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Equilibrium selection under changes in endowments: A geometric approach[☆]

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ABSTRACT

In this paper we propose a geometric approach to the selection of the equilibrium price. After a perturbation of the parameters, the new price is selected thorough the composition of two maps: the projection on the linearization of the equilibrium manifold, a method that underlies econometric modeling, and the exponential map, that associates a tangent vector with a geodesic on the manifold. As a corollary of our main result, we prove the equivalence between zero curvature and uniqueness of equilibrium in the case of an arbitrary number of goods and two consumers, thus extending the previous result by Loi and Matta (2018).

1. Introduction

In a pure exchange economy with two consumers, an arbitrary number l of goods and fixed total resources r distributed across agents, suppose that at the initial endowment allocation the set of corresponding equilibrium prices is not singleton, that is, there are multiple equilibria. As endowments vary, price adjusts towards a new equilibrium. One could explore the out-of-equilibrium dynamics of this adjustment or, less ambitiously, focus on a continuous approximation represented by a sequence of equilibrium changes. Yet this simpler approach, if there is price multiplicity, raises the crucial issue of price selection, that is, of which price will occur after a change in parameters.

In this setting the equilibrium price is often assumed to vary continuously (smoothly) as a result of a continuous (smooth) variation of parameters and discontinuities in prices are usually attributed to singularities, a property called *smooth selection*, despite the lack of a theory behind it and the presence of other potential prices that could occur.

In the present paper we tackle this issue following the equilibrium manifold approach which, from the seminal work of Debreu (1970), found an elegant geometric formulation in Balasko (1988).

We recall that the equilibrium manifold is defined as the pairs of prices and endowments such that aggregate excess demand is zero.

1.1. A stylized representation of the smooth selection

Fig. 1 is a stylized representation of the equilibrium manifold $E(r)$ and the smooth selection (see Balasko (1988), or Balasko (1975) for the original contribution). It depicts a pure exchange economy with multiple prices, where $\Omega(r)$ and S denote the space of endowments and normalized prices, respectively.

Starting from $x = (p, \omega)$, we assume that the endowments are continuously changed to ω'. We observe that ω and ω' belong to a region of the parameters with three equilibria. The curve joining x and x' represents the *equilibrium path*, i.e., prices and endowments consistent with the equilibrium during the transition of the economy from ω to ω'. Such a path is an approximation of the dynamics of price adjustment that ignores the out-of-equilibrium process. In fact, this curve is a subset of the equilibrium manifold. The curve joining ω to ω', an approximation of a discrete sequence of changes of the endowments, is the projection of the equilibrium path onto the space of endowments and it is the cause of the price adjustment.

Following the smooth selection, it is generally accepted that this continuous (smooth) variation of endowments entails a continuous (smooth) change in the supporting equilibrium price vector. In other terms, discontinuities of prices are only attributed to catastrophes, that is, when the endowments change crosses

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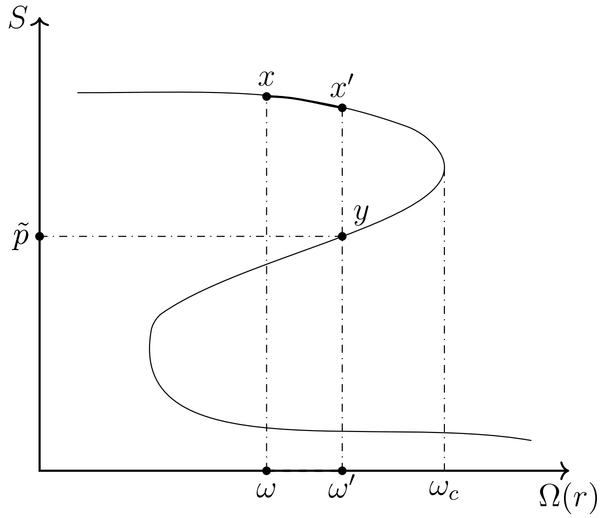

Fig. 1. A redistribution of endowments with smooth selection.

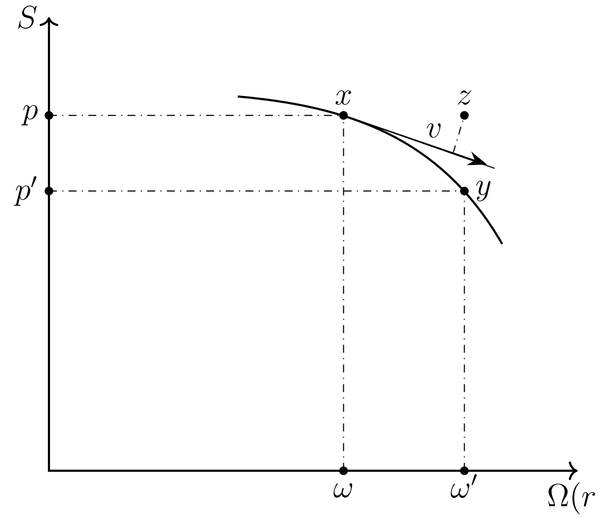

Fig. 2. A geometric approach to the equilibrium selection.

a singular economy, as, e.g., ω_c. Otherwise, prices "should not jump" to different "records", as would occur in $y = (\tilde{p}, \omega')$.

This view of the smooth selection can be seen mathematically equivalent to finding a smooth function $f : S \times \Omega(r) \to E(r)$, such that $f(E(r)) = E(r)$, or to project homotopically (smoothly) $S \times \Omega(r)$ onto $E(r)$ through a deformation retract. If we denote by $\pi : S \times \Omega(r) \to \Omega(r)$ the *natural projection*, the above construction would imply that, given $(p, \omega) \in E(r)$, a neighborhood U_ω containing $\omega = \pi(p, \omega)$, and chosen a point (p', ω') sufficiently close to $E(r)$, we would have $f(p', \omega') \in \pi^{-1}(U_\omega)$.

The knowledgeable reader in the equilibrium manifold could think of an alternative, natural way, to tackle this problem using the unknottedness property of $E(r)$ (DeMichelis and Germano, 2000), that is, by using a diffeomorphism $D : S \times \Omega(r) \to S \times \Omega(r)$ such that $D(E(r)) = \{p_0\} \times \Omega(r)$, where the equilibrium manifold is mapped into an hyperplane. By denoting the projection *proj* : $S \times \Omega(r) \to \{p_0\} \times \Omega(r)$, $(p, \omega) \mapsto (p_0, \omega)$, one could define $f := D^{-1} \circ proj \circ D : S \times \Omega(r) \to S \times \Omega(r)$, but unfortunately, this topological approach would not offer any insight on how to explicitly find the function f.

1.2. The geometric approach

The purpose of the present paper instead is to provide a geometric construction to find the function f to explain the smooth selection phenomenon avoiding "jumping" onto other records.

To provide an insight, let us denote by π_T the projection $\pi_T : S \times \Omega(r) \to TE(r)$, where $TE(r)$ denotes the tangent bundle of $E(r)$, that is, $\{(p, v) | p \in E(r), v \in T_p(E(r))\}$.

Denote the initial equilibrium by $x = (p, \omega)$ and change the endowments to ω'. After this change takes place, the pair $z = (p, \omega')$ is out of the equilibrium manifold. Our approach consists of (1) locally approximating the manifold with its tangent plane at x, $T_xE(r)$, (2) projecting z onto $T_xE(r)$, $\pi_T(z)$, and then (3) mapping the vector $v = \pi_T(z) - x$ into the (unique) geodesic connecting x to y via the exponential map[1] $exp_x : TE(r) \to E(r)$, where $y = (p', \omega')$ represents the new equilibrium that is consistent with the adjustment process. This way, the supporting equilibrium price is unambiguously selected and belongs to the same record. This construction enables us to explicitly define the map f, that is,

$$f := exp_x \circ \pi_T : S \times \Omega(r) \to E(r).$$

Fig. 2 illustrates this approach.

1.3. A numerical example

The metric used to calculate the geodesic is induced by the ambient space $S \times \Omega$: it privileges the perspective of us external observers, who are also the builders of the model. Therefore, this can lead to a difference between ω' and the coordinates of the geodesic related to the final endowments allocation. Such a difference depends on the shape of the manifold. This may result in Fig. 2 presenting inaccuracies and imprecisions due to approximations and excessive simplifications. This should not be interpreted negatively: the purpose of the construction is to ensure that selection occurs in the same record. Furthermore, the analysis is independent of the criticality of equilibria, as selection occurs even at critical equilibria because the exponential map is a global diffeomorphism.

