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MATHEMATISCHE NACHRICHTEN

On the logistic equation for the fractional p-Laplacian

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Abstract
We consider a Dirichlet problem for a nonlinear, nonlocal equation driven by the degenerate fractional p-Laplacian, with a logistic-type reaction depending on a positive parameter. In the subdiffusive and equidiffusive cases, we prove existence and uniqueness of the positive solution when the parameter lies in convenient intervals. In the superdiffusive case, we establish a bifurcation result. A new strong comparison result, of independent interest, plays a crucial role in the proof of such bifurcation result.

KEYWORDS
bifurcation, comparison principle, fractional p-Laplacian, logistic equation

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1 | INTRODUCTION

The paper is devoted to the study of the following nonlinear elliptic equation of fractional order with Dirichlet-type condition:

$$(P_\lambda)\qquad \begin{cases} (-\Delta)_p^s\, u = \lambda u^{q-1} - u^{r-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Here, $\Omega \subset \mathbb{R}^N$ ($N \geqslant 2$) is a bounded domain with $C^{1,1}$ boundary $\partial\Omega$, $s \in (0,1)$, $p \geqslant 2$ are s.t. $ps < N$, and the leading operator is the degenerate fractional p-Laplacian, defined for all $u : \mathbb{R}^N \to \mathbb{R}$ smooth enough and $x \in \mathbb{R}^N$ by

$$(-\Delta)_p^s\, u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\{|x-y|>\varepsilon\}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\, dy$$

(which for $p = 2$ reduces to the linear fractional Laplacian, up to a dimensional constant $C(N,s) > 0$). The reaction is of logistic type, with powers $1 < q < r < p_s^*$, where $p_s^* = Np/(N - ps)$ denotes the critical exponent for fractional Sobolev spaces, and $\lambda > 0$ is a parameter. Problem (P_λ) is classified in three different cases, according to the principal exponent $q > 1$:

(a) subdiffusive, if $q < p < r$;
(b) equidiffusive, if $p = q < r$;
(c) superdiffusive, if $p < q < r$.

Logistic equations are widely studied mainly because of their important applications in mathematical biology. Indeed, the parabolic semilinear logistic equation describes the evolution and spatial distribution of a biological population in the presence of constant rates of reproduction and mortality (Verhulst's law), see [17]. This is the obvious reason why, in the study of logistic-type equations, authors are usually interested in *positive* solutions. More recently, evolutive systems involving logistic terms have been studied as a model for the biological phenomenon of chemotaxis [37], and existence of solutions in the presence of a parameter was studied in [1, 7]. Regarding the elliptic counterpart, it models an equilibrium distribution, see [10]. Several existence results for the equidiffusive case (b), combining variational and topological methods, can be found in [2, 3, 36] (note that multiplicity often includes negative and nodal solutions). Bifurcation results for the superdiffusive case (c) can be found in [23] for the Dirichlet problem, and in [29] for the whole space.

Fractional order equations also have a close connection to mathematical biology. Indeed, since fractional elliptic operators model space diffusion via Lévy-type random motion with jumps, they can be effectively used to describe the movement of populations, see [4, 31]. Studies on logistic equations with several nonlocal operators of fractional order have appeared in recent years, including the square root of the Dirichlet Laplacian [8], the spectral Neumann fractional Laplacian [28], and the fractional Laplacian on the whole space [35].

The operator we consider here is both nonlinear and nonlocal. It represents the nonlinear generalization of the fractional Laplacian, and it can be seen as the gradient of the functional $u \mapsto [u]_{s,p}^p/p$ in the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ (see Section 2), as first pointed out in [5]. The corresponding eigenvalue problem was studied in [26], which led to existence results for general nonlinear reactions via Morse theory in [18]. Due to the nature of the operator, regularity theory for weak solutions required a considerable effort as most of the usual techniques (including the Caffarelli–Silvestre extension method) do not apply here. For any $p > 1$, Hölder continuity of weak solutions in the interior and up to the boundary was studied in [14] and [20], respectively.

In the degenerate case $p > 2$, optimal interior Hölder regularity was proved in [6], while a weighted global Hölder regularity result was proved in [21] (the singular case $p \in (1,2)$ is still open). The result of [21] is the fractional counterpart of Lieberman's $C^{1,\alpha}$-regularity result for the classical p-Laplacian [25] and yields many applications, such as the equivalence of Sobolev and Hölder local minimizers of the energy functional [22], the existence of extremal constant sign solutions [16], and more recently a Brezis–Oswald-type weak comparison principle [19]. We also recall other interesting related results, such as the study of critical growth and singularity performed in [9] and the bifurcation results of [12, 32]. For further information, we refer the reader to the surveys [27, 30].

As far as we know, the present literature includes no specific study on the logistic equation for the fractional p-Laplacian. This paper aims at filling the gap, by presenting the following general result for the existence of solutions to problem (P_λ) (in which $\hat\lambda_1 > 0$ denotes the principal eigenvalue of $(-\Delta)_p^s$ in Ω with Dirichlet conditions, see Equation (2.4)):

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $C^{1,1}$-boundary, $p \geqslant 2$, $s \in (0,1)$ s.t. $ps < N$, and $1 < q < r < p_s^*$. Then, the following hold:*

(a) if $q < p$, then for all $\lambda > 0$ problem (P_λ) has a unique solution $u_\lambda > 0$, with $u_\lambda > u_\mu$ in Ω for all $\lambda > \mu > 0$ and $u_\lambda \to 0$ as $\lambda \searrow 0$;
(b) if $q = p$, then for all $\lambda \in (0, \hat\lambda_1]$ problem (P_λ) has no solution, while for all $\lambda > \hat\lambda_1$ (P_λ) has a unique solution $u_\lambda > 0$, with $u_\lambda > u_\mu$ in Ω for all $\lambda > \mu > \hat\lambda_1$ and $u_\lambda \to 0$ as $\lambda \searrow \hat\lambda_1$;
(c) if $q > p$, then there exists $\lambda_* > 0$ s.t. for all $\lambda \in (0, \lambda_*)$ problem (P_λ) has no solution, while (P_{λ_*}) has at least one solution $u_* > 0$, and for all $\lambda > \lambda_*$ (P_λ) has at least two solutions $u_\lambda > v_\lambda > 0$, with $u_\lambda > u_\mu$ in Ω for all $\lambda > \mu > \lambda_*$ and $u_\lambda \to u_*$ as $\lambda \searrow \lambda_*$.

More precise statements of the results above can be found in Theorems 3.1, 3.2, and 3.7. Our approach is variational, based on critical point theory and comparison–truncation arguments. For the sub- and equidiffusive cases, we apply direct minimization and the weak comparison result of [19] for uniqueness. In the superdiffusive case, we prove a bifurcation result and detect via the mountain pass theorem a second solution for all $\lambda > \lambda_*$.

We remark that our result is new even in the semilinear case $p = 2$ (fractional Laplacian) and in the local case $s = 1$ (classical p-Laplacian). Bifurcation theorems are proved in [8] for the superdiffusive logistic equation driven by the square root of the Laplacian, and in [23] for the classical p-Laplacian, but with no information about monotonicity, order between solutions, and convergence. Also, existence and uniqueness for the equidiffusive case with the fractional Laplacian are proved in [35].

A crucial role in our arguments is played by new strong minimum and comparison principles for weak sub- and supersolutions, including a Hopf-type property (see Theorems 2.6 and 2.7). Previous results of this type were proved in [13, 24], respectively, but our versions involve very general reactions and milder restrictions on the constants p, s and can be of general interest, since they are applicable to a wide class of problems driven by the fractional p-Laplacian.

Structure of the paper: in Section 2, we recall some preliminary results (Section 2.1) and prove new minimum and comparison principles (Section 2.2); in Section 3, we deal with the logistic equation, distinguishing between the subdiffusive case (Section 3.1), the equidiffusive case (Section 3.2), and the superdiffusive case (Section 3.3).

Notation: For any $a \in \mathbb{R}$, $\nu > 0$ we set $a^\nu = |a|^{\nu-1}a$. For any $A \subset \mathbb{R}^N$ we shall set $A^c = \mathbb{R}^N \setminus A$ and denote by $|A|$ the Lebesgue measure of A. For any two measurable functions $u, v : \Omega \to \mathbb{R}$, $u \leqslant v$ will mean that $u(x) \leqslant v(x)$ for a.e. $x \in \Omega$ (and similar expressions). The positive (resp., negative) part of u is denoted as u^+ (resp., u^-). Every function u defined in Ω will be identified with its 0-extension to $\mathbb{R}^N$. If X is an ordered function space, then X_+ will denote its non-negative order cone. For all $\nu \in [1, \infty]$, $\|\cdot\|_\nu$ denotes the standard norm of $L^\nu(\Omega)$ (or $L^\nu(\mathbb{R}^N)$, which will be clear from the context). Moreover, C will denote a positive constant whose value may change case by case.

