Lyapunov-Free Analysis for Consensus of Nonlinear Discrete-Time Multi-Agent Systems

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Abstract— **In this paper we propose a novel method to establish stability and convergence to a consensus state for a class of nonlinear discrete-time Multi-Agent System (MAS) which is not based on Lyapunov function arguments. In particular, we focus on a class of discrete-time multi-agent systems whose global dynamics can be represented by sub-homogeneous and order-preserving nonlinear maps. The preliminary results of this paper directly generalize results for sub-homogeneous and order-preserving linear maps which are shown to be the counterpart to stochastic matrices thanks to nonlinear Perron-Frobenius theory. We provide sufficient conditions on local interaction rules among agents to establish convergence to a fixed point and study the consensus problem in this generalized framework as a particular case. Examples to show the effectiveness of the method are provided to corroborate the theoretical analysis. In these examples, some nonlinear interaction protocols are proved to converge to the consensus state without the use of Lyapunov functions.**

I. INTRODUCTION

The study of complex systems where local interactions between individuals give rise to a global collective behavior has aroused much interest in the control community. In such systems several agents cooperate to achieve a global objective (cooperative systems) or compete pursuing their own interests (competitive systems). A Multi-Agent System (MAS) consists of multiple interacting agents with mutual interactions among them. Specifically, the cooperative control of MAS raises significant theoretical and practical challenges: a topic that recently captured the attention of many researchers is the consensus problem [1], where the objective is to design local interaction rules among agents such that their state variables converge to the same value, the so called agreement or consensus state.

Within the vast literature on multi-agent systems, one of the most popular works is [2], in which the authors established criteria for convergence to a consensus state for MAS whose global dynamics can be represented by linear time-varying systems with row-stochastic state transition matrices. In [2] the authors exploited non-negative matrix theory and algebraic graph theory instead of Lyapunov theory. On of the main reasons why the stability of these systems is appealing is because, as it later became clear by the work in [3], common quadratic Lyapunov functions may not exist in general for these systems and nonlinear and non-smooth Lyapunov functions are usually ad-hoc solutions for particular dynamical systems. In this paper we aim to exploit non-linear Perron-Frobenius theory, a generalization of non-negative matrix theory, to address non-linear interactions in MAS without Lyapunov based arguments. It follows that a MAS modeled by a row-stochastic matrix becomes a particular case of the proposed generalized theory.

The literature on nonlinear consensus problems is vast. It is mostly composed by particular nonlinear consensus protocols which offer advantages such as finite-time convergence [4], [5], resilience to non-uniform time-delays [6] and many more. These protocols are usually proved to converge to the consensus state via ad-hoc Lyapunov functions.

Among the approaches which aim to establish convergence to consensus for some class of nonlinear MAS we mention the work in [7], which is developed for continuous-time MAS, in which the authors identify a class of non-linear interactions denoted as "sub-tangent" and establish necessary and sufficient conditions on the network topology for convergence to consensus. Qualitatively, the class of "sub-tangent" MAS includes those non-linear interaction rules where the direction of movement of each agent is strictly inside the convex hull spanned by the state value of its neighbors.

The **main contribution** of this paper is to identify criteria, not based on Lyapunov arguments, to establish stability and convergence to consensus of a class of discrete-time non-linear MAS. In particular, we take inspiration from non-linear Perron-Frobenius theory and provide sufficient conditions for stability of multi-agent systems whose global dynamics can be represented by sub-homogeneous and order-preserving nonlinear maps. Such maps represent a general class, including maps which are not strictly convex, as the one addressed by [8]. Moreover, the particular case of sub-homogeneous and order-preserving linear maps is equivalent to MAS that evolve according to row-stochastic and row-substochastic matrices. The results in this paper are preliminary in that the sufficient condition provided is restricted to those MAS whose local interaction rules admit fixed points only in the consensus state, this condition will be extended in future works.

This paper is organized as follows. In Section II preliminaries on multi-agent systems and the consensus problem are presented. In Section III an introduction to nonlinear Perron-Frobenius Theory is provided. In Section IV we discuss our main results for the convergence to fixed points and to consensus for nonlinear multi-agent systems. In Section V we provide examples on how to analyse the convergence properties of a nonlinear MAS with our proposed method. Finally, in Section VI concluding remarks and future research

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directions are discussed.

