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SHARP AND UNSHARP STRUCTURES. A UNIFYING FRAMEWORK FOR ALGEBRAIC LOGIC

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A mia madre e mio padre
per il loro sostegno incondizionato.

"C'est le temps que tu as perdu pour ta rose qui fait ta rose si importante."

— Antoine de Saint-Exupéry,
Le Petit Prince

Contents

Abstract

This work lies within the realm of algebraic logic, focusing on structures that are central in their respective fields and extensively studied. In the first part, it investigates the categorical equivalence between (indexed) Boolean algebras and regular double Stone algebras. Despite their strong categorical relation, distinct model-theoretic aspects are identified. However, their structural theories demonstrate a cohesive treatment, preserving significant elementary properties such as finite categoricity. Additionally, pivotal classes in algebraic logic, including Boolean algebras, Stone algebras, Kleene algebras, and 3-valued MV algebras, find ample representation through injective hulls in regular double Stone algebras. In the second part, the focus shifts to structures relevant in the context of quantum logics. Starting from the concept of a block in an orthomodular lattice, the aim is to capture a smooth generalization of the theory of orthomodular lattices in the case of non-orthocomplemented lattices. Under certain conditions, the theory of orthomodular lattices seamlessly extends, elucidating the natural transition in the presence of smooth conditions.

Introduction

From the works of George Boole onwards, the field of logic has seen continuous efforts to model and capture various forms of reasoning. Boole's contributions, notably presented in his major works *The Mathematical Analysis of Logic* (1847, [19]) and *The Laws of Thought* (1854, [20]), marked the formalization of classical logic. This traditional system of formal logic has been foundational in mathematics, philosophy, and computer science. Classical logic is rooted in principles and rules initially formulated by ancient Greek philosophers such as Aristotle and further developed by logicians like Gottlob Frege and Bertrand Russell. Key principles of classical logic include the principle of truth functionality, which dictates that the truth of a compound sentence is determined solely by the truth values of its component sentences, and the principle of bivalence, asserting that every proposition within a theory has precisely one truth value, either true or false.

Classical logic has undeniably played a crucial role in shaping the foundations of reasoning and discourse across numerous disciplines. Its clarity, rigor, and applicability have established it as the default framework for mathematical proofs, philosophical arguments, legal reasoning, and beyond. However, as our interest in complex systems and nuanced phenomena has increased, it has become apparent that classical logic possesses limitations in capturing the intricacies of reasoning within certain contexts.

One notable area where classical logic falls short is in addressing uncertainty, vagueness, and ambiguity. Real-world scenarios frequently entail incomplete information, fuzzy boundaries, or contradictory evidence, challenges which classical logic struggles to accommodate within its binary framework of true or false. In domains like artificial intelligence, decision theory, and linguistics, where uncertainty is inherent, classical logic's strict adherence to bivalence can lead to oversimplification or distortion of complex situations. As early as the 1920s, Polish logician Jan Łukasiewicz raised pertinent questions about the validity of the principle of bivalence in a lecture delivered at the University of Warsaw, later elaborated upon in his article "On Determinism" [79].

Furthermore, classical logic's reliance on the law of excluded middle and the law of non-contradiction can sometimes impede rather than facilitate reasoning in certain philosophical and scientific domains. For instance, in quantum mechanics, where particles may exist in superposition states and events may yield probabilistic outcomes, classical logic's insistence on deterministic truth values may fail to accurately reflect reality.

In response to these challenges, non-classical logics have emerged as alternative formal systems, relaxing or modifying classical principles to better accommodate the complexities of reasoning in diverse contexts. Modal logic, fuzzy logic, intuitionistic logic, and paraconsistent logic are among the non-classical logics offering more flexible and nuanced approaches to reasoning. These logics introduce modalities, truth degrees, constructive reasoning, or tolerance of contradictions, providing formal frameworks that better align with the subtleties of real-world reasoning.

Starting from classical logic, and particularly from its algebraic counterpart, namely Boolean algebras, in this work, we will focus on two different types of generalizations concerning the validity of the law of excluded middle and the law of non-contradiction, as well as the distributive law. Thus, in addition to Boolean algebras (BA), we will deal with regular double Stone algebras ($RDSA$), orthomodular lattices (OML), and unsharp orthomodular lattices ($UOML$). The relationship between these classes of algebraic structures and the two forms of generalization we will focus on is illustrated by the Figure 1.

Throughout the text, we will delve into the specific significance occupied by these algebraic structures. Here, we will provide a brief introduction to contextualize them within a specific framework, reserving a more detailed discussion for the introduction of each chapter.

Double Stone algebras generalize Boolean algebras by splitting the "classical" negation into two, one of which continues to respect the law of non-contradiction, and the other the law of excluded middle. This class of algebras occupies a central

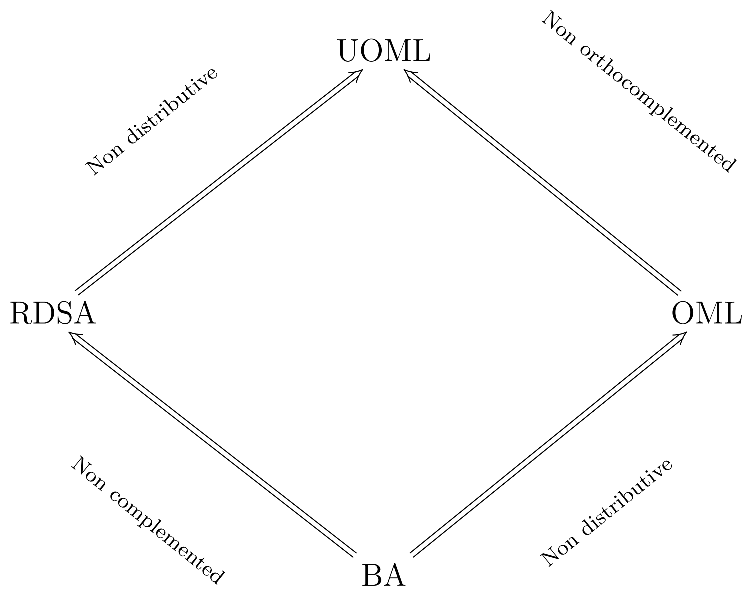


Figure 1: Types of generalization.

position in the field of non-classical logics, as it can be shown to contain important families of “non-classical” structures, such as Kleene algebras, 3-valued MV algebras, and part of Heyting algebras, among others. Furthermore, these can be considered as an algebraic counterpart to rough set theory. Rough set theory is a mathematical framework for dealing with uncertainty and vagueness in data analysis and decision-making. It was introduced by the Polish mathematician Zdzisław Pawlak in the early 1980s [91, 92]. Rough set theory provides a formal method for handling imperfect or incomplete information by dividing data into distinct subsets based on discernibility and indiscernibility relations. At its core, rough set theory is based on the notion of approximation. It allows for the approximation of sets through lower and upper approximations, providing a way to characterize the boundaries of uncertainty in the data. Rough set theory has applications in various fields, including data mining, machine learning, pattern recognition, and decision support systems. It offers a formal framework for handling uncertainty and incompleteness in data, making it a valuable tool for knowledge discovery and decision analysis.

Orthomodular lattices, as indicated by the diagram in Figure 1, generalize

Boolean algebras to a non-distributive case. They owe their name to the orthomodular law, a significant weakening of distributivity. The concept of orthomodular lattices emerged within the context of quantum logic in the 20th century, as researchers sought to develop a suitable logical framework to describe phenomena observed in the quantum world. Garrett Birkhoff, along with John von Neumann, formally introduced the concept of orthomodular lattices in their work in 1936, "The Logic of Quantum Mechanics" [14]. Over the subsequent years, orthomodular lattices have become a fundamental tool for formalizing quantum logic and studying the logical structures present in quantum systems. They have found applications in various fields, including lattice theory, measure theory, theoretical computer science, and quantum information theory.

Unsharp orthomodular lattices generalize Boolean algebras in both directions. This class is relatively less recognized compared to others; detailed elucidation regarding their origins and significance will be provided in subsequent chapters. For now, it suffices to note that they constitute a subclass of PBZ^* lattices (paraorthomodular Brouwer-Zadeh lattices with the star condition), which extend orthomodular lattices to scenarios where the principles of non-contradiction and excluded middle are not universally applicable. The conceptualization of PBZ^* lattices arises within the framework of the so-called unsharp approach to quantum theory. Their formalization stems from the attempt to more precisely capture the set of bounded operators of a Hilbert space.

The aim of this work is to investigate the various forms of generalizations among these structures to achieve a clearer understanding. To accomplish this, we will employ tools from various fields of study, ranging from universal algebra to model theory, and encompassing category theory, among others. The underlying assertion is that there exists an even closer relationship between these structures than initially perceived. Indeed, from a certain perspective, pairs of structures may reveal themselves to be equivalent objects. Therefore, it becomes intriguing to delve into their peculiarities and the manner in which they diverge. In particular, this investigation will focus on examining how similar or divergent characteristics from one perspective relate to, and influence, similar or divergent properties from another perspective.

These objectives are pursued in the second and third chapters, which are based on the works [51] co-authored with Giuntini and Ledda; [76, 77, 78] co-authored with Ledda. In the first chapter, we present an overview of the most relevant notions of the following parts, striving to make this work as self-contained as possible. The second chapter focuses on examining Boolean algebras and regular double Stone algebras, aiming to explore the categorical, algebraic, and model-theoretic relationships inherent in these structures. Specifically, it investigates the equivalence between categories of Boolean algebras with fixed filters and regular double Stone algebras, despite their divergent model theories. Additionally, the chapter delves into the structure theory of regular double Stone algebras and discusses various model-theoretic properties.

While the equivalence of categories between Boolean algebras and regular double Stone algebras is established, certain model-theoretic aspects do not seamlessly transfer between them, such as the notions of atomicity and $\aleph_0$-categoricity. Furthermore, the chapter characterizes algebraic closures within the variety of regular double Stone algebras and elucidates their connections with the algebraic closures of Boolean algebras. Lastly, it provides insights into injective objects within the variety of Stone algebras and examines categoricity results for these classes of injective objects.

In the third chapter, the focus shifts to orthomodular lattices and unsharp orthomodular lattices. Orthomodular lattices are known to be essentially the union of particular distributive subalgebras, known as blocks, which represent classical contexts within the theory. In a more technical sense, they represent σ-algebras of pairwise commuting operators. The objectives of this chapter extend to attempting to generalize this concept to the realm of unsharp orthomodular lattices. It will be demonstrated that this generalization is feasible, and notably, these blocks are identified as regular double Stone algebras, explored in the preceding chapter. Moreover, a significant portion of established results in the domain of orthomodular lattices remains applicable in this broader context. Finally, considering the close relationship between unsharp orthomodular lattices, MV algebras, Heyting algebras, and orthomodular lattices, the inquiry into the implications of residuation in the entire structure naturally arises as a pivotal aspect of investigation.

Chapter 1

Background

In this chapter, we will primarily provide the necessary concepts to ensure the elaboration becomes self-contained as soon as possible. We will assume that the reader has a basic understanding of set theory and arithmetic.

Due to space constraints, we will refrain from providing the proofs of the theorems that we will reference. However, we will indicate possible sources to consult in order to enhance one's understanding.

1.1 Basics on lattice theory and universal algebra

In this brief introduction, we will explore some fundamental concepts from lattice theory and universal algebra. Due to the extensive nature of these fields, we will only cover essential notions to make this text self-contained. Readers interested in further exploration are encouraged to refer to [12, 13, 15, 23, 42].

In particular, the current Section 1.1 is based on [23].

1.1.1 Lattice theory

The origins of lattice theory can be traced back to the works of Boole and Dedekind. However, it was through the works of Birkhoff that it became widely disseminated in a crucial way. Indeed, lattice theory offers elegant and straightforward frameworks for comprehending complex structures. One of the most intriguing aspects of lattice theory is the simplicity of its foundational concepts.

Now, we recall only a handful of basic notions and refer the reader to consult [3, 12, 13, 15, 23, 42, 60] for an extensive treatment.

We start by providing two equivalent definitions of a lattice: one from an order-theoretic standpoint and the other strictly algebraic.

Definition 1. *A pair* $(L, \leq)$*, where* L *is a non-empty set and* $\leq$ *is a binary relation on* L*, is a* partially ordered set (poset) *if the following conditions hold for any* $x, y, z \in L$*:*

1. $x \leq x$*;*

2. $x \leq y$ *and* $y \leq x$ *imply* $x = y$*;*

3. $x \leq y$ *and* $b \leq c$ *imply* $x \leq z$*.*

Definition 2. *A poset* $L = (L, \leq)$ *is a* lattice *if for every pair of elements* $x, y \in L$ *both the least upper bound, denoted* $\sup\{x, y\}$*, and the greatest lower bound, denoted* $\inf\{x, y\}$*, exist.*

Definition 2 is equivalent to Definition 3.

Definition 3. *A structure* $L = (L, \vee, \wedge)$ *is a* lattice *if it satisfies the following conditions:*

1. $x \vee y = y \vee x$ *and* $x \wedge y = y \wedge x$*;*

2. $x \vee (y \vee z) = (x \vee y) \vee z$ *and* $x \wedge (y \wedge z) = (x \wedge y) \wedge z$*;*

3. $x \vee x = x$ *and* $x \wedge x = x$*;*

4. $x \vee (x \wedge y) = x$ *and* $x \wedge (x \vee y) = x$*.*

In general, a subset S of L is referred to as a *sublattice* if S is closed under the lattice operations of L. A lattice is designated as *bounded* if it satisfies $x \wedge 1 = x$ and $x \vee 0 = x$ for any x in L. Finally, a lattice is termed *complete* if, for any $X \subseteq L$, both $\bigvee X$ and $\bigwedge X$ exist.

Within the framework of lattice theory, two fundamental classes of lattices are studied in detail: distributive and modular lattices. These are axiomatized by the distributive law and the modular law, which are crucial conditions in mathematics, and their role will be central in this work.

Definition 4. *A lattice L is* modular *if it satisfies the modular law, namely:*

$$\textit{for} \quad x, y \in L, \quad \textit{if} \quad x \leq y, \quad \textit{then} \quad x \vee (y \wedge z) = y \wedge (x \vee z).$$

Definition 5. *A lattice L is* distributive *if it satisfies either of the distributive laws*[1]*:*

1. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;
2. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

We conclude this section on lattices by mentioning a few relevant theorems concerning forbidden structures without proofs. For detailed definitions and proofs, please refer to [23].

Theorem 1. *Every distributive lattice is a modular lattice.*

Theorem 2 (Dedekind)**.** *A lattice L is a non-modular lattice if and only if N_5 can be embedded into L.*

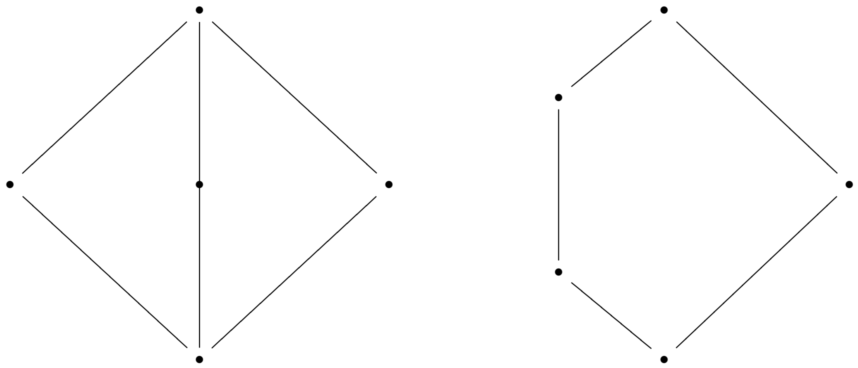
Theorem 3 (Birkhoff)**.** *A lattice L is a non-distributive lattice if and only if M_5 or N_5 can be embedded into L.*

1.1.2 Universal algebra

In the context of algebraic structures, let us define some key concepts. Consider a non-empty set A. An n-ary operation over A is a function $f : A^n \to A$. Here, n represents the arity or rank of f. If $n = 0$, then f is uniquely determined by its image $f(\varnothing)$ and thus can be associated with a distinguished element of A.

Now, let us introduce the notion of a language or type of algebras, denoted by ν. This is a set consisting of function symbols indexed by non-negative integers. Each member f_n in ν is referred to as an n-ary function symbol. We denote

[1] A lattice satisfies one distributive law if and only if it satisfies both laws.



$\mathbf{M}_5$ $\mathbf{N}_5$

Figure 1.1: Lattices M_5 and N_5.

the subset of all n-ary function symbols of ν as ν_n. For any finite language ν, we represent it as an n-tuple of arities of its elements listed in decreasing order. Given a language ν of algebras, an algebra A of type ν is an ordered pair (A, ν), where A is a nonempty set and ν is a family of operations on A, indexed by ν. For each n-ary function symbol $f \in \nu$, there corresponds an n-ary operation f_A on A. The set A is termed the universe of $A = (A, \nu)$, and the f_A's are referred to as the fundamental operations of A. If ν is finite, say $\nu = \{f_1, \ldots, f_n\}$, we often represent A as $(A, f_1, \ldots, f_n)$.

Now, suppose A and B are algebras of the same type. We say that B is a *subalgebra* of A if B is a subset of A and every fundamental operation of B is the restriction of the corresponding operation of A, denoted as $f_B = f \upharpoonright^n_B$. If A has nullary operations, B contains them as well. We write $B \subseteq A$ to denote that B is a subalgebra of A. Additionally, $B \subseteq A$ is simply referred to as a subuniverse of A if, for any n-tuple $(a_1, \ldots, a_n) \in B$, the result of applying any n-ary operation on A yields an element in B.

Now, if A is an algebra and $X \subseteq A$, we define the smallest subalgebra of A containing X as the intersection of all subalgebras containing X. This intersection, which is again provably a subalgebra, is termed the subalgebra generated by X.

Let us define another central concept. Let A and B be algebras of the same type ν. A *homomorphism* $\phi : A \to B$ is a mapping from A to B that preserves operations, i.e., for any $f_n \in \nu$, we have $\phi(f_A(a_1, \ldots, a_n)) = f_B(\phi(a_1), \ldots, \phi(a_n))$.

We refer to a homomorphism $\phi : A \to B$ as an embedding if it is injective. Moreover, if ϕ is onto, then B is called a homomorphic image of A. An injective and surjective homomorphism is an isomorphism, denoted $A \cong B$.

Lemma 1. *If $\phi : A \to B$ is an embedding, then $\phi(A)$ is a subalgebra of B.*

In the entirety of the text, when there is no impending ambiguity, we will synonymously use the terms "structure" and "algebra".

Congruences

A fundamental notion in universal algebra is the concept of *congruence*. Given an algebra A of type ν, an equivalence relation $\sim$ over A is a congruence whenever it has the compatibility property, i.e., for any $f_n \in \nu$ and $a_i \sim b_i$ $(1 \leq i \leq n)$, we have $f_A(a_1, \ldots, a_n) \sim f_A(b_1, \ldots, b_n)$.

In what follows, $\mathrm{Con}(A)$ stands for the set of congruence relations on A.

It can be seen that $\mathrm{Con}(A)$ is a lattice where meets consist of intersections of congruences and joins are the generated congruences, namely the smallest congruences containing the given ones.

According to the literature, we will denote by Δ the identity congruence $\{(x, x) : x \in A\}$, and the universal relation by ∇. It can be readily observed that they represent the bottom and the top elements in $\mathrm{Con}(A)$, respectively.

Theorem 4. *$Con(A) = (Con(A), \wedge, \vee, \Delta, \nabla)$ is a complete lattice.*

Strictly related to notion of congruence we have: *quotient algebras*. Given an algebra A of type ν and $\Theta \in \mathrm{Con}(A)$, the quotient algebra A/Θ of A modulo Θ is the algebra having the quotient set A/Θ as universe and operations defined as follows: for any $f_n \in \nu$, $f_{A/\Theta}(a_1/\Theta, \ldots, a_n/\Theta) = f_A(a_1, \ldots, a_n)/\Theta$.

Let A be an algebra and $\Theta \in \mathrm{Con}(A)$. Then the map $\pi : A \to A/\Theta$, defined by $\pi(a) = a/\Theta$, is called the *natural map*. When no confusion is impending, we will omit unnecessary subscripts.

Lemma 2. *Let A be an algebra and $\Theta \in Con(A)$. Then the natural map $\pi : A \to A/\Theta$ is an onto homomorphism.*

Definition 6. *Let A be an algebra. A congruence $\Theta \in Con(A)$ is a* factor congruence *if there is a congruence $\Theta^* \in Con(A)$ such that:*

$$\Theta \cap \Theta^* = \Delta \quad \Theta \vee \Theta^* = \nabla \quad \textit{and} \quad \Theta \circ \Theta^* = \Theta^* \circ \Theta.$$

In rough terms, factor congruences are complemented elements in $Con(A)$.

Direct products and subdirect products

In addition to the notions of subalgebra and homomorphic image just discussed, we will describe two more constructions that play a central role in universal algebra.

Definition 7. *Let A and B be two algebras of the same type ν. The* direct product *of A and B, indicated as $A \times B$, is the algebra whose universe is the Cartesian product $A \times B$, and for any $f^n \in \nu$, the operations are computed componentwise. Namely:*

$$f^{A\times B} = ((a_1, b_1), \cdots, (a_n, b_n)) = (f^A(a_1, \cdots, a_n), f^B(b_1, \cdots, b_2)).$$

Definition 8. *An algebra A is a* subdirect product *of an indexed family $\{A_i\}_{i\in I}$ of algebras if:*

$$A \leq \textstyle\prod_{i\in I} A_i,$$

and for each $i \in I$, $\pi_i(A) = A_i$.

Theorem 5 (Birkhoff)**.** *Every algebra A is isomorphic to a subdirect product of subdirectly irreducible algebras.*

In more common terms, subdirectly irreducible algebras play a similar role to prime numbers in number theory.

Definition 9. *An algebra A is called* simple *if $Con(A) = \{\Delta, \nabla\}$.*

Class operators and varieties

Now we recall the definition of fundamental *class operators*, useful for studying classes of algebras of the same type closed under one or more constructions.

1. $A \in I(K)$ if and only if A is isomorphic to some member of K;
2. $A \in S(K)$ if and only if A is a subalgebra of some member of K;
3. $A \in H(K)$ if and only if A is a homomorphic image of some member of K;
4. $A \in P(K)$ if and only if A is a direct product of a nonempty family of algebras in K;
5. $A \in P_S(K)$ if and only if A is a subdirect product of a nonempty family of algebras in K.

Through these operators, we can provide the following important definition.

Definition 10. *A non-empty class K of algebras of type ν is called a* variety *if it is closed under subalgebras, homomorphic images, and direct products.*

Finally, we conclude this section by presenting two cardinal theorems in universal algebra, along with one general definition.

Theorem 6 (Tarski)**.** $V = HSP$.

Definition 11. *Let V be a variety. It is*

1. *congruence permutable if for each $A \in V$ and Θ, $\Phi \in Con(A)$, $\Theta \circ \Phi = \Phi \circ \Theta$;*
2. *congruence distributive if $Con(A)$ is a distributive lattice for each $A \in V$;*
3. *congruence regular if for each $A \in V$, $x, y \in A$ and Θ, $\Phi \in Con(A)$, $x/\Theta = y/\Phi$ implies $\Theta = \Phi$;*
4. *congruence uniform if for each $A \in V$ and Θ, $\Phi \in Con(A)$, all Θ-classes have the same cardinality.*

Theorem 7 (Jónsson)**.** *Let $V(K)$ be a congruence distributive variety and K a finite set of finite algebras. Then the subdirectly irreducible algebras of $V(K)$ are in*

$$HS(K),$$

and

$$V(K) = IP_S(HS(K)).$$

1.2 Classical and non-classical algebraic structures

In this section, we provide the indispensable definitions and notions necessary to understand the subsequent chapters. We will indicate specific sources for each topic for further exploration and to see the proofs in detail.

1.2.1 Boolean algebras

Boolean algebras are mathematical structures that play a fundamental role in logic, set theory, computer science, and various branches of mathematics. In the framework of Boolean algebraa, the elements are typically interpreted as truth values—either true or false—and the algebraic operations mirror fundamental logical connectives such as AND, OR, and NOT. One of the key features of Boolean algebras is their capacity to model classical propositional logic, making them essential for understanding and formalizing logical reasoning.

It would be impossible to provide an exhaustive presentation of Boolean algebras; therefore, we recommend interested readers to consult the following works [23, 56, 84].

Definition 12. *A structure $B = (B, \wedge, \vee, ', 0, 1)$ is a* Boolean algebra *if it satisfies the following conditions:*

1. $(B, \wedge, \vee, 0, 1)$ *is a bounded distributive lattice;*
2. $x \vee x' = 1$*;*
3. $x \wedge x' = 0$.

Roughly speaking, the unary operation $'$ represents complementation. The following lemma describes some of its characteristics.

Lemma 3. *Let B be a Boolean algebras, then the following hold:*

1. $x'' = x$;

2. $(x \vee b)' = x' \wedge y'$;

3. $(x \wedge b)' = x' \vee y'$.