To better clarify these points, we introduce the numerical example of a smooth exchange economy with three equilibria proposed by Shapley and Shubik (1977), with two goods, l_1 and l_2, and two consumers, a and b. The vector of total resources is $r = (40, 50)$. Preferences are represented by the utility functions $u^a(l_1, l_2) = l_1 + 100\left(1 - e^{-l_2/10}\right)$ and $u^b(l_1, l_2) = l_2 + 110\left(1 - e^{-l_1/10}\right)$.

Under the standard maximization, the individual demand functions for the first good are $F_1^a = \frac{p\omega_1 + \omega_2 - 10log(10p)}{p}$ and $F_1^b = 10log\left(\frac{11}{p}\right)$, where (ω_1, ω_2) denotes the endowments held by consumer a. Let (ω_1, ω_2) be $(38, 0.9)$. Then the aggregate excess demand $F_1(p)$ for good x has three equilibria, $(0.31, 1.07, 2.94)$.

Let the point $(p, \omega_1, \omega_2) = (1.07, 38, 0.9)$ represent the initial equilibrium point x on the equilibrium manifold. After a perturbation of endowments, let $(38, 1)$ be consumer a's new endowments allocation. The point $z = (1.07, 38, 1)$ does not belong to the equilibrium manifold.

We apply the geometric approach to select the new price. Given the implicit form $F_1^a + F_1^b - 40 = 0$, we can parametrize

[1] Note that there exists a unique geodesic $\gamma : (-a, a) \to E(r)$, such that $\gamma(0) = x$ and $\gamma'(0) = v$, for small $a > 0$ and $\|v\|$. Moreover, because by decreasing its velocity, the interval of definition $(-a, a)$ of the geodesic can be uniformly increased, the exponential map can be defined as $exp_x(v) := \gamma(1)$. This map can be shown to be a (local) diffeomorphism near x. But, in our geometric construction, being $\|v\|$ arbitrary, exp_x must be defined for all $T_xE(r)$, that is, it must be a global diffeomorphism.

the equilibrium manifold in the following way:

$$\tau(u, v) = \left(u, v, 40 - \frac{u}{v} - 10log\left(\frac{11}{v}\right) + \frac{10log(10v)}{v}\right),$$

where u and v denote ω_2 and p, respectively. In order to determine the vector v belonging to $T_xE(r)$, we subtract from the vector $z - x$ its orthogonal component, i.e.,

$$v = (z - x) - \pi_{n_x}(z - x) = (z - x) - ((z - x) \cdot n_x)n_x = (0.08, -0.03, -0.02),$$

where $(z - x) \cdot n_x$ denotes the scalar product and $n_x = (0.41, 0.8, 0.44)$ represents the unit normal vector at x. By denoting with $D\tau$ the differential of the parametrization τ, we set

$$\alpha D\tau(0.9, 1.07)\frac{\partial}{\partial u} + \beta D\tau(0.9, 1.07)\frac{\partial}{\partial v} = v$$

to determine the intrinsic coordinates (α, β) of the tangent vector v. Given the point x and the vector $(\alpha, \beta) = (0.08, -0.03)$, we can numerically[2] compute the unique geodesic $\gamma(t)$, such that $\gamma(0) = x$ and $\gamma'(0) = (\alpha, \beta) = (0.08, -0.03)$. The coordinates of $\gamma(1)$ are $(0.98, 1.04, 37.99)$. Taking care of the change of coordinates of the parametrization, this triple corresponds to $(p, \omega_1, \omega_2) = (1.04, 37.99, 0.98)$ which indeed belongs to the equilibrium manifold. For $(\omega_1, \omega_2) = (37.99, 0.98)$, $F_1(p)$ has roots $(0.32, 1.04, 2.96)$, that is, the algorithm has selected the middle price, as we had set in the starting situation: the new price belongs to the same record as the previous one. Referring to the approximation we were discussing earlier, we note the difference between the point suggested by the geodesic $(1.04, 37.99, 0.98)$ and the one that would correspond to the post-perturbation allocation, $(1.02, 38, 1)$, a part of which is attributable to numerical approximation and a part to the shape of the manifold. To overcome this second aspect opens an interesting and yet largely unexplored research area, that of studying families of metrics with economic meaning, which is beyond the scope of this work (see, e.g., Loi and Matta (2011)).

1.4. A discussion

Despite its seemingly geometric flavor, this construction deeply relies on the economic properties that affect the geometry of the equilibrium manifold. Moreover, it is based on the composition of two natural maps, the projection on the linearization of the equilibrium manifold, a method that underlies regression techniques, and the exponential map that associates a tangent vector with a geodesic on the manifold. Furthermore, our approach differs from the contributions in the literature, which focused on introducing uncertainty through randomization over the equilibra. We refer the reader to Allen et al. (2002) and the references therein.

Generally speaking, the exponential map can be used to define a local chart around any point on a manifold, which allows, using calculus techniques, to study the geometry of the manifold locally, by mapping tangent vectors at one point to points on the manifold near that point.

We recall that a diffeomorphism between manifolds is a smooth and bijective map that preserves smoothness and differentiability. In simpler terms, it is a function that can smoothly stretch, shrink, or bend a manifold into another.

In terms of intuition, a local diffeomorphism can be thought of as a "zoomed-in" view of a manifold, where the local geometry around each point is preserved. It means that the function preserves the smooth structure locally, in a small neighborhood around each point of the open sets. A global diffeomorphism, on the other hand, can be thought of as a "zoomed-out" view of a manifold, where the global geometry and topology are preserved. It means that the function preserves the smooth structure globally, throughout the entire manifold.

If the exponential map is a global diffeomorphism property, then it allows us to extend these local calculations to the entire manifold, giving us a powerful tool for studying the geometry of the manifold, by mapping tangent vectors to points on the entire manifold.

By Hadamard's theorem (see Theorem 2 for details), the exponential map is a global diffeomorphism if the manifold is complete, simply connected with non-positive sectional curvature. It means that the curvature of the manifold in any two-dimensional plane (i.e., a section) is either zero or negative.

This last property is crucial because it implies geodesic completeness, that is, geodesics, which are the shortest paths between two points, can be extended indefinitely in all directions. This ensures that there are no "obstacles" in the space that would prevent us from extending a local result to a global one, because the manifold is complete and simply connected, meaning it has no holes or cavities and no spikes or folds.

This property is precisely what is required to apply Hadamard's theorem: if a space is geodesically complete, then the exponential map is a global diffeomorphism.

And this global property is also crucial for our construction because a local map may not cover the entire record necessary for the selection. It also allows to select a price without ambiguity even in different connected components, that is when crossing catastrophes, which can generally cause price discontinuities.

Since the manifold is complete and simply connected, because it is globally diffeomorphic to a Euclidean space (see Balasko (1988)), all our efforts are aimed at proving that it has non-positive sectional curvature K, with the induced metric. A substantial part of the present paper is devoted to its proof.

There is an interesting connection between curvature of the equilibrium manifold and uniqueness in the literature. Balasko (1988, Theorem 7.3.9) showed that if there is uniqueness of equilibrium for every endowment profile of the commodity space, then the curvature of the equilibrium manifold is zero. It has been shown (Loi and Matta, 2018) that, in the case of two commodities, if the curvature is zero then there is uniqueness of equilibrium. Furthermore, following an information-theoretic approach, Loi and Matta (2021) established a connection between entropy minimization and uniqueness when the equilibrium manifold is a minimal stable submanifold of its ambient space, a property that can be expressed through a minimality condition in terms of the vanishing of the mean curvature. Moreover, Loi and Matta (2018) conjectured that the equivalence between zero curvature property and uniqueness holds for an arbitrary number of goods. As a by-product of our main result, in the present paper we prove (see Corollary 1) this equivalence in the case of an arbitrary number of goods and two consumers, thus extending the previous result in the direction of the conjecture.