II. Preliminaries

A MAS is composed by a set of dynamical systems, i.e., the agents, that interact according to local interaction rules within a communication or sensing network. The interactions among agents of a MAS can be represented by a graph. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists in a set of nodes $\mathcal{V} = \{1, \ldots, n\}$ representing the agents and a set of directed edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ representing the existence of interactions among them. A directed edge $(i, j) \in \mathcal{E}$ exists in the graph if agent i interacts with agent j, i.e., j updates its own state considering the state value of agent i. Two agents i and j are called neighbors if either $(i, j) \in \mathcal{E}$ or $(j, i) \in \mathcal{E}$. Agent i is an in-neighbor of agent j if $(i, j) \in \mathcal{E}$. Agent i is an out-neighbor of agent j if $(j, i) \in \mathcal{E}$. The set of in-neighbors of agent i is denoted as $\mathcal{N}_i^{in} = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$, the set of out-neighbors as $\mathcal{N}_i^{out} = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$.

We consider agents that are modeled as autonomous discrete-time dynamical systems with state $x_i \in \mathbb{R}$ and evolve according to a state transition function, i.e., a map $f_i : \mathbb{R}^n \to \mathbb{R}$ which takes as input the state of agent i and its neighbors, and outputs the new state of the agent i, i.e.,

$$x_i(k+1) = f_i(x_j(k) : j \in \mathcal{N}_i^{in} \cup \{i\}). \tag{1}$$

We call the map f_i a *local interaction rule*. The state of the multi-agent system at time k is then compactly represented as $x(k) = [x_1(k), \ldots, x_i(k), \ldots, x_n(k)]^T$. Thus, the global dynamics of the MAS can be represented as follows

$$x(k+1) = f(x(k)) = \begin{bmatrix} f_1(x_j(k) : j \in \mathcal{N}_1^{in} \cup \{i\}), \\ \vdots \\ f_i(x_j(k) : j \in \mathcal{N}_i^{in} \cup \{i\}), \\ \vdots \\ f_n(x_j(k) : j \in \mathcal{N}_n^{in} \cup \{i\}), \end{bmatrix}. \tag{2}$$

If the dynamics of a MAS consists of linear local interaction rules, the powerful Perron-Frobenius theory, related to non-negative matrices, can be applied to study its asymptotic behavior. An excellent historical account of linear Perron-Frobenius theory is given by Hawkins [9]. A *linear map* $f : \mathbb{R}^n \to \mathbb{R}^n$ can be represented by a square matrix A after choosing a basis for the vector space $\mathbb{R}^n$. Different choices of the basis give rise to equivalent matrices which are related by a similarity transformation. Now, consider the linear version of (2)

$$x(k+1) = f(x(k)) = Ax(k) \tag{3}$$

where A is a $n \times n$ row-stochastic matrix, i.e., a non-negative matrix such that $A\mathbf{1} = \mathbf{1}$ where $\mathbf{1}$ is a vector of n unitary elements.

Theorem 1: Consensus with Linear Interactions
Consider the MAS in (3). If the next statements hold

(i) *Matrix A is non-negative and row-stochastic;*

(ii) *Graph $\mathcal{G}$ has a rooted directed spanning tree and the subgraph of root nodes is aperiodic;*

then the MAS converges asymptotically to the consensus state $x = c\mathbf{1}$ where $c \in \mathbb{R}$.

Proof: See [10] for a detailed discussion on discrete-time consensus with linear-interactions. □

III. Introduction to nonlinear Perron-Frobenius theory

The result in Theorem 1 is built deeply into the linear Perron-Frobenius theory. If we wish to investigate the consensus for nonlinear interaction rules, it is reasonable to start by generalizing such theory. Krein and Rutman [11] placed the Perron-Frobenius theory in the more general context of linear maps, by exploiting a crucial property of non-negative matrices: non-negative matrices leave the set of non-negative vectors $\mathbb{R}^n_{\geq 0} \subset \mathbb{R}^n$ invariant (see Definition 1) where we denote by $\mathbb{R}_{\geq 0}$ the set of positive reals that includes zero. A detailed account of following results from this generalization can be found in Schaefer's book [12].

