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A Matrix Schrödinger Approach to Focusing Nonlinear Schrödinger Equations with Nonvanishing Boundary Conditions

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Abstract

We relate the scattering theory of the focusing AKNS system with equally-sized nonvanishing boundary conditions to that of the matrix Schrödinger equation. This (shifted) Miura transformation converts the focusing matrix nonlinear Schrödinger (NLS) equation into a new nonlocal integrable equation. We apply the matrix triplet method of solving the Marchenko integral equations by separation of variables to derive the multisoliton solutions of this nonlocal equation, thus proposing a method to solve the reflectionless matrix NLS equation.

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1 Introduction

The Nonlinear Schrödinger (NLS) equations have served as the basic models for surface waves on deep waters [1, 4, 82], signals along optical fibres [50, 49, 75], plasma oscillations [79], magnetic spin waves [28, 81], and particle states in Bose-Einstein condensates [70, 71, 55]. The NLS equations with solutions decaying at infinity have been studied in detail [2, 4, 26, 41, 3]. After finding the Peregrine solutions [69], various solutions of the NLS equations with nonvanishing boundary conditions have been presented in [65, 8, 7, 9, 52, 66, 76, 80].

In 1972 in their seminal paper [82], Zakharov and Shabat showed that the NLS equation can be solved by means of the Inverse Scattering Transform (IST) technique. To this aim, they introduced a scattering problem now known as the Zakharov-Shabat (ZS) system. The ZS system was used to solve the scalar NLS system with zero and nonzero boundary conditions [82, 83]. In particular, in [83], Zakharov and Shabat considered the case of nonzero boundary conditions in the defocusing regime, introducing a spectral parameter belonging to a suitable two-sheeted Riemann surface and studying the analyticity properties of the scattering data on this surface. Moreover, in [65], it was proven that, in order to develop the IST for the focusing NLS equation with nonvanishing boundary conditions, the associated ZS system leads to introducing a spectral parameter λ which belongs again to a suitable two-sheeted Riemann surface. The introduction of a two-sheeted Riemann surface evidently makes the study of the NLS equation with nonvanishing boundary conditions via the IST much more complicated with respect to the vanishing case. Furthermore, in 1974 Ablowitz, Kaup, Newell and Segur proposed an alternative but equivalent way to develop the IST for the NLS equation consisting of associating to this equation the so-called AKNS system [2]. In the AKNS system, one (matrix) equation represents the spectral equation, whereas a second (matrix) equation describes the time evolution of the scattering data. Similarly to what happens with the ZS system, developing the IST from the AKNS pairs is significantly more complicated in the nonvanishing cases than in the vanishing case.

Systematic studies of the inverse scattering transform theory (IST) of the (scalar and matrix) NLS equation with nonvanishing boundary conditions have been carried out in the defocusing case in [53, 54, 16, 17, 41, 72, 31] and in the focusing case in [24, 32, 68, 23]. In [22] the IST with full account of the spectral singularities leads to rogue wave solutions of the focusing NLS wth nonvanishing boundary conditions.

In all the papers cited above, a ZS system or an AKNS system is associated to the NLS equation. If one considers the focusing NLS with nonvanishing boundary conditions, it is customary, as we have remarked above, to introduce a new spectral complex parameter, say λ, defined as $\lambda = \sqrt{k^2 + \mu^2}$

(it should be noted that λ is defined through a multivalued function). The study of the analyticity properties of the scattering data with respect to the parameter λ is quite difficult and requires special care. In this article, we show how to associate a Schrödinger equation with a vanishing potential as a spectral problem for the NLS equation with nonzero boundary conditions. In this way, to the best of our knowledge, for the first time we develop the IST for the focusing NLS system with nonzero boundary conditions without associating to it the AKNS system (or the Zakharov-Shabat system). The advantage of associating the Schrödinger equation with vanishing boundary conditions instead of the AKNS system is immediate because the construction of the scattering data for the Schrödinger equation with zero boundary conditions does not require the introduction of a new spectral parameter. Consequently, the study of the analyticity properties of these coefficients can be done in a more transparent way respect to the analogous study while using the AKNS system. In other words, a major obstacle encountered in the above-cited studies of the IST for the non-vanishing NLS systems is the change of variable from the initial spectral parameter k to a new spectral parameter $\lambda = \sqrt{k^2+\mu^2}$ which complicates analyticity issues for Jost solutions and scattering coefficients considerably, especially if such change of variable is considered in the entire complex plane. The main purpose of this article is to greatly simplify these issues by relating the focusing NLS equation to a suitable matrix Schrödinger equation, where the spectral parameter (in this case, λ) is typically chosen in the closed upper half complex half-plane $\mathbb{C}^+ \cup \mathbb{R}$. Here we can rely on a substantial body of knowledge on the direct and inverse scattering theory of the scalar Schrödinger equation on the line [40, 29, 26, 27] and the matrix Schrödinger equation on the half-line [14, 15] and the full-line [78, 12]. In particular, the small λ asymptotics of the scattering data, which is crucial to a rigorous matrix Schrödinger scattering theory, has been developed in detail in [12].

In this article we study the focusing $m+m$ AKNS system

$$v_x = (-ik\boldsymbol{\sigma}_3 + \mathcal{Q})v, \tag{1.1}$$

where $v = v(x,k)$ is a vector function with $n = 2m$ components, I_m is the identity matrix of order m, $\boldsymbol{\sigma}_3 = I_m \oplus (-I_m)$, the potential $\mathcal{Q}$ anticommutes with $\boldsymbol{\sigma}_3$, and the complex conjugate transpose $\mathcal{Q}^\dagger = -\mathcal{Q}$. The potential $\mathcal{Q}$ is to satisfy the integrability condition

$$\int_0^\infty dy\,(1+|y|)\,(\|\mathcal{Q}(-y)-\mathcal{Q}_l\|+\|\mathcal{Q}(y)-\mathcal{Q}_r\|+\|\mathcal{Q}_y(y)\|+\|\mathcal{Q}_y(-y)\|) < +\infty, \tag{1.2}$$

where $\mathcal{Q}_y$ is the y-derivative of $\mathcal{Q}$ and $[\mathcal{Q}_{r,l}]^2 = -\mu^2 I_n$ for some $\mu > 0$.

We pursue an approach that is quite different from the one expounded in [24, 32, 23]. Letting $L = i\boldsymbol{\sigma}_3[\partial_x I_n - \mathcal{Q}]$ stand for the AKNS Hamiltonian, we easily verify that $\boldsymbol{\mathcal{L}} = L^2 + \mu^2\mathbf{1}$ is the matrix Schrödinger Hamiltonian given by

$$\begin{aligned}\mathcal{L}v = (L^2+\mu^2\mathbf{1})v &= -\boldsymbol{\sigma}_3[\partial_x I_n - \mathcal{Q}]\boldsymbol{\sigma}_3[\partial_x I_n - \mathcal{Q}]v + \mu^2 v \\ &= -[\partial_x I_n + \mathcal{Q}][\partial_x I_n - \mathcal{Q}]v + \mu^2 v \\ &= -v_{xx} + \mathcal{Q}^2 v - \mathcal{Q}v_x + (\mathcal{Q}v)_x + \mu^2 v = -v_{xx} + \boldsymbol{Q}v,\end{aligned}$$

where $\mathbf{1}$ stands for the identity operator on a suitable function space and

$$\boldsymbol{Q} = \mathcal{Q}^2 + \mathcal{Q}_x + \mu^2 I_n \tag{1.3}$$

is a matrix Faddeev class Schrödinger potential obtained from $\mathcal{Q}$ by the (shifted) Miura transform [4]. In other words, $\|\boldsymbol{Q}(\cdot)\| \in L^1(\mathbb{R};(1+|x|)dx)$. Then any solution v of the AKNS system (1.1) is also a solution of the matrix Schrödinger equation

$$\mathcal{L}v = (-\partial_x^2 I_n + \boldsymbol{Q})v = \lambda^2 v, \tag{1.4}$$

where

$$\lambda = \sqrt{k^2+\mu^2} \tag{1.5}$$

is the conformal transformation from the complex k-plane $\mathbb{K}$ cut along the segment $[-i\mu, i\mu]$ onto the complex λ-plane satisfying $\lambda \sim k$ at infinity. This transformation provides a $1,1$-correspondence between the open upper/lower half k-plane $\mathbb{K}^{\pm}$ cut along $[-i\mu, i\mu]$ onto the open upper/lower half λ-plane

$\mathbb{C}^{\pm}$ as well as a $1,1$-correspondence between their boundaries $\partial\mathbb{K}^{\pm}$ and $\mathbb{R}$ and their closures $\mathbb{K}^{\pm} \cup \partial\mathbb{K}^{\pm}$ and $\mathbb{C}^{\pm} \cup \mathbb{R}$.

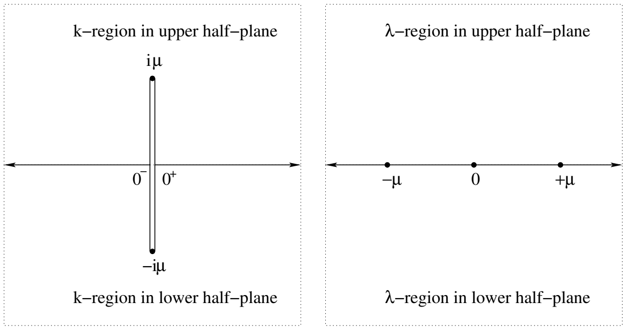

Figure 1.1: The regions $k \in \mathbb{K}^{\pm}$ and $\lambda \in \mathbb{C}^{\pm}$ with manifold boundary.

In this article we wish to take advantage of the well developed direct and inverse scattering theory of the matrix Schrödinger equation with selfadjoint potential ([6, 14, 15] on the half-line, [78, 12] on the full line), especially the established custom of choosing its spectral variable λ in $\mathbb{C}^{+} \cup \mathbb{R}$, in deriving the focusing NLS solutions with nonvanishing boundary conditions. In a previous paper, [35] such full-line theory has been made to fit potentials satisfying

$$\boldsymbol{Q}^{\dagger} = \boldsymbol{\sigma}_3 \boldsymbol{Q} \boldsymbol{\sigma}_3. \tag{1.6}$$

The traditional applications of the matrix Schrödinger equation to quantum graphs, quantum wires, and quantum mechanical scattering of particles with internal structure [18, 19, 20, 21, 25, 39, 43, 44, 45, 46, 47, 48, 57, 58, 59, 60, 61, 62, 63] have led to the almost exclusive development of matrix Schrödinger scattering theory for selfadjoint potentials satisfying $\boldsymbol{Q}^{\dagger} = \boldsymbol{Q}$

(see [6, 14, 15] for the half-line theory and [78, 12] for the full-line theory). Energy losses in such systems naturally lead to potentials whose imaginary part $[\boldsymbol{Q} - \boldsymbol{Q}^\dagger]/2i$ has constant sign. In the present context where $\boldsymbol{Q}$ satisfies (1.6), we thus require the modified matrix Schrödinger scattering theory given in [35] when solving the focusing matrix NLS equation.

Let us discuss the contents of the various sections. In Sec. 2 we introduce the Lax pair $\{\mathcal{L}, A\}$ and the AKNS pair $\{\boldsymbol{X}, \boldsymbol{T}\}$ whose compatibility conditions lead to an integrable nonlocal equation for $\boldsymbol{Q}$. We also relate the solutions of this integrable equation to those of a modified matrix NLS equation which is converted into the usual matrix NLS equation by a trivial gauge transformation. Next, in Sections 3-4 we state the direct and inverse scattering theory of the matrix Schrödinger equation (1.4) with Faddeev class potentials $\boldsymbol{Q}$ satisfying (1.6), disregarding any time dependence. In particular, we introduce the Jost solutions and the scattering coefficients, write them as Fourier transforms of L^1-functions, and state the Marchenko integral equations to solve the inverse scattering problem. We then go on to derive the time evolution of the scattering data [Sec. 5]. In Sec. 6 we apply the so-called matrix triplet method to derive the multisoliton solutions of the nonlocal integrable equation and the focusing matrix NLS equation by separation of variables in the Marchenko integral equations.

We adopt boldface symbols for many of the quantities pertaining to the matrix Schrödinger equation and calligraphic symbols for many of the quantities pertaining to the AKNS system. We deviate from the praxis of [2, 3] in allowing right and left to correspond to the real line endpoints involved in defining the Jost solutions, both in the (matrix) Schrödinger and the AKNS cases. Hence we prioritize traditional notations regarding (matrix) Schrödinger equations [40, 29, 27] over those regarding AKNS systems [2, 3].

2 Lax Pair for the new integrable model

It is well-known that the matrix NLS system is governed by a Lax pair $\{L, A\}$ of linear operators [64, 4, 38]

$$L = i\boldsymbol{\sigma}_3(\partial_x I_n - \mathcal{Q}), \tag{2.1a}$$

$$A = i\boldsymbol{\sigma}_3\left(2\partial_x^2 I_n - 2\mathcal{Q}\partial_x - \boldsymbol{Q}\right), \tag{2.1b}$$

where $\boldsymbol{Q}$ is given by (1.3), $Lv = kv$ is the AKNS eigenvalue problem, and $v_t = Av$ describes the time evolution. Then the zero curvature condition

$$L_t + LA - AL = \mathbf{0},$$

where $\mathbf{0}$ denotes the zero operator on a suitable function space, leads to the integrable PDE

$$i\boldsymbol{\sigma}_3\mathcal{Q}_t + \mathcal{Q}_{xx} - 2\mathcal{Q}^3 - 2\mu^2\mathcal{Q} = 0_{n\times n} \tag{2.2}$$

which coincides with the usual matrix NLS equation, studied in [3, 2], apart from the extra term $-2\mu^2\mathcal{Q}$.