We would like to add one final remark. It is beyond the scope of this work to analyze out-of-equilibrium dynamics. A change in endowments necessarily implies an initial state outside of the equilibrium manifold, and the algorithm presented in this work intercepts such a point by projecting it onto the tangent space and then bringing it back onto the manifold via the exponential map. However, this does not represent a model that aims to explain the endogenous forces at play that bring a point back onto the manifold. The topic is certainly interesting, but any connection to the present work would be misleading and purely speculative.

We could conclude, perhaps in a suggestive way and in the hope of not being misunderstood, that the boundary and point

[2] Geodesics are often computed numerically because finding closed-form solutions is generally very difficult or impossible. In this case we have used (Anon, 2023) and its built-in function *geodesics_numerical*$(x, (\alpha, \beta), interval)$, whose output returns, among other things, coordinates of the geodesic point in the three-dimensional space.

of contact between the known and the unexplored world outside the manifold is represented in our construction by the tangent space. This is not surprising, after all, as linearization generally represents the first step in modeling the unknown.

The rest of this paper is organized as follows. In Section 2 we recall the properties of the equilibrium manifold relevant for our purposes. In Section 3 we introduce the main concepts of differential geometry used in this paper. In Section 4 we prove (see Theorem 3) that the equilibrium manifold has non-positive sectional curvature, a property that legitimates our geometric approach to the equilibrium selection. Moreover, Corollary 1 establishes the equivalence between zero-curvature and uniqueness of equilibrium. Finally, a mathematical appendix contains all the tedious computations and the proof of Theorem 3.

2. The economic setting

We consider a smooth pure exchange economy with L goods and M consumers. The equilibrium manifold E is defined as the set of pairs (p, ω) such that the excess demand function is zero, where p belongs to the set of normalized prices $S = \{p = (p_1, \dots, p_L) \in \mathbb{R}^L | p_l > 0, l = 1, \dots, L, p_L = 1\} \cong \mathbb{R}^{L-1}$ and ω belongs to the space of endowments $\Omega = \mathbb{R}^{LM}$. The equilibrium manifold enjoys very nice geometric properties, being a smooth submanifold of $S \times \Omega$ globally diffeomorphic to $\mathbb{R}^{LM}$ (Balasko, 1988, Lemma 3.2.1).

If total resources $r \in \mathbb{R}^L$ are fixed, the equilibrium manifold, denoted by $E(r)$, is a submanifold of $S \times \Omega(r)$ globally diffeomorphic to $\mathbb{R}^{L(M-1)}$ (Balasko, 1988, Corollary 5.2.5), that is, $E(r) \cong B(r) \times \mathbb{R}^{(L-1)(M-1)}$, where $B(r)$ denotes the *price-income equilibria*, a submanifold of $S \times \mathbb{R}^M$ diffeomorphic to $\mathbb{R}^{M-1}$ (Balasko, 1988, Corollary 5.2.4). If we define this diffeomorphism as

$$\phi : \mathbb{R}^{M-1} \to B(r), \tag{1}$$

$$t = (t_1, \dots, t_{M-1}) \mapsto (p(t), w_1(t), \dots, w_{M-1}(t)),$$

where w_i denotes consumer i's income, a parametrization of $E(r)$ (see formulas (6), (7) and (10) in Loi and Matta (2018)) is given by

$$\Phi : \mathbb{R}^{L(M-1)} \to E(r), \tag{2}$$

$$(t, \bar{\omega}_1, \dots, \bar{\omega}_{M-1}) \mapsto (p(t), \bar{\omega}_1, w_1(t) - p(t)\bar{\omega}_1, \dots \bar{\omega}_{M-1}, w_{M-1}(t) - p(t)\bar{\omega}_{M-1}),$$

where $\bar{\omega}_i$ denotes the first $L-1$ components of ω_i, consumer i's endowments vector.

3. Geometric tools

We refer the reader to Carmo (1992) for a deeper understanding of the concepts of differential geometry used in this paper. In this section we introduce the main tools.

Let V be a submanifold of dimension d in its ambient space $(\mathbb{R}^n, g_{euclid})$. This induces on V a metric in a natural way. In particular, if

$$\psi : \mathbb{R}^d \longrightarrow V \subset \mathbb{R}^n$$
$$(x_1, \dots, x_d) \mapsto (\psi_1, \dots \psi_n)$$

is a parametrization of V, the vector fields $X_1 = (\frac{\partial \psi_1}{\partial x_1}, \dots, \frac{\partial \psi_1}{\partial x_d})$, $X_2 = (\frac{\partial \psi_2}{\partial x_1}, \dots, \frac{\partial \psi_2}{\partial x_d})$ and $X_n = (\frac{\partial \psi_n}{\partial x_1}, \dots, \frac{\partial \psi_n}{\partial x_d})$ form a basis $\{X_1, \dots, X_n\}$ of vector fields of T_qV for $q \in V$. The induced metric is given by

$$ds^2 = \sum_{i,j=1}^{d} g_{ij} dx_i dx_j$$

where $g_{ij} = \langle X_i, X_j \rangle_{g_{euclid}}$. The quantity g_{ij} represents the metric tensor of a Riemannian manifold, which gives a notion of distance or length between two points on the manifold. Intuitively, we can think of g_{ij} as describing how much "stretching" or "squishing" occurs in each direction when we move a small distance on the manifold. In other words, it tells us how the geometry of the manifold changes as we move around on it.

Let us denote by $\mathfrak{X}(V)$ the set of all vector fields of class C^∞ on V. Then there exists an affine connection

$$\nabla : \mathfrak{X}(V) \times \mathfrak{X}(V) \to \mathfrak{X}(V)$$
$$(X, Y) \mapsto \nabla_X Y$$

that satisfies the following properties:

- $\nabla_{fX+gY} = f\nabla_X Z + g\nabla_Y Z$
- $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$
- $\nabla_X(fY) = f\nabla_X Y + X(f)Y$,

with f, g real-valued functions of class C^∞ on V.

Theorem 1 (*Levi-Civita (Carmo, 1992, p.55)*). *Given a Riemannian manifold V, there exists a unique affine connection ∇ on V satisfying the conditions:*

- *∇ is symmetric*
- *∇ is compatible[3] with the Riemannian metric.*

In particular, the Levi-Civita connection can be written, in a coordinate system (U, x), as

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{d} \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

where the coefficients Γ_{ij}^k are called the *Christoffel symbols* and can be computed with the following formula

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{h=1}^{d} g^{hk} \left(\frac{\partial g_{jh}}{\partial x_i} + \frac{\partial g_{hi}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_h} \right),$$

where g^{ij} is the inverse matrix of $g_{ij} = \langle X_i, X_j \rangle$.

The *curvature tensor* intuitively measures the deviation of a manifold from being locally Euclidean.

Definition 1. The curvature tensor R of a Riemannian manifold V is a correspondence that associates to every pair $X, Y \in \mathfrak{X}(V)$ a mapping

$$R(X, Y) : \mathfrak{X}(V) \to \mathfrak{X}(V)$$
$$Z \mapsto R(X, Y)Z,$$

where

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

If $V = \mathbb{R}^n$, then $R(X, Y)Z = 0$ for all $X, Y, Z \in \mathfrak{X}(\mathbb{R}^n)$. It is convenient to express this curvature in a coordinate system (U, x) based at the point $q \in V$. We have

$$R(X_i, X_j)X_k = \sum_l R_{ijk}^l X_l,$$

where the coefficients R_{ijk}^l can be expressed in terms of Γ_{ij}^k

$$R_{ijk}^s = \sum_{l=1}^{d} \Gamma_{ik}^l \Gamma_{jl}^s - \sum_{l=1}^{d} \Gamma_{jk}^l \Gamma_{il}^s + \frac{\partial \Gamma_{ik}^s}{\partial x_j} - \frac{\partial \Gamma_{jk}^s}{\partial x_i}.$$

[3] A connection ∇ on a Riemannian manifold V is compatible with the metric if and only if $X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$ for $X, Y, Z \in \mathfrak{X}(V)$.

