LOCC convertibility of entangled states in infinite-dimensional systems

César Massri[1], Guido Bellomo[2], Hector Freytes[3], Roberto Giuntini[3,4], Giuseppe Sergioli[3] and Gustavo M. Bosyk[5]

[1] Instituto de Investigaciones Matemáticas "Luis A. Santaló", UBA, CONICET, CABA, Argentina
[2] CONICET-Universidad de Buenos Aires, Instituto de Ciencias de la Computación (ICC), CABA, Argentina
[3] Università degli Studi di Cagliari, Cagliari, Italy
[4] Institute for Advanced Study, Technische Universität München, Garching bei Müenchen, Germany
[5] Instituto de Física La Plata, UNLP, CONICET, La Plata, Argentina

E-mail: `gbosyk@fisica.unlp.edu.ar`

Abstract. We advance on the conversion of bipartite quantum states via local operations and classical communication for infinite-dimensional systems. We introduce δ-LOCC convertibility based on the observation that any pure state can be approximated by a state with finite-support Schmidt coefficients. We show that δ-LOCC convertibility of bipartite states is fully characterized by a majorization relation between the sequences of squared Schmidt coefficients, providing a novel extension of Nielsen's theorem for infinite-dimensional systems. Hence, our definition is equivalent to the one of ϵ-LOCC convertibility [Quantum Inf. Comput. **8**, 0030 (2008)], but deals with states having finitely supported sequences of Schmidt coefficients. Additionally, we discuss the notions of optimal common resource and optimal common product in this scenario. The optimal common product always exists, whereas the optimal common resource depends on the existence of a common resource. This highlights a distinction between the resource-theoretic aspects of finite versus infinite-dimensional systems. Our results rely on the order-theoretic properties of majorization for infinite sequences, applicable beyond the LOCC convertibility problem.

1. Introduction

The purpose of this article is to explore the complexities that may arise for the infinite-dimensional quantum systems when dealing with the convertibility of entangled states by local operations and classical communication (LOCC) [1]. For example, it may be the case that a state cannot be converted by LOCC to a target state but can be converted to another state arbitrarily close to the former. To avoid such discontinuity, the notion of ϵ-convertibility under LOCC (ϵ-LOCC) was introduced [2]. Roughly speaking, $|\psi\rangle$ is ϵ-LOCC convertible to $|\phi\rangle$ if, for any neighborhood of $|\phi\rangle$, there exists a LOCC operation that takes $|\psi\rangle$ to a state in that neighborhood of $|\phi\rangle$. Furthermore, ϵ-LOCC convertibility is completely characterized in terms of a majorization relation between the sequences formed by the squared Schmidt coefficients [2, 3], which can be viewed as an extension of Nielsen's theorem [4] to the infinite-dimensional case. Additionally, a generalization of this result is applicable to quantum systems represented by commuting semi-finite von Neumann algebras [5].

Our contribution involves the introduction and discussion of a new definition of approximate LOCC convertibility for infinite-dimensional systems, which we refer to as δ-LOCC *convertibility*. This concept relies on the observation that, for any bipartite pure state, there exists a state that is arbitrarily close to it (in terms of the trace distance) and whose Schmidt coefficients have finite support. We will demonstrate that this approach turns out to be equivalent to ϵ-LOCC convertibility, while offering the added advantage of dealing with states whose sequences of Schmidt coefficients have finite support.

Additionally, we consider the following problem: suppose that two separated parties have to perform a series of quantum information tasks that require different entangled states. Rather than sharing multiple states, they aim to use a single entangled state, manipulating it to suit each task. Thus, the question arises: for any given set of target states, is there a minimal entangled state that can be locally transformed into any other target state using LOCC? This state, if exists, is known as an *optimal common resource* of the set [6]. Similarly, we also explore the existence of a maximal entangled state that can be obtained from any state of the original set by LOCC. This state, if exists, is referred to as an *optimal common product* of the set [7]. Understanding these problems is crucial from the perspective of quantum resource theories and entanglement [8]. By exploring these issues in the infinite-dimensional setting, we gain insights into the fundamental properties of entanglement and its role as a quantum resource.

We recall that, in the case of pure bipartite finite-dimensional systems, the existence of an optimal common resource and an optimal common product has been established using the link between LOCC convertibility and majorization, as shown by Nielsen's theorem [4], and the fact that majorization forms a complete lattice [9, 10].

Here, we exploit the characterization of δ-LOCC (or, equivalently ϵ-LOCC) in terms of majorization in order to describe the optimal common resource and optimal common product for infinite-dimensional systems. Unlike the finite-dimensional case, we obtain that the existence of the optimal common resource is conditioned to the existence of a common resource of the set state under consideration, which not always exist. This poses a novel distinction in the entanglement resource theories of finite versus infinite-dimensional quantum systems. On the other hand, we show that the optimal common product always exists. These results stem from our characterization of the majorization lattice for infinite-dimensional setting, which is a result of mathematical interest in itself and can be applied beyond the scope of the LOCC convertibility problem addressed here.

2. Majorization for infinite sequences

In this section, we present two results regarding the concept of majorization for infinite sequences, which will be useful to discuss the notion of LOCC convertibility. At the same time, they hold mathematical interest in their own right. For references regarding the finite-dimensional case, we recommend consulting the following sources [11, 9, 10].

To ensure clarity in our discussion, we introduce some notations. We consider the space $\ell^1([0,1]) \equiv \ell^1$ of sequences whose series is absolutely convergent, $\ell^1[(0,1)] = \left\{(x_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}} : \sum_{n\in\mathbb{N}} x_n < \infty\right\}$. Additionally, we define the space ℓ_1 (ℓ with one as sub index) as the set of sequences $(x_n)_{n\in\mathbb{N}} \in \ell^1[(0,1)]$ for which some of these sets $\{x_n : x_n = 0\}$ or $\{x_n : x_n > 0\}$ is finite [2, 12]. We deal with this class of sequences since they can be rearranged into monotonically ordered sequences. Accordingly, we define $x^{\downarrow}$ as a sequence whose components are rearranged in non-increasing order, i.e., $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$, and $\ell_1^{\downarrow}$ as the set of correspondingly rearranged sequences.

