

## A CLASS OF FOURTH-ORDER PARABOLIC SYSTEMS WITH TIME DEPENDENT COEFFICIENTS WITH BLOW-UP SOLUTIONS

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**ABSTRACT.** This paper deals with a class of initial-boundary value problems for nonlinear fourth-order parabolic systems with time dependent coefficients in a bounded domain. We establish conditions on the shape of the spatial domain and on data sufficient to guarantee that the solution blows up in finite time  $t^*$ , deriving an upper bound for  $t^*$ . Moreover, by introducing suitable conditions on the source terms, we obtain a lifespan of the solution, i.e. a time interval  $[0, T]$ , where the solution exists and remains bounded, by deriving a lower bound  $T$  of the blow-up time  $t^*$ .

**1. Introduction.** We deal with the following fourth-order parabolic system with time dependent coefficients

$$\begin{aligned} u_t + \delta_1(t)\Delta^2 u - h_1(t)\Delta u &= k_1(t)f_1(v), & x \in \Omega, & t \in (0, t^*), \\ v_t + \delta_2(t)\Delta^2 v - h_2(t)\Delta v &= k_2(t)f_2(u), & x \in \Omega, & t \in (0, t^*), \end{aligned} \quad (1)$$

$$(u, v) = 0, \quad \left( \frac{\partial u}{\partial n}, \frac{\partial v}{\partial n} \right) = 0, \quad x \in \partial\Omega, \quad t \in (0, t^*), \quad (2)$$

$$(u(x, 0), v(x, 0)) = (u_0(x), v_0(x)), \quad x \in \Omega, \quad (3)$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ ,  $N \geq 2$ , with smooth boundary  $\partial\Omega$ ,  $\frac{\partial}{\partial n}$  is the outward normal derivative on the boundary  $\partial\Omega$ ,  $f_1$  and  $f_2$  are non-negative functions,  $t^*$  is the blow-up time of the solution, and  $u_0(x), v_0(x)$  are non-negative and regular functions in  $\Omega$ ,  $\delta_i, k_i, h_i$   $i = 1, 2$ , are in general positive bounded functions of  $t$ .

In the literature, the majority of the results concern the case of only one equation, and this is because fourth-order parabolic equations describe a variety of physical processes (see [6]). In particular, in thin-film theory (see [8], [10], [14]), such problems describe the evolution of the epitaxial growth of thin, and the following equation is introduced:

$$u_t + \Delta^2 u - A_1 \Delta u - A_2 \operatorname{div}(|\nabla u|^2 \nabla u) = g(x, t, u) \quad (4)$$

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where  $u(x, t)$  is the height from the surface of the film,  $\Delta^2 u$  denotes the capillarity-driven surface diffusion,  $A_1 \Delta u$  describes the diffusion due to evaporation-condensation,  $A_2 \operatorname{div}(|\nabla u|^2 \nabla u)$  represents the upward hopping of atoms, and  $g$  corresponds to the mean deposition flux. In [10], King, Stein, and Winkler showed for the solutions of (4) existence, uniqueness, and regularity in an appropriate function space. Xu et al. in [20] investigated equation (4) with  $A_2 = 0$  and  $g(x, t, u) = g(u)$  in a bounded domain in  $\mathbb{R}^N$ , and, by using the potential well method, showed that the solutions exist globally or blow-up in finite time, depending on whether or not the initial data are in the potential well. When the Laplacian term in (4) is replaced by p-Laplacian and  $g(u)$  is of power type, with a modified potential well method, Han in [8] again obtained global existence and finite time blow-up for the solutions when the initial energy is subcritical and critical. By different methods, Philippin in [16], with  $A_1 = A_2 = 0$ ,  $g(u) = k(t)|u|^{p-1}u$ , proved that the solution cannot exist for all time, i.e. it blows up in  $L^2$ -norm, and an upper bound for  $t^*$  is derived; in addition, he constructed (under certain conditions on the data) a lower bound for  $t^*$  by using a first-order differential inequality method. Escudero, Gazzola, and Peral in [4] proved existence and blow-up results for the solutions of the equation

$$u_t + \Delta^2 u = \det(D^2 u) + \lambda h(x, t), \quad (x, t) \in \Omega \times [0, T]$$

under Dirichlet boundary conditions, and  $\Omega \subset \mathbb{R}^2$ , which models epitaxial growth processes, and where the evolution is dictated by the competition between the determinant of the Hessian matrix of the solution and the bilaplacian (see also [3] and [21]).

