Marras, M., Vernier-Piro, S., & Yokota, T. (2022). Blow-up phenomena for a chemotaxis system with flux limitation. *Journal of Mathematical Analysis and Applications, 515*(1), 126376.
https://doi.org/10.1016/j.jmaa.2022.126376

Blow-up phenomena for a chemotaxis system with flux limitation

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Abstract

In this paper we consider nonnegative solutions of the following parabolic-elliptic cross-diffusion system

$$\begin{cases} u_t = \Delta u - \nabla(uf(|\nabla v|^2)\nabla v), \\ 0 = \Delta v - \mu + u, \quad \int_\Omega v = 0, \quad \mu := \frac{1}{|\Omega|}\int_\Omega u dx, \\ u(x,0) = u_0(x), \end{cases}$$

in $\Omega \times (0, \infty)$, with Ω a ball in $\mathbb{R}^N$, $N \geq 3$ under homogeneous Neumann boundary conditions and $f(\xi) = (1+\xi)^{-\alpha}$, $0 < \alpha < \frac{N-2}{2(N-1)}$, which describes gradient-dependent limitation of cross diffusion fluxes. Under conditions on f and initial data, we prove that a solution which blows up in finite time in L^∞-norm, blows up also in L^p-norm for some $p > 1$. Moreover, a lower bound of blow-up time is derived.

AMS Subject Classification Primary: 35B44; Secondary: 35Q92, 92C17.

Key Words: finite-time blow-up; chemotaxis.

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1 Introduction

In this paper we consider the chemotaxis system with flux limitation,

$$
\begin{cases}
u_t = \Delta u - \nabla(uf(|\nabla v|^2)\nabla v), & x \in \Omega,\ t > 0, \\
0 = \Delta v - \mu + u, & x \in \Omega,\ t > 0, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\
u(x,0) = u_0(x), & x \in \Omega,
\end{cases}
\tag{1.1}
$$

with Ω a ball in $\mathbb{R}^N$, $N \geq 3$, $\mu = \frac{1}{|\Omega|}\int u dx > 0$, $\int_\Omega v dx = 0$, $f \in C^2([0,\infty))$.
We assume that the initial data $u_0(x) \in C^0(\overline{\Omega})$, $u_0 \geq 0$.
System (1.1) is a modified version of the well known Keller–Segel model

$$
\begin{cases}
u_t = \Delta u - \nabla(u\nabla v), & x \in \Omega,\ t > 0, \\
v_t = \Delta v - \mu + u, & x \in \Omega,\ t > 0,
\end{cases}
$$

proposed by Keller and Segel [9] in 1970, which is a mathematical model describing aggregation phenomena of organisms due to chemotaxis, i.e., the directed movement of cell density $u(x,t)$ at the position x and at the time t in response to the gradient of a chemical attractant $v(x,t)$. The presence of the elliptic equation in (1.1) instead of the parabolic one reflects the situation where the chemicals diffuse much faster than cells move.
For decades various Keller–Segel type systems have been extensively studied by many authors.
In [1], the authors propose a very exhaustive survey and analysis focused on classical and modified Keller–Segel models. Moreover, other contributions (e.g., [8], [11], [13], [15] and [16]) investigate the behavior of the solutions to chemotaxis systems, specifically boundedness, decay, blow-up properties and non-degeneracy of blow-up points. For more general Keller–Segel systems involving three equations of fully parabolic type or parabolic-elliptic-elliptic type, see [5], [7] and the reference therein.
The finite-time blow-up of nonradial solutions of (1.1) is investigate in [18], where some conditions on the mass and the moment of the initial data are introduced, with $f(|\nabla v|^2) = 1$ on a bounded domain in $\mathbb{R}^2$.

Bellomo and Winkler [3] consider the following chemotaxis system

$$
(1.2) \qquad \begin{cases} u_t = \nabla \cdot \Big(\dfrac{u\nabla u}{\sqrt{u^2 + |\nabla u|^2}} \Big) - \chi \nabla \cdot \Big(\dfrac{u \nabla v}{\sqrt{1 + |\nabla v|^2}} \Big), \\ 0 = \Delta v - \mu + u, \end{cases}
$$

under the initial condition $u_0(x) > 0$ and no-flux boundary conditions, when the spatial domain Ω is a ball in $\mathbb{R}^N$, $N \geq 1$, $\mu := \frac{1}{|\Omega|} \int_\Omega u_0 dx$. The authors prove that if $\chi > 1$ then, for any choice of m with

$$
\begin{cases} m > \frac{1}{\sqrt{\chi^2 - 1}}, & \text{if } N = 1, \\ m > 0 \ \text{ is arbitary}, & \text{if } N \geq 2, \end{cases}
$$

there exist positive initial data $u_0 \in C^3(\overline{\Omega})$, $\int_\Omega u_0 dx = m$, which are such that the problem (1.2) possesses, for some $T_{max} > 0$, a uniquely determined classical solution (u, v) in $\Omega \times (0, T_{max})$, blowing up at time T_{max} in the sense that $\limsup_{t \nearrow T_{max}} \|u(x,t)\|_{L^\infty} = \infty$. These results are a continuation of the analytical study presents in [2] of the flux-limited chemotaxis model (1.2) in which the main results assert the existence of a unique classical solution of (1.2), extensible in time up to a maximal $T_{max} \in (0, \infty]$ which has the property that if $T_{max} < \infty$ then $\limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty} = \infty$.
In [6] Chiyoda et al. consider the system

$$
(1.3) \qquad \begin{cases} u_t = \nabla \cdot \Big(\dfrac{u^p \nabla u}{\sqrt{u^2 + |\nabla u|^2}} \Big) - \chi \nabla \cdot \Big(\dfrac{u^q \nabla v}{\sqrt{1 + |\nabla v|^2}} \Big), \\ 0 = \Delta v - \mu + u, \end{cases}
$$

in a ball in $\mathbb{R}^N$, $N \in \mathbb{N}$, under no-flux boundary conditions and initial condition $u_0(x) > 0$. Assuming suitable conditions for χ and u_0 when $1 \leq p \leq q$, they obtain existence of blow-up solutions of (1.3). When $p = q = 1$ the system (1.3) reduces to (1.2).
In [17] Mizukami et al. for the solutions to the problem (1.3) obtain

- if $p, q \geq 1$, local existence and extensibility criterion ruling out gradient blow-up;
- if $p > q + 1 - \frac{1}{N}$, global existence and boundedness.

