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Loss-ALAE modeling through a copula dependence structure

Abstract

After having described the mathematical background of copula functions we propose a scheme useful to apply a particular family of copulas – the Archimedean copulas – to indemnity payments and loss expenses of an insurance company with the aim of obtaining their joint probability distribution. The joint distribution is used to calculate – via Monte Carlo simulation – the premia of a reinsurance strategy in presence of policy limits and insurer's retentions. Results coming from this strategy are compared with those obtained from the independence hypothesis. We also describe the procedures needed to estimate the parameters of our model. Calculations and estimates are based on a large dataset of an anonymous Italian non-life insurance company. Empirical results show that the correct way to model dependence through copula functions permits to avoid the undervaluation of reinsurance premia. Finally, we observe that the relative simplicity in estimating the right copula from empirical data and the use of algorithms able to be programmed also on a common PC makes this probabilistic instrument easy to be used by insurers and reinsurers to improve their valuation "ability" and to realize more efficient and precise estimation of their assets and liabilities.

Keywords: copula functions, indemnity claims, reinsurance, stochastic simulation.
JEL Classification: G15, G22.

Introduction

Copula functions were introduced in 1959 by Abe Sklar in the framework of "Probabilistic metric Spaces". From 1986 on copula functions are intensively investigated from a statistical point of view due to the impulse of Genest and MacKay's work "The joy of copulas" (1986).

Nevertheless, applications in financial and (in particular) actuarial fields are revealed only in the end of the 90s. We can cite, for example, the papers of Frees and Valdez (1998) in actuarial direction and Embrechts concerning financial applications (Embrechts et al., 2001, 2002).

Copula functions allow to model efficiently the dependence structure between variates, that's why they are assumed in these last years to be an increasingly important tool for investigating problems such as risk measurement in financial and actuarial applications.

In particular, in non-life insurance many processes involve dependent variables. One important example is the relationship between Loss and ALAE whose dynamic has been investigated also by Klugman & Parsa (1999).

In our paper we propose a procedure useful to apply copula functions to indemnity claims with the aim of building a reinsurance strategy in presence of policy limits and insurer's retentions.

The article has the following structure: in section 1 we describe the mathematical background of copula functions and introduce the measure of dependence; we present also a list of the most used copulas and their main characteristics (exhaustive features are revealed in the appendix); in section 2 we propose a scheme useful for fitting copulas to insurance company indemnity claims. Results are used to calculate the premia of the cited reinsurance strategy and these premia are compared with those coming from the independence hypothesis. The last section highlights the main results of the application and concludes.

1. Copula functions: main definitions and properties

Definition 1.1. A bi-dimensional copula ("2-copula") is a function C that satisfies the following properties:

domain $[0,1]\times[0,1]$, (1)

$C(0,u)=C(u,0)=0,$ (2)

$C(u,1)=C(1,u)=u$ for every $u\in[0,1]$

C is a function *2-increasing* that is to say (3)

$C(v_1,v_2)+C(u_1,u_2)\geq C(v_1,u_2)+(u_1,v_2)$

for every $(u_1,u_2)\in[0,1]\times[0,1]$; $(v_1,v_2)\in[0,1]\times[0,1]$ such that $0\leq u_1\leq v_1\leq 1$ and $0\leq u_2\leq v_2\leq 1$.

Consequences:

- C is a distribution function with uniform marginals. Indeed, let's take two uniform variates U_1 and U_2, and construct the vector $U=(U_1,U_2)$.
  We then have:
  $C(u_1,u_2)=Pr\{U_1\leq u_1,U_2\leq u_2\}.$

  From property (2) we get:

  $Pr\{U_1\leq 0,U_2\leq u\}=Pr\{U_1\leq u,U_2\leq 0\}=0.$

Moreover:

$Pr\{U_1 \le 1, U_2 \le u\} = Pr\{U_1 \le u, U_2 \le 1\} = u.$

i.e. the marginals of the joint distribution are uniform.

From property (3) we get finally:

$Pr\{u_1 \le U_1 \le v_1, u_2 \le U_2 \le v_2\} = C(v_1, v_2) - C(v_1, u_2) - C(u_1, v_2) + C(u_1, u_2) \ge 0$

that means C is indeed a probability distribution.