2 | PRELIMINARIES

Problem (P_λ) falls into the following class of Dirichlet problems for the fractional p-Laplacian:

$$\begin{cases} (-\Delta)_p^s\, u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{2.1}$$

Here, $\Omega \subseteq \mathbb{R}^N$ ($N \geqslant 2$) is a bounded domain with $C^{1,1}$ boundary $\partial\Omega$, $s \in (0,1)$, $p > 1$ satisfy $ps < N$. Besides, the general reaction f satisfies the following hypothesis:

H $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function s.t. for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$

$$|f(x,t)| \leqslant c_0(1 + |t|^{r-1}) \qquad (c_0 > 0,\ r \in (p, p_s^*)).$$

In this section, we will collect some old and new properties of the solutions of problem (2.1).

2.1 | Variational formulation and properties of solutions

A variational theory for problem (2.1) was established in the recent literature (see, for instance, [16, 18, 22]). For the reader's convenience, we recall here some of its main features. First, for all measurable $u : \Omega \to \mathbb{R}$, we introduce the Gagliardo seminorm

$$[u]_{s,p,\Omega} = \left[\iint_{\Omega\times\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}}\, dx\, dy \right]^{\frac{1}{p}}$$

setting $[u]_{s,p,\mathbb{R}^N} = [u]_{s,p}$. Then, we define the fractional Sobolev spaces

$$W^{s,p}(\Omega) = \{u \in L^p(\Omega) : [u]_{s,p,\Omega} < \infty\},$$

$$W^{s,p}_0(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ in } \Omega^c\},$$

the latter being a uniformly convex, separable Banach space with norm $\|u\| = [u]_{s,p}$, whose dual space is denoted by $W^{-s,p'}(\Omega)$ (see [15]). The embedding $W^{s,p}_0(\Omega) \hookrightarrow L^\nu(\Omega)$ is continuous for all $\nu \in [1, p^*_s]$ and compact for all $\nu \in [1, p^*_s)$. We also recall from [20, Definition 2.1] the following special space:

$$\widetilde{W}^{s,p}(\Omega) = \Big\{u \in L^p_{\rm loc}(\mathbb{R}^N) : \exists\, U \Supset \Omega \text{ s.t. } u \in W^{s,p}(U), \int_{\mathbb{R}^N} \frac{|u(x)|^{p-1}}{(1+|x|)^{N+ps}}\,dx < \infty\Big\}.$$

By [20, Lemma 2.3], we can define the fractional p-Laplacian as a nonlinear operator $(-\Delta)^s_p : \widetilde{W}^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ by setting for all $u, v \in W^{s,p}_0(\Omega)$

$$\langle(-\Delta)^s_p\, u, v\rangle = \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))^{p-1}(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy$$

(with the convention $a^{p-1} = |a|^{p-2}a$ established above). Such definition is equivalent to the one given in Section 1 as soon as u is smooth enough (for instance, if $u \in \mathcal{S}(\mathbb{R}^N)$).

Clearly $W^{s,p}_0(\Omega) \subset \widetilde{W}^{s,p}(\Omega)$. Also, whenever $u \in \widetilde{W}^{s,p}(\Omega)$ satisfies $u = 0$ in Ω^c, it is easily seen that $u \in W^{s,p}_0(\Omega)$. The restricted operator $(-\Delta)^s_p : W^{s,p}_0(\Omega) \to W^{-s,p'}(\Omega)$ is continuous, maximal monotone, and enjoys the $(S)_+$-property, namely, whenever (u_n) is a sequence in $W^{s,p}_0(\Omega)$ s.t. $u_n \rightharpoonup u$ in $W^{s,p}_0(\Omega)$ and

$$\limsup_n \langle(-\Delta)^s_p\, u_n, u_n - u\rangle \leqslant 0,$$

then $u_n \to u$ in $W^{s,p}_0(\Omega)$ (see [16, Lemma 2.1] for $p \geqslant 2$, with analogous argument for $p \in (1,2)$). For all $u \in W^{s,p}_0(\Omega)$, we have

$$\|u^\pm\|^p \leqslant \langle(-\Delta)^s_p\, u, \pm u^\pm\rangle. \tag{2.2}$$

Another useful property, referred to as strict T-monotonicity, of $(-\Delta)^s_p$ is the following, which holds for *any* $p > 1$ (see [26, proof of Lemma 9]):

Proposition 2.1. *Let $u, v \in \widetilde{W}^{s,p}(\Omega)$ s.t. $(u-v)^+ \in W^{s,p}_0(\Omega)$ satisfy*

$$\langle(-\Delta)^s_p\, u - (-\Delta)^s_p\, v, (u-v)^+\rangle \leqslant 0.$$

Then, $u \leqslant v$ in Ω.

We say that $u \in \widetilde{W}^{s,p}(\Omega)$ is a (weak) supersolution of Equation (2.1) if $u \geqslant 0$ in Ω^c and for all $v \in W^{s,p}_0(\Omega)_+$

$$\langle(-\Delta)^s_p\, u, v\rangle \geqslant \int_\Omega f(x,u)v\,dx,$$

and similarly we define a (weak) subsolution. Finally, $u \in W^{s,p}_0(\Omega)$ is a (weak) solution of Equation (2.1) if it is both a super- and a subsolution, that is, if for all $v \in W^{s,p}_0(\Omega)$

$$\langle(-\Delta)^s_p\, u, v\rangle = \int_\Omega f(x,u)v\,dx.$$

In such cases, we write that weakly in Ω

$$(-\Delta)^s_p\, u = (\geqslant, \leqslant)\, f(x,u).$$

From [9, Theorem 3.3], we have the following a priori bound on the solutions:

Proposition 2.2. *Let* **H** *hold,* $u \in W^{s,p}_0(\Omega)$ *be a solution of Equation (2.1). Then,* $u \in L^\infty(\Omega)$ *with* $\|u\|_\infty \leqslant C(\|u\|)$.

Classical nonlinear regularity theory does not apply to fractional order equations, whose solutions fail to be C^1 in general. Nevertheless, weighted Hölder continuity can replace higher smoothness in most cases. We set $\mathrm{d}_\Omega(x) = \mathrm{dist}(x, \Omega^c)$ for all $x \in \mathbb{R}^N$ and define the following space:

$$C^0_s(\overline{\Omega}) = \left\{ u \in C^0(\overline{\Omega}) : \frac{u}{\mathrm{d}^s_\Omega} \text{ has a continuous extension to } \overline{\Omega} \right\},$$

a Banach space under the norm $\|u\|_{0,s} = \|u/\mathrm{d}^s_\Omega\|_\infty$. By [18, Lemma 5.1], the positive order cone $C^0_s(\overline{\Omega})_+$ has a nonempty interior

$$\mathrm{int}(C^0_s(\overline{\Omega})_+) = \left\{ u \in C^0_s(\overline{\Omega}) : \inf_\Omega \frac{u}{\mathrm{d}^s_\Omega} > 0 \right\}.$$

Similarly, for any $\alpha \in (0,1)$ we set

$$C^\alpha_s(\overline{\Omega}) = \left\{ u \in C^0(\overline{\Omega}) : \frac{u}{\mathrm{d}^s_\Omega} \text{ has a } \alpha\text{-Hölder continuous extension to } \overline{\Omega} \right\},$$

a Banach space under the norm

$$\|u\|_{\alpha,s} = \|u\|_{0,s} + \sup_{x,y\in\Omega,\, x\neq y} \frac{|u(x)/\mathrm{d}^s_\Omega(x) - u(y)/\mathrm{d}^s_\Omega(y)|}{|x-y|^\alpha}.$$

By the Ascoli–Arzelà theorem, $C^\alpha_s(\overline{\Omega}) \hookrightarrow C^0_s(\overline{\Omega})$ with compact embedding for all $\alpha \in (0,1)$. From Proposition 2.2, [20, Theorem 1.1], and [21, Theorem 1.1] we have the following weighted Hölder regularity result:

Proposition 2.3. *Let* **H** *hold,* $u \in W^{s,p}_0(\Omega)$ *be a solution of Equation (2.1). Then, there exists* $\alpha \in (0, s]$*, independent of* u*, s.t.* $u \in C^\alpha(\overline{\Omega})$*. Besides, if* $p \geqslant 2$*, then* $u \in C^\alpha_s(\overline{\Omega})$ *and* $\|u\|_{\alpha,s} \leqslant C(\|u\|)$.

