J Inequal Appl

https://doi.org/10.1186/s13660-026-03523-7

Article in Press

Lattice-ordered structures arising from majorization in $L^1[0, b)$

Received: 2 April 2026

Accepted: 27 July 2026

Published online: 12 August 2026

Cite this article as: Freytes, H., Sergioli, G. Lattice-ordered structures arising from majorization in $L^1[0, b)$. *J Inequal Appl* (2026). https://doi.org/10.1186/s13660-026-03523-7

H. Freytes, G. Sergioli

Lattice-ordered structures arising from majorization in $L^1[0,b)$

H. Freytes[1], G. Sergioli[2,3]

1. Department of Mathematics and Computer Science, University of Cagliari
Palazzo delle Scienze, Via Ospedale 72 - 09124, Cagliari-Italy
2. Department of Philosophy, University of Cagliari,
Via Is Mirrionis 1, 09123, Cagliari-Italy
3. Corresponding author - Email: giuseppe.sergioli@gmail.com

Abstract

This article focuses on characterizing lattice-order structures arising from majorization within the space $L^1[0,b)$. Particular attention is given to the analysis of suprema and infima in the majorization order on continuous strictly unimodal functions.

Mathematics Subject Classification 2020: 06F20, 46E30.
Keywords: Lattices, majorization.

Introduction

Majorization theory is a powerful mathematical framework for comparing probability distributions in terms of disorder or randomness. It allows one to determine which of two distributions exhibits a higher degree of randomness. This theory has been widely applied in several areas, including information theory [17] and economics [11]. In particular, it provides a rigorous tool for the study of quantum information processes [12] such as quantum measurements, entanglement transformations and a broad class of problems in quantum mechanics [13]. In its original formulation, majorization establishes an order relation between probability vectors in finite-dimensional spaces. Subsequently, Hardy, Littlewood, and Pólya [6, 7] extend this theory to continuous probability distributions and Chong [5] later generalized it to the space $L^1(X)$ over an arbitrary measure space X. It is the recent development of quantum information theory in infinite-dimensional spaces [8, 10, 14] that has stimulated growing interest in Chong's extension. Examples include the study of semidoubly stochastic operators on $L^1(X)$ [9] and investigations of information theoretic properties of Wigner functions in phase space [18], among others.

The majorization order also induces a lattice structure on finite-dimensional probability vectors. This structure, known as *majorization lattice*, was first identified by

Bapat [1] and has important applications in several areas of information theory. For instance, in classical information, majorization lattice has been shown to provide an appropriate setting for studying fundamental properties of the Huffman algorithm [15]. In quantum information, it also plays a crucial role in the analysis of local operations with classical communication (LOCC) in the context of quantum resources [2, 3]. More generally, for infinite-dimensional vectors, a majorization lattice can also be defined, playing an analogous role in quantum resource theories within infinite-dimensional quantum information theory [10].

Motivated by the interest in extending majorization to more general settings, this article characterizes several majorization lattices in $L^1[0,b)$, where $0 < b \leq \infty$. In this context, particular attention is given to the majorization lattice operations on strictly unimodal functions, which provide a unifying framework encompassing key families of probability distributions such as Erlang distributions.

The paper is organized as follows. Section 1 introduces the basic notions on lattice-ordered structures and majorization in the discrete setting. In Section 2, we establish technical results concerning lattice operations on concave Lipschitz-continuous functions, which will be instrumental in analyzing majorization lattice structures in $L^1[0,b)$. Section 3 introduces the notion of majorization in $L^1[0,b)$ and presents a comprehensive description of the resulting lattice-ordered structures. A general characterization of the suprema and infima in this context is also provided. Finally, Section 4 is devoted to studying these lattice operations acting on strictly unimodal functions. Erlang distributions are examined as a particular case.

1 Lattice-ordered structures and majorization in discrete settings

We begin this section by introducing basic terminology of lattice theory. Let us consider an ordered set $\langle L, \leq \rangle$. If L has a greatest element (resp. a least element), we denote it by 1 (resp. 0). The ordered set $\langle L, \leq \rangle$ is said to be a *lattice* iff every $a, b \in L$ both a supremum $a \vee b$ and an infimum $a \wedge b$ exist in L. Thus, $\vee$ and $\wedge$ define binary operations on L and this lattice is denoted by $\langle L, \vee, \wedge \rangle$. We denote by $\mathcal{L}$ the class of all lattices. Let L be a lattice. Then L is said to be *distributive* iff, for all $a_1, a_2, a_3 \in L$, the following conditions hold:

$$\begin{aligned} a_1 \wedge (a_2 \vee a_3) &= (a_1 \wedge a_2) \vee (a_1 \wedge a_3), \\ a_1 \vee (a_2 \wedge a_3) &= (a_1 \vee a_2) \wedge (a_1 \vee a_3). \end{aligned} \tag{1}$$

If not, L is referred to as *non-distributive*. If L has a greatest (resp. least) element, it is denoted by $\langle L, \vee, \wedge, 1 \rangle$ (resp. $\langle L, \vee, \wedge, 0 \rangle$). When both extremal elements exist, L is called a *bounded lattice* and denoted by $\langle L, \vee, \wedge, 0, 1 \rangle$. The lattice L is called $\bigvee$-*complete* (resp. $\bigwedge$-*complete*) iff every non-empty subset $(x_i)_{i \in I}$ of L admits a supremum denoted by $\bigvee_{i \in I} x_i$ (resp. an infimum, denoted by $\bigwedge_{i \in I} x_i$). If every subset of L admits both a supremum and an infimum, L is said to be a *complete lattice*. Weaker variants of these notions of completeness are also considered, which, in our setting, emerge as distinctive features of the lattice-ordered structures associated with majorization.

Specifically, L is *conditionally* $\bigvee$*-complete* (resp. *conditionally* $\bigwedge$*-complete*) iff every nonempty subset of L that is bounded above (resp. bounded below) has a supremum (resp. an infimum). If both properties hold, L is said to be *conditionally complete*.

Let X be a countable set whose elements are called variables. We inductively define the notion of $\mathcal{L}$*-term*, also referred to as $\mathcal{L}$*-polynomial symbol*, as follows:

- If $x \in X$, then x is an $\mathcal{L}$-term.
- If t_1 and t_2 are $\mathcal{L}$-terms, then $(t_1 \wedge t_2)$ and $(t_1 \vee t_2)$ are $\mathcal{L}$-terms.
- Only the expressions defined above are $\mathcal{L}$-terms.

We denote by $\mathrm{Term}_{\mathcal{L}}(X)$ the set of all $\mathcal{L}$-terms over X. If $t \in \mathrm{Term}_{\mathcal{L}}(X)$, we use the notation $t(x_1, \ldots, x_n)$ to indicate that the variables involved in t are among $x_1, \ldots, x_n$. The *complexity* of terms is inductively defined as the function $Comp : \mathrm{Term}_{\mathcal{L}}(X) \to \mathbb{N}$ such that

- $Comp(t) = 0$ iff $t \in X$;
- $Comp(t_1 \wedge t_2) = Comp(t_1 \vee t_2) = Comp(t_1) + Comp(t_2) + 1$.

Let L be a lattice. Each term $t(x_1, \ldots, x_n)$ canonically defines an n-ary operation on L, denoted by t^L. For $a_1, \ldots, a_n \in L$, $t^L(a_1, \ldots, a_n)$ denotes the result of applying the term operation t^L to the elements $a_1, \ldots, a_n$. This is generally referred to as a *valuation* of the term t in L.

We now introduce the notion of majorization for discrete structures. A *probability vector* of dimension d is a vector $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ with nonnegative components satisfying $\sum_{i=1}^d x_i = 1$. The *decreasing rearrangement* of $\mathbf{x}$, denoted by $\mathbf{x}^\downarrow = (x_1^\downarrow, \ldots, x_n^\downarrow)$, is the vector consisting of the same components as $\mathbf{x}$ but ordered in non-increasing order i.e., $x_1^\downarrow \geq x_2^\downarrow \geq \cdots \geq x_d^\downarrow \geq 0$. This rearrangement is unique up to permutations of equal components. Given two probability vectors $\mathbf{y} = (y_1, \ldots, y_d)$ and $\mathbf{x} = (x_1, \ldots, x_d)$ in $\mathbb{R}^d$, we say that $\mathbf{y}$ *majorizes* $\mathbf{x}$, denoted by $\mathbf{x} \preceq \mathbf{y}$, iff

$$\sum_{i=1}^{k} x_i^\downarrow \leq \sum_{i=1}^{k} y_i^\downarrow \quad \text{for each } k = 1 \ldots d. \tag{2}$$

The intuitive idea behind majorization is that one probability distribution is more concentrated than another when, after the decreasing rearrangement, more weight is placed in the first coordinates. These first coordinates represent the largest contributions to the total mass, so a distribution that allocates more weight to them concentrates its mass on fewer points, whereas a distribution with weight more evenly spread is less concentrated. By the uniqueness of the decreasing rearrangement, we can restrict our study of majorization to the following set:

$$\Delta_d^\downarrow = \{(x_1, \ldots, x_d) : x_i \geq x_{i+1} \text{ and } \sum_{i=1}^{d} x_i = 1\}. \tag{3}$$

Majorization induces a partial order on $\Delta_d^\downarrow$, with $\mathbf{1}_d = (1, 0, \ldots, 0)$ as the maximal element and $\mathbf{0}_d = \left(\frac{1}{d}, \ldots, \frac{1}{d}\right)$ as the minimal element. Accordingly, $\mathbf{1}_d$ is the most concentrated distribution and thus majorizes all others, while the uniform distribution $\mathbf{0}_d$ is the least concentrated and is majorized by all others. In [1], it is also proven that majorization endows $\Delta_d^{\downarrow}$ with a complete lattice-ordered structure, referred to as *majorization lattice*, and denoted by

$$\langle \Delta_d^{\downarrow}, \wedge, \vee, \mathbf{0}_d, \mathbf{1}_d \rangle. \tag{4}$$

It can be shown that $\Delta_d^{\downarrow}$ is distributive iff $d \leq 3$. In [4], an algorithmic method for computing $\mathbf{x}_1 \wedge \mathbf{x}_2$ and $\mathbf{x}_1 \vee \mathbf{x}_2$ is presented.

The above structure admits a generalization to the set $\ell_{[0,1]}$ of sequences in the real interval $[0, 1]$ [16]. More precisely, consider the set

$$\Delta_\infty^{\downarrow} = \{(x_i)_{i\in\mathbb{N}} \in \ell_{[0,1]} : x_i \geq x_{i+1} \text{ and } \sum_{i=1}^{\infty} x_i = 1\}. \tag{5}$$

For $\mathbf{y} = (y_i)_{i\in\mathbb{N}}$ and $\mathbf{x} = (x_i)_{i\in\mathbb{N}}$ in $\Delta_\infty^{\downarrow}$, we say that $\mathbf{y}$ *majorizes* $\mathbf{x}$, and also write $\mathbf{x} \preceq \mathbf{y}$, iff for each $k \in \mathbb{N}$, $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$. In this setting, $\Delta_\infty^{\downarrow}$ still carries a non-distributive lattice-ordered structure, although it is weaker. For instance, $\Delta_\infty^{\downarrow}$ is bounded above by the sequence $\mathbf{1}_\infty = (1, 0, 0, \ldots)$, but it does not possess a minimal element. Furthermore, completeness is relaxed to $\bigvee$-completeness and conditional $\bigwedge$-completeness. This lattice is denoted by

$$\langle \Delta_\infty^{\downarrow}, \wedge, \vee, \mathbf{1}_\infty \rangle. \tag{6}$$

As in the finite case, this lattice-ordered structure also plays an important role in the theory of quantum resources associated with infinitary LOCC protocols [10, 14].