Moreover, we have

$$\langle R(X_i, X_j)X_k, X_s\rangle = \sum_l R^l_{ijk} g_{ls}.$$

We introduce the *sectional curvature* of V, which generalizes the Gaussian curvature of surfaces. Let V be a Riemannian $n-$manifold and let $q \in V$. If Π is any 2-dimensional subspace of T_qV, and U is a neighborhood of zero on which $\exp_q$ is a diffeomorphism, then $S_\Pi := \exp_q(\Pi \cap U)$ is a 2-dimensional submanifold of V containing q. Then the sectional curvature of V associated with Π is the Gaussian Curvature of S_Π. If $\{X, Y\}$ is any basis for Π, we indicate the sectional curvature as $K(X, Y)$ and we have

Definition 2. If $\{X, Y\}$ is any basis for a 2-plane $\Pi \in T_qV$, then

$$K(X, Y) = \frac{\langle R(X, Y)X, Y\rangle}{|X|^2|Y|^2 - \langle X, Y\rangle^2}.$$

A well-known theorem by Hadamard establishes an important connection between local and global properties of a differential manifold.

Theorem 2 (*Hadamard (Carmo, 1992, p. 149)*)**.** *Let V be a complete Riemannian manifold, simply connected with sectional curvature $K(q, \sigma) \le 0$, for all $q \in V$ and for all $\sigma \in T_q(V)$. Then V is diffeomorphic to $\mathbb{R}^n$, $n = dimV$; more precisely, $exp_q : T_qV \to V$ is a diffeomorphism.*

4. Main results

Theorem 2 represents a key result for our construction, because we need to associate, through the exponential map, a vector belonging to the tangent space of $E(r)$ to a geodesic. Hence, it is crucial for our purposes that the exponential map is a global diffeomorphism. Since $E(r)$ is diffeomorphic to an Euclidean space, and hence complete and simply connected, all we need to prove is that its sectional curvature is non-positive. In the following theorem, we prove this property for the case $M = 2$.

Theorem 3. *Let $M = 2$. Then the equilibrium manifold $E(r)$ has non-positive sectional curvature.*

Proof. See Appendix C. □

As a by-product, the following corollary complements (Loi and Matta, 2018, Theorem 5.1), where they show that for $L = 2$ the zero-curvature condition on the equilibrium manifold is equivalent to the global uniqueness of the equilibrium price.

Corollary 1. *Let $M = 2$. A necessary and sufficient condition for a unique equilibrium price is that the curvature of $E(r)$ is zero.*

Proof. By Balasko (1988, Theorem 7.3.9), if for every $\omega \in \Omega(r)$ there is an unique equilibrium, then the price p associated to ω does not depend on ω, that is, $E(r)$ is an hyperplane and hence its curvature is zero. Conversely, by Theorem 3, if its sectional curvature is zero, then $p_i' = 0$ for all i, that is, the price is constant and hence unique. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A. Christoffel symbols

Type: Γ^k_{ii} where $i, k \neq 0$.

$$\Gamma^k_{ii} = \frac{1}{2}\sum_{h=0}^{L-1} g^{hk}\left(\frac{\partial g_{ih}}{\partial x_i} + \frac{\partial g_{hi}}{\partial x_i} - \frac{\partial g_{ii}}{\partial x_h}\right) = $$
$$= \frac{1}{2}\left[g^{0k}\left(\frac{\partial g_{i0}}{\partial x_i} + \frac{\partial g_{0i}}{\partial x_i} - \frac{\partial g_{ii}}{\partial x_0}\right) + \dots + g^{(L-1)k}\left(\frac{\partial g_{i(L-1)}}{\partial x_i} + \frac{\partial g_{(L-1)i}}{\partial x_i} - \frac{\partial g_{ii}}{\partial x_{L-1}}\right)\right]$$

$$\Gamma^k_{ii} = \frac{1}{2}\left[g^{0k}\left(\frac{\partial g_{i0}}{\partial x_i} + \frac{\partial g_{0i}}{\partial x_i} - \frac{\partial g_{ii}}{\partial x_0}\right) + g^{1k}\left(\cancel{\frac{\partial g_{i1}}{\partial x_i}} + \cancel{\frac{\partial g_{1i}}{\partial x_i}} - \cancel{\frac{\partial g_{ii}}{\partial x_1}}\right) + g^{2k}\left(\cancel{\frac{\partial g_{i2}}{\partial x_i}} + \cancel{\frac{\partial g_{2i}}{\partial x_i}} - \cancel{\frac{\partial g_{ii}}{\partial x_2}}\right) + \cdots\right] =$$
$$= \frac{1}{2}\left[g^{0k}\left(2\frac{\partial g_{i0}}{\partial x_i} - \frac{\partial g_{ii}}{\partial x_0}\right)\right] = \frac{1}{2}\left[g^{0k}\left(2p_ip_i' - 2p_ip_i'\right)\right] = 0$$

Type: Γ^k_{ij} where $i, j, k \neq 0$.

Using $\frac{\partial}{\partial x_i} g_{0j} = p_jp_i'$ and $\frac{\partial}{\partial x_0} g_{ij} = \frac{\partial}{\partial x_0}(p_ip_j) = p_i'p_j + p_ip_j'$.

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{h=0}^{L-1} g^{hk}\left(\frac{\partial g_{jh}}{\partial x_i} + \frac{\partial g_{hi}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_h}\right) =$$
$$= \frac{1}{2}\left[g^{0k}\left(\frac{\partial g_{j0}}{\partial x_i} + \frac{\partial g_{0i}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_0}\right) + g^{1k}\left(\frac{\partial g_{j1}}{\partial x_i} + \frac{\partial g_{1i}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_1}\right) + g^{2k}\left(\frac{\partial g_{j2}}{\partial x_i} + \frac{\partial g_{2i}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_2}\right) + \cdots\right]$$

$$\Gamma^k_{ij} = \frac{1}{2}\left[g^{0k}\left(\frac{\partial g_{j0}}{\partial x_i} + \frac{\partial g_{0i}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_0}\right) + g^{1k}\left(\cancel{\frac{\partial g_{j1}}{\partial x_i}} + \cancel{\frac{\partial g_{1i}}{\partial x_j}} - \cancel{\frac{\partial g_{ij}}{\partial x_1}}\right) + g^{2k}\left(\cancel{\frac{\partial g_{j2}}{\partial x_i}} + \cancel{\frac{\partial g_{2i}}{\partial x_j}} - \cancel{\frac{\partial g_{ij}}{\partial x_2}}\right) + \cdots\right] =$$
$$= \frac{1}{2}\left[g^{0k}\left(\frac{\partial g_{j0}}{\partial x_i} + \frac{\partial g_{0i}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_0}\right)\right]$$
$$= \frac{1}{2}\left[g^{0k}\left(p_jp_i' + p_ip_j' - (p_i'p_j + p_ip_j')\right)\right] = 0$$

Type: Γ^0_{0j} where $j \neq 0$.

Using $\frac{\partial}{\partial x_j} g_{00} = -2Ap_j'$.

$$\Gamma^0_{0j} = \frac{1}{2}\sum_{h=0}^{L-1} g^{h0}\left(\frac{\partial g_{jh}}{\partial x_0} + \frac{\partial g_{h0}}{\partial x_j} - \frac{\partial g_{0j}}{\partial x_h}\right) =$$
$$= \frac{1}{2}\left[g^{00}\left(\frac{\partial g_{j0}}{\partial x_0} + \frac{\partial g_{00}}{\partial x_j} - \frac{\partial g_{0j}}{\partial x_0}\right) + g^{10}\left(\frac{\partial g_{j1}}{\partial x_0} + \frac{\partial g_{10}}{\partial x_j} - \frac{\partial g_{0j}}{\partial x_1}\right) + g^{20}\left(\frac{\partial g_{j2}}{\partial x_0} + \frac{\partial g_{20}}{\partial x_j} - \frac{\partial g_{0j}}{\partial x_2}\right) + \cdots\right]$$

