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An overdetermined problem related to the Finsler p-Laplacian

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Abstract

In this paper, we consider the Finsler p-Laplacian torsion equation. The domain of the problem is bounded by a conical surface supporting a Neumann-type condition, and an unknown surface supporting both a Dirichlet and a Neumann condition. The case when the cone coincides with the punctured space is included. We show that the existence of a weak solution implies that the unknown surface lies on the boundary of a Finsler-ball. Incidentally, some properties of the Finsler–Minkowski norms are proved here under mild smoothness assumptions.

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1 | INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ be a bounded domain. The Finsler p-Laplacian of a twice differentiable function u at a point x with $Du(x) \neq O$ is denoted by $\Delta_{F;p}\, u(x)$ and defined by

$$\Delta_{F;p} u = \operatorname{div}(F(Du)^{p-1} DF(Du)),\ 1 < p < \infty,$$

where $F : \mathbb{R}^n \to [0, \infty)$ is a Finsler–Minkowski norm, which is discussed in the next section. When F is the usual Euclidean norm, the Finsler p-Laplacian becomes the known p-Laplacian

$$\Delta_p u = \operatorname{div}(|Du|^{p-2} Du),\ 1 < p < \infty.$$

Recently, the Finsler p-Laplacian has attracted the attention of researchers. We may mention [4, 5, 7, 9, 10] and [13]. This paper is devoted to the study of problems related to the operator $\Delta_{F;p}$

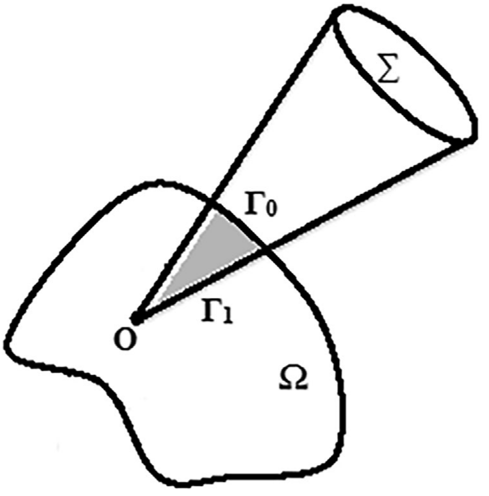

FIGURE 1 $\Omega \cap \Sigma$.

in conjunction with Dirichlet and Neumann boundary conditions. Let the origin O be contained in Ω, and let $\Sigma = \{tx \,:\, x \in \omega,\ t \in \mathbb{R}^+\} \subseteq \mathbb{R}^n$ be a cone for some domain $\omega \subseteq S^{n-1}$ (see Figure 1). We note that when $\omega = S^{n-1}$, the cone Σ becomes the punctured space $\Sigma = \mathbb{R}^n \setminus \{O\}$ (not the whole Euclidean space as mentioned in [8, p. 1026]). We require $\partial\Sigma \setminus \{O\}$ to be a hypersurface of class C^1 when $\omega \subset S^{n-1}$, and we denote the outward normal by ν. Define $\Gamma_0 = \Sigma \cap \partial\Omega$ and $\Gamma_1 = \partial(\Sigma \cap \Omega) \setminus \overline{\Gamma}_0$. As it was investigated in the recent paper [8], for norms H_0 and H (the dual of H_0), if the problem

$$\begin{cases} -\Delta_H u = 1 & \text{in } \Omega \cap \Sigma \\ u = 0 & \text{on } \Gamma_0 \\ \langle DH(Du(x)), \nu \rangle = 0 & \text{on } \Gamma_1 \setminus \{O\} \end{cases} \tag{1.1}$$

has a solution satisfying the condition

$$\lim_{x \to z} H(Du(x)) = q(H_0(z)) \ \forall z \in \overline{\Gamma}_0, \tag{1.2}$$

where $q(r)$ is a positive, real-valued function such that $q(r)/r$ is strictly increasing in r, then $\Omega \cap \Sigma = B_R(O, H_0) \cap \Sigma$ for some $R > 0$. In this paper, we generalize (1.1) and (1.2) by replacing the Finsler Laplace operator Δ_H with the Finsler p-Laplacian operator $\Delta_{F;p}$. In particular, F is not necessarily a norm in the sense of functional analysis, and we also take $p \in (1, \infty)$. We denote by F^* the dual norm of F (see Section 2), and by $B_F^-(O, R) = \{x \in \mathbb{R}^n \,:\, F^*(-x) < R\}$ the corresponding (opposite) ball. Furthermore, let $q(r)$ be a positive, real-valued function such that

$$q(r)/r^{p'-1} \text{ is strictly increasing in } r > 0, \tag{1.3}$$

where $p' = p/(p-1)$. Finally, we define the function space

$$W^{1,p}_{\Gamma_0}(\Omega \cap \Sigma) = \left\{ v : \Omega \cap \Sigma \to \mathbb{R} \text{ with } v = w\chi_{\Omega\cap\Sigma} \text{ for some } w \in W^{1,p}_0(\Omega) \right\}, \tag{1.4}$$

where $\chi_{\Omega\cap\Sigma}$ stands for the characteristic function of $\Omega \cap \Sigma$. If Γ_0 is smooth enough, we may say that functions in $W^{1,p}_{\Gamma_0}$ have a null trace on Γ_0. We are now in a position to state our main result as follows.

Theorem 1.1. *Let $q(r)$ be a positive, real-valued function satisfying (1.3), and let $u \in W^{1,p}_{\Gamma_0}(\Omega \cap \Sigma)$ be a weak solution of the problem*

$$\begin{cases} -\Delta_{F;p} u = 1 & \text{in } \Omega \cap \Sigma; \\ u = 0 & \text{on } \Gamma_0; \\ \langle DF(Du), \nu \rangle = 0 & \text{on } \Gamma_1 \setminus \{O\}. \end{cases} \tag{1.5}$$

If u belongs to the smoothness class $C^1((\Omega \cap \Sigma) \cup (\Gamma_1 \setminus \{O\})) \cap C^0(\overline{\Omega \cap \Sigma} \setminus \{O\})$ and satisfies

$$\lim_{x\to z} F(Du(x)) = q(F^*(-z)) \; \forall z \in \overline{\Gamma}_0, \tag{1.6}$$

then $\Omega \cap \Sigma = B^-_F(O, R) \cap \Sigma$ for some $R > 0$.

Observe that Theorem 1.1 also holds in the case when $\omega = S^{n-1}$. In this case, the intersection $\Omega \cap \Sigma$ becomes the punctured domain $\Omega \setminus \{O\}$. We also prove a result valid in the case when the cone Σ is replaced with the whole Euclidean space $\mathbb{R}^n$:

Theorem 1.2. *Let $q(r)$ be a positive, real-valued function satisfying (1.3), and let $u \in W^{1,p}_0(\Omega)$ be a weak solution of the problem*

$$\begin{cases} -\Delta_{F;p} u = 1 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

If u belongs to $C^1(\Omega) \cap C^0(\overline{\Omega})$ and satisfies

$$\lim_{x\to z} F(Du(x)) = q(F^*(-z)) \; \forall z \in \partial\Omega, \tag{1.8}$$

then $\Omega = B^-_F(O, R)$ for some $R > 0$.