Weighted Hölder continuity is known only for the degenerate case $p \geqslant 2$. This is the main reason why the next results, which use such type of regularity, are only stated for $p \geqslant 2$. From [19, Proposition 2.8], we have the following weak comparison principle under a special monotonicity assumption of Brezis–Oswald type:

Proposition 2.4. *Let* **H** *hold,* $p \geqslant 2$*, and assume that*

$$t \mapsto \frac{f(x,t)}{t^{p-1}}$$

is decreasing in $(0,\infty)$ *for a.e.* $x \in \Omega$*. Let* $u, v \in \mathrm{int}(C^0_s(\overline{\Omega})_+) \cap W^{s,p}_0(\Omega)$ *be a subsolution and a supersolution, respectively, of Equation (2.1). Then,* $u \leqslant v$ *in* Ω.

The energy functional for problem (2.1) is defined by setting for all $u \in W^{s,p}_0(\Omega)$

$$\Phi(u) = \frac{\|u\|^p}{p} - \int_\Omega F(x,u)\,dx,$$

where we have set for all $(x,t) \in \Omega \times \mathbb{R}$

$$F(x,t) = \int_0^t f(x,\tau)\,d\tau.$$

By classical results, we have $\Phi \in C^1(W^{s,p}_0(\Omega))$, and $u \in W^{s,p}_0(\Omega)$ is a solution of Equation (2.1) iff $\Phi'(u) = 0$ in $W^{-s,p'}(\Omega)$. Besides, by [18, Proposition 2.1] Φ satisfies a bounded (PS)-condition, namely, whenever (u_n) is a bounded sequence in $W^{s,p}_0(\Omega)$ s.t. $(\Phi(u_n))$ is bounded in $\mathbb{R}$ and $\Phi'(u_n) \to 0$ in $W^{-s,p'}(\Omega)$, then (u_n) has a convergent subsequence. In this connection, we recall from [22, Theorem 1.1] the following equivalence principle for Sobolev and Hölder local minimizers of Φ:

Proposition 2.5. *Let* **H** *hold,* $p \geqslant 2$, $u \in W^{s,p}_0(\Omega)$. *Then, the following are equivalent:*

(i) there exists $\rho > 0$ *s.t.* $\Phi(u+v) \geqslant \Phi(u)$ *for all* $v \in W^{s,p}_0(\Omega) \cap C^0_s(\overline{\Omega})$, $\|v\|_{0,s} \leqslant \rho$;
(ii) there exists $\sigma > 0$ *s.t.* $\Phi(u+v) \geqslant \Phi(u)$ *for all* $v \in W^{s,p}_0(\Omega)$, $\|v\| \leqslant \sigma$.

Regarding the spectral properties of the fractional p-Laplacian, we refer the reader to [26]. We just recall that the eigenvalue problem is stated as

$$\begin{cases} (-\Delta)^s_p\, u = \lambda u^{p-1} & \text{in } \Omega \\ u = 0 & \text{in } \Omega. \end{cases} \tag{2.3}$$

The principal eigenvalue $\hat{\lambda}_1 > 0$ of Equation (2.3) is simple and isolated, with a unique positive eigenfunction $\hat{u}_1 \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$ s.t. $\|u\|_p = 1$, and both are defined as follows:

$$\hat{\lambda}_1 = \inf_{u \in W^{s,p}_0(\Omega)\setminus\{0\}} \frac{\|u\|^p}{\|u\|_p^p} = \|\hat{u}_1\|^p. \tag{2.4}$$

2.2 | Strong minimum and comparison principles

As mentioned in Section 1, a strong minimum principle and a Hopf-type lemma for the fractional p-Laplacian were proved in [13, Theorems 1.2, 1.5], while a strong comparison principle was obtained in [24, Theorem 1.1]. Nevertheless, the strong comparison principle of [24] does not fit with our purposes for two reasons: first, in the degenerate case $p > 2$ it requires some special relations between the parameters p and s which, combined with the optimal Hölder continuity proved in [6], lead to the quite restrictive condition $s \leqslant 1/p'$; second, the result only ensures that the difference between the super- and the subsolution is positive in Ω, while we need to prove that such difference lies in $\mathrm{int}(C^0_s(\overline{\Omega})_+)$.

Motivated by such difficulties, we present here a new pair of results, following an alternative approach based on the nonlocal superposition principle introduced in [21]. In view of future applications, we will prove such results for *any* $p > 1$.

We begin with a strong minimum principle (including a Hopf-type boundary property):

Theorem 2.6. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain with* $C^{1,1}$ *boundary,* $p > 1$, $s \in (0,1)$ *s.t.* $ps < N$, $g \in C^0(\mathbb{R}) \cap BV_{\mathrm{loc}}(\mathbb{R})$, $u \in \widetilde{W}^{s,p}(\Omega) \cap C^0(\overline{\Omega})$, $u \not\equiv 0$ *s.t.*

$$\begin{cases} (-\Delta)^s_p\, u + g(u) \geqslant g(0) & \text{weakly in } \Omega \\ u \geqslant 0 & \text{in } \mathbb{R}^N. \end{cases}$$

Then,

$$\inf_{\Omega} \frac{u}{\mathrm{d}^s_{\Omega}} > 0.$$

In particular, if $u \in C^0_s(\overline{\Omega})$, *then* $u \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$.

Proof. By Jordan's decomposition, we can find $g_1, g_2 \in C^0(\mathbb{R})$ nondecreasing s.t. $g(t) = g_1(t) - g_2(t)$ for all $t \in \mathbb{R}$, and $g_1(0) = 0$. So, we have weakly in Ω

$$\begin{aligned} (-\Delta)^s_p\, u + g_1(u) &= (-\Delta)^s_p\, u + g(u) + g_2(u) \\ &\geqslant g(0) + g_2(0) = 0. \end{aligned}$$

Thus, without loss of generality we may assume that g is nondecreasing and $g(0) = 0$. In order to prove our assertion, we need a lower barrier for u. Let us consider the following torsion problem:

$$\begin{cases} (-\Delta)_p^s\, v = 1 & \text{in } \Omega \\ v = 0 & \text{in } \Omega^c. \end{cases} \tag{2.5}$$

By convexity, Equation (2.5) has a unique solution $v \in W_0^{s,p}(\Omega)$, which by [21, Lemma 2.3] satisfies $v \geqslant c\,\mathrm{d}_\Omega^s$ in Ω, for some $c > 0$. By Proposition 2.3, we have $v \in C^\alpha(\overline{\Omega})$, in particular v is continuous. So, since $u \not\equiv 0$, we can find $x_0 \in \Omega$, $\rho, \varepsilon > 0$, and $\eta_0 \in (0,1)$ s.t. $\overline{B}_\rho(x_0) \subset \Omega$ and

$$\sup_{\overline{B}_\rho(x_0)} \eta_0 v < \inf_{\overline{B}_\rho(x_0)} u - \varepsilon. \tag{2.6}$$

Set for all $x \in \mathbb{R}^N$, $\eta \in (0, \eta_0]$

$$w_\eta(x) = \begin{cases} \eta v(x) & \text{if } x \in \overline{B}^c_{\rho/2}(x_0) \\ u(x) & \text{if } x \in \overline{B}_{\rho/2}(x_0). \end{cases}$$

First, by Equation (2.6) we have $w_\eta \leqslant u$ in $\overline{B}_\rho(x_0)$. Besides, by the nonlocal superposition principle [21, Proposition 2.6] we have $w_\eta \in \widetilde{W}^{s,p}(\Omega \setminus \overline{B}_\rho(x_0))$ and weakly in $\Omega \setminus \overline{B}_\rho(x_0)$

$$\begin{aligned} (-\Delta)_p^s\, w_\eta(x) &= (-\Delta)_p^s\, (\eta v)(x) + 2\int_{\overline{B}_{\rho/2}(x_0)} \frac{(\eta v(x) - u(y))^{p-1} - (\eta v(x) - \eta v(y))^{p-1}}{|x-y|^{N+ps}}\, dy \\ &\leqslant \eta^{p-1} + 2\int_{\overline{B}_{\rho/2}(x_0)} \frac{(\eta v(x) - u(y))^{p-1} - (\eta v(x) - u(y) + \varepsilon)^{p-1}}{|x-y|^{N+ps}}\, dy, \end{aligned}$$

where we have also used Equation (2.5) and again Inequality (2.6). Now, by continuity we can find $C_\varepsilon > 0$, independent of $\eta \in (0, \eta_0]$, s.t. for all $x \in \Omega \setminus \overline{B}_\rho(x_0)$, $y \in \overline{B}_{\rho/2}(x_0)$

$$(\eta v(x) - u(y))^{p-1} - (\eta v(x) - u(y) + \varepsilon)^{p-1} \leqslant -C_\varepsilon,$$

and $C_\varepsilon \to 0$ as $\varepsilon \searrow 0$. So, we have weakly in $\Omega \setminus \overline{B}_\rho(x_0)$

$$(-\Delta)_p^s\, w_\eta(x) \leqslant \eta^{p-1} - 2\int_{\overline{B}_{\rho/2}(x_0)} \frac{C_\varepsilon}{(\rho/2)^{N+ps}}\, dy \leqslant \eta^{p-1} - \tilde{C}_\varepsilon,$$

with $\tilde{C}_\varepsilon > 0$ independent of η. Choosing $\eta \in (0, \eta_0]$ small enough, we have weakly in $\Omega \setminus \overline{B}_\rho(x_0)$

$$(-\Delta)_p^s\, w_\eta(x) \leqslant -\frac{\tilde{C}_\varepsilon}{2}.$$

Note that $g(w_\eta) \to 0$ uniformly in $\Omega \setminus \overline{B}_\rho(x_0)$ as $\eta \searrow 0$. So, for an even smaller $\eta \in (0, \eta_0]$ we have

$$\begin{cases} (-\Delta)_p^s\, w_\eta + g(w_\eta) \leqslant 0 \leqslant (-\Delta)_p^s\, u + g(u) & \text{weakly in } \Omega \setminus \overline{B}_\rho(x_0) \\ w_\eta \leqslant u & \text{in } (\Omega \setminus \overline{B}_\rho(x_0))^c. \end{cases}$$