- Consider now two uni-dimensional probability distributions, F_1 and F_2, and a bi-dimensional copula C. It is clear that

  $F(x_1, x_2) := C\left(F_1(x_1), F_2(x_2)\right)$

  represents a bi-dimensional distribution with marginals F_1 and F_2.

  Indeed, $U_i = F_i(X_i)$ defines a uniform distribution:

  $Pr\{U_i \le u\} = Pr\{F_i(X_i) \le u\} = Pr\{X_i \le F_i^{-1}(u)\} = F_i(F_i^{-1}(u)) = u.$

  Besides marginals are:

  $C(F_1(x_1), F_2(\infty)) = C(F_1(x_1), 1) = F_1(x_1)$

  $C(F_1(\infty), F_2(x_2)) = C(1, F_2(x_2)) = F_2(x_2)$.

Fortunately, the last result can be inverted; this conduces to the following fundamental theorem demonstrated by Sklar:

Theorem 1.1. Let F be a bi-dimensional distribution, with marginals F_1 and F_2. Then there exists a 2-copula C such that

$F(x_1, x_2) = C(F_1(x_1), F_2(x_2)).$

If the marginals F_1 and F_2 are continuous, then the copula C is unique.

The previous representation is called *canonical representation* of the distribution. Sklar's theorem is then a powerful tool to construct bi-dimensional distributions by using uni-dimensional ones which represent the marginals of the given distribution. Dependence between marginals is then characterized by the copula C. Note moreover, that the construction of multidimensional non-Gaussian models is particularly hard. An approach using copulas permits to simplify this problem; moreover, one can construct multidimensional distributions with different marginals.

1.1. Copulas examples. We present here the copulas involved in the paper.

- **Frank copula.** Frank copula is given by:

  $$C(u_1, u_2; \vartheta) = -\frac{1}{\vartheta} \cdot log\left[1 + \frac{(e^{-\vartheta \cdot u_1} - 1) \cdot (e^{-\vartheta \cdot u_2} - 1)}{e^{-\vartheta} - 1}\right].$$

- **Archimedean copulas.** Let ϕ be a continuous, decreasing and convex function $\phi : [0,1] \to [0, +\infty]$ with $\phi(1) = 0$ and $\phi(u) + \phi(v) \le \phi(0)$. We define an Archimedean copula with generator ϕ in the following way:

  $C(u, v) = \phi^{-1}\left(\phi(u) + \phi(v)\right)$ with $u, v \in [0,1]$.

If we take $\phi(t) = (-\log t)^{\vartheta}$ with $\vartheta \in [1, +\infty)$ we get the Gumble copula. Otherwise, in the case $\phi(t) = \frac{t^{-\vartheta} - 1}{\vartheta}$ with $\vartheta \in [-1, +\infty) \setminus \{0\}$ we get the Clayton copula:

$$C_{\vartheta}(u, v) = \max\left(\left(u^{-\vartheta} + v^{-\vartheta} - 1\right)^{-1/\vartheta}, 0\right).$$

If we take $\phi(t) = -log\, t$ we get the product copula $C^{\perp}$.

If we take $\phi(t) = -log \frac{e^{-\vartheta t} - 1}{e^{-\vartheta} - 1}$ we get Frank copula.

If we take: $\phi(t) = -log(1 - (1 - u)^{\vartheta})$ we get Joe copula $C(u, v) = 1 - \left(u^{-\vartheta} + v^{-\vartheta} - u^{-\vartheta} \cdot v^{-\vartheta}\right)^{1/\vartheta}$.

Finally, Genest and MacKay show that the copula C is Archimedean if it admits partial derivatives and if there exists an integrable function $\xi : (0,1) \to (0, +\infty)$ such that:

$$\xi(v) \cdot \frac{\partial C(u, v)}{\partial u} = \xi(u) \cdot \frac{\partial C(u, v)}{\partial v}$$

for every $u, v \in [0,1] \times [0,1]$.