We also introduce the space Δ_∞ as the set of sequences on ℓ_1 that satisfy the normalization condition $\sum_{n=1}^{\infty} x_n = 1$. This is nothing else that the set of denumerable probability vectors. We use $\Delta_\infty^{\downarrow}$ to denote the set denumerable probability vectors whose components are sorted in non-increasing order. In addition, we consider the subset of denumerable probability vectors with finite support, denoted as Δ'_∞.

We recall the notion of weak submajorization, which is defined as follows [13].

Definition 1. Let $x, y \in \ell_1$. Then, x is said to be weakly submajorized by y, denoted as $x \preceq_w y$, if

$$\sum_{n=1}^{k} x_n^{\downarrow} \leq \sum_{n=1}^{k} y_n^{\downarrow}, \ \forall k \in \mathbb{N}. \tag{1}$$

In addition to weak submajorization, we are interested in the notion of majorization in infinite dimensions. More precisely, if x and y are sequences on Δ_∞ such that $x \preceq_w y$ then x is said to be *majorized* by y, and denoted as $x \preceq y$.

2.1. Majorization lattice for infinite sequences

We now present our first result.

Proposition 2. *The poset* $\left\langle \ell_1^{\downarrow}, \preceq_w, \mathbf{1}, \mathbf{0} \right\rangle$ *is a complete bounded lattice with top element* $\mathbf{1} = (1, 0, \ldots, 0)$ *and bottom element* $\mathbf{0} = (0, 0, \ldots)$.

Proof. Let S be a non-empty subset and let M be a fixed positive integer. Consider the infimum of the M-partial sums of the elements in S,

$$s_M = \inf\{s_M(x) :\ x \in S\}.$$

The sequence of partial sums $\{s_M\}_{M\in\mathbb{N}}$ is increasing and, given that we are dealing with non-increasing ordered sequences, it satisfies [9]

$$2s_M \geq s_{M-1} + s_{M+1}.$$

Also for any $x \in S$, $\lim_{M\to\infty} s_M \leq \lim_{M\to\infty} s_M(x) < \infty$. Hence, the sequence $\{m_M\}_{M\in\mathbb{N}}$, $m_M = s_M - s_{M-1}$ is in $\ell_1^{\downarrow}$ and clearly is the infimum of S.

Now, let $S' \subseteq \ell_1^{\downarrow}$ be another non-empty subset and consider the set $Up(S')$ of all upper bounds of S'. The result follows by recalling that the supremum of S' is equal to the infimum of $Up(S')$. Notice that $Up(S') \neq \emptyset$ since $\mathbf{1} \in Up(S')$.

□

We present the following observation of the order-structure of the set $\Delta_\infty^{\downarrow}$, which arise as a peculiarity in the infinite-dimensional context.

Observation 3. The set $\Delta_\infty^{\downarrow}$ is not bounded from below.

In other words, there is no analog to the uniform probability vector for infinite-dimensional systems. An instance of this situation is presented in the Example 7. On the other side, it can be proved that any finite subset of Δ_∞ is bounded from below.

Lemma 4. *Let us consider the poset* $\langle \Delta_\infty^{\downarrow}, \preceq, \mathbf{1} \rangle$. *Then, for each non-empty finite subset* $\mathcal{S}$ *of* $\Delta_\infty^{\downarrow}$, $\mathcal{S}$ *admits a lower bound, that is, there exist* $z \in \Delta_\infty$ *such that* $z \preceq x$ *for all* $x \in \mathcal{S}$.

Proof. Without loss of generality, we can assume $\mathcal{S} = \{x, y\}$. Let $s_M = \min\{s_M(x), s_M(y)\}$ be the minimum of the M-partial sums. The sequence $\{s_M\}_{M\in\mathbb{N}}$ is increasing and satisfies

$$2s_M \geq s_{M-1} + s_{M+1}.$$

It remains to check that $\lim_{M\to\infty} s_M = 1$, but this is direct since $\lim_{M\to\infty} s_M(x) = 1$ and $\lim_{M\to\infty} s_M(y) = 1$. □

Lemma 5. *Let* $\mathcal{S}$ *be a non-empty subset of* $\Delta_\infty^{\downarrow}$ *and assume that* $z \in \Delta_\infty^{\downarrow}$ *is a lower bound. Then, there exists the infimum of* $\mathcal{S}$.

Proof. Let $s \in \ell_1^{\downarrow}$ be the infimum of $\mathcal{S}$. Let us check that s is in fact in $\Delta_\infty^{\downarrow}$. Given that $z \preceq s$ we have for all $k \in \mathbb{N}$,

$$\sum_{n=1}^{k} z_n \leq \sum_{n=1}^{k} s_n \leq 1.$$

Taking $k \to \infty$, we get $s \in \Delta_\infty^{\downarrow}$. □

We are now able to state the main result of this section, in which we demonstrate the lattice structure of the poset $\langle \Delta^{\downarrow}_{\infty}, \preceq, \mathbf{1} \rangle$, and its completeness properties.

Proposition 6. *The poset* $\langle \Delta^{\downarrow}_{\infty}, \preceq, \mathbf{1} \rangle$ *is a lattice with top element* $\mathbf{1} = (1, 0, \ldots, 0)$. *Moreover, it is* $\bigvee$*-complete and conditionally* $\bigwedge$*-complete.*

Proof. It is straightforward to observe that Lemma 4 guarantees that $\Delta^{\downarrow}_{\infty}$ is a lattice. Moreover, by Lemma 5 the lattice is conditionally $\bigwedge$-complete. Let us prove now that $\Delta^{\downarrow}_{\infty}$ is indeed $\bigvee$-complete. Let $\mathcal{S}'$ be another non-empty subset and let $Up(S')$ be the non-empty set of upper bounds. Notice that any element of $\mathcal{S}'$ is a lower bound of $Up(S')$. Hence, from the previous Lemma, $Up(S')$ has an infimum in $\Delta^{\downarrow}_{\infty}$. In other words, the supremum of S' is in $\Delta^{\downarrow}_{\infty}$. $\square$

Let us explore two illustrative examples that shed light on these results (later on, we will discuss the physical relevance of these examples). In the first case, we present two different families of sequences which infima do not exist, while in the second example, the infimum is clearly defined.