We have $(w_\eta - u)^+ \in \widetilde{W}^{s,p}(\Omega \setminus \overline{B}_\rho(x_0))$ and, by the second inequality above, $(w_\eta - u)^+ = 0$ in $(\Omega \setminus \overline{B}_\rho(x_0)^c$, hence $(w_\eta - u)^+ \in W_0^{s,p}(\Omega \setminus \overline{B}_\rho(x_0))$. So, we can employ such function to test the inequality above. We get

$$\langle (-\Delta)_p^s\, w_\eta - (-\Delta)_p^s\, u, (w_\eta - u)^+ \rangle \leqslant \int_{\Omega \setminus \overline{B}_\rho(x_0)} (g(u) - g(w_\eta))(w_\eta - u)^+\, dx,$$

and the latter is negative by the monotonicity of g. By Proposition 2.1, we have $w_\eta \leqslant u$ in $\Omega \setminus \overline{B}_\rho(x_0)$. Combining with Inequality (2.6) we get in Ω

$$u \geqslant \eta v \geqslant \eta\, c\, \mathrm{d}_\Omega^s,$$

hence the conclusion. In particular, if $u \in C_s^0(\overline{\Omega})$, then clearly we have $u \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$. $\square$

With a similar technique, we prove a strong comparison principle:

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $C^{1,1}$ boundary, $p > 1$, $s \in (0,1)$ s.t. $ps < N$, $g \in C^0(\mathbb{R}) \cap BV_{\mathrm{loc}}(\mathbb{R})$, $u \in \widetilde{W}^{s,p}(\Omega) \cap C^0(\overline{\Omega})$, $v \in W_0^{s,p}(\Omega) \cap C^0(\overline{\Omega})$ s.t. $u \not\equiv v$, $K > 0$ satisfy*

$$\begin{cases} (-\Delta)_p^s\, v + g(v) \leqslant (-\Delta)_p^s\, u + g(u) \leqslant K & \text{weakly in } \Omega \\ 0 < v \leqslant u & \text{in } \Omega \\ u \geqslant 0 & \text{in } \Omega^c. \end{cases}$$

Then, $u > v$ in Ω. In particular, if $u, v \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$, then $u - v \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$.

Proof. As in Theorem 2.6, we may assume g nondecreasing. By continuity, we can find $x_0 \in \Omega$, $\rho, \varepsilon > 0$ s.t. $\overline{B}_\rho(x_0) \subset \Omega$ and

$$\sup_{\overline{B}_\rho(x_0)} v < \inf_{\overline{B}_\rho(x_0)} u - \varepsilon.$$

Hence, for all $\eta \in (1,2)$ close enough to 1 we have

$$\sup_{\overline{B}_\rho(x_0)} \eta v < \inf_{\overline{B}_\rho(x_0)} u - \frac{\varepsilon}{2}. \tag{2.7}$$

Define $w_\eta \in \widetilde{W}^{s,p}(\Omega \setminus \overline{B}_\rho(x_0))$ as in Theorem 2.6, so by Inequality (2.7) we have $w_\eta \leqslant u$ in $(\Omega \setminus \overline{B}_\rho(x_0))^c$. Applying nonlocal superposition as in the previous proof, we have weakly in $\Omega \setminus \overline{B}_\rho(x_0)$

$$(-\Delta)_p^s\, w_\eta \leqslant \eta^{p-1}(-\Delta)_p^s\, v - C_\varepsilon,$$

for some $C_\varepsilon > 0$ independent of $\eta \in (1,2)$. Further, we have weakly in $\Omega \setminus \overline{B}_\rho(x_0)$

$$\begin{aligned} (-\Delta)_p^s\, w_\eta + g(w_\eta) &\leqslant \eta^{p-1}(-\Delta)_p^s\, v + g(w_\eta) - C_\varepsilon \\ &\leqslant \eta^{p-1}\big((-\Delta)_p^s\, v + g(v)\big) + \big(g(w_\eta) - \eta^{p-1}g(v)\big) - C_\varepsilon \\ &\leqslant \eta^{p-1}\big((-\Delta)_p^s\, u + g(u)\big) + \big(g(w_\eta) - \eta^{p-1}g(v)\big) - C_\varepsilon \\ &\leqslant (-\Delta)_p^s\, u + g(u) + K(\eta^{p-1} - 1) + \big(g(w_\eta) - \eta^{p-1}g(v)\big) - C_\varepsilon, \end{aligned}$$

where we have used the hypothesis and the monotonicity of g. Since

$$K(\eta^{p-1} - 1) + \big(g(w_\eta) - \eta^{p-1}g(v)\big) \to 0$$

uniformly in $\Omega \setminus \overline{B}_\rho(x_0)$ as $\eta \searrow 1$, we have for all $\eta > 1$ close enough to 1

$$\begin{cases} (-\Delta)_p^s\, w_\eta + g(w_\eta) \leqslant (-\Delta)_p^s\, u + g(u) & \text{weakly in } \Omega \setminus \overline{B}_\rho(x_0) \\ w_\eta \leqslant u & \text{in } (\Omega \setminus \overline{B}_\rho(x_0))^c. \end{cases}$$

Testing with $(w_\eta - u)^+ \in W^{s,p}_0(\Omega \setminus \overline{B}_\rho(x_0))$, recalling the monotonicity of g, and applying Proposition 2.1 we get $u \geqslant w_\eta$ in $\Omega \setminus \overline{B}_\rho(x_0)$. So we have in Ω

$$u \geqslant \eta v > v,$$

hence the conclusion. In particular, if $u, v \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$, then clearly

$$\inf_\Omega \frac{u-v}{\mathrm{d}^s_\Omega} \geqslant \inf_\Omega \frac{(\eta-1)v}{\mathrm{d}^s_\Omega} > 0,$$

so $u - v \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$. □

Remark 2.8. Both results above exhibit unexpected differences when compared to the corresponding local versions, that is, the case of the classical p-Laplacian. For example, according to Theorem 2.6, the strong minimum principle holds for non-negative supersolutions of the Dirichlet problem

$$\begin{cases} (-\Delta)^s_p\, u + u^\sigma = 0 & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c \end{cases}$$

for any $\sigma > 0$, while for $s = 1$ the same is not true when $\sigma < p - 1$ due to the possible presence of dead cores (see [34, p. 204]). Also, the strong comparison principle of Theorem 2.7 includes cases which are excluded in the local case (see [11, Example 4.1]). This is essentially due to the nonlocal nature of the operator.

3 | THE LOGISTIC EQUATION

In this section, we study problem (P_λ) with $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ bounded domain with a $C^{1,1}$ boundary, $p \geqslant 2$, $s \in (0,1)$ s.t. $ps < N$, and $1 < q < r < p^*_s$. For all $\lambda > 0$, $t \in \mathbb{R}$, we set

$$f_\lambda(t) = \lambda(t^+)^{q-1} - (t^+)^{r-1},$$

$$F_\lambda(t) = \int_0^t f_\lambda(\tau)\, d\tau = \lambda\frac{(t^+)^q}{q} - \frac{(t^+)^r}{r}.$$

Note that $f_\lambda : \mathbb{R} \to \mathbb{R}$ satisfies hypotheses **H** as stated in Section 2. So we may set for all $u \in W^{s,p}_0(\Omega)$

$$\Phi_\lambda(u) = \frac{\|u\|^p}{p} - \int_\Omega F_\lambda(u)\, dx, \tag{3.1}$$

and deduce that $\Phi_\lambda \in C^1(W^{s,p}_0(\Omega))$. As we will see, the positive critical points of Φ_λ coincide with the solutions of (P_λ). In the following subsections, we separately study the different cases according to the position of q.