In this paper, we call the equation $-\Delta_{F;p} u = 1$ Finsler p-Laplacian torsion equation. The proof of Theorem 1.1 is obtained by comparison with solutions in Finsler-balls. The proof of Theorem 1.2 is quite similar. The main difficulty is to manage with condition (1.6), which is given in a limiting form in place of the pointwise form $F(Du(z)) = q(F^*(-z))$ because no regularity assumption is imposed on Γ_0 (to this purpose, cf. [11, Theorem 1] and [8]). In order to overcome such a difficulty, we develop a direct argument, without proceeding by contradiction as done in [8]. All the notations we have used are standard in partial differential equations (PDEs). In the special case when $F(\xi) = |\xi|$, problem (1.7) reduces to

$$\begin{cases} -\Delta_p u = 1 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

If, furthermore, $u \in C^1(\overline{\Omega})$, then (1.8) becomes

$$|Du(z)| = q(|z|)\ \forall z \in \partial\Omega. \tag{1.10}$$

Problem (1.9)–(1.10) is a special case of problem (1.6) in [12], where the more general right-hand side $f(|x|, u)$ is considered in place of the constant 1, and the conclusion is obtained under the assumption that the ratio $q(r)/r^{p'-1}$ is nondecreasing: To see this, just let $\varepsilon_0 = p - 1$ in [12, (1.10)]. The case of the *normalized* p-Laplacian Δ_p^N is considered in [2] and [6].

We now arrange the rest of the paper as follows. In Section 2, we explore some basic concepts about Finsler–Minkowski norms. In Section 3, we see the definition of weak solutions and some properties that are needed in the sequel. In Section 4, the proofs of Theorem 1.1 and Theorem 1.2 are given. We also show the behavior of the operator $\Delta_{F;p} u$ upon the transformation $v(x) = u(-x)$. Note that problem (1.5)–(1.6) under assumption (1.3) is not solvable, in general, even in the case when $\Omega = B_F^-(O, R)$, and a similar remark holds for problem (1.7)–(1.8). It is easy to identify the solvable instances: We do this in the final Section 5, where we also give some examples.

2 | PRELIMINARIES

In this section, we will discuss the definition of a Finsler–Minkowski norm and its dual, and some of their properties in relation to our problem.

2.1 | Finsler–Minkowski norms

We consider a function $F : \mathbb{R}^n \to \mathbb{R}_0^+$ with the following properties.

(**A**) $F \in C^1(\mathbb{R}^n \setminus \{O\})$.
(**B**) $F(t\xi) = tF(\xi)\ \forall \xi \in \mathbb{R}^n$ and $\forall t > 0$.
(**C**) $F(\lambda\xi + (1-\lambda)\zeta) \leqslant \lambda F(\xi) + (1-\lambda)F(\zeta)\ \forall \xi, \zeta \in \mathbb{R}^n$ and $\forall \lambda \in (0, 1)$. Equality holds if and only if $\xi = \kappa\zeta$ or $\zeta = \kappa\xi$ for some $\kappa \geqslant 0$.
(**D**) $F(\xi) > 0\ \forall \xi \in \mathbb{R}^n \setminus \{O\}$.

A function $F : \mathbb{R}^n \to \mathbb{R}_0^+$ that satisfies the above four conditions is called a Finsler–Minkowski norm on $\mathbb{R}^n$. By (**C**), F is convex: Hence $F \in C^0(\mathbb{R}^n)$. Thus, letting $t \to 0^+$ in (**B**), we get $F(O) = 0$. Furthermore, writing $2\xi, 2\zeta$ in place of ξ, ζ in (**C**), and letting $\lambda = \frac{1}{2}$, we obtain $F(\xi + \zeta) \leqslant \frac{1}{2}F(2\xi) + \frac{1}{2}F(2\zeta)$. This and (**B**) imply

$$F(\xi + \zeta) \leqslant F(\xi) + F(\zeta)\ \forall \xi, \zeta \in \mathbb{R}^n. \tag{2.1}$$

Equality holds in (2.1) if and only if $\xi = \kappa\zeta$ or $\zeta = \kappa\xi$ for some $\kappa \geqslant 0$. However, despite the word "norm," F is not necessarily a norm in the sense of functional analysis because a norm must satisfy $F(t\xi) = |t|\, F(\xi)\ \forall \xi \in \mathbb{R}^n$ and $\forall t \in \mathbb{R}$ (*absolute homogeneity*). For instance, the function $F : \mathbb{R}^n \to [0, \infty)$ given by

$$F(\xi) := |\xi| + \langle x, \xi \rangle, \quad \xi \in \mathbb{R}^n$$

for a fixed $x \in \mathbb{R}^n$ satisfying $0 < |x| < 1$ is a Finsler–Minkowski norm but not a norm in the sense of functional analysis (see [15, p. 4]). The following lemma is a consequence of the definition of a Finsler–Minkowski norm given above. It is found in [3] under more restrictive assumptions: In particular, F is taken in $C^\infty(\mathbb{R}^n \setminus \{O\})$, and the Hessian matrix of F^2 is required to be positive definite in $\mathbb{R}^n \setminus \{O\}$. See also [16] for the case when $F \in C^2(\mathbb{R}^n \setminus \{O\})$. We check that such restrictions are not essential.

Lemma 2.1. *Let F be a Finsler–Minkowski norm. The following properties hold.*

(1) $\langle DF(\xi), \zeta\rangle \leqslant F(\zeta)$ $\forall \xi \in \mathbb{R}^n \setminus \{O\}$ *and* $\forall \zeta \in \mathbb{R}^n$. *Equality holds if and only if* $\zeta = \kappa\xi$ *for some* $\kappa \geqslant 0$.
(2) $DF(t\xi) = DF(\xi)$ $\forall \xi \in \mathbb{R}^n \setminus \{O\}$ *and* $\forall t > 0$.

Proof. Let us begin with proving that

$$\langle DF(\xi),\, \xi\rangle = F(\xi) \quad \forall \xi \in \mathbb{R}^n \setminus \{O\}. \tag{2.2}$$

The linear function $\ell(t) = tF(\xi)$ clearly satisfies $\ell(1) = \ell'(1)$. By (**B**) we may write $\ell(t) = F(t\xi)$, and therefore $\ell'(1) = \langle DF(\xi),\, \xi\rangle$, whence (2.2). To prove the inequality (1), recall that F is convex by (**C**), hence its graph lies above its tangent planes:

$$F(\zeta) \geqslant F(\xi) + \langle DF(\xi),\, \zeta - \xi\rangle \quad \forall \xi \in \mathbb{R}^n \setminus \{O\},\ \forall \zeta \in \mathbb{R}^n.$$

By (**C**), and since the case $\xi = O$ is excluded, equality holds if and only if $\zeta = \kappa\xi$ for some $\kappa \geqslant 0$. Using (2.2), inequality (1) follows. Equality (2) is obtained by differentiating (**B**) with respect to ξ. □

It is important to state the following lemma for the purpose of proving the comparison principle (Proposition 3.4). Similar results were presented in [19] for the case when $F \in C^\infty(\mathbb{R}^n \setminus \{O\})$, and the Hessian matrix of F^2 is positive definite in $\mathbb{R}^n \setminus \{O\}$. In [1] the norm F is required, in addition, to be absolutely homogeneous. We show that such restrictions are not necessary.