For the equation

$$u_t + \Delta^2 u = |\det(D^2 u)|^{\frac{1}{2}} + f(u),$$

in [11], the author proved that there exists a time interval  $[0, T]$  where the solution is bounded in  $W_0^{2,2}(\Omega)$ -norm (and in  $L^2(\Omega)$ -norm), when  $\Omega \subset \mathbb{R}^2$ ,  $f(u)$  is of power type and  $\|u_0\|_{W_0^{2,2}} > 0$ ,  $T$  is a lower bound of the blow-up time.

Winkler in [19] considered the equation

$$u_t = -\Delta^2 u - \mu \Delta u - \lambda \Delta(|\nabla u|^2) + f(x)$$

when  $\Omega$  is a bounded convex domain in  $\mathbb{R}^N$  under the conditions  $\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0$  on the boundary  $\partial \Omega$  with bounded initial data, and proved the existence of global weak solutions with spatial dimension  $N \leq 3$  under suitable conditions on data.

To our knowledge, although the fourth-order parabolic equations have been extensively studied, the results on initial value parabolic fourth-order systems with variable coefficients in higher dimensions are still limited. This is the main motivation of the present work.

In a Sobolev-space setting, Au in [1] considered a fourth-order parabolic system of  $M$  equations of the following form:

$$\begin{cases} \frac{\partial}{\partial t} u_i - \nabla \cdot (a_i(x, t) \nabla u_i) + b \Delta^2 u_i \\ = \sum_{j=1}^M (c_{ij} |\nabla u_j|^2 + V_{ij}(u_i) |u_i|^{\alpha_j - 1} u_j), & (x, t) \in \Omega \times (0, T), \\ u_i = \Delta u_i = 0, & (x, t) \in \partial \Omega \times (0, T), \\ u_i = u_{i0}(x), & x \in \Omega, \end{cases} \quad (5)$$

where  $i = 1, \dots, M$ ,  $a_i$  and  $b$  are the diffusion coefficients, and  $u_{i0}$  are given functions. For the power-type nonlinearities containing the sub-critical Sobolev exponents, Au proved the local existence, a unique continuation, global

existence and finite-time blow-up of solutions to the system in a time-weighted Sobolev space. Moreover, the author investigated logarithmic nonlinearities containing critical Sobolev exponents.

Moreover, we cite [17], Section 4, where Philippin and Vernier-Piro investigated a simplified version of the system (1)-(3), i.e. with  $\delta_i(t) = 1, h_i(t) = 0$ , and  $\Omega$  a bounded domain in  $\mathbb{R}^N$ , with  $N = 2$  or  $N = 3$ . They proved that there exists a safe interval of existence  $[0, T]$ , with  $T$  a lower bound of  $t^*$ , by introducing suitable functionals which satisfy first-order differential inequalities, from which upper and/or lower bounds of the blow-up time are derived. The method is also present in the investigation of some classes of second-order parabolic systems (see i.e. [12], [13], [15]).

Our aim is to investigate the behavior near the blow-up time of the solutions in the case of a class of systems as (1)-(3), where the dimension of the spatial domain  $\Omega$  is  $N \geq 2$ : We introduce conditions on data and on a spatial domain sufficient to prove that the solution must blow up in finite time  $t^*$ , deriving an upper bound of  $t^*$ ; moreover, we prove that there exists an interval where the solution remains bounded (lifespan), by deriving a lower bound of  $t^*$ .

In this context, we consider non-negative solutions of the system (1) - (3), motivated by our aim to investigate the behavior of the solutions which approach to  $+\infty$  as  $t$  approaches the finite blow-up time  $t^*$ . For the definition of solutions to the system (1)-(3), we extend the definition in [4], Theorem 3.1: For some  $T > 0$ ,  $f_1$  and  $f_2 \in L^2(0, T; L^2(\Omega))$ , and  $u_0$  and  $v_0 \in W_0^{2,2}(\Omega)$ , the functions  $u$  and  $v$  belong to the space

$$C(0, T; W_0^{2,2}(\Omega)) \cap L^2(0, T; W^{4,2}(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega)).$$

Throughout the paper, we assume that the initial data  $u_0$  and  $v_0 \in W_0^{2,2}(\Omega)$ . We denote by  $\|\cdot\|_r$  the  $L^r(\Omega)$  norm for  $1 \leq r \leq \infty$ , and by  $\|u\|_{W_0^{2,2}(\Omega)} = \|\Delta u\|_2 = (\int_{\Omega} (\Delta u)^2 dx)^{\frac{1}{2}}$  the norm in  $W_0^{2,2}(\Omega)$ .