Definition 1: Invariant Set
A map $f : \mathbb{R}^n \to \mathbb{R}^n$ which maps points of $K \subseteq \mathbb{R}^n$ in K itself, is said to leave the set K invariant. We denote this property by $f(K) \subseteq K$. ■

More precisely, since the set $\mathbb{R}^n_{\geq 0}$ is a cone, a major part of the Perron-Frobenius theory can be generalized to linear maps that leave a cone in a vector space invariant. Let f be a map over a finite-dimensional real vector space X, a subset K of X is called a cone if it is convex, $\alpha K \subseteq K$ for all $\alpha \geq 0$, and $K \cap (-K) = \{0\}$. A cone is *proper* if it is *closed* (it contains all its limit points) and *full* (its largest open set is not empty). A cone K is *polyhedral* if there is some matrix A such that $K = \{x \in \mathbb{R}^n : Ax \geq 0\}$.

In this paper we focus on Euclidean vector spaces $X = \mathbb{R}^n$ (and the proper and polyhedral cone $K = \mathbb{R}^n_{\geq 0}$ of non-negative vectors) which is a partially ordered set with respect to the natural order relation, given $x \in \mathbb{R}^n$ we have

$$x = (x_1, \ldots, x_n) \geq 0 \Leftrightarrow x_i \geq 0 \tag{4}$$

where $i = 1, \ldots, n$. For $u, v \in K = \mathbb{R}^n$, we can write the partial ordering relations as follow

$$\begin{aligned} u &\leq v \Leftrightarrow u_i \leq v_i, \\ u &\lneq v \Leftrightarrow u_i \leq v_i, u \neq v, \\ u &< v \Leftrightarrow u_i < v_i, \end{aligned}$$

where $i = 1, \ldots, n$ and the relation $\leq$ is called the vector order in $\mathbb{R}^n$. Maps which preserve such vector order are said to be order-preserving, and in some special case strictly and strongly order-preserving, as defined below.

Definition 2: (Strict / Strong) order-preservation *Consider the non-negative cone $\mathbb{R}^n_{\geq 0}$, a continuous map $f : \mathbb{R}^n \to \mathbb{R}^m$, is said to be*

- *Order-preserving: for any $x, y \in \mathbb{R}^n_{\geq 0}$ it holds*

$$x \leq y \Leftrightarrow f(x) \leq f(y).$$

- *Strictly order-preserving: for any $x, y \in \mathbb{R}^n_{\geq 0}$ it holds*

$$x \lneq y \Leftrightarrow f(x) \lneq f(y).$$

- *Strongly order-preserving: for any $x, y \in \mathbb{R}^n_{\geq 0}$ it holds*

$$x \lneq y \Leftrightarrow f(x) < f(y). \quad \blacksquare$$

From Definition 2 we immediately obtain the following relationship.

Remark 1: Relationship among order-preserving maps
Strongly order-preserving ⇒ strict order-preserving ⇒ order-preserving. ■

For linear maps $f : \mathbb{R}^n \to \mathbb{R}^n$ the condition $f(\mathbb{R}^n_{\geq 0}) \subseteq \mathbb{R}^n_{\geq 0}$ is equivalent to the property of order-preservation. This is the key concept underlying various detailed extensions of Perron-Frobenius theory for order-preserving nonlinear maps [13]. Order-preserving dynamical systems play an important role for some classes of mathematical models arising in modern natural science [14]. Order-preserving dynamical systems and nonlinear Perron-Frobenius theory are closely related.

In the theory of order-preserving dynamical systems, the emphasis is placed on strong order-preservation. For discrete-time strongly order-preserving dynamical systems one has generic convergence to periodic trajectories under appropriate conditions [15]. An extensive overview of these results was given by Hirsch and Smith [16]. On the other hand, in nonlinear Perron-Frobenius theory one usually considers discrete-time dynamical systems that are only order-preserving, but satisfying an additional concave assumption, obtaining similar results regarding periodic trajectories [17]. We address a class of order-preserving maps, which is somewhere between these two theories while giving sufficient conditions for the convergence to the fixed point set, and avoiding divergent and periodic trajectories.