Putting $\mathcal{L} = L^2 + \mu^2\mathbf{1} = -\partial_x^2 + \boldsymbol{Q}$, we now compute

$$\begin{aligned}
&i\boldsymbol{\sigma}_3[\mathcal{L}_t + \mathcal{L}A - A\mathcal{L}] = i\boldsymbol{\sigma}_3\boldsymbol{Q}_t \\
&\quad - (-\partial_x^2 + \boldsymbol{\sigma}_3\boldsymbol{Q}\boldsymbol{\sigma}_3)\left[2\partial_x^2 - 2\mathcal{Q}\partial_x - \boldsymbol{Q}\right] + \left[2\partial_x^2 - 2\mathcal{Q}\partial_x - \boldsymbol{Q}\right](-\partial_x^2 + \boldsymbol{Q}) \\
&\quad = i\boldsymbol{\sigma}_3\boldsymbol{Q}_t + 4(-\mathcal{Q}_x + \tfrac{1}{2}[\boldsymbol{Q} - \boldsymbol{\sigma}_3\boldsymbol{Q}\boldsymbol{\sigma}_3])\partial_x^2 \\
&\quad + 2(-\mathcal{Q}_{xx} + \boldsymbol{Q}_x + \boldsymbol{\sigma}_3\boldsymbol{Q}\boldsymbol{\sigma}_3\mathcal{Q} - \mathcal{Q}\boldsymbol{Q})\partial_x \\
&\quad + \boldsymbol{Q}_{xx} + \boldsymbol{\sigma}_3\boldsymbol{Q}\boldsymbol{\sigma}_3\boldsymbol{Q} - 2\mathcal{Q}\boldsymbol{Q}_x - \boldsymbol{Q}^2.
\end{aligned}$$

Then the ∂_x^2 term vanishes iff $\boldsymbol{Q} = \boldsymbol{D} + \mathcal{Q}_x$ for some $\boldsymbol{D}$ commuting with $\boldsymbol{\sigma}_3$ and vanishing as $x \to \pm\infty$. Hence the coefficient of the ∂_x term equals $2(\boldsymbol{D} - \mathcal{Q}^2)_x + 2[\boldsymbol{D}, \mathcal{Q}] = 0_{n\times n}$. Putting $\boldsymbol{E} = \boldsymbol{D} - \mathcal{Q}^2 - \mu^2 I_n$ so that $\boldsymbol{E}$ vanishes as $x \to \pm\infty$, we obtain $\boldsymbol{E}_x + [\boldsymbol{E}, \mathcal{Q}] = 0_{n\times n}$. Writing the latter as

$$\left(e^{-x\mathcal{Q}_r}\boldsymbol{E}e^{x\mathcal{Q}_r}\right)_x = -e^{-x\mathcal{Q}_r}[\mathcal{Q}(x) - \mathcal{Q}_r]e^{x\mathcal{Q}_r}$$

and using that $e^{\pm x\mathcal{Q}_r} = \cos(\mu x)I_n \pm \frac{\sin(\mu x)}{\mu}\mathcal{Q}_r$ to arrive at the estimate $\|e^{\pm x\mathcal{Q}_r}\| \le \frac{\sqrt{\mu^2 + \|\mathcal{Q}_r\|^2}}{\mu}$, we can apply Gronwall's inequality to the estimate

$$\|\boldsymbol{E}(x)\| \le \frac{\mu^2 + \|\mathcal{Q}_r\|^2}{\mu^2}\int_x^\infty dy\, \|\boldsymbol{E}(y)\|\|\mathcal{Q}(y) - \mathcal{Q}_r\|,$$

to see that $\boldsymbol{E}$ vanishes identically and therefore $\boldsymbol{D} = \mathcal{Q}^2 + \mu^2 I_n$. Thus, for this particular choice of $\boldsymbol{D}$ we arrive at the nonlinear evolution equation

$$i\boldsymbol{\sigma}_3\boldsymbol{Q}_t + \boldsymbol{Q}_{xx} - \boldsymbol{Q}^2 + \boldsymbol{\sigma}_3\boldsymbol{Q}\boldsymbol{\sigma}_3\boldsymbol{Q} - 2\mathcal{Q}\boldsymbol{Q}_x = 0_{n\times n}, \tag{2.3}$$

where

$$\mathcal{Q}(x;t) = \mathcal{Q}_r - \int_x^\infty dy\, \tfrac{1}{2}\left(\boldsymbol{Q} - \boldsymbol{\sigma}_3\boldsymbol{Q}\boldsymbol{\sigma}_3\right), \tag{2.4a}$$

$$\mathcal{Q}(x;t) = \mathcal{Q}_l + \int_{-\infty}^x dy\, \tfrac{1}{2}\left(\boldsymbol{Q} - \boldsymbol{\sigma}_3\boldsymbol{Q}\boldsymbol{\sigma}_3\right). \tag{2.4b}$$

for time invariant matrices $\mathcal{Q}_{r,l}$ satisfying $[\mathcal{Q}_{r,l}]^2 = -\mu^2 I_n$ for every $t \in \mathbb{R}$.

Conversely, substituting

$$\boldsymbol{Q} = \boldsymbol{D} + \mathcal{Q}_x,$$

where $\boldsymbol{D}$ commutes with $\boldsymbol{\sigma}_3$, $\mathcal{Q}_x$ anticommutes with $\boldsymbol{\sigma}_3$, and $\boldsymbol{D}$ vanishes as $x \to \pm\infty$, into (2.3), we obtain

$$0_{n\times n} = i\boldsymbol{\sigma}_3\boldsymbol{D}_t + (\boldsymbol{D}_x - 2\mathcal{Q}\mathcal{Q}_x)_x + (i\boldsymbol{\sigma}_3\mathcal{Q}_t + \mathcal{Q}_{xx} - 2\mathcal{Q}\boldsymbol{D})_x.$$

Separating the block off-diagonal and block diagonal components we get

$$i\boldsymbol{\sigma}_3\boldsymbol{D}_t + (\boldsymbol{D}_x - 2\mathcal{Q}\mathcal{Q}_x)_x = 0_{n\times n},$$

$$i\boldsymbol{\sigma}_3\mathcal{Q}_t + \mathcal{Q}_{xx} - 2\mathcal{Q}\boldsymbol{D} = 0_{n\times n},$$

where $\mathcal{Q}_t$, $\mathcal{Q}_{xx}$, and $\boldsymbol{D}$ vanish as $x \to \pm\infty$. If there exists a solution $\mathcal{Q}$ of the differential Riccati equation $\mathcal{Q}^2 + \mathcal{Q}_x = \boldsymbol{Q} - \mu^2 I_n$ which anticommutes with $\boldsymbol{\sigma}_3$ and satisfies $\mathcal{Q} \to \mathcal{Q}_{r,l}$ as $x \to \pm\infty$, then $\boldsymbol{D} = \mathcal{Q}^2 + \mu^2 I_n$ and

$$\left[i\boldsymbol{\sigma}_3 \mathcal{Q}_t + \mathcal{Q}_{xx} - 2\mathcal{Q}^3 - 2\mu^2 \mathcal{Q}, \mathcal{Q}\right] = 0_{n\times n}, \tag{2.5a}$$

$$i\boldsymbol{\sigma}_3 \mathcal{Q}_t + \mathcal{Q}_{xx} - 2\mathcal{Q}^3 - 2\mu^2 \mathcal{Q} = 0_{n\times n}, \tag{2.5b}$$

where a matrix commutator appears. The gauge transformation

$$\mathcal{Q}(x;t) = e^{-i\mu^2 t\boldsymbol{\sigma}_3} \mathcal{R}(x;t) e^{i\mu^2 t\boldsymbol{\sigma}_3} \tag{2.6}$$

then converts (2.5b) into the usual matrix NLS equation

$$i\boldsymbol{\sigma}_3 \mathcal{R}_t + \mathcal{R}_{xx} - 2\mathcal{R}^3 = 0_{n\times n},$$

where the limits $\mathcal{R}_{l,r}(t)$ of $\mathcal{R}(x;t)$ as $x \to \pm\infty$ satisfy $[\mathcal{R}_{r,l}]_t = -2i\mu^2 \boldsymbol{\sigma}_3 \mathcal{R}_{r,l}$. This is in agreement with $\mathcal{Q}_t$ vanishing as $x \to \pm\infty$ and with the well-known time evolution (see [32] for $m = 1$)

$$\mathcal{R}_{r,l}(t) = i\mu e^{2i\mu^2 t\boldsymbol{\sigma}_3} e^{i\theta_{r,l}\boldsymbol{\sigma}_3} (\sigma_2 \otimes I_m),$$

where $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ denotes the second Pauli matrix, $\sigma_2 \otimes I_m = \begin{pmatrix} 0_{m\times m} & -iI_m \\ iI_m & 0_{m\times m} \end{pmatrix}$ is a Kronecker product (cf. [51]), and $\theta_{r,l} \in \mathbb{R}$ are phases. Furthermore, (2.5b) and (1.3) imply the nonlinear equation (2.3). In fact,

$$
\begin{aligned}
i\boldsymbol{\sigma}_3\boldsymbol{Q}_t + \boldsymbol{Q}_{xx} - (\boldsymbol{Q} - \boldsymbol{\sigma}_3\boldsymbol{Q}\boldsymbol{\sigma}_3)\boldsymbol{Q} - 2\mathcal{Q}\boldsymbol{Q}_x &= \mathcal{Q}[\mathcal{Q}_{xx} - 2\mathcal{Q}^3 - 2\mu^2\mathcal{Q}] \\
&- [\mathcal{Q}_{xx} - 2\mathcal{Q}^3 - 2\mu^2\mathcal{Q}]\mathcal{Q} - [\mathcal{Q}_{xxx} - 2(\mathcal{Q}^3)_x - 2\mu^2\mathcal{Q}_x] \\
&+ [\mathcal{Q}\mathcal{Q}_{xx} + \mathcal{Q}_{xx}\mathcal{Q} + 2\mathcal{Q}_x^2 + \mathcal{Q}_{xxx}] - [2\mathcal{Q}_x\mathcal{Q}^2 + 2\mathcal{Q}_x^2 + 2\mu^2\mathcal{Q}_x] \\
&- 2[\mathcal{Q}^2\mathcal{Q}_x + \mathcal{Q}\mathcal{Q}_x\mathcal{Q} + \mathcal{Q}\mathcal{Q}_{xx}] = 0_{n\times n}.
\end{aligned}
$$

Recall that the Lax pair $\{\mathcal{L}, A\}$ for the modified nonlinear matrix Schrödinger equation (2.3) is given by (2.1). Let us now derive an AKNS pair $\{\boldsymbol{X}, \boldsymbol{T}\}$ for the same equation. Indeed, (2.3) is compatible with the linear system

$$
\mathcal{L}v = \lambda^2 v, \qquad v_t = Av,
$$

where $\boldsymbol{\mathcal{L}} = -\partial_x^2 + \boldsymbol{Q}$. We may therefore write

$$
\begin{aligned}
v_t = Av &= 2i\boldsymbol{\sigma}_3 v_{xx} - 2i\boldsymbol{\sigma}_3\mathcal{Q}v_x - i\boldsymbol{\sigma}_3\boldsymbol{Q}v \\
&= 2i\boldsymbol{\sigma}_3(\boldsymbol{Q} - \lambda^2\mathbf{1})v - 2i\boldsymbol{\sigma}_3\mathcal{Q}v_x - i\boldsymbol{\sigma}_3\boldsymbol{Q}v \\
&= i\boldsymbol{\sigma}_3\left\{(\boldsymbol{Q} - 2\lambda^2\mathbf{1})v - 2\mathcal{Q}v_x\right\}.
\end{aligned}
$$

Let us compute

$$
\begin{aligned}
(v_x)_t = (Av)_x &= i\boldsymbol{\sigma}_3\left((\boldsymbol{Q} - 2\lambda^2\mathbf{1})v_x + \boldsymbol{Q}_x v - 2\mathcal{Q}_x v_x - 2\mathcal{Q}(\boldsymbol{Q} - \lambda^2\mathbf{1})v\right) \\
&= i\boldsymbol{\sigma}_3\left(\boldsymbol{Q}_x - 2\mathcal{Q}\boldsymbol{Q} + 2\lambda^2\mathcal{Q}\right)v + i\boldsymbol{\sigma}_3(\boldsymbol{Q} - 2\lambda^2\mathbf{1} - 2\mathcal{Q}_x)v_x.
\end{aligned}
$$

Hence, putting $\boldsymbol{V} = \left(\begin{smallmatrix} v \\ v_x \end{smallmatrix}\right)$ we get the linear system

$$\boldsymbol{V}_x = \boldsymbol{X}(x,\lambda;t)\boldsymbol{V}, \qquad \boldsymbol{V}_t = \boldsymbol{T}(x,\lambda;t)\boldsymbol{V},$$

where $\{\boldsymbol{X},\boldsymbol{T}\}$ is the AKNS pair given by

$$\boldsymbol{X}(x,\lambda;t) = \begin{pmatrix} 0_{n\times n} & I_n \\ \boldsymbol{Q}(x;t) - \lambda^2 I_n & 0_{n\times n} \end{pmatrix}, \tag{2.7a}$$

$$\boldsymbol{T}(x,\lambda;t) = \begin{pmatrix} i\boldsymbol{\sigma}_3(\boldsymbol{Q} - 2\lambda^2 I_n) & -2i\boldsymbol{\sigma}_3\mathcal{Q} \\ i\boldsymbol{\sigma}_3(\boldsymbol{Q}_x - 2\mathcal{Q}\boldsymbol{Q} + 2\lambda^2\mathcal{Q}) & i\boldsymbol{\sigma}_3(\boldsymbol{Q} - 2\lambda^2 I_n - 2\mathcal{Q}_x) \end{pmatrix}. \tag{2.7b}$$