Lemma 2.2. *Let $1 < p < \infty$. Then, the following hold true.*

(1) $F(\lambda\xi + (1-\lambda)\zeta)^p \leqslant \lambda F(\xi)^p + (1-\lambda)F(\zeta)^p$ $\forall \xi, \zeta \in \mathbb{R}^n$ *and* $\forall \lambda \in (0,1)$. *Equality holds if and only if* $\xi = \zeta$.
(2) *The scalar function* $\xi \mapsto \frac{1}{p}F(\xi)^p$ *belongs to the smoothness class* $C^1(\mathbb{R}^N)$ *and its gradient* $X(\xi)$ *is given by*

$$X(\xi) = \begin{cases} F(\xi)^{p-1}\, DF(\xi), & \xi \neq O; \\ O, & \xi = O. \end{cases} \tag{2.3}$$

(3) $\langle X(\xi) - X(\zeta),\, \xi - \zeta\rangle \geqslant 0$ $\forall \xi, \zeta \in \mathbb{R}^n$. *Equality holds if and only if* $\xi = \zeta$.

Proof.

(1) Since $p > 1$, the power function $\varphi(t) = t^p$ is strictly increasing and strictly convex over the closed interval $[0, \infty)$. If we take $\lambda \in (0,1)$ and $\xi, \zeta \in \mathbb{R}^n$ with $\xi \neq \zeta$, two cases may occur: Either $F(\xi) = F(\zeta)$ or $F(\xi) \neq F(\zeta)$. In the first case, by the positive homogeneity (**B**), the

distinct points ξ and ζ do not lie on any half-line starting from the origin. Hence, by (**C**), we have the strict inequality $F(\lambda\xi + (1-\lambda)\zeta) < \lambda F(\xi) + (1-\lambda)F(\zeta)$, which reduces to $F(\lambda\xi + (1-\lambda)\zeta) < F(\xi)$. Raising both sides to the power p we obtain

$$\begin{aligned} F(\lambda\xi + (1-\lambda)\zeta)^p &< F(\xi)^p \\ &= \lambda F(\xi)^p + (1-\lambda)F(\zeta)^p. \end{aligned}$$

If, instead, $F(\xi) \neq F(\zeta)$, then $\varphi(\lambda F(\xi) + (1-\lambda)\,F(\zeta)) < \lambda\,\varphi(F(\xi)) + (1-\lambda)\,\varphi(F(\zeta))$ and the conclusion follows.

(2) Since $F \in C^1(\mathbb{R}^n \setminus \{O\})$ by assumption, we obviously have

$$X(\xi) = F(\xi)^{p-1}\,DF(\xi) \text{ for } \xi \neq O. \tag{2.4}$$

Furthermore, since $F(\xi)^p = o(|\xi|)$ as $\xi \to O$, we also have $X(O) = O$. It remains to check the continuity of $X(\xi)$ at $\xi = O$. By Lemma 2.1(2), we may write

$$\sup_{\xi\neq O} |DF(\xi)| = \max_{|\xi|=1} |DF(\xi)|,$$

hence $DF(\xi)$ is bounded in $\mathbb{R}^n \setminus \{O\}$. Letting $\xi \to O$ in (2.4) the claim follows.

(3) Since $\frac{1}{p}\,F^p$ is strictly convex by (1), the graph of $\frac{1}{p}\,F^p$ lies above its tangent planes. Contact occurs only at the point of tangency, hence

$$\frac{1}{p}\,F(\xi)^p \geqslant \frac{1}{p}\,F(\zeta)^p + \langle X(\zeta),\, \xi - \zeta\rangle \text{ for all } \xi, \zeta \in \mathbb{R}^n$$

and equality holds if and only if $\xi = \zeta$. Interchanging ζ with ξ, we obtain $\frac{1}{p}\,F(\zeta)^p \geqslant \frac{1}{p}\,F(\xi)^p + \langle X(\xi),\, \zeta - \xi\rangle$, and summing term to term we obtain (3). □

Remark 2.3. Combining (2.2) with (2.3), we immediately obtain

$$\langle X(\xi), \xi\rangle = F(\xi)^p \text{ for every } \xi \in \mathbb{R}^n. \tag{2.5}$$

2.2 | The dual of *F*

Given a Finsler–Minkowski norm $F : \mathbb{R}^n \to \mathbb{R}_0^+$, we define the dual $F^* : \mathbb{R}^n \to \mathbb{R}_0^+$ by

$$F^*(x) = \sup_{\xi\neq O} \frac{\langle x, \xi\rangle}{F(\xi)}. \tag{2.6}$$

It is easily seen from (2.6) that $F^*(O) = 0$, $F^*(x) > 0$ for $x \neq O$ and F^* is positively homogeneous of degree 1. Furthermore, for every $x_1, x_2 \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n \setminus \{O\}$, and $\lambda \in (0,1)$, we have

$$\begin{aligned} \frac{\langle \lambda x_1 + (1-\lambda)\,x_2,\, \xi\rangle}{F(\xi)} &= \lambda\,\frac{\langle x_1, \xi\rangle}{F(\xi)} + (1-\lambda)\,\frac{\langle x_2, \xi\rangle}{F(\xi)} \\ &\leqslant \lambda F^*(x_1) + (1-\lambda)\,F^*(x_2), \end{aligned}$$

hence F^* is convex, and consequently it is continuous in $\mathbb{R}^n$ and differentiable a.e. in $\mathbb{R}^n \setminus \{O\}$.

Remark 2.4. By positive homogeneity (**B**), there is no loss of generality if we replace the constraint $\xi \neq O$ with $F(\xi) = 1$ in (2.6), and we see that the supremum is in fact attained by compactness. Furthermore, since the set $\{ \xi \in \mathbb{R}^n : F(\xi) \leqslant 1 \}$ is strictly convex by (**C**), if $x \neq O$ the supremum is attained at a unique point ξ_x on the surface $F(\xi) = 1$. When ξ ranges in $\mathbb{R}^n \setminus \{O\}$, the ratio in (2.6) attains its maximum at the point $t\xi_x$ for every $t > 0$.

Let us prove, for completeness, the following known result.