Negreanu and Tello in [19] consider the case when $f(|\nabla v|^2) = \chi|\nabla v|^{p-2}$, i.e.,

$$
\begin{cases} u_t = \Delta u - \nabla \cdot (\chi u|\nabla v|^{p-2}\nabla v), \\ 0 = \Delta v - \mu + u, \end{cases} \tag{1.4}
$$

with homogeneous Neumann boundary conditions and nonnegative initial data $u_0(x)$ with $\mu := \frac{1}{|\Omega|}\int_\Omega u_0 dx$, χ a positive constant and p so that

$$
\begin{cases} p \in (1, \infty), & \text{if} \quad N = 1, \\ p \in \left(1, \frac{N}{N-1}\right), & \text{if} \quad N \geq 2. \end{cases}
$$

Under suitable assumptions on the data, they obtain for the solutions of (1.4) uniform bounds in $L^\infty(\Omega)$ and the global existence, while for the one-dimensional case, the existence of infinitely many non-constant steady-states for $p \in (1, 2)$ for any χ positive and a given positive mass is obtained.
For other related results we refer [10], [21], [24], [25] and [26].

In this paper we focus our attention on blow-up phenomena, extensively studied both in the elliptic and in the parabolic cases (see for instance [12], [14] and references therein).
For the solutions of (1.1), in [23], Winkler proves that, if $f(\xi) \geq (1+\xi)^{-\alpha}$ with $0 < \alpha < \frac{N-2}{2(N-1)}$, then throughout a considerably large set of radially symmetric initial data, the blow-up phenomenon, with respect to the L^∞ norm of u, occurs in finite time.
This result is contained in the following theorem.

Theorem 1.1 ([23] Finite-time blow-up in L^∞-norm)**.** *Let $\Omega \equiv B_R(0) \subset \mathbb{R}^N$, $N \geq 3$ and $R > 0$, and let f satisfy*

$$
f \in C^2([0, \infty)), \text{ as well as } f(\xi) \geq k_f(1+\xi)^{-\alpha} \quad \text{for all } \xi \geq 0 \tag{1.5}
$$

with some $k_f > 0$ and

$$
0 < \alpha < \frac{N-2}{2(N-1)}. \tag{1.6}
$$

Then for any choice of $\mu > 0$ one can find $R_0 = R_0(\mu) \in (0, R)$ with the property that whenever u_0 satisfies

$$
u_0 \in C^0(\overline{\Omega}), \ u_0 \text{ nonnegative with } \frac{1}{|\Omega|}\int_\Omega u_0 dx = \mu > 0 \tag{1.7}
$$

and

$$(1.8)\quad u_0 \text{ is radially symmetric with } \int_{B_r(0)} u_0 dx \geq \int_\Omega u_0 dx,\ \forall\, r \in (0,R)$$

as well as

$$(1.9)\qquad \frac{1}{|\Omega|}\int_{B_{R_0}(0)} u_0 dx \geq \frac{\mu}{2}\Big(\frac{R}{R_0}\Big)^N,$$

the corresponding solution (u,v) *of* (1.1) *blows up in finite time; that is, for the uniquely determined local classical solution, maximally extended up to some time* $T_{max} \in (0,\infty]$ *according to Lemma 2.1 below, we then have* $T_{max} < \infty$ *and*

$$(1.10)\qquad \limsup_{t \nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

The first purpose of this paper is to prove that the solutions of (1.1) blow up in L^p-norm, for some $p > 1$, if they blow up in L^∞-norm.

Theorem 1.2 (Finite-time blow-up in L^p-norm)**.** *Let* $\Omega \equiv B_R(0) \subset \mathbb{R}^N$, $N \geq 3$ *and* $R > 0$*. Then, the classical solution* (u,v) *of* (1.1) *for* $t \in (0,T_{max})$ *and with* $f(\xi) = k_f(1+\xi)^{-\alpha}$ *with some* $k_f > 0$*, provided by Theorem 1.1, is such that for all* $\frac{N}{2} < p < N$,

$$\limsup_{t \nearrow T_{max}} \|u(\cdot,t)\|_{L^p(\Omega)} = \infty.$$

The second purpose of this paper is to study the behavior of the solutions of (1.1) near the blow-up time T_{max}.
Since it is not always possible to compute T_{max}, deriving a lower bound is a matter of great importance, in order to obtain a safe time interval of existence of the solution $[0,T]$ with $T < T_{max}$.
With this aim, we define for all $p > 1$ the auxiliary function

$$(1.11)\qquad \Psi(t) := \frac{1}{p}\|u(\cdot,t)\|^p_{L^p(\Omega)} \quad \text{with} \quad \Psi_0 := \Psi(0) = \frac{1}{p}\|u_0\|^p_{L^p(\Omega)}.$$

Theorem 1.3 (Lower bound of blow-up time)**.** *Let $\Omega \equiv B_R(0) \subset \mathbb{R}^N$, $N \geq 3$, $R > 0$ and let Ψ be defined in* (1.11)*. Then, for all $\frac{N}{2} < p < N$ and some positive constants B_i, $i = 1, \cdots, 4$ defined in* (4.14) *below, the blow-up time T_{max} for* (1.1) *with $f(\xi) = k_f(1+\xi)^{-\alpha}$ with some $k_f > 0$, provided by Theorem 1.1, satisfies the estimate*

$$T_{max} \geq T := \int_{\Psi_0}^{\infty} \frac{d\eta}{B_1\eta + B_2\eta^{\gamma_1} + B_3\eta^{\gamma_2} + B_4\eta^{\gamma_3}}, \tag{1.12}$$

with $\gamma_1 := \frac{p+1}{p}$, $\gamma_2 := \frac{2(p+1)-N}{2p-N}$, $\gamma_3 := \frac{2(p+1)-\frac{N(p+1)(1+\epsilon)}{p+1+\epsilon}}{2p-\frac{N(1+\epsilon)(p+1)}{p+1+\epsilon}}$.