In the course of this work, we will deal with different generalizations of Boolean algebras in which one or more of the conditions in the Lemma 3 and Definition 12 are dropped.

Let us introduce a very basic algebraic concept that will be useful in the subsequent chapters.

Definition 13. *Let B be a Boolean algebra, then a subset F of B is called a* filter *if:*

1. $1 \in F$;

2. $x, y \in F$, *then* $x \wedge y \in F$;

3. $x \in F$ *and* $x \leq y$, *then* $y \in F$.

In general, a filter F is called *prime* if $x \vee y \in F$ implies that either x or y belongs to F. F is said to be *maximal* if there does not exist any filter G which properly contains F. Finally, F is *principal* if F is the upper cone of an element x, in this case we will denote it by either $[x)$ or $\uparrow x$ interchangeably.

In Boolean algebras, there are strong results concerning the set of filters. We will summarize some of them.

Lemma 4. *Let B be a Boolean algebra and F a filter of B. Then:*

1. *F is maximal if and only if F is prime;*

2. *Every infinite Boolean algebra has maximal filters that are non principal.*

In addition, the strong connection between filters and congruences is well known. The following lemma asserts this connection.

Lemma 5. *For any Boolean algebra, the lattice of filters is isomorphic to the lattice of congruences.*

Finally, we report a theorem concerning the generation of the entire variety.

Theorem 8. *In the variety of Boolean algebras, up to isomorphism the unique subdirectly irreducible structures is* $\mathbf{2}$*.*

This implies that the 2-element Boolean algebra generates the variety of Boolean algebras.

1.2.2 Kleene and many valued algebras

In this part, we briefly recall some definitions of possible generalizations of Boolean algebras. For details, the reader is referred to [18, 30, 31, 68].

Definition 14. *[68] A* Kleene algebra $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ *is an algebra where* $(A, \wedge, \vee, 0, 1)$ *is a bounded distributive lattice and* $'$ *is an antitone and involutive operation that satisfies:*

$$x \wedge x' \leq y \vee y'. \tag{Kleene}$$

An intuitive reading of (Kleene) condition is that any "contradiction" is exceeded by a "tautology".

Definition 15. *[31] A structure* $\mathbf{A} = (A, \oplus, \neg, 0)$ *is an* MV algebra *if it satisfies the following conditions:*

(M1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z;$

(M2) $x \oplus y = y \oplus x;$

(M3) $x \oplus 0 = x;$

(M4) $\neg\neg x = x;$

(M5) $x \oplus \neg 0 = \neg 0$;

(M6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ *(Łukasiewicz).*

An MV algebra is called n-valued *if it satisfies*

$$\overbrace{x \oplus \ldots \oplus x}^{n\text{-}times} = \overbrace{x \oplus \ldots \oplus x}^{(n-1)\text{-}times}.$$

In other words, in n-valued MV algebras every element admits a Boolean multiple whose coefficient is at most n.

Definition 16. *[62] A* 3-valued Łukasiewicz algebra *is an algebra* $A = (A, \wedge, \vee, ', \Diamond, 0, 1)$*, where* $= (A, \wedge, \vee, 0, 1)$ *is a bounded distributive lattice and* $'$ *is an antitone and involutive operation, and* A *satisfies:*

1. $\Diamond x \vee x' = 1$;
2. $x' \wedge x = x' \wedge \Diamond x$;
3. $(x \wedge y)' = x' \vee y'$;
4. $\Diamond(x \wedge y) = \Diamond x \wedge \Diamond y$.

We close the preliminaries with the following theorem which is to be credited independently to Mundici [89] and Iorgulescu [64]:

Theorem 9. *3-valued Łukasiewicz algebras and 3-valued MV algebras are term equivalent.*

1.2.3 Pseudocomplemented structures

In the present section we concisely introduce some useful basic facts on pseudocomplemented structures.

Definition 17. *[72] Consider the structure* $L = (L, \wedge, \vee, ^{\sim}, 0, 1)$*, then* L *is a* p-algebra *if it is a lattice and* $^{\sim}$ *is a pseudocomplementation, namely:*

$$x \wedge y = 0 \quad \textit{if and only if} \quad x \leq y^{\sim}.$$

Concerning Definition 17, let us note that some authors also assume distributivity [3].

This definition encompasses a quite number of important algebraic structures. However, our focus will be mainly on double p-algebras and Stone algebras. Additionally, we will extend our discourse to encompass further equations, or conditions. For general and specific accounts we may refer the interested reader to [15, 60] and [72, 98].

Definition 18. *[72] A structure* $L = (L, \wedge, \vee, ^{\sim}, ^{+}, 0, 1)$ *is a* double p-algebra *if the following conditions are satisfied:*

1. $L = (L, \vee, \wedge, ^{\sim}, 0, 1)$ *is a p-algebra. Namely:*

$$x \wedge y = 0 \text{ if and only if } x \leq y^{\sim}.$$

2. $L = (L, \vee, \wedge, ^{+}, 0, 1)$ *is a dual p-algebra. Namely:*

$$x \vee y = 1 \text{ if and only if } x \geq y^{+}.$$

Let us now introduce the notion of Stone algebra:

Definition 19. *[60] A distributive p-algebra* $L = (L, \wedge, \vee, ^{\sim}, 0, 1)$ *is a* Stone algebra *if it satisfies:*

$$x^{\sim} \vee x^{\sim\sim} = 1. \qquad \text{(Stone condition)}$$

As it is well known, every Stone algebra is isomorphic to a subalgebra of the Stone algebra of all ideals of the Boolean algebra of a powerset [13, 57, 59]. This fact will find further discussion in section 2.6.

It can be useful to recall some conditions that are equivalent to (Stone condition).

Theorem 10. *[15] Consider L be a distributive p-algebra, then the following statements are equivalent:*

1. *L satisfies* (Stone condition)*;*

2. $(x \vee y)^{\sim\sim} = x^{\sim\sim} \vee y^{\sim\sim}$ *(strong De Morgan);*

3. $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}$ *(strong De Morgan);*

4. $S_K(L) = \{x^{\sim\sim} : x \in L\}$ *is a Boolean subalgebra.*

Adapting Definition 18, we say that an algebra $L = (L, \wedge, \vee, ^{\sim}, ^{+}, 0, 1)$ is a *double Stone algebra* if $L = (L, \wedge, \vee, ^{\sim}, 0, 1)$ is a Stone algebra, while $L = (L, \wedge, \vee, ^{+}, 0, 1)$ is a dual Stone algebra.

A quite relevant condition regarding double p-algebras is the following:

$$\text{if } x^{\sim} = y^{\sim} \quad \text{and} \quad x^{+} = y^{+} \quad \text{then} \quad x = y. \tag{Regularity}$$

We call *regular* a double p-algebra that satisfies condition (Regularity) (the reader may consult [70] e.g.). In fact, regularity is not a misnomer. Actually, in [98] Varlet proved that (Regularity) condition corresponds to congruence regularity: if two congruences coincide on a congruence class, then they are in fact the same congruence.

In this regard we may recall the following rather strong result proved by Katriňák in [70]:

Theorem 11. *Every double p-algebra L satisfying condition* (Regularity) *is a distributive lattice.*

As observed in [86], it is possible to define two unary operations that behave as the modal operators necessarily ($\Box$) and possibly ($\Diamond$) as follows:

$$\Box x = x^{++} \quad \text{and} \quad \Diamond x = x^{\sim\sim}.$$

Now, given a double p-algebra L, three subsets of the support will assume a definite relevance to the development of our discourse.
We denote by $S_K(L)$ the set of *sharp elements of* L:

$$S_K(L) = \{x \in L : x^{+} = x^{\sim}\} = \{x \in L : x = \Diamond x\} = \{x^{\sim} : x \in L\}.$$

Let us notice that $S_K(L)$ is the largest Boolean subalgebra of L (see e.g. [49]). It may be useful to observe that in $S_K(L)$ the operations $^{+}$ and $^{\sim}$ coincide, and therefore are antitone involutions, and all elements are complemented.

Moreover, all elements in $S_K(L)$ are stable under both $\Diamond$, and $\Box$: $\Diamond x = \Box x = x$. It is straightforward to verify that in $S_K(L)$ condition (Regularity) is nothing but a triviality. Indeed, as we mentioned earlier $S_K(L)$ is in fact a Boolean lattice.

On the one hand, we indicate by $D^{\sim}(L)$ the set of *dense elements of* L:

$$D^{\sim}(L) = \{x \in L : x^{\sim} = 0\}.$$

Let us note that $S_K(L) \cap D^{\sim}(L) = \{1\}$. Furthermore, for all $x \in L$, $x \vee x^{\sim} \in D^{\sim}(L)$.

On the other hand, we define the set of *dually dense elements of* L:

$$D^{+}(L) = \{x \in L : x^{+} = 1\}.$$

Also in this case, we have that $S_K(L) \cap D^{+}(L) = \{0\}$. Moreover, for all $x \in L$, $x \wedge x^{+} \in D^{+}(L)$.

It is possible to define the operation "$'$" in any regular double Stone algebra in order to encompass the Kleene negation as follows:

$$x' = x^{\sim} \vee (x \wedge x^{+}). \tag{1.1}$$

Let us observe that for all $x \in L$, L a regular double p-algebra:

$$\Box x \leq x \leq \Diamond x \text{ and } x^{\sim} \leq x' \leq x^{+}.$$

Moreover, the conditions of Definition 14 are all satisfied.

Let us now introduce a few more notions that will recur in the development of our discourse.

In [99] Varlet and Katriňák in [69] independently proved a result that will be relevant to our discourse:

Theorem 12. *Regular double Stone algebras and* 3*-valued Łukasiewicz algebras are term equivalent.*

Actually, given a regular double Stone algebra A, upon defining, for $x, y \in A$,

$$x \oplus y = (\Diamond x \vee y) \wedge (x \vee \Diamond y)$$

and setting $\neg$ as $'$ defined in Equation (1.1), a term equivalence between regular double Stone algebras and 3-valued MV algebras obtains.

1.2.4 Residuated structures

Residuated structures form a comprehensive algebraic framework that encompasses diverse mathematical structures and establishes deep connections with substructural logics, as well as numerous branches of mathematics. The concept of residuation finds its roots in Dedekind's contributions to ring theory. Originally, residuation was introduced to elucidate the notion of division within rings with a unit, focusing on their ideals ([102]).

In algebra and logic, residuated structures are intricately intertwined with substructural logics, which represent a family of non-classical logics that relax or modify certain structural rules of classical logic. Substructural logics, including linear logic and relevance logic, are primarily concerned with the manipulation of resources and the structure of proofs. Residuated structures provide an inherent algebraic framework for modeling the dynamics of these logics, particularly with respect to implication and entailment relationships.

However, there is a notable exception among the non-classical logics that cannot be categorized under the framework of substructural logics. Quantum structures like orthomodular lattices typically lack a pair of operations that behave similarly to a residuated pair. Consequently, implications within quantum logics do not meet the minimal criteria to be considered as substructural conditionals. To address this challenge, we need to broaden our perspective. Although orthomodular lattices do not fall under the category of residuated lattices or even residuated ℓ-groupoids, recent observations reveal that they can be transformed into a variety of left-residuated ℓ-groupoids with robust properties [22]. Similarly promising outcomes can be achieved for other classes of algebras relevant to quantum logics. For this reason, we will recall the definition of the class of left-residuated ℓ-groupoids.

We cannot even remotely hope to give an account of residuated lattices, or of residuated ℓ-groupoids, that would eventually render the present work self-contained. We just recall some definitions, referring the reader to [16, 17, 47, 66, 75, 81, 97] for extensive treatments of the topic and for properties of these algebras that will be used with no special mention in the sequel.

A binary operation $\cdot$ on a partially ordered set $(P, \leq)$ is said to be residuated provided that there exist binary operations $\backslash$ and $/$ on P such that for all $x, y, z \in P$,

$$x \cdot y \leq z \text{ if and only if } x \leq z/y \text{ iff } y \leq x\backslash z. \tag{Res}$$

We refer to the operations $\backslash$ and $/$ as the right residual and left residual of the operation $\cdot$, respectively. We adopt the convention that, in the absence of parentheses, $\cdot$ binds stronger than the other operations symbols. Any statement about residuated structures has a "mirror image" obtained by reading terms backwards (i.e., replacing $x \cdot y$ by $y \cdot x$ and interchanging x/y with $y\backslash x$). If $\cdot$ is an (associative) operation with two-sided unit 1 and the partial order $\leq$ is a lattice order, the resulting structure $P = (P, \wedge, \vee, \cdot, \backslash, /, 1)$ is called a residuated ℓ-groupoid, resp. a residuated lattice. A pointed residuated ℓ-groupoid, resp. a pointed residuated lattice is an algebra $P = (P, \wedge, \vee, \cdot, \backslash, /, 0, 1)$ such that the reduct $(P, \wedge, \vee, \cdot, \backslash, /, 1)$ is a residuated ℓ-groupoid (a residuated lattice); in other words, nothing is assumed about the additional constant 0.

The existence of both residuals for product implies rather strong properties, that in some of which are listed below, primarily in view of a comparison with the left-residuated structures examined below.

Lemma 6. *In any (pointed) residuated lattice, the following identities and quasi-identities, along with their mirror images, are satisfied.*

1. $(x\backslash y)z = x\backslash yz$;

2. $1\backslash x = x$;

3. $x \leq y \Rightarrow zx \leq zy$ *and* $xz \leq yz$;

4. $x \leq y \Rightarrow y\backslash z \leq x\backslash z$ *and* $z\backslash x \leq z\backslash y$;

5. $(x\backslash y)(y\backslash z) \leq x\backslash z$;

6. $xy\backslash z = y\backslash(x\backslash z)$;

7. $x(x\backslash 1)\ 1 = 1$;

8. $x\backslash(y/z) = (x\backslash y)/z$.

Lemma 7. *The classes of (pointed) residuated ℓ-groupoids and (pointed) residuated lattices belong to finitely based varieties in their respective signatures. This is achieved by substituting the residuation conditions (Res) with the following identities (and their mirror images):*

1. $y \leq x\backslash(xy \vee z)$;

2. $x(y \vee z) = xy \vee xz$;

3. $y(y\backslash x) \leq x$.

The following definition allows us to place orthomodular lattices (and other algebraic structures related to quantum logics) and residuated structures in a similar framework [26, 43].

Definition 20. *A structure $P = (P, \vee, \wedge, \cdot, \rightarrow, 0, 1)$ is a left-residuated ℓ-groupoid lattice if the following are satisfied:*

1. $(P, \vee, \wedge)$ *is a lattice;*

2. $(P, \cdot, \rightarrow)$ *is a left-unital groupoid;*

3. *for all $x, y, z \in P$, $x \cdot y \leq z$ if and only if $x \leq y \rightarrow z$.*

Definition 21. *A pointed left-residuated ℓ-groupoid $P = (P, \vee, \wedge, \cdot, \rightarrow, 0, 1)$ is said to be:*

1. *involutive, if for any $x \in L$ holds $(x \rightarrow 0) \rightarrow 0 = x$;*

2. *antitone, if for any $x, y \in L$, $x \leq y$ implies $y \rightarrow 0 \leq x \rightarrow 0$;*

3. *strongly idempotent, if for any $x, y \in L$ holds $x \cdot (x \vee y) = x$;*

4. *Sasakian, if for any $x, y \in L$ holds $x \cdot y = (x \vee y \rightarrow 0) \wedge y$;*

5. $\to$*-Sasakian, if for any* $x, y \in L$ *holds* $x \to y = (x \wedge y) \vee x \to 0$.

Definition 22. *A pointed left-residuated* ℓ*-groupoid* $P = (P, \vee, \wedge, \cdot, \to, 0, 1)$ *is an orthomodular groupoid if it is involutive, antitone, strongly idempotent, Sasakian and* $\to$*-Sasakian.*

Theorem 13. *[43] Orthomodular groupoids are term equivalent to orthomodular lattices.*

1.2.5 Orthomodular lattices

In this subsection, we will focus on a specific class of algebraic structures related to quantum logics. For general facts about quantum logics and their algebras, the reader is encouraged to consult [11, 38, 42, 71].
In the standard Birkhoff-von Neumann approach to quantum logic, quantum events (or properties) are mathematically represented by projection operators on a complex separable Hilbert space. If H is a Hilbert space and $\Pi(H)$ is the set of all projection operators on H, the structure

$$(\Pi(H), \vee, \wedge, ', 0, 1),$$

where 0, resp. 1 are the projections onto the one-element, resp. total subspaces, $(P_X)'$ is the projection onto the subspace $X^{\perp}$ orthogonal to X and $P_X \vee P_Y = P_{(X \cup Y)^{\perp\perp}}$, is a canonical example of an orthomodular lattice, a structure which is defined hereafter.

Definition 23. 1. *An* orthcolattice *is an algebra* $L = (L, \vee, \wedge, ', 0, 1)$ *such that* $(L, \vee, \wedge, 0, 1)$ *is a bounded lattice and* $'$ *is an orthocomplementation on* L*, namely an antitone involution such that* $x \wedge x' = 0$ *for all* $x \in L$.

2. *An ortholattice* L *is an* orthomodular lattice *iff for all* $x, y \in L$ *such that* $x \leq y$*, we have that* $y = (y \wedge x') \vee x$.

The orthomodular property, for an ortholattice, can be characterized in several equivalent ways. We list some of them in the next theorem, which shows, among other things, that the class of all orthomodular lattices is a variety.

Theorem 14. *Let L be an ortholattice. The following conditions are equivalent:*

1. *L is orthomodular;*

2. *for all $x, y \in L$, $x \wedge ((y \wedge x) \vee x') \leq y$;*

3. *for all $x, y \in L$, $x \vee y = ((x \vee y) \wedge y') \vee y$;*

4. *for all $x, y \in L$, if $x \leq y$ and $x' \wedge y = 0$ then $x = y$.*

In other words, let us briefly emphasize that condition (4) is equivalent to orthomodularity in case the lattice in question is an ortholattice. But it is no longer such if it is not the case.

The reader is referred to [11, 38, 71, 3, 84, 15] for whatever concepts or results are not covered in this concise subsection. We will only provide very basic information on the *commuting relation*, of crucial importance for the development of our discourse.

Definition 24. *Given an orthomodular lattice L and $x, y \in L$, x is said to* commute *with y in case:*

$$(x \wedge y) \vee (x \wedge y') = x.$$

Lemma 8. *Let L be an orthomodular lattice, and let $a, b, c \in L$.*

1. *The commuting relation is reflexive and symmetric.*

2. *If x commutes with y, z, then it commutes with y', z', $y \vee z$ and $y \wedge z$.*

3. *If y commutes with z, then*

$$(((x \vee y') \wedge y) \vee z') \wedge z = (x \vee y') \wedge (y \vee z') \wedge z.$$

4. *If y commutes with z and x commutes with $y \wedge z$, then $x \vee y'$ commutes with z.*

5. *x commutes with y iff $(x \vee y') \wedge y \leq x$.*

The next celebrated result is one of the most useful tools for practitioners in the field [11].

Theorem 15 (Foulis-Holland)**.** *If L is an orthomodular lattice and $x, y, z \in L$ are such that x commutes with both y and z, then the set $\{x, y, z\}$ generates a distributive sublattice of L.*

The next definition captures the notion of *context* inside an orthomodular lattice [71].

Definition 25. *Let L be an orthomodular lattice. A* block *of L is a maximal set M of mutually commuting elements. Moreover, every block is a maximal Boolean subalgebra of L.*

Indeed, as is well known [71], any orthomodular lattice is nothing but the set theoretical union of its blocks.

Chapter 2

Regular double Stone algebras

2.1 Introduction

Several attempts of a mathematical representation of knowledge have been pursued by many authors over the last sixty years. Among them, we may mention the theories of sets, rough sets [91, 95], concept lattices [103], and fuzzy sets [104]. From a parallel perspective, the aim of knowledge representation can be regarded as a problem of classification of the information. A primary example is naturally the theory of sets, whose corresponding algebraic counterpart is obviously the variety of Boolean algebras.

On the one hand, set theory admits only a sharp true/false classification, that would involve a complete knowledge of any property.

On the other hand, many authors over the last century have claimed for either a relative, or an approximate knowledge. This fact has motivated the theories of rough sets, concept lattices, and fuzzy sets. And these theories may model types of non-classical classifications.

Algebraically, if on the one side we have Boolean algebras (and classical logic), on the other we encounter regular double Stone algebras [41], and residuated lattices (and substructural logics) [47, 66]. In fact, it is well known that regular double Stone algebras play a central role with respect to the above mentioned approaches. Indeed, any regular double Stone algebra is isomorphic to an algebra of rough sets of an approximation space [35, 41].

Moreover, the varieties of Boolean algebras, 3-valued MV algebras, Kleene algebras, and Stone algebras find enough injective hulls in the variety of regular double Stone algebras. Actually, this fact seems to clarify and place under a common framework the theories of rough sets and 3-valued logics, whose relations have been widely discussed since the eighties of the last century by several authors, e.g. Ciucci and Dubois [32], Banerjee [7], Walker [101], and Iturrioz [65].

Another classification perspective involves the topologies that these algebraic structures respectively induce: Boolean algebras are (categorically) equivalent to Stone spaces, regular double Stone algebras are continuous sections of a sheaf over a Boolean space [34], and in general distributive lattices can be regarded as Priestly spaces (a reference in this sense is [42]). Remarkably enough, all these algebraic structures share a crucial condition: distributivity[1].

The aim of this chapter is to investigate under a unifying scenario categorical, algebraic and model-theoretic relations and peculiarities of these algebraic structures with respect to each other.

Indeed, taking up a long standing idea, whose roots date back to the work of Katriňák [71], we prove that the categories of Boolean algebras with a fixed filter and regular double Stone algebras, that we will denote by $\mathbf{B_F}$ and $\mathbf{T}$ respectively, are equivalent. Let us emphasize the fact that we will provide a full description of this equivalence, whose intuition was mentioned as a side fact (and in different terms and aims) in a paper by Düntsch [41].

Albeit this strong categorical relation, the above mentioned categories involve definitely different model theories. For a modicum of an example, in the case of Boolean algebras the notions of atomic and atomistic objects are equivalent. This

[1]For this reason, and for readability sake, we will concentrate only on distributive objects, leaving as a future project a possible investigation of non distributive generalizations of classical logic.

fact is no longer true for regular double Stone algebras. Actually, a regular double Stone algebra is atomistic if and only if it is a Boolean algebra. This fact is quite surprising. In fact, let us notice in passing that for orthomodular lattices [11], which are a non distributive generalization of Boolean algebras, the notions of atomic and atomistic coincide. Whilst it is no longer the case for a distributive generalization such as regular double Stone algebras.

In section 2.2 we provide a detailed description of an effective procedure that allows to construct a regular double Stone algebra out of any Boolean algebra. Then, in section 2.3 we capitalize on this construction and we show that it opens the doors to a full fledged categorical equivalence.

In section 2.4, we discuss several aspects of the structure theory of regular double Stone algebras. Apparently, this theory behaves quite smoothly due to the fact that a good deal of relevant features can be carried over from the structure theory of Boolean algebras. With remarkable distinctions, however.

In section 2.5, we consider some model-theoretic properties and relations that the categories $\mathbf{B_F}$ and $\mathbf{T}$ possess, being $\mathbf{B_F}$ and $\mathbf{T}$ tightly related from a categorical point of view.

Surprisingly enough, we may observe that even though $\mathbf{B_F}$ and $\mathbf{T}$ are equivalent as categories, as shown in section 2.3, there are model-theoretic aspects that do not transfer from one category to the other. It is the case of the relation between the concepts of atomic and atomistic or $\aleph_0$-categoricity. We also provide a description of this fact in terms game/model-theoretic by explicitly showing an instance of the Ehrenfeucht-Fraïssé game for Boolean algebras and regular double Stone algebras. This will show that these varieties are in fact not elementary equivalent.

On the positive side, we will completely characterize the algebraic closures in the variety of regular double Stone algebras, and describe the relations that they possess with the algebraic closures of Boolean algebras. To this aim a normal form theorem will be expedient, since this result will allow us to describe and construct the algebraic closures in the variety of regular double Stone algebras.

Finally, we close the section with a few categoricity results that will be enriched at the end of section 2.6, once the notion of injective object is investigated.

In the last section 2.6, we propose, as an application of our construction, an effective description of the injective objects for the variety of Stone algebras. As a direct consequence, an explicit illustration of the injectives for the variety of regular double Stone algebras, Kleene algebras, Stone algebras, and 3-valued Łukasiewicz algebras will be shown.

Finally, we discuss a few categoricity results for these classes of injective objects. In particular we will see that the presence of the fixpoint plays a definite role as regards injectivity and categoricity.

We close the chapter by listing possible future investigations and open problems.

2.2 A representation of regular double Stone algebras

In this section we provide a representation of regular double Stone algebras. Our construction will be as follows. First, we start with a given Boolean algebra B, then we adapt a long dated construction on B, whose roots can be traced back to the early works of Moisil, A. Monteiro and L. Monteiro, Düntsch, and Katriňák [83, 18, 88, 86, 85, 41, 71].