Example 7. Consider the families of sequences $\{x^{(k)}(\lambda)\}_{k\in\mathbb{N}_0}$ and $\{x^{(k)}(\lambda)\}_{\lambda\in(0,1)}$, where

$$x^{(k)}_n(\lambda) = \binom{n+k}{k}(1-\lambda^2)^{k+1}\lambda^{2n}, \; n = 0, 1, \ldots. \tag{2}$$

Let us show that the infima $\bigwedge\{x^{(k)}(\lambda)\}_{k\in\mathbb{N}_0}$ and $\bigwedge\{x^{(k)}(\lambda)\}_{\lambda\in(0,1)}$ do not exist, whereas the suprema are given by $\bigvee\{x^{(k)}(\lambda)\}_{k\in\mathbb{N}_0} = x^{(0)}(\lambda)$ and $\bigvee\{x^{(k)}(\lambda)\}_{\lambda\in(0,1)} = (1, 0, \ldots, 0)$. First, we prove that each component of $x^{(k)}(\lambda)$ tends to zero by proving that some of its factors tends to zero and the others remain bounded. Let $r > n$ be such that $\delta := (1-\lambda^2)(n/r+1) < 1$ and let $k \to \infty$, then

$$\begin{aligned}
(1-\lambda^2)^k\binom{n+k}{k} &= (1-\lambda^2)^k \frac{n+k}{k}\frac{n+k-1}{k-1}\cdots\frac{n+1}{1} \\
&= (1-\lambda^2)^k\left(\frac{n}{k}+1\right)\left(\frac{n}{k-1}+1\right)\cdots\left(\frac{n}{1}+1\right) \\
&\leq (1-\lambda^2)^k\left(\frac{n}{r}+1\right)^{k-r}\left(\frac{n}{1}+1\right)^{r} \\
&= \left((1-\lambda^2)\left(\frac{n}{r}+1\right)\right)^{k-r}\left((1-\lambda^2)\left(\frac{n}{1}+1\right)\right)^{r} \\
&< \delta^{k-r}(n+1)^r \to 0.
\end{aligned}$$

Then $x^{(k)}_n(\lambda) \to 0$ when $k \to \infty$. It is easy to check that $x^{(k)}_n(\lambda) \to 0$ when $\lambda \to 1$. Then, it follows the non-existence of the infima for both sets.

The form of the suprema follows from the fact that $x^{(k+1)}(\lambda) \preceq x^{(k)}(\lambda)$ and $x^{(k)}(\lambda) \preceq x^{(k)}(\lambda')$ with $\lambda' \leq \lambda$, see [14].

The following example is a family of incomparable sequences that admits an infimum.

Example 8. Consider the family of sequences $\{x^{(k)}\}_{k\in\mathbb{N}_{\geq 3}}$ defined as

$$x^{(k)} = \left(1-\frac{1}{\log k}, \overbrace{\frac{1}{k\log k},\dots,\frac{1}{k\log k}}^{k}, 0,\dots,0\right). \tag{3}$$

First, we can prove that the infimum $\bigwedge\{x^{(k)}\}_{k\in\mathbb{N}_{\geq 3}}$ exists. The M-partial sum of the sequence $x^{(k)}$ with $k\geq 3$ is given by

$$s_M(x^{(k)}) = \begin{cases} 1-\frac{1}{\log k}+(M-1)\frac{1}{k\log k} & \text{if } 1\leq M\leq k+1,\\ 1 & \text{if } M\geq k+1.\end{cases}$$

In order to compute $\inf_{k\geq 3} s_M(x^{(k)})$ we are going to use some techniques from calculus. For a fixed M, consider the function $s(\omega)$,

$$s(\omega) = 1-\frac{1}{\log\omega}+(M-1)\frac{1}{\omega\log\omega},\quad \omega\geq 3.$$

For $M=1,2$, we have $s'(\omega)>0$, so the minimum is attained for $\omega=3$. For $M\geq 3$, taking derivatives and equating to 0, it follows that $s(\omega)$ has only one minimum at ω_0, where ω_0 satisfies

$$\frac{\omega_0}{1+\log\omega_0} = M-1.$$

Given that $s(\omega)$ has only one critical point, the number k such that $s_M(x^{(k)})$ is minimum happens at $k=\lfloor\omega_0\rfloor$ or at $k=\lceil\omega_0\rceil$. In other words, given $M\geq 3$, there exists k_0 such that

$$\inf_{k\geq 3} s_M(x^{(k)}) = s_M(x^{(k_0)}).$$

It can be shown directly for $M=1,2,3$ that

$$\inf_{k\geq 3} s_1(x^{(k)}) = 1-\frac{1}{\log 3},\quad \inf_{k\geq 3} s_2(x^{(k)}) = 1-\frac{2}{3\log 3},\quad \inf_{k\geq 3} s_3(x^{(k)}) = 1-\frac{3}{5\log 5}.$$

The value ω_0 can be computed (if necessary) with a fixed-point iteration,

$$r_0=1,\quad r_{i+1}=(M-1)(1+\log r_i),\quad i\geq 1.$$

Notice that $r_1=M-1$ and if $r_i\geq 1$, the value of r_{i+1} is always greater than $M-1$. This implies that the function $(M-1)(1+\log x)$ is a contraction implying the convergence of the method to ω_0.

Finally, it is easy to observe that the supremum of this family is $\bigvee\{x^{(k)}\}_{k\in\mathbb{N}_{\geq 3}} = (1,0,\dots)$.