The scheme of this paper is the following: Section 2 is concerned with preliminaries including the Neumann heat semigroup, in Section 3, since the solution of (1.1) blows up in finite time in L^∞-norm we prove that the solution blows up also in L^p-norm (for some $p > 1$). Section 4 is devoted to find appropriate assumptions on the data, such that the $\|u\|_{L^p(\Omega)}$ remains bounded in $[0, T]$ with $T < T_{max}$. Clearly this value of T provides a lower bound for blow-up time T_{max} of u.

2 Preliminaries

In this section, we present some preliminary lemmata which we shall use in the proof of our main results.

Lemma 2.1 (see [23])**.** *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$ be a bounded domain with smooth boundary, and assume that f and u_0 satisfy* (1.5) *and* (1.7)*. Then there exists $T_{max} \in (0, \infty]$ an a uniquely determined pair (u, v) of functions*

$$u \in C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})),$$
$$v \in \bigcap_{q>N} L^\infty_{loc}([0, T_{max}); W^{1,q}(\Omega)) \cap C^{2,0}(\overline{\Omega} \times (0, T_{max})),$$

with $u \geq 0$ and $v \geq 0$, in $\Omega \times (0, T_{max})$, such that (u, v) solves (1.1) *classically in $\Omega \times (0, T_{max})$, with*

$$\int_\Omega u(\cdot, t) = \int_\Omega u_0 \quad for\ all\ t \in (0, T_{max}) \tag{2.1}$$

and

$$\text{if } T_{max} < \infty, \quad \text{then } \limsup_{t \nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

Moreover, if $\Omega = B_R(0)$ with some $R > 0$ and u_0 is radially simmetric with respect to $x = 0$, then also $u(\cdot,t)$ and $v(\cdot,t)$ are radially symmetric for each $t \in (0, T_{max})$.

We next give some properties of the Neumann heat semigroup which will be used later. For the proof, see [4, Lemma 2.1] and [22, Lemma 1.3].

Lemma 2.2. *Let $(e^{t\Delta})_{t\geq 0}$ be the Neumann heat semigroup in Ω, and let $\mu_1 > 0$ denote the first non zero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then there exist $k_1, k_2 > 0$ which depend only on Ω and have the following properties*:

(i) if $1 \leq q \leq \mathrm{p} \leq \infty$, then

$$\|e^{t\Delta} z\|_{L^{\mathrm{p}}(\Omega)} \leq k_1 t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{\mathrm{p}})} \|z\|_{L^q(\Omega)}, \quad \forall\ t > 0 \tag{2.2}$$

holds for all $z \in L^q(\Omega)$ satisfying $\int_\Omega z = 0$.

(ii) If $1 < q \leq \mathrm{p} \leq \infty$, then

$$\|e^{t\Delta} \nabla \cdot \boldsymbol{z}\|_{L^{\mathrm{p}}(\Omega)} \leq k_2 \big(1 + t^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q}-\frac{1}{\mathrm{p}})}\big) e^{-\mu_1 t} \|\boldsymbol{z}\|_{L^q(\Omega)}, \quad \forall\ t > 0 \tag{2.3}$$

is valid for any $\boldsymbol{z} \in (L^q(\Omega))^N$, where $e^{t\Delta}\nabla\cdot$ is the extension of the operator $e^{t\Delta}\nabla\cdot$ on $(C_0^\infty(\Omega))^N$ to $(L^q(\Omega))^N$.

We observe that since constants are invariant under $e^{t\Delta}$ we can use (2.2) writing $\bar{z} = \frac{1}{|\Omega|}\int_\Omega z$ so that we have $\int_\Omega (z - \bar{z}) = 0$ (see [22]).
In Section 4 we will use the Gagliardo–Nirenberg inequality in the following form.

Lemma 2.3. *Let Ω be a bounded and smooth domain of $\mathbb{R}^N$ with $N \geq 1$. Let $\mathsf{r} \geq 1$, $1 \leq \mathsf{q} < \mathsf{p} \leq \infty$, $\mathsf{s} > 0$. Then there exists a constant $C_{\mathrm{GN}} > 0$ such that*

$$\|f\|^{\mathsf{p}}_{L^{\mathsf{p}}(\Omega)} \leq C_{\mathrm{GN}} \Big(\|\nabla f\|^{\mathsf{p}a}_{L^{\mathsf{r}}(\Omega)} \|f\|^{\mathsf{p}(1-a)}_{L^{\mathsf{q}}(\Omega)} + \|f\|^{\mathsf{p}}_{L^{\mathsf{s}}(\Omega)} \Big) \tag{2.4}$$

for all $f \in L^{\mathsf{q}}(\Omega)$ with $\nabla f \in (L^{\mathsf{r}}(\Omega))^N$ and $a := \frac{\frac{1}{\mathsf{q}}-\frac{1}{\mathsf{p}}}{\frac{1}{\mathsf{q}}+\frac{1}{N}-\frac{1}{\mathsf{r}}} \in (0,1)$.

Proof. Following from the Gagliardo–Nirenberg inequality (see [20] for more details):

$$\|f\|^{\mathsf{p}}_{L^{\mathsf{p}}(\Omega)} \le \left[c_{\mathrm{GN}}\Big(\|\nabla f\|^{a}_{L^{\mathsf{r}}(\Omega)}\|f\|^{1-a}_{L^{\mathsf{q}}(\Omega)} + \|f\|_{L^{\mathsf{s}}(\Omega)}\Big)\right]^{\mathsf{p}},$$

with some $c_{\mathrm{GN}} > 0$, and then from the inequality

$$(\mathsf{a}+\mathsf{b})^{\mathsf{p}} \le 2^{\mathsf{p}}(\mathsf{a}^{\mathsf{p}}+\mathsf{b}^{\mathsf{p}}) \quad \text{for any } \mathsf{a},\mathsf{b}\ge 0,\ \mathsf{p}>0,$$

we arrive to (2.4) with $C_{\mathrm{GN}} = 2^{\mathsf{p}} c^{\mathsf{p}}_{\mathrm{GN}}$. □