In such a case the generator of the copula is:

$$\phi(t) = \int_t^1 \xi(u) du \text{ with } 0 \le t \le 1.$$

The density of the Archimedean copula is:

$$c(u, v) = -\frac{\phi''(C(u, v)) \cdot \phi'(u) \cdot \phi'(v)}{\left[\phi'(C(u, v))\right]^3}.$$

Besides, we can define multidimensional Archimedean copulas setting

$$C(u_1, \ldots, u_n) = \phi^{-1}\left(\phi(u_1) + \cdots + \phi(u_n)\right)$$

with the additional condition for the generator ϕ:

$$(-1)^k \frac{d^k}{du^k} \phi^{-1}(u) \ge 0 ,$$

for $k \geq 1$. We obtain, for example, the multidimensional Gumble copula

$$C(u_1,...,u_n) = Exp\left[-\left((-\log u_1)^{\vartheta} + \cdots + (-\log u_n)^{\vartheta}\right)^{1/\vartheta}\right].$$

1.2. Concordance order. We shall now study some aspects linked to the dependence between variates.

Definition 1.2.1. The distribution F belongs to the Fréchet class $\Im(F_1, F_2)$ if and only if the marginals of F are F_1 and F_2.

The extremal distributions F^- and F^+ in $\Im(F_1, F_2)$ are defined as:

$$F^-(x_1, x_2) = \max\{F_1(x_1) + F_2(x_2) - 1, 0\}$$

$$F^+(x_1, x_2) = \min\{F_1(x_1), F_2(x_2)\}.$$

F^- and F^+ are also called Fréchet lower bound and Fréchet upper bound. We can associate to them the copulas

$$C^-(u_1, u_2) = \max\{u_1 + u_2 - 1, 0\}$$

$$C^+(u_1, u_2) = \min\{u_1, u_2\}.$$

The following relations hold

$$F^-(x_1, x_2) \leq F(x_1, x_2) \leq F^+(x_1, x_2)$$

for every $(x_1, x_2) \in R^2$ and for every $F \in \Im(F_1, F_2)$ or in terms of copulas:

$$C^-(u_1, u_2) \leq C(u_1, u_2) \leq C^+(u_1, u_2).$$

We define now a partial order relation for the set of copulas.

Definition 1.2.2. We say that the copula C_1 is less than the copula C_2 ($C_1 \prec C_2$) if and only if

$$C_1(u_1, u_2) \leq C_2(u_1, u_2)$$

for every $(u_1, u_2) \in [0,1] \times [0,1]$.

The order "$\prec$" is called concordance order and corresponds to the first order stochastic domination for distribution functions. It turns out to be a partial order, indeed not every copula can be confronted. The following still hold: $C^- \prec C \prec C^+$ and $C^- \prec C^{\perp} \prec C^+$. So that we can give the following:

Definition 1.2.3. The copula C represents a positive (negative) dependence structure if $C^{\perp} \prec C \prec C^+$ (if $C^- \prec C \prec C^{\perp}$, respectively).

Remark. A parametric copula $C(u_1, u_2, \vartheta) = C_{\vartheta}(u_1, u_2)$ is said to be totally ordered if we have $C_{\vartheta_2} \succ C_{\vartheta_1}$ for every $\vartheta_2 \geq \vartheta_1$ (positively ordered family) or $C_{\vartheta_2} \prec C_{\vartheta_1}$ (negatively ordered family).

We define besides the positive quadrant dependence ("PQD") in the following way:

Definition 1.2.4. Two variates X_1, X_2 are called PQD if they satisfy:

$$Pr\{X_1 \leq x_1, X_2 \leq x_2\} \geq Pr\{X_1 \leq x_1\} \cdot Pr\{X_2 \leq x_2\}$$

for every $(x_1, x_2) \in R^2$. In terms of copulas: $C(u_1, u_2) \succ C^{\perp}$.

We define analogously the negative quadrant dependence ("NQD") by assuming that $C(u_1, u_2) \prec C^{\perp}$.