2.2. Approximate majorization in terms of finite support probability vectors

We will now proceed to define a notion of majorization in the infinite-dimensional case, based on approximations of the original sequences by sequences with finite support. With that purpose in mind, we first prove Lemma 9 that provides an upper bound for the trace distance between two sequences in $\Delta_\infty^\downarrow$ coinciding in the first N components.

Lemma 9. *Let $x, x' \in \Delta_\infty^\downarrow$ such that $x_n = x'_n$ for all $n \leq N$ and $\sum_{n=1}^N x_n = s_N$. Then,*

$$d_{\mathrm{tr}}(x, x') \leq \sqrt{2(1 - s_N)}, \tag{4}$$

where $d_{\mathrm{tr}}(x, y) = \sqrt{1 - \left(\sqrt{x} \cdot \sqrt{y}\right)^2}$ with $\sqrt{x} = (\sqrt{x_n})_{n\in\mathbb{N}}$ and $\sqrt{y} = (\sqrt{y_n})_{n\in\mathbb{N}}$.

Proof. By direct calculation of the trace distance between x and its finite support counterpart, x', we have

$$\begin{aligned} d_{\mathrm{tr}}(x, x')^2 &= 1 - \left(\sqrt{x} \cdot \sqrt{x'}\right)^2 \\ &= \left(1 + \sqrt{x} \cdot \sqrt{x'}\right)\left(1 - \sqrt{x} \cdot \sqrt{x'}\right) \\ &\leq 2\left(1 - \sqrt{x} \cdot \sqrt{x'}\right) \\ &= 2\left(1 - \sum_{n=1}^{N} x_n - \sum_{n=N+1}^{\infty} \sqrt{x_n}\sqrt{x'_n}\right) \\ &\leq 2\left(1 - \sum_{n=1}^{N} x_n\right). \end{aligned}$$

□

Building on the previous Lemma, we can now demonstrate that any sequence $x \in \Delta_\infty^\downarrow$ can be approximated by another finite-support sequence $x' \in \Delta_\infty'^\downarrow$, which is arbitrarily close to x and majorizes the latter.

Proposition 10. *Let $x \in \Delta_\infty^\downarrow$. For any $\delta \in (0, 1)$, there exists $x' \in \Delta_\infty'^\downarrow$ such that*

$$x \preceq x' \text{ and } d_{\mathrm{tr}}(x, x') \leq \delta. \tag{5}$$

Proof. In order to prove this result, we are going to construct one such x' that fulfills the requirements. Given $x \in \Delta_\infty^\downarrow$, $\delta \in (0, 1)$, let us prove that for any $K \in \mathbb{N}$, there exist $N, M \geq K$ and $x' = (x'_n)_{n\in\mathbb{N}} \in \Delta_\infty'^\downarrow$ where

(i) N is such that $s_N \geq 1 - \frac{\delta^2}{2}$ with $s_N = \sum_{n=1}^N x_n$,

(ii) M is such that $M = \left\lfloor \frac{1 - s_N}{x_N} \right\rfloor + N$,

(iii) $x'_n = x_n$ for $1 \leq n \leq N$,

(iv) $x'_n = x_N$ for $N + 1 \leq n \leq M$,

(v) $x'_n = 1 - s_N - (M - N)x_N$ for $n = M + 1$

(vi) $x'_n = 0$ for $n > M+1$,

First, let us observe that $x' \in \Delta'^{\downarrow}_{\infty}$. By construction, $\sum_{n=1}^{\infty} x'_n = 1$. Thus, all that remains is to demonstrate that $x_N \geq x'_{M+1}$, which is also directly satisfied by construction, since $M+1 \geq \frac{1-s_N}{x_N} + N$. Then, it follows that $s_k(x) = s_k(x')$ for $1 \leq k \leq N$, $s_k(x) \leq s_k(x')$ for $k \geq N+1$. Therefore, $x \preceq x'$. Finally, by Lemma 9, one has that $d_{\text{tr}}(x, x') \leq \delta$. □

It is also interesting to note that, as we demonstrate in the following proposition, the newly introduced approximation scheme preserves the majorization order (see Figure 1 (a)).

Proposition 11. *Let $x, y \in \Delta^{\downarrow}_{\infty}$ be such that $x \preceq y$ and let $\delta > 0$. Then there exist $x', y' \in \Delta'^{\downarrow}_{\infty}$ such that $d_{\text{tr}}(x, x') \leq \delta$ and $d_{\text{tr}}(y, y') \leq \delta$, and $x' \preceq y'$.*

Proof. Given $\delta > 0$, there exists y' such that $y \preceq y'$ and $d_{\text{tr}}(y, y') \leq \delta$, by Proposition 10. Let $K \in \mathbb{N}$ be such that $y'_n = 0$ for all $n \geq K$. For this K and for the given $\delta > 0$, there exists x' such that $x \preceq x'$ and $d_{\text{tr}}(x, x') \leq \delta$, by Proposition 10. Let us see that $x' \preceq y'$. By construction, we have $s_k(x') \leq s_k(y')$ for $1 \leq k \leq K$. For all $k \geq K$, $s_k(x') \geq 1 = s_k(y')$. □

In addition, we have the converse result.

Proposition 12. *Let $x, y \in \Delta^{\downarrow}_{\infty}$. If for all $\delta > 0$, there exists x', y' with finite support such that $d_{\text{tr}}(x, x') < \delta$, $d_{\text{tr}}(y, y') < \delta$ and $x' \preceq y'$. Then, $x \preceq y$.*