3 Blow-up in L^p-norm

The aim of this section is to prove Theorem 1.2. To this end, first we prove the following lemma.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 3$ be a bounded and smooth domain. Let (u, v) be a classical solution of system* (1.1) *with $f(\xi) = k_f(1+\xi)^{-\alpha}$ with some $k_f > 0$. If α satisfies* (1.6) *and if for some $\frac{N}{2} < p < N$ there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C, \qquad \textit{for any } t \in (0, T_{max}),$$

then, for some $\hat{C} > 0$,

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \hat{C}, \qquad \textit{for any } t \in (0, T_{max}). \tag{3.1}$$

Proof. For any $t \in (0, T_{max})$, we set $t_0 := \max\{0, t-1\}$ and we consider the representation formula for u:

$$\begin{aligned} u(\cdot,t) &= e^{(t-t_0)\Delta}u(\cdot,t_0) - k_f\int_{t_0}^{t} e^{(t-s)\Delta}\nabla\cdot\left(u(\cdot,s)\frac{\nabla v(\cdot,s)}{(1+|\nabla v(\cdot,t)|^2)^{\alpha}}\right)ds \\ &=: u_1(\cdot,t) + u_2(\cdot,t) \end{aligned}$$

and

$$\|u(\cdot,t)\|_{L^\infty} \le \|u_1(\cdot,t)\|_{L^\infty(\Omega)} + \|u_2(\cdot,t)\|_{L^\infty(\Omega)}. \tag{3.2}$$

We have

$$\|u_1(\cdot,t)\|_{L^\infty(\Omega)} \le \max\{\|u_0\|_{L^\infty(\Omega)}, \mu|\Omega|k_1\} =: C_1, \tag{3.3}$$

with $k_1 > 0$ and μ defined in (1.7). In fact, if $t \leq 1$, then $t_0 = 0$ and hence the maximum principle yields $u_1(\cdot, t) \leq \|u_0\|_{L^\infty(\Omega)}$. If $t > 1$, then $t - t_0 = 1$ and from (2.1) and (2.2) with p $= \infty$ and $q = 1$, we deduce that $\|u_1(\cdot, t)\|_{L^\infty(\Omega)} \leq k_1(t - t_0)^{-\frac{N}{2}}\|u(\cdot, t_0)\|_{L^1(\Omega)} \leq \mu|\Omega|k_1$.

We next use (2.3) with p $= \infty$, which leads to

$$\begin{aligned}(3.4)\quad &\|u_2(\cdot, t)\|_{L^\infty(\Omega)} \\ &\leq k_2 k_f \int_{t_0}^{t} (1 + (t - s)^{-\frac{1}{2} - \frac{N}{2q}})e^{-\mu_1(t-s)} \|u(\cdot, s)\frac{\nabla v(\cdot, s)}{(1 + |\nabla v|^2)^\alpha}\|_{L^q(\Omega)}\, ds \\ &\leq k \int_{t_0}^{t} (1 + (t - s)^{-\frac{1}{2} - \frac{N}{2q}})e^{-\mu_1(t-s)} \|u(\cdot, s)|\nabla v|^{1-2\alpha}\|_{L^q(\Omega)}\, ds,\end{aligned}$$

with $k := k_2 k_f$ and $\frac{|\nabla v|}{(1+|\nabla v|^2)^\alpha} \leq |\nabla v|^{1-2\alpha}$.

Here, we may assume that $\frac{N}{2} < p < N$, and then we can fix $N < q < \frac{Np}{N-p} = p^*$. Since $2\alpha < 1$, by Hölder's inequality, we can estimate the last term in (3.4) as

$$\begin{aligned}&\|u(\cdot, s)|\nabla v(\cdot, s)|^{1-2\alpha}\|_{L^q(\Omega)} \\ &\leq \|u(\cdot, s)\|_{L^{\frac{q}{2\alpha}}(\Omega)} \|\nabla v(\cdot, s)\|_{L^q(\Omega)}^{1-2\alpha} \\ &\leq C_2 \|u(\cdot, s)\|_{L^{\frac{q}{2\alpha}}(\Omega)} \|\nabla v(\cdot, s)\|_{L^{p^*}(\Omega)}^{1-2\alpha} \quad \text{for all } s \in (0, T_{\max}),\end{aligned}$$

for some $C_2 > 0$. The Sobolev embedding theorem and elliptic regularity theory applied to the second equation in (1.1) tell us that $\|v(\cdot, s)\|_{W^{1,p^*}(\Omega)} \leq C_3\|v(\cdot, s)\|_{W^{2,p}(\Omega)} \leq C_4$ with some $C_3, C_4 > 0$. Thus again by Hölder's inequality, the definition of $\mu = \frac{1}{|\Omega|}\int_\Omega u_0 dx$ and interpolation's inequality, we obtain

$$\begin{aligned}&\|u(\cdot, s)|\nabla v(\cdot, s)|^{1-2\alpha}\|_{L^q(\Omega)} \\ &\leq C_5 \|u(\cdot, s)\|_{L^{\frac{q}{2\alpha}}(\Omega)} \\ &\leq C_5 \|u(\cdot, s)\|_{L^\infty(\Omega)}^{\theta} \|u(\cdot, s)\|_{L^1(\Omega)}^{1-\theta} \\ &\leq C_6 \|u(\cdot, s)\|_{L^\infty(\Omega)}^{\theta} \quad \text{for all } s \in (0, T_{\max}),\end{aligned}$$

with $\theta := 1 - \frac{2\alpha}{q} \in (0, 1)$, $C_5 := C_2 C_4$ and $C_6 := C_5(\mu|\Omega|)^{1-\theta}$. Hence, combining this estimate and (3.4), we infer

$$\|u_2(\cdot,t)\|_{L^\infty(\Omega)} \le C_6 k_2 \int_{t_0}^{t} (1+(t-s)^{-\frac{1}{2}-\frac{N}{2q}}) e^{-\mu_1(t-s)} \|u(\cdot,s)\|^{\theta}_{L^\infty(\Omega)}\, ds.$$

Now fix any $T \in (0, T_{max})$. Then, since $t - t_0 \le 1$, we have

$$\begin{aligned} \|u_2(\cdot,t)\|_{L^\infty(\Omega)} &\le C_6 k_2 \int_{t_0}^{t} (1+(t-s)^{-\frac{1}{2}-\frac{N}{2q}} e^{-\mu_1(t-s)})\, ds \cdot \sup_{t\in[0,T]} \|u(\cdot,t)\|^{\theta}_{L^\infty(\Omega)} \\ &\le C_7 \sup_{t\in[0,T]} \|u(\cdot,t)\|^{\theta}_{L^\infty(\Omega)}, \end{aligned} \tag{3.5}$$

where $C_7 := C_6 k_2 (1 + \mu_1^{\frac{N}{2q}-\frac{1}{2}} \int_0^\infty r^{-\frac{1}{2}-\frac{N}{2q}} e^{-r}\, dr) > 0$ is finite, because $\frac{1}{2}+\frac{N}{2q} < 1$ (i.e., $q > N$).