Among different suitable concave assumptions, we focus on the property of sub-homogeneity. Order-preserving and sub-homogeneous maps arise in a variety of applications, including optimal control and game theory [18], mathematical biology [19], in the analysis of discrete event systems [20] and so on. Dynamical systems ruled by order-preserving and sub-homogeneous maps are non-expansive under some metric, which allows one to prove detailed results concerning their long term behavior, see [21] and also Chapter 8 in [13] and reference therein. The non-expansiveness property severely constrains the complexity of the iterative behavior of a map [22], [23] and makes all fixed points and periodic trajectories Lyapunov stable, i.e. every bounded trajectory in the interior of a polyhedral cone converges to a periodic trajectory. We now define the meaning of sub-homogeneity and non-expansiveness in $\mathbb{R}^n$ with respect to $\mathbb{R}^n \geq 0$, as it is our case of interest.

Definition 3: Sub-homogeneity
With respect to A continuous map $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be sub-homogeneous if $\alpha f(x) \leq f(\alpha x)$ for all $x \in \mathbb{R}^n$ and $\alpha \in [0, 1]$. ■

Definition 4: Non-expansiveness
Let $d(x, y)$ be a metric. A continuous map $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be non-expansive if $x, y \in V$ it holds that $d(f(x), f(y)) \leq d(x, y)$. ■

Concluding this sections, we summon back some basic concepts from dynamical systems theory. Let X be a topological space. A point $x \in X$ is called a *periodic point* of $f : X \to X$ if there exist an integer $p \geq 1$ such that $f^p(x) = x$. The minimal such $p \geq 1$ is called the *period* of x under f. If $f(x) = x$, we call x a fixed point of f. A *fixed point* is a periodic point with period $p = 1$. Fixed points are the equivalent of equilibrium points. We denote $F_f = \{x \in X : f(x) = x\}$ the set of all fixed points of f. A fixed point $x \in int(X)$ is said to be *positive*. A fixed point p is said to be *Lyapunov stable* if for every $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that $x \in K$ and $\|x - p\| < \delta$ implies that $\|f^k(x) - p\| < \varepsilon$ for all $k \in \mathbb{Z}$, where $\|\cdot\|$ denotes the usual norm and $\mathbb{Z}_+ = \{1, 2, ...\}$. The *trajectory* of x under f is given by $\mathcal{T}(x, f) = \{f^k(x) : k \in \mathbb{Z}_+\}$. If f is clear from the context, we simply write $\mathcal{T}(x)$. If x is a periodic point, we say that $\mathcal{T}(x)$ is a periodic trajectory. To understand the long-term behavior of the iterates of a map one has to analyze the structure of its ω-limit sets. For $x \in X$ and a continuous map $f : X \to X$, the *ω-limit set*, denoted $\omega(x, f)$ or simply $\omega(x)$ is defined by

$$\omega(x) = \bigcap_{k \geq 0} cl\left(\{f^m(x) : m \geq k\}\right).$$

IV. Convergence results

In this section we provide our main convergence results. There are not, up to our knowledge, general conditions to ensure convergence to fixed points for a sub-homogeneous order-preserving nonlinear map. Thus, we address a special order-preserving class, but more general than the strongly order-preserving, and we will call it *locally strongly order-preserving* class, as follows.

Definition 5: Local Strong Order-preservation
A continuous map $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be locally strongly order-preserving if, for all $i = 1, \ldots, n$, given $x, y \in \mathbb{R}^n_{\geq 0}$ and $x \lneq y$:

- *f_i is order-preserving when $x_i = y_i$;*
- *f_i is strongly order-preserving when $x_i < y_i$*

where $f_i : \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ is the i-th component of f. ■

Lemma 1: Global and Local Order-preservation
Local strong order-preservation ⇒ strict order-preservation.

Proof: By applying the definition of local strong order-preservation to points $x, y \in K$ such that $x \lneq y$ we obtain

$$f_i(x) \leq f_i(y), \qquad f_j(x) < f_j(y),$$

for all $i \in \{k : x_k = y_k\}$ and $j \in \{k : x_k < y_k\}$. Therefore $f(x) \lneq f(y)$, which is the definition of local strict order-preservation. □

Remark 2: Global and Local Sub-homogeneity

Consider the MAS in eq. (2)*. The global map* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is sub-homogeneous if and only if each local map* $f_i : \mathbb{R}^n \to \mathbb{R}$ *is sub-homogeneous, for all* $i = 1, \ldots, n$. ■

A significant effort has gone into identifying widely applicable and easily verifiable sufficient conditions ensuring that a mapping is non-expansive. The authors of [21] showed how sub-homogeneous order-preserving maps are related to maps that are non-expansive under Thompson's part metric (see [21] for the definition).