3.1 | The subdiffusive case

We assume $1 < q < p < r < p^*_s$. In this case, we have the following global existence and uniqueness result (corresponding to case (a) of Theorem 1.1):

Theorem 3.1. *Let $1 < q < p < r < p^*_s$. Then, for all $\lambda > 0$ problem (P_λ) has a unique solution $u_\lambda \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$, s.t. $u_\lambda - u_\mu \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$ for all $\lambda > \mu > 0$ and $u_\lambda \to 0$ in both $W^{s,p}_0(\Omega)$ and $C^0_s(\overline{\Omega})$ as $\lambda \searrow 0$.*

Proof. Fix any $\lambda > 0$. We will find the solution of (P_λ) by direct minimization. First, we prove that the functional Φ_λ (defined in Equation (3.1)) is coercive. Indeed, since $q < r$, the mapping F_λ is clearly bounded from above, that is, there exists $C > 0$ s.t. $F_\lambda(t) \leqslant C$ for all $t \in \mathbb{R}$. So, for all $u \in W_0^{s,p}(\Omega)$ we have

$$\Phi_\lambda(u) \geqslant \frac{\|u\|^p}{p} - C|\Omega|,$$

and the latter tends to ∞ as $\|u\| \to \infty$. Besides, by the compact embeddings $W_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$, $L^r(\Omega)$, it is easily seen that Φ_λ is sequentially weakly lower semicontinuous in $W_0^{s,p}(\Omega)$. So, there exists $u_\lambda \in W_0^{s,p}(\Omega)$ s.t.

$$\Phi_\lambda(u_\lambda) = \inf_{W_0^{s,p}(\Omega)} \Phi_\lambda =: m_\lambda. \tag{3.2}$$

Besides, let $\hat{u}_1 \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ be defined by Equation (2.4). Then, for all $\tau > 0$

$$\Phi_\lambda(\tau\hat{u}_1) = \tau^p \frac{\|\hat{u}_1\|^p}{p} - \lambda\tau^q \frac{\|\hat{u}_1\|_q^q}{q} + \tau^r \frac{\|\hat{u}_1\|_r^r}{r},$$

and the latter is negative for all $\tau > 0$ small enough (recall that $q < p < r$). So, in Equation (3.2) we have $m_\lambda < 0$, implying $u_\lambda \not\equiv 0$. From Equation (3.2), we deduce that $\Phi'_\lambda(u_\lambda) = 0$ in $W^{-s,p'}(\Omega)$, that is, we have weakly in Ω

$$(-\Delta)_p^s\, u_\lambda = f_\lambda(u_\lambda). \tag{3.3}$$

By Proposition 2.3, we have $u_\lambda \in C_s^\alpha(\overline{\Omega})$. Besides, testing Equation (3.3) with $-u_\lambda^- \in W_0^{s,p}(\Omega)$ and applying Equation (2.2), we have

$$\|u_\lambda^-\|^p \leqslant \langle(-\Delta)_p^s\, u_\lambda, -u_\lambda^-\rangle = \int_\Omega f_\lambda(u_\lambda)(-u_\lambda^-)\, dx = 0,$$

so $u_\lambda \geqslant 0$. Now, Theorem 2.6 implies $u_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$, so u_λ solves (P_λ).

Next, we prove uniqueness. Let $v_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ be another solution of (P_λ). We have for all $t > 0$

$$\frac{f_\lambda(t)}{t^{p-1}} = \lambda t^{q-p} - t^{r-p},$$

and such mapping is decreasing in $(0, \infty)$. Applying Proposition 2.4 twice, we have $u_\lambda = v_\lambda$.

To see monotonicity, let $0 < \mu < \lambda$, and $u_\mu, u_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ be the solutions of (P_μ), (P_λ), respectively. We have weakly in Ω

$$(-\Delta)_p^s\, u_\mu < \lambda u_\mu^{q-1} - u_\mu^{r-1},$$

so u_μ is a strict subsolution of (P_λ). By Proposition 2.4 again, we have $u_\mu \leqslant u_\lambda$ in Ω. This in turn implies that weakly in Ω

$$(-\Delta)_p^s\, u_\mu + u_\mu^{r-1} = \mu u_\mu^{q-1} < \lambda u_\lambda^{q-1} = (-\Delta)_p^s\, u_\lambda + u_\lambda^{r-1}.$$

Since $g(t) = t^{r-1}$ is continuous and with locally bounded variation, we can apply Theorem 2.7 and see that $u_\lambda - u_\mu \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$.

Finally, let (λ_n) be a decreasing sequence in $(0, \infty)$ s.t. $\lambda_n \searrow 0$, and $u_n \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ be the solution of (P_{λ_n}) for all $n \in \mathbb{N}$, that is, we have weakly in Ω

$$(-\Delta)_p^s\, u_n = f_{\lambda_n}(u_n). \tag{3.4}$$

Since $q < p$ and (λ_n) is decreasing, we can find $C > 0$ s.t. for all $n \in \mathbb{N}$, $t \in \mathbb{R}$

$$f_{\lambda_n}(t)t \leqslant C.$$

Testing Equation (3.4) with $u_n \in W_0^{s,p}(\Omega)$, for all $n \in \mathbb{N}$ we have

$$\|u_n\|^p = \langle(-\Delta)_p^s\, u_n, u_n\rangle = \int_\Omega f_{\lambda_n}(u_n)u_n\, dx \leqslant C|\Omega|.$$

So, (u_n) is a bounded sequence in $W^{s,p}_0(\Omega)$. By reflexivity and the compact embeddings $W^{s,p}_0(\Omega) \hookrightarrow L^q(\Omega)$, $L^r(\Omega)$, we can pass to a subsequence s.t. $u_n \rightharpoonup u_0$ in $W^{s,p}_0(\Omega)$ and $u_n \to u_0$ in both $L^q(\Omega)$ and $L^r(\Omega)$. Testing Equation (3.4) with $(u_n - u_0) \in W^{s,p}_0(\Omega)$ and using Hölder's inequality, we have for all $n \in \mathbb{N}$

$$\langle (-\Delta)^s_p\, u_n, u_n - u_0\rangle = \int_\Omega (\lambda_n u_n^{q-1} - u_n^{r-1})(u_n - u_0)\,dx$$
$$\leqslant \lambda_1 \|u_n\|_q^{q-1}\|u_n - u_0\|_q + \|u_n\|_r^{r-1}\|u_n - u_0\|_r,$$

and the latter tends to 0 as $n \to \infty$. By the $(S)_+$-property of $(-\Delta)^s_p$, we have $u_n \to u_0$ in $W^{s,p}_0(\Omega)$. So, we can pass to the limit in Equation (3.4) as $n \to \infty$ and get weakly in Ω

$$(-\Delta)^s_p\, u_0 = -u_0^{r-1}.$$

Testing with $u_0 \in W^{s,p}_0(\Omega)$ we have

$$\|u_0\|^p + \|u_0\|_r^r = 0,$$

that is, $u_0 = 0$. Plus, we note that, by Equation (3.4) and Proposition 2.3, (u_n) is bounded in $C^\alpha_s(\overline{\Omega})$, hence, passing to a further subsequence, $u_n \to 0$ in $C^0_s(\overline{\Omega})$. Recalling that $\lambda \mapsto u_\lambda$ is strictly increasing, we conclude that globally $u_\lambda \to 0$ in both $W^{s,p}_0(\Omega)$ and $C^0_s(\overline{\Omega})$, as $\lambda \searrow 0$. □

3.2 | The equidiffusive case

Now, we assume $2 \leqslant q = p < r < p^*_s$, a case that does not differ too much from the previous one, except that the threshold for the parameter λ turns out to be the principal eigenvalue $\hat{\lambda}_1 > 0$ defined in Equation (2.4). Our existence and uniqueness result (corresponding to case (b) of Theorem 1.1) is the following:

Theorem 3.2. *Let $2 \leqslant q = p < r < p^*_s$. Then, for all $\lambda \in (0, \hat{\lambda}_1]$ problem (P_λ) has no solution, while for all $\lambda > \hat{\lambda}_1$ problem (P_λ) has a unique solution $u_\lambda \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$, s.t. $u_\lambda - u_\mu \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$ for all $\lambda > \mu > \hat{\lambda}_1$ and $u_\lambda \to 0$ in both $W^{s,p}_0(\Omega)$ and $C^0_s(\overline{\Omega})$ as $\lambda \searrow \lambda_1$.*

Proof. First, fix $\lambda \in (0, \hat{\lambda}_1]$. Assume that $u \in W^{s,p}_0(\Omega)_+$ satisfies weakly in Ω

$$(-\Delta)^s_p\, u = \lambda u^{p-1} - u^{r-1}. \tag{3.5}$$

Testing Equation (3.5) with $u \in W^{s,p}_0(\Omega)$ and applying Equation (2.4), we have

$$0 = \|u\|^p - \lambda\|u\|_p^p + \|u\|_r^r \geqslant (\hat{\lambda}_1 - \lambda)\|u\|_p^p + \|u\|_r^r \geqslant \|u\|_r^r,$$

hence $u = 0$. So (P_λ) admits no solution.

