section and cone-shaped tapered beam ($m = n = 1$), corresponding to a linear taper in both x and y directions.

For these two considered cases he observes that the fourth-order ODE can be reduced to two second-order ODEs, and precisely to some particular differential equations whose integral are Bessel functions with real or imaginary argument.

This is a convincing proof of his extraordinary ability as an applied mathematician, as already outlined by Helmholtz [68], but on the other hand, his way of proceeding, even though leads him to the correct result, is nevertheless rather hermetic and obscure, as it has been noticed by Todhunter and Pearson (see [65], page 39: "… it must be confessed that Kirchhoff's methods seem, at least to the Editor of the present work, frequently obscure and occasionally wanting in strictness…") .

3.3 First particular case: wedge shaped beam with rectangular cross-section

For the case $m = 1$ and $n = 0$ (see Figure 3(a) and eq. (B.18), i.e. tapered beam with rectangular cross-section) the ODE (B.11) can be written as eq. (B.19), which may be further expanded as follows

$$\frac{1}{z}\frac{d}{dz}z^2\frac{d}{dz}\frac{1}{z}\frac{d}{dz}z^2\frac{du}{dz} = \left(\frac{1}{z}\frac{d}{dz}z^2\frac{d}{dz}\right)\left(\frac{1}{z}\frac{d}{dz}z^2\frac{d}{dz}\right)(u) = \alpha^2\lambda^2 u, \tag{3.13}$$

which is equivalent to eq. (B.20), when position (3.9) is recalled. Then Kirchhoff shows that eq. (3.13) is satisfied by either of the alternatives shown in eq. (B.21) and eq. (B.22), namely

$$\frac{1}{z}\frac{d}{dz}\left(z^2\frac{du}{dz}\right) = \pm u\alpha\lambda, \tag{3.14}$$

which, with the substitution

$$\zeta = z\alpha\lambda, \tag{3.15}$$

see eq. (B.23), splits into the following two ODEs:

$$\zeta\frac{d^2u}{d\zeta^2} + 2\frac{du}{d\zeta} + u = 0; \tag{3.16}$$

$$\zeta\frac{d^2u}{d\zeta^2} + 2\frac{du}{d\zeta} - u = 0, \tag{3.17}$$

corresponding to eq. (B.25) and eq. (B.24) respectively.

Remark 2.

How Kirchhoff could arrive at this result is not completely clear, but could be somehow explained by exploiting the following semi-inverse approach. Introducing the linear differential operator

$$D_z = \frac{1}{z}\frac{d}{dz}z^2\frac{d}{dz},$$

one can tentatively set

$$D_z(u) = \pm\alpha\lambda u. \tag{3.18}$$

Hence, a further application of the previous definition yields

$$D_z^2(u) = \alpha^2\lambda^2 u, \tag{3.19}$$

that is a more concise way of expressing eq. (3.13).
On the other hand, Todhunter and Pearson (see [65], footnote to page 94) suggest that by the change of variable $z' = 1/z$ eq. (3.14) becomes

$$\frac{d^2u}{dz'^2} = \pm\frac{\beta^2 u}{z'^3}, \qquad \beta^2 = \alpha\lambda$$

a particular case of Riccati's equation which may be solved by Bessel's functions, as shown in Forsyth (see [20], §111). Finally, in a paper bearing the *same* title as Kirchhoff's one and published in 1973, Vdovič [66] was able to reconstruct all this procedure and showed the correctness of the presented results by making use of operator calculus. □

3.3.1 Solution method

In order to solve these equations, namely eq. (3.16) and eq. (3.17), Kirchhoff noticed that if one knows a solution, say ψ, of the following ODE:

$$\zeta\frac{d^2\psi}{d\zeta^2} + \frac{d\psi}{d\zeta} + \psi = 0, \tag{3.20}$$

see eq. (B.27), then the $(p-1)$-th derivative of this function ψ, $w = d^{p-1}\psi/d\zeta^{p-1}$, satisfies the following equation,

$$\zeta\frac{d^2w}{d\zeta^2} + p\frac{dw}{d\zeta} + w = 0, \tag{3.21}$$

for any $p \in \mathbb{N}^+$. In particular, eq. (3.16) is a particular case of eq. (3.21) for $p = 2$: this means that $u = d\psi/d\zeta$ is a solution of eq. (3.16). Similar considerations apply to eq. (3.17): if a solution, e.g. φ, is known for the ODE:

$$\zeta\frac{d^2\varphi}{d\zeta^2} + \frac{d\varphi}{d\zeta} - \varphi = 0, \tag{3.22}$$

see eq. (B.26), then its first derivative, $u = d\varphi/d\zeta$ is a solution of eq. (3.17), as well as, $\forall\, p \in \mathbb{N}^+$, its $(p-1)$-th derivative, $w = d^{p-1}\varphi/d\zeta^{p-1}$ satisfies the general ODE:

$$\zeta\frac{d^2w}{d\zeta^2} + p\frac{dw}{d\zeta} - w = 0. \tag{3.23}$$

Notice that eq. (3.20) becomes a particular case of the following ODE:

$$\frac{d^2\psi}{d\zeta^2} + \frac{1-2\mathfrak{a}}{\zeta}\frac{d\psi}{d\zeta} + \left\{(\mathfrak{b}\mathfrak{c}\zeta^{\mathfrak{c}-1})^2 + \frac{\mathfrak{a}^2 - \nu^2\mathfrak{c}^2}{\zeta^2}\right\}\psi = 0 \tag{3.24}$$

by assuming $\mathfrak{a} = 0$, $\mathfrak{b} = 2$, $\mathfrak{c} = 1/2$, $\nu = 0$. According to von Lommel [46] (see also [72], [24]), the previous ODE can be transformed, by a change of both dependent and independent variables of this kind: $\psi = v\zeta^{\mathfrak{a}}$; $t = \zeta^{\mathfrak{c}}$ into the simpler one:

$$t^2\frac{d^2v}{dt^2} + t\frac{dv}{dt} + \left(\mathfrak{b}^2t^2 - \nu^2\right)v = 0, \tag{3.25}$$

which is Bessel's equation in the argument $\mathfrak{b}t$. The general integral of the previous ODE is a linear combination of two independent solutions:

$$v = C_1 J_\nu(\mathfrak{b}t) + C_2 Y_\nu(\mathfrak{b}t), \tag{3.26}$$

where C_1, C_2 are constants, while J_ν and Y_ν are Bessel functions of the first and second kind of order ν, respectively. For every value of ν both J_ν and Y_ν are linearly independent solutions of Bessel's equation (3.25). The same general integral of eq. (3.24), when expressed in the original independent variable becomes:

$$\psi = \zeta^{\mathfrak{a}}\left[C_1 J_\nu(\mathfrak{b}\zeta^{\mathfrak{c}}) + C_2 Y_\nu(\mathfrak{b}\zeta^{\mathfrak{c}})\right]. \tag{3.27}$$

In this respect we remind that the standard definition of Bessel functions of the first kind of order ν expressed as a series in the argument z, with $z \in \mathbb{C}$ is (see, for instance, [1], [44], [60]):

$$J_\nu(z) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(\nu + r + 1)} \left(\frac{z}{2}\right)^{2r+\nu}, \tag{3.28}$$

where Γ is Euler's gamma function. Similarly, Bessel functions of the second kind (also known as Neumann or Weber functions) of order ν have this series representation in the argument z when $\nu \in \mathbb{N}$ (see, e.g. [24], [44], [60]):

$$\begin{aligned} Y_\nu(z) = & \frac{2}{\pi}\left[\ln\frac{z}{2} + \gamma\right] J_\nu(z) - \frac{1}{\pi}\sum_{r=0}^{\nu-1} \frac{(\nu - r - 1)!}{r!} \left(\frac{z}{2}\right)^{2r-\nu} \\ & - \frac{1}{\pi}\sum_{r=0}^{\infty}(-1)^r \frac{\Phi(r) + \Phi(\nu + r)}{r!(\nu + r)!} \left(\frac{z}{2}\right)^{2r+\nu}, \end{aligned} \tag{3.29}$$

where

$$\gamma = \lim_{r\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{r} - \ln r\right) = 0.5772156\ldots \tag{3.30}$$

is the Euler-Mascheroni constant and Φ corresponds to the harmonic series defined as:

$$\Phi(r) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{r}; \qquad \Phi(0) = 0. \tag{3.31}$$

As a consequence, the complete solution of eq. (3.20) is given by:

$$\psi = \left[C_1 J_0(2\sqrt{\zeta}) + C_2 Y_0(2\sqrt{\zeta})\right] = C_1\psi_1 + C_2\psi_2. \tag{3.32}$$

It should be noticed that Kirchhoff does not use a compact notation like that presented in eq. (3.28) and eq. (3.29), but provides the first few terms of the series; in particular, see eq. (B.29), what he calls ψ, is simply $\psi_1 = J_0(2\sqrt{\zeta})$, while instead of $\psi_2 = Y_0(2\sqrt{\zeta})$, he uses a different solution, which is denoted by ψ'. Indeed eq. (B.31) comes out to be a linear combination of $J_0(2\sqrt{\zeta})$ and $Y_0(2\sqrt{\zeta})$ and, being such, it is again an independent solution of eq. (3.20). As it can be checked, it turns out to be:

$$\psi' = \frac{\pi}{2} Y_0(2\sqrt{\zeta}) - 2\gamma J_0(2\sqrt{\zeta}).$$

Remark 3.

It has to be outlined that throughout the paper Kirchhoff uses a prime to denote a different function, and not the first derivative of the given function with respect to the independent variable. □

Similarly to what has been done in eqs. (3.24)–(3.27) for the *same* suitable values of constant parameters $\mathfrak{a} = 0$, $\mathfrak{b} = 2$, $\mathfrak{c} = 1/2$, $\nu = 0$, eq. (3.22) becomes a particular case of an ODE like this:

$$\frac{d^2\varphi}{d\zeta^2} + \frac{1 - 2\mathfrak{a}}{\zeta}\frac{d\varphi}{d\zeta} - \left\{(\mathfrak{b}\mathfrak{c}\zeta^{\mathfrak{c}-1})^2 + \frac{\nu^2\mathfrak{c}^2 - \mathfrak{a}^2}{\zeta^2}\right\}\varphi = 0 \tag{3.33}$$

which can be transformed again (see [24]), by changing both dependent and independent variables in this way: $\varphi = v\zeta^{\mathfrak{a}}$; $\tau = \zeta^{\mathfrak{c}}$, into this ODE:

$$\tau^2\frac{d^2 v}{d\tau^2} + \tau\frac{dv}{d\tau} - \left(\mathfrak{b}^2\tau^2 + \nu^2\right)v = 0, \tag{3.34}$$

which is the modified Bessel equation in the argument $\mathfrak{b}\tau$.

Remark 4.