$$
\begin{aligned}
\Gamma^0_{0j} =& \frac{1}{2}\left[g^{00}\left(\cancel{\frac{\partial g_{j0}}{\partial x_0}} + \frac{\partial g_{00}}{\partial x_j} - \cancel{\frac{\partial g_{0j}}{\partial x_0}}\right) + g^{10}\left(\frac{\partial g_{j1}}{\partial x_0} + \frac{\partial g_{10}}{\partial x_j} - \frac{\partial g_{0j}}{\partial x_1}\right)\right. \\
&\left. + g^{20}\left(\frac{\partial g_{j2}}{\partial x_0} + \frac{\partial g_{20}}{\partial x_j} - \frac{\partial g_{0j}}{\partial x_2}\right) + \cdots \right] = \\
=& \frac{1}{2}\left[g^{00}\left(\frac{\partial g_{00}}{\partial x_j}\right) + g^{10}\left(\frac{\partial g_{j1}}{\partial x_0} + \frac{\partial g_{10}}{\partial x_j} - \frac{\partial g_{0j}}{\partial x_1}\right)\right. \\
&\left. + g^{20}\left(\frac{\partial g_{j2}}{\partial x_0} + \frac{\partial g_{20}}{\partial x_j} - \frac{\partial g_{0j}}{\partial x_2}\right) + \cdots \right] = \\
=& \frac{1}{2}\left[g^{00}\left(-2Ap'_j\right) + g^{10}\left(p'_1 p_j + p_1 p'_j + p_1 p'_j - p_j p'_1\right)\right. \\
&\left. + g^{20}\left(p'_2 p_j + p_2 p'_j + p_2 p'_j - p_j p'_2\right) + \cdots \right] = \\
=& \frac{1}{2}\left[g^{00}\left(-2Ap'_j\right) + g^{10}\left(2p_1 p'_j\right) + g^{20}\left(2p_2 p'_j\right) + \cdots \right. \\
&\left. + g^{(L-1)0}\left(2p_{n-1}p'_j\right)\right] = \\
=& \frac{p'_j}{\det g}\left[\left(1 + p_1^2 + \cdots + p_{L-1}^2\right)(-A) + p_1 A\left(p_1\right) + p_2 A\left(p_2\right)\right. \\
&\left. + \cdots + p_{L-1}A\left(p_{L-1}\right)\right] = \frac{-p'_j A}{\det g}
\end{aligned}
$$

Type: Γ^k_{0j} **where** $j, k \neq 0$.

$$
\begin{aligned}
\Gamma^k_{0j} =& \frac{1}{2}\sum_{h=0}^{L-1} g^{hk}\left(\frac{\partial g_{jh}}{\partial x_0} + \frac{\partial g_{h0}}{\partial x_j} - \frac{\partial g_{0j}}{\partial x_h}\right) = \\
=& \frac{1}{2}\left[g^{0k}\left(\frac{\partial g_{j0}}{\partial x_0} + \frac{\partial g_{00}}{\partial x_j} - \frac{\partial g_{0j}}{\partial x_0}\right) + g^{1k}\left(\frac{\partial g_{j1}}{\partial x_0} + \frac{\partial g_{10}}{\partial x_j} - \frac{\partial g_{0j}}{\partial x_1}\right)\right. \\
&\left. + g^{2k}\left(\frac{\partial g_{j2}}{\partial x_0} + \frac{\partial g_{20}}{\partial x_j} - \frac{\partial g_{0j}}{\partial x_2}\right) + \cdots \right]
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^k_{0j} =& \frac{1}{2}\left[g^{0k}\left(\cancel{\frac{\partial g_{j0}}{\partial x_0}} + \frac{\partial g_{00}}{\partial x_j} - \cancel{\frac{\partial g_{0j}}{\partial x_0}}\right) + g^{1k}\left(\frac{\partial g_{j1}}{\partial x_0} + \frac{\partial g_{10}}{\partial x_j} - \frac{\partial g_{0j}}{\partial x_1}\right)\right. \\
&\left. + g^{2k}\left(\frac{\partial g_{j2}}{\partial x_0} + \frac{\partial g_{20}}{\partial x_j} - \frac{\partial g_{0j}}{\partial x_2}\right) + \cdots \right] = \\
=& \frac{1}{2}\left[g^{0k}\left(\frac{\partial g_{00}}{\partial x_j}\right) + g^{1k}\left(\frac{\partial g_{j1}}{\partial x_0} + \frac{\partial g_{10}}{\partial x_j} - \frac{\partial g_{0j}}{\partial x_1}\right)\right. \\
&\left. + g^{2k}\left(\frac{\partial g_{j2}}{\partial x_0} + \frac{\partial g_{20}}{\partial x_j} - \frac{\partial g_{0j}}{\partial x_2}\right) + \cdots \right] = \\
=& \frac{1}{2}\left[g^{0k}\left(-2Ap'_j\right) + g^{1k}\left(p'_1 p_j + p_1 p'_j + p_1 p'_j - p_j p'_1\right)\right. \\
&\left. + g^{2k}\left(p'_2 p_j + p_2 p'_j + p_2 p'_j - p_j p'_2\right) + \cdots \right] = \\
=& \frac{1}{2}\left[g^{0k}\left(-2Ap'_j\right) + g^{1k}\left(2p_1 p'_j\right) + g^{2k}\left(2p_2 p'_j\right) + \cdots \right. \\
&\left. + g^{(L-1)k}\left(2p_{L-1}p'_j\right)\right] = \\
=& \frac{p'_j}{\det g}\left[\left(p_k A\left(-A\right) - p_1 p_k B\left(p_1\right) - p_2 p_k B\left(p_2\right) + \cdots\right.\right. \\
&+ \left[\left(1 + p_1^2 + \cdots + p_{k-1}^2 + p_{k+1}^2 + \cdots + p_{L-1}^2\right)B + A^2\right]\left(p_k\right) - \\
&\left.- \cdots - p_k p_{L-1}B\left(p_{L-1}\right)\right] \\
=& \frac{p'_j p_k}{\det g}\left[-A^2 - p_1^2 B - p_2^2 B + \cdots + \left[\left(1 + p_1^2 + \cdots + p_{k-1}^2\right.\right.\right. \\
&\left.\left.\left. + p_{k+1}^2 + \cdots + p_{L-1}^2\right)B + A^2 + \cdots - p_{L-1}^2 B\right)\right] \\
=& \frac{p'_j p_k}{\det g}B
\end{aligned}
$$

Type: Γ^0_{00}.

Using $\frac{\partial}{\partial x_0}g_{0i} = -p'_i A - p_i A'$ and $\frac{\partial}{\partial x_0}g_{00} = -2C + 2AA'$

$$
\begin{aligned}
\Gamma^0_{00} =& \frac{1}{2}\sum_{h=0}^{L-1} g^{h0}\left(\frac{\partial g_{0h}}{\partial x_0} + \frac{\partial g_{h0}}{\partial x_0} - \frac{\partial g_{00}}{\partial x_h}\right) = \\
=& \frac{1}{2}\left[g^{00}\left(\frac{\partial g_{00}}{\partial x_0} + \frac{\partial g_{00}}{\partial x_0} - \frac{\partial g_{00}}{\partial x_0}\right) + g^{10}\left(\frac{\partial g_{01}}{\partial x_0} + \frac{\partial g_{10}}{\partial x_0} - \frac{\partial g_{00}}{\partial x_1}\right)\right. \\
&\left. + g^{20}\left(\frac{\partial g_{02}}{\partial x_0} + \frac{\partial g_{20}}{\partial x_0} - \frac{\partial g_{00}}{\partial x_2}\right) + \cdots \right]
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^0_{00} =& \frac{1}{2}\left[g^{00}\left(\cancel{\frac{\partial g_{00}}{\partial x_0}} + \frac{\partial g_{00}}{\partial x_0} - \cancel{\frac{\partial g_{00}}{\partial x_0}}\right) + g^{10}\left(\frac{\partial g_{01}}{\partial x_0} + \frac{\partial g_{10}}{\partial x_0} - \frac{\partial g_{00}}{\partial x_1}\right)\right. \\
&\left. + g^{20}\left(\frac{\partial g_{02}}{\partial x_0} + \frac{\partial g_{20}}{\partial x_0} - \frac{\partial g_{00}}{\partial x_2}\right) + \cdots \right] = \\
=& \frac{1}{2}\left[g^{00}\left(\frac{\partial g_{00}}{\partial x_0}\right) + g^{10}\left(2\frac{\partial g_{01}}{\partial x_0} - \frac{\partial g_{00}}{\partial x_1}\right)\right. \\
&\left. + g^{20}\left(2\frac{\partial g_{02}}{\partial x_0} - \frac{\partial g_{00}}{\partial x_2}\right) + \cdots \right] = \\
=& \frac{1}{2}\left[g^{00}\left(2C + 2AA'\right) + g^{10}\left(-2p'_1 A - 2p_1 A' + 2Ap'_1\right)\right. \\
&\left. + g^{20}\left(-2p'_2 A - 2p_2 A' + 2Ap'_2\right) + \cdots \right] = \\
=& \frac{1}{2}\left[g^{00}\left(2C + 2AA'\right) + g^{10}\left(\cancel{-2p'_1 A} - 2p_1 A' + \cancel{2Ap'_1}\right)\right. \\
&\left. + g^{20}\left(\cancel{-2p'_2 A} - 2p_2 A' + \cancel{2Ap'_2}\right) + \cdots \right] = \\
=& \frac{1}{\det g}\left[\left(1 + p_1^2 + \cdots + p_{L-1}^2\right)\left(C + AA'\right) + p_1 A\left(-A' p_1\right)\right. \\
&\left. + p_2 A\left(-p_2 A'\right) + \cdots + p_{L-1}A\left(-p_{L-1}A'\right)\right] = \\
=& \frac{1}{\det g}\left[\left(1 + p_1^2 + \cdots + p_{L-1}^2\right)C + AA'\right] = \frac{1}{\det g}\left[\|p\|^2 C + AA'\right]
\end{aligned}
$$