Now, let $\lambda > \hat{\lambda}_1$, and define Φ_λ as in Equation (3.1). Arguing as in Theorem 3.1, we see that Φ_λ has a global minimizer $u_\lambda \in W^{s,p}_0(\Omega)_+$. Besides, let $\hat{u}_1 \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$ be as in Equation (2.4). Then, for all $\tau > 0$ we have

$$\Phi_\lambda(\tau\hat{u}_1) = \tau^p\left[\frac{\|\hat{u}_1\|^p}{p} - \lambda\frac{\|\hat{u}_1\|_p^p}{p}\right] + \tau^r\frac{\|\hat{u}_1\|_r^r}{r}$$
$$= \tau^p\frac{\hat{\lambda}_1 - \lambda}{p} + \tau^r\frac{\|\hat{u}_1\|_r^r}{r},$$

and the latter is negative for $\tau > 0$ small enough (as $p < r$). So, $u_\lambda \not\equiv 0$. The rest of the proof follows exactly as in Theorem 3.1. □

3.3 | The superdiffusive case

In this final case, we assume $2 \leqslant p < q < r < p_s^*$ and define Φ_λ as in Equation (3.1). We will need a more accurate analysis. Let

$$\Lambda = \left\{\lambda > 0 : (P_\lambda) \text{ has a solution } u_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)\right\}.$$

In the following lemmas, we shall investigate the structure of the set Λ and additional properties of solutions. We begin with a lower bound for Λ:

Lemma 3.3. *We have $\Lambda \neq \emptyset$ and $\lambda_* := \inf \Lambda > 0$.*

Proof. Fix $\lambda > 0$. As in the proof of Theorem 3.1, we find $u_\lambda \in W_0^{s,p}(\Omega)_+$ s.t.

$$\Phi_\lambda(u_\lambda) = \inf_{W_0^{s,p}(\Omega)} \Phi_\lambda =: m_\lambda. \tag{3.6}$$

Let $\hat{u}_1 \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ be as in Equation (2.4), then we have

$$\Phi_\lambda(\hat{u}_1) = \frac{\|\hat{u}_1\|^p}{p} - \lambda\frac{\|\hat{u}_1\|_q^q}{q} + \frac{\|\hat{u}_1\|_r^r}{r},$$

which tends to $-\infty$ as $\lambda \to \infty$. So, for all $\lambda > 0$ big enough we have $m_\lambda < 0$ in Equation (3.6), hence $u_\lambda \neq 0$. As in Theorem 3.1 we see that $u_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ and it solves (P_λ). Thus, we have $\Lambda \neq \emptyset$.

We claim that there exists $\lambda_0 > 0$ s.t. for all $t \geqslant 0$

$$f_{\lambda_0}(t) \leqslant \hat{\lambda}_1 t^{p-1}, \tag{3.7}$$

with $\hat{\lambda}_1 > 0$ defined by Equation (2.4). Indeed, since $p < q < r$ we have for any $\lambda > 0$

$$\lim_{t \searrow 0} \frac{f_\lambda(t)}{t^{p-1}} = 0, \quad \lim_{t \to \infty} \frac{f_\lambda(t)}{t^{p-1}} = -\infty.$$

So, we can find $\delta \in (0,1)$ s.t. for all $t \in (0,\delta) \cup (\delta^{-1}, \infty)$ and all $\lambda \in (0,1]$

$$f_\lambda(t) \leqslant \hat{\lambda}_1 t^{p-1}.$$

Now, set

$$\lambda_0 = \min\{\hat{\lambda}_1 \delta^{q-p}, 1\} > 0.$$

Then, for all $t \in [\delta, \delta^{-1}]$ we have

$$f_{\lambda_0}(t) < \lambda_0 t^{q-1} \leqslant \hat{\lambda}_1 t^{p-1},$$

hence Inequality (3.7) holds for all $t \geqslant 0$. We prove that $\inf \Lambda \geqslant \lambda_0$, arguing by contradiction. Assume that for some $\lambda \in (0,\lambda_0)$ problem (P_λ) has a solution $u_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$. Testing with $u_\lambda \in W_0^{s,p}(\Omega)$ and using Equation (3.7), we get

$$\|u_\lambda\|^p = \int_\Omega f_\lambda(u_\lambda)u_\lambda \, dx < \int_\Omega f_{\lambda_0}(u_\lambda)u_\lambda \, dx \leqslant \hat{\lambda}_1 \|u_\lambda\|_p^p,$$

against the characterization of $\hat{\lambda}_1$ in Equation (2.4). □

Next, we prove that Λ is a half-line and the mapping $\lambda \mapsto u_\lambda$ is strictly increasing:

Lemma 3.4. *If $\lambda > \lambda_*$ then $\lambda \in \Lambda$. Besides, for all $\lambda > \mu > \lambda_*$, if $u_\lambda, u_\mu \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ are the solutions of (P_λ), (P_μ) respectively, then $u_\lambda - u_\mu \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$.*

Proof. Fix $\lambda > \lambda_*$. Then, we can find $\mu \in \Lambda$ s.t. $\mu < \lambda$, and a solution $u_\mu \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ of (P_μ). We have weakly in Ω

$$(-\Delta)_p^s\, u_\mu = f_\mu(u_\mu) < f_\lambda(u_\mu), \tag{3.8}$$

that is, u_μ is a strict subsolution of (P_λ). We use u_μ to truncate the reaction f_λ. Set for all $(x,t) \in \Omega \times \mathbb{R}$

$$\hat{f}_\lambda(x,t) = \begin{cases} f_\lambda(u_\mu(x)) & \text{if } t \leqslant u_\mu(x) \\ f_\lambda(t) & \text{if } t > u_\mu(x) \end{cases}$$

and

$$\hat{F}_\lambda(x,t) = \int_0^t \hat{f}_\lambda(x,\tau)\, d\tau.$$

So $\hat{f}_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies **H**. Set for all $u \in W_0^{s,p}(\Omega)$

$$\hat{\Phi}_\lambda(u) = \frac{\|u\|^p}{p} - \int_\Omega \hat{F}_\lambda(x,u)\, dx,$$

then as in Section 2 it is seen that $\hat{\Phi}_\lambda \in C^1(W_0^{s,p}(\Omega))$. Reasoning as in Theorem 3.1 we also see that $\hat{\Phi}_\lambda$ is coercive and sequentially weakly l.s.c., so there exists $u_\lambda \in W_0^{s,p}(\Omega)$ s.t.

$$\hat{\Phi}_\lambda(u_\lambda) = \inf_{W_0^{s,p}(\Omega)} \hat{\Phi}_\lambda.$$

As a consequence, we have $\hat{\Phi}'_\lambda(u_\lambda) = 0$ in $W^{-s,p'}(\Omega)$, that is, weakly in Ω

$$(-\Delta)_p^s\, u_\lambda = \hat{f}_\lambda(x,u). \tag{3.9}$$

Testing Equation (3.9) with $(u_\mu - u_\lambda)^+ \in W_0^{s,p}(\Omega)_+$ we get

$$\begin{aligned} \langle (-\Delta)_p^s\, u_\lambda, (u_\mu - u_\lambda)^+ \rangle &= \int_\Omega \hat{f}_\lambda(x,u_\lambda)(u_\mu - u_\lambda)^+\, dx \\ &= \int_\Omega f_\lambda(u_\mu)(u_\mu - u_\lambda)^+\, dx, \end{aligned}$$

which along with Equation (3.8) gives

$$\langle (-\Delta)_p^s\, u_\mu - (-\Delta)_p^s\, u_\lambda, (u_\mu - u_\lambda)^+ \rangle \leqslant 0.$$

By Proposition 2.1, we have $u_\mu \leqslant u_\lambda$ in Ω. So, Equation (3.9) rephrases as

$$(-\Delta)_p^s\, u_\lambda = f_\lambda(u_\lambda)$$

weakly in Ω, and moreover $u_\lambda > 0$ in Ω. As in Lemma 3.3 we see that $u_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ and it solves (P_λ), so $\lambda \in \Lambda$.

Finally, for all $\lambda > \mu > \lambda_*$ we have $u_\lambda, u_\mu \in W_0^{s,p}(\Omega) \cap C_s^0(\overline{\Omega})$ and

$$\begin{cases} (-\Delta)_p^s\, u_\mu + u_\mu^{r-1} = \mu u_\mu^{q-1} < \lambda u_\lambda^{q-1} = (-\Delta)_p^s\, u_\lambda + u_\lambda^{r-1} & \text{weakly in } \Omega \\ 0 < u_\mu \leqslant u_\lambda & \text{in } \Omega. \end{cases}$$

By Theorem 2.7, we conclude that $u_\lambda - u_\mu \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$. □

Note that in Lemma 3.4 we cannot use Proposition 2.4 to prove the monotonicity of $\lambda \mapsto u_\lambda$, as we did in sub- and equidiffusive cases: this is due to the fact that $t \mapsto f_\lambda(t)/t^{p-1}$ is not a decreasing mapping in $(0,\infty)$ (recall that $q > p$). The same reason prevents the use of Proposition 2.4 to prove uniqueness of the solution.