The modified Bessel equation can be obtained, as a simple check confirms, by substituting in eq. (3.25) $t \to \mathrm{i}\tau$, i.e. by changing the real variable t with the purely imaginary one $\mathrm{i}\tau$; here $\mathrm{i} = \sqrt{-1}$ is the imaginary unit and $\tau \in \mathbb{R}$. Then, as it was recalled by Kirchhoff, the solution of the modified Bessel equation can be thought of as a Bessel function of imaginary argument. □

The general integral of eq. (3.34) is a linear combination of these solutions depending on two constants, D_1 and D_2:

$$v = D_1 I_\nu(\mathfrak{b}\tau) + D_2 K_\nu(\mathfrak{b}\tau), \tag{3.35}$$

or, in the original independent variable,

$$\varphi = \zeta^{\mathfrak{a}} \left[D_1 I_\nu(\mathfrak{b}\zeta^{\mathfrak{c}}) + D_2 K_\nu(\mathfrak{b}\zeta^{\mathfrak{c}}) \right]. \tag{3.36}$$

Differently from eq. (3.27) I_ν and K_ν are *modified* Bessel functions of the first and second kind of order ν, respectively, and, $\forall\,\nu$, are linearly independent solutions of the modified Bessel equation (3.34).

Modified Bessel functions of the first kind of order ν, $I_\nu(z)$, are defined in this standard way (see e.g. [24] or [60]):

$$I_\nu(z) = \sum_{r=0}^{\infty} \frac{1}{r!\Gamma(\nu + r + 1)} \left(\frac{z}{2}\right)^{2r+\nu}, \tag{3.37}$$

and are linked to the corresponding Bessel functions of first kind in this way: $I_\nu(z) = \mathrm{i}^{-\nu} J_\nu(\mathrm{i}z)$. Modified Bessel functions of the second kind of order ν (with $\nu \in \mathbb{N}$), $K_\nu(z)$ are instead defined in this usual way (see e.g. [24] or [60]):

$$\begin{aligned} K_\nu(z) =& (-1)^{\nu+1} \left[\ln\frac{z}{2} + \gamma \right] I_\nu(z) + \frac{1}{2} \sum_{r=0}^{\nu-1} (-1)^r (\nu - r - 1)! \left(\frac{z}{2}\right)^{2r-\nu} \\ &+ \frac{(-1)^\nu}{2} \sum_{r=0}^{\infty} \frac{\Phi(r) + \Phi(\nu + r)}{r!(\nu + r)!} \left(\frac{z}{2}\right)^{2r+\nu}, \end{aligned} \tag{3.38}$$

where γ and $\Phi(r)$ are defined by eqs.(3.30)–(3.31).

In conclusion, the complete solution of eq. (3.22) is given by:

$$\varphi = \left[D_1 I_0(2\sqrt{\zeta}) + D_2 K_0(2\sqrt{\zeta}) \right] = D_1\varphi_1 + D_2\varphi_2. \tag{3.39}$$

As it has been done before, it is possible to check that the first few terms of the series, eq. (B.28) and eq. (B.30) provided by Kirchhoff are related to φ_1 and φ_2 above. Indeed in eq. (B.28), what he simply calls φ, is exactly $\varphi_1 = I_0(2\sqrt{\zeta})$, while instead of $\varphi_2 = K_0(2\sqrt{\zeta})$, he uses a different solution, which is denoted by φ': eq. (B.30) is nothing but a linear combination of $I_0(2\sqrt{\zeta})$ and $K_0(2\sqrt{\zeta})$ and it turns out to be:

$$\varphi' = -Y_0(2\sqrt{\zeta}) - 2\gamma I_0(2\sqrt{\zeta}).$$

which still solves eq. (3.22). At this point, taking advantage of eq. (3.21) and eq. (3.23) Kirchhoff recognizes that the general expression of u, i.e. the solution of eq. (B.20) is given by:

$$u = A_1 \frac{d\varphi}{d\zeta} + A_2 \frac{d\varphi'}{d\zeta} + B_1 \frac{d\psi}{d\zeta} + B_2 \frac{d\psi'}{d\zeta} \tag{3.40}$$

Kirchhoff's solution has been reproduced also by Krienen [45], who in 1959 went through all the derivation by explicitly introducing Bessel functions.

3.3.2 Introduction of boundary conditions

Introducing the boundary conditions in eq. (3.40), Kirchhoff recognizes that, being the pointed edge $\zeta = 0$ free, both bending moment $k(d^2u/d\zeta)$ and shear force $d/d\zeta[k(d^2u/d\zeta)]$ must vanish there, see eq. (B.32); this requires that the two ln-type terms, which are singular at zero, must disappear; hence: $A_2 = 0$ and $B_2 = 0$. Of course this circumstance would not occur in the case of a tapered beam whose shape is a truncated wedge. Then, by setting $A_1 = A$ and $B_1 = B$, u reduces to eq. (B.33). On the other hand at the clamped end $z = l$, i.e. $\zeta = \alpha\lambda l$ both u and $du/d\zeta$ must vanish, see eq. (B.34) and eq. (B.35); however the latter condition, account taken of eq. (B.26) and eq. (B.27), can be replaced by eq. (B.36), and the following homogeneous system of algebraic equations is obtained:

$$\begin{bmatrix} \varphi|_{\zeta=\alpha\lambda l} & -\psi|_{\zeta=\alpha\lambda l} \\ \left.\dfrac{d\varphi}{d\zeta}\right|_{\zeta=\alpha\lambda l} & \left.\dfrac{d\psi}{d\zeta}\right|_{\zeta=\alpha\lambda l} \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{3.41}$$

Non trivial solutions to eq. (3.41) exist provided that the relevant coefficient matrix becomes singular, and this requires this transcendental equation (in the variable λ), which is an equivalent form of eq. (B.37), to be satisfied:

$$\left.\left(\varphi\frac{d\psi}{d\zeta} + \psi\frac{d\varphi}{d\zeta}\right)\right|_{\zeta=\alpha\lambda l} = 0, \tag{3.42}$$

So, eq. (3.42) provides the vibration frequencies λ of the beam; but, as Kirchhoff notices, see eq. (B.38), its l.h.s. can be written also in this way: $d(\varphi\psi)/d\zeta$; as a consequence, vibration modes can be found as the stationary points of the function product $(\varphi\psi)|_{\zeta=\alpha\lambda l}$. However, to avoid multiplying together two power series, Kirchhoff adopts an ingenuous method to find directly the coefficients of the resulting product series. Indeed, see eq. (B.39), he forms the following combinations:

$$\psi\left(\zeta\frac{d^2\varphi}{d\zeta^2} + \frac{d\varphi}{d\zeta} - \varphi\right) - \varphi\left(\zeta\frac{d^2\psi}{d\zeta^2} + \frac{d\psi}{d\zeta} + \psi\right) = 0, \tag{3.43}$$

$$\frac{d\psi}{d\zeta}\left(\zeta\frac{d^2\varphi}{d\zeta^2} + \frac{d\varphi}{d\zeta} - \varphi\right) + \frac{d\varphi}{d\zeta}\left(\zeta\frac{d^2\psi}{d\zeta^2} + \frac{d\psi}{d\zeta} + \psi\right) = 0, \tag{3.44}$$

$$\psi\left(\zeta\frac{d^2\varphi}{d\zeta^2} + \frac{d\varphi}{d\zeta} - \varphi\right) + \varphi\left(\zeta\frac{d^2\psi}{d\zeta^2} + \frac{d\psi}{d\zeta} + \psi\right) = 0, \tag{3.45}$$

and, with some manipulations, he gets respectively eqs. (B.40), (B.41), (B.42). Now, the first two equations (B.40)–(B.41) give immediately, when both sides of the latter are multiplied by ζ and then differentiation with respect to ζ is performed once on both sides:

$$2\varphi\psi = -\frac{d^2}{d\zeta^2}\left(\zeta^2\frac{d\varphi}{d\zeta}\frac{d\psi}{d\zeta}\right), \tag{3.46}$$

while by transforming eq. (3.45) with the help of the identity eq. (B.43), and taking into account that

$$\zeta\frac{d^2}{d\zeta^2}(\varphi\psi) + \frac{d}{d\zeta}(\varphi\psi) = \frac{d}{d\zeta}\left[\zeta\frac{d}{d\zeta}(\varphi\psi)\right],$$

it is possible to express the product $\dfrac{d\varphi}{d\zeta}\dfrac{d\psi}{d\zeta}$, appearing in the r.h.s. of eq. (3.46), as in eq. (B.44), which provides an ODE for the function product $\varphi\psi$. Kirchhoff then looks for a series solution; and inserts an expansion of the kind:

$$\varphi\psi = \sum_{n=0}^{\infty} B_n\zeta^{2n}, \tag{3.47}$$

into eq. (B.45) and then equates the coefficients of the same powers of ζ. Indeed, with the additional assumption $B_0 = 1$, eq. (3.47) coincides with eq. (B.46), where only even powers of the independent variable appear: this is reasonable, since the series expansion of φ, see, e.g., eq. (B.28), only includes terms with alternating signs, while that of ψ, provided by eq. (B.29), only positive terms; hence φ and ψ exhibit, the *same* coefficients (when absolute values are considered) for the corresponding powers of ζ.

The recursion formula which allows one to compute all coefficients B_i, once B_0 is known, is precisely eq. (B.47); hence the sought solution is given by eq. (B.49). Once all terms are multiplied by ζ^2 and eq. (B.45) is fully expanded, it becomes:

$$\zeta^4\frac{d^4(\varphi\psi)}{d\zeta^4} + 5\zeta^3\frac{d^3(\varphi\psi)}{d\zeta^3} + 4\zeta^2\frac{d^2(\varphi\psi)}{d\zeta^2} + 4\zeta^2\varphi\psi = 0. \tag{3.48}$$

This is a fourth order ODE and admits four linearly independent solutions. It is possible to show, however, that only the obtained one is expressible by means of Bessel functions (the other three involve either hypergeometric functions or Meijer G-functions, see e.g. [47] or [3]) and, in particular, it comes out $\varphi\psi = J_0(2\sqrt{\zeta})I_0(2\sqrt{\zeta})$.

The transcendental equation which gives the frequency of vibration is simply obtained by enforcing eq. (B.38); by taking the derivative of eq. (B.49), changing its sign and dividing by ζ to get rid of the physically unfeasible zero solution, it yields, after setting $\zeta_2 = 2\sqrt{\zeta}$:

$$\frac{J_1(\zeta_2)I_0(\zeta_2) - J_0(\zeta_2)I_1(\zeta_2)}{\zeta^{3/2}} = 0, \tag{3.49}$$

whose series expansion is given by eq. (B.50). The smallest positive root of eq. (3.49) gives the fundamental frequency of vibration of the wedge-tapered beam: the value provided by Kirchhoff, $\zeta_0 = \alpha\lambda_0 l = 5.315$ is correct to all four significant digits. This is not always true, as it will appear in subsequent computations: however the lack of any statement about the number of considered series terms, of the number of digits used for performing the computations, etc. makes it impossible to exactly reproduce his way of getting the numerical results.

Consider a rectangular cross-section having at the clamped end depth $2a_0$, and breadth $2b_0$; being

$$q_\ell = q|_{z=l} = 4a_0b_0; \qquad k_\ell = k|_{z=l} = \frac{1}{12}(2a_0)^3 2b_0$$

and $q_\ell = q'l$, $k_\ell = k'l^3$ from eq. (B.10), one has:

$$\frac{q'}{k'} = \alpha^2\frac{E}{\mu} = l^2\frac{q_\ell}{k_\ell} = \frac{3l^2}{a_0^2}, \tag{3.50}$$

taking into account the definition (3.9). Recalling also eq. (B.53), this allows one to express the ratio between the area and the second area moment of the cross-section located at $z = 1$ as a function of the ratio of the corresponding quantities evaluated at the clamped end, $z = l$. Thus, by considering that $\zeta_0 = \alpha\lambda_0 l$, one infers that the fundamental frequency λ_0 can be written as:

$$\lambda_0 = \zeta_0\sqrt{\frac{E}{3\mu}\frac{a_0}{l^2}} \tag{3.51}$$

which corresponds to eq. (B.54).

