mathematics

Article

Symplectic Pairs and Intrinsically Harmonic Forms †

Gianluca Bande

Dipartimento di Matematica e Informatica, Università degli Studi di Cagliari, 09124 Cagliari, Italy; gbande@unica.it

† To the memory of Robert Lutz and Theodor Hangan.

Abstract: In this short note, we prove two properties of symplectic pairs on a four-manifold: firstly we prove that two transversal orientable foliations of codimension two, which are taut for the same Riemannian metric, are the characteristic foliations of a symplectic pair; secondly, we characterize intrinsically harmonic 2-forms of rank two as part of a symplectic pair.

Keywords: symplectic pairs; taut foliations; intrinsically harmonic forms

MSC: 53C15; 57R17; 57R30; 53C12; 53D35; 58A17

check for updates

Citation: Bande, G. Symplectic Pairs and Intrinsically Harmonic Forms. *Mathematics* **2023**, *11*, 2993. https://doi.org/10.3390/math11132993

Academic Editor: Ion Mihai

Received: 7 June 2023
Revised: 1 July 2023
Accepted: 3 July 2023
Published: 5 July 2023

1. Introduction

A symplectic pair on a smooth manifold [1,2] is a pair of non-trivial closed 2-forms (ω, η) of constant and complementary ranks, for which ω restricts to a symplectic form on the leaves of the kernel foliation of η, and vice versa.

On a four-manifold M, a symplectic pair (ω, η) can be equivalently defined by a pair of symplectic forms (Ω_+, Ω_-) satisfying the following conditions:

$$\Omega_+^2 = -\Omega_-^2, \qquad \Omega_+ \wedge \Omega_- = 0. \tag{1}$$

In this case, the forms ω and η are given by $\omega = \frac{1}{2}(\Omega_+ + \Omega_-)$ and $\eta = \frac{1}{2}(\Omega_+ - \Omega_-)$.

Several interesting examples and constructions are given in [1], especially on closed four-manifolds. It is also observed that it is possible to construct *compatible metrics*, making both the characteristic foliations taut (i.e., with minimal leaves).

Recall that a differential form on a manifold is called *intrinsically harmonic* [3] if it is harmonic with respect to some Riemannian metric. With respect to a compatible metric, each 2-form of the pair forming the symplectic pair is harmonic.

The aim of this paper is two-fold: firstly, we prove that on a four-dimensional orientable manifold, two complementary orientable foliations of dimension 2 which are taut for some metric are, in fact, the characteristic foliations of a symplectic pair; secondly, we show that, on a closed four-dimensional manifold, an intrinsically harmonic 2-form of rank 2 is necessarily one of the 2-forms composing a symplectic pair.

All the objects considered in this paper are assumed to be C^∞.

2. Preliminaries on Symplectic Pairs

In this section, we recall the main objects studied in this article and some basic results needed in the next sections.

Definition 1 ([1,2])**.** *Let M be a $2n$-dimensional manifold. A pair of* closed 2-*forms (ω, η) is called a symplectic pair of type $(k, n-k)$ if they have constant ranks $2k$ and $2(n-k)$, respectively, and, moreover, $\omega^{2k} \wedge \eta^{2(n-k)}$ is a volume form.*

A symplectic pair gives rise to two symplectic forms:

$$\Omega_+ = \omega + \eta, \quad \Omega_- = \omega - \eta$$

on M and on $(-1)^{n-p}M$, respectively, where $-M$ denotes the oriented manifold obtained by reversing the orientation of M. To make the definition interesting, we assume that $k > 0$ and $n > k$. Then, when M has dimension four—the case of interest in the present article—a symplectic pair on M can only be of type $(1, 1)$ and, in particular, M is symplectic for both orientations.

In dimension four, a symplectic pair (ω, η) can be equivalently defined by a pair of symplectic forms (Ω_+, Ω_-) satisfying

$$\Omega_+^2 = -\Omega_-^2, \quad \Omega_+ \wedge \Omega_- = 0 \,.$$

The symplectic pair is then given by

$$\Big(\frac{\Omega_+ + \Omega_-}{2}, \frac{\Omega_+ - \Omega_-}{2}\Big),$$

and we say that (Ω_+, Ω_-) *arises from a symplectic pair*.