Type: Γ^k_{00} **where** $k \neq 0$.

$$
\begin{aligned}
\Gamma^k_{00} =& \frac{1}{2}\sum_{h=0}^{L-1} g^{hk}\left(\frac{\partial g_{0h}}{\partial x_0} + \frac{\partial g_{h0}}{\partial x_0} - \frac{\partial g_{00}}{\partial x_h}\right) = \\
=& \frac{1}{2}\left[g^{0k}\left(\frac{\partial g_{00}}{\partial x_0} + \frac{\partial g_{00}}{\partial x_0} - \frac{\partial g_{00}}{\partial x_0}\right)\right. \\
&+ g^{1k}\left(\frac{\partial g_{01}}{\partial x_0} + \frac{\partial g_{10}}{\partial x_0} - \frac{\partial g_{00}}{\partial x_1}\right) \\
&\left. + g^{2k}\left(\frac{\partial g_{02}}{\partial x_0} + \frac{\partial g_{20}}{\partial x_0} - \frac{\partial g_{00}}{\partial x_2}\right) + \cdots \right]
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^k_{00} =& \frac{1}{2}\left[g^{0k}\left(\cancel{\frac{\partial g_{00}}{\partial x_0}} + \frac{\partial g_{00}}{\partial x_0} - \cancel{\frac{\partial g_{00}}{\partial x_0}}\right) + g^{1k}\left(\frac{\partial g_{01}}{\partial x_0} + \frac{\partial g_{10}}{\partial x_0} - \frac{\partial g_{00}}{\partial x_1}\right)\right. \\
&\left. + g^{2k}\left(\frac{\partial g_{02}}{\partial x_0} + \frac{\partial g_{20}}{\partial x_0} - \frac{\partial g_{00}}{\partial x_2}\right) + \cdots \right] = \\
=& \frac{1}{2}\left[g^{0k}\left(\frac{\partial g_{00}}{\partial x_0}\right) + g^{1k}\left(2\frac{\partial g_{01}}{\partial x_0} - \frac{\partial g_{00}}{\partial x_1}\right)\right. \\
&\left. + g^{2k}\left(2\frac{\partial g_{02}}{\partial x_0} - \frac{\partial g_{00}}{\partial x_2}\right) + \cdots \right] = \\
=& \frac{1}{2}\left[g^{0k}\left(2C + 2AA'\right) + g^{1k}\left(-2p'_1 A - 2p_1 A' + 2Ap'_1\right)\right. \\
&\left. + g^{2k}\left(-2p'_2 A - 2p_2 A' + 2Ap'_2\right) + \cdots \right] = \\
=& \frac{1}{2}\left[g^{0k}\left(2C + 2AA'\right) + g^{1k}\left(\cancel{-2p'_1 A} - 2p_1 A' + \cancel{2Ap'_1}\right)\right. \\
&\left. + g^{2k}\left(\cancel{-2p'_2 A} - 2p_2 A' + \cancel{2Ap'_2}\right) + \cdots \right] = \\
=& \frac{1}{\det g}\left[\left(p_k A\left(C + AA'\right) - p_1 p_k B\left(-p_1 A'\right) - p_2 p_k B\left(-p_2 A'\right) + \ldots\right.\right. \\
&+ \left[\left(1 + p_1^2 + \cdots + p_{k-1}^2 + p_{k+1}^2 + \cdots + p_{L-1}^2\right)B + A^2\right] \\
&\left.\left(-p_k A'\right) + \cdots - p_k p_{L-1}B\left(-p_{L-1}A'\right)\right] \\
=& \frac{p_k}{\det g}\left[AC - BA'\right]
\end{aligned}
$$

$$\frac{\partial \Gamma^0_{00}}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{\left[(\|p\|^2)C + AA'\right]}{(\|p\|^2)B + A^2} =$$
$$= \frac{(-p_i'A' - p_i''A)\left[(\|p\|^2)B + A^2\right] + 2p_i'A\left[(\|p\|^2)C + AA'\right]}{(\det g)^2}$$

$$\frac{\partial \Gamma^0_{i0}}{\partial x_0} = \frac{\partial}{\partial x_0}\frac{-p_i'A}{(\|p\|^2)B + A^2} =$$
$$= \frac{(-p_i''A - p_i'A')[(\|p\|^2)B + A^2] + 2p_i'A\left[(p_1p_1' + p_2p_2' + \cdots + p_{L-1}p_{L-1}')B + (\|p\|^2)C + AA'\right]}{(\det g)^2}$$

Box I.

Appendix B. Coefficients of the curvature tensor R

We have

$$\langle R(X_0, X_i)X_0, X_i\rangle = \sum_s R^s_{0i0}g_{si} = R^0_{0i0}g_{0i} + R^1_{0i0}g_{1i} + \cdots + R^{L-1}_{0i0}g_{(L-1)i},$$

where

$$R^s_{0i0} = \sum_{m=0}^{L-1}\Gamma^m_{00}\Gamma^s_{im} - \sum_{m=0}^{L-1}\Gamma^m_{i0}\Gamma^s_{0m} + \frac{\partial \Gamma^s_{00}}{\partial x_i} - \frac{\partial \Gamma^s_{i0}}{\partial x_0}.$$

Recall that $\det g = (1 + p_1^2 + \cdots + p_{L-1}^2)B + A^2 = \|p\|^2B + A^2$ and

$$\frac{\partial \det g}{\partial x_0} = 2\left[(p_1p_1' + p_2p_2' + \cdots + p_{L-1}p_{L-1}')B + \|p\|^2C + AA'\right]$$
$$\frac{\partial \det g}{\partial x_i} = -\,2p_i'A$$

The first addend is

$$R^0_{0i0} = \sum_{m=0}^{L-1}\Gamma^m_{00}\Gamma^0_{im} - \sum_{m=0}^{L-1}\Gamma^m_{i0}\Gamma^0_{0m} + \frac{\partial \Gamma^0_{00}}{\partial x_i} - \frac{\partial \Gamma^0_{i0}}{\partial x_0},$$

that we can expand as

$$R^0_{0i0} = \cancel{\Gamma^0_{00}\Gamma^0_{i0}} + \cancel{\Gamma^1_{00}\Gamma^0_{i1}} + \cancel{\Gamma^2_{00}\Gamma^0_{i2}} + \cdots - \cancel{\Gamma^0_{i0}\Gamma^0_{00}} - \Gamma^1_{i0}\Gamma^0_{01}$$
$$- \Gamma^2_{i0}\Gamma^0_{02} + \cdots + \frac{\partial \Gamma^0_{00}}{\partial x_i} - \frac{\partial \Gamma^0_{i0}}{\partial x_0}.$$