We show that what we obtain is a regular double Stone algebra, and then an exact description of its elements will be discussed. Furthermore, we characterize certain subsets that will play a relevant role in the whole structure of the chapter.

We also prove that given a regular double Stone algebra A we can neatly describe it in terms of elements of a uniquely determined Boolean algebra.

In particular, the case in which a regular double Stone algebra possesses a fixpoint.

The results in this section will be preliminaries to those in Section 2.3 where a categorical equivalence will be established.

To approach the construction, let $B = (B, \wedge, \vee, ', 0, 1)$ be a Boolean algebra, and F a filter on B. Then, let us consider the following set:

$$\mathcal{A}(B,F) = \mathcal{A}(B) = \{(x,y) \in B^2 : (x \leq y) \,\&\, (x \vee y' \in F)\}. \tag{A}$$

In general, since no confusion will be possible, we will denote filter generation by $\uparrow$.

We observe that whenever F is a principal filter of the form $\uparrow z$ then, by residuation, display (A) simplifies to:

$$\mathcal{A}(B,F) = \mathcal{A}(B) = \{(x,y) \in B^2 : (x \leq y) \,\&\, (z \wedge y \leq x)\}.$$

We will treat the improper filter as principal: namely as the 0-generated filter $\uparrow 0$. With a slight abuse of language, when there is no risk of confusion, for the sake of readability, we will write $\mathcal{A}(B)$ instead of $\mathcal{A}(B,F)$. Otherwise, when we want to highlight the Boolean algebra B and the filter F involved in the construction, we will use the following notation $[B,F]$.
On $\mathcal{A}(B)$ we consider $\wedge, \vee$ componentwise, and let us define the following unary operations on $\mathcal{A}(B)$:

$$(x,y)' = (y',x'); \quad (x,y)^{\sim} = (y',y'); \quad (x,y)^{+} = (x',x'). \tag{2.1}$$

Let us observe that this construction is well known in the literature in case the filter is not proper (see e.g. [4]).

Lemma 9 is a first step towards an representation theorem, since it shows that, for any Boolean algebra B and a fixed filter, $\mathcal{A}(B)$ is indeed a Stone algebra.

Lemma 9. *Let B be a Boolean algebra, and F a filter on B. Then $\mathcal{A}(B) = (\mathcal{A}(B), \wedge, \vee, {}^{\sim}, 0, 1)$ is a Stone algebra.*

Proof. We first prove that $\mathcal{A}(B)$ is closed under the operations. Let $(x,y),(z,w) \in \mathcal{A}(B)$, then $(x,y) \wedge (z,w) = (x \wedge z, y \wedge w)$, but $x \wedge z \leq y \wedge w$ by monotonicity, and also we have that:

$$\begin{aligned}(x \wedge z) \vee (y \wedge w)' = & \quad (x \wedge z) \vee (y' \vee w') \\ = & \quad (x \wedge z) \vee y' \vee w' \\ = & \quad ((x \vee y') \wedge (z \vee y')) \vee w' \\ = & \quad (x \vee y' \vee w') \wedge (z \vee y' \vee w').\end{aligned}$$

Upon noticing that $x \vee y', z \vee w' \in F$, and the fact that they are less or equal than $(x \vee y' \vee w'), (z \vee y' \vee w')$, respectively, then $(x \vee y' \vee w') \wedge (z \vee y' \vee w') \in F$. The proof for $\vee$ is completely dual. As regards $^{\sim}$, $(x, y)^{\sim} = (y', y')$ and $y' \vee y'' = 1 \in F$. That $(0, 0), (1, 1) \in \mathcal{A}(B)$ is straightforward. We now show that $\mathcal{A}(B)$ is pseudocomplemented. In fact, if $(x, y) \wedge (z, w) = (0, 0)$, then $x \leq y \leq w'$, and so $(x, y) \leq (w', w') = (z, w)^{\sim}$. Conversely, if $(x, y) \leq (w', w') = (z, w)^{\sim}$, then $x \leq y \leq w'$, thus $w \leq x', y'$, and so $(x, y) \wedge (z, w) = (0, 0)$. Finally, $(x, y)^{\sim} \vee (x, y)^{\sim\sim} = (y', y') \vee (y, y) = (1, 1)$, i.e. Stone condition holds. $\square$

Indeed, as a matter of fact, $\mathcal{A}(B)$ can be regarded as a double Stone algebra, which is also regular.

Lemma 10. *The algebra* $\mathcal{A}(B) = (\mathcal{A}(B), \wedge, \vee, ^{\sim}, 0, 1)$ *is a regular double Stone algebra.*

Proof. Simply set $(x, y)^{+}$ as defined in Equation (2.1). Let us in passing also observe that $(x, y)^{+} = (x, y)'^{\sim\sim}$. The proof that $^{+}$ is a dual pseudocomplementation is analogous to the argument for $^{\sim}$ in Lemma 9. As regards regularity, suppose that $(x, y)^{\sim\sim} = (z, w)^{\sim\sim}$ and $(x, y)^{++} = (z, w)^{++}$, then $(y, y) = (w, w)$ and $(x, x) = (z, z)$, i.e. $(x, y) = (z, w)$. $\square$

We have seen that $\mathcal{A}(B)$ is a regular double Stone algebra. Let us now describe the sets of its sharp, dense and dually dense elements. These sets will play an important role in the representation of any regular double Stone algebra in the next section.

In fact, for any Boolean algebra B and any filter F in B, it can be seen that the set of sharp elements of $\mathcal{A}(B)$ is isomorphic to the Boolean algebra B.

Lemma 11. *Let B be a Boolean algebra. Then $S_K(\mathcal{A}(B))$ is a Boolean algebra isomorphic to B.*

Proof. Consider the mapping $g : B \to S_K(\mathcal{A}(B))$ defined by:

$$g(x) = (x, x). \tag{G}$$

First of all, we notice that $(x, x) \in S_K(\mathcal{A}(B))$. In fact,

$$(x, x) \wedge (x, x)' = (x, x) \wedge (x', x') = (x \wedge x', x \wedge x') = (0, 0).$$

Dually for $(x, x) \vee (x, x)' = 1$.
Clearly, g is a lattice endomorphism that also preserves $'$. We now show that g is surjective. In fact, if $(x, y) \in S_K(\mathcal{A}(B))$, then by definition $0 = (x, y) \wedge (x, y)'$. By construction, $x \leq y$ and $x' \wedge y = 0$ and therefore, since x, y are Boolean elements, we can conclude that $x = y$. □

The next remark points out the fact that, for a regular double Stone algebra A, the sharp elements of A are amenable of a neat characterization.

Remark 16. *From the proof of Lemma 11 it may be worth to observe that:*

$$S_K(\mathcal{A}(B)) = \{(x, y) : x = y\}.$$

Also the set of dense and dually dense elements are tidely representable via the construction displayed in condition (A).

Lemma 12. *Let B be a Boolean algebra and F a filter. Then,*

1. $D^{\sim}(\mathcal{A}(B)) = \{(x, y) \in \mathcal{A}(B) : x \in F \text{ and } y = 1\} = \{(x, 1) : x \in F\}$;
2. $D^{+}(\mathcal{A}(B)) = \{(x, y) \in \mathcal{A}(B) : x = 0 \text{ and } y' \in F\} = \{(0, y) : y' \in F\}$.

Proof. (1) If $(x, y) \in D^{\sim}(\mathcal{A}(B))$, then $(x, y)^{\sim} = (y', y') = (0, 0)$. Then, $y' = 0$, i.e. $y = 1$. Moreover, if $(x, 1) \in \mathcal{A}(B)$ then obviously $x \leq 1$ and $x \vee 1' = x \vee 0 = x \in F$.
(2) is dual. □

From Lemma 12, it is possible to observe that there is a remarkably tight relationship between dense and dually dense elements, and the filter considered in the construction. This fact will be of definite importance in the categorical equivalence that we will discuss in section 2.3.

As a direct corollary, we can easily observe that dense and dually dense elements correspond to specific subsets of the Boolean algebra on which the construction is performed.

Corollary 1. *Let B be a Boolean algebra and F a filter. Then,*

1. *$D^{\sim}(\mathcal{A}(B))$ is a filter;*

2. *$D^{+}(\mathcal{A}(B))$ is a ideal.*

Proof. (1) Follows directly from Lemma 12 by observing that $D^{\sim}(\mathcal{A}(B)) = \{(x, 1) : x \in F\}$. (2) is dual. □

Actually, it can be seen that the correspondence between the set of dense elements and the filter in the construction is in fact a lattice isomorphism. The same result can be proven dually for the set of dually dense elements.

Lemma 13. *Let B be a Boolean algebra and F a filter in B. Then, $D^{\sim}(\mathcal{A}(B))$ is lattice-isomorphic to F.*

Proof. Let us define a mapping $f : D^{\sim}(\mathcal{A}(B)) \to F$ defined by $f((x, 1)) = x$. It is clear that $\wedge$, $\vee$ are preserved, as well as the top element. Clearly, if $D^{\sim}(\mathcal{A}(B))$ has a minimum, then its image via f is the minimum of F. □

Lemma 14 will be important to our discourse. Indeed, it shows that any regular double Stone algebra A has "encoded" in its structure all the information on the Boolean algebra and the filter that via the construction originates A.

To ease the readability, with a slight abuse of language, in the next lemma we set again $f : D^{\sim}(A) \to S_K(A)$. since $B \simeq S_K(A)$.

Lemma 14. *Let A be regular double Stone algebra. Then, $D^{\sim}(A)$ is a sublattice of $S_K(A)$, which is a filter in $S_K(A)$.*

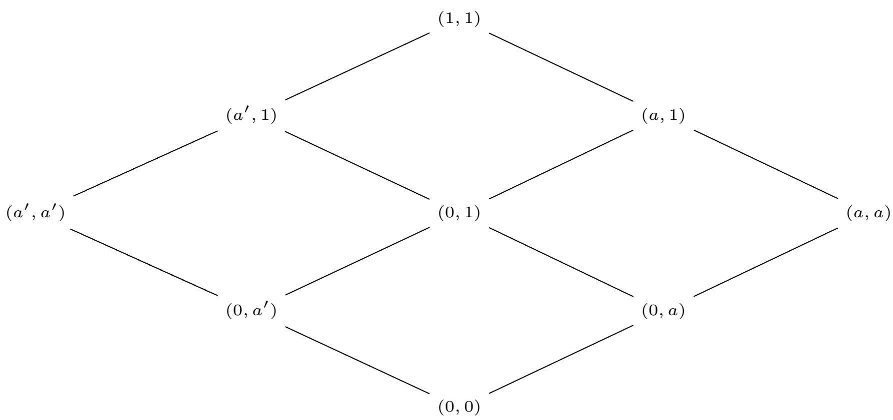
Proof. Let A be regular double Stone algebra. Consider the mapping

$$f : D^{\sim}(A) \to S_K(A) \text{ defined by } f(x) = x^{++} = \Box x.$$

We show that this map is an embedding. Suppose $f(x) = f(y)$, then $x^{++} = y^{++}$. Moreover, $x^{\sim\sim} = y^{\sim\sim} = 1$, since $x, y \in D^{\sim}(A)$, and by regularity $x = y$. Finally, $f(D^{\sim}(A))$ is closed under the lattice operations by virtue of De Morgan.
Now, we prove that $f(D^{\sim}(A))$ is a filter. First, we notice that $1 \in D^{\sim}$ and $\Box 1 = 1$, then $1 \in f(D^{\sim}(A))$. Let $x \in f(D^{\sim}(A))$ and $x \leq y$ in $S_K(A)$. We know that $f(z) = x$ for some $z \in D^{\sim}(A)$, and then we can consider $z \vee y$. Since $(z \vee y)^{\sim} = 0 \wedge y^{\sim} = 0$, then $z \vee y \in D^{\sim}(A)$. Moreover, $f(z \vee y) = z^{++} \vee y^{++} = x \vee y = y$. Therefore $y \in f(D^{\sim}(A))$. $\Box$

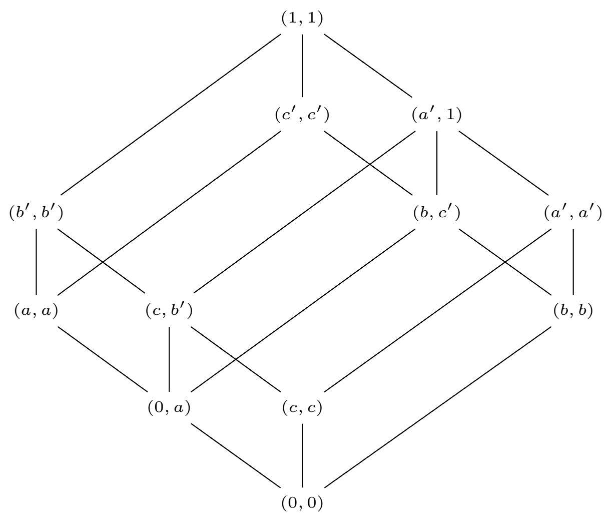
In order to provide some concrete intuitions on the procedure, we now propose a few examples that illustrate how the construction on a Boolean algebra may vary according with the choice of different filters.

Example 17. *Consider the four element Boolean algebra* 4*, whose carrier is* $\{0, a, a', 1\}$*, together with the improper filter. Then,* $\mathcal{A}(4)$ *will be the following:*



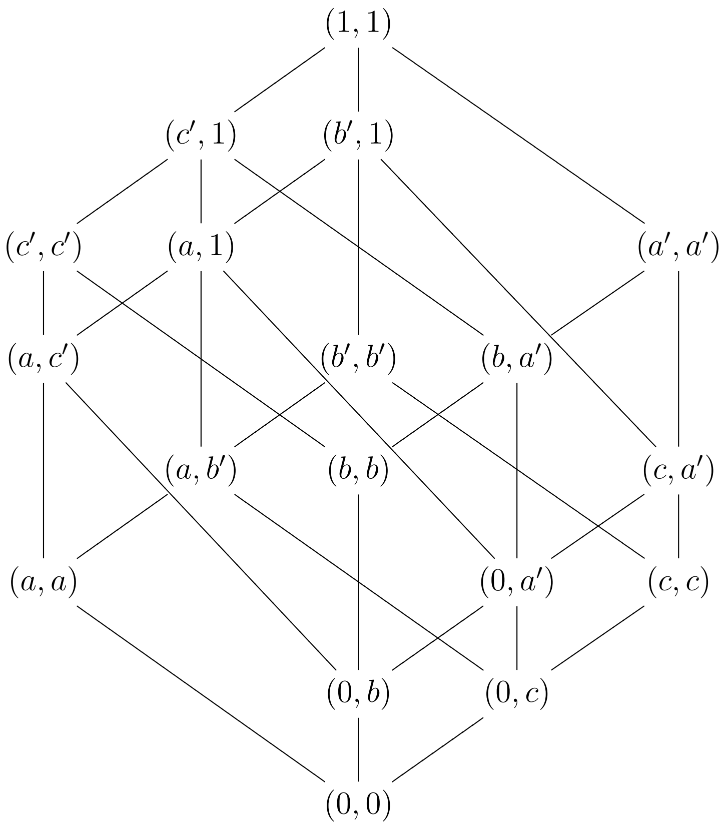
As observed in Lemma 11 and Lemma 12, we can notice that if we consider the algebra in Example 17, then the set of its dense elements is $\{(1,1), (a',1), (a,1), (0,1)\}$, while the set of dually dense elements is $\{(0,1), (0,a'), (0,a), (0,0)\}$, finally the set of its sharp elements is $\{(1,1), (a',a'), (a,a), (0,0)\}$. It can be easily checked that they are all lattices isomorphic to 4 (as a lattice).

Example 18. *Consider the* 8*-element Boolean algebra, whose carrier is* $\{0, a, b, c, a', b', c', 1\}$ *together with the filter* $\{a', 1\}$*. Then,* $\mathcal{A}(8)$ *will be the following:*

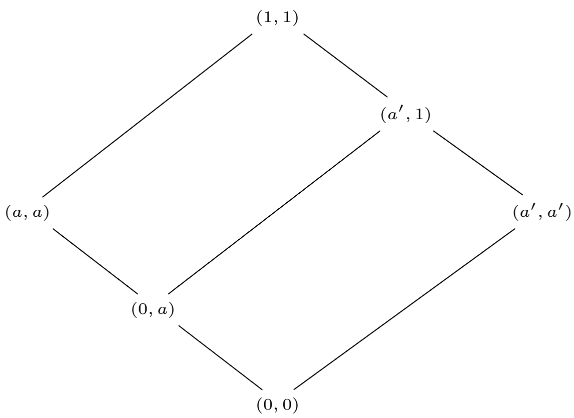


It can be readily seen that in Example 17 and Example 18, the sets of the unsharp elements are $\{(0, a), (0, a'), (0, 1), (a, 1), (a', 1)\}$ and $\{(0, a), (a', 1), (c, b'), (b, c')\}$, respectively.

Example 19. *Consider the* 8*-element Boolean algebra, whose carrier is* $\{0, a, b, c, a', b', c', 1\}$ *together with the filter* $\{a, c', b', 1\}$*. Then,* $\mathcal{A}(8)$ *is the following:*



Example 20. *Consider the* 4*-elements Boolean algebra, whose carrier is* $\{0, a, a', 1\}$ *together with the filter* $\{a', 1\}$*. Then,* $\mathcal{A}(4)$ *will be the following:*



Theorem 21 summarizes the behavior that we have encountered in the previous examples. Actually, it clarifies how the construction differs according with different filters, and the order relations between constructions on a given Boolean algebra.

Theorem 21. *Let B be a Boolean algebra and F and G are filters in B. The following conditions hold:*

1. $[B, \{1\}]$ *is isomorphic to* B*;*
2. *if* $F \subseteq G$*, then* $[B, F]$ *is a subalgebra of* $[B, G]$*;*
3. $[B, B]$ *is the only construction on* B *with a fixpoint for* $'$*;*
4. F *is principal if and only if* $D^{\sim}(\mathcal{A}(B))$ *has a minimum;*
5. *if* F *is maximal, then* $(x, y) \in \mathcal{A}(B)$ *if and only if* $x, y \in F$ *or* $x', y' \in F$*.*

Proof. (1) Let us note that if $(x, y) \in \mathcal{A}(B)$ then $x \leq y$ and $x \vee y' = 1$, i.e. $y \leq x$ and so $x = y$. (2) If $F \subseteq G$, then if $(x, y) \in [B, F]$ we have that $x \vee y' \in F \subseteq G$. Therefore, $(x, y) \in [B, G]$. (3) By item (2), $[B, B]$ is the largest possible construction on B. Moreover, $(0, 1)$ is in $[B, B]$ if and only if the filter is not proper. Routine calculation shows that $(0, 1)$ is the unique possible fixpoint. (4) If F is principal, it is of the form $\uparrow x$ for $x \in B$. Upon recalling that all elements in $D^{\sim}(\mathcal{A}(B))$ are of the form $(y, 1)$ it follows directly that $(x, 1)$ is the minimum in $D^{\sim}(\mathcal{A}(B))$. Conversely, if $D^{\sim}(\mathcal{A}(B))$ has a minimum, it is a principal filter in $[B, F]$, which is isomorphic to F by Lemma 13. (5) Let F be maximal, and $(x, y) \in \mathcal{A}(B)$. Then, $x \leq y$, $x \vee y' \in F$. So, by maximality, either x or y' is in F. If $x \in F$, then $y \in F$. If $y' \in F$, then by upward closure and antitonicity $x' \in F$. □

Let us remark that all the properties that we have described for filters and dense elements can be stated in dual form for ideals and dually dense elements.

Let us now close the present section discussing a particular case of regular double Stone algebras. To this aim we introduce the notion of core of a regular double Stone algebra, see [8, 9, 10].

Definition 26. *Let A be a regular double Stone algebra, we call the* core *of A the intersection between the set of dense and dually dense elements:*

$$D^{\sim}(A) \cap D^{+}(A).$$

Let us notice that if a regular double Stone algebra A has a non-empty core, then it possesses rather remarkable properties. In fact, it can be seen that algebras of this sort are each and all of the form $[B, B]$, for a certain Boolean algebra B. Actually, the element $(0, 1)$ will be in the core of $[B, B]$. Moreover, $(0, 1)' = (0, 1)$, see Equation (1.1). Lemma 15 summarizes these facts.

Lemma 15. *Let A be a regular double Stone algebra. Then,*

1. *if $x \in D^{\sim}(A) \cap D^{+}(A)$, then $x = (0, 1)$.*

2. *the cardinality of the core of A is at most 1.*

3. *x belongs to the core if and only if $x = x'$, i.e. x is a fixed point.*

Proof. 1. Let x be in $D^{\sim}(A) \cap D^{+}(A)$. Then, by Lemma 12, $x = (0, 1)$. Moreover, $A = [B, B]$ by Theorem 21(3). 2. Suppose $x, y \in D^{\sim}(A) \cap D^{+}(A)$, then we have $\Box x = \Box y = 0$ and $\Diamond x = \Diamond y = 1$. Hence, by (Regularity) we conclude that $x = y$. 3. If $x \in D^{\sim}(A) \cap D^{+}(A)$, then $\Box(x') = x^{\sim} = 0 = \Box x$ and $\Diamond(x') = x'^{\sim\sim} = x^{+} = 1 = \Diamond x$. Therefore, $x = x'$. On the other hand, if $x = x'$, we have $\Box x = \Box x' = x^{\sim}$ and $\Diamond x = \Diamond x' = x^{+}$. Then, $x^{\sim} = x^{\sim} \wedge x^{\sim} = x^{\sim} \wedge \Box x = 0$, and $x^{+} = x^{+} \vee x^{+} = x^{+} \vee \Diamond x = 1$. Therefore, $x \in D^{\sim} \cap D^{+}$. $\Box$

In other words, Lemma 15 expresses the fact that a regular double Stone algebra A admits at most one fixed point $k = k'$ which would be dense and dually dense, and moreover $D^{\sim}(A) \cap D^{+}(A) = k$. Also, for a regular double Stone algebra, the concepts of having a non-empty core and possessing a fixpoint of $'$ are equivalent.

Actually, in presence of a fixpoint the following theorem strengthens Lemma 14. Indeed, there is an isomorphism between the lattices of dense and sharp elements of a regular double Stone algebra with a fixpoint.

Theorem 22. *Let A be a regular double Stone algebra with a fixpoint k. Then, $D^{\sim}(A)$, $S_K(A)$ are isomorphic as lattices.*

Proof. Let A be a regular double Stone algebra with a fixpoint k. By Lemma 15, k is both dense and dually dense. Upon setting

$$f : D^{\sim}(A) \to S_K(A) \text{ defined by } f(x) = x^{++} = \Box x$$

that f preserves the operations is proved in Lemma 14. Moreover, $f(k) = k^{++} = 0$. Considering that 0 belongs to the image of f and that this image is a filter (Lemma 14), we can conclude that f is surjective. Injectivity is implied by SK; in fact, if $f(x) = f(y)$, then $\Box x = \Box y$. Moreover, being dense elements, $\Diamond x = \Diamond y = 1$, hence $x = y$, which establishes our claim, i.e. $D^{\sim}(A)$ and $S_K(A)$ are isomorphic as a lattice. Also, it can be readily seen that the mapping $g : S_K(A) \to D^{\sim}(A)$ defined by $g(x) = x \vee k$ is a lattice isomorphism such that $g \circ f = \mathtt{id} = f \circ g$. Hence f is an isomorphism. $\Box$

Let us observe that having a non-empty core yields rather strong properties. Namely, as the next theorem shows, each element can be easily regarded as a combination of sharp elements and the fixpoint.

Theorem 23. *Let A be a regular double Stone algebra with a non-empty core and let $k \in D^{\sim}(A) \cap D^{+}(A)$, then every $x \in A$ can be written as:*

$$x = \Diamond x \wedge (\Box x \vee k);$$
$$x = \Box x \vee (\Diamond x \wedge k).$$

Proof. It is easy to see that the box and the diamond coincide:

$$\begin{aligned} \Box\left(\Diamond x \wedge (\Box x \vee k)\right) = &\quad \Diamond x \wedge (\Box x \vee 0) \\ = &\quad \Box x. \end{aligned}$$

$$\begin{aligned} \Diamond\left(\Diamond x \wedge (\Box x \vee k)\right) = &\quad \Diamond x \wedge (\Box x \vee 1) \\ = &\quad \Diamond x. \end{aligned}$$

Therefore, by (Regularity), $x = \Diamond x \wedge (\Box x \vee k)$. The other equation can be proved dually. □

The next result (which is folklore with respect to the fixpoint case, but it is not so in the other one) shows that the fact of having a fixpoint or a minimal dense element cannot be captured at an equational level.

Theorem 24. *The class of regular double Stone algebras admitting a minimal dense element is not a variety.*

Proof. It is immediate to see that the class of regular double Stone algebras admitting a fixpoint is not closed under the operator S, e.g. consider the 2-elements subalgebra $\{0, 1\}$. In general, if there is a minimal dense element then, by virtue of the results in this section (Theorem 21(4)) we may consider an object of the form $[B, F]$, with F a principal filter.