Once the vibration frequency is known, it is possible to go back to eq. (3.40) in order to evaluate the corresponding vibration mode u. It follows, from the first row of eq. (3.41): $A\,\varphi|_{\zeta=\zeta_0} - B\,\psi|_{\zeta=\zeta_0} = 0$, so that a possible solution is $A = \psi|_{\zeta=\zeta_0} = \psi_0$; $B = \varphi|_{\zeta=\zeta_0} = \varphi_0$. In particular, it follows, with four decimal digits:

$$\varphi_0 = 19.2773; \qquad \psi_0 = -0.2933;$$

which should be compared with Kirchhoff's values of eq. (B.58). Finally, considering that $dJ_0(2\sqrt{\zeta})/d\zeta = -J_1(2\sqrt{\zeta})/\sqrt{\zeta}$; $dI_0(2\sqrt{\zeta})/d\zeta = +I_1(2\sqrt{\zeta})/\sqrt{\zeta}$, the complete solution in terms of the vibration mode can be written as in eq. (B.59), namely:

$$u = -C\left(\frac{\psi_0 I_1(2\sqrt{\alpha\lambda_0 z}) - \varphi_0 J_1(2\sqrt{\alpha\lambda_0 z})}{\sqrt{\alpha\lambda_0 z}}\right), \tag{3.52}$$

where C is a suitable normalization factor.

Remark 5.

Kirchhoff is interested only in evaluating the fundamental frequency and he does not mention higher frequencies of vibration, which can be simply computed by looking for subsequent roots of the same eq. (3.49). This has been done for the first five modes (see Table 1) by means of a Computer Algebra System (CAS), namely Mathematica$^{\text{TM}}$(version 6.0). The roots of the transcendental equation have been computed by using the native function `FindRoot`, [73] which implements a variant of the secants method. Bracketing intervals to isolate roots were defined by properly magnified plots of the corresponding function. The use of a CAS is essential in solving the above mentioned transcendental equation since it exhibits a strongly oscillating behavior, such that a very small deviation in the root value might result in a large error when evaluating the equation itself: this requires algorithms that effectively deal with an extended arbitrary precision. In the present paper, all roots have been computed by assigning variables with 100 digits precision. Moreover, any computed root has been back-substituted in the equation and the associated error ϵ has been checked against a predefined tolerance: it has been verified that all provided roots satisfy the corresponding transcendental equation to within $|\epsilon| \leq 1 \cdot 10^{-100}$. □

For practical reasons, numbers reported hereafter are shown only with 15 significant digits and plots were drawn with the same criterion; interested readers may, however, ask the authors for the original Mathematica notebook to work with an extended arbitrary precision. The corresponding values of φ_0 and ψ_0 entering eq. (3.52) are also given in Table 1, along with the particular value of the normalization factor C which produces, for any vibration mode, a unit deflection at the free end of the beam.

Table 1 First five angular frequencies λ_0 and vibration mode parameters φ_0, ψ_0, C for a tapered beam with one fixed (clamped) and one free end, for the case $m = 1, n = 0$. Results are printed with a precision of 15 digits.

mode	$\lambda_0 = \zeta_0/(\alpha\, l)$	φ_0	ψ_0	C
1	5.31509942365365	19.2773429030318	−0.293327207223605	$-5.10968706930471 \cdot 10^{-2}$
2	15.2071679550051	354.444174527919	+0.215553982937386	$-2.82303558210937 \cdot 10^{-3}$
3	30.0198091456556	7002.87881460655	−0.178464716802568	$-1.42794776640271 \cdot 10^{-4}$
4	49.7633446379036	143701.863210382	+0.155663762623234	$-6.95885955061168 \cdot 10^{-6}$
5	74.4400286512835	3018239.52878180	−0.139836734913753	$-3.31318950710666 \cdot 10^{-7}$

3.3.3 Comparison with a prismatic beam

Vibration frequencies of a clamped-free uniform beam are governed by the transcendental equation (see [4]):

$$\cosh(\sqrt{\alpha\lambda}\, l)\cos(\sqrt{\alpha\lambda}\, l) + 1 = 0. \tag{3.53}$$

Considering a prismatic beam having the same cross-section at the clamped end as the wedge-shaped tapered beam and denoting by $\zeta_0 = \alpha\lambda_0 l$ the smallest root of eq. (3.53), the fundamental frequency is $\lambda_0 = \zeta_0/(\alpha\, l)$; its value, when α is expressed as in eq. (3.50), is given by eq. (B.55), which is correct to four digits.

The corresponding vibration mode is instead:

$$u(z) = C[A_0(\cosh\sqrt{\alpha\lambda_0}\, z + \cos\sqrt{\alpha\lambda_0}\, z) - B_0(\sinh\sqrt{\alpha\lambda_0}\, z + \sin\sqrt{\alpha\lambda_0}\, z)], \tag{3.54}$$

where A_0 and B_0 are amplitude factors, similarly to φ_0 and ψ_0 in eq. (3.52), and C is a normalization factor which has been chosen so as to produce a unit deflection at the free end.

The natural frequencies of the first five vibration modes and the corresponding values of A_0, B_0 and C entering into eq. (3.54) are reported in Table 2. It is apparent that only for the first mode, the only one investigated by Kirchhoff, the frequency of the tapered beam is higher than that of the uniform one.

Table 2 First five angular frequencies λ_0 and vibration mode parameters A_0, B_0, C for a uniform beam (i.e. a tapered beam with $m = 0, n = 0$) having one fixed (clamped) and one free end. Results are printed with a precision of 15 digits.

mode	$\lambda_0 = \zeta_0/(\alpha\, l)$	A_0	B_0	C
1	3.51601526850015	1.00000000000000	.734095513702049	$+5.00000000000000 \cdot 10^{-1}$
2	22.0344915646668	1.00000000000000	1.01846731875921	$-5.00000000000000 \cdot 10^{-1}$
3	61.6972144135547	1.00000000000000	.999224496517428	$+5.00000000000000 \cdot 10^{-1}$
4	120.901916052304	1.00000000000000	1.00003355325171	$-5.00000000000000 \cdot 10^{-1}$
5	199.859530116801	1.00000000000000	0.99999855010865	$+5.00000000000000 \cdot 10^{-1}$

The vibration modes of the wedge-shaped tapered beam and of the prismatic one are compared in Figure 4. Then Kirchhoff addresses another problem, namely that of finding the *maximum amplitude of vibration at the free end such that the longitudinal elastic strain never exceeds the limit value ε_{max} within the beam, when the beam is vibrating at the fundamental frequency*. For a prismatic beam (having the same cross-section as that at the fixed one of the tapered beam, namely with a cross-section whose half-depth is equal to a_0), it is an easy task to show that the maximum strain occurs at top/bottom fibres of the cross-section located at the clamped end. For a wedge-shaped tapered beam, this maximum longitudinal strain occurs still at the top/bottom fibers of the particular cross-section where the following expression attains its maximum value:

$$\varepsilon_{\max} = \frac{d^2u}{dz^2} x_{\max} = \frac{d^2u}{dz^2}\frac{a_0 z}{l}, \tag{3.55}$$

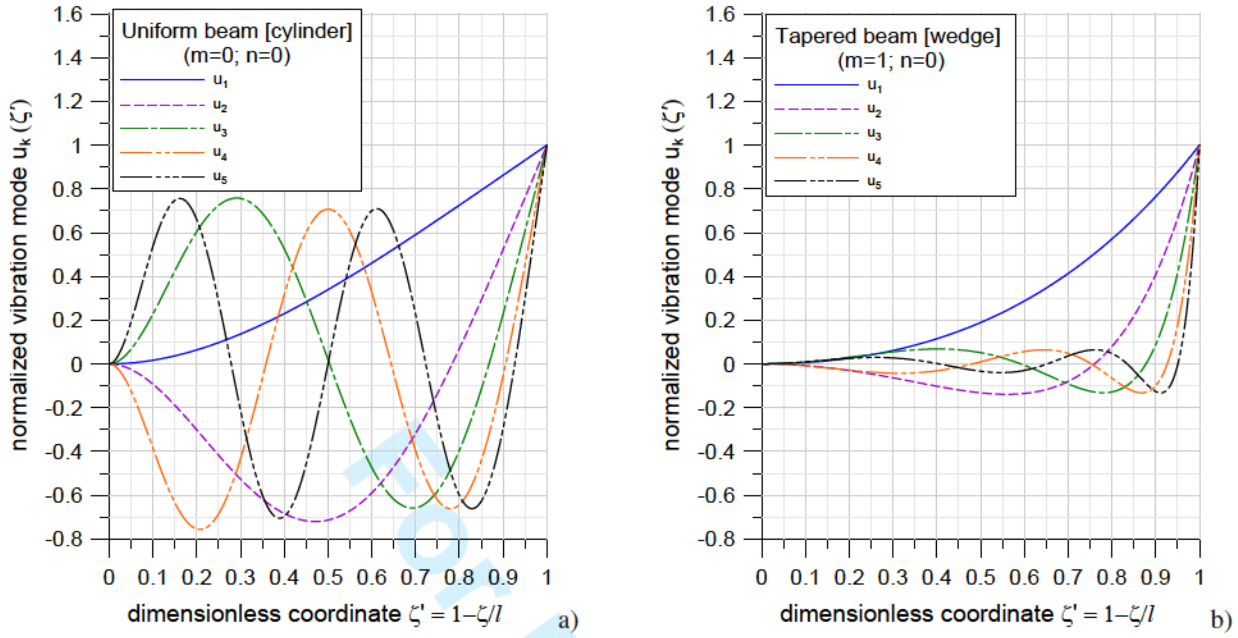

Fig. 4 Normalized vibration shapes corresponding to modes 1–5 for a beam clamped at the left end ($\zeta' = 0$) and free at the right one, ($\zeta' = 1$); (a): prismatic beam; (b): wedge-shaped tapered beam. In both cases the normalization factor has been chosen such that it produces a unit displacement at the free end.

where d^2u/dz^2 is the curvature of the beam, according to Euler-Bernoulli's theory, and $x_{\max} = a_0(z/l)$ is the absolute value of the distance, measured along the x-axis of the top/bottom fiber from the cross-section centroid, see eq. (B.56). So the position, along the beam axis, of the particular cross-section where $\varepsilon_{\max}$ occurs, is defined by the condition:

$$\frac{d}{dz}\left(\frac{a_0 z}{l}\frac{d^2u}{dz^2}\right) = 0, \tag{3.56}$$

where u is defined by eq. (3.52). Then, since $d^2u/dz^2 = (\alpha\lambda_0)d^2u/d\zeta^2$ and $\lambda_0 = \zeta_0/(\alpha l)$, see eq. (B.57), it follows that eq. (3.56) becomes:

$$\frac{d}{d\zeta}\left(\frac{a_0\zeta\zeta_0}{l^2}\frac{d^2u}{d\zeta^2}\right) = 0. \tag{3.57}$$