In this section, we have highlighted the most relevant features and applications of majorization lattices in the discrete setting. This provides a clearer understanding of the ideas underlying the extension of majorization to $L^1[0, b)$.

2 Concavity, Lipschitz continuity and lattice operations

This section gathers the technical results needed to study lattice-ordered structures associated with the extension of majorization to Lebesgue integrable functions. Throughout the paper, λ denotes the Lebesgue measure on $\mathbb{R}$. In view of the mathematical framework considered here, we confine our developments to functions defined on real intervals.

Let $I \subseteq \mathbb{R}$ be an interval. We denote by I° the interior of I. For measurable functions $f, g : I \to \mathbb{R}$, we write $f = g \quad$ a.e. on I iff $f(x) = g(x)$ for λ-almost every $x \in I$. We denote by $L^1(I)$ the set of real-valued Lebesgue integrable functions on I. We also denote by $L^1_{\text{loc}}(I)$ the set of real-valued locally integrable functions, namely, those functions that are integrable on every compact subset of I.

A function f is said to be *concave* on I iff, for every $x, y \in I$ and every $\beta \in [0, 1]$, $f(\beta x + (1-\beta)y) \geq \beta f(x) + (1-\beta)f(y)$. An equivalent characterization of concavity is that, for every $x < y < z$ in I,

$$\frac{f(y) - f(x)}{y - x} \geq \frac{f(z) - f(y)}{z - y}. \tag{7}$$

We denote by $Conc(I)$ the set of all concave functions on I. The following proposition states a well known lattice-theoretic property of concave functions.

Proposition 2.1 *Let $I \subseteq \mathbb{R}$ be an interval and $(f_j)_{j\in J}$ be a subset of $Conc(I)$. Assume that, for all $x \in I$, $-\infty < \bigwedge_{j\in J} f_j(x)$. Then $\bigwedge_{j\in J} f_j \in Conc(I)$.*

$\square$

Definition 2.2 Let $I \subseteq \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be a function. The *concave envelope* of f on I is defined as $\widehat{f} = \bigwedge\{g \in Conc(I) : f \leq g\}$ whenever this infimum exists.

Let $f \in Conc[a, b)$ with $a < b \leq \infty$. A standard fact in convex analysis is that, for every $x_0 \in (a, b)$, there exists at least one real number p, called a *supergradient*, such that

$$f(x) \leq p(x - x_0) + f(x_0) \quad \text{for all } x \in [a, b). \tag{8}$$

Let $f_1, f_2 \in Conc[a, b)$ and $x_0 \in (a, b)$, with $a < b \leq \infty$. By Eq. (8) consider supergradients p_1, p_2 such that

$$f_i(x) \leq p_i(x - x_0) + f_i(x_0), \quad i = 1, 2.$$

Set $m = \max\{p_1, p_2\}$ and $h = \max\{f_1(x_0), f_2(x_0)\}$. Then $f(x) = m(x - x_0) + h \in Conc[a, b)$ and dominates f_1 and f_2 on $[a, b)$. Hence,

$$C_{f_1,f_2} = \{h \in Conc[a, b) : f_i \leq h \text{ for } i = 1, 2\} \neq \emptyset.$$

Consequently, it follows from Proposition 2.1, that

$$f_1 \widehat{\vee} f_2 = \bigwedge C_{f_1,f_2} \tag{9}$$

is well defined as a binary operation on $Conc[a, b)$. By considering this operation, the following proposition describes the natural lattice-ordered structure of $Conc[a, b)$ based on the concave envelope. As we will see in Theorem 3.8 and Example 3.9, it will be crucial for describing the majorization lattices in $L^1[0, b)$ and, in particular, for the valuation of terms in these lattices.

Proposition 2.3 *Let $0 < a \leq b \leq \infty$. Then*

1. *$\langle Conc[a, b), \wedge, \widehat{\vee}\rangle$ is a conditionally complete lattice.*
2. *If $(f_j)_{j\in J}$ is a subset in $Conc[a, b)$ such that there exists the supremum $\widehat{\bigvee}_{j\in J} f_j$ in $Conc[a, b)$ then $\widehat{\bigvee}_{j\in J} f_j = \widehat{\bigvee_{j\in J} f_j}$.*

Proof: 1) This follows directly from Proposition 2.1 and Eq. (9).

2) Since $\widehat{\bigvee_{j\in J}} f_j$ exists in $Conc[a,b)$, it is also an upper bound of the family $(f_j)_{j\in J}$. Hence $f_j \leq \bigvee_{j\in J} f_j \leq \widehat{\bigvee_{j\in J}} f_j$ for every $j \in J$. Thus, by Proposition 2.1, the concave envelope of $\bigvee_{j\in J} f_j$ exists and it satisfies $\bigvee_{j\in J} f_j \leq \widehat{\bigvee_{j\in J} f_j} \leq \widehat{\bigvee_{j\in J}} f_j$. Therefore, by the definition of the supremum in $Conc[a,b)$, we obtain $\widehat{\bigvee_{j\in J}} f_j = \widehat{\bigvee_{j\in J} f_j}$.

□

A function $f : I \to \mathbb{R}$ is said to be *Lipschitz continuous* on I iff there exists a constant $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in I$. Any such M is referred to as a *Lipschitz constant* for the function f. We denote by $Lip(I)$ the sets of Lipschitz continuous functions on I. By the Rademacher theorem, if $f \in Lip(I)$, then f is differentiable λ-almost everywhere on I°, and its derivative f' satisfies

$$f' \leq M. \tag{10}$$

$$\int_x^y f'(t)\,dt = f(y) - f(x) \quad \text{for all } x, y \in I \text{ with } x \leq y. \tag{11}$$

The following proposition provides a classical characterization of concavity for Lipschitz continuous functions.

Proposition 2.4 *Let $I \subseteq \mathbb{R}$ be an interval and $f \in Lip(I)$. Then, $f \in Conc(I^\circ)$ iff there exists a decreasing function $g : I^\circ \to \mathbb{R}$ such that $f' = g \quad$ a.e. on I°.*

□

Let us consider the interval $[a,b)$ with $a < b \leq \infty$. We then define

$$\widehat{Lip}[a,b) = Conc[a,b) \cap Lip[a,b).$$

Combining Eq. (10) and Proposition 2.4, if $f \in \widehat{Lip}[a,b)$, then the decreasing function g such that $f' = g \quad$ a.e. on (a,b) can be extended to the left endpoint a by setting $g(a) = \bigvee_{x\in(a,b)} g(x)$. The resulting extension, which we denote by Df, is decreasing on $[a,b)$ and right-continuous at a. This observation allows us to introduce the following differential-type operator, which will be useful for simplifying the notation and subsequent developments in the next sections.

$$\widehat{Lip}[a,b) \ni f \longmapsto Df \tag{12}$$

which satisfies the following properties:

D1. Df is a decreasing function on $[a,b)$.

D2. $Df = f' \quad$ a.e. on (a,b). Consequently, if $a \leq x \leq y < b$ then

$$\int_x^y Df(t)\,dt = f(y) - f(x).$$

D3. Df is right-continuous at a.

Proposition 2.5 *Let $f : [a,b) \to [0,\infty)$ where $a < b \leq \infty$ and assume that $f \in Conc(a,b)$.*

1. *If f is right-continuous at a then $f \in Conc[a,b)$.*
2. *If $f(x) \leq M(x-a)$ then $f \in \widehat{Lip}[a,b)$.*

Proof: 1) Let $\epsilon > 0$ be such that $a < a+\epsilon < x < y < b$. Since $f \in Conc(a,b)$ and is right-continuous at a, by Eq. (7) we have

$$\frac{f(x)-f(a)}{x-a} = \lim_{\epsilon\to 0^+} \frac{f(x)-f(a+\epsilon)}{x-(a+\epsilon)} \geq \frac{f(y)-f(x)}{y-x}.$$

Hence, $f \in Conc[a,b)$.

2) Since $f(x) \leq M(x-a)$, it follows that $f(a) = 0$ and that f is right-continuous at a. Therefore, by item 1, we have $f \in Conc[a,b)$. Let $a < x < y < b$. Then, by Eq. (7),

$$M = \frac{M(x-a)}{x-a} \geq \frac{f(x)}{x-a} = \frac{f(x)-f(a)}{x-a} \geq \frac{f(y)-f(x)}{y-x}.$$

Hence $f \in \widehat{Lip}[a,b)$.

□

Proposition 2.6 *Let $f : [0,b) \to [0,\infty)$ be an increasing bounded function, where $0 < b \leq \infty$. Then the concave envelope $\widehat{f}$ exists and satisfies*

$$\lim_{x\to b} \widehat{f}(x) = \lim_{x\to b} f(x).$$

Moreover, if f is right-continuous at 0, the following hold:

1. *$\widehat{f}(0) = f(0)$.*
2. *$\widehat{f}$ is increasing on $[0,b)$.*

Proof: Since f is bounded and increasing on $[0,b)$, we have $m = \bigvee_{x\in[0,b)} f(x) = \lim_{x\to b} f(x) < \infty$. The constant function m is concave and dominates f on $[0,b)$. Hence,

$$C_f = \{g : [0,b) \to [0,\infty) : g \text{ is concave and } f \leq g\}$$

is nonempty and Proposition 2.1 yields $\widehat{f}(x) = \bigwedge_{g\in C_f} g(x)$. Let $n_0 \in \mathbb{N}$ be such that $\frac{1}{n_0} < b$. For each $n > n_0$, we define the function

$$g_n(x) = \begin{cases} n(m - f(\frac{1}{n}))x + f(\frac{1}{n}), & \text{if } 0 \leq x \leq \frac{1}{n}, \\ m, & \text{if } \frac{1}{n} < x < b, \end{cases}$$

i.e., on $[0,\frac{1}{n}]$ the function g_n coincides with the line segment joining $(0, f(\frac{1}{n}))$ and $(\frac{1}{n}, m)$, elsewhere it equals m. Observe that $g_n \in C_f$ and $f \leq g_n$. Thus, for each $n \geq n_0$,

$$f \leq \widehat{f} \leq g_n \leq m. \tag{13}$$

By Eq. (13) and the Squeeze Theorem, it follows that $m = \lim_{x\to b} f(x) \le \lim_{x\to b} \widehat{f}(x) \le \lim_{x\to b} g_n(x) = m$ and hence $\lim_{x\to b} \widehat{f}(x) = \lim_{x\to b} f(x)$. Let us now suppose that f is right-continuous at 0.

1) By Eq. (13) and the right-continuity of f at 0, we obtain $\widehat{f}(0) \le \lim_{n\to\infty} g_n(0) = \lim_{n\to\infty} f\left(\frac{1}{n}\right) = f(0) = 0$.

2) Suppose that $\widehat{f}$ is strictly decreasing on $[0, b)$. From item 1, we have $f(0) = \widehat{f}(0) = 0$. Since $f \le \widehat{f}$, it would follow that there exists $x_0 \in [0, b)$ such that $f(x_0) \le \widehat{f}(x_0) < 0$, which contradicts the non-negativity of f. Therefore, $\widehat{f}$ must be increasing on $[0, b)$.