Without any loss of generality, we can consider A of the form $[B, B]$, with B an infinite Boolean algebra. Then, we can freely consider a non-principal filter G on B, whose existence is proven in e.g. [56]. Therefore, by Theorem 21(4) $[B, G]$ has no minimal dense elements. As a consequence, the class of regular double Stone algebras admitting a minimal dense element is not closed under the class operator S, i.e. they do not form a variety. □

Let us remark though that the class of regular double Stone algebras admitting a minimal dense element is obviously closed under the operators H, P.

2.3 A categorical equivalence

We start our discourse by focusing on a rather reasonable category, whose objects are Boolean algebras indexed by a given filter (quotient). On the one hand, as natural as it is, this category properly contains the category of Boolean algebras, which serves as paramount models of classical reasoning, on which all contemporary mathematics is based [56]. On the other hand, we consider another category whose objects are regular double Stone algebras, indexed by the filter of dense elements. In general, for regular double Stone algebras we may dispense of these indexes. However, this notation will be expedient in easing our arguments.

Regular double Stone algebras are a natural generalization of Boolean algebras both to the case in which *tertium non datur*, symbolized as $x \vee x' = 1$, and *non contradiction principle*, symbolized as $x \wedge x' = 0$, need not hold for $^{\sim},^{+}$, respectively (see [59, 3, 60]). Precisely, $^{\sim}$ does not satisfy tertium non datur, and $^{+}$ does not satisfy non contradiction principles. As observed by Katriňák and Varlet in [70, 98], regular double Stone algebras form a variety that includes (as reducts) Kleene and Stone algebras, and they are term equivalent with 3-valued MV algebras [62] and 3-valued Łukasiewicz algebras [31, 64].

Surprisingly enough, we will see that the categories of Boolean algebras indexed by a filter and regular double Stone algebras (indexed by their dense elements) are from an abstract point of view different perspectives on the same concept: they will be shown to be equivalent. We will elaborate explicitly on a long standing construction, which traces back to the work of Moisil [83]. This idea has attracted the work of many authors in various contexts, from general algebra to rough sets, see e.g. [41] and [71].

Let us now explicitly introduce the two categories that will play a quite central role in this whole chapter.

Definition 27. *We call* Boolean algebras with fixed filter*, and denote by* $\mathbf{B_F}$*, the category defined as follows:*

1. *the* objects *are pairs* $[B, F]$*, where* B *is a Boolean algebra and* F *is a filter of* B*;*
2. *the* morphisms *are Boolean homomorphisms that preserve the filters, that is, if* $f : [A, F] \to [B, G]$ *in* $\mathbf{B_F}$*, then* $f(F) \subseteq G$*.*

Namely, f *is a Boolean homomorphism such that*

Since no danger of confusion will be impending, to ease the discourse we will sometimes call $\mathbf{B_F}$ the category of *indexed Boolean algebras*. Let us remark that this naming is not to be referred by any mean to the concepts of indexed category or comma category [1].

We will also denote by **B** the category of Boolean algebras, with the usual morphisms. It will be shown in due course that **B** can be regarded as a full subcategory of $\mathbf{B_F}$.

Definition 28. *The category* twist*, denoted by* **T***, is defined as follows:*

1. *the* objects *are pairs* $(A, D^{\sim}(A))$*, where* A *is a regular double Stone algebra and* $D^{\sim}(A)$ *is the filter of dense elements of* A*;*

2. *the* morphisms *are homomorphisms.*

Let us remark that in Definition 28, the homomorphisms always preserve $D^{\sim}(A)$. Furthermore, we recall that to define the category **T**, it would be sufficient to consider regular double Stone algebras and homomorphisms, as the moment you pick a regular double Stone algebra A, $D^{\sim}(A)$ is already determined.

Theorem 25 relies heavily on the representation of regular double Stone algebras, and it shows that the information in the category $\mathbf{B_F}$ can be faithfully shifted by the functor Ψ in the category **T**.

Theorem 25. *Let* $[B_1, F_1]$, $[B_2, F_2]$ *objects in* $\mathbf{B_F}$ *and* $g : [B_1, F_1] \to [B_2, F_2]$ *a morphism. Upon defining the functor* $\Psi : \mathbf{B_F} \to \mathbf{T}$ *by setting:*

$$\Psi([B, F]) = \mathcal{A}(B) = [B, F], \text{ and } (\Psi(g))(x, y) = (g(x), g(y)),$$

the following diagram commutes:

$$\begin{array}{ccc} [B_1, F_1] & \xrightarrow{\quad g \quad} & [B_2, F_2] \\ \big\downarrow{\scriptstyle \Psi} & & \big\downarrow{\scriptstyle \Psi} \\ (A_1, D^{\sim}(A_1)) & \xrightarrow{\quad \Psi(g) \quad} & (A_2, D^{\sim}(A_2)) \end{array} \qquad (\star)$$

Proof. To ease the notation, since no confusion will be impending, we leave the filters in the category $\mathbf{B_F}$ implicit. The fact that $\Psi([B, F]) = \mathcal{A}(B)$ is a regular double Stone algebra is from Lemma 9. It can be seen that for all $(x, y) \in A_1$, $(\Psi(g))(x, y) = (g(x), g(y))$ is such that $g(x) \leq g(y)$ and since g is a morphism, it follows that $g(x) \vee g(y)' \in g(F_1) \subseteq F_2$, i.e. $\Psi(g)$ is well defined. That $\Psi(g)$ preserves the lattice operations and bounds is immediate. As regards $^\sim$, let $x = (x_1, x_2) \in A_1$, then:

$$\begin{aligned}(\Psi(g))(x^\sim) = &\ (\Psi(g))((x_1, x_2)^\sim) \\ = &\ (\Psi(g))((x_2', x_2')) \\ = &\ (g(x_2'), g(x_2')) \\ = &\ (g(x_2)', g(x_2)') \\ = &\ ((\Psi(g))(x))^\sim.\end{aligned}$$

The argument for $^+$ is dual. For the commutativity of diagram $(\star)$, it can be seen that $\Psi(g(x)) = (g(x), g(x)) = (\Psi(g))(\Psi(x))$. □

Let us observe that, in general, the category **B** can be regarded as a full subcategory of $\mathbf{B_F}$ and **T**. In fact, for any triple (B_1, B_2, f) in **B**, with $f : B_1 \to B_2$, there always exists a natural transformation. Specifically,

Theorem 26. *Let $f : B_1 \to B_2$ be a triple in* **B**. *Then,*

1. *there always exists a natural transformation by considering as objects $[B_1, f^{-1}(1)]$, $[B_2, \{1\}]$, and $\Psi(f(x)) = \Psi(f(x_1, x_2)) = (f(x_1), f(x_2))$ componentwise;*

2. *for any G filter on B_2, there always exists a natural transformation by considering as objects $[B_1, f^{-1}(G)]$, $[B_2, G]$, and $\Psi(f(x)) = \Psi(f(x_1, x_2)) = (f(x_1), f(x_2))$ componentwise.*

Proof. 1. Consider the system $f : B_1 \to B_2$ in **B**. By Lemma 10 we can map $[B_1, f^{-1}(1)]$ in $\mathcal{A}(B_1, f^{-1}(1))$ and $[B_2, \{1\}]$ in $\mathcal{A}(B_2, 1)$. Then, we define $\Psi : \mathcal{A}(B_1, f^{-1}(1)) \to \mathcal{A}(B_2, 1)$ componetwise as follows:

$$\Psi(f(x)) = \Psi(f(x_1, x_2)) = (f(x_1), f(x_2)).$$

The rest follows from Theorem 25.

2. It is analogous to the first point.

□

Let us observe that given a triple (B_1, B_2, f) in **B** the conditions in Theorem 26 are necessary.

In fact consider the Boolean algebras 4 in Example 20 and the 8-elements Boolean algebra 8. Let the morphism $f : 4 \to 8$ defined by $f(0) = f(a) = 0$. In Example 20 the construction is performed with the filter $\{a', 1\}$. Then, condition (1) of Theorem 26 applies and therefore the system finds a natural transformation in $[8, \{1\}]$. Instead, if we consider $\{a, 1\}$ as filter in 4, then it is not possible to construct a natural transformation in $[8, \{1\}]$, since $\{a, 1\}$ is not $f^{-1}(1)$, and therefore it would not be possible to define (f, f) componentwise.

Theorem 27 refines the representation presented in section 2.2. In fact, it is shown that the representation is exact for any regular double Stone algebra: the Boolean algebra and the fixed filter in the construction must be unique.

Therefore, any regular double Stone algebra A can be thought of as a pair either of two copies of the same Boolean algebra, if A has a fixpoint, or as a pair of Boolean algebras, if the filter is principal, or as a pair composed by a Boolean algebra and a (dual) generalized Boolean algebra [48]. Let us note that the ideas elaborated in the present section are mentioned as a side fact, and with different aims and techniques, by Düntsch in [41].

Theorem 27. *Every regular double Stone algebra is of the form $\mathcal{A}(B)$, for a unique Boolean algebra with a fixed filter $[B, F]$.*

Proof. Let A be a regular double Stone algebra, and consider $S_K(A)$ and $D^{\sim}(A)$. Clearly, $B = S_K(A)$ is a Boolean algebra and by Lemma 14 (the image of) $D^{\sim}(A)$ is a filter $F = f(D^{\sim}(A))$ on such set. Fixing F, we can perform $\mathcal{A}(B, F)$ and define a mapping $\phi : A \to \mathcal{A}(B)$ by $\phi(x) = (x^{++}, x^{\sim\sim})$. It is evident that ϕ is well defined. By (Regularity), if $\phi(x) = (x^{++}, x^{\sim\sim}) = (y^{++}, y^{\sim\sim}) = \phi(y)$ then $x = y$. Therefore, ϕ is injective. Due to Theorem 10, $\wedge, \vee, 0, 1$ are preserved. For $^{\sim}$:

$$\begin{aligned}\phi(x^{\sim}) &= (x^{\sim++}, x^{\sim\sim\sim}) \\ &= (x^{\sim}, x^{\sim}) \\ &= (x^{++}, x^{\sim\sim})^{\sim} \\ &= \phi(x)^{\sim}.\end{aligned}$$

Let us observe that the equality $(x^{\sim}, x^{\sim}) = (x^{++}, x^{\sim\sim})^{\sim}$ is justified by the definition of $^{\sim}$ in display (2.1).

A dual argument applies to $^{+}$. We now claim that ϕ is onto. Let $(x, y) \in \mathcal{A}(B)$. By construction $x, y \in S_K(A)$, $x \leq y$ and $x \vee y' \in f(D^{\sim}(A))$. Since f is injective, $x \vee y'$ can be thought of as an element of $D^{\sim}(A)$. To ease the arguments we can denote by $f^{-1}(x \vee y')$ (see Lemma 13 and Lemma 14) the preimage of $x \vee y'$. Then, consider $z = y \wedge f^{-1}(x \vee y')$. On the one hand,

$$\begin{aligned} z^{\sim\sim} = &\quad (y \wedge f^{-1}(x \vee y'))^{\sim\sim} \\ = &\quad y^{\sim\sim} \wedge (f^{-1}(x \vee y'))^{\sim\sim} \\ = &\quad y \wedge 1 \\ = &\quad y \end{aligned}$$

because $f^{-1}(x \vee y')$ is a dense element, then $f^{-1}(x \vee y') = 1$ and y is in $S_K(A)$, then $y^{\sim\sim} = y$. On the other hand:

$$\begin{aligned} z^{++} = &\quad (y \wedge f^{-1}(x \vee y'))^{++} \\ = &\quad y^{++} \wedge (f^{-1}(x \vee y'))^{++} \\ = &\quad y \wedge f \circ f^{-1}(x \vee y') \\ = &\quad y \wedge (x \vee y') \\ = &\quad (y \wedge x) \vee (y \wedge y') \\ = &\quad (y \wedge x) \vee 0 \\ = &\quad y \wedge x \\ = &\quad x. \end{aligned}$$

Thus, $\phi(z) = (x, y)$, which establishes our claim that ϕ is onto. □

As a consequence of Theorem 27 we readily obtain that if any pair of regular double Stone algebras share isomorphic dense and sharp sets, they are indeed isomorphic as structures. In fact,

Corollary 2. *Any regular double Stone algebra A is uniquely determined by the system $[S_K(A), f(D^{\sim}(A))]$, where f is as in Lemma 13 and Lemma 14.*

Also, combining Theorem 27 and Corollary 2 with Theorem 22, it is immediate to observe that:

Corollary 3. *Any regular double Stone algebra A with a fixpoint is uniquely determined by $S_K(A)$, as well as by $D^{\sim}(A)$.*

Remark 28. *In spite of Corollary 2 and Corollary 3, we note that there are regular double Stone algebras that share the same dense and dually dense elements, but their sharp elements are not isomorphic as Boolean algebras, as Example 18 and Example 20 show.*

By virtue of the Lemma 10, Lemma 11, Lemma 14 and Theorem 27, we can state as a consequence Theorem 29 which describes a functor Ξ converse to Ψ, that faithfully lifts the information in $\mathbf{T}$ to the category $\mathbf{B_F}$.

Theorem 29. *Let Ψ be the functor as defined in Theorem 25. Define the functor $\Xi : \mathbf{T} \to \mathbf{B_F}$ by setting:*

$$\Xi((A, D^{\sim}(A))) = (S_K(A), f(D^{\sim}(A))), \ \text{and } \Xi(h) = h \upharpoonright_{S_K(A)},$$

the following diagram commutes:

$$\begin{array}{ccc} (A_1, D^{\sim}(A_1)) & \xrightarrow{\quad h \quad} & (A_2, D^{\sim}(A_2)) \\ \Big\downarrow \Xi & & \Big\downarrow \Xi \\ [S_K(A_1), f(D^{\sim}(A_1))] & \xrightarrow{\Xi(h)} & [S_K(A_2), f(D^{\sim}(A_2))] \end{array} \qquad (\star\star)$$

Moreover, Ψ, Ξ are mutually inverse functors.

Combining Theorem 25 and Theorem 29 we obtain the following major result:

Corollary 4. *The categories $\mathbf{B_F}$ and $\mathbf{T}$ are equivalent.*

2.4 Structure theory and standard completeness

In this section we concentrate on the structure theory of regular double Stone algebras. This theory behaves quite smoothly due to the several relevant features are carried over from the structure theory of Boolean algebras, with remarkable distinctions, however. For regular double Stone algebras e.g. model categoricity is valid only on finite objects, being the presence of a fixpoint a discriminating fact as we shall observe.

Let us remark that several results on the structure theory of the classes in question may be also viewed as consequences of general Mal'cev conditions (see e.g. [23, 80]). Nonetheless, on the one hand, in our opinion it may be worth to study the structure theory of these classes from a unifying point of view, which makes a transparent use of the categorial relation between the varieties that we are considering. On the other hand, the tight correspondence that holds between the structure theories of Boolean algebras and regular double Stone algebras will be definitely expedient in the investigation of their model-theoretic relations, which will be discussed in section 2.5 and section 2.6.

Following [70] (see also [98]) we now introduce the notion of B-filter (briefly, filter) in the variety of regular double Stone algebras. This concept will be expedient as a bridge between Boolean algebras and regular double Stone algebras. Making use of the results in section 2.2 and section 2.3, we will show that the whole theory of regular double Stone algebras can be traced back to the theory of the Boolean algebras they are determined by. Albeit major distinctions arising when leaving finite model theory to infinite contexts.

Definition 29. *Let A be a regular double Stone algebra. A lattice filter $F \subseteq A$ is* B-filter *if and only if whenever $a \in F$, then $\Box a \in F$, i.e. F is closed under "$\Box$".*

B-filters are adequate in describing the congruence lattice of a regular double Stone algebra. In fact, B-filters enjoy a large deal of the properties that the Boolean filters possess. As an example, the notions of prime B-filter and maximal B-filter are equivalent. If we relax the condition in Definition 29 to the case of lattice filters, then it is no longer so. Actually, if a lattice filter is maximal then it is also prime. But the converse need not hold. For instance, we may consider Example 17, and the filter $F = \{(a, a), (a, 1), (1, 1)\}$. Clearly, F is a prime lattice filter (in fact, a is join-irreducible as sharp element). However, F is not maximal as a lattice filter.

We will denote by $Fil(A)$ *the lattice of B-filters of A.* As in the Boolean case, it is known that the notion of B-filter is perfectly adequate to describe the congruence lattice of a regular double Stone algebra. Furthermore, we will show that the whole congruence lattice of a regular double Stone algebra corresponds precisely to the congruence lattice of its sharp elements, and they form a Boolean subalgebra.

Theorem 30. *[70, 98] Let A be a regular double Stone algebra. Then, $Fil(A)$ and $Con(A)$ are isomorphic.*

Making use of this general result we can derive the following theorem, which explicitly shows that, in any regular double Stone algebra A, the lattices of filters in $S_K(A)$ and the B-filters of A are isomorphic.

Theorem 31. *Let A be a regular double Stone algebra. Then, the lattices $Con(A)$, $Con(S_K(A))$ are isomorphic.*

Proof. Let A be a double regular Stone algebra. By Theorem 30 we can confine ourselves in considering $Fil(A)$ and $Fil(S_K(A))$. Set $f : Fil(S_K(A)) \to Fil(A)$ as follows:

$$f(F) = \uparrow F = \{x \in A : \exists y \in F \text{ such that } (y \leq x)\}, \text{ for } F \in Fil(S_K(A)).$$

We first prove that $\uparrow F$ is in $Fil(A)$. Indeed, if $x \in \uparrow F$, then $x \geq y \in F$. So, by monotonicity $\Box y = y \leq \Box x \in \uparrow F$, that $\uparrow F$ is a lattice filter is straightforward. We now show that the mapping is a one to one homomorphism. Let us suppose that $f(F_1) = \uparrow F_1 = f(F_2) = \uparrow F_2$. Then, $(\uparrow F_1) \cap S_K(A) = (\uparrow F_2) \cap S_K(A)$, i.e. $F_1 = F_2$. As regards the operations:

$$\begin{aligned} f(F_1 \cap F_2) &= \uparrow (F_1 \cap F_2) \\ &= \{x \in A : \exists y \in F_1 \cap F_2 \text{ such that } (y \leq x)\} \\ &= (\uparrow F_1) \cap (\uparrow F_2) \\ &= f(F_1) \cap f(F_2). \end{aligned}$$

Concerning the join we have that:

$$f(F_1 \vee F_2) = \uparrow (F_1 \vee F_2) = \{x \in A : \exists y \in F_1 \vee F_2 \text{ such that } (y \leq x)\}.$$

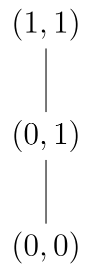
By the definition of join in the lattice of filters, $y \in F_1 \vee F_2$ implies that there exist $f_1 \in F_1$, $f_2 \in F_2$ such that $f_1 \wedge f_2 \leq y$. If $y \leq x \in A$, then $x \geq y = y \vee (f_1 \wedge f_2) = (y \vee f_1) \wedge (y \vee f_2)$, with $y \vee f_1 \in \uparrow F_1$, $y \vee f_2 \in \uparrow F_2$. And so $x \in \uparrow F_1 \vee \uparrow F_2$. The converse inclusion is obvious. Now, set $g : Fil(A) \to Fil(S_K(A))$ as follows:

$$g(F) = \{\Box x \in S_K(A) : x \in F\} = F \cap S_K(A).$$

That $g(F) \in Fil(S_K(A))$ is immediate. If $g(F_1) = g(F_2)$, then for $a \in F_1$, $a \geq \Box a \in F_1 \cap F_2$, and then $a \in F_2$. As regards the operations it is obvious that $g(F_1 \cap F_2) = (F_1 \cap F_2) \cap S_K(A) = g(F_1) \cap g(F_2)$, and $g(F_1 \vee F_2) = (F_1 \vee F_2) \cap S_K(A) = g(F_1) \vee g(F_2)$, by congruence distributivity. Therefore by Theorem 30, $Con(S_K(A))$ and $Con(A)$ are isomorphic lattices. $\Box$

We will see soon that a major consequence of Theorem 31 is the well-known fact that the variety of regular double Stone algebras is generated by the three elements chain viewed as a regular double Stone algebra. This fact is transferred from the theory of Boolean algebras.

We will denote by 3 the three elements regular double Stone algebra obtained from the costruction $\mathcal{A}(2, 2)$. Graphically:



(3)

In parallel with the variety of Boolean algebras, also in the variety of regular double Stone algebras we have a standard completeness theorem. To this aim, let us observe that it is possible to provide a new proof of the follwing well-known fact:

Theorem 32. *The algebras* 2 *and* 3 *are the only subdirectly irreducible (simple) in the variety of regular double Stone algebras.*

Proof. For a modicum of a proof, let us observe first that 3 is obviously subdirectly irreducible, actually 3 is simple. Moreover, if A is not 3 then $S_K(A)$ is not 2. Hence, $Con(S_K(A))$ is non-trivial. Therefore, there are factor congruences θ_1, θ_2 in $Con(S_K(A))$, and by virtue of Theorem 31 $f(\theta_1)$, $f(\theta_2)$ are factor congruences in $Con(A)$. □

A byproduct of the previous fact is that a major deal of the Boolean structure theory freely lifts over regular double Stone algebras.

Corollary 5. *[3] Let A be a regular double Stone algebra. Then,*

1. *every congruence on the lattice reduct of A is a congruence on A;*
2. $Con(A)$ *is arithmetic;*
3. $Con(A)$ *is factorable;*
4. $Con(A)$ *is regular;*
5. $Con(A)$ *is uniform.*

It is well known that the congruence lattice of a finite Boolean algebra is a Boolean algebra. If an infinite Boolean algebra is complete then it is a Stone algebra, and in general it is a pseudocomplemented lattice [56].

We will see that similar achievements are available for the variety of regular double Stone algebras.

In fact, taking advantage of Theorem 31, we can transfer several results from Balbes and Grätzer to the context of regular double Stone algebras.

Corollary 6. *[15, 60] Let A be a regular double Stone algebra. Then,*

1. *if A is finite, then* $Con(A)$ *is a Boolean algebra;*
2. *if A is infinite and complete, then* $Con(A)$ *is a proper Stone algebra;*

3. in general, $Con(A)$ is pseudocomplemented.

We can now provide an equivalent characterization (see [4]) of the pseudocomplement in the Boolean ideal lattice of a regular double Stone algebra. Upon denoting by $\downarrow$ the ideal generation operator, we have that:

Theorem 33. *Let A be a complete regular double Stone algebra. Then, upon setting for any Boolean ideal I:*

$$\Diamond I = \downarrow \bigvee_{i \in I} i,$$

then $I^{\sim} = (\Diamond I)'$, where, for any principal ideal $\downarrow a$, $(\downarrow a)' = \downarrow a'$.

Proof. First, let us note that $\Diamond I$ is a principal ideal, i.e. it is of the form $\downarrow a$, for some a in A. It can be seen that, for all $y \in I^{\sim}$, $y \leq a'$, and then, for all $x \in I$, $y \wedge x \leq a' \wedge x \leq a' \wedge a = 0$. Consequently, $I \cap I^{\sim} = \{0\}$. Also, let Y be an ideal such that $Y \cap I = \{0\}$, Then, for all $y \in Y$, $x \in I$, $x \wedge y = 0$. Therefore, $y \leq x'$ for any x. But then $y \leq a'$, which is the greatest upper bound. Therefore, $Y \subseteq I^{\sim}$ and our claim follows. $\square$

As a direct application of the construction in section 2.2 we can state, without proof, a completeness result for regular double Stone algebras, equivalent to what has been shown by Kumar and Banerjee, and Kumar and Kumari in [73] and [74], respectively.

Corollary 7. *The variety of regular double Stone algebras is $HSP(3) = HSP([2, 2])$.*

A direct application of Jónsson's Lemma 7 and the representation theorem yields directly the following:

Theorem 34. *The variety of regular double Stone algebras is $P_S(3) = P_S([2, 2])$.*

Theorem 34 is closely related to the description through the construction of regular double Stone algebras; however, it is possible to assert an even stronger, well-known result [36].

Theorem 35. *Every regular double Stone algebra is a Boolean product of simple algebras.*

We note that if we restrict our attention to the context of Kleene algebras or Stone algebras, then Theorem 34 is no longer valid. As a counterexample consider the 4-elements Boolean algebra and 4-elements chain.

Let us observe that Theorem 24 is a direct consequence of Corollary 7. In fact, the class of regular double Stone algebras admitting a minimal dense element contains the algebra 3, which generates the whole variety.

However, there are regular double Stone algebras that do not admit a minimal dense element. Actually, even more can be said. In fact, since Di Nola's Theorem [31] the algebra 3 generates the variety of regular double Stone algebras as a quasivariety, meaning that any quasi-equation has solutions in the real interval $[0, 1]$.

Theorem 36 (Di Nola)**.** *A Horn clause* Ψ *holds in* MV *if and only if* Ψ *holds in* $[0, 1]$*, i.e.* MV *as quasivariety is* $ISPP_r([0, 1])$*.*

A byproduct of this fact is that the class of regular double Stone algebras admitting a minimal dense element is not even definable in terms of quasi-identities. That is to say, the condition

$$\forall x \exists ! y ((x^{\sim} = 0) \,\&\, (y^{\sim} = 0) \Rightarrow (y \leq x))$$

is not even expressible as a universal Horn clause.