By expanding eq. (3.57) Kirchhoff provides eq. (B.61), which, once common factors are simplified, is equivalent to:

$$\frac{\varphi_0}{8\zeta^{5/2}}\left[(3\zeta-9)\sqrt{\zeta}J_0(\zeta_2) - (12\zeta-9)J_1(\zeta_2) - (4\zeta-9)\sqrt{\zeta}J_2(\zeta_2) + 4\sqrt{\zeta}J_3(\zeta_2) + \zeta J_4(\zeta_2)\right] +$$
$$\frac{\psi_0}{8\zeta^{5/2}}\left[(3\zeta+9)\sqrt{\zeta}I_0(\zeta_2) - (12\zeta+9)I_1(\zeta_2) + (4\zeta+9)\sqrt{\zeta}I_2(\zeta_2) - 4\sqrt{\zeta}I_3(\zeta_2) + \zeta I_4(\zeta_2)\right] = 0, \tag{3.58}$$

where the shorthand notation $\zeta_2 = 2\sqrt{\zeta}$ has been adopted again. By solving eq. (3.58) it is found that the maximum strain occurs at a position defined by $\zeta_\varepsilon = 3.710$, which has to be compared with Kirchhoff's value, eq. (B.62). In particular, it results $\zeta_\varepsilon/\zeta_0 = 0.698\,l$. The resulting largest strain is then given by:

$$\varepsilon_{\max} = \left(\frac{a_0\zeta\zeta_0}{l^2}\frac{d^2u}{d\zeta^2}\right)\bigg|_{\zeta=\zeta_\varepsilon} = 4.649\,C\frac{a_0\zeta_0}{l^2}, \tag{3.59}$$

compared to which, Kirchhoff's value, provided by eq. (B.63) or eq. (B.64), has almost a 7% relative error. On the other hand, the longitudinal strain of the top/bottom fiber at the *clamped* end, is given by:

$$\varepsilon_{\zeta_0} = \left(\frac{a_0\zeta\zeta_0}{l^2}\frac{d^2u}{d\zeta^2}\right)\bigg|_{\zeta=\zeta_0} = 4.334\,C\frac{a_0\zeta_0}{l^2}, \tag{3.60}$$

and is therefore lower than $\varepsilon_{\max}$. Finally, Kirchhoff evaluates the maximum deflection at the free end, U, corresponding to this maximum strain, and since by eq. (3.52)

$$U = \lim_{z\to 0} u = C(\varphi_0 - \psi_0) = 19.571\, C,$$

he finds that it is possible to eliminate $C = U/(\varphi_0 - \psi_0)$ from eq. (3.59); then it follows:

$$U = 4.209\,\frac{\varepsilon l^2}{a_0\zeta_0}, \tag{3.61}$$

and this has to be compared with Kirchhoff's value, eq. (B.67), which is affected again by a relative error around 7%. In any case Kirchhoff's conclusion that the maximum deflection (corresponding to the same value of the maximum longitudinal strain) of the tapered beam, see eq. (B.68) is about four times larger than that of the prismatic beam is *a fortiori* confirmed.

3.4 Second particular case: cone/pyramid-shaped beam with generic cross-section

For the case $m = 1$ and $n = 1$ (see Figure 3(b) and eq. (B.70), i.e. tapered beam with conical shape) the ODE (B.11) can be written as

$$\alpha^2\lambda^2 u = \frac{1}{z^2}\frac{d}{dz}z^3\frac{d}{dz}\frac{1}{z^2}\frac{d}{dz}z^3\frac{du}{dz}, \tag{3.62}$$

which is equivalent to eq. (B.71), when position (3.9) is recalled. Then Kirchhoff shows that eq. (3.62) is satisfied by either of the alternatives shown in eq. (B.72), namely

$$\frac{1}{z^2}\frac{d}{dz}\left(z^3\frac{du}{dz}\right) = \pm u\alpha\lambda, \tag{3.63}$$

which, with the substitution eq. (3.15), see eq. (B.73), splits into these two ODEs:

$$\zeta\frac{d^2u}{d\zeta^2} + 3\frac{du}{d\zeta} + u = 0; \tag{3.64}$$

$$\zeta\frac{d^2u}{d\zeta^2} + 3\frac{du}{d\zeta} - u = 0, \tag{3.65}$$

corresponding to the alternatives of eq. (B.74).

3.4.1 Solution method

It is possible to recognize that eq. (3.21) and eq. (3.23), for the particular value $p = 3$, coincide with eqs. (3.64)–(3.65); this means that the second derivatives of functions ψ and φ defined by eq. (3.20) and eq. (3.22) respectively do satisfy the same eqs. (3.64)–(3.65). As a consequence, by following the procedure presented in Section 3.3.1 it possible to construct the general solution to eq. (3.62):

$$u = A_1\frac{d^2\varphi}{d\zeta^2} + A_2\frac{d^2\varphi'}{d\zeta^2} + B_1\frac{d^2\psi}{d\zeta^2} + B_2\frac{d^2\psi'}{d\zeta^2} \tag{3.66}$$

3.4.2 Introduction of boundary conditions

Since the pointed edge $\zeta = 0$ is free, both bending moment $k(d^2u/d\zeta)$ and shear force $d/d\zeta[(k(d^2u/d\zeta)]$ must vanish there, see eq. (B.75). As a consequence, the two ln-type terms, which survive to differentiation and are singular at zero, must disappear: this implies: $A_2 = 0$ and $B_2 = 0$. Hence u reduces to eq. (B.76) by setting $A_1 = A$ and $B_1 = B$. However, if the free end $z = l$ (or $\zeta = \alpha\lambda l$) is clamped, both u and $du/d\zeta$ must vanish there, as eq. (B.77) and eq. (B.78) require. On the other hand, by taking the first derivatives of eq. (B.26) and eq. (B.27), it is possible to replace eq. (B.78) with eq. (B.79), and the following homogeneous system of algebraic equations is obtained:

$$\begin{bmatrix} \left.\dfrac{d\varphi}{d\zeta}\right|_{\zeta=\alpha\lambda l} & -\left.\dfrac{d\psi}{d\zeta}\right|_{\zeta=\alpha\lambda l} \\[2ex] \left.\dfrac{d^2\varphi}{d\zeta^2}\right|_{\zeta=\alpha\lambda l} & \left.\dfrac{d^2\psi}{d\zeta^2}\right|_{\zeta=\alpha\lambda l} \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{3.67}$$

Non trivial solutions to eq. (3.67) do exist provided that the relevant coefficient matrix becomes singular, and this requires this transcendental equation (in the variable λ), which is equivalent to eq. (B.80), to be satisfied:

$$\left(\frac{d\varphi}{d\zeta}\frac{d^2\psi}{d\zeta^2}+\frac{d\psi}{d\zeta}\frac{d^2\varphi}{d\zeta^2}\right)\bigg|_{\zeta=\alpha\lambda l}=0, \tag{3.68}$$

eq. (3.68) provides the vibration frequencies λ of the beam but, as Kirchhoff notices, see, e.g., eq. (B.81), its l.h.s. can be written also in this way: $d/d\zeta[(d\varphi/d\zeta)(d\psi/d\zeta)]$; hence, vibration frequencies are the stationary points of the function product $[(d\varphi/d\zeta)(d\psi/d\zeta)]|_{\zeta=\alpha\lambda l}$.

Again, to avoid multiplying two power series, Kirchhoff makes use of eq. (B.44), with the function product $\varphi\psi$ defined by eq. (3.47) and eq. (B.46). Indeed, it follows:

$$\left(\frac{d\varphi}{d\zeta}\frac{d\psi}{d\zeta}\right)=-\frac{I_1(2\sqrt{\zeta})J_1(2\sqrt{\zeta})}{\zeta} \tag{3.69}$$

which is equivalent to eq. (B.82). The transcendental equation which gives the frequency of vibration is obtained by enforcing eq. (B.81); by taking the derivative of eq. (B.82), changing its sign, dividing by 2ζ to get rid of the physically unfeasible zero solution, and adopting the shortcut notation $\zeta_2 = 2\sqrt{\zeta}$, one has

$$-\frac{1}{4\zeta^3}\left\{\sqrt{\zeta}[J_1(\zeta_2)(I_0(\zeta_2)+I_2(\zeta_2))+I_1(\zeta_2)(J_0(\zeta_2)-J_2(\zeta_2))]-2J_1(\zeta_2)I_1(\zeta_2)\right\}=0, \tag{3.70}$$

whose series expansion is given by eq. (B.83). The smallest positive root of eq. (3.70) gives the fundamental frequency of vibration of the cone-tapered beam: the correct value with four significant digits is $\zeta_0 = \alpha\lambda_0 l = 8.719$, while Kirchhoff provides eq. (B.84), a slightly different value. The angular frequency λ_0 is then simply computed by making use of eq. (B.85). Notice that, at the clamped end $z = l$, the outer fibres of the beam cross-section lie at a distance a_0, measured in the direction of the oscillation, from the cross-section centroid. Hence considering also that $q_\ell = q|_{z=l} = q'l^2$; $k_\ell = k|_{z=l} = k'l^4$ on account of eq. (B.10), one has

$$\frac{q'}{k'}=\frac{1}{l^2}\frac{q_\ell}{k_\ell}, \tag{3.71}$$

which corresponds to eq. (B.86) since Kirchhoff defines $q_0 = q_\ell$ and $k_0 = k_\ell$. This allows expressing again the ratio between the area and the second area moment of the cross-section located at $z = 1$ as a function of the ratio of the corresponding quantities evaluated at the clamped end, $z = l$. Thus, by considering that $\zeta_0 = \alpha\lambda_0 l$, it follows that the fundamental frequency λ_0 can be written as in eq. (B.87), showing that it is inversely proportional to the square of the beam length.

Once vibration frequency is known, one may evaluate the corresponding vibration mode, u going back to eq. (3.66). It follows, from the first row of eq. (3.67): $A\ (d\varphi/d\zeta)|_{\zeta=\zeta_0} - B\ (d\psi/d\zeta)|_{\zeta=\zeta_0} = 0$. A possible solution is then $A = (d\psi/d\zeta)|_{\zeta=\zeta_0} = (d\psi/d\zeta)_0$; $B = (d\varphi/d\zeta)|_{\zeta=\zeta_0} = (d\varphi/d\zeta)_0$. In particular, it follows, assuming five significant digits:

$$\left(\frac{d\varphi}{d\zeta}\right)_0 = 19.031; \qquad \left(\frac{d\psi}{d\zeta}\right)_0 = 0.099620; \tag{3.72}$$

which should be compared with Kirchhoff's values of eq. (B.91). Then, considering that

$$\frac{d^2\varphi}{d\zeta^2}=\frac{-I_1(2\sqrt{\zeta})+\sqrt{\zeta}[I_2(2\sqrt{\zeta})+I_0(2\sqrt{\zeta})]}{2\zeta^{3/2}}; \qquad \frac{d^2\psi}{d\zeta^2}=\frac{J_1(2\sqrt{\zeta})+\sqrt{\zeta}[J_2(2\sqrt{\zeta})-J_0(2\sqrt{\zeta})]}{2\zeta^{3/2}}, \tag{3.73}$$

the complete solution in terms of the vibration mode can be written as in eq. (B.92), more precisely:

$$\begin{aligned} u=\frac{C}{2(\alpha\lambda_0 z)^{3/2}}\Bigg\{&\left(\frac{d\psi}{d\zeta}\right)_0[-I_1(2\sqrt{\alpha\lambda_0 z})+\sqrt{\alpha\lambda_0 z}(I_2(2\sqrt{\alpha\lambda_0 z})+I_0(2\sqrt{\alpha\lambda_0 z}))]+\\ &\left(\frac{d\varphi}{d\zeta}\right)_0[J_1(2\sqrt{\alpha\lambda_0 z})+\sqrt{\alpha\lambda_0 z}(J_2(2\sqrt{\alpha\lambda_0 z})-J_0(2\sqrt{\alpha\lambda_0 z}))]\Bigg\}. \end{aligned} \tag{3.74}$$

where C is a suitable normalization factor.