In fact, for $\lambda > \lambda_*$ we detect at least one more solution beside u_λ:

Lemma 3.5. *For all $\lambda > \lambda_*$ there exists a second solution $v_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ of (P_λ) s.t. $u_\lambda - v_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$.*

Proof. Fix $\lambda > \lambda_*$. As in Lemma 3.4 we pick $\mu \in \Lambda$ s.t. $\lambda_* < \mu < \lambda$, define $\hat{\Phi}_\lambda \in C^1(W_0^{s,p}(\Omega))$, and find a global minimizer $u_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$, which solves (P_λ) and satisfies $u_\lambda - u_\mu \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$. Set now

$$V = \{u_\mu + v : v \in \mathrm{int}(C_s^0(\overline{\Omega})_+)\},$$

which is an open set in $C_s^0(\overline{\Omega})$ containing u_λ. For all $x \in \Omega$, $t > u_\mu(x)$, we have

$$\begin{aligned} \hat{F}_\lambda(x,t) &= \int_0^{u_\mu(x)} f_\lambda(u_\mu(x))\,d\tau + \int_{u_\mu(x)}^t f_\lambda(\tau)\,d\tau \\ &= F_\lambda(t) + \big[f_\lambda(u_\mu(x))u_\mu(x) - F_\lambda(u_\mu(x))\big], \end{aligned}$$

hence for all $u \in V \cap W_0^{s,p}(\Omega)$ (note that $u > u_\mu$ in Ω)

$$\hat{\Phi}_\lambda(u) = \frac{\|u\|^p}{p} - \int_\Omega F_\lambda(u)\,dx - \int_\Omega \big[f_\lambda(u_\mu)u_\mu - F_\lambda(u_\mu)\big]\,dx = \Phi_\lambda(u) - C,$$

with $C \in \mathbb{R}$ independent of u. So, recalling that u_λ minimizes $\hat{\Phi}_\lambda$ over $W_0^{s,p}(\Omega)$, for all $u \in V \cap W_0^{s,p}(\Omega)$ we have

$$\Phi_\lambda(u) \geqslant \Phi_\lambda(u_\lambda),$$

that is, u_λ is a local minimizer of Φ_λ in $C_s^0(\overline{\Omega})$. By Proposition 2.5, u_λ is as well a local minimizer of Φ_λ in $W_0^{s,p}(\Omega)$. To proceed with the proof, we need to perform a different truncation on the reaction. Set for all $(x,t) \in \Omega \times \mathbb{R}$

$$\tilde{f}_\lambda(x,t) = \begin{cases} f_\lambda(t) & \text{if } t \leqslant u_\lambda(x) \\ \lambda u_\lambda^{q-1}(x) - t^{r-1} & \text{if } t > u_\lambda(x) \end{cases}$$

and as usual

$$\tilde{F}_\lambda(x,t) = \int_0^t \tilde{f}_\lambda(x,\tau)\,d\tau.$$

Clearly $\tilde{f}_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies **H**. So, we set for all $u \in W_0^{s,p}(\Omega)$

$$\tilde{\Phi}_\lambda(u) = \frac{\|u\|^p}{p} - \int_\Omega \tilde{F}_\lambda(x,u)\,dx$$

and thus define a functional $\tilde{\Phi}_\lambda \in C^1(W_0^{s,p}(\Omega))$. We note that for all $(x,t) \in \Omega \times \mathbb{R}$ we have $\tilde{f}_\lambda(x,t) \leqslant f_\lambda(t)$ and hence $\tilde{F}_\lambda(x,t) \leqslant F_\lambda(t)$. This in turn implies for all $u \in W_0^{s,p}(\Omega)$

$$\tilde{\Phi}_\lambda(u) \geqslant \Phi_\lambda(u). \tag{3.10}$$

Since u_λ is a local minimizer of Φ_λ, we can find $\rho > 0$ s.t. $\Phi_\lambda(u) \geqslant \Phi_\lambda(u_\lambda)$ for all $u \in B_\rho(u_\lambda)$, hence by Inequality (3.10)

$$\tilde{\Phi}_\lambda(u) \geqslant \Phi_\lambda(u) \geqslant \Phi_\lambda(u_\lambda) = \tilde{\Phi}_\lambda(u_\lambda).$$

So, u_λ is as well a local minimizer of $\tilde{\Phi}_\lambda$. Besides, fix $\varepsilon \in (0, \hat{\lambda}_1)$ (with $\hat{\lambda}_1 > 0$ defined by Equation (2.4)), then we can find $\delta > 0$ s.t. for all $x \in \mathbb{R}$, $|t| \leqslant \delta$

$$\tilde{F}_\lambda(x,t) \leqslant F_\lambda(t) \leqslant \varepsilon \frac{(t^+)^p}{p}.$$

Since Ω is bounded, we can find $\sigma > 0$ s.t. $\|u\|_\infty \leqslant \delta$ for all $u \in C_s^0(\overline{\Omega})$, $\|u\|_{0,s} \leqslant \sigma$. Then, using also Equation (2.4), for all $u \in W_0^{s,p}(\Omega) \cap C_s^0(\overline{\Omega})$ with $0 < \|u\|_{0,s} \leqslant \sigma$ we have

$$\tilde{\Phi}_\lambda(u) \geqslant \frac{\|u\|^p}{p} - \int_\Omega \varepsilon \frac{(u^+)^p}{p}\,dx \geqslant (\hat{\lambda}_1 - \varepsilon)\frac{\|u\|_p^p}{p} > 0.$$

So, 0 is a strict local minimizer of $\tilde{\Phi}_\lambda$ in $C_s^0(\overline{\Omega})$. By Proposition 2.5 again, 0 is as well a local minimizer of $\tilde{\Phi}_\lambda$ in $W_0^{s,p}(\Omega)$. From Lemma 3.3, we know that Φ_λ is coercive in $W_0^{s,p}(\Omega)$, so by Inequality (3.10) $\tilde{\Phi}_\lambda$ is coercive as well. As recalled in Section 2, $\tilde{\Phi}_\lambda$ then satisfies the (PS)-condition. Thus, we may apply the mountain pass theorem (see [33, Theorem 2.1]) and deduce the existence of $v_\lambda \in W_0^{s,p}(\Omega) \setminus \{0, u_\lambda\}$ s.t. $\tilde{\Phi}'_\lambda(v_\lambda) = 0$ in $W^{-s,p'}(\Omega)$. So, we have weakly in Ω

$$(-\Delta)_p^s\, v_\lambda = \tilde{f}_\lambda(x, v_\lambda). \tag{3.11}$$

Testing Equation (3.11) with $-v_\lambda^- \in W_0^{s,p}(\Omega)$ and applying Equation (2.2) we have

$$\|v_\lambda^-\|^p \leqslant \langle (-\Delta)_p^s\, v_\lambda, -v_\lambda^- \rangle = \int_\Omega \tilde{f}_\lambda(x, v_\lambda)(-v_\lambda^-)\,dx = 0,$$

so $v_\lambda \in W_0^{s,p}(\Omega)_+ \setminus \{0\}$. Recalling the definition of $\tilde{f}_\lambda$ and testing Equation (3.11) with $(v_\lambda - u_\lambda)^+ \in W_0^{s,p}(\Omega)$, we have

$$\begin{aligned}\langle (-\Delta)_p^s\, v_\lambda, (v_\lambda - u_\lambda)^+ \rangle &= \int_\Omega \tilde{f}_\lambda(x, v_\lambda)(v_\lambda - u_\lambda)^+\,dx \\ &\leqslant \int_\Omega f_\lambda(u_\lambda)(v_\lambda - u_\lambda)^+\,dx \\ &= \langle (-\Delta)_p^s\, u_\lambda, (v_\lambda - u_\lambda)^+ \rangle,\end{aligned}$$

which by Proposition 2.1 implies $v_\lambda \leqslant u_\lambda$ in Ω. So, Equation (3.11) rephrases as

$$(-\Delta)_p^s\, v_\lambda = f_\lambda(v_\lambda)$$

weakly in Ω. Using Theorem 2.6 as in Theorem 3.1, we see that $v_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ and it solves (P_λ). So we have

$$\begin{cases} (-\Delta)_p^s\, v_\lambda + v_\lambda^{r-1} = \lambda v_\lambda^{q-1} \leqslant \lambda u_\lambda^{q-1} = (-\Delta)_p^s\, u_\lambda + u_\lambda^{r-1} & \text{weakly in } \Omega \\ v_\lambda \leqslant u_\lambda & \text{in } \Omega, \end{cases}$$

while $v_\lambda \not\equiv u_\lambda$. By Theorem 2.7, we have $u_\lambda - v_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$. □