2.5 A model-theoretic view

In the present section we discuss some model-theoretic properties and relations that the categories $\mathbf{B_F}$ and $\mathbf{T}$ possess. In section 2.3, we have discussed the fact that $\mathbf{B_F}$ and $\mathbf{T}$ are tightly related from a categorical point of view.

First of all, let us observe that even though $\mathbf{B_F}$ and $\mathbf{T}$ are equivalent as categories, as shown in section 2.3, there are model-theoretic properties that do not transfer from one category to the other. It is the case of the relation between the concepts of atomic and atomistic, which are equivalent in $\mathbf{B_F}$, but not in $\mathbf{T}$. We will also provide a description of this fact in game/model-theoretic terms by explicitly describing an Ehrenfeucht-Fraïssé game for Boolean algebras and regular double Stone algebras. This will show that these varieties are in fact not elementary equivalent.

On the positive side, we will completely characterize the algebraic closures in the variety of regular double Stone algebras, and describe the relations they possess with algebraic closures of Boolean algebras. This result refines Clark's Theorem for double Stone algebras [33]. Moreover, these facts are relevant to the work of Schmid on model companions of distributive p-algebras [94].

Let us recall that an algebra B in a variety V is *algebraically closed* if every conjunction p of equations with parameters in B which is satisfiable in some extension of B in V is also satisfiable in B.

To this aim a normal form theorem will be proven. This result will be helpful for describing the algebraic hulls (and closures) in the variety of regular double Stone algebras, which are minimal among the algebraic closures.

Finally, we close the section with a few categoricity results that will be enriched at the end of section 2.6, once the notion of injective object is investigated.
To keep the work self contained, we will provide in this same section the basic model-theoretic notions. For a general account we may refer the reader to [28, 63, 93].

Let us recall that a bounded lattice L is *atomic* in case every element in L is greater than some atom. Also, L is *atomistic* if every element in L can be written as a supremum of the atoms it exceeds.

It may be worth noticing that for Boolean algebras and orthomodular lattices the concepts of atomic and atomistic coincide [60], while this is no longer the case for ortholattices. We encounter a similar scenario here. Rather surprisingly indeed, when these concepts coincide the regular double Stone algebras is deemed to be a Boolean algebra. The interested reader may find relevant and deep results on atomic regular double Stone algebras in [36].

Theorem 37. *For a regular double Stone algebra atomic is equivalent to atomistic* *if and only if it is Boolean.*

Proof. Let A be a regular double Stone algebra, and suppose that A is atomic. Then, by Theorem 27, A can be regarded as a pair $[B, F]$, for a Boolean algebra B and a filter F. Let us notice that B must be atomic as well. In fact, suppose that $y \geq x$ and x is an atom in A. Then, $\Diamond y \geq \Diamond x$ and $\Diamond x$ must be an atom in $S_K(A)$, which is a subalgebra of A. Consider x an atom of B. The image of x in $S_K(A)$ is (x, x). If A is a non-Boolean Stone algebra, for some $x \in B$ there must exist $(0, x)$. Otherwise $F = \{1\}$, a contradiction. We observe that by construction $(0, x)$ is an unsharp atom exceeded by (x, x). Now, by definition, (x, x) cannot be any join of atoms. Therefore, a proper regular double Stone algebra, which is atomic, cannot be atomistic. $\square$

As a concrete case, we may consider Example 20, where (a, a) is not expressible as the supremum of the atoms it exceeds. Actually, (a, a) is join-irreducible, since a is an atom in the Boolean algebra 4.

We now introduce a bridge-notion that will be helpful in clarifying the relation between the concepts of atomic and atomistic for regular double Stone algebras with a fixpoint.

Definition 30. *Let L be a regular double Stone algebra with a fixed point $k \in L$. Then L is called k-*atomistic *if any element $a \in L$ can be written as a lattice polynomial of atoms of $S_K(L)$ and k, for a fixed $k \in L$.*

Theorem 38. *If A is an atomic regular double Stone algebra with a fixpoint k then A is k-atomistic.*

Proof. Let A be an atomic regular double Stone algebra with a fixpoint k. Then by Theorem 23, any $a \in A$ is of the form $\Diamond a \wedge (\Box a \vee k)$. However, because $\Diamond a$, $\Box a$ are in $S_K(A)$, and A is atomic, $S_K(A)$ is atomistic. Hence, both $\Diamond a$ and $\Box a$ are suprema of their atoms, and our result readily follows. $\square$

Let us remark that we cannot extend Definition 30 to the case of atoms in general. For a countermodel, consider Example 17, where (a', a') is not expressible as a combination of atoms of the algebra and the fixpoint. In fact, (a', a') is join and meet-irreducible.

Actually it is immediate to verify the following:

Theorem 39. *Let A be an atomic regular double Stone algebra. Then, an element $a \in A$ is meet (join)-irreducible if both coordinates of its representation $(a_1, a_2) \in [S_K(A), f(D^{\sim}(A))]$ are meet (join)-irreducible.*

Proof. Follows by construction from Lemma 9. □

It is well known that in the case of an atomic Boolean algebra B the compact elements are exactly the atoms of B. In other words, the sole join-irreducible elements of B are its atoms. As we have seen, this is no longer the case for regular double Stone algebras. From Theorem 39, we can characterize the compact elements of an atomic regular double Stone algebra.

Theorem 40. *Let A be an atomic regular double Stone algebra. Then, an element is compact if and only if it is join-irreducible.*

Proof. Straightforward from Theorem 39 and Theorem 27. □

Indeed, in Boolean algebras the notions of atomic and atomistic coincide, and the compact elements are the atoms, for regular double Stone algebras the concepts of atomic and atomistic do not coincide. However, if the algebra is atomic then its compact elements are the join-irreducible elements (which need not be atoms, in general).

We now discuss the fact that, in general, for a Boolean algebra B, its first order theory is not entirely preserved by the construction presented in section 2.2 and elaborated as a categorical equivalence in section 2.3.

In fact, consider the property $x \in At(y)$ (whose intuition reads as "x is an atom under y") defined by

$$\forall y \exists x((x \leq y) \& (x \neq 0) \Rightarrow \forall z((z < x) \Rightarrow z = 0).$$

Clearly, $At(y)$ is a first order property, which in the case of Boolean algebras is equivalent to atomicity, i.e.

$$y = \bigvee (At(y)).$$

Then, as a consequence of Theorem 37,

Theorem 41. *Boolean algebras and regular double Stone algebras are not elementary equivalent. Moreover, their lattice reducts are not elementary equivalent either.*

As a concrete case, consider Example 20. It can be readily seen that (a, a) cannot be expressed as $\bigvee (At((a, a)))$. However, there is a tight connection between atomic Boolean algebras and atomic regular double Stone algebras.

Theorem 42. *A Boolean algebra B is atomic if and only if $[B, F]$ is atomic, for any filter F.*

Proof. It is straightforward to verify that if B is atomic, then it is not possible that $[B, F]$ contains a descending chain with no minimal element. Conversely, if $[B, F]$ is atomic, for any filter F, then $[B, \{1\}] \simeq B$ is such. $\square$

In other words, Theorem 37 and Theorem 42 reflect the fact that there are no regular double Stone algebras that are not Boolean algebras, which are simultaneously both atomic and atomistic.

We now prove a useful technical lemma.

Lemma 16. *Let A be a regular double Stone algebra, then a is an atom in A if and only if $\Diamond x$ is an atom in $S_K(A)$.*

Proof. Suppose x is an atom in A. Due to the construction $x = (0, a)$. By way of contradiction, assume that $\Diamond x$ is not an atom of $S_K(A)$. Then, there is a $y = (b, b) \in S_K(A)$ and $\Diamond x = \Diamond(0, a) = (a, a)$. Therefore, $b \leq a$. And then $a' \leq b'$, and so $A \ni (0, b) \leq (0, a)$, which is not possible.

Conversely, suppose that $x = (a, a)$ is an atom in $S_K(A)$. But if $(b, c) \leq (a, a)$ in A, then $b \leq c \leq a$. But then since a is an atom either $b = c = 0$, or $b = c = a$, or $b = 0$, $c = a$. As a consequence in the latter case $(0, a)$ is an atom. $\square$

The next normal form theorem will be expedient in constructing the algebraic closures in the variety of regular double Stone algebras, and relies deeply on the categorical equivalence presented in section 2.3. In fact, it will be proven that any polynomial in the language of regular double Stone algebras can be written as a conjunction of disjunctions of finite literals, or negated literals.

Theorem 43 (Normal Form)**.** *Any term $p(\vec{x}) = p(x_1, \ldots, x_n)$ in a regular double Stone algebra $[B, F]$ is of the form*

$$\bigwedge_{i,j \leq n} (\bigvee_{i,j \leq n} (x_i, x_j)),$$

where x_i, x_j are either literals or negated literals.

Proof. Induction on the complexity of the polynomial, in symbols $||p(\vec{x})||$ for a polynomial $p(\vec{x})$.

Base: for $||p(\vec{x})|| = 1$ it is immediate.

Suppose that the claim holds for $||p(\vec{x})|| = n$. Two cases are possible. (1) $p(\vec{x}) = \sigma(\vec{x}) * \tau(\vec{x})$, for $* \in \{\wedge, \vee\}$. Then the claim follows by induction hypothesis and distributivity.

(2) $p(\vec{x}) = \sigma(\vec{x})^*$, for $^* \in \{^+, ^\sim\}$. Then the claim follows by strong De Morgan (see Theorem 10 for the definitions) and distributivity. □

Theorem 43 yields rather strong applications to the first order theory of Stone algebras, Kleene algebras, regular double Stone algebras, and 3-valued MV algebras, and will find several applications also in the discussion in section 2.6.

Example 44. *Consider the algebra $[4, \{a', 1\}]$ in Example 20. It can be verified that the polynomial $(0, a) \vee x' = x$ with parameters has no solution in $[4, \{a', 1\}]$, but it finds a solution in $[4, 4]$ of Example 17, which extends $[4, \{a', 1\}]$.*

Let us remark that the same construction of Example 44 can be extended to any regular double Stone algebra with no fixpoint, since it can be embedded into a regular double Stone algebra with a fixpoint.

Remark 45. *Let us notice that, in general, given a polynomial* $p(\vec{a}, \vec{x})$ *in the language of double Stone algebras, with parameters* $\vec{a} \in A$*, then in presence of regularity by Theorem 43 and the results in section 2.2 we can regard* $p(\vec{a}, \vec{x})$ *as a pair of "Boolean polynomials" as follows:*

$$p(\vec{a}, \vec{x}) = p(\overrightarrow{(a_i, a_j)}, \overrightarrow{(x_i, x_j)}) = (q_i(\vec{a}_i, \vec{x}_i), q_j(\vec{a}_j, \vec{x}_j)). \tag{2.2}$$

This is a direct consequence of the representation of regular double Stone algebras given in Section 2.2 and Theorem 27, which allows us to consider each element in regular double Stone algebras as a pair of elements in a Boolean algebra with specific characteristics. Therefore, for every polynomial $p(\vec{a}, \vec{x})$ *in the language of double Stone algebras, with parameters* $\vec{a} \in A$*, computing componentwise yields a pair of Boolean polynomials* $q_i(\vec{a}_i, \vec{x}_i), q_j(\vec{a}_j, \vec{x}_j)$*.*

As a matter of fact, making use of the normal form Theorem 43, it is possible to characterize the algebraically closed algebras in the variety of regular double Stone algebras. Let us recall the well known Clark's theorem. Theorem 47 is a refinement of this.

Theorem 46 (Clark)**.** *For a double Stone algebra* A*, the following are equivalent:*

1. *A is algebraically closed.*
2. *A satisfies the following:*
   - *the core* $D^{\sim}(A) \cap D^{+}(A)$ *of* A *is nonempty and forms a relatively complemented sublattice of* A*;*
   - *for every* $x, y \in D^{\sim}(A) \cap D^{+}(A)$*, there is a* $z \in S_K(A)$ *such that*

$$(x \wedge z) \vee (y \wedge z^{\sim}) = x \vee y.$$

Theorem 47 (Algebraic Closures)**.** *A regular double Stone algebra* A *is algebraically closed if and only if* A *is of the form* $[B, B]$*, with* B *an atomless Boolean algebra.*

Proof. Let A be an algebraically closed regular double Stone algebra. If a polynomial with parameters in A $p(\vec{a}, \vec{x}) = q(\vec{a}, \vec{x})$ finds a solution in an extension, then it finds a solution in A. We observe that any Boolean polynomial can be written in the language of double Stone algebras. And therefore $S_K(A)$ is atomless. Therefore, as a consequence of Lemma 16 and the construction A is atomless. Suppose that A has no fixpoint. Due to Theorem 34, we can freely assume that A embeds into 3^k, for a suitable k. Then, there is a coordinate in the representation of A which is the two elements Boolean algebra 2. Thus, as shown in Example 44 there is a polynomial with parameters in A which solves in 3^k, but not in A.

Conversely, let A be of the form $[B, B]$, with B an atomless Boolean algebra. Then, $B \simeq S_K(A)$ is algebraically closed [93]. Consider a polynomial $p(\vec{a}, \vec{x})$ with parameters in A. Since Equation (2.2), $p(\vec{a}, \vec{x}) = (q_i(\vec{a}_i, \vec{x}_i), q_j(\vec{a}_j, \vec{x}_j))$. Several cases are possible.

(1) $q_i(\vec{a}_i, \vec{x}_i), q_j(\vec{a}_j, \vec{x}_j)$ find solutions d, e, respectively, in $B \simeq S_K(A)$, but $d \not\leq e$. Consider the polynomial $q_i(\vec{a}_i, \vec{x}_i) \vee q_j(\vec{a}_j, \vec{x}_j) = q_j(\vec{a}_j, \vec{x}_j)$. Then, this polynomial does not find solution in $S_K(A)$, and in any of its Boolean extensions, because $S_K(A)$ is algebraically closed. If $p(\vec{a}, \vec{x})$ has a solution in some extension D of A, then $S_K(D)$ extends $S_K(A)$ and in $S_K(D)$ $q_i(\vec{a}_i, \vec{x}_i) \vee q_j(\vec{a}_j, \vec{x}_j) = q_j(\vec{a}_j, \vec{x}_j)$ has a solution, a contradiction.

(2) At least one of the polynomials $q_i(\vec{a}_i, \vec{x}_i), q_j(\vec{a}_j, \vec{x}_j)$ does not find a solution in $B \simeq S_K(A)$. Then, it does not exist any possible extension in which this polynomial finds a solution. As a consequence, there is no extension of A in which $p(\vec{a}, \vec{x})$ resolves.

(3) $q_i(\vec{a}_i, \vec{x}_i), q_j(\vec{a}_j, \vec{x}_j)$ find solutions d, e, respectively, in $S_K(A)$, and $d \leq e$, then they are in A, since it is of the form $[B, B]$.

Therefore, A is algebraically closed. □

We can refine this result by describing that the algebraic closures of a regular double Stone algebra A admit an *algebraically closed hull*.

Theorem 48. *Any regular double Stone algebra A admits an algebraically closed hull.*

Proof. Let A be a regular double Stone algebra. By Theorem 27, A is completely described by $[S_K(A), f(D^{\sim}(A))]$. By construction (or by the generation of the variety), any A can be embedded into a C with fixpoint. To this aim, we now use a version of Cantor's Theorem to show how to construct an atomless regular double Stone algebra in which C embeds. Consider the limit

$$D = \lim \prod^{\omega} C,$$

and embeds C diagonally, i.e. $C \ni c \mapsto (c, c, \ldots, c \ldots,)$. Suppose c is an atom in C. Then,

$$\begin{matrix} (c, & c, & \ldots, & c, & \ldots) \\ (0, & c, & \ldots, & c, & \ldots) \\ (0, & 0, & \ldots, & c, & \ldots) \\ (0, & 0, & 0, & c, & \ldots) \\ & & \vdots & & \end{matrix}$$

is an infinite matrix of strictly decreasing elements of D. And therefore D is atomless and possesses a fixpoint. Therefore D is algebraically closed by Theorem 47. □

We close this section by showing a finite categoricity result for the variety of double regular Stone algebras. However, as we shall discuss in section 2.6, this result does not hold if we stick on infinite model theory. Theorem 49 is well-known, however we provide a new proof.

Theorem 49. *For any finite n, there is at most one regular double Stone algebras A with $|A| = n$.*

Proof. Suppose that there are finite regular double Stone algebras A, B such that $|A| = |B| = n$. By standard completeness, Corollary 7, $A, B \in HSP(3)$. Let us observe that by finiteness $A, B \in HSP_{fin}(3)$. Therefore, $A, B \in SP_{fin}(3)$ since regular double Stone algebras enjoy congruence factorability, Corollary 5. Again by standard completeness $A, B \in P_{fin}(2, 3)$. But then for cardinality reasons and up to factors permutation A, B are isomorphic. □

2.5.1 The Ehrenfeucht-Fraïssé game

In this subsection we propose a game-theoretic description of the fact that Boolean algebras and regular double Stone algebras are not elementary equivalent. To this aim, we describe an application of the Ehrenfeucht-Fraïssé game for these structures [45, 46, 44].
The Ehrenfeucht-Fraïssé game is an effective game-theoretic procedure that allows to determine the expressive power of a language. It makes its first appearance in first-order logic, but finds applications in many other contexts and it plays a role in computer science, having applications in process logics, query languages, logics that capture complexity classes, and regular-like expressions. If one values the expressive power of a logic by considering its ability to distinguish between structures, then measuring this capacity of a logic means to describe the equivalence between structures that holds if they are indistinguishable by formulas of this logic.

The Ehrenfeucht-Fraïssé game is one of the few methods from model theory which works effectively well for finite structures, hence to many definability questions in computer science. Actually, within the developing area of finite model theory, the Ehrenfeucht-Fraïssé method plays a relevant part. This procedure is used so frequently in theoretical computer science since it applies transparently to relational structures like graphs, linear orderings, and partially and lattice ordered structures. Structures of this type are often encountered in many fields of computer science and algebraic logic.

Quoting Hodges [63],

> The theory behind back-and-forth games [*Ehrenfeucht-Fraïssé games*] uses very few assumptions about the logic in question. As a result, these games are one of the few model-theoretic techniques that apply as well to finite structures as they do to infinite ones, and this makes them one of the cornerstones of theoretical computer science. One can use them to measure the expressive strength of formal languages, for example database query languages. A typical result might say, for example, that a certain language cannot distinguish between "even" and

"odd".

For readability convenience, let us first recall the notion of Ehrenfeucht-Fraïssé game [63, 93].

Definition 31. *Consider two structures A and B, each with no function symbols and the same set of relation symbols, and fix $n \in N$. We can then define the* Ehrenfeucht-Fraïssé game *$G_n(A, B)$ to be a game between two players, Spoiler and Duplicator, played as follows:*

- *The first player, Spoiler, picks either a member a_1 of A or a member b_1 of B.*

- *If Spoiler picked a member of A, Duplicator picks a member b_1 of B; otherwise, Duplicator picks a member a_1 of A.*

- *Spoiler picks either a member a_2 of A or a member b_2 of B. Duplicator picks an element a_2 or b_2 in the model from which Spoiler did not pick.*

- *Spoiler and Duplicator continue to pick members of A and B for $n-2$ further steps.*

- *At the conclusion of the game, they have chosen distinct elements $a_1, \ldots, a_n$ of A and $b_1, \ldots, b_n$ of B. We therefore have two structures on the set $\{1, \ldots, n\}$, one induced from A via the map sending i to a_i, and the other induced analogously from B.*

Duplicator wins if these structures are the same.
Spoiler wins if they are not.

It is well known that in case Spoiler wins the game, then the structures in question are not elementary equivalent [93]. We can then re-formulate Theorem 41 in terms of an application of the Ehrenfeucht-Fraïssé game as described in Game 1.

In other words, the Ehrenfeucht-Fraïssé Game 1 means that there will be no manner (that preserves the elementary properties!) to embed a 4-elements chain into the 4-elements Boolean algebra 4. And therefore the first-order theories of

Game 1 Ehrenfeucht-Fraïssé

Require: Fix the 4-elements Boolean algebra 4, and the 6-elements algebra $[4, \{a', 1\}]$ in Example 20.

(i) Spoiler calls $(0, 0) \in [4, \{a', 1\}]$.

return Duplicator has to choose $0 \in 4$.

(ii) Spoiler calls $(1, 1) \in [4, \{a', 1\}]$.

return Duplicator has to choose $1 \in 4$.

(iii) Spoiler calls $(a, a) \in [4, \{a', 1\}]$.

return Duplicator has to choose $a \in 4$.

(iv) Spoiler calls $(0, a) \in [4, \{a', 1\}]$.

print Duplicator has lost the game.

Boolean algebras and regular double Stone algebras are not elementary equivalent, in spite of the fact that they are equivalent as categories with indexes, as discussed in section 2.3. In fact, albeit the structure theories of Boolean algebras and regular double Stone algebras share rather strong structural properties (as we have seen in section 2.4), their model theories present remarkable distinct aspects.

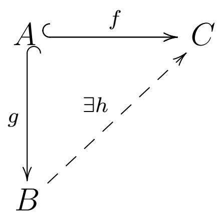
2.6 Injective Stone algebras

In this section we propose, as an application of our construction, an effective description of the injective objects for the variety of Stone algebras. As a direct consequence, an explicit illustration of the injective algebras for the variety of regular double Stone algebras, Kleene algebras and three valued Łukasiewicz algebras will be shown.

Finally, we discuss a few categoricity results for these classes of injective objects. In particular we will see that the presence of the fixpoint plays an important role as regards injectivity and categoricity.

For readability convenience, let us recall a few notions that will be expedient to our discourse in this section.

Definition 32. *Let A, B, C be algebras in a category. We say that C is* injective *for A in case, if A embeds via f into C, and via g into B, then there is a morphism h that renders the following diagram commutative:*



Also, it might be useful to recall that any Boolean algebra B admits a minimal complete Boolean algebra, that we will denote $\overline{B}$, in which B is embeddable: the *Dedekind MacNeille completion of B* (the interested reader may consult [3] for a general introduction and the works of Banaschewski for a specific account [5, 6]). It is folklore that $\overline{B}$ is the smallest injective object for B in the category of Boolean algebras.

Let us recall a useful result from [3] and [4]:

Theorem 50. *Any Stone algebra L is isomorphic to a subalgebra of $Con(B)$, the congruence lattice of some complete Boolean algebra B. Moreover, $Con(B)$ is itself a Stone algebra.*

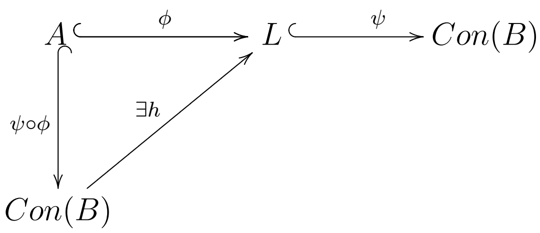
Theorem 51. *[4] A Stone algebra L is injective if and only if the following conditions hold:*

1. *L is complete;*
2. *$D^{\sim}(L)$ has a smallest element;*
3. *L is a double Stone algebra;*
4. *L is regular.*

Combining the previous results we easily note the following observation:

Lemma 17. *Any injective Stone algebra L is a retract, as a Stone algebra, of the congruence lattice of a Boolean algebra.*

Proof. The proof is straightforward since we have the following commuting diagram, by virtue of Theorem 50, for a given Stone algebra A:



And then clearly $Con(B)/h \cong L$. □

We now use the information from Lemma 9 to provide an explicit construction of the injective objects for the variety of Stone algebras.

Let us now present without the easy proof a useful lemma. We recall the definition of complete filter.

Definition 33. *A* complete filter *F on a Boolean algebra B is a filter that satisfies the following condition: if $\{a_i\}_{i \in I}$ is a family of elements in F and the infimum (greatest lower bound) x of $\{a_i\}_{i \in I}$ exists in B then x is also in F.*

Lemma 18. *A regular double Stone algebra is complete if and only if it is of the form $\mathcal{A}(B, F)$, with B a complete Boolean algebra and F a complete filter.*

Theorem 52 (Injective Stone algebras)**.** *A Stone algebra L is injective if and only if it is of the form $[B, F]$ for a complete Boolean algebra B and F a principal filter.*

Proof. First, let us observe that any principal filter is also complete.

If a Stone algebra L is injective then it is of the form $[B, F]$ by Theorem 27. It follows from Theorem 51(1)-(2) that B is complete and, because $D^{\sim}(L)$ is isomorphic to F, as a consequence of Lemma 13, we claim that F has a smallest element and therefore is principal.

Conversely, it can be seen that $[B, F]$ is complete by construction. Moreover, by Lemma 13, F is isomorphic to $D^{\sim}(L)$, and since F is principal $D^{\sim}(L)$ possesses a smallest element. Also, by Lemma 10, L is indeed a regular double Stone algebra. Therefore, Theorem 51 applies and our claim follows. □

In other words injective objects for Stone algebras (which are enough as proven in [3]) are of the form $[B, C]$, where B, C are Boolean algebras such that C embeds (as a lattice) into B as a filter.