Remark 6.

Kirchhoff is interested only in evaluating the fundamental frequency and he does not mention higher frequencies of vibration, which can be simply computed by looking for subsequent roots of eq. (3.70). This has been done for the first five modes (see Table 3), as in the previously presented case. Reference values may be compared with those provided by [15]. In Table 3 also the corresponding values of $(d\varphi/d\zeta)_0$ and $(d\psi/d\zeta)_0$ entering eq. (3.74) are given, and the particular value of the normalization factor C which produces, for any vibration mode, a unit deflection at the free end of the beam. □

Table 3 First five angular frequencies λ_0 and vibration mode parameters $(d\varphi/d\zeta)_0$, $(d\psi/d\zeta)_0$, C for a tapered beam with one fixed (clamped) and one free end, for the case $m = 1$, $n = 1$. Results are printed with a precision of 15 digits.

mode	$\lambda_0 = \zeta_0/(\alpha l)$	$(d\varphi/d\zeta)_0$	$(d\psi/d\zeta)_0$	C
1	8.71925885507992	19.0311180121041	+0.0996198251914283	$+1.04543798415395 \cdot 10^{-1}$
2	21.1456623878687	270.306035232624	−0.0473872881082891	$+7.40031823296399 \cdot 10^{-3}$
3	38.4537712277326	4307.29019431664	+0.0290899050614729	$+4.64325922465574 \cdot 10^{-4}$
4	60.6801387750973	73856.3296625232	−0.0201780837819134	$+2.70796092298841 \cdot 10^{-5}$
5	87.8339912946009	1330802.38808128	+0.0150508382920721	$+1.50285271148661 \cdot 10^{-6}$

3.4.3 Comparison with a cylindrical beam

For a prismatic or cylindrical beam having the same cross-section at the clamped end as the cone-shaped tapered beam the fundamental frequency is simply: $\lambda_0 = \zeta_0/(\alpha\, l)$, if $\zeta_0 = \alpha\lambda_0 l$ denotes the smallest root of eq. (3.53). When α, provided by eq. (3.9), is expressed through eq. (3.71), the value of λ_0 is given by eq. (B.88).

In order to evaluate again the maximum amplitude of vibration at the free end such that *maximum longitudinal strain never exceeds the elastic limit value within the beam*, it is found that such maximum strain, defined by eq. (B.89) does not occur at the clamped end, but at a position ζ_ε defined by the condition (B.90), which, again, depends on the vibration frequency; the position of the cross-section where the maximum strain is attained is defined by eq. (B.93), which, once common factors are simplified, becomes, when $\zeta_2 = 2\sqrt{\zeta}$:

$$\begin{aligned}\left(\frac{d\psi}{d\zeta}\right)_0 \frac{1}{16\zeta^{7/2}}\{(-3\sqrt{\zeta}(25+8\zeta)I_0(\zeta_2)+(75+99\zeta+10\zeta^2)I_1(\zeta_2)-\sqrt{\zeta}(75+32\zeta)I_2(\zeta_2)+\\ (33\zeta+55\zeta^2)I_3(\zeta_2)-8\zeta^{3/2}I_4(\zeta_2)+\zeta^2 I_5(\zeta_2))\}+\\ \left(\frac{d\varphi}{d\zeta}\right)_0 \frac{1}{16\zeta^{7/2}}\{(3\sqrt{\zeta}(25-8\zeta)J_0(\zeta_2)-(75-99\zeta+10\zeta^2)J_1(\zeta_2)-\sqrt{\zeta}(75-32\zeta)J_2(\zeta_2)-\\ (33\zeta-5\zeta^2)J_3(\zeta_2)-8\zeta^{3/2}J_4(\zeta_2)-\zeta^2 J_5(\zeta_2))\}=0\end{aligned} \tag{3.75}$$

By solving eq. (3.75) it is found that the maximum strain occurs at a position defined by $\zeta_\varepsilon = 4.402$; this has to be compared with Kirchhoff's value, eq. (B.94), which is affected by a relative error around 1%. It follows that the position of the cross-section where maximum longitudinal strain occurs is defined by the ratio $\zeta_\varepsilon/\zeta_0 = 0.505l$. The resulting largest longitudinal strain is then, see eq. (B.95):

$$\varepsilon_{\max} = \left(\frac{a_0\zeta_0}{l^2}\zeta\frac{d^2u}{d\zeta^2}\right)\Bigg|_{\zeta=\zeta_\varepsilon} = 1.380\, C\frac{a_0\zeta_0}{l^2}, \tag{3.76}$$

which is comparable with Kirchhoff's value, eq. (B.95). The longitudinal strain of the top/bottom fibre at the *clamped* end, is instead given by:

$$\varepsilon_{\zeta_0} = \left(\frac{a_0\zeta_0}{l^2}\zeta\frac{d^2u}{d\zeta^2}\right)\Bigg|_{\zeta=\zeta_0} = 0.9749\, C\frac{a_0\zeta_0}{l^2}, \tag{3.77}$$

and is therefore lower than $\varepsilon_{\max}$. Finally, Kirchhoff evaluates the maximum deflection at the free end, U, corresponding to this maximum strain, and since by eq. (3.72) and eq. (3.74):

$$U = \lim_{z\to 0} u = (C/2)[(d\varphi/d\zeta)_0 + (d\psi/d\zeta)_0] = 9.565\, C,$$

he finds that it is possible to eliminate $C = 2U/[(d\varphi/d\zeta)_0 + (d\psi/d\zeta)_0]$ from eq. (3.76). It follows, then:

$$U = 6.933\,\frac{\varepsilon l^2}{a_0\zeta_0}, \tag{3.78}$$

and this has to be compared with Kirchhoff's value, eq. (B.98), which is affected by a relative error less than 1%. So Kirchhoff's conclusion that the maximum deflection (corresponding to the same value of the maximum longitudinal strain) of the conical tapered beam, see eq. (B.100), is about seven times larger than that of the cylindrical beam, eq. (B.101), is precisely confirmed. In Figure 5 (*left*) the normalized shapes of vibration for the first five modes are presented for the cone-shaped tapered beam: these shapes should be compared with those, shown in Figure 4, of the uniform beam and of the wedge-shaped tapered beam. In Figure 5 (*right*) the maximal longitudinal strain at each cross-section (as a function of the normalized coordinate ζ/ζ_0) has been plotted for the wedge- and for the cone-shaped tapered beam, when they are oscillating according to the fundamental mode (i.e. the first vibration mode). It is apparent that the ***maximum*** of such longitudinal strains does not occur at the clamped end, corresponding to $\zeta/\zeta_0 = 1$ but at a specific location, ζ_ε, which is different for the two considered cases.

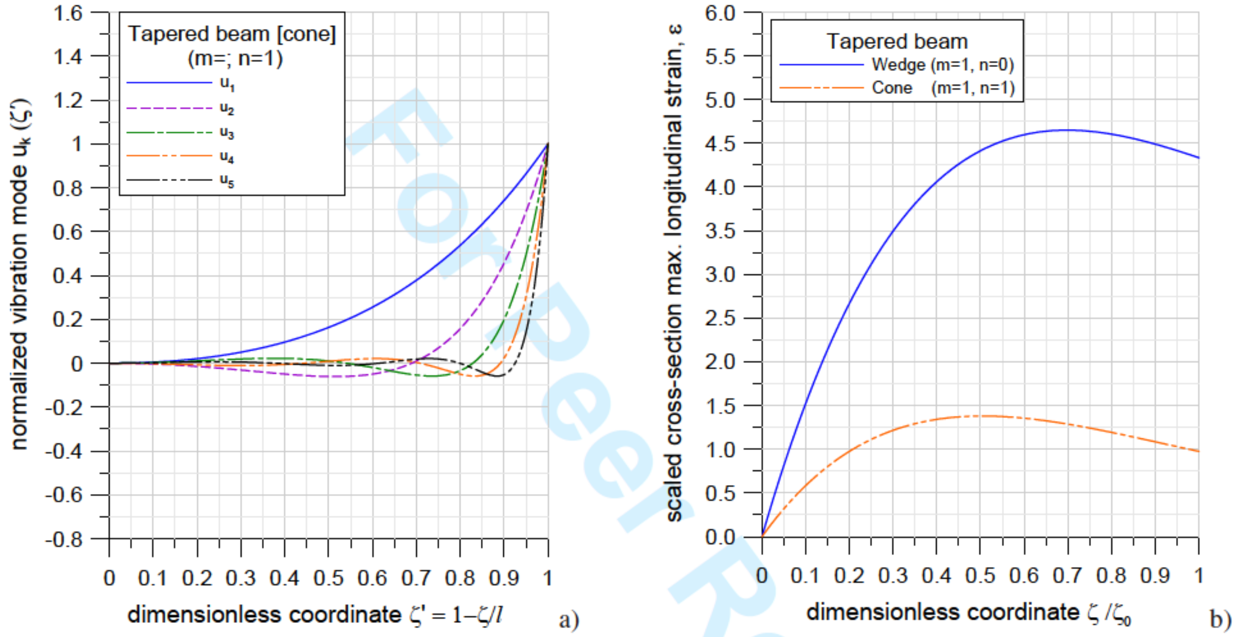

Fig. 5 (a): Normalized vibration shapes corresponding to modes 1–5 for a cone-shaped tapered beam clamped at the left end, ($\zeta' = 0$), and free at the right one, ($\zeta' = 1$); the normalization factor has been chosen such that it produces a unit displacement at the free end. (b): Comparison of maximal longitudinal strain at each cross-section as a function of the normalized coordinate ζ/ζ_0 for a wedge- and for a cone-shaped tapered beam when they oscillate according to the first i.e. the *fundamental* vibration mode.

4 Kirchhoff legacy in the theory of vibration of tapered beams

In the 90 years after 1880, when his contribution was published for the first time, many extensions to Kirchhoff's theory have been presented: a partial list of the more interesting ones is briefly discussed in the sequel. The interested reader can find a short but rather complete historic excursus up to 1965 in the paper by Wang [69]. In particular, in the first years after the appearance of Kirchhoff's essay, tapered beams, whether pointed or truncated, like in the case of a frustum, had been mainly a research topic for Mathematical Physicists; instead, in the years following WWII the prevalent interest of aircraft applications led many engineers to deal with this challenging topic, which is still an active area of research.

The first known contribution after Kirchhoff's appeared in 1888 and was authored by F. Meyer zur Capellen [49], who studied some other particular cases, like that of a beam with constant depth and variable width, and provided also the vibration frequencies of higher-order modes. Other noteworthy contributions in the field of Mathematical Physics came from Morrow [50], Ward [71], Nicholson [52] and [53], Wrinch [74] and [75], and Ono [54]. Among them a particular mention deserves, Dorothy Wrinch (1894–1976), a female scientist and the first woman to receive a D.Sc. from Oxford; her fame is mostly due to the research work she did after 1932 on the the mathematical modeling of the structure of proteins and cells, but in the early years she worked mainly on classical topics like mathematical logic and applied mathematics.