We now state a known inequality, which will be needed in the proofs of our results. From the Rellich-Kondrachov theorem on the embedding  $W^{p,m} \subset L^r$  (see [5] Theorem 2.4), the following lemma is derived for  $p = 2$ .

**Lemma 1.1.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ . Let  $r$  be an arbitrary number with  $2 \leq r < +\infty$  if  $N \leq 4$ , and  $2 \leq r < \frac{2N}{N-4}$  if  $N > 4$ . Then, for any  $w \in W_0^{2,2}(\Omega)$ , there exists a constant  $S = S(r, \Omega)$  such that*

$$\|w\|_r \leq S \|\Delta w\|_2. \quad (6)$$

In the first part of this paper, with the aim to derive conditions to have blow-up phenomena for the solution, we introduce the functional

$$\Psi(t) := \int_{\Omega} u \phi_1 dx + \int_{\Omega} v \phi_1 dx = \Psi_1(t) + \Psi_2(t), \quad (7)$$

with

$$\Psi(0) = \Psi_0 = \int_{\Omega} u_0 \phi_1 dx + \int_{\Omega} v_0 \phi_1 dx = \Psi_1(0) + \Psi_2(0) \quad (8)$$

and  $\phi_1$  the first eigenfunction associate to the first eigenvalue  $\Lambda_1$  of the biharmonic eigenvalue problem with Dirichlet boundary conditions:

$$\Delta^2 \phi = \Lambda \phi, \quad x \in \Omega \subset \mathbb{R}^N, \quad N \geq 2, \quad (9)$$

$$\phi = 0, \quad \frac{\partial \phi}{\partial n} = 0, \quad x \in \partial \Omega, \quad (10)$$

with  $\phi$  normalized by

$$\|\phi\|_2^2 = 1. \quad (11)$$

For all  $\phi \neq 0$ ,  $\Lambda_1$  satisfies

$$\|\phi\|_2^2 \leq \Lambda_1^{-1} \|\Delta\phi\|_2^2, \quad (12)$$

(see [9]). Problem (9) is closely related to the biharmonic differential equation

$$\Delta^2\phi = f$$

with the same boundary conditions (10), which describes characteristic vibrations of a clamped plate. It is well-known that the biharmonic operator under Dirichlet boundary conditions does not satisfy the positivity preserving property in general domains; nevertheless, it holds when  $\Omega = B_R$  the ball in  $\mathbb{R}^N$  with radius  $R$ . We remark that the balls are not the only sets where the property holds. (See e.g. [5], [7] and [18]). As a consequence, the functional  $\Psi(t)$  defined in (7) and (8) is non-negative.

**Remark 1.2.** We note that, although fourth-order parabolic equations do not satisfy a classical maximum principle, non-negativity of solutions is preserved under the present boundary conditions. Indeed, the fourth-order operator with Dirichlet boundary conditions generates a positivity-preserving semigroup, and since the nonlinearities  $f_1, f_2$  are non-negative (with polynomial growth) on  $[0, \infty)$ , non-negative initial data yield non-negative solutions for all  $t > 0$  (see [2]).

We now prove the following theorem where for technical reasons we consider a domain  $\Omega = B_R$ , but the argument extends to any domain where the required positivity-preserving property holds (see [7]).