Plugging (3.3) and (3.5), into (3.2), we see that

$$\|u(\cdot,t)\|_{L^\infty} \le C_1 + C_7 \sup_{t\in[0,T]} \|u(\cdot,t)\|^{\theta}_{L^\infty(\Omega)},$$

which implies

$$\sup_{t\in[0,T]} \|u(\cdot,t)\|_{L^\infty(\Omega)} \le C_1 + C_7 \Big(\sup_{t\in[0,T]} \|u(\cdot,t)\|_{L^\infty(\Omega)} \Big)^{\theta} \quad \text{for all } T \in (0, T_{max}).$$

From this inequality with $\theta \in (0,1)$, we arrive at (3.1). □

Proof of Theorem 1.2. Since Theorem 1.1 holds, the unique local classical solution of (1.1) blows up at $t = T_{max}$ in the sense of (1.10), that is, $\limsup_{t \nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$. We prove that it blows up also in L^p-norm by contradiction.

In fact, if one supposes that there exist $p > \frac{N}{2}$ and $C > 0$ such that

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C, \quad \text{for all } t \in (0, T_{max}),$$

then, from Lemma 3.1, it would exist $\hat{C} > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \hat{C}, \quad \text{for all } t \in (0, T_{max}),$$

which contradics (1.10). Thus, if u blows up in L^∞-norm, then u blows up also in L^p-norm for all $p > \frac{N}{2}$. □

4 Lower bound of the blow-up time $T_{\max}$

Throughout this section we assume that Theorem 1.2 holds.

We want to derive a lower bound T of the blow up time T_{max}: in this way we obtain that the solution exists in the safe interval $[0, T]$. To this end, first we construct a first order differential inequality for Ψ defined in (1.11) and by integration we get the lower bound.

Proof of Theorem 1.3. By differentiating (1.11) we have

$$\Psi'(t) = \int_\Omega u^{p-1}\Delta u\,dx - \int_\Omega u^{p-1}\nabla\cdot(u\nabla v f(|\nabla v|^2)\,dx =: \mathcal{I}_1 + \mathcal{I}_2, \tag{4.1}$$

with

$$\mathcal{I}_1 = \int_\Omega u^{p-1}\Delta u\,dx = \int_\Omega \nabla\cdot\big(u^{p-1}\nabla u\big)dx - (p-1)\int_\Omega u^{p-2}|\nabla u|^2 dx = -\frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx. \tag{4.2}$$

In the second term of (4.1), integrating by parts and using the boundary conditions in (1.1), $\forall\, t\in[0,T_{max})$ we obtain

$$\begin{aligned}\mathcal{I}_2 &= -\int_\Omega u^{p-1}\nabla\cdot(u\nabla v f(|\nabla v|^2)\,dx\\ &= (p-1)\int_\Omega f(|\nabla v|^2)u^{p-1}\nabla u\nabla v dx\\ &= \frac{p-1}{p}\int_\Omega \nabla u^p\cdot\nabla v f(|\nabla v|^2)dx\\ &= -\frac{p-1}{p}\int_\Omega u^p\nabla\cdot[\nabla v f(|\nabla v|^2)]dx\\ &= -\frac{p-1}{p}\int_\Omega u^p[\Delta v f(|\nabla v|^2)]dx\\ &\quad -\frac{p-1}{p}\int_\Omega u^p f'(|\nabla v|^2)\nabla v\cdot\nabla(|\nabla v|^2)dx.\end{aligned} \tag{4.3}$$

Using the second equation of (1.1) and taking into account that $f(\xi) = k_f(1+\xi)^{-\alpha}$, $f'(\xi) = -\alpha k_f(1+\xi)^{-\alpha-1}$ in (4.3), we have

$$
\begin{aligned}
\mathcal{I}_2 &= -k_f\frac{p-1}{p}\int_\Omega u^p\frac{\mu-u}{(1+|\nabla v|^2)^\alpha}dx \\
&+ \alpha k_f\frac{p-1}{p}\int_\Omega u^p\frac{\nabla v\cdot\nabla(|\nabla v|^2)}{(1+|\nabla v|^2)^{\alpha+1}}dx \\
&\le k_f\frac{p-1}{p}\int_\Omega u^{p+1}dx + \alpha k_f\frac{p-1}{p}\int_\Omega u^p\frac{\nabla v\cdot\nabla(|\nabla v|^2)}{(1+|\nabla v|^2)^{\alpha+1}}dx,
\end{aligned}
\tag{4.4}
$$

where we dropped the negative term $-k_f\frac{p-1}{p}\int_\Omega u^p\frac{\mu}{(1+|\nabla v|^2)^\alpha}dx$ and used the inequality $\frac{1}{(1+|\nabla v|^2)^\alpha} \le 1$ as $\alpha > 0$.
In order to estimate the second term of (4.4) we recall the radially symmetric setting to obtain (with ω_N the surface area of the unit sphere in N dimension)

$$
\begin{aligned}
&\int_\Omega u^p\frac{\nabla v\cdot\nabla(|\nabla v|^2)}{(1+|\nabla v|^2)^{\alpha+1}}dx = \omega_N\int_0^R u^p\frac{Nv_r(v_r^2)_r}{(1+v_r^2)^{\alpha+1}}r^{N-1}dr \\
&= 2N\omega_N\int_0^R u^p\frac{v_r^2v_{rr}}{(1+v_r^2)^{\alpha+1}}r^{N-1}dr,
\end{aligned}
$$