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A Kirchhoff's paper on the dynamics of tapered beams: versions, structure and translation notes

In this Section some notes about the different versions of the paper, as well about its structure, translation and editing are provided, to allow the interested reader to compare the original German texts and the English translation.

There exist three versions of the same paper with minimal differences, mostly misprint corrections, but having a different number of pages, due to different typing and compositions, namely:

1. the 1879 version [34] (14 pages), which appeared on the "Monatsberichte der königlich preussischen Akademie der Wissenschaften zu Berlin", (*Monthly reports of the Royal Prussian Academy of Sciences at Berlin*), and will be referred shortly by MAW; the full-text[6] can be freely downloaded by following the given link.

2. the 1880 version [35] (12 pages), which appeared in the "Annalen der Physik und Chemie", (*Annals of Physics and Chemistry*), also known (between 1877 and 1899) by the name of the editor-in-chief, Gustav Heinrich Wiedemann (1826–1899), as "Wiedemanns Annalen"; note that volume 237 of the whole collection corresponds to Wid. Ann. 1, while the last one, Wid. Ann. 69 corresponds to volume 305. This version will be referred shortly by AdP; its full-text[7] may be freely retrieved by following the given link.

3. the 1882 version [37] (13 pages), which was included by Kirchhoff himself in his "Gesammelte Abhandlungen", (*Collected essays*), pages 339–351; this last version will be simply referred to as GA and its full-text[8],[9] may be retrieved at one of the given links.

The three versions exhibit very small differences, mostly related to different typographic conventions: for instance, MAW and GA do not have punctuation marks before displayed equations, while AdP does. In the translation the convention used by MAW has been adopted. In any case, there is no equation numbering, no subdivisions into sections, and only one interruption is marked; moreover, only two references are mentioned: J.W. Strutt (Lord Rayleigh: 1842–1919) [63] and a work by Kirchhoff himself [27].

Language recalls often acoustic or music theory expressions (*Quinte* = fifth, *Grundton* = fundamental tone, etc.) since most motivations for studying structural vibration problems were coming from the need of understanding the production of sound: this was indeed the first aim of both Chladni [8] and Lord Rayleigh [63].

The German language has steadily evolved since Kirchhoff's times, and the spelling of some words has changed. To provide some examples, *Theil* is now spelled *Teil*, *Hülfe* is replaced by *Hilfe*, *cylindrisch* is written as *zylindrisch*, *Coordinatensystem*, *Excursion* are substituted by *Koordinatensystem*, *Exkursion* and *Coëfficient* becomes *Koeffizient*. Similarly, verbal forms like *variirt* are now spelled as *variiert*, etc.

For ease of reference, all beginnings of a new page have been marked with the source text (within brackets) followed by the page number, e.g. [MAW: 817] denotes the beginning of page 817 in the MAW text. In the presented translation, again for the sake of convenience, all equations, which are unnumbered in the original text, have been given a number. Some minor misprints, which are still standing in all versions of the paper, have been corrected, e.g. the wrong use of partial derivatives instead of ordinary ones in MAW: 820, lines 2 and 3 from top, the missing index i in symbol B, MAW: 821, one line above eq. (B.46), the missing denominator in the r.h.s of eq. (B.47), or the denominator of the last term appearing in the r.h.s. of eq. (B.93). Additions to the text to make it more intelligible are denoted by angle brackets, as follows: <. . .>. For the same reason, the original footnotes by Kirchhoff are numbered with *letters*, while footnotes denoted by *arabic numbers* are comments added by the authors.

A different problem arises since Kirchhoff used the same symbol x with two different meanings: in the text and in eqs. (B.1), (B.9) as a coordinate measured along the principal inertia axis corresponding to the direction of oscillation; in eq. (B.23) and following as a properly scaled coordinate measured along the beam length. Of course, the use of different meaning for the same symbol might create confusion in the reader and for this reason, following the notation employed by Todhunter and Pearson [65] the scaled coordinate defined by the above mentioned eq. (B.23) has been substituted by the symbol ζ, which replaces x in all following occurrences. For similar reason, to avoid using with different meaning the same symbol in the comments, what Kirchhoff denotes by a (e.g. the half-depth of the cross-section at the clamped end $z = l$), see eq. (B.53) and following, has been replaced by a_0.

Finally, the standard dot notation has been adopted for decimal numbers, i.e. one tenth is represented as 0.10, while in all versions of the original paper Kirchhoff made use of the German comma notation ($1/10 = 0{,}10$).

[6] https://de.wikisource.org/wiki/Monatsberichte_der_Königlich_Preussischen_Akademie.

[7] https://de.wikisource.org/wiki/Annalen_der_Physik.

[8] https://books.google.it/books?isbn=1143798961.

[9] https://archive.org/details/gesammelteabhan01unkngoog.

B On the transversal vibrations of a beam of variable cross-section by G. Kirchhoff

The transversal vibrations of cylindrical beams are theoretically and experimentally treated in detail; the vibrations of a beam whose cross-section is variable are not however, [GA: 340] up to now, more closely investigated, even though, besides the mathematical interest which they deserve, they possess in this respect a practical one, too, because for a beam which oscillates with a free end, the amplitude of vibration of this end can be much larger, without exceeding the elasticity limit, when toward this end the beam is tapered, than when the cross-section is everywhere the same. The following considerations are referred to a beam which forms a prism or a cone with an extremely small angle, with the edge or the sharp tip at the free end.

For the moment a beam is taken into consideration, whose cross-section, which has arbitrary shape, only varies in the direction of the length such that cross-sections become infinitesimal, their centroids lie along a straight line and their principal axes have the same directions. A beam like that can carry out small oscillations, by which displacements in one of these two directions <namely x or y> occur; on such oscillations attention is focused; the differential equation itself is known[a] and is easily deduced with the help of Hamilton's principle.

Let the line, which the centroids of the cross-sections form in the equilibrium position, be the z-axis of an orthogonal coordinate system, and let the direction of the principal axis of a cross-section, which happens to be parallel to the oscillations, be the direction of the x-axis. Let moreover be [MAW: 816; AdP: 502]

$$q = \iint dx dy \ , \quad k = \iint x^2 dx dy \ , \tag{B.1}$$

the integrations extended over the corresponding cross-section depending on the variable z, ξ the displacement of the centroid of this cross-section as a function of time t, μ the density, E the elastic coefficient of the material of the beam; then the living force[10] is

$$\frac{\mu}{2} \int dz \, q \left(\frac{\partial \xi}{\partial t} \right)^2 \tag{B.2}$$

and the potential energy of the beam

$$\frac{E}{2} \int dz k \left(\frac{\partial^2 \xi}{\partial z^2} \right)^2 , \tag{B.3}$$

[GA: 341] the integrations being extended along the length of the beam. It follows from here the partial differential equation

$$q\mu \frac{\partial^2 \xi}{\partial t^2} = -E \frac{\partial^2}{\partial z^2} \left(k \frac{\partial^2 \xi}{\partial z^2} \right) , \tag{B.4}$$

and, if at both ends of the beam no forces act, which produce work, i.e., when the ends are free or fixed, it follows further, that for each end

$$\frac{\partial}{\partial z} \left(k \frac{\partial^2 \xi}{\partial z^2} \right) \delta \xi \quad \text{and} \quad k \frac{\partial^2 \xi}{\partial z^2} \delta \frac{\partial \xi}{\partial z} \tag{B.5}$$

do vanish.

We limit ourselves to the analysis of oscillations by which the beam produces one simple vibration mode, hence one can set

$$\xi = u \sin \lambda t \ , \tag{B.6}$$

where u represents a function of z, and λ is a constant.

For u one has therefore the ordinary differential equation

$$q\mu\lambda^2 u = E \frac{d^2}{dz^2} \left(k \frac{d^2 u}{dz^2} \right) \tag{B.7}$$

[a] The theory of sound by John William Strutt, London 1877, Vol. I, page 240.

[10] "Lebendige Kraft" i.e. *Living force* is the old-fashioned name for *kinetic energy* in German. From a historic point of view, it is very interesting that Kirchhoff still used this expression instead of kinetic energy.

and the <boundary> condition, that at each end

$$\frac{d}{dz}\left(k\frac{d^2u}{dz^2}\right)\delta u \quad \text{and} \quad k\frac{d^2u}{dz^2}\delta\frac{du}{dz} \tag{B.8}$$

do vanish.

[MAW: 817] The general integral of this differential equation is obtained without difficulty when the change of the cross-section is such that the equation of its contour is an equation between these variables

$$\frac{x}{z^m} \quad \text{and} \quad \frac{y}{z^n}\,, \tag{B.9}$$

where m and n represent two constants. If the values of q and k for $z = 1$ are defined by q' and k', then

$$q = q'z^{m+n}, \qquad k = k'z^{3m+n}, \tag{B.10}$$

[AdP: 503] hence the differential equation

$$q'\mu\lambda^2 z^{m+n}u = Ek'\frac{d^2}{dz^2}\left(z^{3m+n}\frac{d^2u}{dz^2}\right). \tag{B.11}$$

An integral of this equation is obtained by setting

$$u = Az^h + A_1 z^{h+(4-2m)} + A_2 z^{h+2(4-2m)} + \cdots \tag{B.12}$$

where h is determined by the 4th-degree equation

$$h(h-1)(h-2+3m+n)(h-3+3m+n) = 0 \tag{B.13}$$

[GA: 342] and the coefficients $A_1, A_2, \ldots$ by the equations

$$\begin{aligned}\frac{q'u\lambda^2}{k'E}A =& A_1(h+4-2m)(h-1+4-2m)\\ &(h-2+4-2m+3m+n)(h-3+4-2m+3m+n)\end{aligned} \tag{B.14}$$

$$\begin{aligned}\frac{q'u\lambda^2}{k'E}A_1 =& A_2(h+2(4-2m))(h-1+2(4-2m))\\ &(h-2+2(4-2m)+3m+n)(h-3+2(4-2m)+3m+n)\end{aligned} \tag{B.15}$$

and so on. If one chooses the values for h one after another according to the 4 values $0, 1, 2-3m-n, 3-3m-n$, gives to the arbitrary constant A different values, and forms the sum of the obtained expressions for u, then one gets the general integral of the mentioned differential equation. The convergent series by which the same general integral is represented proceed by increasing or decreasing powers of z, according to m being smaller or larger [MAW: 818] than 2. In the limiting case $m = 2$, u is equal to the sum of the 4 values which the expression

$$Az^h \tag{B.16}$$

takes, when one places inside h a root of the 4th-degree equation

$$h(h-1)(h+4+n)(h+3+n) = \frac{q'\mu\lambda^2}{k'E} \tag{B.17}$$

and chooses the arbitrary constant A always different. Even in other cases the developed form of the general integral of the differential equation loses its validity, i.e. when two of the indicated values for h become equal to each other, or when one of the factors within brackets, which appear with $A_1, A_2, \ldots$ in the equations <which have been> established for these quantities, [AdP: 504] disappears. A valid form of the integral is obtained then, when one thinks of the value of m changing by an extremely small amount; then one finds it as a sum of power-series which are partly multiplied by $\ln z$; the coefficients can be found as well directly from the differential equation.