**Theorem 1.3.** *(Conditions for blow-up and an upper bound).*

Let  $\Omega = B_R$  be the  $N$ -dimensional ball,  $N \geq 2$ . Let  $(u, v)$  be a non-negative solution of the system (1)-(3) with  $h_i = 0$ , and let  $\Psi(t)$  and  $\Psi_0$  be defined in (7)-(8). Assume

$$f_1(s) \geq s^p, \quad f_2(s) \geq s^q, \quad s > 0, \quad \text{with } p > q > 1, \quad (13)$$

and let  $c, \delta$ , and  $Q$  be positive constants such that the initial data are so large that

$$H(\Psi_0) > 0, \quad \text{with } H(\Psi(t)) := -\Lambda_1\delta\Psi + 2^{1-q}c\Psi^q - cQ, \quad (14)$$

where  $\Lambda_1$  is the first eigenvalue of the problem (9)-(11). Then,  $\Psi$  blows up in a finite time  $t^*$  in the sense that

$$\lim_{t \rightarrow t^*} \Psi(t) = \infty,$$

with the upper bound

$$T_0 := \int_{\Psi_0}^{\infty} \frac{d\eta}{H(\eta)} \geq t^*. \quad (15)$$

In the particular case  $q = p > 1$ , we achieve the following corollary.

**Corollary 1.4.** *Under the hypotheses of Theorem 1.3, with  $q = p > 1$ , if the initial data  $(u_0, v_0)$  satisfy the condition*

$$\Psi_0 > \left( \frac{\delta\Lambda_1}{2^{1-p}c} \right)^{\frac{1}{p-1}}, \quad (16)$$

then  $\Psi(t)$  must blow up at time  $t^* \leq \bar{T}$  with

$$\bar{T} := -\frac{1}{(p-1)\delta\Lambda_1} \log \left\{ 1 - \frac{\delta\Lambda_1}{2^{1-p}c\Psi_0^{p-1}} \right\}. \quad (17)$$

The second investigation of the paper is to determine a safe time interval of existence of the solution  $(u, v)$  of (1)-(3) (before the blow-up time), say  $[0, T]$ , with  $T$  a lower bound of  $t^*$ . We remark that in this investigation,  $\Omega$  is not assumed to be a ball, only a bounded domain in  $\mathbb{R}^N, N \geq 2$ . We introduce the functional

$$\Phi(t) = \|\Delta u\|_2^2 + \|\Delta v\|_2^2 = \Phi_1(t) + \Phi_2(t), \quad (18)$$

with initial value

$$\Phi(0) := \Phi_0 = \|\Delta u_0\|_2^2 + \|\Delta v_0\|_2^2. \quad (19)$$

**Theorem 1.5.** (*Lower Bound*). *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ . Let  $(u, v)$  be a non-negative solution of the system (1)-(3), which blows up in finite time  $t^*$ , and let  $\Phi(t)$  be defined in (18)-(19). Assume*

$$f_1(s) \leq s^p, \quad f_2(s) \leq s^q, \quad s > 0, \quad \text{with } p \geq q > 1, \quad (20)$$

where  $p, q$  satisfy condition (6) for the index  $r$  in Lemma 1.1. Then, there exist positive constants  $A, \bar{A}, B$  (defined in (49), (51) and (46)) such that  $\Phi(t)$  remains bounded in the interval  $[0, T]$ , with  $T < t^*$  ( $T$  is a lower bound of  $t^*$ ) and

$$\begin{cases} T := \frac{1}{B(p-1)} \ln \left[ 1 + \frac{B}{2\bar{A}} \Phi_0^{1-p} \right], & \text{if } p > q, \\ T := \frac{1}{B(p-1)} \ln \left[ 1 + \frac{B}{\bar{A}} \Phi_0^{1-p} \right], & \text{if } q = p. \end{cases} \quad (21)$$

**Remark 1.6.** We point out that the constants present in Theorem 1.3, Corollary 1.4, and Theorem 1.5 depend on the coefficients in system (1), as well as on the constants in the Hölder, Young and Rellich inequalities. These constants are fundamental in the proof of the results.

The scheme of this paper is the following: In Section 2, we consider system (1)-(3) when the spatial domain is an  $N$ -dimensional ball  $B_R(0)$ ,  $R > 0$  the radius of the ball centered at the origin. We introduce conditions on the data sufficient to guarantee that blow-up will occur and under these conditions we derive an upper bound for the blow-up in finite time. We prove Theorem 1.3 and Corollary (1.4).