Lemma 2.5. *If F is a Finsler–Minkowski norm, then*

$$F(\xi) = \sup_{x \neq O} \frac{\langle x, \xi \rangle}{F^*(x)}. \tag{2.7}$$

Proof. Since $F(O) = 0$, and by the definition of F^*, we get

$$\langle x, \xi \rangle \leqslant F^*(x) F(\xi) \quad \forall x, \xi \in \mathbb{R}^n.$$

Thus, given $\xi \in \mathbb{R}^n$ we have

$$\frac{\langle x, \xi \rangle}{F^*(x)} \leqslant F(\xi) \quad \forall x \in \mathbb{R}^n \setminus \{O\}. \tag{2.8}$$

We note that equality holds in (2.8) if $\xi = O$, as well as if we take $\xi \neq O$ and $x = DF(\xi)$ on the left-hand side: In fact, by (2.2), we have $\langle x, \xi \rangle = \langle DF(\xi), \xi \rangle = F(\xi)$. It remains to check that

$$F^*(DF(\xi)) = 1 \quad \forall \xi \in \mathbb{R}^n \setminus \{O\}. \tag{2.9}$$

We have seen before that there exists a unique ξ_x satisfying $F(\xi_x) = 1$ and such that $F^*(x) = \langle x, \xi_x \rangle$. Here $x = DF(\xi)$ for a given $\xi \in \mathbb{R}^n \setminus \{O\}$. Hence, by Lemma 2.1, claim (1), we may write

$$\begin{aligned} F^*(DF(\xi)) &= \langle DF(\xi), \xi_x \rangle \\ &\leqslant F(\xi_x) = 1. \end{aligned}$$

This and (2.8) for $x = DF(\xi)$ imply

$$\langle DF(\xi), \xi \rangle \leqslant \frac{\langle DF(\xi), \xi \rangle}{F^*(DF(\xi))} \leqslant F(\xi) = \langle DF(\xi), \xi \rangle,$$

which proves (2.9), and (2.7) follows. □

Remark 2.6. By property (**C**) in the definition of a Finsler–Minkowski norm, it follows that the set $\{ \xi \in \mathbb{R}^n : F(\xi) \leqslant 1 \}$ is strictly convex, hence F^* is differentiable in $\mathbb{R}^n \setminus \{O\}$ [18, Corollary 1.7.3]. Since F^* is also convex, DF^* is continuous in $\mathbb{R}^n \setminus \{O\}$ [18, Theorem 1.5.2]. Finally, since $F \in C^1(\mathbb{R}^n \setminus \{O\})$ by property (**A**) in the definition, and in view of (2.7), we may apply [18, Corollary 1.7.3] again and conclude that the set $\{ x \in \mathbb{R}^n : F^*(x) \leqslant 1 \}$ is strictly convex: Hence, F^* fully satisfies the definition of a Finsler–Minkowski norm given in Section 2.1. Now the statement of

Lemma 2.5 may be shortly expressed by

$$(F^*)^* = F.$$

We note that whenever $x \in \mathbb{R}^n \setminus \{O\}$, the inequalities

$$\frac{|x|^2}{F(x)} \leqslant \sup_{\xi \neq O} \frac{\langle x, \xi \rangle}{F(\xi)} \leqslant \sup_{\xi \neq O} \frac{|\xi|}{F(\xi)} |x|$$

hold, and therefore we may write

$$\alpha|x| \leqslant F^*(x) \leqslant \beta|x| \quad \forall x \in \mathbb{R}^n, \tag{2.10}$$

where

$$\alpha = \inf_{\xi \neq O} \frac{|\xi|}{F(\xi)} \text{ and } \beta = \sup_{\xi \neq O} \frac{|\xi|}{F(\xi)}.$$

Similarly, we may write

$$\rho|\xi| \leqslant F(\xi) \leqslant \sigma|\xi| \quad \forall \xi \in \mathbb{R}^n, \tag{2.11}$$

where $\rho = \inf_{x \neq O} \frac{|x|}{F^*(x)}$ and $\sigma = \sup_{x \neq O} \frac{|x|}{F^*(x)}$. We also remark that the following inequality is a direct consequence of (2.10):

$$\frac{\alpha}{\beta} F^*(x) \leqslant F^*(-x) \leqslant \frac{\beta}{\alpha} F^*(x) \quad \forall x \in \mathbb{R}^n.$$

Given $x_0 \in \mathbb{R}^n$ and $R > 0$, the opposite Finsler-ball is defined by

$$B_F^-(x_0, R) := \{x \in \mathbb{R}^n : F^*(x_0 - x) < R\},$$

which is, in general, different from $B_F^+(x_0, R) := \{x \in \mathbb{R}^n : F^*(x - x_0) < R\}$. Of course, the equality $B_F^+(O, R) = -B_F^-(O, R)$ holds. We adopt here the notation of [15, p. 3] and [16, p. 1143]. By contrast, the notation in [9, (1.5)] is different. As a consequence of (2.10), we note that both $B_F^+(x_0, R)$ and $B_F^-(x_0, R)$ are bounded: In fact we find that

$$B\left(x_0, \frac{R}{\beta}\right) \subseteq B_F^{\pm}(x_0, R) \subseteq B\left(x_0, \frac{R}{\alpha}\right).$$

We conclude this section by recalling two known properties that are needed in the sequel.

Lemma 2.7. *Let F be a Finsler–Minkowski norm. Then,* $DF(DF^*(x)) = \frac{x}{F^*(x)}$ $\forall x \in \mathbb{R}^n \setminus \{O\}$.

Proof. We have observed in Remark 2.4 that for each $x \in \mathbb{R}^n \setminus \{O\}$ there exists a unique $\xi_x \in \mathbb{R}^n \setminus \{O\}$ such that $F(\xi_x) = 1$ and

$$F^*(x) = \langle x, \xi_x \rangle = \max_{\xi \neq O} \frac{\langle x, \xi \rangle}{F(\xi)}. \tag{2.12}$$

We can also write (2.12) as

$$\frac{\langle x, \xi_x \rangle}{F^*(x)} = 1 = F(\xi_x). \tag{2.13}$$

This and (2.7) imply

$$\frac{\langle x, \xi_x \rangle}{F^*(x)} = \max_{y \neq O} \frac{\langle y, \xi_x \rangle}{F^*(y)}.$$

Since the gradient DL of the linear function $L(\xi) = \langle x, \xi \rangle$ is $DL = x$, the extremality conditions

$$D_\xi \left(\frac{\langle x, \xi \rangle}{F(\xi)} \right) \bigg|_{\xi = \xi_x} = O \quad \text{and} \quad D_y \left(\frac{\langle y, \xi_x \rangle}{F^*(y)} \right) \bigg|_{y = x} = O$$

give, respectively,

$$x = \langle x, \xi_x \rangle \, DF(\xi_x) \quad \text{and} \quad \xi_x \, F^*(x) = \langle x, \xi_x \rangle \, DF^*(x).$$

Using (2.13), these are transformed into

$$x = F^*(x) DF(\xi_x) \tag{2.14}$$

and

$$\xi_x = DF^*(x). \tag{2.15}$$

Substituting (2.15) into (2.14), we obtain the desired result. □

Remark 2.8. By (2.15) we immediately obtain

$$F(DF^*(x)) = F(\xi_x) = 1 \quad \forall x \in \mathbb{R}^n \setminus \{O\}, \tag{2.16}$$

which is the dual of (2.9).