Lemma 2: Non-expansiveness of Sub-homogeneous Order-preserving maps

If $f : \mathbb{R}^n \to \mathbb{R}^m$ *is an order-preserving map, then it is sub-homogeneous if and only if is non-expansive with respect to Thompson's part metric.*

Proof: See Lemma 3.3 in [21]. □

This result can be generalized to polyhedral cones by saying that Thompson's metric spaces can be isometrically embedded into $(\mathbb{R}^n, \|\cdot\|_\infty)$ if the underlying cone is polyhedral. Therefore we recall the following result due to [24].

Remark 3: Sup-norm Non-expansiveness

All maps which are non-expansive under the Thompson's part metric are also non-expansive under the sup-norm $\|\cdot\|_\infty$. ■

This allows us to apply the result shown next to sub-homogeneous order-preserving maps on the standard positive cone, which is polyhedral.

Lemma 3: Trajectory Periodicity

If $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ *is a sup-norm non-expansive map, then either all trajectories* $\mathcal{T}(x)$ *of* f *are unbounded, i.e.,* $\omega(x) = \emptyset$ *for all* $x \in K$*, or for each* $x \in K$ *there exists an integer* $p \geq 1$ *and a periodic point* $y_x \in K$ *of* f *with period* p*, i.e.,* $|\omega(x)| = p$*, such that* $\lim_{k\to\infty} f^{kp}(x) = y_x$.

Proof: It is a particular case of Theorem 4.2.1 in [13]. □

Lemma 3 ensures that if there exists at least one bounded trajectory, then all trajectories converge to the same periodic trajectory, while if there exists at least one unbounded trajectory, then all trajectories diverges. As mentioned above, in this work we focus on the convergence problem and then we aim to systems where all trajectories approach special periodic points with period $p = 1$, called fixed points.

In [25], Jiang studied sub-homogeneous and locally strongly order-preserving maps and gave a theorem which ensures their convergence in the standard positive cone.

Lemma 4: Periodic Trajectory Convergence

Let $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ *be sub-homogeneous and locally strongly order-preserving. If every trajectory is bounded in* $\mathbb{R}^n_{\geq 0}$*, then every trajectory converges, that is* $\omega(x)$ *is a singleton for each* $x \in \mathbb{R}^n_{\geq 0}$.

Proof: See Theorem 2.3 in [25]. □

We now state our main results.

Theorem 2: Convergence to Fixed Points

If the map $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ *is locally strongly order-preserving and sub-homogeneous and if* f *has a Lyapunov stable fixed point, then every trajectory converges to a fixed point.*

Proof: Since f is locally is locally strongly order-preserving, it is also strictly order preserving (see Lemma 1) and then order-preserving (see Remark 1). Therefore, f is an order-preserving sub-homogeneous map which is non-expansive under the Thompson's part metric (see Lemma 2) and also under the sup-norm (see Remark 3).

Let $\bar{x}$ be a Lyapunov stable fixed point. Lyapunov stability of a fixed point $\bar{x}$ means that for any $\varepsilon > 0$ arbitrary small there exists $\delta_\varepsilon > 0$ such that solutions starting from x_0 sufficiently close to $\bar{x}$, i.e., $\|x_0 - \bar{x}\| < \delta_\varepsilon$, remain in a bounded domain forever, i.e., $\left\|f^k(x_0) - \bar{x}\right\| < \varepsilon$ for $k \in \mathbb{Z}_+$. This means that the trajectory $\mathcal{T}(x_0, f)$ is enclosed in a ball of center x_0 and ray ε, i.e., it is bounded.

Since f leaves the cone $\mathbb{R}^n_{\geq 0}$ invariant as in Definition 1 and it is a sup-norm non-expansive map, Lemma 3 ensures that if at least one trajectory is bounded, as it is our case, then all trajectories of f are bounded. Therefore, map f satisfies all conditions of Lemma 4, which ensures that all trajectories converges to a fixed point. □

To put it another way, a sup-norm non-expansive map with a stable point guarantees that all trajectories do not diverge, in fact, they converge to periodic points. By requiring the additional property of local order-preservation we prevent the system to have periodic trajectories, forcing it to converge to the fixed point set.