Using the information in section 2.2 and section 2.3, together with Theorem 52, we can constructively describe in a unifying framework the structure of all injective objects for the varieties of Boolean algebras, Stone algebras, regular double Stone algebras, Kleene algebras, and 3-valued Łukasiewicz algebras (for details, we refer the reader to section 1.2).

The following result proposes a novel easy characterization of the injective objects in regular double Stone algebras, Kleene algebras, 3-valued Łukasiewicz algebras, and 3-valued MV algebras, whose descriptions are scattered in the literature, see e.g. [29, 87].

Theorem 53. *Let B be a Boolean algebra, and consider the system $(\overline{B}, F_i)_{i\in I}$, where $\overline{B}$ is the Dedekind MacNeille completion of B and the F_i are filters on B. Then:*

1. *$[\overline{B}, \{1\}]$ is injective for B;*
2. *for Stone algebras, $[\overline{B}, \overline{B}]$ is the largest injective object with sharp elements of cardinality $|\overline{B}|$. Furthermore, $\overline{B}$ is the smallest injective Boolean algebra for B;*
3. *$[\overline{B}, \overline{B}]$ is injective in the varieties of: regular double Stone algebras, Kleene algebras, 3-valued Łukasiewicz algebras, and 3-valued MV algebras.*

   Moreover, in the variety of regular double Stone algebras all injective objects are of the form $[\overline{B}, \overline{B}]$, for a Boolean algebra B.

Proof. (1) It is well known that $\overline{B}$ is injective for B and $[\overline{B}, \{1\}]$ is isomorphic to B, by Theorem 21.

(2) Straightforward.

(3) Follows by the previous item and [29]. □

The following corollary is folklore, however to the best of our knowledge it has never been established in this form. It is in fact a straightforward consequence of Theorem 53.

Corollary 8. *Both varieties of Kleene and Stone algebras can be embedded in the variety of regular double Stone algebras.*

Now, we take advantage of the results and notions introduced in section 2.3 and section 2.5 to describe some categoricity results that are related to the construction of injective objects in the variety of regular double Stone algebras.

We will see in fact that injective objects in regular double Stone algebras enjoy rather strong properties: there is a unique atomless and denumerable regular double Stone algebra, which is an injective object.

Let us recall that a class is $\aleph_0$ *categorical* in case any object with denumerable cardinality is unique up to isomorphism.

Let us first present the following useful observation.

Remark 54. *Let us note that infinite injective Stone algebras are not categorical. Indeed, consider the infinite denumerable Boolean algebra B and let F be a proper principal filter. Then, $|[B, F]| = |[B, B]| = \aleph_0$, but $[B, B]$ has a fixpoint whilst $[B, F]$ does not.*

The next theorem makes large use of Theorem 53.

Theorem 55. *The class of atomless regular double Stone algebras with a fixpoint is $\aleph_0$ categorical.*

Proof. Let us suppose that $|A_1| = |A_2| = \aleph_0$, and both A_1 and A_2 possess a fixpoint. Then, $S_K(A_1) \simeq S_K(A_2)$. Also by Theorem 21(3), $A_1 = [S_K(A_1), S_K(A_1)] \simeq [S_K(A_2), S_K(A_2)] = A_2$. □

As a direct corollary of Theorem 55 and Theorem 47 we obtain that:

Corollary 9. *The class of regular double Stone algebras whose cardinality does not exceed* $\aleph_0$ *finds a categorical algebraic closure in the unique atomless and denumerable regular double Stone algebra.*

It is well known that the Boolean algebra of the sharp elements of the unique atomless and denumerable regular double Stone algebra is in fact the interval algebra consisting of finite unions of half closed intervals whose endpoints are rational (or $\pm\infty$) [56].

Making use of Theorem 55, a neat characterization of categoricity for injective (regular double) Stone algebras will be at hand.

Theorem 56. *If* A, B *are regular double Stone algebras such that* $|S_K(A)| = |S_K(B)|$ *and* $|D^{\sim}(A)| = |D^{\sim}(B)|$*, then* $|A| = |B|$.

Proof. We reason by cases. If A, B are finite and $|S_K(A)| = |S_K(B)|$ and $|D^{\sim}(A)| = |D^{\sim}(B)|$, then $S_K(A) = S_K(B)$, and $D^{\sim}(A) = D^{\sim}(B)$, since they are principal filters. And therefore by Lemma 9, A and B are the same object up to isomorphism.

If A, B are infinite, then

$$|S_K(A)| \leq |[S_K(A), D^{\sim}(A)]| \leq |[S_K(A), S_K(A)]| \leq |(S_K(A) \times S_K(A))|.$$

However, $|S_K(A)| = |(S_K(A) \times S_K(A))|$, whence our claim follows. □

Corollary 10. *Finite injective Stone algebras are categorical.*

Proof. Straightforward by Theorem 56. □

Corollary 11. *Injective atomless Kleene algebras are* $\aleph_0$*-categorical.*

Proof. Follows from 56, and [29, 87] □

Therefore, even all finite regular double Stone algebras are categorical (Theorem 49), Remark 54 observes that this is no longer the case for infinite cardinalities.

As direct consequences of Theorem 24 we can readily state that:

Theorem 57. *Injective Stone algebras, injective Kleene algebras and injective 3-valued MV algebras do not form a variety.*

2.7 Future works and developments

We conclude the chapter by listing several possible future developments and investigations.

- Is it possible to describe an algorithm that, given a Boolean algebra B and F a filter, tells the value of $|[B, F]|$?
- Is it possible to drop the indexes in the categories presented in section 2.3? If so, one may wonder whether the categorical equivalence still holds true, and eventually in which form.
- Is it possible to describe a class of Heyting algebras admitting enough injective objects?
- By virtue of the construction described in this chapter, is it possible to characterize free products in (regular double) Stone algebras?
- It may be interesting to characterize regular double Stone algebras in terms of ring-like structures, and compare them with the theory of Boolean rings.
- Would it be possible to generalize the construction to other domains?
- What is the role of the fixpoint in regular double Stone algebras?
- It may be interesting to investigate possible generalizations of Stone representation Theorem.
- It may be interesting to discuss functional completeness.
- It may be interesting to discuss well known constructions such as e.g. Boolean powers and ultraproducts of regular double Stone algebras.
- It may be interesting to discuss Fraïssé and Ramsey theories in the context of the variety of regular double Stone algebras.

Chapter 3

Unsharp orthomodular lattices

3.1 Introduction

Orthomodular lattices are structures of paramount importance in several fields of mathematics (for an account we may refer the reader to [11, 22]). This chapter essentially originates from the notion of a *block* in an orthomodular lattice [71]. Actually, if L is an orthomodular lattice, a block B is a maximal Boolean subalgebra of L, and L is just the union of its blocks. From a foundational point of view, a block represents a "classical context" that reflects in an abstract setting the behavior of commuting projectors on a Hilbert space [11]. In fact, a block is a *distributive* subalgebra. This concept has been widely investigated over the last fifty years, see e.g. the works of Birkhoff [13], Bruns [21], Greechie [61] etc.

In [49], the authors consider certain Brouwer-Zadeh lattices [25, 24], named PBZ^* lattices, where P stands for paraorthomodular (section 3.2) (briefly, PBZ^*) that serve as abstract counterparts of lattices of effects in Hilbert spaces under the spectral ordering [40, 90] (see also [25, 24]). These algebras generalize the concept of orthomodularity to the "unsharp realm" in which the orthocomplement is no longer such, but it just satisfies the Kleene condition. Remarkably, in this context the orthomodular law is no longer equivalent to the condition:

$$\text{if } x \leq y \text{ and } x' \wedge y = 0 \text{ implies } x = y,$$

which still maintains true in PBZ^* lattices (for posets the reader may consult e.g. [27]). These lattices are equipped with two unary operations: a fuzzy like complement $'$, and an intuitionistic like complement $^\sim$, which, if I is the identity operator and E an effect on a Hilbert space, stand for $I - E$ and $\ker(E)$ (and $E' = E^\sim$ if E is a projector), respectively.

In this chapter, we will consider the variety that we call unsharp orthomodular lattices, denoted by UOM, which is axiomatized with respect to PBZ^* by (SDM) and (SK) (see section 3.2). Although Equations (SDM) and (SK) need not be satisfied in the set of effects of any Hilbert space, they are natural conditions (actually they are trivial in the lattice of projectors of any Hilbert space) that generalize smoothly the theory of orthomodular lattices.

In fact, we will see that in this context a proper direct generalization of the orthomodular law obtains, and if one considers the elements for which the operations $'$ and $^\sim$ coincide, then they satisfy the orthomodular law. Moreover, our generalized orthomodularity plays precisely the same role that orthomodularity assumes with orthomodular lattices: it completely determines the form of the building blocks. As we shall see, this fact renders the theory of UOM definitely manageable.

This seemingly harmless splitting of the former orthomodular complement into two different unary operations yields rather strong structural properties. Indeed, as orthomodular lattices generalize Boolean algebras to the non-distributive case, UOM, by virtue of the paraorthomodular law, generalizes orthomodular lattices to the case in which only for the sharp elements (that form an orthomodular sublattice) the operations $'$, $^\sim$ coincide. From a logical perspective, this fact can be regarded as a generalization of orthomodular lattices to the context in which *non-contradiction* and *tertium non datur* principles need not be satisfied.

In other words, we will see that relaxing the orthocomplement condition induces consequences somehow similar to what happens in residuated lattices (or substructural logics) when a structural rule is remitted: new, “multiplicative” connectives appear [47]. Indeed, in the present case the orthocomplementation is no longer such, but it splits into two different negation connectives: a fuzzy-like $'$, and an intuitionistic-like $^{\sim}$.

It is well known that orthomodular lattices are essentially the union of classical contexts, i.e. their building blocks are Boolean algebras that represent σ-algebras of pairwise commuting operators. In the same guise, for the case of UOM we have perfectly specular results which claim that a UOM lattice L is the union of its blocks, or contexts, that can be equivalently regarded as Heyting algebras [13], regular double Stone algebras [2], or finitely-valued MV algebras [31, 62, 101, 96] (Łukasiewicz algebras), and these blocks, in turn, contain a classical Boolean context. Actually, UOM can be regarded as a common ground in which structures, such as e.g. Boolean algebras, finite-valued MV algebras, Stone algebras, regular double Stone algebras, Kleene lattices, orthomodular lattices etc., which are of paramount relevance to algebraic logic, may find a unifying general frame. We shall see that in the context of UOM many properties of the above mentioned classes that are specifically investigated in the same class are lifted to a general encompassing level.

In UOM it will be quite tame to characterize several properties of commutation, and, surprisingly enough, we will find out that a great deal of well known results that hold true in the realm of orthomodular lattices maintain true in this general context. Now, considering the tight connection between UOM lattices, MV algebras, Heyting algebras and orthomodular lattices, a question as genuine as important naturally arises: what happens about residuation in the whole structure. In our opinion, Theorem 77 will propose a reasonable answer.

This chapter is structured as follows. In section 3.2 we dispatch (or refer to) all the required notions and results that are necessary to keep this chapter self-contained. In section 3.3, we introduce and widely discuss the notion of blocks in UOM. In section 3.4, we introduce a new operation “$\oplus$” with an aim in mind: showing that, given $A \in UOM$, each block M of A can be equivalently regarded as an Heyting algebra, or a regular double Stone algebra, or a (three-valued) MV algebra. In section 3.5, we carry over in UOM the investigation of the notions of commutation and block. We characterize several properties of commutation, and we generalize a number of well known results that hold true for orthomodular lattices. In section 3.6, we discuss what happens to the notion of residuation if we generalize the orthomodular law. Finally, we close the chapter with section 3.7, where we present a number of possible future topics of investigation.

3.2 PBZ^* lattices

For the convenience of the reader, we have decided to include the preliminaries concerning PBZ^* lattices directly in this chapter.

While the association with the set of all projection operators has sparked considerable research into orthomodular lattices, it is widely recognized that the class of orthomodular lattices derived from lattices of projection operators does not encompass the entire range of orthomodular lattices. In other words, there are equational properties that are not accurately captured by this proposed mathematical abstraction.

A distinct formal representation of the set of quantum events has emerged within the unsharp approach to quantum theory, where the concept of a quantum event is extended to include all effects, representing the set of all positive linear operators on a complex separable Hilbert space. Moreover, Olson [90] and de Groote [40] have shown that under the spectral ordering, the set of all effects possesses a lattice structure. PBZ^* lattices serve as abstract counterparts to lattices of effects in Hilbert spaces under the spectral ordering.

Another noteworthy observation concerns the orthomodular law. As we mentioned earlier, in the context of ortholattices the orthomodular law is equivalent to the quasi-equation:

$$x \leq y \text{ and } x' \wedge y = 0 \quad \text{then} \quad x = y. \tag{3.1}$$

In [49] the authors named this quasi-equation *paraorthomodularity*. In fact, if the lattice is not orthocomplemented, then paraorthomodularity and orthomodularity are no longer equivalent. Condition (3.1) will play a crucial role in this chapter because it ensures that the set of *sharp elements*, i.e. those elements satisfying $x' = x^{\sim}$, form an orthomodular sublattice. Under pain of repetition, let us remember that condition (3.1) is equivalent to orthomodularity, if the equation $x' = x^{\sim}$ holds, i.e. orthomodular lattices form a (*proper*) subvariety of UOM. For reader's convenience, we will provide the required definitions in due course.

Concerning the unsharp approach to quantum theory, PBZ^* lattices offer a wide range of advantages. For instance, they form a variety, and there is a good formal description of the sharp elements. Moreover, they can be considered as a rather neat generalization of orthomodular lattices. For more information the reader may consult [49, 50, 52, 53, 54].

Definition 34. *A* pseudo Kleene lattice $L = (L, \vee, \wedge, ', 0, 1)$ *is an algebra where* $L = (L, \vee, \wedge, 0, 1)$ *is a bounded lattice and* $'$ *is an antitone and involutive operation that satisfies Condition Kleene:*

$$x \wedge x' \leq y \vee y'.$$

Moreover, L is a paraorthomodular pseudo Kleene lattice *if it satisfies the paraorthomodular law:*

$$x \leq y \quad and \quad x' \wedge y = 0 \quad then \quad x = y.$$

Definition 35. *A structure* $L = (L, \vee, \wedge, ', \sim, 0, 1)$ *is a* PBZ^* *lattice if the following are satisfied:*

1. $(L, \vee, \wedge, ', 0, 1)$ *is a paraorthomodular pseudo Kleene lattice;*

2. $x^{\sim\sim} \geq x$*;*

3. $x \leq y$ *implies* $y^{\sim} \leq x^{\sim}$*;*

4. $x \wedge x^{\sim} = 0$*;*

5. $x^{\sim\prime} = x^{\sim\sim}$*;*

6. $(x \wedge x')^{\sim} = x^{\sim} \vee x'^{\sim}$ *(star condition, see [49]).*

Moreover, it is possible to define two unary operations to behave as the modal operators necessarily ($\Box$) and possibly ($\Diamond$) as follows:

$$\Box x = x^{'\sim} \quad \text{and} \quad \Diamond x = x^{\sim\sim}.$$

For notational convenience we set:

$$x^{+} = x^{\prime\sim\sim} = x^{\prime\sim\prime}.$$

Let us observe that for any $L \in PBZ^{*}$, for all $x \in L$:

$$\Box x \leq x \leq \Diamond x \text{ and } x^{\sim} \leq x' \leq x^{+}.$$

It will be helpful to note that in general $x \vee x^{\sim} \neq 1$. Also, let us state, without proof, a number of remarkable properties of the operators we introduced:

Lemma 19. *Let $L \in PBZ^{*}$, then for all $x \in L$ the following are satisfied:*

1. $\Box x \vee x^{+} = 1$;
2. $\Box x \wedge x^{+} = 0$;
3. $\Diamond x \vee x^{\sim} = 1$;
4. $\Diamond x \wedge x^{\sim} = 0$;
5. $\Box x \wedge x^{\sim} = 0$;
6. $\Diamond x \vee x^{+} = 1$;
7. $\Box x \wedge x' = 0$;
8. $\Diamond x \vee x' = 1$.

In plain words, Lemma 19 shows that the complement of $\Box x$ is x^{+}, and the complement of $\Diamond x$ is $x^{\sim}$.

Let us now present two more axioms that we assume in this work in order to work in specific subvarieties:

$$x^{\sim} \vee y^{\sim} = (x \wedge y)^{\sim}. \qquad \text{(SDM)}$$

$$\text{If } \Box x = \Box y \text{ and } \Diamond x = \Diamond y, \text{ then } x = y. \qquad \text{(SK)}$$

Let us remark that, under condition (SDM) [52], condition (SK) is indeed equationally expressible, as Lemma 20 shows:

Lemma 20. *[52] Let L be a PBZ^* lattice that satisfies condition* (SDM)*. Then, the following are equivalent in L:*

1. (SK)*;*
2. $(x \wedge \Diamond y) \wedge (\Box x \vee y) = x \wedge \Diamond y$.

Following the notation for PBZ^* lattices, we will sometimes indicate by PBZ^*_{SDM} the subvariety satisfying (SDM).

As a side remark, let us note that axiom (SDM) is the *strong de Morgan property*, that expresses the fact that de Morgan condition works also for the Brower negation $^\sim$. In general, it is possible to show that the class of all effects of a finite-dimensional Hilbert space determines a de Morgan BZ poset [25].

For our purposes, this axiom is crucial to conclude that the blocks are subalgebras.

Concerning axiom (SK), it implies rather strong consequences in our framework. For instance, the partial order is completely determined by modalities. In general, (SK) induces a strong dependence of the whole structure on the orthomodular subalgebra formed by the sharp elements [25].

Moreover, condition (SK) enables us to maintain a strong connection with the orthomodular case and its associated properties, thereby ensuring an intriguing symmetry with more specific structures. In fact, it is straightforward to check that orthomodular lattices satisfy $x' = x^\sim$, and therefore condition (SK) holds trivially true.

Let us recall that we will denote by UOM the variety of PBZ^* lattices that satisfies Equation (SDM) and Equation (SK). By the previous observations and Lemma 20, we will equivalently treat condition (SK) both as a quasi-equation and an equation.

We now close the present section by mentioning, given a PBZ^* lattice L, three subsets of the support that will assume a definite relevance to the development of our discourse.

We denote by $S_K(L)$ the set of *sharp elements of* L:

$$S_K(L) = \{x \in L : x' = x^{\sim}\} = \{x \in L : x = \Diamond x\} = \{x^{\sim} : x \in L\}.$$

Let us notice that (see e.g. [49]) $S_K(L)$ is the largest orthomodular subalgebra of L. It can be seen that in $S_K(L)$ the operations $'$ and $^{\sim}$ coincide, and therefore are antitone involutions, and all elements are complemented. Moreover, all elements are stable under $\Diamond$, $\Box$, i.e. $\Diamond x = \Box x = x$. It is straightforward that in $S_K(L)$ Condition (SK) is nothing but a triviality. Indeed, as we mentioned earlier $S_K(L)$ do form an orthomodular lattice.

We observe that given $x \in L$, $\Box x$, $\Diamond x$ are its *sharp approximations*. Namely, $\Box x$ is the largest element in $S_K(L)$ under x, and dually $\Diamond x$ is the smallest element in $S_K(L)$ over x. It goes without saying that the sharp approximations of an element of the form $x' \in L$ are

$$x^{\sim} \leq x' \leq x^{+}.$$

We indicate by $D^{\sim}(L)$ the set of *dense elements of* L:

$$D^{\sim}(L) = \{x \in L : x^{\sim} = 0\}.$$

We note that if $x \in S_K(L) \cap D^{\sim}(L)$, then $x = 1$. Furthermore, for all $x \in L$, $x \vee x^{\sim} \in D^{\sim}(L)$.

Dually, we define the set of *dually dense elements of* L:

$$D^{+}(L) = \{x \in L : x^{+} = 1\}.$$

We note that if $x \in S_K(L) \cap D^{+}(L)$, then $x = 0$. Moreover, for all $x \in L$, $x \wedge x^{+} \in D^{+}(L)$.

3.3 On the structure theory of UOM: blocks

In this chapter, we will consider the variety that we call unsharp orthomodular lattices, denoted by UOM, which is axiomatized with respect to PBZ^* by conditions (SDM) and (SK) (see section 3.2). Precisely,

Definition 36. *An* unsharp orthomodular lattice *is a PBZ^* lattice* $A = (A, \vee, \wedge, ', ^\sim, 0, 1)$ *(Definition 35, see also [49]) that satisfies conditions* (SDM) *and* (SK).

As observed in section 3.2, in presence of Equation (SDM) condition (SK) is equationally expressible, and therefore, UOM form a variety.

As observed in [49], PBZ^* lattices are a rather natural generalization of the notion of orthomodular lattice to a non orthocomplemented case. We will see in the present section that UOM will be a very natural generalization of orthomodular lattices to the case in which many structures of prominent relevance to algebraic logic (Boolean algebras, Kleene algebras, Stone algebras, MV algebras, regular double Stone algebras etc.) find a unifying treatment.

Paralleling what happens for orthomodular lattices, in this section we introduce a rather smooth notion of commutativity, that plainly generalize the same notion for orthomodular structures. From this relation emerges directly a general notion of block. It is well known that in the orthomodular case every block can be regarded as a classical context, namely a Boolean algebra. We shall see that in case of UOM the situation is perfectly reflected: every block corresponds to a "non classical" context, namely it can be regarded as a Kleene lattice. Indeed, we have a natural generalization of the concept of Boolean block of an orthomodular lattice, to a non orthocomplemented case. Equivalently, any block can be also considered as an MV algebra, or equivalently again as a regular double Stone algebra.

Let us also notice that, as we will see, these non classical contexts still maintain a "classical flavour", that we characterize in several ways.

We start with a quite general lemma that describes cases in which strong de Morgan holds in PBZ^* lattices.

Lemma 21. *Let A be a PBZ^*_{SDM} lattice. Then, for any non-zero $x \in A$,*

1. *if $y \in D^{\sim}(A)$, then*
$$x \wedge y \neq 0.$$

2. *if $y \in D^{+}(A)$, then*
$$x \vee y \neq 1.$$

Proof. We confine ourselves in proving item (1), being item (2) dual. Suppose $x \neq 0$ in A, and $x \wedge y = 0$. If $y \in D^{\sim}(A)$, then it can be seen that

$$1 = (x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim} = x^{\sim} \vee 0 = x^{\sim}.$$

Now, $0 < x \leq x^{\sim\sim} = 0$, impossible. And therefore we have that strong de Morgan fails in A. □

We now concentrate on the variety of UOM lattices. Given the relevance of the sets of dense and dually dense elements, it is natural to consider and describe their intersection.

Definition 37. *Let A be a UOM lattice, we call the* core *of A the intersection between the set of dense elements and the set of dual dense: $D^{\sim}(A) \cap D^{+}(A)$.*

Lemma 22. *Let A be a UOM lattice, then*

1. *the cardinality of the core of A is at most 1.*

2. *$x \in D^{\sim}(A) \cap D^{+}(A)$ if and only if $x = x'$, i.e. x is a fixed point for $'$.*

Proof. (1) Suppose $x, y \in D^{\sim}(A) \cap D^{+}(A)$, then we have $\Box x = \Box y = 0$ and $\Diamond x = \Diamond y = 1$. Hence, by (SK) we conclude that $x = y$.
(2) If $x \in D^{\sim}(A) \cap D^{+}(A)$, then $\Box(x') = x^{\sim} = 0 = \Box x$ and $\Diamond(x') = x'^{\sim\sim} = x^{+} = 1 = \Diamond x$. Therefore, $x = x'$. On the other hand, if $x = x'$, we have $\Box x = \Box x' = x^{\sim}$ and $\Diamond x = \Diamond x' = x^{+}$. Then, $x^{\sim} = x^{\sim} \wedge x^{\sim} = x^{\sim} \wedge \Box x = 0$, and $x^{+} = x^{+} \vee x^{+} = x^{+} \vee \Diamond x = 1$. Therefore, $x \in D^{\sim} \cap D^{+}$. □

In other words, Lemma 22 expresses the fact that if $A \in UOM$, then A admits at most a fixed point $k = k'$ which would be dense and dually dense, and moreover $D^{\sim}(A) \cap D^{+}(A) = \{k\}$.

Let us note that Lemma 22 will be useful in proving a sort of decomposition theorem that tells us that, if k is a fixed point, then every element can be viewed as a lattice combination of k with sharp elements. Indeed,

Theorem 58. *Let A be a UOM lattice with a non-empty core and let $k \in D^{\sim}(A) \cap D^{+}(A)$, then every $x \in A$ can be written as:*

$$x = \Diamond x \wedge (\Box x \vee k);$$
$$x = \Box x \vee (\Diamond x \wedge k).$$

Proof. It is easy to see that the box and the diamond coincide:

$$\begin{aligned} \Box\left(\Diamond x \wedge (\Box x \vee k)\right) = &\quad \Diamond x \wedge (\Box x \vee 0) \\ = &\quad \Box x, \end{aligned}$$

since $\Box$ and $\Diamond$ are mutually idempotent; and $k \in D^{\sim}(A) \cap D^{+}(A)$ implies $\Box k = 0$. Moreover $\Box x \leq \Diamond x$. And also:

$$\begin{aligned} \Diamond\left(\Diamond x \wedge (\Box x \vee k)\right) = &\quad \Diamond x \wedge (\Box x \vee 1) \\ = &\quad \Diamond x. \end{aligned}$$

Therefore, by (SK), $x = \Diamond x \wedge (\Box x \vee k)$. The other equation can be proved dually. □

We now introduce one of the key concept on which the entire chapter is based: commutativity. Let us anticipate that, in order to introduce our concept, the commutativity relation between sharp elements plays a crucial role.