In Section 3, we obtain in Theorem 1.5 a lifespan of the solution, showing that it remains bounded in  $W_0^{2,2}$ -norm on some time interval  $(0, T)$ .  $T$  provides a lower bound for blow-up time  $t^*$ . There are particular cases where  $T$  may be explicitly computed in terms of the data of system (1). We point out that as a consequence, the  $L^2$ -norm of the solution also remains bounded in  $(0, T)$ , since from the eigenvalue theory (see [8]), we have

$$\|u\|_2^2 + \|v\|_2^2 \leq \Lambda_1^{-1} (\|\Delta u\|_2^2 + \|\Delta v\|_2^2),$$

where  $\Lambda_1$  is the first eigenvalue of the problem (9)-(11) applied to  $u$  and  $v$ .

**2. Conditions for blow-up and an upper bound of the blow up time.** In this section, the system (1)-(3) is simplified by the assumption  $h_i = 0$ . We recall that, as already observed in the introduction, the functional  $\Psi(t)$  defined in (7)-(8) is non-negative.

We derive sufficient conditions on the data which guarantee that the solution  $(u, v)$  blows up in finite time.

*Proof of Theorem 1.3.* Testing the first equation with  $h_1 = 0$  in (1) with  $\phi_1$ , we have

$$\int_{\Omega} u_t \phi_1 dx + \delta_1 \int_{\Omega} (\Delta^2 u) \phi_1 dx = k_1 \int_{\Omega} f_1 \phi_1 dx. \quad (22)$$

For the first term on the left in (22), with  $\Psi_1$  defined in (7),

$$\int_{\Omega} u_t \phi_1 = \frac{d}{dt} \int_{\Omega} u \phi_1 = \Psi_1'(t)$$

since  $\phi_1$  does not depend on  $t$ . Then,

$$\frac{d}{dt} \int_{\Omega} u \phi_1 dx = -\delta_1 \int_{\Omega} (\Delta^2 u) \phi_1 dx + k_1 \int_{\Omega} f_1 \phi_1 dx.$$

Using the hypothesis (13), we obtain

$$\Psi_1' = \int_{\Omega} (k_1 f_1 - \delta_1 \Delta^2 u) \phi_1 dx \geq k_1 \int_{\Omega} v^p \phi_1 dx - \delta_1 \int_{\Omega} \Delta^2 u \phi_1 dx. \quad (23)$$

In the first term on the right of (23), we use Hölder's inequality and (11), leading to

$$\int_{\Omega} v^p \phi_1 dx \geq |\Omega|^{-\frac{p-1}{2}} \Psi_2^p. \quad (24)$$

Using the second Green's identity and (9), the second term on the right of (23) can be estimated as follows:

$$-\int_{\Omega} \phi_1 \Delta^2 u dx = -\int_{\Omega} u \Delta^2 \phi_1 dx = -\Lambda_1 \int_{\Omega} u \phi_1 dx = -\Lambda_1 \Psi_1. \quad (25)$$

Substituting (24) and (25) in (23), we arrive at

$$\Psi_1'(t) \geq k_1 |\Omega|^{-\frac{p-1}{2}} \Psi_2^p(t) - \delta_1(t) \Lambda_1 \Psi_1(t). \quad (26)$$

Testing the second equation with  $h_2 = 0$  in (1) with  $\phi_1$ , a computation similar to the previous leads to

$$\Psi_2'(t) \geq k_2(t) |\Omega|^{-\frac{q-1}{2}} \Psi_1^q - \delta_2(t) \Lambda_1 \Psi_2. \quad (27)$$

Adding (26) and (27), recalling the boundedness of the coefficients, there exist two positive constants  $\delta$  and  $c$  such that

$$\Psi' \geq -\delta \Lambda_1 (\Psi_1 + \Psi_2) + c(\Psi_2^p + \Psi_1^q), \quad (28)$$

Since  $p > q > 1$ , we make use of the inequality

$$\Psi_2^q = (a \Psi_2^p)^{\frac{q}{p}} \left( a^{-\frac{q}{p-q}} \right)^{\frac{p-q}{p}} \leq \frac{q}{p} (a \Psi_2^p) + \frac{p-q}{p} a^{-\frac{q}{p-q}},$$

valid for arbitrary  $a > 0$ .