Corollary 1: Convergence to Positive Fixed Point

If the map $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ *is locally strongly order-preserving and sub-homogeneous and if* f *has a positive fixed point, then every trajectory converges to a fixed point.*

Proof: It is convenient to recall a preliminary result of [25] given in Lemma 3.3: a positive fixed point $p \in int(\mathbb{R}^n_{\geq})$, i.e., p is a vector with all entries which are strictly positive, is Lyapunov stable if f is sub-homogeneous order-preserving. This means that f has one Lyapunov stable point. The rest of the proof is similar to the one of Theorem 2. □

Now we are interested in understanding when the system converges to consensus when the fixed point set is a subset of all vectors with all entries equal. Before stating the theorem, we introduce the concept of graph for a nonlinear map, which was introduced in [26] for homogeneous order-preserving maps.

Theorem 3: Nonlinear Consensus

If a set of (possibly heterogeneous) local interaction rules $f_i(x_j : j \in \mathcal{N}_i^{in})$ *with* $i \in \mathcal{V}$ *satisfies the next properties:*

(*i*) $f_i(x) \geq 0$ *for all* $x \in \mathbb{R}^m_{\geq 0}$;

(*ii*) $\partial f_i/\partial x_i > 0$ *and* $\partial f_i/\partial x_j \geq 0$ *for* $i \neq j$;

(iii) $\alpha f_i(x) \le f_i(\alpha x)$ *for all* $\alpha \in [0,1]$ *and* $x \in \mathbb{R}^n_{\ge 0}$*;*
(iv) $f_i(x_j : j \in \mathcal{N}_i^{in} \cup \{i\}) = x_i$ *if and only if* $x_i = x_j$ *for all* $j \in \mathcal{N}_i^{in}$*;*
(v) *Graph* $\mathcal{G}$ *has a rooted directed spanning tree;*

then the multi-agent system in (2) converges asymptotically to the consensus state.

Sketch of the proof: This proof is based upon the equivalence relationship between properties $(i),(ii),(iii),(iv),(v)$ and the following (a,b,c,d):

(a) f leaves the cone $\mathbb{R}^n_{\ge 0}$ invariant;
(b) $f : \mathbb{R}^n_{\ge 0} \to \mathbb{R}^n_{\ge 0}$ is locally strongly order-preserving;
(c) $f : \mathbb{R}^n_{\ge 0} \to \mathbb{R}^n_{\ge 0}$ is sub-homogeneous;
(d) $F_f = \{x \in \mathbb{R}^n_{\ge 0} : x = c\mathbf{1}, \quad c \in \mathbb{R}^+\}$.

Due to space limitations, we give now, in broad outline, the reasoning behind the proof. It holds:

- $(i) \Leftrightarrow (a)$;
- $(ii) \Leftrightarrow (b)$: it is due to Definitions 2 and 5;
- $(iii) \Leftrightarrow (c)$: can be derived from Remark 2;
- $((iv)+(v)) \Rightarrow (d)$: First, note that all vectors $v = c\mathbf{1}$ where $c \in \mathbb{R}$ are a feasible solution for (iv), therefore

$$F_f \supseteq \{x \in \mathbb{R}^n_{\ge 0} : x = c\mathbf{1}, \quad c \in \mathbb{R}^+\}.$$

Then, we show that if there exists a fixed point $v \in F_f$ such that $v_i \neq v_j$, then (iv) is false and thus we find a contradiction. Since (v) holds by assumption, select a directed path from the root node of the spanning tree to a leaf node such that nodes i, j with $v_i \neq v_j$ are included in the path. For each edge (k,l) in the path, due to (iv), it holds $v_k = v_l$. Therefore, also $v_i = v_j$ which is contradiction. Thus, if (iv) and (v) hold, the only fixed points of the MAS are those of the consensus state

$$F_f = \{x \in \mathbb{R}^n_{\ge 0} : x = c\mathbf{1}, \quad c \in \mathbb{R}^+\}.$$

Thus, if the (possibly different) local interaction rules executed by the agents of the MAS satisfy conditions $(i),(ii),(iii),(iv),(v)$, then $(a),(b),(c),(d)$ hold, thus due to Theorem 2 the MAS converges to the fixed point set which corresponds to the consensus state. □

V. Convergence analysis examples

Consider a MAS described by a graph $\mathcal{G}$ as in Figure 1 and a generic nonlinear local rule as defined in (5).

$$f_i = x_i + \varepsilon \sum_{j \in \mathcal{N}_i^{in}} g_{ij}(x) \tag{5}$$

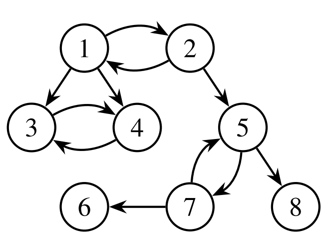

Fig. 1. Graph with two rooted directed spanning trees, where root nodes are 1 and 2.