3 | WEAK SOLUTIONS

In this part, we examine the nonnegativity of a weak solution, and the comparison between two weak solutions in nested domains. We also recall the explicit solution of (1.5) in the case when $\Omega = B_F^-(O, R)$. We commence by the following definition.

Definition 3.1. Let $f \in L^{p'}(\Omega \cap \Sigma)$, and denote by $X(\xi)$ the vector field in (2.3). A weak solution of the boundary value problem

$$\begin{cases} -\Delta_{F;p} u = f(x) & \text{in } \Omega \cap \Sigma; \\ u = 0 & \text{on } \Gamma_0; \\ \langle DF(Du), \nu \rangle = 0 & \text{on } \Gamma_1 \setminus \{O\} \end{cases} \tag{3.1}$$

is a function u belonging to the function space $W^{1,p}_{\Gamma_0}(\Omega\cap\Sigma)$ in (1.4) and satisfying

$$\int_{\Omega\cap\Sigma}\langle X(Du(x)),\, Dv(x)\rangle dx = \int_{\Omega\cap\Sigma} f(x)\,v(x)\,dx \tag{3.2}$$

for every $v\in W^{1,p}_{\Gamma_0}(\Omega\cap\Sigma)$. Similarly, if $f\in L^{p'}(\Omega)$, then a weak solution of the boundary value problem

$$\begin{cases} -\Delta_{F;p}\, u = f(x) & \text{in } \Omega;\\ u = 0 & \text{on } \partial\Omega; \end{cases} \tag{3.3}$$

is a function $u\in W^{1,p}_0(\Omega)$ such that

$$\int_{\Omega}\langle X(Du(x)),\, Dv(x)\rangle dx = \int_{\Omega} f(x)\,v(x)\,dx \tag{3.4}$$

for every $v\in W^{1,p}_0(\Omega)$.

The well-posedness of problem (3.1) is readily established by the direct method of the calculus of variations:

Lemma 3.2. *Problem (3.1) (respectively, problem (3.3)) has a unique weak solution for every $f\in L^{p'}(\Omega\cap\Sigma)$ (respectively, $f\in L^{p'}(\Omega)$).*

Proof. Define the functional $J : W^{1,p}_{\Gamma_0}(\Omega\cap\Sigma)\to\mathbb{R}$ by

$$J(u) = \int_{\Omega\cap\Sigma} L(x,u(x),Du(x))\,dx,$$

where $L(x,u,\xi)=\frac{1}{p}F(\xi)^p - f(x)\,u$. Due to (2.11), we may use the Poincaré inequality in $W^{1,p}_{\Gamma_0}$ (see [17, Theorem 7.91, p. 488] for the special case when $p=2$), and we can easily show that J is coercive. By Lemma 2.2, the function $L(x,u,\xi)$ is strictly convex in $\xi\in\mathbb{R}^n$, hence J has a unique minimizer $u\in W^{1,p}_{\Gamma_0}(\Omega\cap\Sigma)$. Since J is differentiable, the minimizer u is the unique solution of (3.1). Finally, if we replace the cone Σ with the whole Euclidean space $\mathbb{R}^n$, and the Sobolev space $W^{1,p}_{\Gamma_0}(\Omega\cap\Sigma)$ with $W^{1,p}_0(\Omega)$, the same argument still applies, thus proving the well-posedness of problem (3.3). The usual Poincaré's inequality in $W^{1,p}_0$ is needed in this case. □

In the next lemma, we state and prove the nonnegativity of weak solutions of (3.1) and (3.3) in the case when $f\geqslant 0$.

Lemma 3.3. *If f is a nonnegative function in $L^{p'}(\Omega\cap\Sigma)$, then the weak solution of (3.1) is nonnegative. If f is a nonnegative function in $L^{p'}(\Omega)$, then the weak solution of (3.3) is nonnegative.*

Proof. We give details for the first claim, the second one being analogous. Take

$$v(x) = \begin{cases} 0 & \text{if } u\geqslant 0\\ u & \text{if } u<0 \end{cases}$$

as a test function in (3.2). Using (2.5), almost everywhere, we have

$$\langle X(Du(x)), Dv(x)\rangle = \begin{cases} 0, & u \geqslant 0; \\ F(Du(x))^p, & u < 0. \end{cases}$$

Let $\Omega^- = \{x \in \Omega \cap \Sigma : u(x) < 0\}$. Then

$$0 \leqslant \int_{\Omega^-} F(Du(x))^p \, dx = \int_{\Omega^-} \langle X(Du(x)), Dv(x)\rangle dx$$
$$= \int_{\Omega^-} f(x)\, u(x)\, dx \leqslant 0.$$

We conclude that

$$\int_{\Omega^-} F(Du(x))^p \, dx = 0$$

and therefore $u \geqslant 0$ a.e. in $\Omega \cap \Sigma$. □

Next, we compare two solutions in nested domains (cf. [9, Lemma 2.4]).

Proposition 3.4. *Let Ω_i, $i = 1, 2$ be two bounded domains in $\mathbb{R}^n$, $n \geqslant 2$, containing the origin and satisfying $\Omega_1 \cap \Sigma \subseteq \Omega_2 \cap \Sigma$. Choose a nonnegative function $f \in L^{p'}(\Omega_2 \cap \Sigma)$, and denote by u_i the weak solution of problem (3.1) with $\Omega = \Omega_i$. Then, $u_1 \leqslant u_2$ a.e. in $\Omega_1 \cap \Sigma$. In the case when $\Sigma = \mathbb{R}^n$, the statement continues to hold for problem (3.3).*

Proof. Let $\Gamma_{0i} = \Sigma \cap \partial\Omega_i$, $i = 1, 2$. Since $f \geqslant 0$, from Lemma 3.3, we have $u_2 \geqslant 0$ a.e. in $\Omega_2 \cap \Sigma$. Hence the function

$$v = \begin{cases} u_1 - u_2, & u_1 > u_2 \\ 0, & u_1 \leqslant u_2 \end{cases}$$

belongs to $W^{1,p}_{\Gamma_{01}}(\Omega_1 \cap \Sigma)$ and has an extension, still denoted by v, to $W^{1,p}_{\Gamma_{02}}(\Omega_2 \cap \Sigma)$ vanishing identically outside $\Omega_1 \cap \Sigma$. Therefore, v is an admissible test function in Definition 3.1 for $\Omega = \Omega_i$, $i = 1, 2$, and we may write

$$\int_{\Omega_1 \cap \Sigma} \langle X(Du_1(x)), Dv(x)\rangle dx = \int_{\Omega_1 \cap \Sigma} f(x)\, v(x)\, dx$$

and

$$\int_{\Omega_2 \cap \Sigma} \langle X(Du_2(x)), Dv(x)\rangle dx = \int_{\Omega_2 \cap \Sigma} f(x)\, v(x)\, dx.$$