□

Let $[0, b)$ be an interval with $0 < b \le \infty$. In order to simplify notation and streamline the developments in the subsequent sections, we consider the *Volterra operator* defined on $L^1_{\text{loc}}[0, b)$, namely

$$L^1_{\text{loc}}[0, b) \ni f \mapsto (\mathcal{V}f)(x) = \int_0^x f(t)\, dt, \qquad 0 \le x < b. \tag{14}$$

Since $\widehat{Lip}[0, b) \subseteq L^1_{\text{loc}}[0, b)$ and, considering the operator D introduced in Eq. (12), it follows from condition D2 that

$$f \in \widehat{Lip}[0, b) \;\mapsto\; (\mathcal{V}Df)(x) = f(x) - f(0). \tag{15}$$

Note that if $f : [0, b) \to [0, \infty)$ is decreasing, then $f \in L^1_{\text{loc}}[0, b)$, so that the function $(\mathcal{V}f)(x)$ is well defined on $[0, b)$. Based on this observation, we state the following result:

Proposition 2.7 *Let $(f_i)_{i\in I}$ be a subset of non negative decreasing functions on $[0, b)$ where $0 < b \le \infty$. Then*

$$\bigwedge_{i\in I} \mathcal{V}f_i \in \widehat{Lip}[0, b)$$

Proof: For each $i \in I$ and all $0 \le x < y < b$, we have $\left|(\mathcal{V}f_i)(y) - (\mathcal{V}f_i)(x)\right| = \left|\int_x^y f_i(t)\, dt\right| \le f_i(0)\, |y - x|$, so that $\mathcal{V}f_i \in Lip[0, b)$. Moreover, since $(\mathcal{V}f_i)'(x) = f_i(x)$ and f_i is decreasing on $[0, b)$, it follows from Proposition 2.4 that $\mathcal{V}f_i \in Conc(0, b)$. Thus, by Proposition 2.1, $\bigwedge_{i\in I} \mathcal{V}f_i \in Conc(0, b)$. Since $f_i(0) = \max f_i$ and $f_i \ge 0$ there exists $M = \bigwedge_{i\in I} f_i(0)$. Thus,

$$\bigwedge_{i\in I} (\mathcal{V}f_i)(x) = \bigwedge_{i\in I} \int_0^x f_i(t)\, dt \le \bigwedge_{i\in I} \int_0^x f_i(0)\, dt = \bigwedge_{i\in I} f_i(0)x \le Mx.$$

Hence, by Proposition 2.5-2, it follows that $\bigwedge_{i\in I} \mathcal{V}f_i \in \widehat{Lip}[0, b)$.

□

Proposition 2.8 *Let $(f_i)_{i\in I}$ be a subset of nonnegative functions in $L^1_{\mathrm{loc}}([0,b))$, with $0 < b \le \infty$. Assume that there exists a constant $M > 0$ such that $f_i \le M$ on $(0,b)$ for all $i \in I$. Then the concave envelope $\widehat{\bigvee_{i\in I} \mathcal{V} f_i}$ exists and satisfies*

$$\widehat{\bigvee_{i\in I} \mathcal{V} f_i} \in \widehat{Lip}[0,b).$$

Proof: Since $\bigvee_{i\in I} (\mathcal{V} f_i)(x) = \bigvee_{i\in I} \int_0^x f_i(t)\,dt \le Mx \in Conc[0,b)$ it follows that the concave envelope $\widehat{\bigvee_{i\in I} \mathcal{V} f_i}$ exists and also satisfies $\widehat{\bigvee_{i\in I} \mathcal{V} f_i}(x) \le Mx$. Hence, by Proposition 2.5-2, it follows that $\widehat{\bigvee_{i\in I} \mathcal{V} f_i} \in \widehat{Lip}[0,b)$.

□

3 Majorization lattices in $L^1[0,b)$

Following Chong's generalization of majorization to arbitrary measure spaces [5], we begin by introducing preliminaries and basic terminology adapted to the case of $L^1[0,b)$.

Let $f \in L^1[0,b)$ where $0 < b \le \infty$ and $\alpha \in \mathbb{R}$. The set

$$S^f_\alpha = \{x \in [0,b) : f(x) > \alpha\} \tag{16}$$

is referred to as the *α-superlevel set of f*. The *distribution function associated with f* is defined as the function

$$[0,\infty) \ni \alpha \mapsto \delta_f(\alpha) = \lambda(S^f_\alpha). \tag{17}$$

Definition 3.1 Let $f \in L^1[0,b)$ with $f \ge 0$ and $0 < b \le \infty$. The *decreasing rearrangement $f^\downarrow$* of f is equivalently defined (see [9, Theorem 2.1]) as the function

$$[0,b) \ni x \mapsto f^\downarrow(x) = \inf\{\alpha \ge 0 : \delta_f(\alpha) \le x\} = \sup\{\alpha \ge 0 : \delta_f(\alpha) > x\}.$$

The decreasing rearrangement of a function plays a role analogous to that of the decreasing rearrangement of a probability vector. Let $f \in L^1[0,b)$ with $f \ge 0$ and $0 < b \le \infty$. A characteristic property of $f^\downarrow$ is that

$$\int_0^b f(t)\,dt = \int_0^b f^\downarrow(t)\,dt. \tag{18}$$

Moreover, if f is also decreasing, then

$$f = f^\downarrow \quad \text{a.e. on } [0,b). \tag{19}$$

Definition 3.2 Let $f, g \in L^1[0,b)$ with $0 < b \le \infty$, $f, g \ge 0$, and $\int_0^b f(t)\,dt = \int_0^b g(t)\,dt$. We say that *$f$ is λ-majorized by g*, also written $f \preceq_\lambda g$, iff

$$\mathcal{V} f^\downarrow \le \mathcal{V} g^\downarrow \quad \text{on } [0,b).$$

An immediate consequence of Definition 3.2, Eq. (18) and Eq. (19) is that the study of λ-majorization may be restricted to the following subsets of $L^1[0, b)$.

Definition 3.3 Let $0 < b \leq \infty$ and $0 < \alpha < \infty$. Then we define

$$L_\alpha^\downarrow[0, b) = \left\{ f \in L^1[0, b) : f \geq 0,\ f \text{ is decreasing, and } \int_0^b f(t)\,dt = \alpha \right\}.$$

We observe that $L_\alpha^\downarrow[0, b)$ provides a natural generalization of the set of probability vectors $\Delta_d^\downarrow$ defined in Eq. (3), where the λ-majorization relation can be established.

Let $f \in L_\alpha^\downarrow[0, b)$. Since f is decreasing, it is bounded above by $f(0)$. Hence we may define

$$f(0^+) = \bigvee_{x \in (0,b)} f(x). \tag{20}$$

Proposition 3.4 *Let $f_1, f_2 \in L_\alpha^\downarrow[0, b)$ such that $f_1 \preceq_\lambda f_2$. Then $f_1(0^+) \leq f_2(0^+)$.*

Proof: For $i = 1, 2$, we define the function

$$f_{i0}(x) = \begin{cases} f_i(0^+), & \text{if } x = 0, \\ f_i(x), & \text{if } x \in (0, b). \end{cases}$$

Note that f_{i0} is right-continuous at 0. Hence, for every $\epsilon > 0$ there exists $\delta > 0$ such that $f_{i0}(0) - \epsilon \leq f_{i0}(x) \leq f_{i0}(0) + \epsilon$ for $x \in [0, \delta)$. Consequently, for $i = 1, 2$ and $x \in [0, \delta)$, we have

$$(f_{i0}(0) - \epsilon)x = (\mathcal{V}(f_{i0}(0) - \epsilon))(x) \leq (\mathcal{V} f_{i0})(x) \leq (\mathcal{V}(f_{i0}(0) + \epsilon))(x) = (f_{i0}(0) + \epsilon)x.$$

It shows that $f_i(0^+) = \lim_{x \to 0^+} \frac{1}{x} (\mathcal{V} f_{i0})(x)$ for $i = 1, 2$.

Since $f_i = f_{i0}$ a.e. on $[0, b)$ it follows from the hypothesis and the preceding result that $f_1(0^+) = \lim_{x \to 0^+} \frac{1}{x} (\mathcal{V} f_{10})(x) = \lim_{x \to 0^+} \frac{1}{x} (\mathcal{V} f_1)(x) \leq \lim_{x \to 0^+} \frac{1}{x} (\mathcal{V} f_2)(x) = \lim_{x \to 0^+} \frac{1}{x} (\mathcal{V} f_{20})(x) = f_2(0^+)$.

□

Proposition 3.5 *Let $f, g \in L_\alpha^\downarrow[0, b)$ with $0 < b \leq \infty$. Then*

$$f \preceq_\lambda g \text{ and } g \preceq_\lambda f \quad \text{iff} \quad f = g \quad \text{a.e. on } [0, b).$$

Proof: Suppose that $f \preceq_\lambda g$ and $g \preceq_\lambda f$. Then $(\mathcal{V} f)(x) = (\mathcal{V} g)(x)$ for all $x \in [0, b)$. Since both functions are differentiable on $(0, b)$, it follows that $f = (\mathcal{V} f)' = (\mathcal{V} g)' = g$ a.e. on $[0, b)$. The converse is immediate.

□

Note that the relation $=$ a.e. defines an equivalence on $L_\alpha^\downarrow[0, b)$. Consequently, Proposition 3.5 establishes the antisymmetry of the relation $\preceq_\lambda$ in the quotient set $L_\alpha^\downarrow[0, b)/_{=\text{a.e.}}$. Since $\preceq_\lambda$ is clearly reflexive and transitive on $L_\alpha^\downarrow[0, b)$, it therefore induces a partial order on the corresponding quotient set. For the sake of simplicity, and in view of the above argument, we shall regard $\preceq_\lambda$ as an order on $L_\alpha^\downarrow[0, b)$, where equality between two functions $f, g \in L^{\downarrow}_{\alpha}[0,b)$ is understood modulo the equivalence $=$ a.e.. This partially ordered set will be denoted by

$$\langle L^{\downarrow}_{\alpha}[0,b), \preceq_{\lambda} \rangle. \tag{21}$$

To study the lattice-ordered structure of $\langle L^{\downarrow}_{\alpha}[0,b), \preceq_{\lambda} \rangle$ we denote by $(\bigwedge^{\lambda}, \wedge^{\lambda})$ and $(\bigvee^{\lambda}, \vee^{\lambda})$ the corresponding infima and suprema with respect to the λ-majorization order.

By considering the operator D defined in Eq. (12) we establish the following result.

Theorem 3.6 *Let $f_1, f_2 \in L^{\downarrow}_{\alpha}[0,b)$ with $0 < b \leq \infty$. Then*

1. $f_1 \wedge^{\lambda} f_2 = D(\mathcal{V} f_1 \wedge \mathcal{V} f_2)$.
2. $f_1 \vee^{\lambda} f_2 = D(\mathcal{V} f_1 \widehat{\vee} \mathcal{V} f_2)$.

Thus, $\langle L^{\downarrow}_{\alpha}[0,b), \wedge^{\lambda}, \vee^{\lambda} \rangle$ is a lattice.

Proof: 1) By Proposition 2.7, $\mathcal{V} f_1 \wedge \mathcal{V} f_2 \in \widehat{Lip}[0,b)$. Thus, $D(\mathcal{V} f_1 \wedge \mathcal{V} f_2)$ is well defined on $[0,b)$. We first prove that $D(\mathcal{V} f_1 \wedge \mathcal{V} f_2) \in L^{\downarrow}_{\alpha}[0,b)$. By condition D1 in Eq. (12), $D(\mathcal{V} f_1 \wedge \mathcal{V} f_2)$ is decreasing on $[0,b)$. Then we only need to show that $\int_0^b D(\mathcal{V} f_1 \wedge \mathcal{V} f_2)(t)\, dt = \alpha$. Since $\wedge$ is continuous, Eq. (15) yields

$$\begin{aligned}\int_0^b D(\mathcal{V} f_1 \wedge \mathcal{V} f_2)(t)\, dt &= \lim_{x \to b} \mathcal{V} D(\mathcal{V} f_1 \wedge \mathcal{V} f_2)(x) \\ &= \lim_{x \to b} (\mathcal{V} f_1 \wedge \mathcal{V} f_2)(x) - (\mathcal{V} f_1 \wedge \mathcal{V} f_2)(0) \\ &= \int_0^b f_1(t)\, dt \wedge \int_0^b f_2(t)\, dt - 0 = \alpha.\end{aligned}$$

Hence, $D(\mathcal{V} f_1 \wedge \mathcal{V} f_2) \in L^{\downarrow}_{\alpha}[0,b)$.