Then we easily compute

$$i(\boldsymbol{\sigma}_3 \oplus \boldsymbol{\sigma}_3)\left(\boldsymbol{X}_t - \boldsymbol{T}_x + \boldsymbol{X}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{X}\right) = \begin{pmatrix} 0_{n\times n} & 0_{n\times n} \\ \boldsymbol{E}_{21} & 0_{n\times n} \end{pmatrix},$$

where

$$\begin{aligned}
\boldsymbol{E}_{21} &= i\boldsymbol{\sigma}_3\boldsymbol{Q}_t + \boldsymbol{Q}_{xx} - 2(\mathcal{Q}\boldsymbol{Q})_x + 2\lambda^2\mathcal{Q}_x \\
&\quad - \boldsymbol{\sigma}_3(\boldsymbol{Q} - \lambda^2 I_n)\boldsymbol{\sigma}_3(\boldsymbol{Q} - 2\lambda^2 I_n) + (\boldsymbol{Q} - 2\lambda^2 I_n - 2\mathcal{Q}_x)(\boldsymbol{Q} - \lambda^2 I_n) \\
&= i\boldsymbol{\sigma}_3\boldsymbol{Q}_t + \boldsymbol{Q}_{xx} + \boldsymbol{Q}^2 - \boldsymbol{\sigma}_3\boldsymbol{Q}\boldsymbol{\sigma}_3\boldsymbol{Q} - 2\mathcal{Q}\boldsymbol{Q}_x - 4\mathcal{Q}_x\boldsymbol{Q} \\
&\quad + \lambda^2\left(2\mathcal{Q}_x + 2\boldsymbol{\sigma}_3\boldsymbol{Q}\boldsymbol{\sigma}_3 + \boldsymbol{Q} - \boldsymbol{Q} - 2\boldsymbol{Q} + 2\mathcal{Q}_x\right) \\
&= i\boldsymbol{\sigma}_3\boldsymbol{Q}_t + \boldsymbol{Q}_{xx} + \boldsymbol{Q}^2 - \boldsymbol{\sigma}_3\boldsymbol{Q}\boldsymbol{\sigma}_3\boldsymbol{Q} - 2\mathcal{Q}\boldsymbol{Q}_x - 2\left(\boldsymbol{Q}^2 - \boldsymbol{\sigma}_3\boldsymbol{Q}\boldsymbol{\sigma}_3\boldsymbol{Q}\right) \\
&= i\boldsymbol{\sigma}_3\boldsymbol{Q}_t + \boldsymbol{Q}_{xx} - \boldsymbol{Q}^2 + \boldsymbol{\sigma}_3\boldsymbol{Q}\boldsymbol{\sigma}_3\boldsymbol{Q} - 2\mathcal{Q}\boldsymbol{Q}_x,
\end{aligned}$$

as claimed. Thus the zero curvature condition for the AKNS pair $\{\boldsymbol{X}, \boldsymbol{T}\}$ is equivalent to the nonlinear evolution equation (2.3).

3 Direct scattering theory

In this section we introduce the Jost solutions and scattering coefficients for the matrix Schrödinger equation (1.4) with Faddeev class potential $\boldsymbol{Q}$ satisfying (1.6). For the scalar Schrödinger equation with real Faddeev class potential the direct scattering theory is well documented [40, 29, 26, 67, 27]. The matrix theory is discussed at great length in [14, 15] for the half-line and in [78, 12] for the full line. Here [12] contains the essential small λ asymptotics of scattering coefficients that is lacking in [78]. The adjoint symmetry $\boldsymbol{Q}$ requires some modifications of existing theory (cf. [35]).

3.1 Jost solutions of the matrix Schrödinger equation

a. $n \times n$ **Jost solutions**.

Let us define the *Jost solution from the left* $F_l(x, \lambda)$ and the *Jost solution from the right* $F_r(x, \lambda)$ as those solutions of the matrix Schrödinger equation (1.4) which satisfy the asymptotic conditions

$$F_l(x, \lambda) = e^{i\lambda x} \left[I_n + o(1) \right], \qquad x \to +\infty, \tag{3.1a}$$

$$F_r(x, \lambda) = e^{-i\lambda x} \left[I_n + o(1) \right], \qquad x \to -\infty, \tag{3.1b}$$

where $n = 2m$. Calling $m_l(x,\lambda) = e^{-i\lambda x}F_l(x,\lambda)$ and $m_r(x,\lambda) = e^{i\lambda x}F_r(x,\lambda)$ *Faddeev functions*, we easily define them as the unique solutions of the Volterra integral equations

$$m_l(x,\lambda) = I_n + \int_x^\infty dy\, \frac{e^{2i\lambda(y-x)} - 1}{2i\lambda} \boldsymbol{Q}(y) m_l(y,\lambda), \tag{3.2a}$$

$$m_r(x,\lambda) = I_n + \int_{-\infty}^x dy\, \frac{e^{2i\lambda(x-y)} - 1}{2i\lambda} \boldsymbol{Q}(y) m_r(y,\lambda). \tag{3.2b}$$

Then, for each $x \in \mathbb{R}$, $m_l(x,\lambda)$ and $m_r(x,\lambda)$ are continuous in $\lambda \in \mathbb{C}^+ \cup \mathbb{R}$, are analytic in $\lambda \in \mathbb{C}^+$, and tend to I_n as $\lambda \to \infty$ from within $\mathbb{C}^+ \cup \mathbb{R}$. For $0 \neq \lambda \in \mathbb{R}$ we can reshuffle (3.2) and arrive at the asymptotic relations

$$F_l(x,\lambda) = e^{i\lambda x}A_l(\lambda) + e^{-i\lambda x}B_l(\lambda) + o(1), \qquad x \to -\infty, \tag{3.3a}$$

$$F_r(x,\lambda) = e^{-i\lambda x}A_r(\lambda) + e^{i\lambda x}B_r(\lambda) + o(1), \qquad x \to +\infty, \tag{3.3b}$$

where

$$A_{r,l}(\lambda) = I_n - \frac{1}{2i\lambda} \int_{-\infty}^\infty dy\, \boldsymbol{Q}(y) m_{r,l}(y,\lambda), \tag{3.4a}$$

$$B_{r,l}(\lambda) = \frac{1}{2i\lambda} \int_{-\infty}^\infty dy\, e^{\mp 2i\lambda y} \boldsymbol{Q}(y) m_{r,l}(y,\lambda). \tag{3.4b}$$

Then $A_{r,l}(\lambda)$ is continuous in $0 \neq \lambda \in \mathbb{C}^+ \cup \mathbb{R}$, is analytic in $\lambda \in \mathbb{C}^+$, and tends to I_n as $\lambda \to \infty$ from within $\mathbb{C}^+ \cup \mathbb{R}$, while $2i\lambda[I_n - A_{r,l}(\lambda)]$ has the finite limit $-\boldsymbol{\Delta}_{r,l} = \int_{-\infty}^\infty dy\, \boldsymbol{Q}(y) m_{r,l}(y,\lambda)$ as $\lambda \to 0$ from within $\mathbb{C}^+ \cup \mathbb{R}$. By the same token, $B_{r,l}(\lambda)$ is continuous in $0 \neq \lambda \in \mathbb{R}$, vanishes as $\lambda \to \pm\infty$, and satisfies $2i\lambda B_{r,l}(\lambda) \to -\boldsymbol{\Delta}_{r,l}$ as $\lambda \to 0$ along the real λ-axis.

b. $2n \times 2n$ **Jost solutions**.

Putting

$$\boldsymbol{F}_l(x,\lambda) = \begin{pmatrix} F_l(x,-\lambda) & F_l(x,\lambda) \\ F_l'(x,-\lambda) & F_l'(x,\lambda) \end{pmatrix}, \ \boldsymbol{F}_r(x,\lambda) = \begin{pmatrix} F_r(x,\lambda) & F_r(x,-\lambda) \\ F_r'(x,\lambda) & F_r'(x,-\lambda) \end{pmatrix}, \tag{3.5}$$

where the prime denotes differentiation with respect to x, we obtain

$$\boldsymbol{F}_r(x,\lambda) = \boldsymbol{F}_l(x,\lambda) \begin{pmatrix} A_r(\lambda) & B_r(-\lambda) \\ B_r(\lambda) & A_r(-\lambda) \end{pmatrix}, \tag{3.6a}$$

$$\boldsymbol{F}_l(x,\lambda) = \boldsymbol{F}_r(x,\lambda) \begin{pmatrix} A_l(-\lambda) & B_l(\lambda) \\ B_l(-\lambda) & A_l(\lambda) \end{pmatrix}, \tag{3.6b}$$

where $0 \neq \lambda \in \mathbb{R}$. Using that $\boldsymbol{F}_{r,l}(x,\lambda)$ satisfies the linear first order system

$$\begin{pmatrix} V \\ V' \end{pmatrix}' = \begin{pmatrix} 0_{n\times n} & I_n \\ \boldsymbol{Q}(x) - \lambda^2 I_n & 0_{n\times n} \end{pmatrix} \begin{pmatrix} V \\ V' \end{pmatrix} \tag{3.7}$$

with traceless system matrix, we see that, for $0 \neq \lambda \in \mathbb{R}$, $\boldsymbol{F}_{r,l}(x,\lambda)$ has a determinant not depending on $x \in \mathbb{R}$. Using (3.1) we easily verify that $\det \boldsymbol{F}_{r,l}(x,\lambda) = (2i\lambda)^n$ for $0 \neq \lambda \in \mathbb{R}$.

Let us now apply the x-independence (A proof of this property will be given in Appendix A) of $\boldsymbol{W}(x,\lambda)^\dagger(\sigma_2 \otimes \boldsymbol{\sigma}_3)\boldsymbol{V}(x,\lambda)$, where $\sigma_2 \otimes \boldsymbol{\sigma}_3 =$

$\begin{pmatrix} 0_{n\times n} & -i\boldsymbol{\sigma}_3 \\ i\boldsymbol{\sigma}_3 & 0_{n\times n} \end{pmatrix}$, for any two square matrix solutions $\boldsymbol{V}$ and $\boldsymbol{W}$ of (3.7) to derive identities for the A and B coefficients by equating the asymptotics as $x \to +\infty$ to the asymptotics as $x \to -\infty$. Using $\boldsymbol{V} = \boldsymbol{W} = \boldsymbol{\Phi} = \boldsymbol{F}_r\boldsymbol{e}_1 + \boldsymbol{F}_l\boldsymbol{e}_2$, where $\boldsymbol{e}_1 = I_n \oplus 0_{n\times n}$ and $\boldsymbol{e}_2 = 0_{n\times n} \oplus I_n$, we get

$$A_{r,l}(\lambda)^\dagger\boldsymbol{\sigma}_3 A_{r,l}(\lambda) - B_{r,l}(\lambda)^\dagger\boldsymbol{\sigma}_3 B_{r,l}(\lambda) = \boldsymbol{\sigma}_3, \tag{3.8a}$$

$$B_{r,l}(\lambda)^\dagger = -\boldsymbol{\sigma}_3 B_{l,r}(\lambda)\boldsymbol{\sigma}_3, \tag{3.8b}$$

where $0 \neq \lambda \in \mathbb{R}$. Using $\boldsymbol{V} = \boldsymbol{W} = \boldsymbol{F}_{r,l}$, we get

$$A_{r,l}(\lambda)^\dagger\boldsymbol{\sigma}_3 B_{r,l}(-\lambda) = B_{r,l}(\lambda)^\dagger\boldsymbol{\sigma}_3 A_{r,l}(-\lambda), \tag{3.9}$$

where $0 \neq \lambda \in \mathbb{R}$. Using the x-independence of $\boldsymbol{W}(x, -\lambda^*)^\dagger(\sigma_2 \otimes \boldsymbol{\sigma}_3)\boldsymbol{V}(x, \lambda)$ for $\boldsymbol{V} = \boldsymbol{F}_l$ and $\boldsymbol{W} = \boldsymbol{F}_r$ we obtain

$$A_r(\lambda)^\dagger = \boldsymbol{\sigma}_3 A_l(-\lambda)\boldsymbol{\sigma}_3, \qquad B_r(\lambda)^\dagger = -\boldsymbol{\sigma}_3 B_l(\lambda)\boldsymbol{\sigma}_3, \tag{3.10}$$

where $0 \neq \lambda \in \mathbb{R}$. Finally, for $\boldsymbol{V} = \boldsymbol{W} = \boldsymbol{\Phi}$ we get

$$A_{r,l}(-\lambda^*)^\dagger = \boldsymbol{\sigma}_3 A_{l,r}(\lambda)\boldsymbol{\sigma}_3, \tag{3.11}$$

where $0 \neq \lambda \in \mathbb{C}^+ \cup \mathbb{R}$.

c. Reflection coefficients.