Proof. By hypothesis we can construct sequences $\{x'_m\}_{m=1}^{\infty}$ and $\{y'_m\}_{m=1}^{\infty}$ such that $d_{\text{tr}}(x, x'_m) < 1/m$ and $d_{\text{tr}}(y, y'_m) < 1/m$ for all $m \in \mathbb{N}$. Notice that the first k coordinates of x'_m (resp. y'_m) converge to the first k coordinates of x (resp. y). Indeed, let $\overline{x} = (x_1, \ldots, x_k)$ and $\overline{x'_m} = (x'_{m1}, \ldots, x'_{mk})$. Then,

$$\begin{aligned}
\|\sqrt{\overline{x}} - \sqrt{\overline{x'_m}}\|_2^2 &= \sum_{n=1}^{k} (\sqrt{x_n} - \sqrt{x'_{mn}})^2 \\
&\leq \sum_{n=1}^{\infty} (\sqrt{x_n} - \sqrt{x'_{mn}})^2 \\
&= 2 - 2 \sum_{n=1}^{\infty} \sqrt{x_n}\sqrt{x'_{mn}} \\
&= 2 - 2(x \cdot x'_m) \\
&\leq 2 - 2(x \cdot x'_m)^2 \\
&= 2\, d_{\text{tr}}(x, x'_m)^2 \\
&< \frac{2}{m^2}.
\end{aligned}$$

Given that all norms are equivalent in $\mathbb{R}^k$, we get that $\overline{x'_m}$ converges to $\overline{x}$ and in particular, being s_k a continuous function, $s_k(\overline{x'_m})$ converges to $s_k(\overline{x})$. But we have

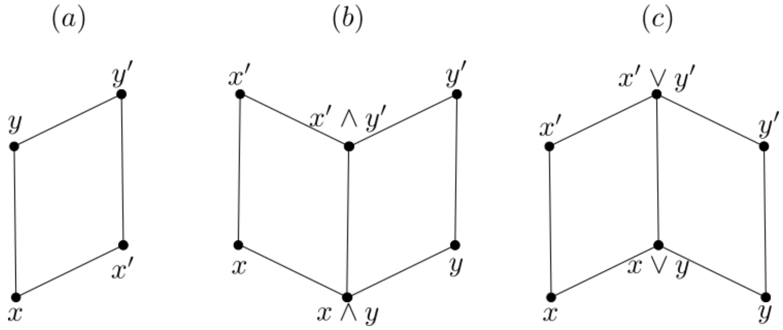

Figure 1. Hasse diagrams showing the majorization relations proved in: (a) Prop. 11; (b) and (c) Obs. 13.

the equalities $s_k(\overline{x'_m}) = s_k(x'_m)$ and $s_k(\overline{x}) = s_k(x)$ (same for $s_k(y'_m)$ and $s_k(y)$). Then, taking limit in m to the relation $s_k(x'_m) \leq s_k(y'_m)$ it follows that $s_k(x) \leq s_k(y)$. □

Finally, from Proposition 11 and appealing to the properties of the lattice, the following observation about the infimum and supremum elements for infinite sequences and their finite-support counterparts follows (see Figure 1 (b) and (c)).

Observation 13. Consider $x, y \in \Delta^{\downarrow}_{\infty}$ with x' and y' representing their respective approximated finite-support sequences. Then, one has $x \wedge y \preceq x' \wedge y'$. Similarly, for the supremum, one can establish $x \vee y \preceq x' \vee y'$.

3. LOCC convertibility for infinite-dimensional systems

In this section, we present a new definition of LOCC convertibility for infinite-dimensional systems. Later on, we will prove that our definition coincides with the one already defined by Owari et al [2], which is known as ϵ-LOCC convertibility.

In what follows, we consider composite systems that consists of two parties, A and B, such that the Hilbert space of the joint system is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, with $\dim \mathcal{H}_A = \infty$ and $\dim \mathcal{H}_B = \infty$.

3.1. ϵ-LOCC convertibility

First, let us recall the Schmidt decomposition of bipartite pure states in the infinite-dimensional case [2].

Theorem 14. *[2, th.4] For any $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, there exist orthonormal sets (but not necessarily basis sets) $\{|a_n\rangle\}_{n\in\mathbb{N}}$ and $\{|b_n\rangle\}_{n\in\mathbb{N}}$ of $\mathcal{H}_A$, and $\mathcal{H}_B$, respectively, such that*

$$|\psi\rangle = \sum_{n=1}^{\infty} \sqrt{\psi_n}\, |a_n\rangle\, |b_n\rangle\,, \tag{6}$$

where $\psi = (\psi_i)_{i\in\mathbb{N}} \in \Delta^{\downarrow}_{\infty}$.

Second, we review the notion of ϵ-LOCC convertibility introduced in [2], defined in terms of the trace distance $\| \, |\psi\rangle - |\phi\rangle \, \|_{\mathrm{tr}} = \sqrt{1 - |\, \langle\psi|\phi\rangle \,|^2}$ between pure states.

Definition 15. [2, def.1] $|\psi\rangle$ is ϵ-convertible to $|\phi\rangle$ by LOCC, denoted as $|\psi\rangle \underset{\epsilon-\mathrm{LOCC}}{\longrightarrow} |\psi\rangle$, if for any $\epsilon > 0$, there exists a LOCC operation Λ_ϵ such that $\|\Lambda_\epsilon(|\psi\rangle) - |\phi\rangle \, \|_{\mathrm{tr}} < \epsilon$.

Finally, we recall the following theorem stating the equivalence between ϵ-LOCC and majorization of the squared Schmidt coefficients, which can be viewed as the infinite-dimensional version of Nielsen's theorem [4].

Theorem 16. *[2, th.1] Let $|\psi\rangle$ and $|\psi\rangle$ bipartite pure states belonging to $\mathcal{H}_A \otimes \mathcal{H}_B$. Then, $|\psi\rangle \underset{\epsilon-\mathrm{LOCC}}{\longrightarrow} |\phi\rangle$ if and only if $\psi \preceq \phi$, where ψ and ϕ are the sequences formed by the squared Schmidt coefficients of $|\psi\rangle$ and $|\phi\rangle$, respectively.*

3.2. δ-LOCC convertibility

Our definition of LOCC convertibility makes use of the construction proposed in Proposition 11, where we demonstrated how to approximate infinite sequences by finite-support probability vectors while preserving the majorization order. Before introducing the new definition, we provide two observations. First, for any pair of pure states $|\psi\rangle$ and $|\phi\rangle$ with same orthonormal Schmidt sets the trace distance is

$$\| \, |\psi\rangle - |\phi\rangle \, \|_{\mathrm{tr}} = d_{\mathrm{tr}}(\psi, \phi), \tag{7}$$

where ψ and ϕ are the corresponding sequences formed by the squared Schmidt coefficients of the states. Second, for any bipartite pure state $|\psi\rangle$ belonging to $\mathcal{H}_A \otimes \mathcal{H}_B$, there exists another pure state $|\psi'\rangle$ with same Schmidt set than $|\psi\rangle$ and such that $\| \, |\psi\rangle - |\psi'\rangle \, \|_{\mathrm{tr}} = d_{\mathrm{tr}}(\psi, \psi') \leq \delta$, with ψ' having finite support. The state $|\psi'\rangle$ can be obtained, for instance, from Proposition 10. With this in mind, we define the concept of δ-LOCC as follows.