The kernels of ω and η are integrable complementary distributions and therefore integrate to a pair of transverse foliations $\mathcal{F}_\omega$ and $\mathcal{F}_\eta$ called *characteristic foliations* [1] such that

$$T\mathcal{F}_\omega = \ker\omega \quad \text{and} \quad T\mathcal{F}_\eta = \ker\eta.$$

Each form is symplectic on the leaves of the foliation induced by the other form and, moreover, $\mathcal{F}_\omega$ and $\mathcal{F}_\eta$ are symplectically orthogonal with respect to both the symplectic forms Ω_+ and Ω_-.

By Rummler and Sullivan's criterion (see [4]), the characteristic foliation of a closed 2-form is taut, which means that there exists a Riemannian metric for which the leaves are minimal. For a symplectic pair, it is possible to construct a Riemannian metric, making the foliations orthogonal and both with minimal leaves (see [1] and the discussion after Definition 5.6 in [5]).

3. Taut Foliations and Symplectic Pairs

It is shown in [6] that on a four-dimensional orientable manifold "two taut make one symplectic", which means that the existence of two complementary orientable 2-dimensional taut foliations implies that the manifold is symplectic.

In this section, we see that "two taut make a symplectic pair", proving that the two foliations are, in fact, the characteristic foliations of a symplectic pair as shown in the following result:

Theorem 1. *Let M be an orientable four-dimensional manifold endowed with two transverse and complementary orientable foliations $\mathcal{F}$ and $\mathcal{G}$ of dimension 2. If $\mathcal{F}$ and $\mathcal{G}$ are orthogonal and have minimal leaves for some Riemannian metric on M, then they are the characteristic foliations of a symplectic pair.*

Proof. Let g be a metric for which $\mathcal{F}$ and $\mathcal{G}$ are orthogonal and have minimal leaves. Consider g-orthogonal almost complex structures J_1, J_2, respectively, on $T\mathcal{F}$ and $T\mathcal{G}$. Then, we have two almost complex structures on TM given by

$$J_\pm = J_1 \oplus (\pm J_2).$$

Let $\Omega_\pm(X, Y) = g(X, J_\pm Y)$. For ∇, the Levi–Civita connection of g and X, Z vector fields on M, we have (see Appendix in [6] for a proof):

$$d\Omega_\pm(X, J_\pm X, Z) = g([X, J_\pm X], J_\pm Z) - g(\nabla_X X + \nabla_{J_\pm X} J_\pm X, Z) \,.$$

To prove that $d\Omega_{\pm} = 0$, it is enough to prove that $d\Omega_{\pm}$ vanishes when calculated on any triple of linearly independent vector fields. We can choose a local basis such that $X, J_{\pm}X$ are tangent to $\mathcal{F}$ and $Z, J_{\pm}Z$ are tangent to $\mathcal{G}$. Any triple of vectors fields of the local basis has the form $(U, J_{\pm}U, V)$ for some U, V in the local basis.

By the minimality of the leaves, we have $g(\nabla_X X + \nabla_{J_{\pm}X} J_{\pm}X, Z) = 0$ (see [7] for example). Frobenius' theorem and the orthogonality of $\mathcal{F}$ and $\mathcal{G}$ imply $g([X, J_{\pm}X], J_{\pm}Z) = 0$. This implies that the 2-forms $\Omega_{\pm}$ are closed and therefore symplectic.

Since $\Omega_{\pm}$ are symplectic, there is a unique isomorphism A of the tangent bundle of M, called *recursion operator*, such that

$$\Omega_{+}(X, Y) = \Omega_{-}(AX, Y).$$

In fact, A is the composition of the usual musical isomorphisms.