Introducing the *reflection coefficients*

$$R_{r,l}(\lambda) = B_{r,l}(\lambda)A_{r,l}(\lambda)^{-1} = -A_{l,r}(\lambda)^{-1}B_{l,r}(-\lambda) \tag{3.12}$$

and the *transmission coefficients* $A_{r,l}(\lambda)^{-1}$, we obtain the *Riemann-Hilbert problem*

$$\begin{pmatrix} F_l(x,-\lambda) & F_r(x,-\lambda) \end{pmatrix} = \begin{pmatrix} F_r(x,\lambda) & F_l(x,\lambda) \end{pmatrix} \begin{pmatrix} A_r(\lambda)^{-1} & -R_l(\lambda) \\ -R_r(\lambda) & A_l(\lambda)^{-1} \end{pmatrix}, \tag{3.13}$$

where the matrix $\boldsymbol{S}(\lambda)$ containing the A and R quantities is called the *scattering matrix* and a discussion of the nonsingularity of $A_{r,l}(\lambda)$ will be presented shortly. Then it is easily verified that

$$R_{r,l}(\lambda)^\dagger = \boldsymbol{\sigma}_3 R_{r,l}(-\lambda)\boldsymbol{\sigma}_3, \tag{3.14a}$$

and

$$\boldsymbol{S}(\lambda)^\dagger(\boldsymbol{\sigma}_3 \oplus \boldsymbol{\sigma}_3)\boldsymbol{S}(\lambda) = \boldsymbol{\sigma}_3 \oplus \boldsymbol{\sigma}_3, \tag{3.14b}$$

provided $0 \neq \lambda \in \mathbb{R}$ and $\det A_{r,l}(\lambda) \neq 0$.

Above we have defined $\boldsymbol{\Delta}_{r,l}$ as follows:

$$\boldsymbol{\Delta}_{r,l} = \lim_{\lambda\to 0} 2i\lambda A_{r,l}(\lambda) = -\lim_{\lambda\to 0^\pm} 2i\lambda B_{r,l}(\lambda),$$

where the first limit may be taken from the closed upper half-plane. Then the matrices $\boldsymbol{\Delta}_{r,l}$ have the same determinant. If $\boldsymbol{\Delta}_{r,l}$ is nonsingular, we are said to be in the *generic case*; if instead $\boldsymbol{\Delta}_{r,l}$ is singular, we are said to be in the *exceptional case* cf. [12]. We are said to be in the *superexceptional case* if $\boldsymbol{\Delta}_{r,l} = 0_{n\times n}$ and $A_{r,l}(\lambda)$ tends to a nonsingular matrix, $A_{r,l}(0)$ say, as $\lambda \to 0$ from within $\mathbb{C}^+ \cup \mathbb{R}$.

Throughout this article, we assume the absence of *spectral singularities*, i.e., the absence of nonzero real λ for which $\det A_{r,l}(\lambda) = 0$. Under this condition the reflection coefficients $R_{r,l}(\lambda)$ are continuous in $0 \neq \lambda \in \mathbb{R}$.

d. Triangular representations.

The Jost solutions allow the triangular representations

$$F_l(x,\lambda) = e^{i\lambda x} I_n + \int_x^\infty dy\, e^{i\lambda y} K(x,y), \tag{3.15a}$$

$$F_r(x,\lambda) = e^{-i\lambda x} I_n + \int_{-\infty}^x dy\, e^{-i\lambda y} J(x,y), \tag{3.15b}$$

where for every $x \in \mathbb{R}$

$$\int_x^\infty dy\, \|K(x,y)\| + \int_{-\infty}^x dy\, \|J(x,y)\| < +\infty. \tag{3.16}$$

The integral equations satisfied by the auxiliary matrix functions $K(x,y)$ and $J(x,y)$ derived in [35] imply that

$$K(x,x) = \tfrac{1}{2} \int_x^\infty dy\, \boldsymbol{Q}(y), \qquad J(x,x) = \tfrac{1}{2} \int_{-\infty}^x dy\, \boldsymbol{Q}(y). \tag{3.17}$$

e. Wiener algebras.

For convenience we introduce the well-known Wiener algebra [42]. By the (continuous) Wiener algebra $\mathcal{W}$ we mean the complex vector space of constants plus Fourier transforms of L^1-functions

$$\mathcal{W} = \{c + \hat{h} : c \in \mathbb{C},\ h \in L^1(\mathbb{R})\}$$

endowed with the norm $|c| + \|h\|_1$. Here we define the Fourier transform as follows: $\hat{h}(k) = \int_{-\infty}^{\infty} dy\, e^{iky} h(y)$. The invertible elements of the commutative Banach algebra $\mathcal{W}$ with unit element are exactly those $c + \hat{h} \in \mathcal{W}$ for which $c \neq 0$ and $c + \hat{h}(k) \neq 0$ for each $k \in \mathbb{R}$ [42].

The algebra $\mathcal{W}$ has the two closed subalgebras $\mathcal{W}^+$ and $\mathcal{W}^-$ consisting of those $c + \hat{h} \in \mathcal{W}$ for which h is supported on $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively. The invertible elements of $\mathcal{W}^\pm$ are exactly those $c + \hat{h} \in \mathcal{W}^\pm$ for which $c \neq 0$ and $c + \hat{h}(k) \neq 0$ for each $k \in \mathbb{C}^\pm \cup \mathbb{R}$ [42]. Letting $\mathcal{W}_0^\pm$ and $\mathcal{W}_0$ stand for the (nonunital) closed subalgebras of $\mathcal{W}^\pm$ and $\mathcal{W}$ consisting of those $c + \hat{h}$ for which $c = 0$, we obtain the direct sum decompositions

$$\mathcal{W} = \mathbb{C} \oplus \mathcal{W}_0^+ \oplus \mathcal{W}_0^-, \qquad \mathcal{W}_0 = \mathcal{W}_0^+ \oplus \mathcal{W}_0^-.$$

We denote the (bounded) projections of $\mathcal{W}$ onto $\mathcal{W}_0^\pm$ along $\mathcal{W}^\mp$ by $\Pi_\pm$.

Throughout this article we denote the vector spaces of $n \times m$ matrices with entries in $\mathcal{W}$, $\mathcal{W}^\pm$, and $\mathcal{W}_0^\pm$ by $\mathcal{W}^{n\times m}$, $\mathcal{W}^{\pm\, n\times m}$, and $\mathcal{W}_0^{\pm\, n\times m}$, respectively. We write $L^1(\mathbb{R})^{n\times m}$ and $L^1(\mathbb{R}^\pm)^{n\times m}$ for the vector spaces of $n \times m$ matrices with entries in $L^1(\mathbb{R})$ and $L^1(\mathbb{R}^\pm)$, respectively. Using a submultiplicative matrix norm, we can turn all of these vector spaces into Banach spaces. It is then clear that $\mathcal{W}^{n\times n}$ and $\mathcal{W}^{\pm n\times n}$ are noncommutative Banach algebras with unit element and $\mathcal{W}_0^{\pm n\times n}$ are (nonunital) noncommutative Banach algebras. As above, we then define $\Pi^\pm$ as the (bounded) projections of $\mathcal{W}^{n\times m}$ onto $\mathcal{W}_0^{\pm n\times m}$ along $\mathcal{W}^{\mp n\times m}$. The invertible elements of $\mathcal{W}^{n\times n}$ and $\mathcal{W}^{\pm n\times n}$ are exactly those elements whose determinants are invertible elements of $\mathcal{W}$ and $\mathcal{W}^\pm$, respectively. Hence, according to (3.15) and (3.16), for each $x \in \mathbb{R}$ the Faddeev functions $m_{r,l}(x,\cdot) \in \mathcal{W}_+^{n\times n}$. We then easily prove with the help of (3.4) that $2i\lambda[I_n - A_{r,l}(\lambda)]$ belong to $\mathcal{W}_+^{n\times n}$ and $2i\lambda B_{r,l}(\lambda)$ belong to $\mathcal{W}^{n\times n}$.

Assuming the absence of spectral singularities and to be in the generic case, we proved in [35] that the reflection coefficients $R_{r,l}(\lambda)$ belong to $\mathcal{W}_0^{n\times n}$ and the transmission coefficients $A_{r,l}(\lambda)^{-1}$ to $\mathcal{W}_+^{n\times n}$. In the superexceptional case, where $\boldsymbol{\Delta}_{r,l} = 0_{n\times n}$, we proved in [35] that $A_{r,l} \in \mathcal{W}_+^{n\times n}$, provided $\boldsymbol{Q} \in L^1(\mathbb{R};(1+|x|)^2dx)$; assuming the absence of spectral singularities and using the nonsingularity of $A_{r,l}(0)$, we see that the reflection coefficients $R_{r,l}(\lambda)$ and the transmission coefficients $A_{r,l}(\lambda)^{-1}$ belong to $\mathcal{W}^{n\times n}$.

At present it is not known if, under the absence of spectral singularities, the reflection and transmission coefficients belong to $\mathcal{W}^{n\times n}$ in any other exceptional case and for general $\boldsymbol{Q} \in L^1(\mathbb{R};(1+|x|)dx)$. Under the condition $\boldsymbol{Q} \in L^1(\mathbb{R};(1+|x|)dx)$, the continuity of the reflection and transmission coefficients at $\lambda = 0$ is known for $n = 1$ [56] and for selfadjoint potentials [12]. In neither case is it known if these continuous functions belong to $\mathcal{W}$.

4 Inverse scattering theory

In this section we introduce the Marchenko integral equations for the matrix Schrödinger equation (1.4) with Faddeev class potential $\boldsymbol{Q}$ satisfying (1.6). We make use of the hypothesis that the reflection coefficients $R_{r,l} \in \mathcal{W}_0^{n\times n}$, something proved in the generic case but not in the most general exceptional case. For the sake of simplicity we assume that the poles of $A_{r,l}(\lambda)^{-1}$ in $\mathbb{C}^+$ are simple. The extension to multiple pole situations is rather technical but straightforward [33]. Inverse scattering theory is well documented in the scalar case [40, 29, 26, 27], in the matrix half-line case [14, 15], and in the matrix full-line case [78, 12]. The adjoint symmetry (1.6) requires some modifications to existing theory (cf. [35]).

Let us write the *transmission coefficients* in the form

$$A_r(\lambda)^{-1} = A_{r0}(\lambda) + \sum_{s=1}^{N} \frac{\tau_{r;s}}{\lambda - \lambda_s}, \quad A_l(\lambda)^{-1} = A_{l0}(\lambda) + \sum_{s=1}^{N} \frac{\tau_{l;s}}{\lambda - \lambda_s}, \quad (4.1)$$

where $\lambda_1, \ldots, \lambda_N$ are the distinct simple poles of $A_{r,l}(\lambda)^{-1}$ in $\mathbb{C}^+$, $\tau_{r;s}$ and $\tau_{l;s}$ are the residues of $A_r(\lambda)^{-1}$ and $A_l(\lambda)^{-1}$ at $\lambda = \lambda_s$ ($s = 1, \ldots, N$), and $A_{r0}(\lambda)$ and $A_{l0}(\lambda)$ are continuous in $\lambda \in \mathbb{C}^+ \cup \mathbb{R}$, are analytic in $\lambda \in \mathbb{C}^+$, and tend to I_n as $\lambda \to \infty$ from within $\mathbb{C}^+ \cup \mathbb{R}$. Then it is easily proved that $\tau_{r;\bar{s}} = -\boldsymbol{\sigma}_3 \tau_{l;s}^\dagger \boldsymbol{\sigma}_3$ and $\tau_{l;\bar{s}} = -\boldsymbol{\sigma}_3 \tau_{r;s}^\dagger \boldsymbol{\sigma}_3$ whenever $\lambda_{\bar{s}} = -\lambda_s^*$ (cf. [35]).

Let us write

$$R_r(\lambda) = \int_{-\infty}^{\infty} d\alpha\, e^{-i\lambda\alpha} \hat{R}_r(\alpha), \qquad R_l(\lambda) = \int_{-\infty}^{\infty} d\alpha\, e^{i\lambda\alpha} \hat{R}_l(\alpha), \qquad (4.2)$$

where $\hat{R}_{r,l} \in L^1(\mathbb{R})^{n\times n}$. In fact, this has only been proved in the generic case and, under the condition that $\boldsymbol{Q} \in L^1(\mathbb{R}; (1+|x|)^2 dx)$, in the superexceptional case. Using (3.14a) it follows that $\hat{R}_{r,l}(\alpha; t)$ are $\boldsymbol{\sigma}_3$-hermitian matrices. Then the following Marchenko integral equations can be derived (see [35] for details):

$$K(x,y) + \Omega_r(x+y) + \int_x^\infty dz\, K(x,z)\Omega_r(z+y) = 0_{n\times n}, \tag{4.3a}$$

$$J(x,y) + \Omega_l(x+y) + \int_{-\infty}^x dz\, J(x,z)\Omega_l(z+y) = 0_{n\times n}, \tag{4.3b}$$

where the *Marchenko integral kernels* are given by

$$\Omega_r(w) = \hat{R}_r(w) + \sum_{s=1}^N e^{i\lambda_s w} N_{r;s}, \tag{4.4a}$$

$$\Omega_l(w) = \hat{R}_l(w) + \sum_{s=1}^N e^{-i\lambda_s w} N_{l;s}. \tag{4.4b}$$

Here $N_{r;s}$ and $N_{l;s}$ are the so-called norming constants defined by

$$F_r(x,\lambda_s)\tau_{r;s} = iF_l(x,\lambda_s)N_{r;s}, \tag{4.5a}$$

$$F_l(x,\lambda_s)\tau_{l;s} = iF_r(x,\lambda_s)N_{l;s}, \tag{4.5b}$$

where λ_s is a (simple) pole of $A_{r,l}(\lambda)^{-1}$ in $\mathbb{C}^+$ ($s = 1, 2, \ldots, N$). Then $\tau_{r;s}$ and $N_{r;s}$ have the same rank and the same null space; the same thing is true for $\tau_{l;s}$ and $N_{l;s}$. As in [33], we can prove the adjoint symmetry relations

$$\Omega_{r,l}(w) = \boldsymbol{\sigma}_3 \Omega_{r,l}(w)^\dagger \boldsymbol{\sigma}_3, \tag{4.6}$$

thus implying the following symmetry relations for the norming constants:

$$N_{r;s} = \boldsymbol{\sigma}_3 N_{r;\bar{s}}^\dagger \boldsymbol{\sigma}_3, \qquad N_{l;s} = \boldsymbol{\sigma}_3 N_{l;\bar{s}}^\dagger \boldsymbol{\sigma}_3, \tag{4.7}$$

provided $\lambda_s = -\lambda_{\bar{s}}^*$ is a simple pole of $A_{r,l}(\lambda)^{-1}$. For the rather tedious details we refer to [35, App. B].