By subtracting the second equality from the first one, we obtain

$$\int_{\Omega_1 \cap \Sigma} \langle X(Du_1(x)) - X(Du_2(x)), Dv(x)\rangle dx = 0.$$

Observe that the function under the sign of integral is nonnegative by claim (3) of Lemma 2.2. Since the integral vanishes, the function under the sign of integral must vanish almost everywhere. Using claim (3) of Lemma 2.2 again, we deduce $Dv = O$ a.e. in $\Omega_1 \cap \Sigma$ and therefore $u_1 \leqslant u_2$ a.e. in $\Omega_1 \cap \Sigma$. In the case when $\Sigma = \mathbb{R}^n$, the argument proceeds identically after replacing $W^{1,p}_{\Gamma_{0i}}(\Omega_i \cap \Sigma)$ with $W^{1,p}_0(\Omega_i)$ for $i = 1, 2$. □

Proposition 3.5. *The weak solution of (1.5) in the case when* $\Omega = B^-_F(O, R)$ *is*

$$u_R(x) = \frac{1}{p'\,N^{p'-1}}\left(R^{p'} - F^*(-x)^{p'}\right), \quad p' = \frac{p}{p-1}.$$

Proof. Observe that $u_R(x)$ is well defined for all $x \in \mathbb{R}^n$, and clearly vanishes on $\partial B^-_F(O, R)$. Being $p' > 1$, the function $F^*(x)^{p'}$ belongs to $C^1(\mathbb{R}^n)$: See Lemma 2.2(2) and Remark 2.6. Hence, $u_R \in C^1(\mathbb{R}^n)$. Furthermore, by differentiation, we obtain

$$Du_R(x) = \frac{1}{N^{p'-1}}\,F^*(-x)^{p'-1}\,DF^*(-x) \text{ for } x \neq O \tag{3.5}$$

and so by the positive homogeneity condition (**B**) and by Lemma 2.1 (2), we have, respectively,

$$F(Du_R(x)) = \frac{1}{N^{p'-1}}\,F^*(-x)^{p'-1}\,F(DF^*(-x)) \quad \text{for } x \neq O$$

and

$$DF(Du_R(x)) = DF(DF^*(-x)) \quad \text{for } x \neq O.$$

Consequently by (2.16) and Lemma 2.7, we get

$$F(Du_R(x)) = \frac{F^*(-x)^{p'-1}}{N^{p'-1}},\ x \in \mathbb{R}^n, \quad \text{and} \quad DF(Du_R(x)) = \frac{-x}{F^*(-x)},\ x \neq O.$$

The last equality implies $\langle DF(Du_R), \nu\rangle = 0$ on $\Gamma_1 \setminus \{O\}$ because $\langle x, \nu\rangle = 0$ there, hence the third condition in (1.1) is pointwise satisfied. Finally, by (2.3), we get $X(Du_R(x)) = -x/N$ for $x \in \mathbb{R}^n$, and therefore

$$\begin{aligned}
\int_{B^-_F(O,R)\cap\Sigma} \langle X(Du_R(x)), Dv(x)\rangle\,dx &= \frac{-1}{N}\int_{B^-_F(O,R)\cap\Sigma} \langle x, Dv(x)\rangle\,dx \\
&= \frac{1}{N}\int_{B^-_F(O,R)\cap\Sigma} v(x)\,\mathrm{div}(x)\,dx \\
&= \int_{B^-_F(O,R)\cap\Sigma} v(x)\,dx
\end{aligned}$$

for every $v \in W^{1,p}_{\Gamma_0}(\Omega \cap \Sigma)$, and the proof is complete. □

It is readily seen that if u is harmonic in a domain Ω, then the function $\check{u}(y) = u(-y)$ is harmonic in the set $-\Omega = \{y : -y \in \Omega\}$. In the present case, we may prove the following three-minuses formula:

$$-\Delta_{F;p} u(x) = \Delta_{F;p}(-u(-x)). \tag{3.6}$$

To be more specific, we confine ourselves to problem (3.1). We have the following:

Lemma 3.6. *Let $f \in L^{p'}(\Omega \cap \Sigma)$, and denote by $X(\xi)$ the vector field in (2.3). A function $u \in W^{1,p}_{\Gamma_0}(\Omega \cap \Sigma)$ is a weak solution of (3.1) if and only if $\check{u}$ solves the boundary value problem*

$$\begin{cases} \Delta_{F;p}(-\check{u}) = \check{f}(y) & \text{in } -(\Omega \cap \Sigma); \\ \check{u} = 0 & \text{on } -\Gamma_0; \\ \langle DF(-D\check{u}), \nu\rangle = 0 & \text{on } -\Gamma_1 \setminus \{O\}. \end{cases} \tag{3.7}$$

Proof. Suppose $\check{u} \in W^{1,p}_{-\Gamma_0}(-(\Omega \cap \Sigma))$ is a weak solution of (3.7). For every $v \in W^{1,p}_{\Gamma_0}(\Omega \cap \Sigma)$, the function $\check{v}(y) = v(-y)$ belongs to $W^{1,p}_{-\Gamma_0}(-(\Omega \cap \Sigma))$ and therefore we may write

$$-\int_{-\Omega\cap\Sigma} \langle X(-D\check{u}(y)), D\check{v}(y)\rangle dy = \int_{-\Omega\cap\Sigma} \check{f}(y)\, \check{v}(y)\, dy.$$

Taking into account that $-D\check{u}(y) = Du(-y)$ and $-D\check{v}(y) = Dv(-y)$, by the change of variable $x = -y$, we immediately obtain

$$\int_{\Omega\cap\Sigma} \langle X(Du(x)), Dv(x)\rangle dx = \int_{\Omega\cap\Sigma} f(x)\, v(x)\, dx,$$

hence u is a weak solution of (3.1). The converse is proved similarly. □

4 | PROOF OF THE MAIN RESULTS AND ALTERNATIVE FORMULATION

In the present section, we prove the main results of this paper. We also derive an alternative formulation of Theorem 1.1 using the three-minuses formula (3.6).