Definition 38. *Let A be a PBZ^{*} lattice, and $x, y \in A$. We say that x* commutes *with y, in symbols xy, if the following conditions are satisfied:*

1. *$\Box x C \Box y$ and $\Diamond x C \Diamond y$;*

2. *$\Box x C \Diamond y$ and $\Diamond x C \Box y$;*

where the relation C is the commutativity relation as defined for the variety of orthomodular lattices (see Definition 24).

Let us observe that if $k \in A$ is a fixed point, then $\Box k = 0$ and $\Diamond k = 1$, and therefore $k\bar{C}x$, for any $x \in A$.

By virtue of Definition 38, we can define a corresponding notion of block that neatly generalizes the crucial notion of a block in the orthomodular case. In fact, it is possible to consider a block M of A as a maximal subset, where if $x, y \in M$ then $x\bar{C}y$.
In addition, we give another explicit definition of a block. Obviously, they are equivalent, however as we shall see the latter description of a block M will be more useful at an operational level.

Definition 39. *Let A be in PBZ^*_{SDM}. Given a block B in the orthomodular lattice $S_k(A)$ of sharp elements of A, a B-*block *is a set M of A such that*

$$M = \{x \in A : \Box x \quad and \quad \Diamond x \ \in \ B\}.$$

Whenever no confusion will be possible, we simply call a B-block a *block*.

Let us notice that the following conditions can be easily derived using general facts on orthomodular lattices.

Remark 59. *Let $x\bar{C}y$, then the following hold:*

1. $\Box x C \Box y$;
2. $x^+ C \Box y$;
3. $\Box x C \Diamond y$;
4. $x^+ C \Diamond y$;
5. $\Diamond x C \Box y$;
6. $x^\sim C \Box y$;
7. $\Diamond x C \Diamond y$;
8. $x^\sim C \Diamond y$.

Theorem 60. *Let M be a block of $A \in PBZ^*_{SDM}$. Then, M is a subalgebra of A.*

Proof. By Definition 39, M is determined by a Boolean algebra B, which is a subalgebra of $S_K(A)$, and a fortiori of A. Let $a, b \in M$. As regards lattice operations, by the previous observation $\Box a \wedge \Box b = \Box(a \wedge b) \in B$, and in the same B is $\Diamond a \wedge \Diamond b = \Diamond(a \wedge b)$, which implies that $\Diamond(a \wedge b) \in B$. Dually for $\vee$.

If $x \in M$, then $\Box x$, $\Diamond x$ and their negation are in the same block B by definition. Consider x', we have to show that both $\Box(x')$ and $\Diamond(x')$ belong to B. In fact,

$\Box(x') = x''^{\sim} = x^{\sim} = x^{\sim\sim\sim} = (\Diamond x)'$, and then $\Box(x') \in B$. Moreover, $\Diamond(x') = x'^{\sim\sim} = x'^{\sim'} = (\Box x)'$, and then $\Diamond(x') \in B$. Therefore, we can conclude that $x' \in M$.
As regards $x^{\sim}$, we have that $\Box(x^{\sim}) = x^{\sim'\sim} = x^{\sim\sim\sim} = (\Diamond x)'$, and then $\Box(x^{\sim}) \in B$. Also, $\Diamond(x^{\sim}) = x^{\sim\sim\sim} = (\Diamond x)'$, and then $\Box(x^{\sim}) \in B$. These conditions imply that $x^{\sim} \in M$.
Therefore, we conclude that M is a subalgebra of A. □

Now, we can state our first result that will be fundamental to our discourse. It paves the way to new perspectives about the connections between blocks in PBZ^* lattices and other structures.
We now prove that in a block of a PBZ^*_{SDM} lattice, the Brouwer negation is the pseudocomplementation. However, this need not be true in the general case of the whole structure.

Theorem 61. *Let A be a PBZ^*_{SDM} lattice and M one of its blocks. Then, the reduct of M in the language $(\vee, \wedge, ^{\sim}, 0, 1)$ is a pseudocomplementated lattice.*

Proof. We have to show that:

$$\text{for } x, y \in M,\ x \wedge y = 0 \text{ if and only if } y \leq x^{\sim}.$$

From right to left it is consequence of a general property of PBZ^* lattices. Indeed, if $y \leq x^{\sim}$, then $0 = x \wedge x^{\sim} \geq x \wedge y = 0$.
Conversely, consider $x \wedge y = 0$. Due to the fact both x and y are elements of a common block M, then $\Box x, \Box y, \Diamond x, \Diamond y$ are elements of a common Boolean algebra. Hence, $x \wedge y = 0$ implies that:

$$1 = 0^{\sim} = (x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim};$$

but $x^{\sim}$, $y^{\sim}$ are elements of a Boolean algebra, then:

$$x^{\sim} \geq y^{\sim\sim} \geq y.$$

It is worth to notice that in the first point the equivalence $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}$ is a consequence of the strong de Morgan axiom. □

Moreover, we can see that an analogous result holds true if we consider the reduct with the operation $^+$. Namely,

Theorem 62. *Let M be a block of $A \in PBZ^*_{SDM}$. Then, M in the signature $(\vee, \wedge, ^+, 0, 1)$ is a dual pseudocomplemented lattice.*

Proof. We have to show that:

$$\text{for } x, y \in M,\ x \vee y = 1 \text{ if and only if } y \geq x^+.$$

From left to right. Consider $x \vee y = 1$, then:

$$0 = (x \vee y)^+ = x^+ \wedge y^+;$$

therefore:

$$x^+ \leq y^{+'} = y^{'\sim''} = \Box y \leq y.$$

From right to left. Consider $y \geq x^+$, then:

$$1 = x \vee x^+ \leq x \vee y = 1.$$

□

In other words, if $A \in PBZ^*_{SDM}$, then Theorem 61 and Theorem 62 show that the blocks of A (see Definition 38 and Definition 39) are subalgebras of A that are doubly pseudocomplemented.

Now, if we narrow our focus to UOM, then we may take advantage of a result by Katriňak,

Theorem 63. *[70] Every double p-algebra A that satisfies* (SK) *is a distributive lattice.*

And we immediately obtain the following:

Theorem 64. *Let $A \in UOM$. If M is a block of A, then M is a distributive subalgebra.*

As a consequence,

Corollary 12. *Let A be in UOM. If M is a block of A, then M is a double Stone algebra.*

Proof. By Theorems 61, 62 64, we know that M is a doubly pseudocomplemented distributive lattice, that trivially satisfies

$$x^{\sim\sim} \vee x^{\sim} = 1 \tag{Stone}$$

□

We recall the following well-known theorems, [70].

Theorem 65. *Every double Stone algebra A satisfying* (SK) *is a double Heyting algebra.*

Theorem 66. *Every double Stone algebra A satisfying* (SK) *is a three-valued Łukasiewicz algebra.*

By virtue of the previous statements, if A is in UOM, and M a block, then a suitable reduct of M can be regarded as a Kleene lattice, whose lattice reduct, we recall, is distributive. Let us emphasize that this fact is perfectly symmetric to what happens with orthomodular lattices and their Boolean blocks. Moreover, according to Theorem 65 and Theorem 66 any block in UOM can be viewed as a double Heyting algebra or a three-valued Łukasiewicz algebra.

As a side remark, let us notice that for $n \leq 4$, an n-valued Łukasiewicz algebra is indeed an n-valued MV algebra [64]. This observation will be relevant to the content of section 3.4.

3.4 A new operation in UOM

The present section builds on the results presented in section 3.3. Specifically, we introduce a new operation "$\oplus$" with an aim in mind: showing that, given $A \in UOM$, each block M of A can be equivalently regarded as a (3-valued) MV algebra, which in turn may be regarded either as regular Heyting algebras or regular double Stone algebras.

First of all, we introduce this new operation:

Definition 40. *Let $A \in UOM$. We define the following term operation on A:*

$$x \oplus y = (x \vee \Diamond y) \wedge (\Diamond x \vee y) \, .$$

For reader's convenience we now recall a useful notion:

Definition 41. *[31] A structure $A = (A, \oplus, \neg, 0)$ is an* MV algebra *if it satisfies the following conditions:*

M1 $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

M2 $x \oplus y = y \oplus x$;

M3 $x \oplus 0 = 0$;

M4 $\neg\neg x = x$;

M5 $x \oplus \neg 0 = \neg 0$;

M6 $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ *(Łukasiewicz).*

Let us notice that M is a distributive subalgebra of A, by Theorem 60 and Theorem 64. We can now show that if we consider a block M in the language $(\oplus, ', 0)$, then it is in fact an MV algebra.

Theorem 67. *Let $A \in UOM$, and M one of its blocks. Then, $M = (M, \oplus, ', 0)$ is an MV algebra, if we set $\neg x = x'$.*

Proof. First of all, let us notice that in this proof we will make large use of the fact that M is distributive, Corollary 12. Concerning M2:

$$x \oplus y = (x \vee \Diamond y) \wedge (\Diamond x \vee y) = (y \vee \Diamond x) \wedge (\Diamond y \vee x) = y \oplus x,$$

i.e. $x \oplus y = y \oplus x$.

For M1:

$$\begin{aligned}
x \oplus (y \oplus z) &= x \oplus (y \vee \Diamond z) \wedge (\Diamond y \vee z) \\
&= (x \vee ((\Diamond y \vee \Diamond z) \wedge (\Diamond y \vee \Diamond z))) \wedge (\Diamond x \vee ((y \vee \Diamond z) \wedge (\Diamond y \vee z))) \\
&= (x \vee \Diamond y \vee \Diamond z) \wedge (\Diamond x \vee y \vee \Diamond z) \wedge (\Diamond x \vee \Diamond y \vee z) \\
&= (\Diamond z \vee ((x \vee \Diamond y) \wedge (\Diamond x \vee y))) \wedge (z \vee ((\Diamond x \vee \Diamond y) \wedge (\Diamond x \vee \Diamond y))) \\
&= ((x \vee \Diamond y) \wedge (\Diamond x \vee y)) \oplus z \\
&= (x \oplus y) \oplus z,
\end{aligned}$$

and so $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.

As regards M3:

$$x \oplus 0 = (x \vee \Diamond 0) \wedge (\Diamond x \vee 0) = x \wedge \Diamond x = x,$$

i.e. $x \oplus 0 = x$.

M4 is immediate.

For M5:

$$x \oplus \neg 0 = (x \vee \Diamond 0') \wedge (\Diamond x \vee 0') = (x \vee 1) \wedge (\Diamond x \vee 1) = \neg 0,$$

then $x \oplus \neg 0 = \neg 0$.

Finally for Łukasiewicz condition, $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$, we compute:

$$\begin{aligned}
\neg(\neg x \oplus y) \oplus y &= ((x \wedge y^{\sim}) \vee (\Box x \wedge y')) \oplus y = \\
&= ((x \wedge y^{\sim}) \vee (\Box x \wedge y') \vee \Diamond y) \wedge \big((\Diamond x \wedge y^{\sim}) \vee (\Box x \wedge y^{+}) \vee y\big)\,;
\end{aligned}$$

and

$$\begin{aligned}
\neg(\neg y \oplus x) \oplus x &= ((y \wedge x^{\sim}) \vee (\Box y \wedge x')) \oplus x = \\
&= ((y \wedge x^{\sim}) \vee (\Box y \wedge x') \vee \Diamond x) \wedge \big((\Diamond y \wedge x^{\sim}) \vee (\Box y \wedge x^{+}) \vee x\big)\,.
\end{aligned}$$

Now we can use the (SK) axiom in order to show that this identity holds:

$$
\begin{aligned}
&\Box(\neg(\neg y \oplus x) \oplus x) \\
&= \Box(((y \wedge x^{\sim}) \vee (\Box y \wedge x') \vee \Diamond x) \wedge \big((\Diamond y \wedge x^{\sim}) \vee (\Box y \wedge x^{+}) \vee x\big)) \\
&= ((\Box y \wedge x^{\sim}) \vee (\Box y \wedge x^{\sim}) \vee \Diamond x) \wedge \big((\Diamond y \wedge x^{\sim}) \vee (\Box y \wedge x^{+}) \vee \Box x\big) \\
&= (\Box y \vee \Diamond x) \wedge ((\Box y \vee \Box x) \vee (\Diamond y \wedge x^{\sim})) \\
&= (\Box y \vee \Box x) \vee ((\Box y \vee \Diamond x) \wedge x^{\sim} \wedge \Diamond y) \\
&= \Box y \vee \Box x \vee (\Box y \wedge x^{\sim} \wedge \Diamond y) \\
&= \Box x \vee \Box y \vee (\Box y \wedge x^{\sim}) \\
&= \Box x \vee \Box y.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\Box(\neg(\neg x \oplus y) \oplus y) \\
&= \Box(((x \wedge y^{\sim}) \vee (\Box x \wedge y') \vee \Diamond y) \wedge \big((\Diamond x \wedge y^{\sim}) \vee (\Box x \wedge y^{+}) \vee y\big)) \\
&= ((\Box x \wedge y^{\sim}) \vee (\Box x \wedge y^{\sim}) \vee \Diamond y) \wedge \big((\Diamond x \wedge y^{\sim}) \vee (\Box x \wedge y^{+}) \vee \Box y\big) \\
&= (\Box x \vee \Diamond y) \wedge ((\Box x \vee \Box y) \vee (\Diamond x \wedge y^{\sim})) \\
&= (\Box x \vee \Box y) \vee ((\Box x \vee \Diamond y) \wedge y^{\sim} \wedge \Diamond x) \\
&= \Box x \vee \Box y \vee (\Box x \wedge y^{\sim} \wedge \Diamond x) \\
&= \Box y \vee \Box x \vee (\Box x \wedge y^{\sim}) \\
&= \Box x \vee \Box y.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\Diamond(((y \wedge x^{\sim}) \vee (\Box y \wedge x') \vee \Diamond x) \wedge \big((\Diamond y \wedge x^{\sim}) \vee (\Box y \wedge x^{+}) \vee x\big)) \\
&= \big((\Diamond y \wedge x^{\sim}) \vee (\Box y \wedge x^{+}) \vee \Diamond x\big) \wedge \big((\Diamond y \wedge x^{\sim}) \vee (\Box y \wedge x^{+}) \vee \Diamond x\big) \\
&= (\Diamond y \vee \Diamond x) \wedge (\Diamond y \vee \Diamond x) \\
&= (\Diamond y \vee \Diamond x);
\end{aligned}
$$

and

$$\begin{aligned}
&\Diamond(((x \wedge y^{\sim}) \vee (\Box x \wedge y') \vee \Diamond y) \wedge \big((\Diamond x \wedge y^{\sim}) \vee (\Box x \wedge y^{+}) \vee y\big))\\
&\quad = \big((\Diamond x \wedge y^{\sim}) \vee (\Box x \wedge y^{+}) \vee \Diamond y\big) \wedge \big((\Diamond x \wedge y^{\sim}) \vee (\Box x \wedge y^{+}) \vee \Diamond y\big)\\
&\quad = (\Diamond x \vee \Diamond y) \wedge (\Diamond x \vee \Diamond y)\\
&\quad = (\Diamond y \vee \Diamond x).
\end{aligned}$$

Therefore, $\Box(\neg(\neg x \oplus y) \oplus y) = \Box(\neg(\neg y \oplus x) \oplus x)$, and $\Diamond(\neg((\neg x \oplus y) \oplus y)) = \Diamond(\neg(\neg y \oplus x) \oplus x)$, and by (SK) our claim follows. □

We briefly recall that an n-valued MV algebra, $0 < n$, is the variety generated by the chain of length $n + 1$ and the algebras belonging to this variety are the algebraic models of the $(n + 1)$-valued Łukasiewicz propositional calculus. An algebraic characterization was proposed by Grigolia in [62].

Theorem 68. *If M is a block of a UOM lattice A, then the MV algebra $M = (M, \oplus, \neg, 0)$ is 3-valued, i.e. M satisfies the following identity*

$$x \oplus x \oplus x = x \oplus x.$$

Proof. Observe that $x \oplus x = \Diamond x$, then:

$$\begin{aligned}
x \oplus x \oplus x =& ((\Diamond x \vee x) \wedge (x \vee \Diamond x)) \oplus x\\
=& \Diamond x \oplus x\\
=& (\Diamond x \vee x) \wedge (\Diamond x \vee \Diamond x)\\
=& \Diamond x.
\end{aligned}$$

□

Obviously, any Boolean algebra can be regarded as a 2-valued MV algebra.

3.5 Commutation

In section 3.3, we introduced the notion of blocks and commutation in the case of UOM. In the present section, we carry over the investigation of these notions.

We characterize several properties of commutation, and, surprisingly enough, we will find out that a great deal of well known results that holds true in the realm of orthomodular lattices maintains true in this general context of UOM.

First, we start with a technical lemma.

Lemma 23. *Let $A \in UOM$. The following conditions are equivalent:*

1. $x\bar{C}y$;
2. $y\bar{C}x$;
3. $x\bar{C}y'$.

Proof. The proof is straightforward, by Remark 59. $\square$

We now prove a result that parallels what happens between blocks and commutation in orthomodular lattices. Indeed, we will provide two equational conditions equivalent to the notion of commutation, and so to the notion of block. Let us emphasize that in the case of orthomodular lattices a single equation, or its dual, would suffice to fully capture commutation. In UOM, a modicum of elaboration will be in order. In fact, we do have two equivalent equational conditions that characterize commutation, the one "dual" with respect to the other up to switching $^{\sim}$ with $^{+}$ (Corollary 13).

Theorem 69. *Let $A \in UOM$. The following are equivalent:*

1. *A possesses a unique block;*
2. $A \models (x \wedge y) \vee (x \wedge y^{+}) = x$;
3. $A \models y \vee (y^{+} \wedge x) = x \vee y$.

Proof. (1) implies (2) by Theorem 64. (2) implies (3) by absorption. (3) implies (1): by straightforward properties of orthomodular lattices we obtain that $\Box x \vee \Box y = \Box y \vee (y^{+} \wedge \Box x)$, i.e. $\Box x C \Box y$. Now, if we set $x = \Diamond x$, $y = \Diamond y$ we readily have that $\Diamond x \vee \Diamond y = \Diamond y \vee ((\Diamond y)^{+} \wedge \Diamond x) = \Diamond y \vee (y^{\sim} \wedge \Diamond x)$, and therefore $\Diamond x C \Diamond y$. If we cross change either $x = \Diamond x$, $y = \Box y$, or $x = \Box x$, $y = \Diamond y$ we analogously get that $\Diamond x C \Box y$ and $\Box x C \Diamond y$. Hence, $x\bar{C}y$, and our claim follows. $\square$

Corollary 13. *Let $A \in UOM$. The following are equivalent:*

1. $(x \wedge y) \vee (x \wedge y^+) = x$;

2. $(x \vee y) \wedge (x \vee y^\sim) = x$.

Proof. Straightforward from the fact that $x\bar{C}y$, and Theorem 64. □

Lemma 24. *Let $A \in UOM$. The following are equivalent:*

1. $y \wedge (y^\sim \vee x) = y \wedge x$;

2. $y \vee (y^+ \wedge x) = x \vee y$.

Proof. We can easily see that $x \vee y = (x' \wedge y')' = ((y' \wedge (x' \vee (y')^\sim)))' = y \vee (x \wedge y^+)$. The converse direction dually follows. □

By virtue of Corollary 13 and Lemma 24, Theorem 69 can be re-stated as

Theorem 70. *Let $A \in UOM$. The following are equivalent:*

1. $x\bar{C}y$;

2. $(x \vee y) \wedge (x \vee y^\sim) = x$;

3. $y \wedge (y^\sim \vee x) = y \wedge x$.

Let us notice that Theorem 69 and Theorem 70 smootly generalize well known properties of orthomodular lattices. However, we may observe that if for the orthomodular case the equivalences are carried over by virtue of the lattice operations and the orthocomplement, in the case of UOM both negations $^\sim$, $^+$ are necessary in the following sense. In case of orthomodular lattices, it is well known that xCy iff $x = (x \wedge y) \vee (x \wedge y')$ iff $x = (x \vee y) \wedge (x \vee y')$. In the case of UOM, in order to preserve this lattice-theoretic duality we have to consider both $^+$ and $^\sim$ (Corollary 13).[1]

The following lemma provides several useful conditions that are sufficient for elements to commute. Let us observe that, in spite of the orthomodular case, the condition $x \leq y$ is not sufficient for x, y to commute in UOM. We may refer the reader to Example 75 and the comment on page 114.

[1]Let us remark that choosing $^+$ instead of $'$ is nothing but a matter of choice, since they are each other term definable.

Lemma 25. *Let $A \in UOM$. Then, if*

1. $x \leq \Box y$ *then* $x\bar{C}y$*;*
2. $\Diamond y \leq x$ *then* $x\bar{C}y$*;*
3. $x \leq y^{\sim}$ *then* $x\bar{C}y$*;*
4. $y^{+} \leq x$ *then* $x\bar{C}y$*.*

Proof. We confine ourselves in proving (1) and (3). As regards (1), Let $x \leq \Box y$, then by monotonicity $\Diamond x \leq \Box y$. Then, $\Box x \leq \Box y$, and so *a fortiori* $\Box x, \Diamond x \leq \Diamond y$. So all these sharp elements commute in $S_K(A)$, and then they commute in the sense of Definition 38. For item (3), if $x \leq y^{\sim}$ then $\Diamond x \leq y^{\sim}$. So, $\Box x \leq y^{\sim} \leq y^{+}$, and therefore $\Box x C y^{+}$, $\Diamond x C y^{+}$ and by Lemma 23 our claim follows. □

Theorem 71 presents a neat generalization of orthomodularity in the context of UOM, i.e. in case we are dealing with a bounded de Morgan lattice (not necessarily distributive, thou!) whose blocks are 3-valued MV algebras.

Theorem 71. *Let $A \in UOM$. Then,*

1. $\Diamond x \leq y \Rightarrow x \vee (x^{+} \wedge y) = y$*;*
2. $y \leq \Box x \Rightarrow x \wedge (x^{\sim} \vee y) = y$*.*

Proof. (1) If $\Diamond x \leq y$ then $x\bar{C}y$, by Lemma 25. And then by distributivity of blocks, $x \vee (x^{+} \wedge y) = (x \vee x^{+}) \wedge (x \vee y) = 1 \wedge (x \vee y) = x \vee y = y$. (2) It is dual. □

In other words, Theorem 71 says that if $x\bar{C}y$ ($\Diamond x \leq y$ and $y \leq \Box x$), then a generalized form of orthomodularity holds true.
Let us observe that in the context of UOM a full fledged version of Foulis-Holland Theorem 15 can be proven.

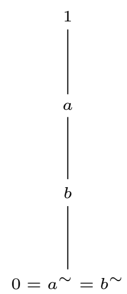
Theorem 72 (Unsharp Foulis-Holland)**.** *If L is in UOM and $x, y, z \in L$ are such that both $x\bar{C}y$ and $x\bar{C}z$, then the set $\{x, y, z\}$ generates a distributive sublattice of L.*

Proof. Suppose that both $x\bar{C}y$ and $x\bar{C}z$. Then, $\Box(x \wedge (y \vee z)) = \Box(x) \wedge \Box(y \vee z) = \Box x \wedge (\Box y \vee \Box z) = (\Box x \wedge \Box y) \vee (\Box x \wedge \Box z)$, by condition (SDM), and analogously $\Diamond(x \wedge (y \vee z)) = (\Diamond x \wedge \Diamond y) \vee (\Diamond x \wedge \Diamond z)$. Then, condition (SK) applies, and therefore $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. $\Box$

Finally, we can observe without a straightforward proof that other well known facts that are true for orthomodular lattices can be generalized in UOM, see in the preliminaries Lemma 8-(2), (4).

Let us observe that the conditions in Theorem 71 are weaker than condition (SK). However, if we extend the axioms of PBZ^*_{SDM} with those conditions, axiom (SK) still does not hold, as the next example shows.

Example 73. *Consider the lattice L whose Hasse diagram is the following:*



Clearly, L satisfies strong de Morgan and the conditions in Theorem 71, but not (SK)*.*

We leave for a future work the investigation of the variety of PBZ^* axiomatized by strong de Morgan and the conditions of Theorem 71. In fact, these conditions allow a tighter relation with orthomodularity, and therefore the structures in such variety would be smoother and more tractable than those in PBZ^*_{SDM}.

From Theorem 71 we easily derive the last result of the present section. The corollary that follows in fact generalize the notion of paraorthomodularity, which in the context of PBZ^*_{SDM} is no longer equivalent to the conditions in Theorem 71.