From here on, only the cases with $m = 1, n = 0$ or $m = 1, n = 1$ will be treated. In any of these cases the 4th-order differential equation can be reduced to 2nd-order differential equations [GA: 343] whose integral are Bessel's functions with real or imaginary argument.

Let be now

$$m = 1\,, \qquad n = 0\,; \tag{B.18}$$

this occurs when the beam is delimited in the width direction by 2 parallel planes, and in the thickness[11] by 2 planes making each other an infinitesimal angle at the tip, hence when the beam forms a very sharp prism. The differential equation is then

$$\frac{q'\mu\lambda^2}{k'E} z u = \frac{d^2}{dz^2} z^3 \frac{d^2 u}{dz^2} \tag{B.19}$$

or, what is the same,

$$\frac{q'\mu\lambda^2}{k'E} u = \frac{1}{z}\frac{d}{dz} z^2 \frac{d}{dz}\frac{1}{z}\frac{d}{dz} z^2 \frac{du}{dz}. \tag{B.20}$$

[MAW: 819] <The equation> holds, when

$$\frac{1}{z}\frac{d}{dz} z^2 \frac{du}{dz} = u\lambda\sqrt{\frac{q'\mu}{k'E}}\,, \tag{B.21}$$

and also, when

$$\frac{1}{z}\frac{d}{dz} z^2 \frac{du}{dz} = -u\lambda\sqrt{\frac{q'\mu}{k'E}}\,. \tag{B.22}$$

It follows from here that, setting

$$z\lambda\sqrt{\frac{q'\mu}{k'E}} = \zeta\,, \tag{B.23}$$

the general integral of the differential equation valid for u is equal to the sum of the general integrals of the differential equations [AdP: 505]

$$\zeta\frac{d^2u}{d\zeta^2} + 2\frac{du}{d\zeta} = u \tag{B.24}$$

$$\zeta\frac{d^2u}{d\zeta^2} + 2\frac{du}{d\zeta} = -u\,. \tag{B.25}$$

Now let φ and ψ be certain integrals of the equations

$$\zeta\frac{d^2\varphi}{d\zeta^2} + \frac{d\varphi}{d\zeta} = \varphi \tag{B.26}$$

$$\zeta\frac{d^2\psi}{d\zeta^2} + \frac{d\psi}{d\zeta} = -\psi\,, \tag{B.27}$$

with

$$\varphi = 1 + \frac{\zeta}{1^2} + \frac{\zeta^2}{(1\cdot 2)^2} + \frac{\zeta^3}{(1\cdot 2\cdot 3)^2} + \cdots \tag{B.28}$$

$$\psi = 1 - \frac{\zeta}{1^2} + \frac{\zeta^2}{(1\cdot 2)^2} - \frac{\zeta^3}{(1\cdot 2\cdot 3)^2} + \cdots\,, \tag{B.29}$$

[GA: 344] let φ' and ψ' be additional integrals of the same equation, namely

$$\varphi' = \varphi\ln\zeta - 2\left(\frac{\zeta}{1^2} + \frac{\zeta^2(1+\frac{1}{2})}{(1\cdot 2)^2} + \frac{\zeta^3(1+\frac{1}{2}+\frac{1}{3})}{(1\cdot 2\cdot 3)^2} + \cdot\right) \tag{B.30}$$

$$\psi' = \psi\ln\zeta + 2\left(\frac{\zeta}{1^2} - \frac{\zeta^2(1+\frac{1}{2})}{(1\cdot 2)^2} + \frac{\zeta^3(1+\frac{1}{2}+\frac{1}{3})}{(1\cdot 2\cdot 3)^2} - \cdot\right)\,; \tag{B.31}$$

[11] Kirchhoff uses the German word "Dicke" i.e. *thickness* to denote the depth of the beam, namely the dimension of the beam cross-section measured in the plane where vibrations occur.

[MAW: 820] the general expression for u is then the sum of the differential quotients $\frac{d\varphi}{d\zeta}$, $\frac{d\varphi'}{d\zeta}$, $\frac{d\psi}{d\zeta}$, $\frac{d\psi'}{d\zeta}$, which are multiplied by arbitrary constants.

For one end of the beam let z, and hence ζ, be infinitesimally small, and let this end be free; then for an infinitesimally small ζ

$$\zeta^3 \frac{d^2 u}{d\zeta^2} \qquad \text{and} \qquad \frac{d}{d\zeta}\zeta^3 \frac{d^2 u}{d\zeta^2} \tag{B.32}$$

must vanish; this occurs, when the coefficients of $\frac{d\varphi'}{d\zeta}$, $\frac{d\psi'}{d\zeta}$ in the expression of u are set equal to zero, hence u appears as

$$u = A\frac{d\varphi}{d\zeta} + B\frac{d\psi}{d\zeta}. \tag{B.33}$$

Let the second end of the beam be constrained in such a way, that for it u and $\frac{du}{dz}$, hence also $\frac{du}{d\zeta}$, must vanish; for this end it is then

$$0 = A\frac{d\varphi}{d\zeta} + B\frac{d\psi}{d\zeta} \tag{B.34}$$

and

$$0 = A\frac{d^2\varphi}{d\zeta^2} + B\frac{d^2\psi}{d\zeta^2}\ , \tag{B.35}$$

[AdP: 506] hence also, according to the differential equations, which φ and ψ satisfy,

$$0 = A\varphi - B\psi\ , \tag{B.36}$$

therefore

$$0 = \varphi\frac{d\psi}{d\zeta} + \psi\frac{d\varphi}{d\zeta} \tag{B.37}$$

or

$$0 = \frac{d(\varphi\psi)}{d\zeta}\ . \tag{B.38}$$

This is the equation from which the values of λ are to be determined, i.e. the oscillation numbers of the vibration modes[12] which the beam [MAW: 821; GA: 345] can produce.

For this development it can be useful <adopting> the method which I have used in a more general case in my work on the vibrations of a circular plate[b]. If the differential equations for φ and ψ are multiplied

$$\begin{array}{cccccc} \text{by} & \psi & \text{or by} & \frac{d\psi}{d\zeta} & \text{or by} & \psi \\ & -\varphi & & \frac{d\varphi}{d\zeta} & & \varphi \end{array} \tag{B.39}$$

and added every time, then one obtains

$$2\varphi\psi = \frac{d}{d\zeta}\zeta\left(\psi\frac{d\varphi}{d\zeta} - \varphi\frac{d\psi}{d\zeta}\right) \tag{B.40}$$

$$\psi\frac{d\varphi}{d\zeta} - \varphi\frac{d\psi}{d\zeta} = -\frac{1}{\zeta}\frac{d}{d\zeta}\zeta^2\frac{d\varphi}{d\zeta}\frac{d\psi}{d\zeta} \tag{B.41}$$

$$\zeta\left(\psi\frac{d^2\varphi}{d\zeta^2} + \varphi\frac{d^2\psi}{d\zeta^2}\right) + \frac{d\varphi\psi}{d\zeta} = 0\ . \tag{B.42}$$

[12] Kirchhoff uses here the word “Töne” i.e. *tones, sounds*, which is borrowed from music to denote vibration modes corresponding to different frequencies.

[b] Crelle’s Journal, Vol. 40. <page 51, 1850>.

If the last of these equations is transformed by means of the identity

$$\frac{d^2(\varphi\psi)}{d\zeta^2} = \psi\frac{d^2\varphi}{d\zeta^2} + \varphi\frac{d^2\psi}{d\zeta^2} + 2\frac{d\varphi}{d\zeta}\frac{d\psi}{d\zeta}\,, \tag{B.43}$$

then it becomes

$$\frac{d\varphi}{d\zeta}\frac{d\psi}{d\zeta} = \frac{1}{2\zeta}\frac{d}{d\zeta}\zeta\frac{d(\varphi\psi)}{d\zeta}\,. \tag{B.44}$$

Thereof, it results for $\varphi\psi$ the fourth-order differential equation

$$4\varphi\psi = -\frac{d^2}{d\zeta^2}\zeta\frac{d}{d\zeta}\zeta\frac{d(\varphi\psi)}{d\zeta}\,, \tag{B.45}$$

and this determines the coefficients B_i in the equation

$$\varphi\psi = 1 + B_1\zeta^2 + B_2\zeta^4 + B_3\zeta^6 + \cdots\,, \tag{B.46}$$

which immediately follows from the expressions of φ and ψ. [AdP: 507] One finds [MAW: 822]

$$B_n = -\frac{B_{n-1}}{n^2\cdot(2n-1)\cdot 2n}\,, \tag{B.47}$$

and, if one defines

$$1{\cdot}2{\cdot}3{\cdot}\ldots n \quad \text{through} \quad n! \tag{B.48}$$

<it follows>

$$\varphi\psi = 1 - \frac{\zeta^2}{(1!)^2 2!} + \frac{\zeta^4}{(2!)^2 4!} - \frac{\zeta^6}{(3!)^2 6!} + \cdots\,. \tag{B.49}$$

[GA: 346] The equation, which has to be used for the determination of the vibration frequencies, is therefore

$$0 = 1 - \frac{\zeta^2}{(2!)^2 3!} + \frac{\zeta^4}{(3!)^2 5!} - \frac{\zeta^6}{(4!)^2 7!} + \cdots\,. \tag{B.50}$$

Let ζ_0 be the smallest positive root of this equation, which provides the fundamental frequency[13] of the beam. Without difficulty one finds:

$$\zeta_0 = 5.315\,. \tag{B.51}$$

The length of the beam is l, so that

$$l\lambda\sqrt{\frac{q'\mu}{k'E}} = \zeta_0\,; \tag{B.52}$$

from which the value of λ for the fundamental frequency can be computed. Let $2a_0$ be the thickness of the beam at the clamped end; it is then

$$\frac{q'}{k'} = \frac{3l^2}{a_0^2}\,, \tag{B.53}$$

and hence

$$\lambda = 5.315\sqrt{\frac{E}{3\mu}\frac{a_0}{l^2}}\,. \tag{B.54}$$

For the prism-shaped beam therefore, like for the parallelepiped one, the oscillation number of the fundamental frequency is inversely proportional to the square of the length and directly proportional to the thickness, when the thickness is measured

[13] Again, reference is made to music, as Kirchhoff writes literally "Grundton", i.e. *fundamental sound.*

at the clamped end. For equal values of a_0 and l the fundamental frequency of the prismatic beam is higher than that of the parallelepipedal; for the latter it is indeed

$$\lambda = 3.516\sqrt{\frac{E}{3\mu}\frac{a_0}{l^2}} \, , \tag{B.55}$$

so that the fundamental frequency of the prismatic beam is approximately the fifth[14] of the fundamental frequency of the parallelepiped.

[MAW: 823] Now it will be examined how large the amplitudes of oscillation of the free end of the prismatic end might be, when [AdP: 508] the magnitude of strain must not exceed anywhere a given limit.