Choosing  $a = \frac{p}{q}$ , we arrive at

$$\Psi_2^q \geq \Psi_2^p - Q, \quad (29)$$

with

$$Q := \frac{p-q}{p} \left( \frac{q}{p} \right)^{\frac{q}{p-q}}. \quad (30)$$

Inserting (29) in (28), we obtain the first-order differential inequality

$$\Psi' \geq -\delta \Lambda_1 \Psi + c(\Psi_1^q + \Psi_2^q) - cQ \geq -\delta \Lambda_1 \Psi + 2^{1-q} c \Psi^q - cQ := H(\Psi), \quad (31)$$

where in the second term of (31) we used the arithmetic inequality

$$X^q + Y^q \geq 2^{1-q} (X + Y)^q, \quad q > 1, \quad (32)$$

with  $X \geq 0$ ,  $Y \geq 0$ .

Since the initial data satisfy (14), then  $\Psi(t)$  is increasing for small values of  $t$ . We have that  $H(\Psi)$  is increasing in  $\Psi$  from its negative minimum, and it follows that  $H(\Psi(t))$  is increasing for  $t > 0$ . Thus,  $\Psi'(t)$  remains positive as a consequence  $\Psi(t)$  blows up at time  $t^*$ .

Integrating (31), we obtain the upper bound  $T_0$  in (15).  $\square$

We consider now the case  $q = p$ : We prove that an explicit estimate from above for  $t^*$  can be obtained.

*Proof of Corollary 1.4.* We restart from the inequality (28), and set  $q = p$ :

$$\Psi'(t) \geq -\delta\Lambda_1\Psi + c(\Psi_1^p + \Psi_2^p), \quad (33)$$

Since  $p > 1$ , in (33), we apply inequality (32) with  $q$  replaced by  $p$ . Then,

$$\Psi'(t) \geq -\delta\Lambda_1\Psi + 2^{1-p}c\Psi^p, \quad (34)$$

Integrating (34) from 0 to  $t$ , we arrive at

$$\Psi^{1-p}(t) \leq e^{(p-1)\delta\Lambda_1 t} \left( \Psi_0^{1-p} - \frac{2^{1-p}c}{\delta\Lambda_1} \right) + \frac{2^{1-p}c}{\delta\Lambda_1} =: \mathcal{H}(t). \quad (35)$$

Since the initial data satisfy (16), then  $\mathcal{H}(t)$  vanishes at some time  $\bar{T}$ . As a consequence,  $\Psi(t)$  must blow up at some time  $t^* \leq \bar{T}$  with  $\bar{T}$  the desired upper bound (17).  $\square$

**3. Lower bound of  $t^*$ .** In this section, we consider the system (1)-(3), and we assume that the solution blows up at finite time  $t^*$  in the sense that  $\lim_{t \rightarrow t^*} \Phi(t) = \infty$  with  $\Phi$  defined in (18)-(19). To derive a lower bound of the blow up time  $t^*$  we construct a first-order differential inequality for  $\Phi$  defined on (18) and prove Theorem 1.5.

*Proof of Theorem 1.5.* Testing the first equation in (1), with  $\Delta\Delta u$ , we have

$$\int_{\Omega} u_t \Delta^2 u + \delta_1 \int_{\Omega} (\Delta^2 u)^2 - h_1 \int_{\Omega} \Delta u \Delta^2 u = k_1 \int_{\Omega} f_1 \Delta^2 u. \quad (36)$$

By twice using the  $\epsilon$ -Young inequality, we get

$$\int_{\Omega} f_1 \Delta^2 u \leq \frac{\epsilon_1}{2} \int_{\Omega} f_1^2 + \frac{1}{2\epsilon_1} \int_{\Omega} (\Delta^2 u)^2, \quad (37)$$

and

$$\int_{\Omega} \Delta u \Delta^2 u \leq \frac{\epsilon_2}{2} \int_{\Omega} |\Delta u|^2 + \frac{1}{2\epsilon_2} \int_{\Omega} (\Delta^2 u)^2 \quad (38)$$

with  $\epsilon_1 = \epsilon_1(t)$  and  $\epsilon_2 = \epsilon_2(t)$  two arbitrary bounded, continuous, and uniformly positive functions  $\forall t \in (0, t^*)$ .