Corollary 14. *Any $A \in UOM$ satisfies the following:*

1. *if $\Diamond y \leq x$ and $y^+ \wedge x = 0$, then $x = \Diamond y$;*

2. *if $x \leq \Box y$ and $x \vee y^\sim = 1$, then $x = \Box y$.*

Proof. For a modicum of a proof, let us observe that $\Diamond y \leq x$ implies that $x\bar{C}y$, and by pseudocomplementation $y^+ \wedge x = 0$ yields that $x = \Diamond y$. The other condition is dual. □

Finally, we can sum up the previous results within the next theorem, that perfectly generalizes the state of affairs in orthomodular lattices, see e.g. [11, Theorem 6.11].

Theorem 74. *Let $A \in UOM$. Then, the following conditions are equivalent:*

1. *A is a distributive UOM, i.e. it is a regular double Stone algebra;*

2. *A is an MV algebra;*

3. *for all x, y, $x\bar{C}y$;*

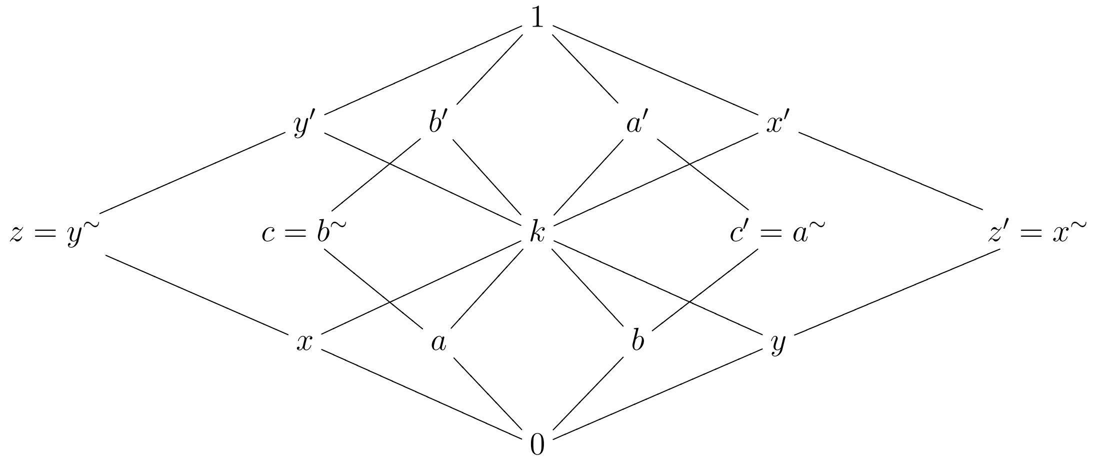
4. *$\bar{C}$ is transitive;*

5. *$\bar{C}$ is an equivalence relation;*

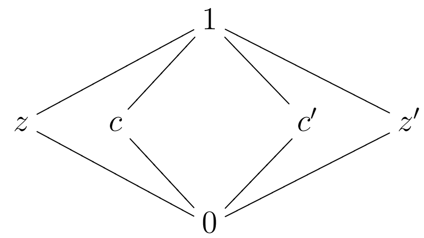
6. *$^\sim$ is a pseudocomplement;*

7. *$S_K(A)$ is a Boolean algebra.*

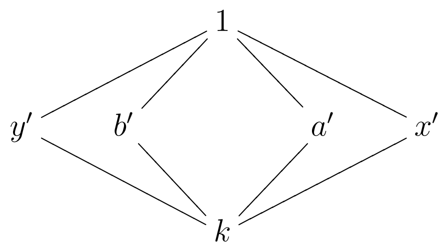
Example 75. *Let us call* **15** *the UOM lattice whose structure is described by the following Hasse diagram:*

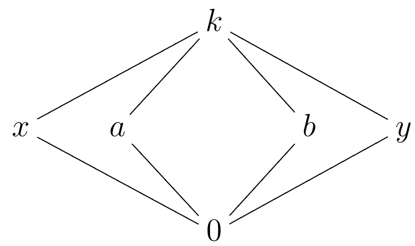


We can see that $S_K(\mathbf{15})$ is the following lattice:

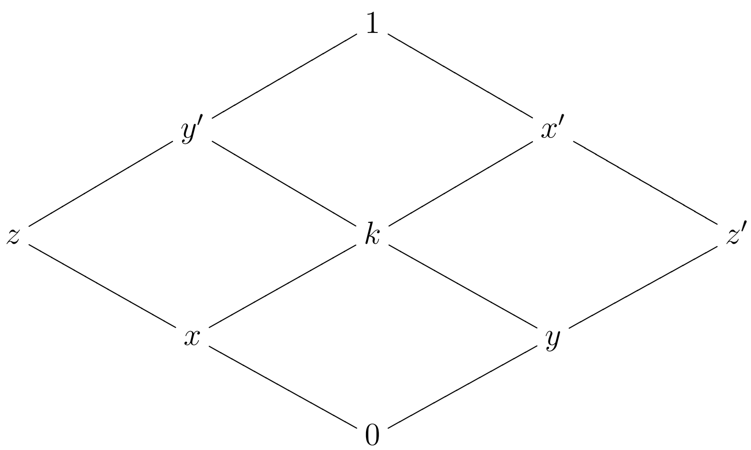


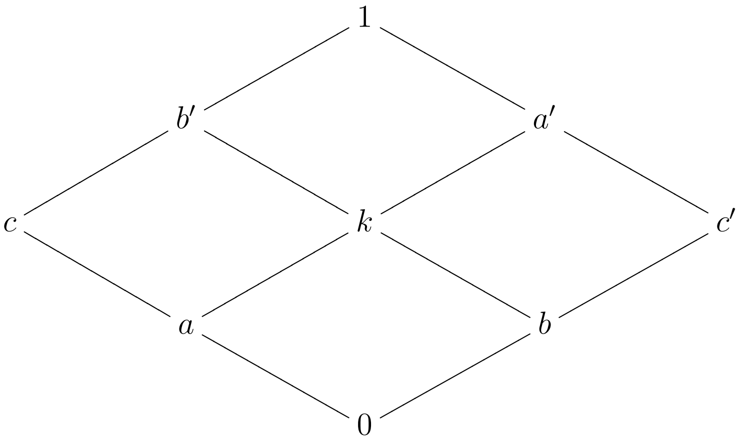
while the dense elements $D^{\sim}(\mathbf{15})$ and the dually dense elements $D^{+}(\mathbf{15})$ are respectively the lattices:





Also the blocks of **15** *are isomorphic copies of the* 9*-element Kleene lattice. Namely,*





Note that $k = k'$. Then, by Lemma 22, we may observe that **15** *is the smallest lattice in UOM such that $S_K(\mathbf{15})$ is a proper orthomodular lattice that is not a Boolean algebra, namely MO_2.*

Let us remark that Example 75 falsifies Equation 3 in Theorem 69, just set $y = a'$. In fact, x and a' belong to different blocks.

We can now propose necessary and sufficient conditions for arbitrary elements to commute.

Theorem 76. *Let $A \in UOM$. The following are equivalent:*

1. $x\bar{C}y$;
2. $(x \vee y) \wedge (x \vee y^{\sim}) = x = (x \wedge y) \vee (x \wedge y^{+})$.

Proof. We just observe that, if $x\bar{C}y$, the equations in (2) are trivially satisfied. Conversely, suppose that (2) holds and x, y do not commute. Then, $y \in D^{\sim}(A) \cap D^{+}(A)$. And so y is the fixed point by Lemma 22, which commutes with any element. A contradiction. □

We notice that the class of UOM lattices that possess a fixed point do not form a proper variety in the language $(\vee, \wedge, ', ^{\sim}, 0, 1)$. Indeed, MO_2 is a proper subalgebra of **15** in Example 75, and it does not admit any fixed point.

In passing let us observe that in [55] the quasi-equation

$$y^{\sim} \leq x \text{ implies } y^{\sim} \vee (\Diamond y \wedge x) = x \qquad \text{(Semiorthomodularity)}$$

is introduced as a generalization of orthomodularity in the context of PBZ^* lattices. This condition may appear similar to the quasi-equations in Theorem 71. However, condition (Semiorthomodularity) does not capture the notion of block in Definition 39. In fact, in Example 75, if we consider a', x', then both $a'^{\sim} \leq x'$ and condition (Semiorthomodularity) hold true, but a', x' do not commute in the sense of Definition 39.

3.6 Residuation in UOM

Considering the tight connection between UOM lattices, MV algebras, Heyting algebras and orthomodular lattices, a question naturally arises: what happens to residuation in the whole structure.
In section 3.3, we have observed that any block M of a UOM lattice A can be regarded as a regular double Stone algebra, or as a regular double Heyting algebra, or as a 3-valued MV algebra at the same time. These structures are paramount examples of residuated lattices. The original connection between double Heyting algebras and double regular p-algebras was firstly formulated by Katriňák in [70], and then elaborated very recently by Conejo, Kinyom, and Sankappanavar in [37].

Katriňák's translation is:

$$x \to y = (x^{\sim} \vee y^{\sim\sim})^{\sim\sim} \wedge ((x \vee x^{\sim})^{+} \vee x^{\sim} \vee y \vee y^{\sim}).$$

Using distributivity, de Morgan, and (Stone condition) Katriňák definition rewrites into:

$$x \otimes y = x \wedge y \quad \text{and} \quad x \to y = (x^{\sim} \vee \Diamond y) \wedge (x^{+} \vee y \vee y^{\sim}).$$

Would it be possible to define a general form of residuation in a UOM lattice A that reasonably reflects the full residuation that can be encountered in any block of A or in $S_K(A)$?

In our opinion, the next theorem proposes a reasonable answer.

Theorem 77. *Let $A \in UOM$. Then, if we define $\otimes, \to$ as follows:*

$$x \otimes y = \Diamond y \wedge (y^{\sim} \vee x) \quad \textit{and} \quad x \to y = x^{\sim} \vee (\Diamond x \wedge y),$$

we have that:

$$x \otimes y \leq z \ \textit{ if and only if } \ x \leq y \to z.$$

Proof. Consider $x, y \in A$ and $x \otimes y \leq z$. Then,

$$x \leq x \vee y^{\sim}.$$

Let us notice that $(x \vee y^{\sim})$, $y^{\sim}$ and $\Diamond y$ are in the same block (see Definition 39), because:

$$\Box(x \vee y^{\sim}) \geq y^{\sim} \text{ and } \Diamond(x \vee y^{\sim}) \geq y^{\sim}.$$

Then by properties of orthomodular lattices,

$$\Box(x \vee y^{\sim}) \ C \ y^{\sim} \text{ and } \Diamond(x \vee y^{\sim}) \ C \ y^{\sim}.$$

Moreover,

$$\Box(x \vee y^{\sim}) \ C \ (y^{\sim})^{\sim} = \Diamond y \text{ and } \Diamond(x \vee y^{\sim}) \ C \ (y^{\sim})^{\sim} = \Diamond y.$$

Hence, $(x \vee y^{\sim})\bar{C}y^{\sim}$ and $(x \vee y^{\sim})\bar{C}\Diamond y$; therefore, we can use distributivity by Theorem 72.

So, using distributivity:

$$\begin{aligned}
x \leq x \vee y^{\sim} &= ((x \vee y^{\sim}) \vee y^{\sim}) \wedge 1 \\
&= ((x \vee y^{\sim}) \vee y^{\sim}) \wedge (\Diamond y \vee y^{\sim}) \\
&= ((x \vee y^{\sim}) \wedge \Diamond y) \vee y^{\sim} \\
&= ((x \vee y^{\sim}) \wedge \Diamond y \wedge \Diamond y) \vee y^{\sim} \\
&= ((x \otimes y) \wedge \Diamond y) \vee y^{\sim} \\
&\leq (z \wedge \Diamond y) \vee y^{\sim} \\
&= y \to z.
\end{aligned}$$

Otherwise, suppose $x \leq y \to z$. Then:

$$\begin{aligned} x \otimes y &= \Diamond y \wedge (y^{\sim} \vee x) \\ &\leq \Diamond y \wedge (y^{\sim} \vee (y \to z)) \\ &= \Diamond y \wedge (y^{\sim} \vee (y^{\sim} \vee (\Diamond y \wedge z))) \\ &= \Diamond y \wedge (y^{\sim} \vee (\Diamond y \wedge z)) \\ &= (\Diamond y \wedge y^{\sim}) \vee (\Diamond y \wedge (\Diamond y \wedge z)) \\ &= 0 \vee (\Diamond y \wedge (\Diamond y \wedge z)) \\ &= (\Diamond y \wedge z) \\ &\leq z. \end{aligned}$$

□

Let us remark that a result similar to Theorem 77 is present in [55], albeit in the context of semiorthomodular BZ^*-lattices.

Perhaps, a few observations about these operations may be useful. To begin with, if we consider x, y elements of a common block, then the operations that we have defined behave "almost" as Heyting or MV operations. Indeed, the operations turn into:

$$x \otimes y = x \wedge \Diamond y \quad \text{and} \quad x \to y = x^{\sim} \vee y.$$

In addition, they are similar to the orthomodular lattices case. In fact, in order to show that orthomodular lattices can be converted into left residuated ℓ-groupoids Chajda and Länger use the Sasaki projection and the Sasaki hook [26], namely:

$$x \otimes y = (x \vee y') \wedge y \text{ and } x \to y = (y \wedge x) \vee x'.$$

Let us observe that, in case x, y are sharp elements of an $A \in UOM$, then the definitions coincide.

In case $x \leq y^{\sim}$, then $\otimes$ coincides with $\wedge$, and therefore $\to$ is its residual. In fact, if $x \leq y^{\sim}$, then they commute, and so:

$$x \otimes y = x \wedge \Diamond y \leq y^{\sim} \wedge \Diamond y = 0.$$

Therefore, $x \wedge y \leq x \wedge \Diamond y \leq y^{\sim} \wedge \Diamond y = 0$, i.e. $x \wedge y = x \wedge \Diamond y$.

We now state a technical lemma that will be useful in continuing our arguments.

Lemma 26. *Consider A be a UOM lattice. Then, $(A, \vee, \wedge, \otimes, \rightarrow)$ satisfies the following properties:*

1. *$x \otimes 1 = x$ (right unital);*
2. *for all x, $x \leq 1$ (integral);*
3. *for all x, $0 \leq x$ (0-bounded);*
4. *$x \otimes (x \vee y) = x$ (strongly idempotent);*
5. *$(x \rightarrow (x \wedge y)) \otimes x = x \wedge y$ (divisible);*
6. *$x \leq y$ implies $y \rightarrow 0 \leq x \rightarrow 0$ (antitone);*
7. *$x \otimes \Diamond y = (x \vee (y \rightarrow 0)) \wedge \Diamond y$ ($\Diamond$-Sasakian);*
8. *$\Diamond x \rightarrow y = (\Diamond x \wedge y) \vee (x \rightarrow 0)$ ($\Diamond \rightarrow$-Sasakian).*

Proof. We confine ourselves in proving only (1) (4) (6) and (7), leaving the other properties to the interested reader.

(1) $x \otimes 1 = \Diamond 1 \wedge (1^{\sim} \vee x) = 1 \wedge x = x$.

(4) It can be seen that:

$$\begin{aligned} x \otimes (x \vee y) &= \Diamond(x \vee y) \wedge ((x \vee y)^{\sim} \vee x) \\ &= (\Diamond(x \vee y) \wedge (x \vee y)^{\sim}) \vee \Diamond((x \vee y) \wedge x)) \\ &= x. \end{aligned}$$

(6) Suppose $x \leq y$. Then,

$$\begin{aligned} y \rightarrow 0 &= y^{\sim} \vee (\Diamond y \wedge 0) \\ &= y^{\sim} \vee 0 \\ &= y^{\sim} \\ &\leq x^{\sim} \\ &= x \rightarrow 0. \end{aligned}$$

(7) $x \otimes \Diamond y = \Diamond y \wedge (y^{\sim} \vee x) = \Diamond y \wedge ((y \to 0) \vee x)$. □

We now introduce the notion of a right double residuated ℓ-groupoid. This notion will define our target structure for obtaining a term equivalence with UOM lattices.

Definition 42. *A* right double residuated ℓ-groupoid *is an algebra*

$$A = (A, \vee, \wedge, \otimes_1, \to_1, \otimes_2, \to_2, 0, 1),$$

such that:

1. $(A, \vee, \wedge)$ *is a lattice;*

2. $(A, \otimes_1, \to_1)$ *is a right-unital groupoid;*

3. $(A, \otimes_2, \to_2)$ *is a right-unital groupoid;*

4. *for all* $x, y, z \in A$*:*

$$x \otimes_1 y \leq z \text{ if and only if } x \leq y \to_1 z.$$

5. *for all* $x, y, z \in A$*:*

$$x \otimes_2 y \leq z \text{ if and only if } x \leq y \to_2 z.$$

Finally, we close the present section by stating and proving our term equivalence result.

Theorem 78. *Consider* $A = (A, \vee, \wedge, \otimes_1, \to_1, \otimes_2, \to_2, 0, 1)$ *right double residuated* ℓ*-groupoid, which satisfies the following properties:*

1. $x \otimes_1 y = \Diamond y \wedge (y^{\sim} \vee x)$*;*

2. $x \to_1 y = x^{\sim} \vee (\Diamond x \wedge y)$*;*

3. $x \otimes_2 y = \Box y \wedge (y^{+} \vee x)$*;*

4. $x \to_2 y = x^{+} \vee (\Box x \wedge y)$*;*

5. $x \leq (x \rightarrow_1 0) \rightarrow_1 0$*;*

6. $(x \rightarrow_2 0) \rightarrow_2 0 \leq x$*;*

7. $x \leq y$ *then* $y \rightarrow_1 0 \leq x \rightarrow_1 0$ *and* $y \rightarrow_2 0 \leq x \rightarrow_2 0$*;*

8. $x \otimes_1 (x \vee y) = x$*;*

9. $x \otimes_1 x = x$*;*

10. $x \wedge \Diamond y \leq \Box x \vee y$*;*

11. $(x \rightarrow_2 0) \rightarrow_1 0 = (x \rightarrow_2 0) \rightarrow_2 0$*;*

12. $(x \rightarrow_1 0) \rightarrow_2 0 = (x \rightarrow_1 0) \rightarrow_1 0$*.*

where

$$\Diamond x = (x \rightarrow_1 0) \rightarrow_1 0 \quad \textit{and} \quad \Box x = (x \rightarrow_2 0) \rightarrow_2 0.$$

Then $\mathbb{F}(A) = (A, \vee, \wedge, ', ^{\sim}, 0, 1)$ *is a* UOM *lattice, where the negations are defined as follows:*

$$x^{\sim} = x \rightarrow_1 0, \quad x^{+} = x \rightarrow_2 0, \quad x' = \left(x \wedge x^{+}\right) \vee x^{\sim}.$$

Conversely, if $A = (A, \vee, \wedge, ', ^{\sim}, 0, 1)$ *is a* UOM *lattice, then* $\mathbb{G}(A) = (A, \vee, \wedge, \otimes_1, \rightarrow_1, \otimes_2, \rightarrow_2, 0, 1)$ *is a right double residuated* ℓ*-groupoid. Moreover,* $\mathbb{F}$*,* $\mathbb{G}$ *are mutually inverse mappings.*

Proof. Obviously, $(A, \vee, \wedge)$ is a lattice. To begin with, notice that the negations are also involutive if they are applied on elements of the form $\Box x$, $\Diamond x$, $x^{\sim}$, x^{+}. By definition, we have that:

$$\begin{aligned} x \leq \Diamond x = &\ (x \rightarrow_1 0) \rightarrow_1 0 \text{ (square extensive)}; \\ \Diamond x \rightarrow_1 0 \leq &\ x \rightarrow_1 0 \text{ (antitone)}. \end{aligned}$$

However:

$$x \rightarrow_1 0 \leq ((x \rightarrow_1 0) \rightarrow_1 0) \rightarrow_1 0 = \Diamond x \rightarrow_1 0.$$

Therefore, it follows that

$$x^{\sim} = x \to_1 0 = ((x \to_1 0) \to_1 0) \to_1 0 = x^{\sim\sim\sim}.$$

Moreover, $\Diamond x = (\Diamond x)^{\sim\sim}$.
The cases of $\Box$, $^+$ are dealt with similarly. Let us notice that on elements of the form $\Box x$, $\Diamond x$, $x^{\sim}$, x^+ the operations $^{\sim}$, $^+$ coincide, by conditions (11) and (12). Therefore, the set of these elements, that we call unimaginatively $S_K(\mathbb{F}(A))$, is an involutive and antitone lattice.

In addition, we can show that $S_K(\mathbb{F}(A))$ is an orthomodular lattice.
Consider $x, y \in S_K(\mathbb{F}(A))$ and $x \leq y$. Then:

$$\begin{aligned} y &= (y^{\sim})^{\sim} \\ &= (y^{\sim} \otimes_1 (y^{\sim} \vee x^{\sim}))^{\sim} \\ &= (y^{\sim} \otimes_1 x^{\sim})^{\sim} \\ &= (x^{\sim} \wedge (x \vee y^{\sim}))^{\sim} \\ &= x \vee (x^{\sim} \wedge y). \end{aligned}$$

Hence, it satisfies the orthomodular law. Obviously, $x \vee x^{\sim} = x \vee (1 \wedge x^{\sim}) = 1$, i.e. it is complemented. Besides, we can prove: $x \wedge x^{\sim} = 0$ and $x \vee x^+ = 1$. In fact,

$$\begin{aligned} &x \wedge x^{\sim} \leq \Diamond x \wedge x^{\sim} = 0; \\ &x \vee x^+ \geq \Box x \vee x^+ = 1. \end{aligned}$$

Now, it is necessary to prove that $'$ is a Kleene negation. We can use axiom (SK), which is equivalent to axiom (10) (Lemma 20) : $x \wedge \Diamond y \leq \Box x \vee y$.
For antitonicity, suppose $x \leq y$, then:

$$\begin{aligned} \Box(y') &= \Box((y \wedge y^+) \vee y^{\sim}) \\ &= y^{\sim} \leq x^{\sim} \\ &= \Box(x'). \end{aligned}$$

Also:

$$\begin{aligned}\Diamond(y') = &\quad \Diamond((y \wedge y^{+}) \vee y^{\sim}) \\ = &\quad y^{+} \leq x^{+} \\ = &\quad \Diamond(x').\end{aligned}$$

Hence, we can conclude that $y' \leq x'$. For involutivity, consider $x \in A$, then

$$\begin{aligned}(x')' &= ((x \wedge x^{+}) \vee x^{\sim})' \\ &= (((x \wedge x^{+}) \vee x^{\sim}) \wedge ((x \wedge x^{+}) \vee x^{\sim})^{+}) \vee ((x \wedge x^{+}) \vee x^{\sim})^{\sim} \\ &= (((x \wedge x^{+}) \vee x^{\sim}) \wedge ((x^{+} \vee \Box x) \wedge \Diamond x)) \vee ((x^{\sim} \vee \Box x) \wedge \Diamond x) \\ &= (((x \wedge x^{+}) \vee x^{\sim}) \wedge ((0) \wedge \Diamond x)) \vee \Box x.\end{aligned}$$

We take advantage of (SK) axiom again and we obtain the following:

$$\Box([((x \wedge x^{+}) \vee x^{\sim}) \wedge ((0) \wedge \Diamond x)] \vee \Box x) = \Box x;$$
$$\Diamond([((x \wedge x^{+}) \vee x^{\sim}) \wedge ((0) \wedge \Diamond x)] \vee \Box x) = \Diamond x.$$

And then $(x')' = x$. Finally, we check Kleene. Take $x, y \in A$, then:

$$\Box(x \wedge x') = 0 \leq \Box(y \vee y').$$

Also:

$$\begin{aligned}\Diamond(x \wedge x') = &\quad \Diamond x \wedge x^{+} \\ \leq &\quad 1 \\ = &\quad \Diamond(y \vee y').\end{aligned}$$

Therefore, $x \wedge x' \leq y \vee y'$ for any $x, y \in A$. The converse and the fact that $\mathbb{F}$, $\mathbb{G}$ define a term-equivalence are straightforward and left to the reader. $\Box$

3.7 Future work

As we have seen in this chapter there are profound connections between UOM lattices and varieties of paramount relevance to classical mathematics, such as e.g. Boolean algebras, Heyting algebras, MV algebras, orthomodular lattices, Kleene algebras, Stone algebras etc. And the significance of all of them is, we hope, harmonically merged into the variety of UOM lattices.

In the near future we are planning to deepen the investigation of the themes introduced in this work. In particular, hopefully we will

1. investigate to what extent the relations between Boolean algebras, Heyting algebras, MV algebras, orthomodular lattices can be carried to a general level;

2. investigate the properties and the general behavior of the operation $\oplus$ regarded as an operation on the whole structure $A \in UOM$;

3. discuss the interplays between MV algebras and double Stone algebras;

4. enrich a linguistic extension of the language of UOM lattices by a fixed point k and study its features;

5. investigate the variety of PBZ^* axiomatized by strong de Morgan and the generalized orthomodular laws;

6. study possible generalizations of classical theorems such as Greechie's pasting Theorems;

7. investigate the notion of center.

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