The maximum of the strain in any cross-section occurs when the beam has experienced its largest bending deflection at the upper or [GA: 347] at the lower side, and this maximum is equal to the absolute value of

$$\frac{a_0 z}{l}\frac{d^2u}{dz^2} \tag{B.56}$$

i.e. of

$$\frac{a_0\zeta_0}{l^2}\zeta\frac{d^2u}{d\zeta^2} \, . \tag{B.57}$$

This expression gets, when ζ increases from 0 to ζ_0, a maximum for a particular value of ζ which must be computed. Denote the values of φ and ψ for $\zeta = \zeta_0$ by φ_0 and ψ_0; it is then

$$\varphi_0 = 19.2772 \qquad \psi_0 = -0.2934 \tag{B.58}$$

and one can set

$$u = -C\left(\varphi_0\frac{d\psi}{d\zeta} + \psi_0\frac{d\varphi}{d\zeta}\right) \, , \tag{B.59}$$

where C is a constant. The condition for the sought maximum is therefore

$$0 = \varphi_0\frac{d}{d\zeta}\zeta\frac{d^3\psi}{d\zeta^3} + \psi_0\frac{d}{d\zeta}\zeta\frac{d^3\varphi}{d\zeta^3} \tag{B.60}$$

or

$$0 = \varphi_0\left(\frac{1}{3!} - \frac{2\zeta}{1!4!} + \frac{3\zeta^2}{2!5!} - \frac{4\zeta^3}{3!6!} + \cdots\right) - \psi_0\left(\frac{1}{3!} + \frac{2\zeta}{1!4!} + \frac{3\zeta^2}{2!5!} + \frac{4\zeta^3}{3!6!} + \cdots\right) \, . \tag{B.61}$$

The smallest root of this equation, and the only one lying between 0 and ζ_0, is

$$= 3.688 \, . \tag{B.62}$$

[MAW: 824] For this value of ζ it is

$$\zeta\left(\varphi_0\frac{d^3\psi}{d\zeta^3} + \psi_0\frac{d^3\varphi}{d\zeta^3}\right) = -4.992 \, . \tag{B.63}$$

For $\zeta = \zeta_0$ the same expression is $= -4.333$ .. If the largest strain is denoted by ε, then it is

$$\varepsilon = C\frac{a_0\zeta_0}{l^2}4.992 \, . \tag{B.64}$$

[AdP: 509] Now let U be the largest elongation[15] of the free end of the beam; hence

$$U = C(\varphi_0 - \psi_0) \tag{B.65}$$

[14] Namely, with terms borrowed again from music, in a 3 : 2 ratio to the fundamental frequency of the parallelepipedal beam.

[15] Kirchhoff denotes by "Elongation" i.e. *elongation* what is better defined by the word *deflection*.

[GA: 348] which means

$$= C{\cdot}19.563\,, \tag{B.66}$$

so that

$$U = \varepsilon\frac{l^2}{a_0\zeta_0}3.919\,, \tag{B.67}$$

or by substitution into the equation which determines λ,

$$U = \varepsilon\frac{1}{\lambda}\sqrt{\frac{E}{3\mu}}{\cdot}3.919\,. \tag{B.68}$$

For the vibrations corresponding to the fundamental frequency of the parallelepipedal beam one finds the maximum strain at the clamped end, and between this maximum and the largest deflection at the free end there exists the relationship

$$U = \varepsilon\frac{1}{\lambda}\sqrt{\frac{E}{3\mu}}\,. \tag{B.69}$$

From here one sees, that for equal material and equal period of oscillation, the prismatic beam can produce deflection amplitudes about 4 times larger than the parallelepiped.

—ooo—

[MAW: 825] Now, in a similar way, it will be treated the case in which the beam forms a very pointed cone. The differential equation of its vibrations is then, according to the previous observations

$$\frac{q'\mu\lambda^2}{k'E}z^2u = \frac{d^2}{dz^2}z^4\frac{d^2u}{dz^2}\,. \tag{B.70}$$

This can be written as

$$\frac{q'\mu\lambda^2}{k'E}u = \frac{1}{z^2}\frac{d}{dz}z^3\frac{d}{dz}\frac{1}{z^2}\frac{d}{dz}z^3\frac{du}{dz}\,, \tag{B.71}$$

and it is satisfied when one sets

$$\frac{1}{z^2}\frac{d}{dz}z^3\frac{du}{dz} = \pm u\lambda\sqrt{\frac{q'\mu}{k'E}}\,. \tag{B.72}$$

If once more ζ is like

$$\zeta = z\lambda\sqrt{\frac{q'\mu}{k'E}}\,, \tag{B.73}$$

[GA: 349] then the general expression of u is the sum of the general integrals of the two differential equations

$$\zeta\frac{d^2u}{d\zeta^2} + 3\frac{du}{d\zeta} = \pm u\,. \tag{B.74}$$

[AdP: 510] If the symbols φ, ψ, φ', ψ' are used with the same meaning as above, then u is a homogeneous linear function of $\frac{d^2\varphi}{d\zeta^2}$, $\frac{d^2\psi}{d\zeta^2}$, $\frac{d^2\varphi'}{d\zeta^2}$, $\frac{d^2\psi'}{d\zeta^2}$, whose coefficients are arbitrary constants. But one end of the beam has to be free and z must be infinitesimally small at that end; consequently, for an infinitesimally small ζ

$$\zeta^4\frac{d^2u}{d\zeta^2} \quad\text{and}\quad \frac{d}{d\zeta}\zeta^4\frac{d^2u}{d\zeta^2} \tag{B.75}$$

must vanish; this requires that the coefficients of $\frac{d^2\varphi'}{d\zeta^2}$ and of $\frac{d^2\psi'}{d\zeta^2}$ are set equal to zero. From that one has

$$u = A\frac{d^2\varphi}{d\zeta^2} + B\frac{d^2\psi}{d\zeta^2}\,. \tag{B.76}$$

[MAW: 826] For the second end of the beam let again be $u = 0$ and $\frac{du}{dz} = 0$, which means

$$A\frac{d^2\varphi}{d\zeta^2} + B\frac{d^2\psi}{d\zeta^2} = 0 \tag{B.77}$$

$$A\frac{d^3\varphi}{d\zeta^3} + B\frac{d^3\psi}{d\zeta^3} = 0\ ; \tag{B.78}$$

for the same <end> it must be also

$$A\frac{d\varphi}{d\zeta} - B\frac{d\psi}{d\zeta} = 0\ , \tag{B.79}$$

so that

$$\frac{d\varphi}{d\zeta}\frac{d^2\psi}{d\zeta^2} + \frac{d\psi}{d\zeta}\frac{d^2\varphi}{d\zeta^2} = 0 \tag{B.80}$$

or

$$\frac{d}{d\zeta}\frac{d\varphi}{d\zeta}\frac{d\psi}{d\zeta} = 0\ . \tag{B.81}$$

For the given development of $\varphi\psi$ it follows then

$$-\frac{d\varphi}{d\zeta}\frac{d\psi}{d\zeta} = 1 - \frac{\zeta^2}{1!2!3!} + \frac{\zeta^4}{2!3!5!} - \frac{\zeta^6}{3!4!7!} + \cdots\ ; \tag{B.82}$$

and hence the equation to be satisfied for the clamped end is [GA: 350]

$$0 = \frac{1}{2!3!} - \frac{\zeta^2}{1!3!5!} + \frac{\zeta^4}{2!4!7!} - \frac{\zeta^6}{3!5!9!} + \cdots\ . \tag{B.83}$$

If again ζ_0 is defined as the smallest root of this equation, then the fundamental frequency of the beam gives

$$\zeta_0 = 8.718\ . \tag{B.84}$$

The value of z for the clamped end of the beam is again l; hence, also here one has

$$l\lambda\sqrt{\frac{q'\mu}{k'E}} = \zeta_0\ . \tag{B.85}$$

[AdP: 511] If the values of q and k for $z = l$ are defined by q_0 and k_0, then

$$\frac{q'}{k'} = \frac{q_0}{k_0}l^2. \tag{B.86}$$

[MAW: 827] From here it follows that

$$\lambda = 8.718\sqrt{\frac{k_0E}{q_0\mu}\frac{1}{l^2}}\ . \tag{B.87}$$

Therefore also here the frequency of oscillations of the fundamental mode is inversely proportional to the square of the length, provided that the cross-sections at the clamped end are equal in both cases. For a cylindrical beam, clamped only at one end, for which q and k assume the values q_0 and k_0, and having length l, the fundamental frequency is

$$\lambda = 3.516\sqrt{\frac{k_0E}{q_0\mu}\frac{1}{l^2}}\ , \tag{B.88}$$

so that the frequencies of the fundamental mode for the conical and the cylindrical beam behave like $8.718 : 3.516$.

For what concerns the strains in the conical beam, their maximum in any cross-section is

$$\frac{a_0\zeta_0}{l^2}\zeta\frac{d^2u}{d\zeta^2}\,, \tag{B.89}$$

if a_0 denotes the maximum distance, in the direction of the oscillation, of the outer fibre of the cross-section from its centroid. Hence the maximum occurs for a value of ζ which satisfies the equation

$$0 = \frac{d}{d\zeta}\zeta\frac{d^2u}{d\zeta^2}\,. \tag{B.90}$$

For $\zeta = \zeta_0$ it is [GA: 351]

$$\frac{d\varphi}{d\zeta} = 19.024 \qquad \frac{d\psi}{d\zeta} = 0.099534\,, \tag{B.91}$$

from here one has

$$u = C\left(0.09953\frac{d^2\varphi}{d\zeta^2} + 19.024\frac{d^2\psi}{d\zeta^2}\right), \tag{B.92}$$

and that equation <namely (B.90)> is

$$0 = 0.09953\left(\frac{1}{4!} + \frac{2\zeta}{1!5!} + \frac{3\zeta^2}{2!6!} + \cdots\right) + 19.024\left(\frac{1}{4!} - \frac{2\zeta}{1!5!} + \frac{3\zeta^2}{2!6!} - \cdots\right). \tag{B.93}$$

[MAW: 828; AdP: 512] The smallest root of this equation is

$$\zeta = 4.464\,. \tag{B.94}$$

For this value of ζ it is

$$\frac{1}{C}\zeta\frac{d^2u}{d\zeta^2} = 1.388\,. \tag{B.95}$$

For $\zeta = \zeta_0$ the same expression is $= 0.9734$. Again, let ε denote the maximum magnitude of strain, then one gets

$$\varepsilon = C\cdot\frac{a_0\zeta_0}{l^2}\cdot 1.338\,. \tag{B.96}$$

Let U be the largest deflection of the free end of the beam, so it is

$$U = C\cdot 9.592\,, \tag{B.97}$$

therefore

$$U = \varepsilon\cdot\frac{l^2}{a_0\zeta_0}\cdot 6.889 \tag{B.98}$$

or, since

$$\frac{l^2}{\zeta_0} = \frac{1}{\lambda}\sqrt{\frac{k_0E}{q_0\mu}}\,, \tag{B.99}$$

$$U = \varepsilon\frac{1}{\lambda}\frac{1}{a_0}\sqrt{\frac{k_0E}{q_0\mu}}\cdot 6.889\,. \tag{B.100}$$

For a cylindrical beam, whose clamped end has the same dimensions, <the largest deflection at the free end> for the fundamental frequency is

$$U = \varepsilon\frac{1}{\lambda}\frac{1}{a_0}\sqrt{\frac{k_0E}{q_0\mu}}\,, \tag{B.101}$$

such that for equal materials and equal periods of the oscillations the conical beam might produce amplitudes of oscillation at the free end about 7 times larger than the cylindrical one.

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