In our case, the recursion operator A is the identity on one foliation and minus the identity on the other one. In particular, A is not the identity itself, but its square is the identity. Thus, by Theorem 3 in [8] the pair (Ω_{+}, Ω_{-}) arises from a symplectic pair. □

4. Intrinsically Harmonic 2-Forms

A differential form ω on an n-dimensional manifold is called *intrinsically harmonic* [3] if it is harmonic with respect to some Riemannian metric. An intrinsically harmonic form is a fortiori closed, and then the main problem is to give necessary and sufficient conditions under which a closed form is intrinsically harmonic.

In fact, only the forms of degrees 1 and $n-1$ are quite well understood. A classical theorem of Calabi [3] answers the question for 1-forms with non-degenerate zeros, and Honda [9] proves the dual case of $(n-1)$-form.

In 2007, Volkov [10] was able to drop the condition on the zeros of the 1-form, giving a complete characterization.

In general, the forms of degrees strictly between 1 and $n-1$ present additional problems. One of this difficulties is illustrated in [10], where the author gives an example of a closed 2-form of rank 2 on a 4-dimensional manifold, which is not intrinsically harmonic.

Observe that any symplectic form is harmonic with respect to a compatible metric because its Hodge dual is, up to a constant, a power of the symplectic form.

On a 4-dimensional manifold, a non-vanishing 2-form of constant rank has either rank 2 or 4. Vanishing forms and symplectic forms (of rank 4) are intrinsically harmonic, so let us consider the 2-forms of rank 2. We start with the following result of linear algebra.

Lemma 1. *Let ω be a 2-form of rank 2 on a four-dimensional Euclidean vector space (W, g) and let $\star$ be the Hodge operator with respect to g. Then $\star\omega$ also has rank 2.*

Proof. Since the rank of ω is 2, then it has 2-dimensional kernel. Let $W \subset W$ be the kernel of ω and let $\{e_1, e_2\}$ be an othonormal basis of V. Complete $\{e_1, e_2\}$ to an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of W. Then ω is (up to a non-zero constant) equal to $e_3 \wedge e_4$ and $\star\omega$ is equal (up to a non-zero constant) to $e_1 \wedge e_2$. □

We can now give the proof of the main theorem:

Theorem 2. *A closed 2-form of constant rank 2 on an orientable closed four-dimensional manifold M is intrinsically harmonic if and only if it is part of a symplectic pair.*

Proof. We already observed that on a manifold endowed with a symplectic pair, there exist compatible metrics. With respect to these metrics, each form of the symplectic pair is then closed and co-closed.

On the other end, let ω be a 2-form of rank 2 which is intrinsically harmonic for some metric g on M. Then $\star\omega$ has also rank 2 at each point by Lemma 1. Moreover, for Vol_g, the volume form associated with g, we have

$$\omega \wedge \star\omega = ||\omega||^2 Vol_g$$

which vanishes in a point p only if ω_p does. Then $(\omega\,,\,\star\omega)$ is a symplectic pair on M. □

5. Conclusions

Theorem 2 is somehow suggested at the end of the last section of [10], but symplectic pairs are not mentioned there. Moreover, the author seems to relegate this possible link to a mere tautological definition. We think instead, that this point of view could reveal some interesting aspects.

Let us consider, for example, the case of $\mathbb{C}P^2$. Since it is symplectic, it admits intrinsically harmonic 2-forms of constant rank 4. The existence of a symplectic pair on a manifold implies that its second Betti number b_2 satisfies $b_2 \geq 2$ and, therefore, no intrinsically harmonic 2-form of constant rank 2 exists.

A more subtle example is given by $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ (the non-trivial $\mathbb{C}P^1$-bundle over $\mathbb{C}P^1$), which fulfills all the basic topological obstructions to the existence of a symplectic pair. In Example 1 of [10], the author considers, on $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, the pullback to the total space of a volume form on the base (which has constant rank 2) and proves that it is not intrinsically harmonic. But we can say much more, because, by the results in [11], $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ admits no symplectic pair at all and thus, by Theorem 2, we have the following.

Corollary 1. *$\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ admits no intrinsically harmonic 2-form of constant rank 2.*

Recall that $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ admits a symplectic form and hence the only intrinsically harmonic 2-forms of constant rank, can have rank 4 or 0.