To complete the picture, we examine the limiting case $\lambda = \lambda_*$. In such case, we can prove existence of at least one solution, to which all principal solutions u_λ converge:

Lemma 3.6. *There exists a solution $u_* \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ of (P_{λ_*}). Besides, if $u_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ is the solution given in Lemma 3.4, then $u_\lambda \to u_*$ in both $W_0^{s,p}(\Omega)$ and $C_s^0(\overline{\Omega})$ as $\lambda \searrow \lambda_*$.*

Proof. We prove a slightly more general assertion. Let (λ_n) be a decreasing sequence s.t. $\lambda_n \searrow \lambda_*$, and denote by $u_n \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ any solution of (P_{λ_n}), then up to a subsequence $u_n \to u_*$ in both $W_0^{s,p}(\Omega)$ and $C_s^0(\overline{\Omega})$ as $n \to \infty$, being $u_* \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ a solution of (P_{λ_*}). First, for all $n \in \mathbb{N}$ we have weakly in Ω

$$(-\Delta)_p^s\, u_n = f_{\lambda_n}(u_n). \tag{3.12}$$

Arguing as in the proof of Theorem 3.1, we find $u_* \in W_0^{s,p}(\Omega)_+$ s.t. up to a subsequence $u_n \to u_*$ in both $W_0^{s,p}(\Omega)$ and $C_s^0(\overline{\Omega})$, hence we can pass to the limit in Equation (3.12) and get weakly in Ω

$$(-\Delta)_p^s\, u_* = f_{\lambda_*}(u_*). \tag{3.13}$$

We claim that $u_* \not\equiv 0$. Arguing by contradiction, assume that $u_n \to 0$ in both $W_0^{s,p}(\Omega)$ and $C_s^0(\overline{\Omega})$, hence in particular $u_n \to 0$ uniformly in Ω. Then, for all $n \in \mathbb{N}$ big enough we have $0 < u_n \leqslant 1$ in Ω. Set for all $n \in \mathbb{N}$

$$v_n = \frac{u_n}{\|u_n\|} \in W_0^{s,p}(\Omega) \cap \mathrm{int}(C_s^0(\overline{\Omega})_+).$$

The sequence (v_n) is obviously bounded in $W_0^{s,p}(\Omega)$. By reflexivity and the compact embedding $W_0^{s,p}(\Omega) \hookrightarrow L^p(\Omega)$, passing to a subsequence we have $v_n \rightharpoonup v$ in $W_0^{s,p}(\Omega)$, $v_n \to v$ in $L^p(\Omega)$. Besides, by Equation (3.12), for all $n \in \mathbb{N}$ we have weakly in Ω

$$(-\Delta)_p^s\, v_n = \lambda_n \frac{u_n^{q-1}}{\|u_n\|^{p-1}} - \frac{u_n^{r-1}}{\|u_n\|^{p-1}}. \tag{3.14}$$

Consider the first term in the right-hand side of Equation (3.14). Since $0 < u_n \leqslant 1$ in Ω and $p < q$, we have

$$0 < \frac{u_n^{q-1}}{\|u_n\|^{p-1}} \leqslant \frac{u_n^{p-1}}{\|u_n\|^{p-1}} = v_n^{p-1},$$

so $(u_n^{q-1}/\|u_n\|^{p-1})$ is bounded in $L^{p'}(\Omega)$. Passing to a subsequence, we have $u_n^{q-1}/\|u_n\|^{p-1} \rightharpoonup w$ in $L^{p'}(\Omega)$, hence a fortiori in $L^1(\Omega)$. By Hölder's inequality and the continuous embedding $W_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$, we have

$$\begin{aligned}\|w\|_1 &\leqslant \liminf_n \int_\Omega \frac{u_n^{q-1}}{\|u_n\|^{p-1}}\, dx \\ &\leqslant \limsup_n \frac{\|u_n\|_q^{q-1}|\Omega|^{\frac{1}{q}}}{\|u_n\|^{p-1}} \\ &\leqslant C \limsup_n \|u_n\|^{q-p} = 0.\end{aligned}$$

So, we get $w = 0$, that is,

$$\frac{u_n^{q-1}}{\|u_n\|^{p-1}} \rightharpoonup 0 \text{ in } L^{p'}(\Omega). \tag{3.15}$$

An entirely similar argument proves that $(u_n^{r-1}/\|u_n\|^{p-1})$ is bounded in $L^{p'}(\Omega)$ and, up to a subsequence,

$$\frac{u_n^{r-1}}{\|u_n\|^{p-1}} \rightharpoonup 0 \text{ in } L^{p'}(\Omega). \tag{3.16}$$

Testing Equation (3.14) with $(v_n - v) \in W_0^{s,p}(\Omega)$ and using Hölder's inequality, we have for all $n \in \mathbb{N}$

$$\begin{aligned}\langle(-\Delta)_p^s\, v_n, v_n - v\rangle &= \int_\Omega \left[\lambda_n \frac{u_n^{q-1}}{\|u_n\|^{p-1}} - \frac{u_n^{r-1}}{\|u_n\|^{p-1}}\right](v_n - v)\, dx \\ &\leqslant \lambda_1 \left\|\frac{u_n^{q-1}}{\|u_n\|^{p-1}}\right\|_{p'} \|v_n - v\|_p - \left\|\frac{u_n^{r-1}}{\|u_n\|^{p-1}}\right\|_{p'} \|v_n - v\|_p,\end{aligned}$$

and the latter tends to 0 as $n \to \infty$ by the relations above. By the $(S)_+$-property of $(-\Delta)_p^s$ we have $v_n \to v$ in $W_0^{s,p}(\Omega)$, hence $\|v\| = 1$. On the other hand, testing Equation (3.14) with $v \in W_0^{s,p}(\Omega)$, we have for all $n \in \mathbb{N}$

$$\langle(-\Delta)_p^s\, v_n, v\rangle = \int_\Omega \left[\lambda_n \frac{u_n^{q-1}}{\|u_n\|^{p-1}} - \frac{u_n^{r-1}}{\|u_n\|^{p-1}}\right] v\, dx.$$

Passing to the limit as $n \to \infty$ and recalling Equations (3.15) and (3.16), we get $\|v\|^p = 0$, a contradiction. Summarizing, $u_* \in W_0^{s,p}(\Omega)_+ \setminus \{0\}$ and satisfies Equation (3.13). As in Lemma 3.3 we see that $u_* \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ solves (P_{λ_*}).

Finally, taking into account the monotonicity property of Lemma 3.4, we conclude that globally $u_\lambda \to u_*$ in both $W_0^{s,p}(\Omega)$ and $C_s^0(\overline{\Omega})$, with monotone convergence, as $\lambda \searrow \lambda_*$, for some $u_* \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ solving (P_{λ_*}). □

Looking at the proof of Lemma 3.6 above, we can easily argue that, for any sequence (λ_n) s.t. $\lambda_n \searrow \lambda_*$, the sequence of solutions (v_{λ_n}) provided by Lemma 3.5 has a subsequence which converges to a solution of (P_{λ_*}), which might differ from the global limit of u_λ.

Combining Lemmas 3.3–3.6, we obtain the following bifurcation result for the superdiffusive case (corresponding to case (c) of Theorem 1.1):

Theorem 3.7. *Let $2 \leqslant p < q < r < p_s^*$. Then, there exists $\lambda_* > 0$ with the following properties: for all $\lambda \in (0, \lambda_*)$ problem (P_λ) has no solution; (P_{λ_*}) has at least one solution $u_* \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$; and for all $\lambda > \lambda_*$ problem (P_λ) has at least two solutions $u_\lambda, v_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ s.t. $u_\lambda - v_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$, $u_\lambda - u_\mu \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ for all $\lambda > \mu > \lambda_*$, and $u_\lambda \to u_*$ in both $W_0^{s,p}(\Omega)$ and $C_s^0(\overline{\Omega})$ as $\lambda \searrow \lambda_*$.*

Remark 3.8. For simplicity, we confined our study to the pure power logistic reactions. Nevertheless, most of our Theorem 3.7 can be extended to the following generalized logistic equation:

$$\begin{cases} (-\Delta)_p^s\, u = \lambda f(x,u) - g(x,u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases}$$

where $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory mappings, both $(p-1)$-superlinear at ∞ and at 0, satisfying a subcritical growth condition like **H**, and jointly satisfying a pseudo-monotonicity condition (see [23] for the case of the p-Laplacian).

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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