Example. Let us now solve the Marchenko integral equations (4.3) in the one-soliton case, where $\Omega_r(w;t) = e^{-a_0 w} N_{r;0}(t)$ and $\Omega_l(w;t) = e^{a_0 w} N_{l;0}(t)$ for a suitable eigenvalue $\lambda_0 = ia_0 \in \mathbb{C}^+$. Then separation of variables yields

$$K(x,y;t) = -e^{-a_0(x+y)} \left[I_n + \frac{1}{2a_0} e^{-2a_0 x} N_{r;0}(t) \right]^{-1} N_{r;0}(t), \tag{4.8a}$$

$$J(x,y;t) = -e^{a_0(x+y)} \left[I_n + \frac{1}{2a_0} e^{2a_0 x} N_{l;0}(t) \right]^{-1} N_{l;0}(t). \tag{4.8b}$$

so that

$$\int_x^\infty dy\, \boldsymbol{Q}(y;t) = -2 \left[e^{2a_0 x} I_n + \frac{1}{2a_0} N_{r;0}(t) \right]^{-1} N_{r;0}(t), \tag{4.9a}$$

$$\int_{-\infty}^x dy\, \boldsymbol{Q}(y;t) = -2 \left[e^{-2a_0 x} I_n + \frac{1}{2a_0} N_{l;0}(t) \right]^{-1} N_{l;0}(t), \tag{4.9b}$$

where the $\boldsymbol{\sigma}_3$-hermitian norming constants $N_{r;0}(t)$ and $N_{l;0}(t)$ will be expressed in their initial values shortly. The off-diagonal parts of these expressions yield explicit expressions for $\mathcal{Q}_r - \mathcal{Q}(x;t)$ and $\mathcal{Q}(x;t) - \mathcal{Q}_l$, respectively.

5 Time evolution of the scattering data

In this section we establish the time evolution of the scattering data of the matrix Schrödinger equation. We then go on to derive the Marchenko integral kernels as a function of time. These results allow us, in Sec. 6, to derive the reflectionless solutions of the integrable nonlocal equation (2.3) and hence of the focusing matrix NLS equation.

Recall that the integrable equation (2.3) arises as the zero curvature condition of the AKNS pair $\{\boldsymbol{X}, \boldsymbol{T}\}$ given by (2.7). Thus there exist nonsingular matrices $C_{\boldsymbol{F}_r}(\lambda;t)$ and $C_{\boldsymbol{F}_l}(\lambda;t)$ not depending on $x \in \mathbb{R}$ such that

$$\boldsymbol{F}_r(x,\lambda;t) = \boldsymbol{V}(x,\lambda;t) C_{\boldsymbol{F}_r}(\lambda;t)^{-1}, \qquad \boldsymbol{F}_l(x,\lambda;t) = \boldsymbol{V}(x,\lambda;t) C_{\boldsymbol{F}_l}(\lambda;t)^{-1}.$$

Then a simple differentiation yields

$$[C_{\boldsymbol{F}_r}(\lambda;t)]_t \, C_{\boldsymbol{F}_r}(\lambda;t)^{-1} = \boldsymbol{F}_r^{-1} \boldsymbol{T} \boldsymbol{F}_r - \boldsymbol{F}_r^{-1} [\boldsymbol{F}_r]_t, \tag{5.1a}$$

$$[C_{\boldsymbol{F}_l}(\lambda;t)]_t \, C_{\boldsymbol{F}_l}(\lambda;t)^{-1} = \boldsymbol{F}_l^{-1} \boldsymbol{T} \boldsymbol{F}_l - \boldsymbol{F}_l^{-1} [\boldsymbol{F}_l]_t, \tag{5.1b}$$

where the two left-hand sides do not depend on $x \in \mathbb{R}$. Using (2.4) we now compute the $x \to \pm\infty$ limits of the two right-hand sides by evaluating the matrix product

$$\frac{1}{2i\lambda}\begin{pmatrix} i\lambda e^{i\lambda x}I_n & -e^{i\lambda x}I_n \\ i\lambda e^{-i\lambda x}I_n & e^{-i\lambda x}I_n \end{pmatrix}\begin{pmatrix} -2i\lambda^2\boldsymbol{\sigma}_3 & -2i\boldsymbol{\sigma}_3\mathcal{Q}_{r,l} \\ 2i\lambda^2\boldsymbol{\sigma}_3\mathcal{Q}_{r,l} & -2i\lambda^2\boldsymbol{\sigma}_3 \end{pmatrix}\begin{pmatrix} e^{-i\lambda x}I_n & e^{i\lambda x}I_n \\ -i\lambda e^{-i\lambda x}I_n & i\lambda e^{i\lambda x}I_n \end{pmatrix}$$

and obtain

$$[C_{\boldsymbol{F}_r}(\lambda;t)]_t\, C_{\boldsymbol{F}_r}(\lambda;t)^{-1} = \begin{pmatrix} -\Lambda_r^{\mathrm{up}}(\lambda) & 0_{n\times n} \\ 0_{n\times n} & -\Lambda_r^{\mathrm{dn}}(\lambda) \end{pmatrix}, \tag{5.2a}$$

$$[C_{\boldsymbol{F}_l}(\lambda;t)]_t\, C_{\boldsymbol{F}_l}(\lambda;t)^{-1} = \begin{pmatrix} -\Lambda_l^{\mathrm{up}}(\lambda) & 0_{n\times n} \\ 0_{n\times n} & -\Lambda_l^{\mathrm{dn}}(\lambda) \end{pmatrix}, \tag{5.2b}$$

where

$$\Lambda_{r,l}^{\mathrm{up}}(\lambda) = 2i\lambda^2\boldsymbol{\sigma}_3 + 2\lambda\boldsymbol{\sigma}_3\mathcal{Q}_{r,l}, \tag{5.3a}$$

$$\Lambda_{r,l}^{\mathrm{dn}}(\lambda) = 2i\lambda^2\boldsymbol{\sigma}_3 - 2\lambda\boldsymbol{\sigma}_3\mathcal{Q}_{r,l}, \tag{5.3b}$$

are time invariant. Then, using that $\mathcal{Q}^\dagger = -\mathcal{Q}$ and $\mathcal{Q}\boldsymbol{\sigma}_3 = -\boldsymbol{\sigma}_3\mathcal{Q}$, we arrive at the symmetry relations

$$\Lambda_{r,l}^{\mathrm{up}}(\lambda) = \boldsymbol{\sigma}_3\Lambda_{r,l}^{\mathrm{dn}}(\lambda)\boldsymbol{\sigma}_3, \tag{5.4a}$$

$$\Lambda_{r,l}^{\mathrm{up}}(-\lambda^*)^\dagger = -\Lambda_{r,l}^{\mathrm{up}}(\lambda), \tag{5.4b}$$

$$\Lambda_{r,l}^{\mathrm{dn}}(-\lambda^*)^\dagger = -\Lambda_{r,l}^{\mathrm{dn}}(\lambda), \tag{5.4c}$$

where $\lambda \in \mathbb{C}^+ \cup \mathbb{R}$. Relating $\boldsymbol{F}_{r,l}(x,\lambda;t)$ by means of the equalities [cf. (3.6)]

$$\boldsymbol{F}_r(x,\lambda;t) = \boldsymbol{F}_l(x,\lambda;t)\boldsymbol{A}_r(\lambda;t), \qquad \boldsymbol{F}_l(x,\lambda;t) = \boldsymbol{F}_r(x,\lambda;t)\boldsymbol{A}_l(\lambda;t),$$

where the factors $\boldsymbol{A}_{r,l}(\lambda;t)$ are given by the matrices

$$\boldsymbol{A}_r(\lambda;t) = \begin{pmatrix} A_r(\lambda;t) & B_r(-\lambda;t) \\ B_r(\lambda;t) & A_r(-\lambda;t) \end{pmatrix}, \ \boldsymbol{A}_l(\lambda;t) = \begin{pmatrix} A_l(-\lambda;t) & B_l(\lambda;t) \\ B_l(-\lambda;t) & A_l(\lambda;t) \end{pmatrix}, \tag{5.5}$$

for $0 \neq \lambda \in \mathbb{R}$ we compute

$$\begin{aligned} [\boldsymbol{A}_r]_t &= -\boldsymbol{F}_l^{-1}[\boldsymbol{F}_l]_t\boldsymbol{F}_l^{-1}\boldsymbol{F}_r + \boldsymbol{F}_l^{-1}[\boldsymbol{F}_r]_t \\ &= -\boldsymbol{F}_l^{-1}\left(\boldsymbol{T}\boldsymbol{F}_l - \boldsymbol{F}_l\,[C_{\boldsymbol{F}_l}(\lambda;t)]_t\,C_{\boldsymbol{F}_l}(\lambda;t)^{-1}\right)\boldsymbol{A}_r \\ &+ \boldsymbol{F}_l^{-1}\left(\boldsymbol{T}\boldsymbol{F}_r - \boldsymbol{F}_r\,[C_{\boldsymbol{F}_r}(\lambda;t)]_t\,C_{\boldsymbol{F}_r}(\lambda;t)^{-1}\right) \\ &= [C_{\boldsymbol{F}_l}(\lambda;t)]_t\,C_{\boldsymbol{F}_l}(\lambda;t)^{-1}\boldsymbol{A}_r - \boldsymbol{A}_r\,[C_{\boldsymbol{F}_r}(\lambda;t)]_t\,C_{\boldsymbol{F}_r}(\lambda;t)^{-1} \\ &= \boldsymbol{A}_r \begin{pmatrix} \Lambda_r^{\mathrm{up}}(\lambda) & 0_{n\times n} \\ 0_{n\times n} & \Lambda_r^{\mathrm{dn}}(\lambda) \end{pmatrix} - \begin{pmatrix} \Lambda_l^{\mathrm{up}}(\lambda) & 0_{n\times n} \\ 0_{n\times n} & \Lambda_l^{\mathrm{dn}}(\lambda) \end{pmatrix} \boldsymbol{A}_r. \end{aligned} \tag{5.6}$$

Using that $\boldsymbol{A}_l(\lambda;t) = \boldsymbol{A}_r(\lambda;t)^{-1}$, we obtain from (5.6)

$$[\boldsymbol{A}_l]_t = \boldsymbol{A}_l \begin{pmatrix} \Lambda_l^{\mathrm{up}}(\lambda) & 0_{n\times n} \\ 0_{n\times n} & \Lambda_l^{\mathrm{dn}}(\lambda) \end{pmatrix} - \begin{pmatrix} \Lambda_r^{\mathrm{up}}(\lambda) & 0_{n\times n} \\ 0_{n\times n} & \Lambda_r^{\mathrm{dn}}(\lambda) \end{pmatrix} \boldsymbol{A}_l. \tag{5.7}$$

Therefore, (5.5), (5.6), and (5.7) imply

$$[A_r]_t = A_r(\lambda;t)\Lambda_r^{\mathrm{up}}(\lambda) - \Lambda_l^{\mathrm{up}}(\lambda)A_r(\lambda;t), \tag{5.8a}$$

$$[A_l]_t = A_l(\lambda;t)\Lambda_l^{\mathrm{dn}}(\lambda) - \Lambda_r^{\mathrm{dn}}(\lambda)A_l(\lambda;t), \tag{5.8b}$$

where $0 \neq \lambda \in \mathbb{C}^+ \cup \mathbb{R}$, and

$$[B_r]_t = B_r(\lambda;t)\Lambda_r^{\mathrm{up}}(\lambda) - \Lambda_l^{\mathrm{dn}}(\lambda)B_r(\lambda;t), \tag{5.8c}$$

$$[B_l]_t = B_l(\lambda;t)\Lambda_l^{\mathrm{dn}}(\lambda) - \Lambda_r^{\mathrm{up}}(\lambda)B_l(\lambda;t), \tag{5.8d}$$

where $0 \neq \lambda \in \mathbb{R}$.