which together with $v_{rr} = \frac{\mu}{N} - u + \frac{N-1}{r^N}\int_0^r \rho^{N-1}u\ d\rho$ implies

$$
\begin{aligned}
&\int_\Omega u^p\frac{\nabla v\cdot\nabla(|\nabla v|^2)}{(1+|\nabla v|^2)^{\alpha+1}}dx \\
&= 2\mu\omega_N\int_0^R u^p\frac{v_r^2}{(1+v_r^2)^{\alpha+1}}r^{N-1}dr \\
&- 2N\omega_N\int_0^R u^{p+1}\frac{v_r^2}{(1+v_r^2)^{\alpha+1}}r^{N-1}dr \\
&+ 2N(N-1)\omega_N\int_0^R u^p\frac{v_r^2}{(1+v_r^2)^{\alpha+1}}\frac{1}{r}\Big(\int_0^r\rho^{N-1}ud\rho\Big)dr \\
&\le 2\mu\omega_N\int_0^R u^pr^{N-1}dr + 2N(N-1)\omega_N\int_0^R u^p\frac{1}{r}\Big(\int_0^r\rho^{N-1}ud\rho\Big)dr,
\end{aligned}
\tag{4.5}
$$

where we dropped the negative term $-2N\omega_N\int_0^R u^{p+1}\frac{v_r^2}{(1+v_r^2)^{\alpha+1}}r^{N-1}dr$ and used the inequality $\frac{v_r^2}{(1+v_r^2)^{\alpha+1}} \le 1$.

In the second term of (4.5), Hölder's inequality yelds that for all $\epsilon > 0$ there exists $c = c(\epsilon, N, p)$ such that

(4.6)

$$
\begin{aligned}
&\omega_N \int_0^R u^p \frac{1}{r}\Big(\int_0^r \rho^{N-1} u d\rho\Big) dr \\
&\leq \omega_N \int_0^R u^p \frac{1}{r}\Big(\int_0^r \rho^{N-1} d\rho\Big)^{\frac{p}{p+1}}\Big(\int_0^r u^{p+1}\rho^{N-1} d\rho\Big)^{\frac{1}{p+1}} dr \\
&\leq \Big(\frac{1}{N}\Big)^{\frac{p}{p+1}}\Big(\int_\Omega u^{p+1} dx\Big)^{\frac{1}{p+1}} \omega_N^{\frac{p}{p+1}} \int_0^R u^p r^{\frac{Np}{p+1}-1} dr \\
&\leq \Big(\frac{1}{N}\Big)^{\frac{p}{p+1}}\Big(\int_\Omega u^{p+1} dx\Big)^{\frac{1}{p+1}} \omega_N^{\frac{p}{p+1}}\Big(\int_0^R u^{p+1+\epsilon} r^{N-1} dr\Big)^{\frac{p}{p+1+\epsilon}}\Big(\int_0^R r^{\frac{\epsilon N p}{p+1}-1} dr\Big)^{\frac{1+\epsilon}{p+1+\epsilon}} \\
&= c\Big(\int_\Omega u^{p+1} dx\Big)^{\frac{1}{p+1}}\Big(\int_\Omega u^{p+1+\epsilon} dx\Big)^{\frac{p}{p+1+\epsilon}}.
\end{aligned}
$$

Combining (4.6) and (4.5) with (4.4) we obtain

$$
\begin{aligned}
\mathcal{I}_2 &\leq 2\alpha\mu k_f \frac{p-1}{p}\int_\Omega u^p dx + k_f\frac{p-1}{p}\int_\Omega u^{p+1} dx \\
&\quad + 2\alpha N(N-1)ck_f\ \frac{p-1}{p}\Big(\int_\Omega u^{p+1} dx\Big)^{\frac{1}{p+1}}\Big(\int_\Omega u^{p+1+\epsilon} dx\Big)^{\frac{p}{p+1+\epsilon}} \\
&\leq c_1\int_\Omega u^p dx + c_2\int_\Omega u^{p+1} dx + c_3\Big(\int_\Omega u^{p+1+\epsilon} dx\Big)^{\frac{p+1}{p+1+\epsilon}}
\end{aligned} \tag{4.7}
$$

where, in the last term, we used Young's inequality with $c_1 = 2\alpha\mu k_f\frac{p-1}{p},\ c_2 = k_f\frac{p-1}{p} + 2\alpha N(N-1)ck_f\ \frac{p-1}{p(p+1)},\ \ c_3 = 2\alpha N(N-1)ck_f\ \frac{p-1}{p+1}$.
Thanks to the Gagliardo–Nirenberg inequality (2.4), with $\mathsf{p} = 2\frac{p+1}{p},\ \mathsf{r} = \mathsf{q} = \mathsf{s} = 2,\ a = \theta_0 := \frac{N}{2(p+1)} \in (0,1)$ for all $p > \frac{N}{2}$, we see that

$$
\begin{aligned}
\int_\Omega u^{p+1} dx &= \|u^{\frac{p}{2}}\|^{2\frac{p+1}{p}}_{L^{2\frac{p+1}{p}}(\Omega)} \\
&\leq C_{GN}\|\nabla u^{\frac{p}{2}}\|^{2\frac{p+1}{p}\theta_0}_{L^2(\Omega)}\|u^{\frac{p}{2}}\|^{2\frac{p+1}{p}(1-\theta_0)}_{L^2(\Omega)} + C_{GN}\|u^{\frac{p}{2}}\|^{2\frac{p+1}{p}}_{L^2(\Omega)} \\
&= C_{GN}\Big(\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx\Big)^{\frac{N}{2p}}\Big(\int_\Omega u^p dx\Big)^{\frac{2(p+1)-N}{2p}} + C_{GN}\Big(\int_\Omega u^p dx\Big)^{\frac{p+1}{p}}.
\end{aligned} \tag{4.8}
$$