Definition 17. $|\psi\rangle$ is δ-convertible to $|\phi\rangle$ by LOCC, denoted as $|\psi\rangle \underset{\delta-\mathrm{LOCC}}{\longrightarrow} |\psi\rangle$, if for any $\delta > 0$, there exist states $|\psi'_\delta\rangle$ and $|\phi'_\delta\rangle$, both with sequences of Schmidt coefficients with finite support, such that $\| \, |\psi\rangle - |\psi'_\delta\rangle \, \|_{\mathrm{tr}} < \delta$, $\| \, |\phi\rangle - |\phi'_\delta\rangle \, \|_{\mathrm{tr}} < \delta$ and there exists a LOCC operation Λ_δ such that $|\phi'_\delta\rangle = \Lambda_\delta(|\psi'_\delta\rangle)$.

From this definition, we can state the following version of Nielsen's theorem in the context of infinite-dimensional systems.

Proposition 18. *Let $|\psi\rangle$ and $|\phi\rangle$ be bipartite pure states belonging to $\mathcal{H}_A \otimes \mathcal{H}_B$. Then, $|\psi\rangle \underset{\delta-\mathrm{LOCC}}{\longrightarrow} |\phi\rangle$ if and only if $\psi \preceq \phi$, where ψ and ϕ are the sequences formed by the squared Schmidt coefficients of $|\psi\rangle$ and $|\phi\rangle$, respectively.*

Proof. ($\Longrightarrow$)

By hypothesis, $|\psi\rangle \underset{\delta-\mathrm{LOCC}}{\longrightarrow} |\phi\rangle$, then for any $\delta > 0$ there exist states $|\psi_\delta\rangle$ and $|\phi_\delta\rangle$ with sequences of Schmidt coefficients of finite support such that $\| \, |\psi\rangle - |\psi_\delta\rangle \, \|_{\mathrm{tr}} < \delta$,

$\| |\phi\rangle - |\phi_\delta\rangle \|_{\mathrm{tr}} < \delta$ and there exists a LOCC operation Λ_δ such that $|\phi_\delta\rangle = \Lambda_\delta(|\psi_\delta\rangle)$. Let $\psi, \phi, \psi', \phi' \in \Delta_\infty^\downarrow$ be the Schmidt coefficients of $|\psi\rangle, |\phi\rangle, |\psi_\delta\rangle, |\phi_\delta\rangle$, respectively. Then, we have that ψ', ϕ' have finite support, $d_{\mathrm{tr}}(\psi, \psi') < \delta$, $d_{\mathrm{tr}}(\phi, \phi') < \delta$ and by Nielsen's Theorem $\psi' \preceq \phi'$. Then, by Proposition 12, $\psi \preceq \phi$.

($\Longleftarrow$) Let $|\psi\rangle$ and $|\phi\rangle$ such that $\psi \preceq \phi$. Then, from Proposition 11, for any $\delta > 0$, we can obtain states $|\psi'\rangle$ and $|\phi'\rangle$ with $\psi', \phi' \in \Delta_\infty'^\downarrow$ such that $\psi' \preceq \phi'$ (hence $|\psi'\rangle \underset{\mathrm{LOCC}}{\rightarrow} |\phi'\rangle$ by Nielsen's theorem), and $\| |\psi\rangle - |\psi'\rangle \|_{\mathrm{tr}} = d_{\mathrm{tr}}(\psi, \psi') < \delta$ and $\| |\phi\rangle - |\phi'\rangle \|_{\mathrm{tr}} = d_{\mathrm{tr}}(\phi, \phi') < \delta$. Therefore, $|\psi\rangle \underset{\delta-\mathrm{LOCC}}{\longrightarrow} |\phi\rangle$.

□

As a corollary, we obtain that ϵ-LOCC and δ-LOCC are equivalent notions:

Corollary 19. Given two bipartite pure states $|\psi\rangle$ and $|\phi\rangle$, belonging to $\mathcal{H}_A \otimes \mathcal{H}_B$, the following three statements are equivalent:

- $|\psi\rangle \underset{\delta-\mathrm{LOCC}}{\longrightarrow} |\phi\rangle$
- $|\psi\rangle \underset{\epsilon-\mathrm{LOCC}}{\longrightarrow} |\phi\rangle$
- $\psi \preceq \phi$

4. Optimal common resource and optimal common product

We have already studied the convertibility between infinite-dimensional entangled states via LOCC, and we have seen how this operation is subject to a majorization relationship between the sequences of Schmidt coefficients. We now introduce the notions of optimal common resource and optimal common product. Both concepts are related to the completeness of the majorization lattice, i.e., on the ability to define supremum and infimum elements for any subset of sequences. We will formulate these definitions in terms of δ-LOCC convertibility, but they can be formulated in an equivalent way in the ϵ-LOCC setting.

4.1. Optimal common resource

First, let us introduce the definitions of common resource and optimal common resource.