We now show that $D(\mathcal{V} f_1 \wedge \mathcal{V} f_2)$ is the infimum of $\{f_1, f_2\}$ with respect to λ-majorization. Since $(\mathcal{V} f_1 \wedge \mathcal{V} f_2)(0) = 0$, it follows from Eq. (15) that

$$\mathcal{V} f_i \geq \mathcal{V} f_1 \wedge \mathcal{V} f_2 = \mathcal{V} D(\mathcal{V} f_1 \wedge \mathcal{V} f_2), \qquad i = 1, 2 \tag{22}$$

which shows that $D(\mathcal{V} f_1 \wedge \mathcal{V} f_2)$ is a lower bound of $\{f_1, f_2\}$. Moreover, if $g \in L^{\downarrow}_{\alpha}[0,b)$ satisfies $g \preceq_{\lambda} f_i$ for $i = 1, 2$ then Eq. (15) again yields

$$\mathcal{V} g \leq \mathcal{V} f_1 \wedge \mathcal{V} f_2 = \mathcal{V} D(\mathcal{V} f_1 \wedge \mathcal{V} f_2). \tag{23}$$

This shows that $g \preceq_{\lambda} D(\mathcal{V} f_1 \wedge \mathcal{V} f_2)$ i.e., $D(\mathcal{V} f_1 \wedge \mathcal{V} f_2)$ is the greatest lower bound of $\{f_1, f_2\}$ in the sense of λ-majorization. Hence, $f_1 \wedge^{\lambda} f_2 = D(\mathcal{V} f_1 \wedge \mathcal{V} f_2)$.

2) Since f_1 and f_2 are both decreasing on $[0,b)$, we have

$$f_i \leq f_1(0) \vee f_2(0), \qquad i = 1, 2. \tag{24}$$

It then follows from Proposition 2.8 that $\widehat{\mathcal{V}f_1 \vee \mathcal{V}f_2} \in \widehat{Lip}[0,b)$. Hence, by Eq. (12), $D(\widehat{\mathcal{V}f_1 \vee \mathcal{V}f_2})$ is well defined on $[0,b)$. We first show that $D(\widehat{\mathcal{V}f_1 \vee \mathcal{V}f_2}) \in L_\alpha^{\downarrow}[0,b)$. By condition D1 in Eq. (12), $D(\widehat{\mathcal{V}f_1 \vee \mathcal{V}f_2})$ is decreasing on $[0,b)$. Then we only need to show that $\int_0^b D(\widehat{\mathcal{V}f_1 \vee \mathcal{V}f_2})(t)\,dt = \alpha$. Note that $\mathcal{V}f_1 \vee \mathcal{V}f_2$ is a non-negative, increasing function defined on $[0,b)$ and bounded in view of Eq. (24). Moreover, as $\vee$ is continuous, $\mathcal{V}f_1 \vee \mathcal{V}f_2$ is also right-continuous at 0. Then, by Proposition 2.6, the following two results hold:

$$\widehat{\mathcal{V}f_1 \vee \mathcal{V}f_2}(0) = (\mathcal{V}f_1 \vee \mathcal{V}f_2)(0) = 0, \tag{25}$$

$$\lim_{x\to b} \widehat{\mathcal{V}f_1 \vee \mathcal{V}f_2}(x) = \lim_{x\to b} (\mathcal{V}f_1 \vee \mathcal{V}f_2)(x) = \int_0^b f_1(t)\,dt \vee \int_0^b f_2(t)\,dt = \alpha \tag{26}$$

Combining the above results and using Eq. (15), we have

$$\begin{aligned}\int_0^b D(\widehat{\mathcal{V}f_1 \vee \mathcal{V}f_2})(t)\,dt &= \lim_{x\to b} \mathcal{V}D(\widehat{\mathcal{V}f_1 \vee \mathcal{V}f_2})(x) \\ &= \lim_{x\to b} (\widehat{\mathcal{V}f_1 \vee \mathcal{V}f_2})(x) - (\widehat{\mathcal{V}f_1 \vee \mathcal{V}f_2})(0) = \alpha.\end{aligned}$$

Hence $D(\widehat{\mathcal{V}f_1 \vee \mathcal{V}f_2}) \in L_\alpha^{\downarrow}[0,b)$.

We now show that $D(\widehat{\mathcal{V}f_1 \vee \mathcal{V}f_2})$ is the supremum of $\{f_1, f_2\}$ with respect to λ-majorization. By Eq. (15) and Eq. (25), we have

$$\mathcal{V}f_i \le \mathcal{V}f_1 \vee \mathcal{V}f_2 \le \widehat{\mathcal{V}f_1 \vee \mathcal{V}f_2} = \mathcal{V}D(\widehat{\mathcal{V}f_1 \vee \mathcal{V}f_2}), \qquad i = 1,2 \tag{27}$$

which shows that $D(\mathcal{V}f_1 \vee \mathcal{V}f_2)$ is an upper bound of $\{f_1, f_2\}$. Let $g \in L_\alpha^{\downarrow}[0,b)$ be such that $f_i \preceq_\lambda g$ for $i = 1,2$. Then

$$\mathcal{V}f_1 \vee \mathcal{V}f_2 \le \mathcal{V}g. \tag{28}$$

Since g is decreasing and non-negative on $[0,b)$, by Proposition 2.7, $\mathcal{V}g \in \widehat{Lip}[0,b)$, so that $\mathcal{V}g$ is concave on $[0,b)$. Then, by combining Eq. (15), Eq. (25) and Eq. (28), we obtain

$$\mathcal{V}D(\widehat{\mathcal{V}f_1 \vee \mathcal{V}f_2}) = \widehat{\mathcal{V}f_1 \vee \mathcal{V}f_2} \le \mathcal{V}g. \tag{29}$$

This shows that $D(\mathcal{V}f_1 \vee \mathcal{V}f_2) \preceq_\lambda g$ i.e., $D(\mathcal{V}f_1 \vee \mathcal{V}f_2)$ is the least upper bound of $\{f_1, f_2\}$ in the sense of λ-majorization. Thus $f_1 \vee^\lambda f_2 = D(\widehat{\mathcal{V}f_1 \vee \mathcal{V}f_2})$. Since $\mathcal{V}f_1, \mathcal{V}f_2 \in Conc[0,b)$, by Proposition 2.3-2, we have $\mathcal{V}f_1 \widehat{\vee} \mathcal{V}f_2 = \widehat{\mathcal{V}f_1 \vee \mathcal{V}f_2}$. This completes the proof of the item.

Hence, $\langle L_\alpha^{\downarrow}[0,b), \wedge^\lambda, \vee^\lambda\rangle$ is a lattice.

□

The following theorem gives conditions ensuring the existence of arbitrary infima and suprema in $L_\alpha^{\downarrow}[0,b)$ with respect to λ-majorization.

Theorem 3.7 *Let $0 < b \le \infty$, $g \in L_\alpha^{\downarrow}[0,b)$ and $(f_i)_{i\in I} \subseteq L_\alpha^{\downarrow}[0,b)$. Then:*

1. *If $g \preceq_\lambda f_i$ for every $i \in I$, then the family $(f_i)_{i\in I}$ admits an infimum, given by*

$$\bigwedge_{i\in I}^{\lambda} f_i = D \bigwedge_{i\in I} \mathcal{V} f_i.$$

2. *If $f_i \preceq_\lambda g$ for every $i \in I$, then the family $(f_i)_{i\in I}$ admits a supremum, given by*

$$\bigvee_{i\in I}^{\lambda} f_i = D \widehat{\bigvee_{i\in I} \mathcal{V} f_i}.$$

Proof: 1) By Proposition 2.7, $\bigwedge_{i\in I} \mathcal{V} f_i \in \widehat{Lip}[0,b)$. Thus, $D\bigwedge_{i\in I} \mathcal{V} f_i$ is well defined on $[0,b)$. We first prove that $D\bigwedge_{i\in I} \mathcal{V} f_i \in L^{\downarrow}_{\alpha}[0,b)$. By condition D1 in Eq. (12), $D\bigwedge_{i\in I} \mathcal{V} f_i$ is decreasing on $[0,b)$. Then we only need to show that $\int_0^b D\bigwedge_{i\in I} \mathcal{V} f_i(t)\,dt = \alpha$. By hypothesis we have $\alpha = \lim_{x\to b}(\mathcal{V}g)(x) \leq \lim_{x\to b} \bigwedge_{i\in I}(\mathcal{V} f_i)(x) \leq \lim_{x\to b}(\mathcal{V} f_i)(x) = \alpha$. Thus,

$$\lim_{x\to b} \bigwedge_{i\in I}(\mathcal{V} f_i)(x) = \alpha. \tag{30}$$

Since $\bigwedge_{i\in I} \mathcal{V} f_i(0) = 0$, by Eq. (15) and Eq. (30) we have

$$\int_0^b D \bigwedge_{i\in I} \mathcal{V} f_i(t)\,dt = \lim_{x\to b} \mathcal{V} D \bigwedge_{i\in I} \mathcal{V} f_i(x) = \lim_{x\to b} \bigwedge_{i\in I} \mathcal{V} f_i(x) = \alpha.$$

Hence $D\bigwedge_{i\in I} \mathcal{V} f_i \in L^{\downarrow}_{\alpha}[0,b)$. Finally, to prove that $D\bigwedge_{i\in I} \mathcal{V} f_i$ is the infimum, in the sense of λ-majorization, of the set $(f_i)_{i\in I}$, one can proceed by arguments analogous to those used to prove Eq. (22) and Eq. (23) in Theorem 3.6.

2) Since $f_i \preceq_\lambda g$, by Proposition 3.4 we have $f_i(0^+) \leq g(0^+)$. Thus, for all $i \in I$, $f_i \leq g(0^+)$ on $(0,b)$, which shows that $g(0^+)$ is an upper bound of all f_i on $(0,b)$. It then follows from Proposition 2.8 that $\widehat{\bigvee_{i\in I} \mathcal{V} f_i} \in \widehat{Lip}[0,b)$. Hence, by Eq. (12), $D\widehat{\bigvee_{i\in I} \mathcal{V} f_i}$ is well defined on $[0,b)$. We first show that $D\widehat{\bigvee_{i\in I} \mathcal{V} f_i} \in L^{\downarrow}_{\alpha}[0,b)$. By condition D1 in Eq. (12), $D\widehat{\bigvee_{i\in I} \mathcal{V} f_i}$ is decreasing on $[0,b)$. Then we only need to show that $\int_0^b D\widehat{\bigvee_{i\in I} \mathcal{V} f_i}(t)\,dt = \alpha$. Since g is decreasing and non-negative on $[0,b)$, it follows from Proposition 2.7 that $\mathcal{V}g \in \widehat{Lip}[0,b)$, so that $\mathcal{V}g$ is concave on $[0,b)$. As a consequence of this and the hypothesis, we have

$$\bigvee_{i\in I} \mathcal{V} f_i \leq \widehat{\bigvee_{i\in I} \mathcal{V} f_i} \leq \mathcal{V} g. \tag{31}$$

By Eq. (31), we have $\alpha = \lim_{x\to b}(\mathcal{V} f_i)(x) \le \lim_{x\to b} \bigvee_{i\in I}(\mathcal{V} f_i)(x) \le \lim_{x\to b} \widehat{\bigvee_{i\in I}\mathcal{V} f_i}(x) \le \lim_{x\to b}(\mathcal{V} g)(x) = \int_0^b g(t)\,dt = \alpha$. Thus,

$$\lim_{x\to b} \widehat{\bigvee_{i\in I} \mathcal{V} f_i}(x) = \alpha \tag{32}$$

Considering Eq. (31) again, it follows that $\widehat{\bigvee_{i\in I}\mathcal{V} f_i}(0) = 0$. From this and, by combining Eq. (15) and Eq. (32), we obtain

$$\int_0^b D\widehat{\bigvee_{i\in I} \mathcal{V} f_i}(t)\,dt = \lim_{x\to b} \mathcal{V} D\widehat{\bigvee_{i\in I} \mathcal{V} f_i}(x) = \lim_{x\to b} \widehat{\bigvee_{i\in I} \mathcal{V} f_i}(x) = \alpha$$

Hence $D\widehat{\bigvee_{i\in I}\mathcal{V} f_i} \in L^{\downarrow}_{\alpha}[0,b)$. To prove that $D\widehat{\bigvee_{i\in I}\mathcal{V} f_i}$ is the supremum, in the sense of λ-majorization, of the set $(f_i)_{i\in I}$, one can proceed by arguments analogous to those used to prove Eq. (27) and Eq. (29) in Theorem 3.6. Finally, by Proposition 2.3-2, we have $\widehat{\bigvee_{j\in J}} f_j = \widehat{\bigvee_{j\in J} f_j}$. This completes the proof of this item.