Applying Young's inequality at the first term of (4.8) we have

$$\int_\Omega u^{p+1}dx \le \frac{N}{2p}\epsilon_1 C_{GN}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx + C_{GN}\frac{2p-N}{2p\epsilon_1^{\frac{N}{2p-N}}}\Big(\int_\Omega u^p dx\Big)^{\frac{2(p+1)-N}{2p-N}} + C_{GN}\Big(\int_\Omega u^p dx\Big)^{\frac{p+1}{p}} \tag{4.9}$$

with $\epsilon_1 > 0$ to be choose later on, and also

$$\begin{aligned}\Big(\int_\Omega u^{p+1+\epsilon}dx\Big)^{\frac{p+1}{p+1+\epsilon}} &= \|u^{\frac{p}{2}}\|^{2\frac{p+1}{p}}_{L^{2\frac{p+1+\epsilon}{p}}(\Omega)}\\ &\le C_{GN}\|\nabla u^{\frac{p}{2}}\|^{2\frac{p+1}{p}\theta_\epsilon}_{L^2(\Omega)}\|u^{\frac{p}{2}}\|^{2\frac{p+1}{p}(1-\theta_\epsilon)}_{L^2(\Omega)} + C_{GN}\|u^{\frac{p}{2}}\|^{2\frac{p+1}{p}}_{L^2(\Omega)}\\ &= C_{GN}\Big(\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx\Big)^{\frac{p+1}{p}\theta_\epsilon}\Big(\int_\Omega u^p dx\Big)^{\frac{p+1}{p}(1-\theta_\epsilon)}\\ &\quad + C_{GN}\Big(\int_\Omega u^p dx\Big)^{\frac{p+1}{p}},\end{aligned} \tag{4.10}$$

with $\mathsf{p} = 2\frac{p+1}{p}$, $\mathsf{r} = \mathsf{q} = \mathsf{s} = 2$, $a = \theta_\epsilon := \frac{N(1+\epsilon)}{2(p+1+\epsilon)} \in (0,1)$ for all $p > \frac{N}{2}$ and sufficiently small $\epsilon > 0$.
Now, in the first term of (4.10), we apply the Young's inequality to obtain

$$\begin{aligned}&\Big(\int_\Omega u^{p+1+\epsilon}dx\Big)^{\frac{p+1}{p+1+\epsilon}}\\ &\le c_4\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx + c_5\Big(\int_\Omega u^p dx\Big)^{\sigma} + C_{GN}\Big(\int_\Omega u^p dx\Big)^{\frac{p+1}{p}},\end{aligned} \tag{4.11}$$

with

$$c_4 := \frac{N(1+\epsilon)(p+1)}{2p(p+1+\epsilon)}C_{GN}, \quad c_5 := C_{GN}\Big(\frac{2p(p+1+\epsilon) - N(p+1)(1+\epsilon)}{2p(p+1+\epsilon)}\Big),$$
$$\sigma := \frac{2(p+1) - \frac{N(p+1)(1+\epsilon)}{p+1+\epsilon}}{2p - \frac{N(1+\epsilon)(p+1)}{p+1+\epsilon}}.$$

Note that we can fix $\epsilon > 0$ such that $2p - N(1+\epsilon) > 0$.

Plugging (4.9) and (4.11) into (4.7) leads to

$$
\begin{aligned}
(4.12) \qquad \mathcal{I}_2 \le C\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx + c_1 \int_\Omega u^p dx + C_{GN}\Big(\int_\Omega u^p dx\Big)^{\frac{p+1}{p}} \\
+ \tilde{c}_1\Big(\int_\Omega u^p dx\Big)^{\frac{2(p+1)-N}{2p-N}} + c_5\Big(\int_\Omega u^p dx\Big)^{\sigma}
\end{aligned}
$$

with $C := \frac{N}{2p}\epsilon_1 C_{GN} + c_1,\ \epsilon_1 > 0, \quad \tilde{c}_1 := C_{GN}\frac{2p-N}{2p\epsilon_1^{\frac{N}{2p-N}}}c_2.$

Finally, combining (4.12) with (4.1) and (4.2) and choosing ϵ_1 such that the term containing $\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx$ vanishes, we arrive at

$$
(4.13) \qquad \Psi' \le c_1\Psi + C_{GN}\Psi^{\frac{p+1}{p}} + \tilde{c}_1\Psi^{\frac{2(p+1)-N}{2p-N}} + c_5\Psi^{\sigma}.
$$

Integrating (4.13) from 0 to T_{max}, we arrive at the desired lower bound (1.12) with

$$
\begin{aligned}
(4.14) \qquad & B_1 := c_1, \quad B_2 := C_{GN}, \\
& B_3 := \tilde{c}_1, \quad B_4 := c_5
\end{aligned}
$$

and $\gamma_1 := \frac{p+1}{p},\ \gamma_2 := \frac{2(p+1)-N}{2p-N},\ \gamma_3 := \sigma.$ □

Remark 4.1. We note that it is possible to reduce (4.13) so as to have an explicit expression of the lower bound T of T_{max}. In fact, since $\Psi(t)$ blows up at time T_{max}, there exists a time $t_1 \in (0, T_{max})$ such that $\Psi(t) \ge \Psi_0$ for all $t \in (t_1, T_{max})$. Thus, taking into account that

$$
1 < \gamma_1 < \gamma_2
$$

and putting $\gamma := \max\{\gamma_2, \gamma_3\}$ we have

$$
\begin{aligned}
(4.15) \qquad & \Psi \le \Psi^{\gamma}\Psi_0^{1-\gamma}, \\
& \Psi^{\gamma_i} \le \Psi^{\gamma}\Psi_0^{\gamma_i-\gamma}, \quad i = 1, 2, 3.
\end{aligned}
$$

From (4.13) and (4.15) we arrive at

$$
(4.16) \qquad \Psi' \le A\Psi^{\gamma}, \quad \forall t \in (t_1, T_{max}),
$$

with $A := B_1\Psi_0^{1-\gamma} + B_2\Psi_0^{\gamma_1-\gamma} + B_3\Psi_0^{\gamma_2-\gamma} + B_4\Psi_0^{\gamma_3-\gamma}$, and Ψ_0 in (1.11).