Definition 20. Let $\mathcal{P}$ be an arbitrary set of bipartite pure states in $\mathcal{H}_A \otimes \mathcal{H}_B$. The state $|\psi^{\mathrm{cr}}\rangle$ is said to be a common resource of $\mathcal{P}$, if

$$|\psi^{\mathrm{cr}}\rangle \underset{\delta-\mathrm{LOCC}}{\longrightarrow} |\phi\rangle \quad \forall\ |\phi\rangle \in \mathcal{P}. \tag{8}$$

Moreover, the state $|\psi^{\mathrm{ocr}}\rangle$ is said to be an optimal common resource of $\mathcal{P}$, if $|\psi^{\mathrm{ocr}}\rangle$ is a common resource and for any other common resource $|\psi^{\mathrm{cr}}\rangle$, one has

$$|\psi^{\mathrm{cr}}\rangle \underset{\delta-\mathrm{LOCC}}{\longrightarrow} |\psi^{\mathrm{ocr}}\rangle\,. \tag{9}$$

For finite-dimensional $\mathcal{P}$, there always exists an optimal common resource. Unlike the finite-dimensional case, the existence of an optimal common resource for infinite-dimensional systems is conditioned to the existence of a common resource. At the same time, this is subject to the completeness of the lattice of sequences discussed in Proposition 6. More precisely,

Proposition 21. *Let $\mathcal{P}$ be an arbitrary set of bipartite pure states in $\mathcal{H}_A \otimes \mathcal{H}_B$. Then, if there exists a common resource $|\psi^{\mathrm{cr}}\rangle$ of $\mathcal{P}$, there also exists an optimal common resource $|\psi^{\mathrm{ocr}}\rangle$ of $\mathcal{P}$.*

Proof. Let $|\psi^{\mathrm{cr}}\rangle$ be a common resource for the set $\mathcal{P}$, as in Definition 20. In that case, Proposition 18 says that $\psi^{\mathrm{cr}} \preceq \phi \;\forall\, |\phi\rangle \in \mathcal{P}$, with ψ^{cr}, ϕ the corresponding sequences of Schmidt coefficients. Thus, ψ^{cr} is a lower bound for all the considered sequences and, by Lemma 5, there exists an infimum. That infimum gives us the sequence of Schmidt coefficients associated with the optimal common resource $|\psi^{\mathrm{ocr}}\rangle$. □

Let us see two examples, where in the first an optimal common resource does not exit whereas in the second it does. In particular, the next example was introduced in [14] in the context of the Gaussian channel minimum entropy conjecture. Here, we use this example in order to illustrate two sets of bipartite pure sates that do not admit an optimal common resource.

Example 22. Let a two-mode squeezer of parameter r, that is, $U(r) = \exp\left[r(ab - a^\dagger b^\dagger)\right]$ where $a, b, a^\dagger$ and $b^\dagger$ are the creation and annihilation operator of the inputs modes, respectively. The action of the two-mode squeezer over the input state $|k\rangle\,|0\rangle$ can be expressed in the Schmidt decomposition as $|\psi_\lambda^{(k)}\rangle = U(r)\,|k\rangle\,|0\rangle = \sum_{n=0}^{\infty} \sqrt{x_n^{(k)}(\lambda)}\,|n+k\rangle\,|n\rangle$ with $x_n^{(k)}(\lambda)$ given by Eq. (2) and $\lambda = \tanh r$ [14]. Let consider the sets $\left\{|\psi_\lambda^{(k)}\rangle\right\}_{k\in\mathbb{N}_0}$ and $\left\{|\psi_\lambda^{(k)}\rangle\right\}_{\lambda\in(0,1)}$, which have the peculiarities that $|\psi_\lambda^{(k+1)}\rangle \underset{\delta-\mathrm{LOCC}}{\longrightarrow} |\psi_\lambda^{(k)}\rangle$ and $|\psi_\lambda^{(k)}\rangle \underset{\delta-\mathrm{LOCC}}{\longrightarrow} |\psi_{\lambda'}^{(k)}\rangle$ for $\lambda' \leq \lambda$. The corresponding sets of sequences of Schmidt coefficients were studied in Example 7, in which we showed they do not have infima. Then, optimal common resources for these sets do not exist.

The following example was introduced in [15] in order to show that the entropy of entanglement for infinite-dimensional quantum systems is not necessarily continuous in the trace-norm. We use this example in order to illustrate the case of a set of bipartite pure sates having an optimal common resource.

Example 23. Let consider the set of bipartite pure sates $\left\{|\psi^{(k)}\rangle\right\}_{k\in\mathbb{N}_{\geq 3}}$, where $|\psi^{(k)}\rangle = \sum_{n=1}^{k+1} \sqrt{x_n^{(k)}}\,|a_n\rangle\,|b_n\rangle$ with $x^{(k)}$ given by Eq. (3). In particular, we have that all the states are not LOCC convertible to each other, that is, $|\psi^{(k)}\rangle \underset{\delta-\mathrm{LOCC}}{\not\longleftrightarrow} |\psi^{(k')}\rangle$ for all $k \neq k'$. However, an optimal common resource of the set $\left\{|\psi^{(k)}\rangle\right\}_{k\in\mathbb{N}_{\geq 3}}$ exists and its Schmidt coefficients can be computed algorithmically as shown in Example 8.

4.2. Optimal common product

We introduce now the notion of common product and optimal common product of a set of states.

Definition 24. Let $\mathcal{P}$ an arbitrary set of bipartite pure states in $\mathcal{H}_A \otimes \mathcal{H}_B$. The state $|\psi^{\rm cp}\rangle$ is said to be a common product of $\mathcal{P}$, if

$$|\phi\rangle \underset{\delta-\rm LOCC}{\longrightarrow} |\psi^{\rm cp}\rangle \ \forall\, |\phi\rangle \in \mathcal{P}. \tag{10}$$

Moreover, the state $|\psi^{\rm ocp}\rangle$ is said to be an optimal common product of $\mathcal{P}$, if $|\psi^{\rm ocp}\rangle$ is a common product and for any other common product $|\psi^{\rm cp}\rangle$, one has

$$|\psi^{\rm ocp}\rangle \underset{\delta-\rm LOCC}{\longrightarrow} |\psi^{\rm cp}\rangle\,. \tag{11}$$

Just as the common resource problem is associated with the existence of lower bounds in the space of Schmidt sequences, the common product problem is linked to the existence of upper bounds. In that sense, given that the majorization lattice is $\bigvee$-complete, there always exists an optimal common product.