□

The following proposition establishes a useful relation between the valuations of $\mathcal{L}$-terms in $\langle L^{\downarrow}_{\alpha}[0,b), \wedge^{\lambda}, \vee^{\lambda}\rangle$ and in $\langle Conc[0,b), \wedge, \widehat{\vee}\rangle$.

Proposition 3.8 *Let* $t(x_1,\dots,x_n) \in \mathrm{Term}_{\mathcal{L}}(X)$ *and* $0 < b \le \infty$. *For* $f_1,\dots,f_n \in L^{\downarrow}_{\alpha}[0,b)$ *the following statements hold:*

1. $t^{Conc[0,b)}(\mathcal{V} f_1,\dots,\mathcal{V} f_n) \in \widehat{Lip}[0,b)$.
2. $t^{L^{\downarrow}_{\alpha}[0,b)}(f_1,\dots,f_n) = D\,t^{Conc[0,b)}(\mathcal{V} f_1,\dots,\mathcal{V} f_n)$.

Proof: 1) We first prove that there exists $M > 0$ such that $t^{Conc[0,b)}(\mathcal{V} f_1,\dots,\mathcal{V} f_n)(x) \le Mx$. We proceed by induction on the complexity of $\mathcal{L}$-terms. If $Comp(t) = 0$, then t is a variable. In this case, $t^{Conc[0,b)}(\mathcal{V} f_1,\dots,\mathcal{V} f_n) = \mathcal{V} f_i$ for some $i = 1,\dots,n$. Since f_i is decreasing on $[0,b)$, it follows that $(\mathcal{V} f_i)(x) \le f_i(0)x$. Assume that the result holds for all terms with $Comp(t) < n$, and suppose that $Comp(t) = n$. There are two cases to consider.
Case 1: $t^{Conc[0,b)}(\mathcal{V} f_1,\dots,\mathcal{V} f_n) = t_1^{Conc[0,b)}(\mathcal{V} f_1,\dots,\mathcal{V} f_n) \wedge t_2^{Conc[0,b)}(\mathcal{V} f_1,\dots,\mathcal{V} f_n)$.

$$\begin{aligned} t^{Conc[0,b)}(\mathcal{V} f_1,\dots,\mathcal{V} f_n) &= t_1^{Conc[0,b)}(\mathcal{V} f_1,\dots,\mathcal{V} f_n) \wedge t_2^{Conc[0,b)}(\mathcal{V} f_1,\dots,\mathcal{V} f_n) \\ &\le M_1 x \wedge M_2 x \le \max\{M_1, M_2\}x. \end{aligned}$$

Case 2: $t^{Conc[0,b)}(\mathcal{V} f_1,\dots,\mathcal{V} f_n) = t_1^{Conc[0,b)}(\mathcal{V} f_1,\dots,\mathcal{V} f_n)\widehat{\vee} t_2^{Conc[0,b)}(\mathcal{V} f_1,\dots,\mathcal{V} f_n)$. In this case, observe that

$$t_1^{Conc[0,b)}(\mathcal{V} f_1,\dots,\mathcal{V} f_n) \vee t_2^{Conc[0,b)}(\mathcal{V} f_1,\dots,\mathcal{V} f_n) \le M_1 x \vee M_2 x \le \max\{M_1, M_2\}x = Mx.$$

Since Mx is concave, by the definition of concave envelopes and Proposition 2.3, it follows that

$$t_1^{Conc[0,b)}(\mathcal{V}f_1,\ldots,\mathcal{V}f_n)\widehat{\vee}t_2^{Conc[0,b)}(\mathcal{V}f_1,\ldots,\mathcal{V}f_n)\leq Mx.$$

Hence, by Proposition 2.5, $t^{Conc[0,b)}(\mathcal{V}f_1,\ldots,\mathcal{V}f_n)\in\widehat{Lip}[0,b)$.

2) We first observe that item 1 together with condition D2 in Eq. (12) guarantee that

$$\mathcal{V}Dt^{Conc[0,b)}(\mathcal{V}f_1,\ldots,\mathcal{V}f_n)=t^{Conc[0,b)}(\mathcal{V}f_1,\ldots,\mathcal{V}f_n). \tag{33}$$

To prove this item, we again proceed by induction on the complexity of $\mathcal{L}$-terms. If $Comp(t)=0$, then t is a variable, and this case follows immediately. Assume the result holds for all terms with $Comp(t)<n$, and suppose that $Comp(t)=n$. Taking into account Theorem 3.6-1 and Eq. (33), we consider two cases.

If $t(x_1,\ldots,x_n)=t_1(x_1,\ldots,x_n)\wedge t_2(x_1,\ldots,x_n)$, then $Comp(t_i)<n$ for each $i=1,2$. Then

$$\begin{aligned}
t^{L^{\downarrow}_{\alpha}[0,b)}(f_1,\ldots,f_n)&=t_1^{L^{\downarrow}_{\alpha}[0,b)}(f_1,\ldots,f_n)\wedge^{\lambda}t_2^{L^{\downarrow}_{\alpha}[0,b)}(f_1,\ldots,f_n)\\
&=D\left(\mathcal{V}t_1^{L^{\downarrow}_{\alpha}[0,b)}(f_1,\ldots,f_n)\wedge\mathcal{V}t_2^{L^{\downarrow}_{\alpha}[0,b)}(f_1,\ldots,f_n)\right)\\
&=D\left(\mathcal{V}D\,t_1^{Conc[0,b)}(\mathcal{V}f_1,\ldots,\mathcal{V}f_n)\wedge\mathcal{V}D\,t_2^{Conc[0,b)}(\mathcal{V}f_1,\ldots,\mathcal{V}f_n)\right)\\
&=D\left(t_1^{Conc[0,b)}(\mathcal{V}f_1,\ldots,\mathcal{V}f_n)\wedge t_2^{Conc[0,b)}(\mathcal{V}f_1,\ldots,\mathcal{V}f_n)\right)\\
&=D\,t^{Conc[0,b)}(\mathcal{V}f_1,\ldots,\mathcal{V}f_n).
\end{aligned}$$

If $t(x_1,\ldots,x_n)=t_1(x_1,\ldots,x_n)\vee t_2(x_1,\ldots,x_n)$, once more, we have $Comp(t_i)<n$ for each $i=1,2$. Then

$$\begin{aligned}
t^{L^{\downarrow}_{\alpha}[0,b)}(f_1,\ldots,f_n)&=t_1^{L^{\downarrow}_{\alpha}[0,b)}(f_1,\ldots,f_n)\vee^{\lambda}t_2^{L^{\downarrow}_{\alpha}[0,b)}(f_1,\ldots,f_n)\\
&=D\left(\mathcal{V}t_1^{L^{\downarrow}_{\alpha}[0,b)}(f_1,\ldots,f_n)\widehat{\vee}\mathcal{V}t_2^{L^{\downarrow}_{\alpha}[0,b)}(f_1,\ldots,f_n)\right)\\
&=D\left(\mathcal{V}D\,t_1^{Conc[0,b)}(\mathcal{V}f_1,\ldots,\mathcal{V}f_n)\widehat{\vee}\mathcal{V}D\,t_2^{Conc[0,b)}(\mathcal{V}f_1,\ldots,\mathcal{V}f_n)\right)\\
&=D\left(t_1^{Conc[0,b)}(\mathcal{V}f_1,\ldots,\mathcal{V}f_n)\widehat{\vee}t_2^{Conc[0,b)}(\mathcal{V}f_1,\ldots,\mathcal{V}f_n)\right)\\
&=D\,t^{Conc[0,b)}(\mathcal{V}f_1,\ldots,\mathcal{V}f_n).
\end{aligned}$$

Hence, $t^{L^{\downarrow}_{\alpha}[0,b)}(f_1,\ldots,f_n)=D\,t^{Conc[0,b)}(\mathcal{V}f_1,\ldots,\mathcal{V}f_n)$.

□

Item 2 in Proposition 3.8 provides a streamlined method for evaluating $\mathcal{L}$-terms in $L^{\downarrow}_{\alpha}[0,b)$. Specifically, if $t(x_1,\ldots,x_n)\in\mathrm{Term}_{\mathcal{L}}$ and $f_1,\ldots,f_n\in L^{\downarrow}_{\alpha}[0,b)$, then to compute $t^{L^{\downarrow}_{\alpha}[0,b)}(f_1,\ldots,f_n)$ in $L^{\downarrow}_{\alpha}[0,b)$, it suffices to evaluate $t^{Conc[0,b)}(\mathcal{V}f_1,\ldots,\mathcal{V}f_n)$ in $Conc[0,b)$ and then apply the operator D to the resulting function. As an application of this procedure, in the following example we study the non-distributivity of the lattice $L^{\downarrow}_{\alpha}[0,b)$ for $0<b\leq\infty$.