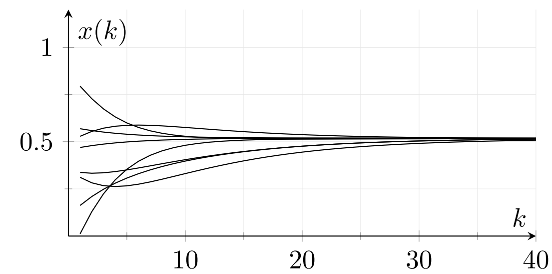

Fig. 2. Evolution of a MAS with local interaction rule as in (5) and (6), for $\varepsilon = 0.1$.

Example 1: Nonlinear interaction based on arctangent function

Consider a MAS as in (2) ruled by a function f as in (5) where

$$g_{ij} = \text{atan}(x_j - x_i). \tag{6}$$

We now evaluate conditions $(i)-(v)$ of Theorem 3 to establish the convergence of the associated MAS.

$$(i) \qquad x_i + \varepsilon \sum_{j \in \mathcal{N}_i^{in}} \text{atan}(x_j - x_i) \ge 0, \quad \forall x \in \mathbb{R}^n_{\ge 0},$$

thus f leaves the positive cone $\mathbb{R}^n_{\ge 0}$ invariant.

$$(ii) \qquad \frac{\partial}{\partial x_j} f_i = \frac{\varepsilon}{1 + (x_j - x_i)^2} \ge 0, \quad \forall x \in \mathbb{R}^n_{\ge 0}.$$

$$\frac{\partial}{\partial x_i} f_i = 1 - \frac{\varepsilon \cdot |\mathcal{N}_i^{in}|}{1 + (x_j - x_i)^2} > 0, \quad \forall x \in \mathbb{R}^n_{\ge 0},$$

and for any $\varepsilon < \min_{i \in \mathcal{V}} 1/|\mathcal{N}_i^{in}|$.

$$(iii) \ \alpha \sum_{j \in \mathcal{N}_i^{in}} \text{atan}(x_j - x_i) \le \sum_{j \in \mathcal{N}_i^{in}} \text{atan}(\alpha x_j - \alpha x_i), \forall x \in \mathbb{R}^n_{\ge 0}.$$

$$(iv) \qquad v_i + \alpha \sum_{j \in \mathcal{N}_i^{in}} \text{atan}(v_j - v_i) = v_i,$$

for $x_i = x_j$ for all $i, j = 1, \ldots, n$, i.e., $v^* = c\mathbf{1}$ with $c \in \mathbb{R}^n_{\ge 0}$ is the unique solution.

(v) Graph $\mathcal{G}$ has a rooted directed spanning tree.

Since all conditions are satisfied, then, due to Theorem 3, the MAS converges to consensus for all $x \in \mathbb{R}^n_{\ge 0}$. A numerical simulation is given in Figure 2.

Example 2: Piece-wise linear local interaction rules

Consider a MAS as in (2) ruled by a function f as in (5) where g_{ij} is piece-wise linear, for example

$$g_{ij} = \begin{cases} a_{ij}(x_j - x_i) & \text{if } x \in D \subseteq \mathbb{R}^n_{\ge 0} \\ b_{ij}(x_j - x_i) & \text{if } x \in \mathbb{R}^n_{\ge 0} \setminus D \end{cases} \tag{7}$$

We now evaluate conditions $(i)-(v)$ of Theorem 3 to establish the convergence of the associated MAS.

$$(i) \qquad \begin{cases} x_i + \varepsilon \sum_{j \in \mathcal{N}_i^{in}} a_{ij}(x_j - x_i) & \ge 0 \\ x_i + \varepsilon \sum_{j \in \mathcal{N}_i^{in}} b_{ij}(x_j - x_i) & \ge 0 \end{cases}, \quad \forall x \in \mathbb{R}^n_{\ge 0},$$