Proposition 25. *Let $\mathcal{P}$ an arbitrary set of bipartite pure states in $\mathcal{H}_A \otimes \mathcal{H}_B$. Then, there exists an optimal common product $|\psi^{\rm ocp}\rangle$ of $\mathcal{P}$.*

Proof. It follows directly from Proposition 6, noting that there always exists a supremum for the corresponding set of sequences of squared Schmidt coefficients. □

Reviewing the Examples 22 and 23 just discussed, it is evident that in both cases there exist the so-called optimal common products, whose Schmidt coefficients are determined by the suprema outlined in Examples 7 and 8.

5. Concluding remarks

In conclusion, this article delves into the intricacies of infinite-dimensional systems, specifically focusing on the convertibility of entangled states through local operations and classical communication (LOCC). In particular, we have introduced a new definition of LOCC convertibility for infinite-dimensional systems, termed δ-LOCC *convertibility*, which is fully characterized by a majorization relation between sequences of squared Schmidt coefficients and proves to be equivalent to ϵ-LOCC convertibility. Notably, this definition offers the mathematical advantage of dealing with finitely supported sequences.

Moreover, we have explored the LOCC convertibility problem in practical situations involving two parties aiming to perform various quantum information tasks using a single entangled state. In these scenarios, the concepts of optimal common resource and optimal common product for a given set of infinite-dimensional target states naturally arise. While the existence of an optimal common product is always guaranteed, an optimal common resource is conditionally dependent to the existence of a common resource, highlighting a novel difference in the entanglement properties between finite and infinite-dimensional systems.

We have leveraged the majorization lattice characterization for infinite sequences to establish these results. This not only contributes to the understanding of the LOCC paradigm in the infinite-dimensional case, but also presents mathematical insights with broader applicability beyond the specific scope of the addressed problem. Moreover, our results can be applied to other majorization-based resource theories. Overall, the exploration of these issues for infinite-dimensional systems enhances our comprehension of the fundamental properties of entanglement and its role as a quantum resource.

Acknowledgments

This work was partially supported by the project PIBAA 0718 funded by Consejo Nacional de Investigaciones Científicas y Técnicas CONICET (Argentina). GMB acknowledges to the Università di Cagliari (Italy) for their warm hospitality during his stay research supported by the Visiting Professor/Scientist Program 2022 sponsored by LR 7/2007 Regione Autonoma della Sardegna. H. F., R.G. and G.S. are partially supported by the project "Ubiquitous Quantum Reality (UQR): understanding the natural processes under the light of quantum-like structures" funded by Fondazione di Sardegna. R.G. and G.S. are also partially supported by the project PRIN-PNRR "Quantum Models for Logic, Computation and Natural Processes (Qm4Np)". They are also partially supported by the project PRIN2022 "CORTEX The COst of Reasoning: Theory and EXperiments". R.G. is also funded by the TÜV SÜD Foundation, the Federal Ministry of Education and Research (BMBF) and the Free State of Bavaria under the Excellence Strategy of the Federal Government and the Länder, as well as by the Technical University of Munich, Institute for Advanced Study.

References

[1] E. Chitambar, D. Leung, L. Maninska, M. Ozols, A. Winter, Everything you always wanted to know about LOCC (but were afraid to ask), Commun. Math. Phys. **328**, 303 (2014).
[2] M. Owari, S. L. Braunstein, K. Nemoto, M. Murao, ϵ-convertibility of entangled states and extension of Schmidt rank in infinite-dimensional systems, Quantum Inf. Comput. **8**, 0030 (2008).
[3] S. M. Manjegania, S. Moein, Quasi doubly stochastic operator on l^1 and Nielsen's theorem, J. Math. Phys. **60**, 103508 (2019).
[4] M.A. Nielsen, Conditions for a class of entanglement transformations, Phys. Rev. Lett. **83**, 436 (1999).
[5] J. Crann, D. W. Kribs, R. H. Levene, and I. G. Todorov, State Convertibility in the von Neumann Algebra Framework, Commun. Math. Phys. **378**, 1123 (2020).
[6] C. Guo, E. Chitambar, R. Duan, Common Resource State for Preparing Multipartite Quantum Systems via Local Operations and Classical Communication, arXiv:1601.06220 (2018).
[7] S. Deside, M. Arnhem, C. Griffet, N. J. Cerf, Probabilistic pure state conversion on the majorization lattice, arXiv:2303.10086 (2023).
[8] E. Chitambar and G. Gour, Quantum resource theories, Rev. Mod. Phys. **91**, 025001 (2019).
[9] R. B. Bapat, Majorization and singular values III, Linear Algebra Appl. **145**, 59 (1991).

[10] G. M. Bosyk, G. Bellomo, F. Holik, H. Freytes, G. Sergioli, Optimal common resource in majorization-based resource theories, New J. Phys. **21**, 083028 (2019).

[11] A. W. Marshall, I. Olkin, *Inequalities: Theory of Majorization and its Applications* (Academic Press, New York, 1979).

[12] R. Pereira, S. Plosker, Extending a characterization of majorization to infinite dimensions, Linear Algebra Appl. **468**, 80 (2015).

[13] V. Kaftal and G. Weiss, An infinite dimensional Schur Horn Theorem and majorization theory, J. Funct. Anal. **259**, 3115 (2010).

[14] R. García-Patrón, C. Navarrete-Benlloch,S. Lloyd, J. H. Shapiro, N. J. Cerf, Majorization Theory Approach to the Gaussian Channel Minimum Entropy Conjecture, Phys. Rev. Lett. **108**, 110505 (2012).

[15] J. Eisert, C. Simon, M. B. Plenio, On the quantification of entanglement in infinitedimensional quantum systems, J. Phys. A: Math. Gen. **35**, 3911 (2002).