Taking into account (20), we get

$$\frac{\epsilon_1}{2} \int_{\Omega} f_1^2 + \frac{1}{2\epsilon_1} \int_{\Omega} (\Delta^2 u)^2 dx \leq \frac{\epsilon_1}{2} \|v^p\|_2^2 + \frac{1}{2\epsilon_1} \|\Delta^2 u\|_2^2. \quad (39)$$

Inserting (37), (38), and (39) into (36), it follows

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\Delta u|^2 \leq -\delta_1 \|\Delta^2 u\|_2^2 + \frac{h_1 \epsilon_2}{2} \|\Delta u\|_2^2 \quad (40)$$

$$+ \frac{h_1}{2\epsilon_2} \|\Delta^2 u\|_2^2 + \frac{k_1 \epsilon_1}{2} \|v^p\|_2^2 + \frac{k_1}{2\epsilon_1} \|\Delta^2 u\|_2^2. \quad (41)$$

Now, using (6) in Lemma 1.1 with  $r = 2p$ , we obtain

$$\|v^p\|_2^2 \leq S^{2p} \left( \|\Delta v\|_2^2 \right)^p. \quad (42)$$

Finally, we can write

$$\Phi'_1(t) \leq \left( \frac{h_1}{\epsilon_2} + \frac{k_1}{\epsilon_1} - 2\delta_1 \right) \|\Delta^2 u\|_2^2 + k_1 \epsilon_1 S^{2p} \left( \|\Delta v\|_2^2 \right)^p + h_1 \epsilon_2 \|\Delta u\|_2^2. \quad (43)$$

In the same way, we derive

$$\Phi'_2(t) \leq \left( \frac{h_2}{\epsilon_4} + \frac{k_2}{\epsilon_3} - 2\delta_2 \right) \|\Delta^2 v\|_2^2 + k_2 \epsilon_3 S^{2q} \left( \|\Delta u\|_2^2 \right)^q + h_2 \epsilon_4 \|\Delta v\|_2^2 \quad (44)$$

where  $\epsilon_3 = \epsilon_3(t)$  and  $\epsilon_4 = \epsilon_4(t)$  are two arbitrary bounded, continuous, and uniformly positive functions  $\forall t \in (0, t^*)$ . Since  $\Phi' = \Phi'_1 + \Phi'_2$ , using (43) and (44) and choosing the functions  $\epsilon_i$ ,  $i = 1, \dots, 4$  such that the first terms in (43) and (44) vanish, we arrive at

$$\begin{aligned} \Phi'(t) &\leq k_1 \epsilon_1 S^{2p} \left( \|\Delta v\|_2^2 \right)^p + k_2 \epsilon_3 S^{2q} \left( \|\Delta u\|_2^2 \right)^q + h_1 \epsilon_2 \|\Delta u\|_2^2 \\ &\quad + h_2 \epsilon_4 \|\Delta v\|_2^2 = k_2 \epsilon_3 S^{2q} \Phi_1^q + k_1 \epsilon_1 S^{2p} \Phi_2^p + h_1 \epsilon_2 \Phi_1 + h_2 \epsilon_4 \Phi_2 \\ &\leq k_2 \epsilon_3 S^{2q} \Phi^q + k_1 \epsilon_1 S^{2p} \Phi^p + B\Phi, \end{aligned} \quad (45)$$

with

$$B := \max\{B_1, B_2\}, \quad (46)$$

$$B_1 = \sup_t \{h_1(t) \epsilon_2(t)\}, \quad B_2 = \sup_t \{h_2(t) \epsilon_4(t)\}.$$

Since  $\Phi(t)$  blows up in a finite time  $t^*$ ,  $\Phi(t)$  can be non-decreasing such that  $\Phi(t) \geq \Phi(0) = \Phi_0$  with  $t \in [0, t^*)$ , or non-increasing (possibly with some kind of oscillations), in which case there exists a time  $t_1 > 0$  where  $\Phi(t_1)$  coincides with the initial value  $\Phi_0$ . As a consequence,  $\Phi(t) \geq \Phi_0$  for all  $t \in [t_1, t^*)$ , and we can write with  $p > q$ ,

$$\Phi^q(t) \leq \Phi^p(t) \Phi_0^{q-p}, \quad \forall t \in (t_1, t^*). \quad (47)$$