Proposition 5.1 *The reflection coefficients satisfy the following differential equations:*

$$[R_r]_t = R_r(\lambda;t)\Lambda_l^{up}(\lambda) - \Lambda_l^{dn}(\lambda)R_r(\lambda;t), \tag{5.9a}$$

$$[R_l]_t = R_l(\lambda;t)\Lambda_r^{dn}(\lambda) - \Lambda_r^{up}(\lambda)R_l(\lambda;t), \tag{5.9b}$$

where $0 \neq \lambda \in \mathbb{R}$*. Moreover, for fixed* λ *the matrices* $\boldsymbol{\sigma}_3 R_{r,l}(\lambda;t)$ *have time invariant traces.*

Proof. Using (3.12) we compute

$$\begin{aligned}[R_r]_t &= [B_r A_r^{-1}]_t = [B_r]_t A_r^{-1} - B_r A_r^{-1}[A_r]_t A_r^{-1} \\ &= \left(B_r \Lambda_r^{\text{up}} - \Lambda_l^{\text{dn}} B_r\right) A_r^{-1} - B_r A_r^{-1}\left(A_r \Lambda_r^{\text{up}} - \Lambda_l^{\text{up}} A_r\right) A_r^{-1} \\ &= B_r A_r^{-1} \Lambda_l^{\text{up}} - \Lambda_l^{\text{dn}} B_r A_r^{-1} = R_r \Lambda_l^{\text{up}} - \Lambda_l^{\text{dn}} R_r,\end{aligned}$$

where we have not written the dependence on $(\lambda; t)$. Similarly, we compute

$$\begin{aligned}[R_l]_t &= [B_l A_l^{-1}]_t = [B_l]_t A_l^{-1} - B_l A_l^{-1}[A_l]_t A_l^{-1} \\ &= \left(B_l \Lambda_l^{\text{dn}} - \Lambda_r^{\text{up}} B_l\right) A_l^{-1} - B_l A_l^{-1}\left(A_l \Lambda_l^{\text{dn}} - \Lambda_r^{\text{dn}} A_l\right) A_l^{-1} \\ &= B_l A_l^{-1} \Lambda_r^{\text{dn}} - \Lambda_r^{\text{up}} B_l A_l^{-1} = R_l \Lambda_r^{\text{dn}} - \Lambda_r^{\text{up}} R_l.\end{aligned}$$

Finally, since $\boldsymbol{\sigma}_3 \Lambda_{r,l}^{\text{up}}(\lambda) \boldsymbol{\sigma}_3 = \Lambda_{r,l}^{\text{dn}}(\lambda)$, we see that

$$[\boldsymbol{\sigma}_3 R_r]_t = [\boldsymbol{\sigma}_3 R_r(\lambda; t), \Lambda_l^{\text{up}}(\lambda)], \quad [\boldsymbol{\sigma}_3 R_l]_t = [\boldsymbol{\sigma}_3 R_l(\lambda; t), \Lambda_r^{\text{dn}}(\lambda)], \tag{5.10}$$

where the square brackets in the right-hand sides are matrix commutators. Consequently, $[\boldsymbol{\sigma}_3 R_{r,l}]_t$ are traceless matrices. ■

Let us now derive the time evolution equations for the norming constants. First, writing (5.8) in the form

$$\begin{aligned}[A_r^{-1}]_t &= A_r(\lambda; t)^{-1} \Lambda_l^{\text{up}}(\lambda) - \Lambda_r^{\text{up}}(\lambda) A_r(\lambda; t)^{-1}, \\ [A_l^{-1}]_t &= A_l(\lambda; t)^{-1} \Lambda_r^{\text{dn}}(\lambda) - \Lambda_l^{\text{dn}}(\lambda) A_l(\lambda; t)^{-1},\end{aligned}$$

and computing the residues at the simple poles λ_s, we get

$$[\tau_{r;s}]_t = \tau_{r;s}(t)\Lambda_l^{\mathrm{up}}(\lambda_s) - \Lambda_r^{\mathrm{up}}(\lambda_s)\tau_{r;s}(t), \tag{5.11a}$$

$$[\tau_{l;s}]_t = \tau_{l;s}(t)\Lambda_r^{\mathrm{dn}}(\lambda_s) - \Lambda_l^{\mathrm{dn}}(\lambda_s)\tau_{l;s}(t). \tag{5.11b}$$

Next, using (5.2) we write (5.1) in the form

$$[\boldsymbol{F}_{r,l}]_t = \boldsymbol{T}(x,\lambda;t)\boldsymbol{F}_{r,l}(x,\lambda;t) + \boldsymbol{F}_{r,l}(x,\lambda;t)\begin{pmatrix} \Lambda_{r,l}^{\mathrm{up}}(\lambda) & 0_{n\times n} \\ 0_{n\times n} & \Lambda_{r,l}^{\mathrm{dn}}(\lambda)\end{pmatrix}. \tag{5.12}$$

Using the standard block structure $\boldsymbol{T} = \left(\begin{smallmatrix}\boldsymbol{T}_1 & \boldsymbol{T}_2 \\ \boldsymbol{T}_3 & \boldsymbol{T}_4\end{smallmatrix}\right)$ as a 2×2 matrix having $m\times m$ entries, from (5.12) we easily arrive at the identities

$$\begin{aligned}[F_l(x,\lambda_s;t)]_t &= \boldsymbol{T}_1(x,\lambda_s;t)F_l(x,\lambda_s;t) + \boldsymbol{T}_2(x,\lambda_s;t)F_l'(x,\lambda_s;t) \\ &\quad + F_l(x,\lambda_s;t)\Lambda_l^{\mathrm{dn}}(\lambda_s), \qquad (5.13a) \\ [F_r(x,\lambda_s;t)]_t &= \boldsymbol{T}_1(x,\lambda_s;t)F_r(x,\lambda_s;t) + \boldsymbol{T}_2(x,\lambda_s;t)F_r'(x,\lambda_s;t) \\ &\quad + F_r(x,\lambda_s;t)\Lambda_r^{\mathrm{up}}(\lambda_s). \qquad (5.13b)\end{aligned}$$

Differentiating (4.5a) with respect to t, utilizing both of (5.13), and applying (4.5a) as well as its derivative with respect to x, we obtain

$$F_r\left(\Lambda_r^{\mathrm{up}}\tau_{r;s} + [\tau_{r;s}]_t\right) = iF_l\left(\Lambda_l^{\mathrm{dn}}N_{r;s} + [N_{r;s}]_t\right),$$

where we have omitted the arguments $(x,\lambda_s;t)$, λ_s, and t. With the help of

(5.11a) we write the latter in the form

$$F_r \tau_{r;s} \Lambda_l^{\mathrm{up}} = iF_l \left(\Lambda_l^{\mathrm{dn}} N_{r;s} + [N_{r;s}]_t \right).$$

Using (4.5a) once again and considering the $x \to +\infty$ asymptotics of the resulting expression to lose the resulting common factors iF_l, we obtain

$$[N_{r;s}]_t = N_{r;s}(t) \Lambda_l^{\mathrm{up}}(\lambda_s) - \Lambda_l^{\mathrm{dn}}(\lambda_s) N_{r;s}(t). \tag{5.14}$$

Analogously, differentiating (4.5b) with respect to t, utilizing both of (5.13), and applying (4.5b) as well as its derivative with respect to x, we obtain

$$F_l \left(\Lambda_l^{\mathrm{dn}} \tau_{l;s} + [\tau_{l;s}]_t \right) = iF_r \left(\Lambda_r^{\mathrm{up}} N_{l;s} + [N_{l;s}]_t \right),$$

where we have omitted the arguments $(x, \lambda_s; t)$, λ_s, and t. With the help of (5.11b) we write the latter in the form

$$F_r \tau_{l;s} \Lambda_r^{\mathrm{dn}} = iF_l \left(\Lambda_r^{\mathrm{up}} N_{l;s} + [N_{l;s}]_t \right).$$

Using (4.5b) once again and considering the $x \to -\infty$ asymptotics of the resulting expression to lose the resulting common factors iF_r, we obtain

$$[N_{l;s}]_t = N_{l;s}(t) \Lambda_r^{\mathrm{dn}}(\lambda_s) - \Lambda_r^{\mathrm{up}}(\lambda_s) N_{l;s}(t). \tag{5.15}$$

As in the proof of Proposition 5.1, we can prove that for each λ the matrices

$\boldsymbol{\sigma}_3 N_{r;s}(t)$ are similar and the matrices $\boldsymbol{\sigma}_3 N_{l;s}(t)$ are similar. Hence the traces of $\boldsymbol{\sigma}_3 N_{r;s}(t)$ and $\boldsymbol{\sigma}_3 N_{l;s}(t)$ are time independent. Thus the ranks of $N_{r;s}(t)$ and $N_{l;s}(t)$ are time independent. We recall [see (4.7)] that the norming constants corresponding to eigenvalues symmetrically located with respect to the imaginary axis are each other's $\boldsymbol{\sigma}_3$-adjoints.

Let us now derive the differential equations for the Marchenko integral kernels. Using (4.2) and (5.3) we obtain the PDEs

$$[\hat{R}_r]_t = -2i\left([\hat{R}_r]_{\alpha\alpha}\boldsymbol{\sigma}_3 - \boldsymbol{\sigma}_3[\hat{R}_r]_{\alpha\alpha} + [\hat{R}_r]_\alpha\boldsymbol{\sigma}_3\mathcal{Q}_l - \mathcal{Q}_l\boldsymbol{\sigma}_3[\hat{R}_r]_\alpha\right), \tag{5.16a}$$

$$[\hat{R}_l]_t = -2i\left([\hat{R}_l]_{\alpha\alpha}\boldsymbol{\sigma}_3 - \boldsymbol{\sigma}_3[\hat{R}_l]_{\alpha\alpha} + [\hat{R}_l]_\alpha\boldsymbol{\sigma}_3\mathcal{Q}_r - \mathcal{Q}_r\boldsymbol{\sigma}_3[\hat{R}_l]_\alpha\right), \tag{5.16b}$$

provided $\int_{-\infty}^{\infty} d\alpha\,(1+\alpha^2)\|\hat{R}_{r,l}(\alpha;t)\|$ converges for every $t \in \mathbb{R}$. Here we recall that $\hat{R}_{r,l}(\alpha;t)$ are $\boldsymbol{\sigma}_3$-hermitian for all $(\alpha,t) \in \mathbb{R}^2$. Using (4.2) and Proposition 5.1 we see that the traces of $\boldsymbol{\sigma}_3\hat{R}_{r,l}(\alpha;t)$ are time independent. Using (5.16) and (4.4) to derive PDEs for the Marchenko integral kernels $\Omega_{r,l}(w;t)$, we obtain with the help of (5.14) and (5.15)

$$[\Omega_r]_t = -2i\left([\Omega_r]_{ww}\boldsymbol{\sigma}_3 - \boldsymbol{\sigma}_3[\Omega_r]_{ww} + [\Omega_r]_w\boldsymbol{\sigma}_3\mathcal{Q}_l - \mathcal{Q}_l\boldsymbol{\sigma}_3[\Omega_r]_w\right), \tag{5.17a}$$

$$[\Omega_l]_t = -2i\left([\Omega_l]_{ww}\boldsymbol{\sigma}_3 - \boldsymbol{\sigma}_3[\Omega_l]_{ww} + [\Omega_l]_w\boldsymbol{\sigma}_3\mathcal{Q}_r - \mathcal{Q}_r\boldsymbol{\sigma}_3[\Omega_l]_w\right), \tag{5.17b}$$

where $\Omega_{r,l}(w;t)$ are $\boldsymbol{\sigma}_3$-hermitian for all $(w,t) \in \mathbb{R}^2$. Hence, the reflection kernels $\hat{R}_{r,l}(w;t)$ and the Marchenko integral kernels $\Omega_{r,l}(w;t)$ satisfy the same PDEs. Finally, the traces of $\boldsymbol{\sigma}_3\Omega_{r,l}(w;t)$ are time independent.

Recalling that $\mathcal{Q}_{r,l}$ are time independent, we observe that the matrices $\Lambda^{\text{up}}_{r,l}(\lambda)$ and $\Lambda^{\text{dn}}_{r,l}(\lambda)$ are time independent as well. We easily compute

$$e^{t\Lambda^{\text{up}}_{r,l}(\lambda)} = \cos(2\lambda kt)I_n + \frac{\sin(2\lambda kt)}{2\lambda k}\left[2i\lambda^2\boldsymbol{\sigma}_3 + 2\lambda\mathcal{Q}_{r,l}\right], \tag{5.18a}$$

$$e^{t\Lambda^{\text{dn}}_{r,l}(\lambda)} = \cos(2\lambda kt)I_n + \frac{\sin(2\lambda kt)}{2\lambda k}\left[2i\lambda^2\boldsymbol{\sigma}_3 - 2\lambda\mathcal{Q}_{r,l}\right], \tag{5.18b}$$

where $k^2 = \lambda^2 - \mu^2$ and the expressions (5.18) are even functions of k for fixed λ (cf. [32] where these matrix groups also appear). Using that the initial value problem for the matrix differential equation

$$F_t = \boldsymbol{B}_1 F(t) - F(t)\boldsymbol{B}_2$$

has the unique solution

$$F(t) = e^{t\boldsymbol{B}_1}F(0)e^{-t\boldsymbol{B}_2},$$

we obtain for the solutions of (5.9a) and (5.9b)

$$R_r(\lambda;t) = e^{-t\Lambda^{\text{dn}}_l(\lambda)}R_r(\lambda;0)e^{t\Lambda^{\text{up}}_l(\lambda)}, \tag{5.19a}$$

$$R_l(\lambda;t) = e^{-t\Lambda^{\text{up}}_r(\lambda)}R_l(\lambda;0)e^{t\Lambda^{\text{dn}}_r(\lambda)}. \tag{5.19b}$$

Because of (5.4a), the matrices $\boldsymbol{\sigma}_3 R_r(\lambda;t)$ are similar and so are the matrices $\boldsymbol{\sigma}_3 R_l(\lambda;t)$. In the same way we get for the time evolution of the norming constants

$$N_{r;s}(t) = e^{-t\Lambda_l^{\mathrm{dn}}(\lambda_s)} N_{r;s}(0) e^{t\Lambda_l^{\mathrm{up}}(\lambda_s)}, \tag{5.20a}$$

$$N_{l;s}(t) = e^{-t\Lambda_r^{\mathrm{up}}(\lambda_s)} N_{l;s}(0) e^{t\Lambda_r^{\mathrm{dn}}(\lambda_s)}, \tag{5.20b}$$

where $k_s^2 = \lambda_s^2 - \mu^2$ and the expressions (5.20) are even functions of k_s for fixed λ_s. Because of (5.4a), the matrices $\boldsymbol{\sigma}_3 N_{r;s}(t)$ are similar and so are the matrices $\boldsymbol{\sigma}_3 N_{l;s}(t)$. In the same way we derive from (5.8) the identities

$$A_r(\lambda;t) = e^{-t\Lambda_l^{\mathrm{up}}(\lambda)} A_r(\lambda;0) e^{t\Lambda_r^{\mathrm{up}}(\lambda)}, \tag{5.21a}$$

$$A_l(\lambda;t) = e^{-t\Lambda_r^{\mathrm{dn}}(\lambda)} A_l(\lambda;0) e^{t\Lambda_l^{\mathrm{dn}}(\lambda)}, \tag{5.21b}$$

where $0 \neq \lambda \in \mathbb{C}^+ \cup \mathbb{R}$, and

$$B_r(\lambda;t) = e^{-t\Lambda_l^{\mathrm{dn}}(\lambda)} B_r(\lambda;0) e^{t\Lambda_r^{\mathrm{up}}(\lambda)}, \tag{5.21c}$$

$$B_l(\lambda;t) = e^{-t\Lambda_r^{\mathrm{up}}(\lambda)} B_l(\lambda;0) e^{t\Lambda_l^{\mathrm{dn}}(\lambda)}, \tag{5.21d}$$

where $0 \neq \lambda \in \mathbb{R}$. Observe that (5.21) and (3.12) imply (5.19).