We thus have the following natural question.

Question 1. *Does $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ admit an intrinsically harmonic 2-form of non-constant rank, which is not symplectic and has at least rank 2 in a point?*

The answer is positive, and an example can be constructed as follows.

Example 1. *Let ω be the pullback to $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ of the Fubini–Study volume form on $\mathbb{C}P^1$, and fix any Riemannian metric g on $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. By the Hodge theorem, the cohomology class of ω has a unique harmonic representative, let us say $\omega + d\alpha$ for some 1-form α. Since ω is not exact (because it is one of the generators of the second cohomology group of $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$), there is a point q, where $\omega + d\alpha$ is non-zero. Then $\omega + d\alpha$ is non-trivial and, in particular, its rank r at q is $r_q \geq 2$. On the other hand, because $\omega^2 = 0$, we have $(\omega + d\alpha)^2 = 2\omega \wedge d\alpha + d\alpha^2$, which is exact and, thus, it cannot be a volume form by Stokes' theorem. This means that there is a point p, where $(\omega + d\alpha)^2 = 0$ and then the rank of $(\omega + d\alpha)_p$ is $r_p \leq 2$. Therefore, $\omega + d\alpha$ is non-trivial, g-harmonic, cannot have constant rank 2 and cannot be symplectic.*

One can try to seek more restricted ranks and ask the following questions:

Question 2. *Does $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ admit an intrinsically harmonic 2-form of non-constant rank $r \leq 4$, which has rank 4 at least in a point?*

Question 3. *Does $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ admit an intrinsically harmonic 2-form of non-constant rank r such that $2 \leq r \leq 4$ and there are points where $r = 2$ and $r = 4$?*

Of course, $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ can be replaced by the blow-up of $\mathbb{C}P^2$ in the k point or, more generally, by a closed orientable symplectic 4-manifold, which is non-minimal.

Funding: The author is supported by the project "GOACT"—CUP: F75F21001210007—Fondazione di Sardegna—2020.

Data Availability Statement: Data sharing not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Bande, G.; Kotschick, D. The Geometry of Symplectic pairs. *Trans. Am. Math. Soc.* **2006**, *358*, 1643–1655. [CrossRef]
2. Kotschick, D.; Morita, S. Signatures of foliated surface bundles and the symplectomorphism groups of surfaces. *Topology* **2005**, *44*, 131–149. [CrossRef]
3. Calabi, E. An intrinsic characterization of harmonic one-forms. In *Global Analysis (Papers in Honor of K. Kodaira)*; Princeton University Press: Tokyo, Japan, 1969; pp. 101–117.
4. Rummler, H. Quelques notions simples en géométrie riemannienne et leurs applications aux feuilletages compacts. *Comment. Math. Helv.* **1979**, *54*, 224–239. [CrossRef]
5. Bande, G.; Kotschick, D. Contact pairs and locally conformally symplectic structures. *Harmon. Maps Differ. Geom. Contemp. Math. Am. Math. Soc. Provid. RI* **2011**, *542*, 85–98.
6. Scorpan, A. Existence of foliations on 4-manifold. *Algebr. Geom. Topol.* **2003**, *3*, 1225–1256. [CrossRef]
7. do Carmo, M.P. *Riemannian Geometry*, *Mathematics: Theory & Applications*; Birkhäuser Boston, Inc.: Boston, MA, USA, 1992.
8. Bande, G.; Kotschick, D. The geometry of recursion operators. *Comm. Math. Phys.* **2008**, *280*, 737–749. [CrossRef]
9. Honda, K. On Harmonic Forms for Generic Metrics. Ph.D. Thesis, Princeton University, Princeton, NJ, USA, 1997.
10. Volkov, E. Characterization of intrinsically harmonic forms. *J. Topol.* **2008**, *1*, 643–650. [CrossRef]
11. Bande, G.; Ghiggini, P. Holomorphic spheres and four-dimensional symplectic pairs. *J. Geom. Anal.* **2020**, *30*, 861–873. [CrossRef]