Integrating (4.16) from $t = 0$ to $t = T_{max}$, we obtain

$$\frac{1}{(\gamma-1)\Psi_0^{\gamma-1}} = \int_{\Psi_0}^{\infty} \frac{d\eta}{\eta^{\gamma}} \leq A \int_{t_1}^{T_{max}} d\tau \leq A \int_0^{T_{max}} d\tau = AT_{max}. \tag{4.17}$$

We conclude, by (4.17), that the solution of (1.1) is bounded in $[0, T]$ with $T := \frac{1}{A(\gamma-1)\Psi_0^{\gamma-1}}$.

Acknowledgments

M. Marras and S. Vernier-Piro are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Financial disclosure

M. Marras is partially supported by the research project: *Evolutive and stationary Partial Differential Equations with a focus on biomathematics* (Fondazione di Sardegna 2019), and by the grant PRIN n. PRIN-2017AYM8XW: *Non-linear Differential Problems via Variational, Topological and Set-valued Methods*.
T. Yokota is partially supported by Grant-in-Aid for Scientific Research (C), No. 21K03278, and by Tokyo University of Science Grant for International Joint Research.

References

[1] N. Bellomo, A. Belloquid, Y. Tao and M. Winkler, Toward a mathematical theory of Keller–Segel model of pattern formation in biological tissues. *Math. Mod. Meth. Appl. Sci.* 25 (9) (2015), 1663–1763.

[2] N. Bellomo and M. Winkler, A degenerate chemotaxis system with flux limitation: maximally extended solutions and absence of gradient blow-up, *Comm. Partial Differential Equations* 42 (2017), no. 3, 436–473.

[3] N. Bellomo and M. Winkler, Finite-time blow-up in a degenerate chemotaxis system with flux limitation, *Trans. Amer. Math. Soc. Ser. B* 4 (2017) 31–67.

[4] X. Cao, Global bounded solutions of the higher-dimensional Keller–Segel system under smallness conditions in optimal spaces, *Discrete Contin. Dyn. Syst. Series A* 35 (5) (2015), 1891–1904.

[5] Y. Chiyo and T. Yokota, Remarks on finite-time blow-up in a fully parabolic attraction-repulsion chemotaxis system via reduction to the Keller–Segel system, *arXiv:2103.02241v1*

[6] Y. Chiyoda, M. Mizukami and T. Yokota, Finite-time blow-up in a quasilinear degenerate chemotaxis system with flux limitation, *Acta Appl. Math.* 167 (2020), 231–259.

[7] Y. Chiyo, Y. Tanaka, M. Marras and T. Yokota, Blow-up phenomena in a parabolic–elliptic–elliptic attraction-repulsion chemotaxis system with superlinear logistic degradation, *Nonlinear Anlysis* 212 (2021), 112550.

[8] S. Ishida and T. Yokota, Boundedness in a quasilinear fully parabolic Keller–Segel system via maximal Sobolev regularity, *Discrete Contin. Dyn. Syst. Series S*, 13 (2) (2020), 211–232.

[9] E.F. Keller and L.A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theoret. Biol.* 26 (1970), 399–415.

[10] J. Liu, Influence of Flux Limitation on Large Time Behavior in a Three-Dimensional Chemotaxis-Stokes System Modeling Coral Fertilization, *Acta Applicandae Mathematicae* 174:9 (2021).

[11] M. Marras, T. Nishino and G. Viglialoro, A refined criterion and lower bounds for the blow-up time in a parabolic–elliptic chemotaxis system with nonlinear diffusion, *Nonlinear Analysis* (2019) 111725, DOI 10.1016/j.na.2019.111725.

[12] M. Marras and G. Porru, Estimates and uniqueness for boundary blow-up solutions of p-Laplace equations, *Electronic Journal of Differential Equations* **119** (2011), 1–10.

[13] M. Marras and S. Vernier-Piro, Finite time collapse in chemotaxis systems with logistic-type superlinear source, *Math. Meth. Appl. Sci.* 43 (17) (2020), 10027–10040.

[14] M. Marras and S. Vernier-Piro, Blow-up time estimates in nonlocal reaction-diffusion systems under various boundary conditions, *Boundary value problems* 2017 (2) (2017), DOI 10.1186/s13661-016-0732-2.

[15] M. Marras, S. Vernier-Piro and G. Viglialoro, Decay in chemotaxis systems with a logistic term, *Discrete Contin. Dyn. Syst. Series S* 13 (2) (2020), 257–268.

[16] N. Mizoguchi and P. Souplet, Nondegeneracy of blow-up points for the parabolic Keller–Segel system, *Ann. I. H. Poincaré - AN* 31 (4) (2014), 851–875.

[17] M. Mizukami, T. Ono and T, Yokota, Extensibility criterion ruling out gradient blow-up in a quasilinear degenerate chemotaxis system with flux limitation, *J. Differential Equations* 267 (9) (2019), 5115–5164.

[18] T. Nagai, Blowup of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains, *J. Inequal. Appl.* 6, (2001), 37–55

[19] M. Negreanu and J.I. Tello, On a parabolic-elliptic system with gradient dependent chemotactic coefficient, *J. Differential Equations* 265, (2018), 733–751

[20] L. Nirenberg, On elliptic partial differential equations, *Ann. Scuola Norm. Sup.Pisa* 3 (1959), 13:115–162.

[21] H. Wang, P. Zheng and J. Xing, Boundedness in a chemotaxis-haptotaxis model with gradient-dependent flux limitation, *Applied Mathematics Letters*, 122 (2021), 107505.

[22] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller–Segel model, *J. Differential Equations*, 248 (12) (2010), 2889–2905.

[23] M. Winkler, A critical blow-up exponent for flux limitation in a Keller–Segel system, *Indiana Univ. Math. J., https://arxiv.org/abs/2010.01553*

[24] M. Winkler, Global weak solutions in a three-dimensional Keller-Segel-Navier-Stokes system with gradient-dependent flux limitation, *Nonlinear Analysis: Real World Applications*, 59 (2021), 103257.

[25] M. Winkler, Conditional estimates in three-dimensional chemotaxis-Stokes systems and application to a Keller-Segel-fluid model accounting for gradient-dependent flux limitation, *Journal of Differential Equations* 281 (2021), 33–57.

[26] M. Winkler, Suppressing blow-up by gradient-dependent flux limitation in a planar Keller-Segel-Navier-Stokes system, *Zeitschrift fur Angewandte Mathematik und Physik*, 72 (2021)