In view of (47), inequality (45) becomes

$$\Phi'(t) \leq 2A\Phi^p + B\Phi, \quad (48)$$

with

$$A := \max\{A_1, A_2\} \quad (49)$$

$$A_1 = \sup_t \{k_2(t) \epsilon_3(t) S^{2q} \Phi_0^{q-p}\}, \quad A_2 = \sup_t \{k_1(t) \epsilon_1(t) S^{2p}\}.$$

Integrating (48), we arrive at

$$\begin{aligned} &e^{B(p-1)t} \Phi^{1-p}(t) - \Phi_0^{1-p} \\ &\geq -2A(p-1) \int_0^t e^{B(p-1)\tau} d\tau \\ &= \frac{-2A}{B} \left[ e^{B(p-1)t} - 1 \right]. \end{aligned}$$

Finally, letting  $t \rightarrow t^*$ , we obtain a lower bound  $T$  of the blow-up time with

$$T := \frac{1}{B(p-1)} \ln \left[ 1 + \frac{B}{2A} \Phi_0^{1-p} \right] \leq t^*.$$

In the particular case  $p = q$ , we replace (45) with

$$\Phi' \leq k_2 \epsilon_3 S^{2p} \Phi_1^p + k_1 \epsilon_1 \Phi_2^p + B\Phi$$

and by using the basic inequality  $a^\gamma + b^\gamma \leq (a + b)^\gamma$ ,  $\gamma > 1$ ,  $a, b > 0$ , we get

$$\Phi' \leq \tilde{A}\Phi^p + B\Phi, \quad (50)$$

with

$$\tilde{A} = \max\{S^{2p}\tilde{A}_1, \tilde{A}_2\} \quad (51)$$

$$\tilde{A}_1 = \sup_t \{k_2(t)\epsilon_3(t)\}, \quad \tilde{A}_2 = \sup_t \{k_1(t)\epsilon_1(t)\}.$$

Integrating (50) leads to the lower bound  $T \leq t^*$ , with

$$T := \frac{1}{B(p-1)} \ln \left[ 1 + \frac{B}{\tilde{A}} \Phi_0^{1-p} \right].$$

We have obtained a lifespan of the solution  $[0, T]$ , with  $T$  defined (21), and Theorem 1.5 is proved.  $\square$

**Remark 3.1.** The existence of a lower bound  $T$  for the blow-up time to the functional  $\Phi(t)$  has a consequence that the interval  $[0, T]$  is a safe interval of existence of the solution, since, as observed in the introduction,

$$\|u\|_2^2 + \|v\|_2^2 \leq \Lambda_1^{-1} (\|\Delta u\|_2^2 + \|\Delta v\|_2^2).$$

**Remark 3.2.** We may obtain a further lower bound of the blow up time  $t^*$  (easier to be computed) by estimating  $\Phi(t)$  on the right of (50) in the following way: Arguing as in (47) with  $q = 1$ , we get

$$\Phi(t) \leq \Phi^p(t) \Phi_0^{1-p}, \quad \forall t \in (t_1, t^*). \quad (52)$$

By replacing (52) in (48), we arrive at

$$\Phi'(t) \leq K\Phi^p, \quad (53)$$

with  $K := 2A + B\Phi_0^{1-p}$ . Integrating (53) in the interval  $(t_1, t)$  and letting  $t \rightarrow t^*$ , we have

$$\int_{\Phi_0}^{\infty} \frac{d\eta}{\eta^p} = \int_{\Phi(t_1)}^{\infty} \frac{d\eta}{\eta^p} \leq K \int_{t_1}^{t^*} d\tau \leq K \int_0^{t^*} d\tau = Kt^*,$$

from which we obtain the following lower bound:

$$\tilde{T} = \frac{\Phi_0^{1-p}}{(p-1)K} \leq t^*.$$

**Remark 3.3.** In [4] (Theorem 4.7 and Theorem 4.8), the authors proved that the blow-up of the solution in  $W_0^{2,2}(\Omega)$ -norm implies the blow-up also in  $W_0^{1,4}(\Omega)$ -norm. In our case, since we have proved that the solution blows up in  $W_0^{2,2}(\Omega)$ -norm, by imposing proper conditions on the initial data and following the method in [4] and [21], we obtain that the solution blows up also in  $W_0^{1,4}(\Omega)$ -norm.

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