Proof of Theorem 1.1. Define $R_1 = \min_{z\in\overline{\Gamma}_0} F^*(-z)$ and $R_2 = \max_{z\in\overline{\Gamma}_0} F^*(-z)$. Let u_i, $i = 1, 2$, be the weak solution of the Dirichlet problem in the opposite Finsler-ball $\Omega_i = B^-_F(O, R_i)$, $i = 1, 2$. Then, $\Omega_1 \cap \Sigma \subseteq \Omega \cap \Sigma \subseteq \Omega_2 \cap \Sigma$. We want to show $\Omega_1 = \Omega_2$. By Lemma 3.3 and Proposition 3.4, we have

$$u_1 \leqslant u \text{ a.e. in } \Omega_1 \cap \Sigma, \quad u \leqslant u_2 \text{ a.e. in } \Omega \cap \Sigma.$$

Let us take $z_i \in \overline{\Gamma}_0 \cap \partial\Omega_i$ and observe that $R_i = F^*(-z_i)$, $i = 1, 2$. Furthermore, we have $u_i(z_i) = u(z_i) = 0$, and taking into account that u_i is continuously differentiable up to z_i, we will show the following two inequalities.

$$\frac{R_1^{p'-1}}{N^{p'-1}} = F(Du_1(z_1)) \leqslant q(R_1) \tag{4.1}$$

and

$$q(R_2) \leqslant F(Du_2(z_2)) = \frac{R_2^{p'-1}}{N^{p'-1}}. \tag{4.2}$$

We first prove (4.1). Letting $x(t) = z_1 - t\,|z_1|^{-1} z_1 \in \Omega_1 \cap \overline{\Sigma}$ for $t \in (0, |z_1|)$, we compute the limit

$$l = \lim_{t\to 0^+} \frac{u_1(x(t))}{t}$$

following two different arguments.

(i) Since u_1 is differentiable at z_1, we may write $l = -\left\langle |z_1|^{-1} z_1, Du_1(z_1)\right\rangle$. But using (3.5), we get $Du_1(z_1) = \dfrac{R_1^{p'-1}}{N^{p'-1}} DF^*(-z_1)$. Hence applying (2.2) for F^*, we have

$$l = -\frac{R_1^{p'-1}}{N^{p'-1}} \left\langle |z_1|^{-1} z_1, DF^*(-z_1)\right\rangle = \frac{|z_1|^{-1} R_1^{p'}}{N^{p'-1}}. \tag{4.3}$$

(ii) By the mean value theorem, we have $u(x(t)) = -\left\langle t|z_1|^{-1} z_1, Du(\tilde{x})\right\rangle$ for a convenient point $\tilde{x}$ on the segment from z_1 to $x(t)$. Letting $\xi = Du(\tilde{x})$ and $x = -t|z_1|^{-1} z_1$ in (2.7), and since $F^*(-z_1) = R_1$, we may estimate

$$u(x(t)) \leqslant tR_1 |z_1|^{-1} F(Du(\tilde{x})).$$

As $t \to 0^+$, using assumption (1.6), we obtain

$$l \leqslant R_1\, |z_1|^{-1}\, q(R_1). \tag{4.4}$$

By comparing (4.3) with (4.4), the inequality in (4.1) follows. We now prove the inequality in (4.2). Take $\epsilon \in (0, q(R_2))$. By (1.6), there exists $\delta > 0$ such that for every $x \in \Omega \cap \Sigma$ satisfying $|x - z_2| < \delta$, we have

$$0 < q(R_2) - \epsilon < F(Du(x)). \tag{4.5}$$

Without loss of generality we may take $\delta < \epsilon$ and we define $U_\delta = \{x \in \Omega \cap \Sigma : |x - z_2| < \delta\} = \Omega \cap \Sigma \cap B(z_2, \delta)$. Pick $x_0 \in U_\delta$ and consider the initial value problem

$$\begin{cases} x'(t) = DF(Du(x(t))) \\ x(0) = x_0. \end{cases} \tag{4.6}$$

Note that (4.5) implies $Du \neq O$ in U_δ. Since DF and Du are continuous vector fields, by Peano's theorem [14] the initial value problem (4.6) has a local solution (possibly many). Let us denote by

$x(t)$ a maximal extension of a local solution, subject to $x(t) \in \overline{U_\delta} \setminus \partial B(z_2, \delta)$ for $t \in [0, T)$. Observe that $DF(Du(x))$ is bounded by Lemma 2.1(2) and so is $|x'(t)|$. Furthermore, $|x'(t)|$ keeps also away from zero. By differentiation, and using (2.2), we find

$$\begin{aligned}\frac{d}{dt}\, u(x(t)) &= \big\langle Du(x(t)),\, x'(t)\big\rangle \\ &= \langle Du(x(t)),\, DF(Du(x(t)))\rangle \\ &= F(Du(x(t))).\end{aligned}$$

Hence, $\frac{d}{dt}\, u(x(t))$ is positive and keeps away from zero: This and $u(x_0) > 0$ prevent $x(t)$ from approaching Γ_0. Note, finally that $DF(Du(x))$ is tangent to $\partial\Sigma$ whenever $x \in \Gamma_1 \setminus \{O\}$ as a consequence of the Neumann condition in (1.5). Therefore, the maximal solution $x(t)$ not only does not approach Γ_0, but it will also proceed further even in case $x(t) \in \Gamma_1$ for some $t > 0$ and it will eventually satisfy $|x(T) - z_2| = \delta$ for some finite T. Let us estimate $u(x(T))$. On the one side, since $\frac{d}{dt}\, u(x(t)) = F(Du(x(t))) > q(R_2) - \epsilon$ by (4.5), we may write

$$u(x(T)) = u(x_0) + \int_0^T \frac{d}{dt}\, u(x(t))\, dt > u(x_0) + (q(R_2) - \epsilon)\, T.$$

On the other side, by differentiation and using Lemma 2.1(1), we find

$$\begin{aligned}\frac{d}{dt}\, u_2(x(t)) &= \big\langle Du_2(x(t)),\, x'(t)\big\rangle \\ &= \langle Du_2(x(t)),\, DF(Du(x(t)))\rangle \\ &\leqslant F(Du_2(x(t))).\end{aligned} \tag{4.7}$$

Since $F(Du_2(x)) < F(Du_2(z_2))$ in Ω_2 by Proposition 3.5, and using (4.7), we have

$$u_2(x(T)) = u_2(x_0) + \int_0^T \frac{d}{dt}\, u_2(x(t))\, dt < u_2(x_0) + TF(Du_2(z_2))$$

and consequently

$$\begin{aligned}(q(R_2) - \epsilon)\, T &< u(x(T)) - u(x_0) \\ &\leqslant u_2(x(T)) - u(x_0) \\ &< u_2(x_0) - u(x_0) + TF(Du_2(z_2)),\end{aligned}$$

whence

$$q(R_2) - F(Du_2(z_2)) - \epsilon < \frac{u_2(x_0) - u(x_0)}{T}. \tag{4.8}$$

Let us estimate T from below. Since $|x'(t)| \leqslant M = \max_{\xi \neq O} |DF(\xi)|$, we obtain

$$|x(T) - x_0| \leqslant \int_0^T |x'(t)|\, dt \leqslant MT.$$

Letting $x_0 \to z_2$ in (4.8), the value of T varies as well as the point $x(T)$, however $\lim_{x_0\to z_2} |x(T) - x_0| = \delta$ and therefore $\liminf_{x_0\to z_2} T \geqslant \dfrac{\delta}{M}$. Hence, we have $\dfrac{1}{T} \leqslant \dfrac{2M}{\delta}$ for x_0 close to z_2. Using this last estimate in (4.8), we obtain

$$q(R_2) - F(Du_2(z_2)) - \epsilon < (u_2(x_0) - u(x_0))\,\frac{2M}{\delta}. \tag{4.9}$$