We determine the derivative of Christoffel' symbols (see Box I). By subtracting and simplifying,

$$\frac{\partial \Gamma^0_{00}}{\partial x_i} - \frac{\partial \Gamma^0_{i0}}{\partial x_0} = \frac{-2p_i'AB(p_1p_1' + \cdots + p_{L-1}p_{L-1}')}{(\det g)^2}.$$

hence

$$R^0_{0i0} = -\,\Gamma^1_{i0}\Gamma^0_{01} - \Gamma^2_{i0}\Gamma^0_{02} + \cdots + \frac{\partial \Gamma^0_{00}}{\partial x_i} - \frac{\partial \Gamma^0_{i0}}{\partial x_0} =$$
$$= -\,\frac{p_i'p_1B}{\det g}\cdot\frac{-p_1'A}{\det g} - \frac{p_i'p_2B}{\det g}\cdot\frac{-p_2'A}{\det g} + \cdots$$
$$+ \frac{-2p_i'AB(p_1p_1' + \cdots + p_{L-1}p_{L-1}')}{(\det g)^2} =$$
$$= \frac{p_i'AB(p_1p_1' + \cdots + p_{L-1}p_{L-1}')}{(\det g)^2}$$
$$- \frac{2p_i'AB(p_1p_1' + \cdots + p_{L-1}p_{L-1}')}{(\det g)^2} =$$
$$= -\,\frac{p_i'AB(p_1p_1' + \cdots + p_{L-1}p_{L-1}')}{(\det g)^2}.$$

We calculate the coefficients for $k, i \neq 0$

$$R^k_{0i0} = \Gamma^0_{00}\Gamma^k_{i0} + \Gamma^1_{00}\cancel{\Gamma^k_{i1}} + \Gamma^2_{00}\cancel{\Gamma^k_{i2}} + \cdots - \Gamma^0_{i0}\Gamma^k_{00} - \Gamma^1_{i0}\Gamma^k_{01}$$
$$- \Gamma^2_{i0}\Gamma^k_{02}\cdots + \frac{\partial \Gamma^k_{00}}{\partial x_i} - \frac{\partial \Gamma^k_{i0}}{\partial x_0}.$$

$$\frac{\partial \Gamma^k_{00}}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{p_k\left[AC - BA'\right]}{(\|p\|^2)B + A^2} =$$
$$= \frac{p_k(-p_i'C + p_i''B)\left[(\|p\|^2)B + A^2\right] + 2p_kp_i'\left[AC - BA'\right]A}{(\det g)^2} =$$
$$= \frac{-p_kp_i'C(\|p\|^2)B + p_kp_i''B(\|p\|^2)B - p_kp_i'CA^2 + p_kp_i''BA^2 + 2p_kp_i'A^2C - 2p_kp_i'BAA'}{(\det g)^2}$$

See the equations $\frac{\partial \Gamma^k_{i0}}{\partial x_0}$ and $-\frac{\partial \Gamma^k_{i0}}{\partial x_0}$ given in Box II.

Hence the difference is given in Box III, and finally we obtain

$$R^k_{0i0} = \Gamma^0_{00}\Gamma^k_{i0} - \Gamma^0_{i0}\Gamma^k_{00} - \Gamma^1_{i0}\Gamma^k_{01} - \Gamma^2_{i0}\Gamma^k_{02} + \cdots + \frac{\partial \Gamma^k_{00}}{\partial x_i} - \frac{\partial \Gamma^k_{i0}}{\partial x_0}$$
$$= \frac{[\|p\|^2C + AA']}{\det g}\frac{p_i'p_kB}{\det g} + \frac{p_i'A}{\det g}\frac{p_k\left[AC - A'B\right]}{\det g}$$
$$- \frac{p_i'p_1B}{\det g}\frac{p_1'p_kB}{\det g} - \frac{p_i'p_2B}{\det g}\frac{p_2'p_kB}{\det g} + \cdots + \frac{\partial \Gamma^k_{00}}{\partial x_i} - \frac{\partial \Gamma^k_{i0}}{\partial x_0} =$$
$$= \frac{p_i'p_kB[\|p\|^2C + AA']}{(\det g)^2} + \frac{p_i'p_kA\left[AC - A'B\right]}{(\det g)^2}$$
$$- \frac{p_i'p_kB^2(p_1p_1' + \cdots + p_{L-1}p_{L-1}')}{(\det g)^2} + \frac{\partial \Gamma^k_{00}}{\partial x_i} - \frac{\partial \Gamma^k_{i0}}{\partial x_0} =$$
$$= \frac{p_i'p_kB^2(p_1p_1' + p_2p_2' + \cdots + p_{L-1}p_{L-1}') - p_i'p_k'B[(\|p\|^2)B + A^2]}{(\det g)^2}$$

Combining all the terms, we get the equation given in Box IV.

Appendix C. Proof of Theorem 3

Proof. If $M = 2$, the manifold $B(r)$ is diffeomorphic to $\mathbb{R}$ through the map (see (1) above)

$$\phi : \mathbb{R} \longrightarrow B(r) \subset S \times \mathbb{R}^{M-1} = S \times \mathbb{R}$$
$$t \longmapsto (p(t), w(t)),$$

and $E(r)$ is a submanifold of dimension L in a space of dimension $2L - 1$. By setting $\alpha_i := \omega^i$, a parametrization of $E(r)$ is given by (see (2) above)

$$\Phi : \mathbb{R}^L \longrightarrow E(r),$$
$$(t, \alpha_1, \ldots, \alpha_{L-1}) \longmapsto (p_1(t), \ldots, p_{L-1}(t), \alpha_1, \ldots, \alpha_{L-1},$$
$$w(t) - p_1(t)\alpha_1 - \cdots - p_{L-1}(t)\alpha_{L-1}).$$

Consider a basis of a vector field of $T_xE(r)$ given by

$$\Phi_0 = \left(\frac{\partial p_1}{\partial t}, \ldots, \frac{\partial p_{L-1}}{\partial t}, 0, \ldots, 0, \frac{\partial w}{\partial t} - \frac{\partial p_1}{\partial t}\alpha_1\right.$$

$$\frac{\partial \Gamma^k_{i0}}{\partial x_0} = \frac{\partial}{\partial x_0}\frac{p'_i p_k B}{(\|p\|^2)B + A^2} =$$
$$= \frac{(p_k p''_i B + p'_i p'_k B + 2p'_i p_k C)[(\|p\|^2)B + A^2] - 2p'_i p_k B\left[(p_1p'_1 + p_2p'_2 + \cdots + p_{L-1}p'_{L-1})B + (\|p\|^2)C + AA'\right]}{(\det g)^2}$$
$$-\frac{\partial \Gamma^k_{i0}}{\partial x_0} =$$
$$= \frac{-p_k p''_i B^2(\|p\|^2) - p'_i p'_k(\|p\|^2)B^2 - 2p'_i p_k C(\|p\|^2)B - p_k p''_i BA^2 - p'_i p'_k BA^2 - 2p'_i p_k CA^2}{(\det g)^2}$$
$$+ \frac{+2p'_i p_k B^2(p_1p'_1 + p_2p'_2 + \cdots + p_{L-1}p'_{L-1}) + 2p'_i p_k B(\|p\|^2)C + 2p'_i p_k BAA'}{(\det g)^2}.$$

Box II.

$$\frac{\partial \Gamma^k_{00}}{\partial x_i} - \frac{\partial \Gamma^k_{i0}}{\partial x_0} = \frac{p'_i p_k\left[-BC\|p\|^2 - CA^2 + 2B^2(p_1p'_1 + p_2p'_2 + \cdots + p_{L-1}p'_{L-1})\right] - p'_i p'_k B[(\|p\|^2)B + A^2]}{(\det g)^2}$$

Box III.