Example 3.9 [*Failure of distributivity in* $L^{\downarrow}_{\alpha}[0,b)$] To prove the failure of distributivity in $L^{\downarrow}_{\alpha}[0,b)$ for all $0 < b \le \infty$, we begin by showing that the first condition in Eq.(1) does not hold in the particular case of $L^{\downarrow}_{2}[0,4)$. Consider the following decreasing step functions on $L^{\downarrow}_{2}[0,4)$:

$$f_1(x) = \begin{cases} 1, & 0 \le x < 1, \\ \frac{1}{3}, & 1 \le x < 4. \end{cases} \quad f_2(x) = \begin{cases} \frac{4}{5}, & 0 \le x < 1, \\ \frac{8}{15}, & 1 \le x < 2, \\ \frac{7}{15}, & 2 \le x < 3, \\ \frac{1}{5}, & 3 \le x < 4. \end{cases} \quad f_3(x) = \begin{cases} \frac{2}{3}, & 0 \le x < 3, \\ 0, & 3 \le x < 4. \end{cases}$$

Then

$$\mathcal{V}f_1(x) = \begin{cases} x, & 0 \le x < 1, \\ \frac{1}{3}x + \frac{2}{3}, & 1 \le x < 4. \end{cases} \quad \mathcal{V}f_2(x) = \begin{cases} \frac{4}{5}x, & 0 \le x < 1, \\ \frac{8}{15}x + \frac{4}{15}, & 1 \le x < 2, \\ \frac{7}{15}x + \frac{2}{5}, & 2 \le x < 3, \\ \frac{1}{5}x + \frac{6}{5}, & 3 \le x < 4. \end{cases}$$

$$\mathcal{V}f_3(x) = \begin{cases} \frac{2}{3}x, & 0 \le x < 3, \\ 2, & 3 \le x < 4. \end{cases}$$

By a straightforward calculation, we obtain

$$(\mathcal{V}f_1 \wedge (\mathcal{V}f_2 \widehat{\vee} \mathcal{V}f_3))(x) = \begin{cases} \frac{4}{5}x + \frac{4}{15}, & 0 \le x < 1, \\ \frac{3}{5}x + \frac{1}{5}, & 1 \le x < \frac{7}{4}, \\ \frac{1}{3}x + \frac{2}{3}, & \frac{7}{4} \le x < 4. \end{cases}$$

Thus, by Theorem 3.8-2 we have that

$$f_1 \wedge^{\lambda} (f_2 \vee^{\lambda} f_3)(x) = D\,(\mathcal{V}f_1 \wedge (\mathcal{V}f_2 \widehat{\vee} \mathcal{V}f_3))(x) = \begin{cases} \frac{4}{5}, & 0 \le x < 1, \\ \frac{3}{5}, & 1 \le x < \frac{7}{4}, \\ \frac{1}{3}, & \frac{7}{4} \le x < 4. \end{cases} \tag{34}$$

By a straightforward calculation, we also obtain

$$\Big((\mathcal{V}f_1 \wedge \mathcal{V}f_2) \widehat{\vee} (\mathcal{V}f_1 \wedge \mathcal{V}f_3)\Big)(x) = \begin{cases} \frac{4}{5}x, & 0 \le x < 1, \\ \frac{8}{15}x + \frac{4}{15}, & 1 \le x < 2, \\ \frac{1}{3}x + \frac{2}{3}, & 2 \le x < 4. \end{cases}$$

Thus, by Theorem 3.8-2 we have that

$$\begin{aligned} (f_1 \wedge^{\lambda} f_2) \vee^{\lambda} (f_1 \wedge^{\lambda} f_3)(x) &= D\Big((\mathcal{V}f_1 \wedge \mathcal{V}f_2) \widehat{\vee} (\mathcal{V}f_1 \wedge \mathcal{V}f_3)\Big)(x) \\ &= \begin{cases} \frac{4}{5} & 0 \le x < 1, \\ \frac{8}{15}, & 1 \le x < 2, \\ \frac{1}{3}, & 2 \le x < 4. \end{cases} \end{aligned} \tag{35}$$

Then, by Eq.(34) and Eq.(35), we have

$$f_1 \wedge^\lambda (f_2 \vee^\lambda f_3) \neq (f_1 \wedge^\lambda f_2) \vee^\lambda (f_1 \wedge^\lambda f_3).$$

which shows that $L_2^\downarrow[0,4)$ is not a distributive lattice.

We now use f_1, f_2, and f_3 to extend the above argument and establish the failure of distributivity in $L_\alpha^\downarrow[0,b)$ for any $0 < b \leq \infty$ and $\alpha > 0$. We distinguish two cases.

> *Case*: $L_\alpha^\downarrow[0,\infty)$. Extend f_i by zero on $[4,\infty)$ for $i = 1,2,3$ and consider the functions $(\frac{\alpha}{2} f_i)_{i=1,2,3}$. Observe that $\frac{\alpha}{2} f_i \in L_\alpha^\downarrow[0,\infty)$ and, by an argument analogous to the case of $L_2^\downarrow[0,4)$, one can show that $\frac{\alpha}{2} f_1$, $\frac{\alpha}{2} f_2$, $\frac{\alpha}{2} f_3$ do not satisfy the distributivity condition in $L_\alpha^\downarrow[0,\infty)$.
>
> *Case*: $L_\alpha^\downarrow[0,b)$ where $0 < b < \infty$. Consider $(\frac{\alpha b}{8} f_i(\frac{4}{b}x))_{i=1,2,3}$. It is straightforward to verify that $\frac{\alpha b}{8} f_i(\frac{4}{b}x) \in L_\alpha^\downarrow[0,b)$ for $i = 1,2,3$ and, by an argument analogous to the case of $L_2^\downarrow[0,4)$, one can show that $\frac{\alpha b}{8} f_1(\frac{4}{b}x)$, $\frac{\alpha b}{8} f_2(\frac{4}{b}x)$, $\frac{\alpha b}{8} f_3(\frac{4}{b}x)$ do not satisfy the distributivity condition in $L_\alpha^\downarrow[0,b)$.

Hence, $L_\alpha^\downarrow[0,b)$ is not a distributive lattice for any $0 < b \leq \infty$ and $\alpha > 0$.

In the final part of the section, we focus on the extremal elements of the lattice $\langle L_\alpha^\downarrow[0,b), \wedge^\lambda, \vee^\lambda\rangle$.

Proposition 3.10 *For $0 < b \leq \infty$, the lattice $\langle L_\alpha^\downarrow[0,b), \wedge^\lambda, \vee^\lambda\rangle$ has no maximum element.*

Proof: Suppose that m is the maximum element of $\langle L_\alpha^\downarrow[0,b), \preceq_\lambda\rangle$. We first prove that

$$\frac{\alpha}{m(0^+)} \leq b. \tag{36}$$

Assume first that $b < \infty$. Since m is decreasing, we have $m(x) \leq m(0^+)$ for all $x \in (0,b)$, and therefore $\alpha = \int_0^b m(t)\,dt \leq m(0^+)\,b$. The case $b = \infty$ is immediate. In view of Eq. (36), we may now define the following step function:

$$m_2(x) = \begin{cases} 2m(0^+), & \text{if} \quad 0 \leq x \leq \frac{\alpha}{2m(0)}, \\ 0, & \text{if} \quad \frac{\alpha}{2m(0)} < x < b. \end{cases}$$

Note that $m_2(0^+) = m_2(0) \nleq m(0^+)$. Therefore, by Proposition 3.4, $m_2 \npreceq_\lambda m$, which contradicts the maximality of m. Hence, $L_\alpha^\downarrow[0,b)$ does not admit a maximum.

□

Proposition 3.11 *1. If $0 < b < \infty$ then the constant function $\mathbf{0}_\alpha(x) = \frac{\alpha}{b}$ in $[0,b)$ is the minimum element of $\langle L_\alpha^\downarrow[0,b), \wedge^\lambda, \vee^\lambda\rangle$.*

2. $\langle L_\alpha^\downarrow[0,\infty), \wedge^\lambda, \vee^\lambda\rangle$ has no minimum element.

Proof: 1) We first observe that $\mathbf{0}_\alpha \in L^\downarrow_\alpha[0,b)$. Let $f \in L^\downarrow_\alpha[0,b)$. Since $b < \infty$, by Proposition 2.7, it follows that $\mathcal{V}f$ is a concave function on $[0,b]$. Let $x \in [0,b]$. Then we can write $x = \beta b + (1-\beta)0$ and $\beta = \frac{x}{b}$ where $\beta \in [0,1]$. By the concavity of $\mathcal{V}f$, we obtain

$$\begin{aligned}(\mathcal{V}f)(x) &= (\mathcal{V}f)(\beta b + (1-\beta)0)\\ &\geq \beta(\mathcal{V}f)(b) = \frac{x}{b}\alpha = \int_0^x \frac{\alpha}{b}\,dt = (\mathcal{V}\mathbf{0}_\alpha)(x).\end{aligned}$$

Thus $\mathbf{0}_\alpha \preceq_\lambda f$ and therefore, $\mathbf{0}_\alpha$ is the minimum element in $L^\downarrow_\alpha[0,b)$.

2) For each $n \in \mathbb{N}$, define

$$f_n(x) = \begin{cases} -\frac{2\alpha}{n^2}x + \frac{2\alpha}{n}, & \text{if } \ 0 \leq x \leq n,\\ 0, & \text{if } \ x > n.\end{cases}$$

Observe that $f_n \in L^\downarrow_\alpha[0,\infty)$. Suppose that there exists $h \in L^\downarrow_\alpha[0,\infty)$ such that $h \preceq_\lambda f_n$ for all $n \in \mathbb{N}$. Then, by Proposition 3.4, it follows that $h(0^+) \leq f_n(0^+) = f_n(0) = \frac{2\alpha}{n}$ for all $n \in \mathbb{N}$. Consequently, $h(0^+) = 0$ and, since h is decreasing on $[0,\infty)$, we must have $h = 0$ on $(0,\infty)$. Hence $h \notin L^\downarrow_\alpha[0,\infty)$, which is a contradiction. Therefore, the sequence $(f_n)_{n\in\mathbb{N}}$ does not admit a lower bound in $L^\downarrow_\alpha[0,\infty)$. This shows that $L^\downarrow_\alpha[0,\infty)$ is not bounded below.

□

To summarize the properties established in Theorem 3.7, Propositions 3.10, Proposition 3.11 and to compare them with the majorization lattices $\Delta^\downarrow_d$ and $\Delta^\downarrow_\infty$, we present the following synthesis:

- □ For $0 < b < \infty$, $\langle L^\downarrow_\alpha[0,b), \wedge^\lambda, \vee^\lambda, \mathbf{0}_\alpha\rangle$ is a non-distributive, $\bigwedge^\lambda$-complete and conditionally $\bigvee^\lambda$-complete lattice that is bounded below but not bounded above.
- □ $\langle L^\downarrow_\alpha[0,\infty), \wedge^\lambda, \vee^\lambda\rangle$ is a non-distributive, unbounded, conditionally complete lattice.

4 $\langle \wedge^\lambda, \vee^\lambda\rangle$ on continuous strictly unimodal functions

A difficulty in handling $\wedge^\lambda$ and $\vee^\lambda$ in concrete settings, such as probability distributions, is that their computation requires an estimate of the decreasing rearrangement, which can be difficult to obtain. The case of continuous strictly unimodal functions is particularly interesting, since it allows us to bypass the decreasing rearrangement and, moreover, provides an abstract framework encompassing important families of probability distributions, such as the Erlang distributions.

We begin by presenting specific technical results on the decreasing rearrangement, which will be important throughout the section.

Theorem 4.1 [Layer-cake representation] *Let $f \in L^1[0,b)$ with $0 < b \leq \infty$ and $f \geq 0$, and let $E \subseteq [0,b)$ be a Lebesgue measurable set. Then*

$$\int_E f(t)\,dt = \int_0^\infty \lambda(E \cap S^f_\alpha)\,d\alpha.$$

□

Proposition 4.2 *Let $f \in L^1[0,b)$ where $f \geq 0$ and $0 < b \leq \infty$. Then for each $x \in [0,b)$ we have*

$$(\mathcal{V}f^{\downarrow})(x) \;=\; \int_0^{\infty} \min\{x, \delta_f(\alpha)\}\, d\alpha.$$

Proof: By Definition (3.1) we have that $f^{\downarrow}(t) > \alpha$ iff $\sup\{\beta : \delta_f(\beta) > t\} > \alpha$ iff $\delta_f(\alpha) > t$. Therefore, $\{t \geq 0 : \; f^{\downarrow}(t) > \alpha\} \;=\; \{t \geq 0 : \; \delta_f(\alpha) > t\} \;=\; (0, \delta_f(\alpha))$ and consequently

$$\begin{aligned} \lambda\Big([0,x] \cap \{t \geq 0 : \; f^{\downarrow}(t) > \alpha\}\Big) \;&=\; \lambda\Big([0,x] \cap (0, \delta_f(\alpha))\Big) \\ &=\; \min\{x, \delta_f(\alpha)\}. \end{aligned} \tag{37}$$

Thus, by Theorem 4.1 and Eq.(37) we have

$$(\mathcal{V}f^{\downarrow})(x) = \int_0^x f^{\downarrow}(t)\, dt = \int_0^{\infty} \lambda\Big([0,x] \cap (0, \delta_f(\alpha))\Big)\, d\alpha = \int_0^{\infty} \min\{x, \delta_f(\alpha)\}\, d\alpha.$$

□

Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function. The function f is said to be *unimodal* iff there exists a point $x_0 \in A$ such that f is non-decreasing on $\{x \in A : x \leq x_0\}$ and non-increasing on $\{x \in A : x \geq x_0\}$. In other words, f increases up to a single mode (or peak) at x_0 and decreases thereafter. If, moreover, x_0 is unique *and* f is strictly increasing on $\{x \in A : x < x_0\}$ and strictly decreasing on $\{x \in A : x > x_0\}$, then f is said to be *strictly unimodal*.