Keeping ϵ fixed and letting $x_0 \to z_2$, we arrive at $q(R_2) - F(Du_2(z_2)) \leqslant \epsilon$. Since ϵ is arbitrary, the inequality in (4.2) follows. Using condition (1.3), and inequalities (4.1) and (4.2), we obtain $R_1 = R_2$. □

Proof of Theorem 1.2. We argue as in the preceding proof. The definitions of R_i, Ω_i and u_i, $i = 1, 2$ are identical. It follows that $\Omega_1 \subseteq \Omega_2$, and we prove that in fact $\Omega_1 = \Omega_2$. Since Lemma 3.3 and Proposition 3.4 are applicable to the case when $\Sigma = \mathbb{R}^n$, we have now

$$u_1 \leqslant u \text{ a.e. in } \Omega_1, \quad u \leqslant u_2 \text{ a.e. in } \Omega.$$

We take $z_i \in \partial\Omega \cap \partial\Omega_i$ and prove the inequalities (4.1) and (4.2). The derivation of (4.1) is identical, apart from the fact that we use condition (1.8) in place of (1.6), and the point $x(t) = z_1 - t\,|z_1|^{-1} z_1$ now ranges in Ω_1, being $\overline{\Sigma} = \mathbb{R}^n$. For proving (4.2), we take ϵ as before, and using (1.8), we determine $\delta \in (0, \epsilon)$ such that (4.5) holds for every $x \in \Omega$ satisfying $|x - z_2| < \delta$. Next we pick $x_0 \in U_\delta = \{x \in \Omega : |x - z_2| < \delta\}$ and consider a maximal solution $x(t)$ of the initial value problem (4.6): Since in the present case $\Gamma_1 = \emptyset$, the solution $x(t)$ will eventually satisfy $|x(T) - z_2| = \delta$, as before, for some finite T. The estimate of T is identical, and we arrive again at (4.9), whence the inequality in (4.2) is obtained. Using the assumption (1.3), and the inequalities (4.1) and (4.2), we finally prove $R_1 = R_2$ and the theorem follows. □

As stated at the beginning of this section, we now derive an alternative formulation of Theorem 1.1 using the three-minuses formula (3.6). Here, $B_F^+(O, R) = \{x \in \mathbb{R}^n : F^*(x) < R\}$.

Theorem 4.1. *Let $q(r)$ be a positive, real-valued function satisfying (1.3), and let $v \in W^{1,p}_{\Gamma_0}(\Omega \cap \Sigma)$ be a weak solution of the problem*

$$\begin{cases} \Delta_{F;p}(-v) = 1 & \text{in } \Omega \cap \Sigma; \\ v = 0 & \text{on } \Gamma_0; \\ \langle DF(-Dv), \nu\rangle = 0 & \text{on } \Gamma_1 \setminus \{O\}. \end{cases} \tag{4.10}$$

If v belongs to the smoothness class $C^1((\Omega \cap \Sigma) \cup (\Gamma_1 \setminus \{O\})) \cap C^0(\overline{\Omega \cap \Sigma} \setminus \{O\})$ and satisfies

$$\lim_{x\to z} F(-Dv(x)) = q(F^*(z)) \;\forall z \in \overline{\Gamma}_0, \tag{4.11}$$

then $\Omega \cap \Sigma = B_F^+(O, R) \cap \Sigma$ for some $R > 0$.

Proof. By Lemma 3.6, the function $u = \check{v}$ is a solution of

$$\begin{cases} -\Delta_{F;p} u = 1 & \text{in } -(\Omega \cap \Sigma); \\ u = 0 & \text{on } -\Gamma_0; \\ \langle DF(Du), \nu \rangle = 0 & \text{on } -\Gamma_1 \setminus \{O\}. \end{cases}$$

Furthermore, if we pick $z \in -\overline{\Gamma}_0$, we obviously have $-z \in \overline{\Gamma}_0$ and by (4.11)

$$\lim_{x \to -z} F(-Dv(x)) = q(F^*(-z)).$$

Since $-Dv(x) = Du(-x)$, letting $y = -x$ we arrive at

$$\lim_{y \to z} F(Du(y)) = q(F^*(-z)) \, \forall z \in -\overline{\Gamma}_0,$$

which is (1.6) in different notation. Therefore, by Theorem 1.5 we may conclude that $-(\Omega \cap \Sigma) = B_F^-(O,R) \cap (-\Sigma)$ for some $R > 0$, hence $\Omega \cap \Sigma = B_F^+(O,R) \cap \Sigma$ as claimed. □

5 | EXAMPLES

As mentioned at the end of the introduction, problem (1.5)–(1.6) under assumption (1.3) is not solvable, in general, even in the case when $\Omega = B_F^-(O,R)$, and a similar remark holds for problem (1.7)–(1.8). Let us identify the solvable instances. For simplicity, we focus on the last problem.

Proposition 5.1. *Let $q(r)$ be a positive, real-valued function, and let $\Omega = B_F^-(O,R)$. Problem (1.7) has a weak solution $u \in W_0^{1,p}(\Omega) \cap C^1(\Omega) \cap C^0(\overline{\Omega})$ satisfying (1.8) if and only if*

$$\left(\frac{R}{N}\right)^{p'-1} = q(R). \tag{5.1}$$

Proof. The unique solution $u_R \in W_0^{1,p}(\Omega)$ of problem (1.7) in the ball $\Omega = B_F^-(O,R)$ is given in Proposition 3.5, and has an extension to $C^1(\mathbb{R}^n)$. The gradient Du_R is found in (3.5). Taking (2.16) into account, condition (1.8) reduces to (5.1). □

Therefore, if the function $q(r)$ satisfies (1.3), then the solvability of the overdetermined problem (1.7)–(1.8) depends on the existence of a solution $r_0 = R > 0$ to the equation

$$\left(\frac{r}{N}\right)^{p'-1} = q(r). \tag{5.2}$$

By (1.3), the equation above may have at most one solution, but it may well happen that (5.2) is unsolvable. On the basis of this discussion, we exhibit the following examples.

Example 5.2. If we choose $q(r) = (r^{p'-1} + r^{p'})/N^{p'-1}$, then assumption (1.3) is satisfied. However, Equation (5.2) has no positive solutions, hence the overdetermined problem

(1.7)–(1.8) is unsolvable. If, instead, we take $q(r) = r^{p'}$, then assumption (1.3) is still satisfied, and Equation (5.2) has the unique solution $r_0 = 1/N^{p'-1}$. Consequently, the overdetermined problem (1.7)–(1.8) is solvable if and only if $\Omega = B_F^-(O,R)$ with $R = 1/N^{p'-1}$.

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