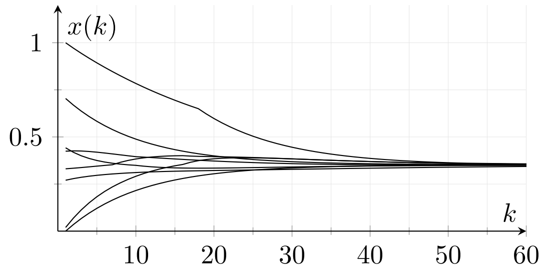

Fig. 3. Evolution of a MAS with local interaction rule as in (5) and (7), for $\varepsilon = 0.1$, $a_{ij} = 1$, $b_{ij} = 0.5$ $\forall i, j \in \mathcal{V}$ and $D = \{x \in \mathbb{R}^n_{\geq 0} : 0.35 \leq x_i \leq 0.65, \; \forall i \in \mathcal{V} \leq \}$.

if $\varepsilon \sum_{j \in \mathcal{N}_i^{in}} a_{ij} \leq 1$, $\varepsilon \sum_{j \in \mathcal{N}_i^{in}} b_{ij} \leq 1$ and $a_{i,j}, b_{i,j} \geq 0$.

$$(ii) \quad \begin{array}{l} \frac{\partial}{\partial x_j} f_i = \begin{cases} \varepsilon a_{ij} \geq 0 & \text{if } x \in D \subseteq \mathbb{R}^n_{\geq 0} \\ \varepsilon b_{ij} \geq 0 & \text{if } x \in \mathbb{R}^n_{\geq 0} \setminus D \end{cases}, \quad \forall x \in \mathbb{R}^n_{\geq 0}, \\ \frac{\partial}{\partial x_i} f_i = \begin{cases} 1 - \varepsilon \sum_{j \in \mathcal{N}_i^{in}} a_{ij} > 0 & \text{if } x \in D \subseteq \mathbb{R}^n_{\geq 0} \\ 1 - \varepsilon \sum_{j \in \mathcal{N}_i^{in}} b_{ij} > 0 & \text{if } x \in \mathbb{R}^n_{\geq 0} \setminus D \end{cases}, \end{array}$$

if $\varepsilon \sum_{j \in \mathcal{N}_i^{in}} a_{ij} \leq 1$ and $\varepsilon \sum_{j \in \mathcal{N}_i^{in}} b_{ij} \leq 1$.

$$(iii) \quad \begin{array}{l} \alpha a_{ij}(x_j - x_i) \leq a_{ij}(\alpha x_j - \alpha x_i), \\ \alpha b_{ij}(x_j - x_i) \leq b_{ij}(\alpha x_j - \alpha x_i), \end{array} \quad x \in \mathbb{R}^n_{\geq 0}.$$

$$(iv) \quad \begin{array}{l} x_i + \varepsilon \sum_{j \in \mathcal{N}_i^{in}} a_{ij}(x_j - x_i) = x_i \\ x_i + \varepsilon \sum_{j \in \mathcal{N}_i^{in}} b_{ij}(x_j - x_i) = x_i, \end{array}$$

for $x_i = x_j$ for all $i, j = 1, \dots, n$, i.e., $v^* = c\mathbf{1}$ with $c \in \mathbb{R}^n_{\geq 0}$ is the unique solution.

(v) Graph $\mathcal{G}$ has a rooted directed spanning tree.

Since all conditions are satisfied, then, due to Theorem 3, the MAS converges to consensus for all $x \in \mathbb{R}^n_{\geq 0}$. A numerical simulation is given in Figure 3.

VI. Conclusions and future works

In this paper we proposed a novel method to establish the convergence properties of nonlinear discrete-time multi-agent systems whose global dynamics can be represented by sub-homogeneous and order-preserving maps. The proposed method is inspired by non-linear Perron-Frobenius theory and is not based on Lyapunov arguments. It directly generalizes results on stability and convergence to consensus of MAS whose global dynamics can be represented by row-stochastic and row-substochastic matrices. Examples are provided to corroborate the description of our method.

Future work will aim to generalize the proposed method to non-linear discrete-time MAS with time-varying network topology.

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