Proposition 4.3 *Let $f : [0,b) \to [0,\infty)$ be a continuous and strictly unimodal function with a unique maximum at x_0, where $0 < b \leq \infty$. For each $\alpha \geq 0$ consider S^f_α the α-superlevel set of f Then:*

1. $S^f_\alpha = \begin{cases} \emptyset, & \text{if } \alpha \geq f(x_0), \\ (a_\alpha, b_\alpha) & \text{if } \alpha < f(x_0) \qquad \text{where } 0 \leq a_\alpha < x_0 < b_\alpha \leq b. \end{cases}$

2. *$\delta_f(\alpha) = b_\alpha - a_\alpha$ and the restriction $\delta_f|_{(\lim_{x\to b} f(x), f(x_0)]}$ define a strictly decreasing homeomorphism onto $[0,b)$.*

3. *$f(a_\alpha) = f(b_\alpha) = \alpha$.*

Proof: 1) If $\alpha \geq f(x_0)$, it follows directly from Eq. (16) that $S^f_\alpha = \emptyset$. To analyze the case $\alpha < f(x_0)$, we consider the restrictions $f|_{[0,x_0]} : [0,x_0] \to [f(0), f(x_0)]$ and $f|_{[x_0,b)} : [x_0,b) \to (\lim_{x\to b} f(x), f(x_0)]$. Because f is continuous and strictly monotone on each side of x_0, the mappings $f|_{[0,x_0]}$ and $f|_{[x_0,b)}$ are both homeomorphisms, and their inverses

$$\begin{aligned} [f(0), f(x_0)] \ni \alpha \quad &\mapsto \quad f|^{-1}_{[0,x_0]}(\alpha) = a_\alpha, \\ (\lim_{x\to b} f(x), f(x_0)] \ni \alpha \quad &\mapsto \quad f|^{-1}_{[x_0,b)}(\alpha) = b_\alpha \end{aligned} \tag{38}$$

exist, are continuous and strictly monotone. We next distinguish three possible subcases:

Subcase 1: $\alpha \in [f(0), f(x_0)) \cap (\lim_{x\to b} f(x), f(x_0))$, that is, α corresponds to a horizontal line intersecting the graph of f at two distinct points, one on each side of x_0. By Eq. (38), there exist unique points $a_\alpha \in [0, x_0]$ and $b_\alpha \in [x_0, b)$, that is, $0 \le a_\alpha < x_0 < b_\alpha < b$, such that $f(a_\alpha) = f(b_\alpha) = \alpha$. Thus, from the definition of S^f_α, together with the continuity of f and the fact that it is strictly increasing on $(a_\alpha, x_0]$ and strictly decreasing on $[x_0, b_\alpha)$, it follows that $S^f_\alpha = (a_\alpha, b_\alpha)$.

Subcase 2: $\alpha < f(0)$, that is, α corresponds to a horizontal line intersecting the graph of f at most once, to the right of x_0. In this case $(0, x_0] \subseteq S^f_\alpha$. If $\alpha < \lim_{x\to b} f(x)$, it follows immediately that $S^f_\alpha = (0, b)$. If instead $\alpha \in (\lim_{x\to b} f(x), f(x_0))$, then by Eq. (38) there exists a unique $b_\alpha \in [x_0, b)$ such that $\alpha = f(b_\alpha)$. Hence, from the definition of S^f_α, together with the continuity of f and the fact that f is strictly decreasing on $[x_0, b_\alpha)$, we conclude that $S^f_\alpha = (0, b_\alpha)$.

Subcase 3: $\alpha < \lim_{x\to b} f(x)$, that is, α corresponds to a horizontal line intersecting the graph of f at most once, to the left of x_0. In this case, $[x_0, b) \subseteq S^f_\alpha$. If $\alpha < f(0)$ then, by Subcase 2, it follows that $S^f_\alpha = (0, b)$. For $\alpha \in [f(0), f(x_0))$, Eq. (38) ensures the existence of a unique $a_\alpha \in [0, x_0]$ such that $f(a_\alpha) = \alpha$. Therefore, by the definition of S^f_α, together with the continuity of f and its strict monotonicity on $[0, x_0]$, we obtain $S^f_\alpha = (a_\alpha, b)$.

2) By item 1, it follows immediately that the image of δ_f is contained in $[0, b)$, and moreover,

$$\delta_f(\alpha) = b_\alpha - a_\alpha. \tag{39}$$

As shown in Eq. (38), $f|^{-1}_{[0,x_0]}(\alpha) = a_\alpha$ is continuous and satisfies $f|^{-1}_{[0,x_0]}(f(0)) = 0$. We may therefore extend $f|^{-1}_{[0,x_0]}$ to the entire interval $[0, f(x_0)]$ by defining $f|^{-1}_{[0,x_0]}(\alpha) = a_\alpha = 0$ for all $0 \le \alpha < f(0)$. Accordingly, the mapping

$$[0, f(x_0)] \ni \alpha \mapsto a_\alpha = \begin{cases} 0, & \text{if } 0 \le \alpha < f(0), \\ f|^{-1}_{[0,x_0]}(\alpha), & \text{if } f(0) \le \alpha < f(x_0), \end{cases} \tag{40}$$

is continuous. If $f(0) \le \lim_{x\to b} f(x)$, then by Eqs. (39) and (38) it follows immediately that $\delta_f|_{(\lim_{x\to b} f(x), f(x_0)]}$ is continuous. If instead $f(0) > \lim_{x\to b} f(x)$, the same conclusion follows from Eqs. (40) and (38). Hence, this restriction defines a continuous mapping of the form

$$\delta_f : (\lim_{x\to b} f(x), f(x_0)] \to [0, b).$$

For any $\lim_{x\to b} f(x) < \alpha_1 < \alpha_2 \le f(x_0)$ we have $a_{\alpha_1} < a_{\alpha_2}$ and $b_{\alpha_1} > b_{\alpha_2}$. Hence, $\delta_f(\alpha_1) = b_{\alpha_1} - a_{\alpha_1} > b_{\alpha_2} - a_{\alpha_2} = \delta_f(\alpha_2)$, showing that δ_f is strictly decreasing on $(\lim_{x\to b} f(x), f(x_0)]$. Moreover, $\lim_{\alpha\nearrow f(x_0)} a_\alpha = x_0$ and $\lim_{\alpha\nearrow f(x_0)} b_\alpha = x_0$, and therefore $\lim_{\alpha\nearrow f(x_0)} \delta_f(\alpha) = 0$. We thus conclude that the restriction $\delta_f|_{(\lim x\to b f(x), f(x_0)]}$ is bijective, and hence a homeomorphism.

3) Since f is strictly increasing on $[0, x_0]$, with $f|[0, a_\alpha) < \alpha$ and $f|(a_\alpha, x_0] > \alpha$, the continuity of f implies that $f(a_\alpha) = \alpha$. Similarly, as f is strictly decreasing on $[x_0, b)$, an analogous argument shows that $f(b_\alpha) = \alpha$ as well.

□

Theorem 4.4 *Let $f : [0, b) \to [0, \infty)$ be a continuous, strictly unimodal function with a unique maximum at x_0, where $0 < b \leq \infty$. Then, for each $x \in (0, b)$, there exists a unique $a_x \in [0, x_0]$ such that $f(a_x) = f(a_x + x)$. Moreover,*

$$(\mathcal{V} f^{\downarrow})(x) = \int_{a_x}^{a_x+x} f(t)\, dt.$$

Proof: Let $x \in [0, b)$. By Proposition 4.3-2, there exists a unique $\alpha_x \in (\lim_{t\to b} f(t),\ f(x_0)]$ such that $\delta_f(\alpha_x) = x$. Then, by Proposition 4.3-1, S_{α_x} is an interval of the form $S_{\alpha_x} = (a_x, b_x)$ satisfying $x = b_x - a_x$, i.e., $b_x = a_x + x$. By Proposition 4.3-3, we have $f(a_x) = f(a_x + x)$. Since $a_x \leq x_0 \leq a_x + x$ and, f is strictly increasing on $[0, x_0]$ and strictly decreasing on $[x_0, b)$, it follows that a_x is the unique point in $[0, x_0]$ satisfying this equality. We first prove that

$$\lambda([a_x, a_x + x] \cap S^f_\alpha) = \begin{cases} \delta_f(\alpha), & \text{if } \delta_f(\alpha) \leq x, \\ x, & \text{if } \delta_f(\alpha) > x. \end{cases}$$

If $\delta_f(\alpha) \leq x = \delta_f(\alpha_x)$ then $S^f_\alpha \subseteq S_{\alpha_x} = [a_x, a_x + x]$. Thus, $\lambda([a_x, a_x + x] \cap S^f_\alpha) = \lambda(S^f_\alpha) = \delta_f(\alpha)$. While if $\delta_f(\alpha) > x = \delta_f(\alpha_x)$ then $[a_x, a_x + x] = S_{\alpha_x} \subseteq S^f_\alpha$. Thus, $\lambda([a_x, a_x + x] \cap S^f_\alpha) = \lambda([a_x, a_x + x]) = x$. It shows that

$$\lambda([a_x, a_x + x] \cap S^f_\alpha) = \min\{x, \delta_f(\alpha)\}. \tag{41}$$

Hence, by Eq.(41) and Proposition 4.2 we have

$$\int_{a_x}^{a_x+x} f(t)\, dt = \int_0^\infty \lambda([a_x, a_x + x] \cap S^f_\alpha)\, d\alpha = \int_0^\infty \min\{x, \delta_f(\alpha)\}\, d\alpha = (\mathcal{V} f^{\downarrow})(x).$$

□

Note that Theorem 4.4 provides an expression for $\mathcal{V} f^{\downarrow}$ without relying on the decreasing rearrangement. Combining this result with Theorem 3.6 yields an explicit form of $\langle \wedge^\lambda, \vee^\lambda \rangle$ for non negative, continuous, strictly unimodal functions, as stated in the following theorem.

Theorem 4.5 *Let $f_1, f_2 : [0, b) \to [0, \infty)$, where $0 < b \leq \infty$, be continuous, strictly unimodal functions with unique maxima x_{0_1} and x_{0_2} in $(0, b)$, respectively. For each $x \in [0, b)$, let $a_{x_i} \in [0, x_{0_i}]$ be such that $f_i(a_{x_i}) = f_i(a_{x_i} + x)$ for $i = 1, 2$. Then*

$$\begin{aligned} (f_1 \wedge^\lambda f_2)(x) &= D \bigwedge_{i=1,2} \int_{a_{x_i}}^{a_{x_i}+x} f_i(t)\, dt, \\ (f_1 \vee^\lambda f_2)(x) &= D \widehat{\bigvee_{i=1,2}} \int_{a_{x_i}}^{a_{x_i}+x} f_i(t)\, dt. \end{aligned} \tag{42}$$

□

In the above theorem, when a_{x_1} and a_{x_2} can be explicitly determined or accurately estimated from f_1 and f_2, the computation of $f_1 \wedge^\lambda f_2$ and $f_1 \vee^\lambda f_2$ is greatly simplified, as it avoids resorting to the decreasing rearrangements $f_1^{\downarrow}$ and $f_2^{\downarrow}$.