6 Multisoliton solutions

In this section we apply the matrix triplet method to write the reflectionless Marchenko integral kernels in separated form and solve the Marchenko equations by separation of variables. This method has been successfully applied to the Korteweg-de Vries (KdV) equation [5, 13], the NLS equation [10, 34], the sine-Gordon equation [74, 11], the modified Korteweg-de Vries (mKdV) equation [30], the Toda lattice equation [73], and the Heisenberg Ferromagnet Equation [36, 37]. An introduction to this method can be found in [77]. In contrast to earlier work, we allow the time factors in these triplets to be absorbed by both the input and output matrices.

Before solving the Marchenko integral equations (4.4), we write the reflectionless Marchenko integral kernels in the form

$$\Omega_r(w;t) = \sum_{s=1}^{N} e^{-a_s w} N_{r;s}(t), \qquad \Omega_l(w;t) = \sum_{s=1}^{N} e^{a_s w} N_{l;s}(t), \tag{6.1}$$

where $a_s = -i\lambda_s$ $(s = 1, \ldots, N)$. Then it is easily proved that, for $s = 1, \ldots, N$, the norming constants $N_{r;s}(t)$ and $N_{l;s}(t)$ both have the same time independent rank r_s. In fact, r_s coincides with the ranks of the residues $\tau_{r;s}$ and $\tau_{l;s}$ of $A_{r,l}(\lambda;t)^{-1}$ at $\lambda = \lambda_s$. Since $\boldsymbol{\sigma}_3 N_{r;s}(t)$ and $\boldsymbol{\sigma}_3 N_{l;s}(t)$ have $\boldsymbol{\sigma}_3 N_{r;\overline{s}}(t)$ and $\boldsymbol{\sigma}_3 N_{l;\overline{s}}(t)$ as their respective complex conjugate transposes whenever $\lambda_s = -\lambda_{\overline{s}}^*$, there exist $n \times r_s$ matrices $\boldsymbol{e}_{r;s}(t)$ and $\boldsymbol{e}_{l;s}(t)$ having $r_s = r_{\overline{s}}$ orthonormal columns and spanning the ranges of $\boldsymbol{\sigma}_3 N_{r;s}(t)$ and $\boldsymbol{\sigma}_3 N_{l;s}(t)$ and time independent diagonal $r_s \times r_s$ matrices $\boldsymbol{d}_{r;s} = \boldsymbol{d}_{r;\overline{s}}^\dagger$ and $\boldsymbol{d}_{l;s} = \boldsymbol{d}_{l;\overline{s}}^\dagger$ having only nonzero diagonal entries such that

$$\boldsymbol{\sigma}_3 N_{r;s}(t) = \boldsymbol{e}_{r;s}(t) \boldsymbol{d}_{r;s} \boldsymbol{e}_{r;\overline{s}}(t)^\dagger, \qquad \boldsymbol{\sigma}_3 N_{l;s}(t) = \boldsymbol{e}_{l;s}(t) \boldsymbol{d}_{l;s} \boldsymbol{e}_{l;\overline{s}}(t)^\dagger, \tag{6.2}$$

whenever $\lambda_s = -\lambda_{\bar{s}}^*$. Furthermore,

$$\boldsymbol{e}_{r;s}(t) = e^{-t\Lambda_l^{\text{up}}(\lambda_s)}\boldsymbol{e}_{r;s}(0), \qquad \boldsymbol{e}_{l;s}(t) = e^{-t\Lambda_r^{\text{dn}}(\lambda_s)}\boldsymbol{e}_{l;s}(0). \tag{6.3}$$

If $\lambda_s = -\lambda_s^*$ is purely imaginary and therefore $\boldsymbol{\sigma}_3 N_{r;s}(t)$ and $\boldsymbol{\sigma}_3 N_{l;s}(t)$ are hermitian matrices, the number of positive and negative diagonal entries of $\boldsymbol{d}_{r;s}$ and $\boldsymbol{d}_{l;s}$ corresponds to the (time independent) number of positive and negative eigenvalues of $\boldsymbol{\sigma}_3 N_{r;s}(t)$ and $\boldsymbol{\sigma}_3 N_{l;s}(t)$.

Now define the matrix triplets as follows:

$$\boldsymbol{A}_r = \boldsymbol{A}_l = a_1 I_{r_1} \oplus \ldots \oplus a_N I_{r_N}, \tag{6.4}$$

where $\boldsymbol{A}_{r,l}$ are diagonal matrices of order $q = r_1 + \ldots + r_N$ having r_s copies of $a_s = -i\lambda_s$ on the diagonal. Next, we define

$$\boldsymbol{B}_r = \begin{pmatrix} \boldsymbol{d}_{r;1}\boldsymbol{e}_{r;\bar{1}}^\dagger \\ \vdots \\ \boldsymbol{d}_{r;N}\boldsymbol{e}_{r;\overline{N}}^\dagger \end{pmatrix}, \qquad \boldsymbol{B}_l = \begin{pmatrix} \boldsymbol{d}_{l;1}\boldsymbol{e}_{l;\bar{1}}^\dagger \\ \vdots \\ \boldsymbol{d}_{l;N}\boldsymbol{e}_{l;\overline{N}}^\dagger \end{pmatrix}, \tag{6.5a}$$

$$\boldsymbol{C}_r = \begin{pmatrix} \boldsymbol{\sigma}_3\boldsymbol{e}_{r;1} & \ldots & \boldsymbol{\sigma}_3\boldsymbol{e}_{r;N} \end{pmatrix}, \quad \boldsymbol{C}_l = \begin{pmatrix} \boldsymbol{\sigma}_3\boldsymbol{e}_{l;1} & \ldots & \boldsymbol{\sigma}_3\boldsymbol{e}_{l;N} \end{pmatrix}, \tag{6.5b}$$

where we have not written the time dependence. Then the Marchenko integral kernels in (6.1) are given by

$$\Omega_r(w;t) = \boldsymbol{C}_r(t)e^{-w\boldsymbol{A}_r}\boldsymbol{B}_r(t), \tag{6.6a}$$

$$\Omega_l(w;t) = \boldsymbol{C}_l(t)e^{w\boldsymbol{A}_l}\boldsymbol{B}_l(t), \tag{6.6b}$$

where the $q \times q$ matrices $\boldsymbol{A}_{r,l}$ have only eigenvalues with positive real parts, $\boldsymbol{B}_{r,l}(t)$ are $q \times n$ matrices, and $\boldsymbol{C}_{r,l}(t)$ are $n \times q$ matrices.

Let us now depart from arbitrary Marchenko integral kernels (6.6), where the $q \times q$ matrices $\boldsymbol{A}_{r,l}$ have only eigenvalues with positive real parts, $\boldsymbol{B}_{r,l}(t)$ are $q \times n$ matrices, $\boldsymbol{C}_{r,l}(t)$ are $n \times q$ matrices, and the specific expressions (6.4) and (6.5) need not be applied. Solving the Marchenko integral equations (4.3) we get

$$K(x,y;t) = -\boldsymbol{W}_r(x;t)e^{-y\boldsymbol{A}_r}\boldsymbol{B}_r(t), \tag{6.7a}$$

$$J(x,y;t) = -\boldsymbol{W}_l(x;t)e^{y\boldsymbol{A}_l}\boldsymbol{B}_l(t), \tag{6.7b}$$

where

$$\begin{aligned}\boldsymbol{W}_r(x;t) &= \boldsymbol{C}_r e^{-x\boldsymbol{A}_r} + \int_x^\infty dz\, K(x,z;t)\boldsymbol{C}_r(t)e^{-z\boldsymbol{A}_r},\\ \boldsymbol{W}_l(x;t) &= \boldsymbol{C}_l e^{x\boldsymbol{A}_l} + \int_{-\infty}^x dz\, J(x,z;t)\boldsymbol{C}_l(t)e^{z\boldsymbol{A}_l}.\end{aligned}$$

Substituting (6.7) into (4.3) and solving for $\boldsymbol{W}_{r,l}(x;t)$ we get

$$\begin{aligned}
\boldsymbol{W}_r(x;t) &= \boldsymbol{C}_r(t)e^{-x\boldsymbol{A}_r}\left[I_q + e^{-x\boldsymbol{A}_r}\boldsymbol{P}_r(t)e^{-x\boldsymbol{A}_r}\right]^{-1},\\
\boldsymbol{W}_l(x;t) &= \boldsymbol{C}_l(t)e^{x\boldsymbol{A}_l}\left[I_q + e^{x\boldsymbol{A}_l}\boldsymbol{P}_l(t)e^{x\boldsymbol{A}_l}\right]^{-1},
\end{aligned}$$

provided the inverse matrices exist. Here

$$\boldsymbol{P}_{r,l}(t) = \int_0^\infty dz\, e^{-z\boldsymbol{A}_{r,l}}\boldsymbol{B}_{r,l}(t)\boldsymbol{C}_{r,l}(t)e^{-z\boldsymbol{A}_{r,l}}$$

are the unique solutions of the Sylvester equations

$$\boldsymbol{A}_{r,l}\boldsymbol{P}_{r,l}(t) + \boldsymbol{P}_{r,l}(t)\boldsymbol{A}_{r,l} = \boldsymbol{B}_{r,l}(t)\boldsymbol{C}_{r,l}(t).$$

More precisely, given $(x,t)\in\mathbb{R}^2$, the Marchenko integral equations (4.3) are uniquely solvable (in an L^1-setting) iff the algebraic equations for $\boldsymbol{W}_{r,l}(x;t)$ are uniquely solvable. Consequently,

$$\begin{aligned}
K(x,y;t) &= -\boldsymbol{C}_r(t)e^{-x\boldsymbol{A}_r}\left[I_q + e^{-x\boldsymbol{A}_r}\boldsymbol{P}_r(t)e^{-x\boldsymbol{A}_r}\right]^{-1}e^{-y\boldsymbol{A}_r}\boldsymbol{B}_r(t)\\
&= -\boldsymbol{C}_r(t)\left[I_q + e^{-2x\boldsymbol{A}_r}\boldsymbol{P}_r(t)\right]^{-1}e^{-(x+y)\boldsymbol{A}_r}\boldsymbol{B}_r(t)\\
&= -\boldsymbol{C}_r(t)\left[e^{2x\boldsymbol{A}_r} + \boldsymbol{P}_r(t)\right]^{-1}e^{-(y-x)\boldsymbol{A}_r}\boldsymbol{B}_r(t), \qquad (6.8a)\\
J(x,y;t) &= -\boldsymbol{C}_l(t)e^{x\boldsymbol{A}_l}\left[I_q + e^{x\boldsymbol{A}_l}\boldsymbol{P}_l(t)e^{x\boldsymbol{A}_l}\right]^{-1}e^{y\boldsymbol{A}_l}\boldsymbol{B}_l(t)\\
&= -\boldsymbol{C}_l(t)\left[I_q + e^{2x\boldsymbol{A}_l}\boldsymbol{P}_l(t)\right]^{-1}e^{(x+y)\boldsymbol{A}_l}\boldsymbol{B}_l(t)\\
&= -\boldsymbol{C}_l(t)\left[e^{-2x\boldsymbol{A}_l} + \boldsymbol{P}_l(t)\right]^{-1}e^{-(x-y)\boldsymbol{A}_l}\boldsymbol{B}_l(t). \qquad (6.8b)
\end{aligned}$$

Using (3.17) we obtain

$$\int_x^\infty dy\, \boldsymbol{Q}(y;t) = -2\boldsymbol{C}_r(t) \left[e^{2x\boldsymbol{A}_r} + \boldsymbol{P}_r(t)\right]^{-1} \boldsymbol{B}_r(t), \tag{6.9a}$$

$$\int_{-\infty}^x dy\, \boldsymbol{Q}(y;t) = -2\boldsymbol{C}_l(t) \left[e^{-2x\boldsymbol{A}_l} + \boldsymbol{P}_l(t)\right]^{-1} \boldsymbol{B}_l(t). \tag{6.9b}$$