$$\langle R(X_0, X_i)X_0, X_i\rangle = \sum_s R^s_{0i0}g_{si} = R^0_{0i0}g_{0i} + R^1_{0i0}g_{1i} + \cdots + R^{l-1}_{0i0}g_{(L-1)i} =$$
$$= \frac{p'_i AB(p_1p'_1 + \cdots + p'_{L-1}p_{L-1})}{(\det g)^2}\cdot p_i A + \frac{p'_i p_1 B^2(p_1p'_1 + \cdots + p_{L-1}p'_{L-1}) - p'_i p'_1 B[(\|p\|^2)B + A^2]}{(\det g)^2}\cdot p_1 p_i + \cdots +$$
$$+ \frac{p'_i p_i B^2(p_1p'_1 + \cdots + p_{L-1}p'_{L-1}) - p'_i p'_i B[(\|p\|^2)B + A^2]}{(\det g)^2}\cdot (1 + p_i^2) + \cdots +$$
$$+ \frac{p'_i p_{L-1} B^2(p_1p'_1 + \cdots + p_{L-1}p'_{L-1}) - p'_i p'_{L-1} B[(\|p\|^2)B + A^2]}{(\det g)^2}\cdot p_{L-1}p_i =$$
$$= \frac{p'_i p_i B(p_1p'_1 + \cdots + p_{L-1}p'_{L-1})\left[A^2 + (\|p\|)^2 B\right] - p'_i p'_1 B[(\|p\|^2)B + A^2](p_1p'_1 + \cdots + p_{L-1}p'_{L-1}) - (p'_i)^2 B[(\|p\|^2)B + A^2]}{(\det g)^2} =$$
$$= \frac{-(p'_i)^2 B[(\|p\|^2)B + A^2]}{(\det g)^2} =$$
$$= \frac{-(p'_i)^2 B}{(\det g)}.$$

Box IV.

$$-\cdots - \frac{\partial p_{L-1}}{\partial t}\alpha_{L-1}\Bigg) = (p'_1, \ldots, p'_{L-1}, 0, \ldots, 0, A)$$

$$\Phi_1 = (0, \ldots, 0, 1, 0, 0, -p_1(t))$$

$$\Phi_i = (0, \ldots, 0, 0, 1, 0, -p_i(t))$$

where 1 is in the $L-1+i$ position and where $A = \frac{\partial w}{\partial t} - \frac{\partial p_1}{\partial t}\alpha_1 - \cdots - \frac{\partial p_{L-1}}{\partial t}\alpha_{L-1}$ or, more compactly,[4]

$$A = w' - p'_1 x_1 - \cdots - p'_{L-1}x_{L-1}.$$

Clearly, we have $A(x_0, x_1, \ldots, x_{L-1})$ and $\frac{\partial}{\partial x_i}A = -p'_i$, for all $i \neq 0$.

We set $x_0 = t,\ \ x_1 = \alpha_1,\ \ x_2 = \alpha_2,\ \ x_{L-1} = \alpha_{L-1}$ and $X_i = \Phi_i$.

The induced metric on $E(r)$ is given by

$$ds^2 = \sum_{i,j=0}^{L-1} g_{ij}dx_i dx_j.$$

Set $B(x_0) = \left(\frac{\partial p_1}{\partial x_0}\right)^2 + \cdots + \left(\frac{\partial p_{L-1}}{\partial x_0}\right)^2$ or, more compactly,

$$B(x_0) = (p'_1)^2 + \cdots + (p'_{L-1})^2.$$

An easy calculation gives

$$\langle \Phi_0, \Phi_0\rangle = B + A^2$$
$$\langle \Phi_0, \Phi_i\rangle = -p_i A$$
$$\langle \Phi_i, \Phi_i\rangle = 1 + p_i^2$$
$$\langle \Phi_i, \Phi_j\rangle = p_i p_j$$

4 $A = w' - \langle p', \alpha\rangle$ with $p' = (p'_1, \ldots, p'_{L-1})$ and $\alpha = (\alpha_1, \ldots, \alpha_{L-1})$.

$$\frac{1}{g}\begin{pmatrix} 1+p_1^2+\cdots+p_{L-1}^2 & p_1A & \dots & p_iA & \dots & p_{L-1}A \\ p_1A & (1+p_2^2+\cdots+p_{L-1}^2)B+A^2 & \dots & -p_1p_iB & \dots & -p_1p_{L-1}B \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_iA & -p_1p_iB & \dots & (1+p_1^2+\cdots+p_{i-1}^2+p_{i+1}^2+\cdots+p_{L-1}^2)B+A^2 & \dots & -p_ip_{L-1}B \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_{L-1}A & -p_1p_{L-1}B & \dots & -p_ip_{L-1}B & \dots & (1+p_1^2+\cdots+p_{L-2}^2)B+A^2 \end{pmatrix}$$

Box V.

$$g_{ij}=\begin{pmatrix} B+A^2 & -p_1A & \dots & -p_iA & \dots & -p_{L-1}A \\ -p_1A & 1+p_1^2 & \dots & p_1p_i & \dots & p_1p_{L-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -p_iA & p_1p_i & \dots & 1+p_i^2 & \dots & p_ip_{L-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -p_{L-1}A & p_1p_{L-1} & \dots & p_ip_{L-1} & \dots & 1+p_{L-1}^2 \end{pmatrix}.$$

Setting

$$g := (1+p_1^2+\cdots+p_{L-1}^2)B+A^2 = \|p\|^2B+A^2,$$

where

$$\boxed{\|p\|^2=(1+p_1^2+\cdots+p_{L-1}^2)},$$

the inverse matrix g^{ij} can be written as given in Box V.

To compute the Christoffel symbols

$$\Gamma_{ij}^k=\frac{1}{2}\sum_{h=1}^{L-1}g^{hk}\left(\frac{\partial g_{jh}}{\partial x_i}+\frac{\partial g_{hi}}{\partial x_j}-\frac{\partial g_{ij}}{\partial x_h}\right),$$

observe that the entries g_{ij} with i and j both different from 0 do not depend on the x_i, so their derivatives with respect to x_k, with $k\neq 0$, are zero.

Moreover, $\frac{\partial}{\partial x_i}g_{0i} = -p_i(-p_i') = p_ip_i'$, while $\frac{\partial}{\partial x_0}g_{ii} = 2p_ip_i'$. Hence all Christoffel symbols with subscript different from 0 vanish. The others symbols can be obtained through a long but straightforward calculation (see Appendix A). Observe that $\frac{\partial B}{\partial x_0} = 2(p_1'p_1''+p_2'p_2''+\cdots+p_{L-1}'p_{L-1}'')$, so for convenience we set

$$C := p_1'p_1''+p_2'p_2''+\cdots+p_{L-1}'p_{L-1}''$$

and

$$A' := w''-p_1''x_1-\cdots-p_{L-1}''x_{L-1}$$

The Christoffel symbols are

$$\Gamma_{00}^0=\frac{[\|p\|^2C+AA']}{\|p\|^2B+A^2},$$
$$\Gamma_{00}^k=\frac{p_k\left[AC-A'B\right]}{\|p\|^2B+A^2},$$
$$\Gamma_{0j}^0=\frac{-p_j'A}{\|p\|^2B+A^2},$$
$$\Gamma_{0j}^k=\frac{p_j'p_kB}{\|p\|^2B+A^2},$$
$$\Gamma_{ij}^0 = \Gamma_{ii}^0=0,$$
$$\Gamma_{ij}^k = \Gamma_{ii}^k=0.$$

To determine the sectional curvature

$$K(X,Y)=\frac{\langle R(X,Y)X,Y\rangle}{|X\wedge Y|},$$

we need to calculate the coefficients of the tensor R using

$$R_{ijk}^s=\sum_{m=0}^{l-1}\Gamma_{ik}^m\Gamma_{jm}^s-\sum_{m=0}^{l-1}\Gamma_{jk}^m\Gamma_{im}^s+\frac{\partial\Gamma_{ik}^s}{\partial x_j}-\frac{\partial\Gamma_{jk}^s}{\partial x_i}. \quad (3)$$

In particular, we have that $K(X_i,X_j)=0$ for all $i,j\neq 0$.

Since

$$K(X_0,X_i)=\frac{\langle R(X_0,X_i)X_0,X_i\rangle}{|X_0\wedge X_i|}$$

we use (3) to compute $\langle R(X_0,X_i)X_0,X_i\rangle=\sum_s R_{0i0}^s g_{si}$.

After a long but straightforward calculation (see Appendix B), we obtain

$$\langle R(X_0,X_i)X_0,X_i\rangle=\sum_s R_{0i0}^s g_{si}=R_{0i0}^0g_{0i}+R_{0i1}^1g_{1i}+\cdots+R_{0i0}^{l-1}g_{(l-1)i}=$$
$$=-\frac{(p_i')^2B}{(\det g)}\leq 0.$$

□

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