Example 4.6 [*Erlang distributions*] The Erlang distributions form a family of probability distributions that generalize the exponential distribution. While an exponential random variable models the waiting time between consecutive events, an Erlang random variable describes the waiting time between an event and the k-th subsequent event. Erlang distributions arise naturally in Poisson processes and are widely used in queueing theory, mathematical biology, and other applied areas. The probability density function of an Erlang distribution with parameters $k \in \mathbb{N}_{\geq 1}$ and $l > 0$ is given by

$$f_{k,l}(x) = \frac{l^k}{(k-1)!}\, x^{k-1} e^{-lx}, \qquad x \geq 0.$$

where k denotes the *shape parameter* and l the *rate parameter*.

It is well known that the family $(f_{k,l})_{k>1}$ constitutes a particular class of strictly unimodal functions. Consequently, Theorem 4.5 can be applied to derive explicit expressions for $\wedge^\lambda$ and $\vee^\lambda$. For fixed $k > 1$ and l, using standard arguments, we can see that the maximum of $f_{k,l}(x)$ is attained at

$$x_0 = \frac{k-1}{l}, \qquad f_{k,l}(x_0) = l\frac{(k-1)^{k-1}}{(k-1)!} e^{-(k-1)}. \tag{43}$$

For each $x \in [0, \infty)$, a straightforward calculation shows that the unique $a_x \in [0, x_0]$ such that $f_{k,l}(a_x) = f_{k,l}(a_x + x)$ is given by

$$a_x = \begin{cases} 0, & \text{if } k = 1, \\ \frac{x}{e^{\frac{lx}{k-1}}-1}, & \text{if } k > 1. \end{cases} \tag{44}$$

Thus, for $f_{k_1,l_1}(x)$ and $f_{k_2,l_2}(x)$ with $k_1, k_2 > 1$, Theorem 4.5 and Eq.(44) yields

$$\begin{aligned} (f_{k_1,l_1} \wedge^\lambda f_{k_2,l_2})(x) &= D \bigwedge_{i=1,2} \int_{a_{x_i}}^{a_{x_i}+x_i} \frac{l_i^{k_i}}{(k_i-1)!}\, t^{k_i-1} e^{-l_i t}\, dt, \\ (f_{k_1,l_1} \vee^\lambda f_{k_2,l_2})(x) &= D \widehat{\bigvee_{i=1,2}} \int_{a_{x_i}}^{a_{x_i}+x_i} \frac{l_i^{k_i}}{(k_i-1)!}\, t^{k_i-1} e^{-l_i t}\, dt. \end{aligned} \tag{45}$$

Now, by considering Eq.(45), we will calculate $\wedge^\lambda$ and $\vee^\lambda$ for the specific case of $f_{1,\frac{1}{4}}$ and $f_{2,\frac{1}{2}}$. Note that

$$(\mathcal{V} f^{\downarrow}_{1,\frac{1}{4}})(x) = \frac{1}{4}\int_0^x e^{-\frac{t}{4}}\, dt = 1 - e^{-\frac{x}{4}} \tag{46}$$

$$(\mathcal{V} f^{\downarrow}_{2,\frac{1}{2}})(x) = \frac{1}{4}\int_{\frac{x}{e^{x/2}-1}}^{\frac{x}{e^{x/2}-1}+x} t e^{-\frac{t}{2}}\, dt = e^{-\frac{x}{2}\left(1+\frac{1}{e^{x/2}-1}\right)}\,(e^{x/2}-1). \tag{47}$$

We can verify that the only intersection of the graphs of the two functions in $[0, \infty)$ occurs at

$$a_0 \approx 3.9936354442081754.$$

Thus, to compute $f_{1,\frac{1}{4}} \wedge^{\lambda} f_{2,\frac{1}{2}}$, we first observe that

$$(\mathcal{V}f^{\downarrow}_{1,\frac{1}{4}} \wedge \mathcal{V}f^{\downarrow}_{2,\frac{1}{2}})(x) = \begin{cases} e^{-\frac{x}{2}\left(1+\frac{1}{e^{x/2}-1}\right)}(e^{x/2}-1) & 0 \le x < a_0, \\ 1-e^{-x/4}, & a_0 \le x. \end{cases}$$

Hence $f_{1,\frac{1}{4}} \wedge^{\lambda} f_{2,\frac{1}{2}}$ is give by

$$(f_{1,\frac{1}{4}} \wedge^{\lambda} f_{2,\frac{1}{2}})(x) = D\,(\mathcal{V}f^{\downarrow}_{1,\frac{1}{4}} \wedge \mathcal{V}f^{\downarrow}_{2,\frac{1}{2}})(x) = \begin{cases} \frac{1}{4}, & x = 0, \\ \dfrac{xe^{-\frac{x}{2(e^{x/2}-1)}}}{4\,(e^{x/2}-1)^2}, & 0 < x < a_0, \\ \frac{1}{4}\,e^{-x/4}, & a_0 < x. \end{cases} \tag{48}$$

Next, in order to compute $f_{1,\frac{1}{4}} \vee^{\lambda} f_{2,\frac{1}{2}}$, we begin by observing

$$(\mathcal{V}f^{\downarrow}_{1,\frac{1}{4}} \vee \mathcal{V}f^{\downarrow}_{2,\frac{1}{2}})(x) = \begin{cases} 1-e^{-x/4}, & 0 \le x \le a_0, \\ e^{-\frac{x}{2}\left(1+\frac{1}{e^{x/2}-1}\right)}(e^{x/2}-1), & a_0 < x. \end{cases}$$

In this way the concave envelope of $\mathcal{V}f^{\downarrow}_{1,\frac{1}{4}} \vee \mathcal{V}f^{\downarrow}_{2,\frac{1}{2}}$ is given by

$$(\mathcal{V}f^{\downarrow}_{1,\frac{1}{4}} \mathbin{\widehat{\vee}} \mathcal{V}f^{\downarrow}_{2,\frac{1}{2}})(x) = \begin{cases} 1-e^{-x/4}, & 0 \le x < a_1, \\ m\,x+b, & a_1 \le x \le a_2, \\ e^{-\frac{x}{2}\left(1+\frac{1}{e^{x/2}-1}\right)}(e^{x/2}-1), & a_2 < x. \end{cases}$$

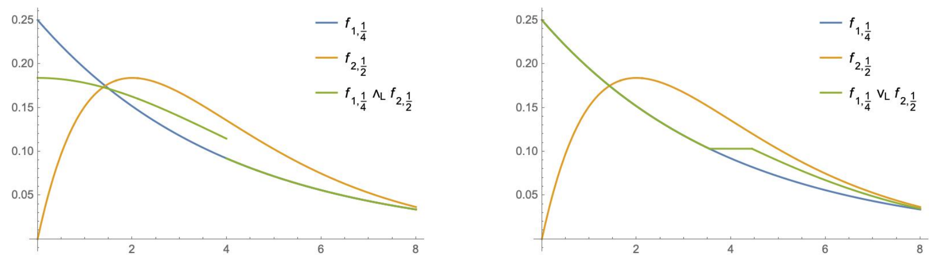
where

$$a_1 \approx 3.54205, \quad m \approx 0.10316718375,$$
$$a_2 \approx 4.440225, \quad b \approx 0.22207396534.$$

Hence $f_{1,\frac{1}{4}} \vee^{\lambda} f_{2,\frac{1}{2}}$ is give by

$$(f_{1,\frac{1}{4}} \vee^{\lambda} f_{2,\frac{1}{2}})(x) = D\,(\mathcal{V}f^{\downarrow}_{1,\frac{1}{4}} \mathbin{\widehat{\vee}} \mathcal{V}f^{\downarrow}_{2,\frac{1}{2}})(x) = \begin{cases} \frac{1}{4}\,e^{-x/4}, & 0 \le x < a_1, \\ m, & a_1 \le x \le a_2 \\ \dfrac{xe^{-\frac{x}{2(e^{x/2}-1)}}}{4\,(e^{x/2}-1)^2}, & a_2 < x. \end{cases} \tag{49}$$

The following figures show the plots of $f_{1,\frac{1}{4}} \wedge^{\lambda} f_{2,\frac{1}{2}}$ and $f_{1,\frac{1}{4}} \vee^{\lambda} f_{2,\frac{1}{2}}$, respectively.



Declarations

Ethical Approval: not applicable.
Funding: This work is supported by the following projects: DeKLA: Developing Kleene Logics and their Applications - Code: 2022SM4XC8- CUP: F53D23004840006, and Q-NACS – Quantumness in Natural and Cognitive Structures (Fondazione di Sardegna) - CUP: F83C26000350007.
Availability of data and materials: not applicable.

References

[1] Bapat R.B., Majorization and singular values. Linear Algebra and its Applications **145**, 59–70 (1991).

[2] Bosyk G.M., Freytes H., Bellomo G., Sergioli G.: The lattice of trumping majorization for 4D probability vectors and 2D catalysts. Sci Rep 8, 3671 (2018).

[3] Bosyk G.M., Bellomo G., Holik F., H. Freytes, Sergioli G.: Optimal common resource in majorization-based resource theories: New Journal of Physics 21(8) (2019).

[4] Cicalese F., Gargano L., Vaccaro U.: Information theoretic measures of distances and their econometric applications. Proceedings of ISIT 2013, 40 (2013).

[5] Chong K.M.: Some extensions of a theorem of Hardy, Littlewood and Pólya and their applications. Can. J. Math. **26**, 1321-1340 (1974).

[6] Hardy, G.H.: Some simple inequalities satisfied by convex functions. Messenger Math. **58**, 145-152 (1929).

[7] Hardy G.H., Littlewood J.E., Pólya G.: Inequalities. University Press, Edinburgh, UK, (1952).

[8] Manjegani S.M., Moein S.: Quasi doubly stochastic operator on l1 and Nielsen's theorem. J. Math. Phys. **60**, 103508 (2019)

[9] Manjegani S.M., Moein S.: Majorization and semidoubly stochastic operators on $L^1(X)$. J Inequal Appl **27** (2023).

[10] Masari C., Bellomo G., Freytes H., Giuntini R., Sergioli G., Bosyk G.M.: LOCC convertibility of entangled states in infinite-dimensional systems. New Journal of Physics **26**(6) (2024).

[11] Mosler K.: Majorization in economic disparity measures. Linear Algebra and its Applications, **199**, 91-114 (1994).

[12] Nielsen M., Vidal G.: Majorization and the interconversion of bipartite states. Quant. Inf. Comput. **1**, 76–93 (2001).

[13] Nielsen, M.: An introduction to majorization and its applications to quantum mechanics. Unpublished manuscript, (2002). Available at: https://michaelnielsen.org/papers/maj-book-notes.pdf

[14] Owari M., Braunstein S.L., Nemoto K., Murao M.: ϵ-convertibility of entangled states and extension of Schmidt rank in infinite-dimensional systems. Information and Computation, Vol 81, 30-52 (2008).

[15] Parker D. S., Ram P.: The construction of Huffman codes is a submodular (convex) optimization problem over a lattice of binary trees. SIAM J. Comput. **28**, 1875-1905 (1999).

[16] Pereira R., Plosker S.: Extending a characterization of majorization to infinite dimensions. Linear Algebra and its Applications, **468**, 80–86 (2015).

[17] Van Erven T., Harremoes P.: Rényi divergence and majorization. In 2010 IEEE International Symposium on Information Theory, 1335-1339, IEEE (2010).

[18] Van Herstraeten Z., Jabbour M. G., Cerf N. J.: Continuous majorization in quantum phase space, Quantum **7**, 1021 (2023).