Consequently,

$$\boldsymbol{Q}(x;t) = -4\boldsymbol{C}_r(t) \left[e^{2x\boldsymbol{A}_r} + \boldsymbol{P}_r(t)\right]^{-1} \boldsymbol{A}_r e^{2x\boldsymbol{A}_r} \left[e^{2x\boldsymbol{A}_r} + \boldsymbol{P}_r(t)\right]^{-1} \boldsymbol{B}_r(t), \tag{6.10a}$$

$$\boldsymbol{Q}(x;t) = -4\boldsymbol{C}_l(t) \left[e^{-2x\boldsymbol{A}_l} + \boldsymbol{P}_l(t)\right]^{-1} \boldsymbol{A}_l e^{-2x\boldsymbol{A}_l} \left[e^{-2x\boldsymbol{A}_l} + \boldsymbol{P}_l(t)\right]^{-1} \boldsymbol{B}_l(t). \tag{6.10b}$$

Using the partitioning

$$\boldsymbol{C}_{r,l}(t) = \begin{pmatrix} \boldsymbol{C}^{\mathrm{up}}_{r,l}(t) \\ \boldsymbol{C}^{\mathrm{dn}}_{r,l}(t) \end{pmatrix}, \qquad \boldsymbol{B}_{r,l}(t) = \begin{pmatrix} \boldsymbol{B}^{\mathrm{lt}}_{r,l}(t) & \boldsymbol{B}^{\mathrm{rt}}_{r,l}(t) \end{pmatrix},$$

and assuming the nonsingularity of $\boldsymbol{P}_{r,l}(t)$, we obtain

$$\begin{aligned}\mathcal{Q}(x;t) &= \mathcal{Q}_r + 2\boldsymbol{C}_r^{\mathrm{up}}(t)\boldsymbol{P}_r(t)^{-1}\boldsymbol{B}_r^{\mathrm{rt}}(t) + 2\boldsymbol{C}_r^{\mathrm{dn}}(t)\boldsymbol{P}_r(t)^{-1}\boldsymbol{B}_r^{\mathrm{lt}}(t)\\ &\quad - 2\boldsymbol{C}_r^{\mathrm{up}}(t)\left[e^{2x\boldsymbol{A}_r} + \boldsymbol{P}_r(t)\right]^{-1}\boldsymbol{B}_r^{\mathrm{rt}}(t)\\ &\quad - 2\boldsymbol{C}_r^{\mathrm{dn}}(t)\left[e^{2x\boldsymbol{A}_r} + \boldsymbol{P}_r(t)\right]^{-1}\boldsymbol{B}_r^{\mathrm{lt}}(t), \qquad (6.11a)\\ \mathcal{Q}(x;t) &= \mathcal{Q}_l + 2\boldsymbol{C}_l^{\mathrm{up}}(t)\boldsymbol{P}_l(t)^{-1}\boldsymbol{B}_l^{\mathrm{rt}}(t) + 2\boldsymbol{C}_l^{\mathrm{dn}}(t)\boldsymbol{P}_l(t)^{-1}\boldsymbol{B}_l^{\mathrm{lt}}(t)\\ &\quad - 2\boldsymbol{C}_l^{\mathrm{up}}(t)\left[e^{-2x\boldsymbol{A}_l} + \boldsymbol{P}_l(t)\right]^{-1}\boldsymbol{B}_l^{\mathrm{rt}}(t)\\ &\quad - 2\boldsymbol{C}_l^{\mathrm{dn}}(t)\left[e^{-2x\boldsymbol{A}_l} + \boldsymbol{P}_l(t)\right]^{-1}\boldsymbol{B}_l^{\mathrm{lt}}(t). \qquad (6.11b)\end{aligned}$$

If (6.11) are solutions of the differential Riccati equation $\mathcal{Q}^2+\mathcal{Q}_x = \boldsymbol{Q}-\mu^2 I_n$, then they represent the multisoliton solutions of the focusing matrix NLS equation with extra term (2.2). Using the gauge transformation (2.6) we then get the multisoliton solutions of the usual focusing matrix NLS equation.

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A Wronskian relations

In this appendix we give the details of the proofs of the identities (3.8), (3.9), (3.10), (3.11). First of all, we prove the following

Proposition A.1 *For $\lambda \in \mathbb{R}$, let $\boldsymbol{V}(x,\lambda)$ and $\boldsymbol{W}(x,\lambda)$ be two $2n\times 2n$ matrix solutions of the first order system* (3.7)*. Then*

$$\boldsymbol{W}(x,\lambda)^\dagger(\sigma_2\otimes\boldsymbol{\sigma}_3)\boldsymbol{V}(x,\lambda)$$

is independent of $x \in \mathbb{R}$. In particular, its asymptotic forms as $x \to \pm\infty$ coincide.

Proof. It is easily verified by using (1.6) that for $\lambda \in \mathbb{R}$ we have

$$(\sigma_2\otimes\boldsymbol{\sigma}_3)\begin{bmatrix} 0_{n\times n} & I_n \\ \boldsymbol{Q}(x)-\lambda^2 I_n & 0_{n\times n}\end{bmatrix}(\sigma_2\otimes\boldsymbol{\sigma}_3) = -\begin{bmatrix} 0_{n\times n} & I_n \\ \boldsymbol{Q}(x)-\lambda^2 I_n & 0_{n\times n}\end{bmatrix}^\dagger.$$

Then

$$\begin{aligned}
&\frac{\partial}{\partial x}\left[\boldsymbol{W}(x,\lambda)^\dagger(\sigma_2\otimes\boldsymbol{\sigma}_3)\boldsymbol{V}(x,\lambda)\right] \\
&\quad= \boldsymbol{W}(x,\lambda)^\dagger\begin{bmatrix} 0_{n\times n} & I_n \\ \boldsymbol{Q}(x)-\lambda^2 I_n & 0_{n\times n}\end{bmatrix}^\dagger(\sigma_2\otimes\boldsymbol{\sigma}_3)\boldsymbol{V}(x,\lambda) \\
&\quad+ \boldsymbol{W}(x,\lambda)^\dagger(\sigma_2\otimes\boldsymbol{\sigma}_3)\begin{bmatrix} 0_{n\times n} & I_n \\ \boldsymbol{Q}(x)-\lambda^2 I_n & 0_{n\times n}\end{bmatrix}\boldsymbol{V}(x,\lambda) = 0_{2n\times 2n},
\end{aligned}$$

as claimed. ■

Let us first apply Proposition A.1 to $\boldsymbol{V}(x,\lambda)=\boldsymbol{W}(x,\lambda)=\boldsymbol{\Phi}(x,\lambda)$ and divide the resulting equation by 2λ. For $0\neq\lambda\in\mathbb{R}$ we get by equating the $x\to+\infty$ asymptotics to the $x\to-\infty$ asymptotics and dividing the resulting equation by 2λ

$$\begin{bmatrix}-A_r^\dagger\boldsymbol{\sigma}_3A_r+B_r^\dagger\boldsymbol{\sigma}_3B_r & B_r^\dagger\boldsymbol{\sigma}_3\\ \boldsymbol{\sigma}_3B_r & \boldsymbol{\sigma}_3\end{bmatrix}=\begin{bmatrix}-\boldsymbol{\sigma}_3 & -\boldsymbol{\sigma}_3B_l\\ -B_l^\dagger\boldsymbol{\sigma}_3 & A_l^\dagger\boldsymbol{\sigma}_3A_l-B_l^\dagger\boldsymbol{\sigma}_3B_l\end{bmatrix},$$

where we have not written the λ-dependence. Consequently, for $0\neq\lambda\in\mathbb{R}$ we have the equalities

$$A_r(\lambda)^\dagger\boldsymbol{\sigma}_3A_r(\lambda)-B_r(\lambda)^\dagger\boldsymbol{\sigma}_3B_r(\lambda)=\boldsymbol{\sigma}_3, \tag{A.1a}$$

$$A_l(\lambda)^\dagger\boldsymbol{\sigma}_3A_l(\lambda)-B_l(\lambda)^\dagger\boldsymbol{\sigma}_3B_l(\lambda)=\boldsymbol{\sigma}_3, \tag{A.1b}$$

$$B_r(\lambda)^\dagger=-\boldsymbol{\sigma}_3B_l(\lambda)\boldsymbol{\sigma}_3,\qquad B_l(\lambda)^\dagger=-\boldsymbol{\sigma}_3B_r(\lambda)\boldsymbol{\sigma}_3. \tag{A.1c}$$

We observe that (A.1) coincide with (3.8).

Let us now apply Proposition A.1 to $\boldsymbol{V}(x,\lambda)=\boldsymbol{W}(x,\lambda)=\boldsymbol{F}_{r,\lambda}(x,\lambda)$ and divide the resulting equation by 2λ. For $0\neq\lambda\in\mathbb{R}$ we get by equating the

$x \to +\infty$ asymptotics to the $x \to -\infty$ asymptotics and dividing by 2λ

$$\begin{bmatrix} -A_r^\dagger \boldsymbol{\sigma}_3 A_r + B_r^\dagger \boldsymbol{\sigma}_3 B_r & -A_r^\dagger \boldsymbol{\sigma}_3 B_r^\# + B_r^\dagger \boldsymbol{\sigma}_3 A_r^\# \\ A_r^{\#^\dagger} \boldsymbol{\sigma}_3 B_r - B_r^{\#^\dagger} \boldsymbol{\sigma}_3 A_r & A_r^{\#^\dagger} \boldsymbol{\sigma}_3 A_r^\# - B_r^{\#^\dagger} \boldsymbol{\sigma}_3 B_r^\# \end{bmatrix} = \begin{bmatrix} -\boldsymbol{\sigma}_3 & 0_{n\times n} \\ 0_{n\times n} & \boldsymbol{\sigma}_3 \end{bmatrix},$$

$$\begin{bmatrix} -\boldsymbol{\sigma}_3 & 0_{n\times n} \\ 0_{n\times n} & \boldsymbol{\sigma}_3 \end{bmatrix} = \begin{bmatrix} -A_l^{\#^\dagger} \boldsymbol{\sigma}_3 A_l^\# + B_l^{\#^\dagger} \boldsymbol{\sigma}_3 B_l^\# & -A_l^{\#^\dagger} \boldsymbol{\sigma}_3 B_l + B_l^{\#^\dagger} \boldsymbol{\sigma}_3 A_l \\ A_l^\dagger \boldsymbol{\sigma}_3 B_l^\# - B_l^\dagger \boldsymbol{\sigma}_3 A_l^\# & A_l^\dagger \boldsymbol{\sigma}_3 A_l - B_l^\# \boldsymbol{\sigma}_3 B_l \end{bmatrix},$$

respectively, where the short-hand notation $C^\#(k) = C(-k)$ is adopted. Equating the block diagonal entries implies (A.1a) and (A.1b). Equating the block off-diagonal entries implies

$$A_r(\lambda)^\dagger \boldsymbol{\sigma}_3 B_r(-\lambda) = B_r(\lambda)^\dagger \boldsymbol{\sigma}_3 A_r(-\lambda), \tag{A.2a}$$

$$A_l(\lambda)^\dagger \boldsymbol{\sigma}_3 B_l(-\lambda) = B_l(\lambda)^\dagger \boldsymbol{\sigma}_3 A_l(-\lambda), \tag{A.2b}$$

and these equalities coincide with (3.9)

Finally, let us now apply Proposition A.1 to $\boldsymbol{V}(x,\lambda) = \boldsymbol{F}_l(x,\lambda)$ and $\boldsymbol{W}(x,\lambda) = \boldsymbol{F}_r(x,\lambda)$ and divide the resulting equation by 2λ. For $0 \neq \lambda \in \mathbb{R}$ we get by equating the $x \to +\infty$ asymptotics to the $x \to -\infty$ asymptotics and dividing the resulting equation by 2λ

$$\begin{bmatrix} -A_r(\lambda)^\dagger \boldsymbol{\sigma}_3 & B_r(\lambda)^\dagger \boldsymbol{\sigma}_3 \\ -B_r(-\lambda)^\dagger \boldsymbol{\sigma}_3 & A_r(-\lambda)^\dagger \boldsymbol{\sigma}_3 \end{bmatrix} = \begin{bmatrix} -\boldsymbol{\sigma}_3 A_l(-\lambda) & -\boldsymbol{\sigma}_3 B_l(\lambda) \\ \boldsymbol{\sigma}_3 B_l(-\lambda) & \boldsymbol{\sigma}_3 A_l(\lambda) \end{bmatrix}.$$

As a result, we arrive at the two identities

$$A_r(\lambda)^\dagger = \boldsymbol{\sigma}_3 A_l(-\lambda)\boldsymbol{\sigma}_3, \qquad B_r(\lambda)^\dagger = -\boldsymbol{\sigma}_3 B_l(\lambda)\boldsymbol{\sigma}_3. \tag{A.3}$$

Identities (A.3) coincide with (3.10).

Equation (3.11) can easily be derived from the x-independence of

$$\boldsymbol{W}(x,-\lambda^*)^\dagger(\sigma_2 \otimes \boldsymbol{\sigma}_3)\boldsymbol{V}(x,\lambda)$$

for given solutions $\boldsymbol{V}(x,\lambda)$ and $\boldsymbol{W}(x,\lambda)$ of (3.7).

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