1.3. Measure of dependence. We introduce now another dependence concept. Recall that:

$$C^-(u_1, u_2) = max\{u_1 + u_2 - 1, 0\}$$

$$C^+(u_1, u_2) = min\{u_1, u_2\}$$

with the relation

$$C^-(u_1, u_2) \leq C(u_1, u_2) \leq C^+(u_1, u_2).$$

If we denote $U \approx U(0,1)$ the following also hold:

$$C^-(u_1, u_2) = Pr\{U \leq u_1, 1 - U \leq u_2\}$$

$$C^+(u_1, u_2) = Pr\{U \leq u_1, U \leq u_2\}$$

One can prove the following:

Theorem 1.3.1. Suppose that the bivariate (X, Y) has a copula C^- or C^+. So there exist two monotonous functions $u, v : R \to R$ and a variate Z such that $(X, Y) = (u(Z), v(Z))$ with u increasing and v decreasing in the case of the copula C^-; u and v decreasing in the case of the copula C^+ (the converse is true).

Using this result we can introduce the following:

Definition 1.3.1. If the couple (X, Y) admits copula C^+, the variates X and Y are called comonotonous; in the case of a copula C^- they are called countermonotonous.

When the distributions F_1 and F_2 are continuous, the last theorem can be strengthened in the following manner:

$C = C^- \Leftrightarrow Y = T(X)$ with $T = F_2^{-1} \cdot (1 - F_1)$ decreasing;

$C = C^+ \Leftrightarrow Y = T(X)$ with $T = F_2^{-1} \cdot F_1$ increasing.

We conclude with a list of suitable properties which should satisfy a good dependence measure between variates:

Definition 1.3.2. A dependence measure δ is an application which associates to a couple of variates (X,Y) a real number $\delta(X,Y)$ such that:

$$\delta(X,Y)=\delta(Y,X) \text{ (symmetry)}, \quad (4)$$

$$-1 \le \delta(X,Y) \le 1 \text{ (normalization)}, \quad (5)$$

$$\delta(X,Y)=1, \quad (6)$$

if and only if X, Y are comonotonous;

$$\delta(X,Y)=-1, \quad (7)$$

if and only if X, Y are countermonotonous;
for every monotonous application $T:R\to R$ we have:
$\delta(T(X),Y)=\delta(X,Y)$ for T increasing,
$\delta(T(X),Y)=-\delta(X,Y)$ for T decreasing. (8)

Linear correlation satisfies properties (4) and (5); we shall see later on (see the appendix) that rank correlation satisfies also properties (6) and (8).

Remark. We may want to introduce a property of the form $\delta(X,Y)=0$ if and only if X and Y are independent, unfortunately it can be proved that such a property is incompatible with (8). These results can be found in Roncalli (2000) and in Embrechts et al. (2001, 2002).

2. An application to indemnity claims

In this section we apply Archimedean copulas to the estimation of the joint probability distribution of losses and expenses of an Insurance Company. The section is divided into four parts.

In the first part we briefly present the input data of the application; in the second we describe the methods used to fit the marginal distribution functions; in the third part we compare some copulas to individuate the most appropriate to represent the dependence of the empirical data; in the fourth part, after having identified the joint distribution of the two variables, we examine the expected value of the payment of a reinsurance strategy. The valuation of the reinsurer's expected payment is made in different hypotheses:

- using the joint distribution coming from the estimated copulas;
- in the usual condition of independence between the two variables.

The results coming from the two hypotheses are compared.

2.1. The input data. The data of the application comprise about 5.880 liability claims provided by an anonymous Insurance Company; these claims are a significative sample of the whole set of claims of the Company that are about 16.000.

Each claim consists of:

- an indemnity payment (Loss), X;
- an allocated loss adjustment expense (ALAE[1]), Y.

The main features of the sample data are summarized in Table 1.

Table 1. Descriptive statistics of losses and expenses

	Losses	Expenses
Mean	10,496	3,289
Standard deviation	40,132	11,899
Min	75	120
Max	2,075,300	380,430
Median	4,100	1,340
Mode	1,000	670
Kurtosis	1,301	314
Skewness	29	15
25th percentile	1,489	850
50th percentile	4,100	1,340
75th percentile	10,588	2,470
95th percentile	29,505	7,000

Figure 1 and Figure 2 show the frequency distribution of the empirical data for X and Y.

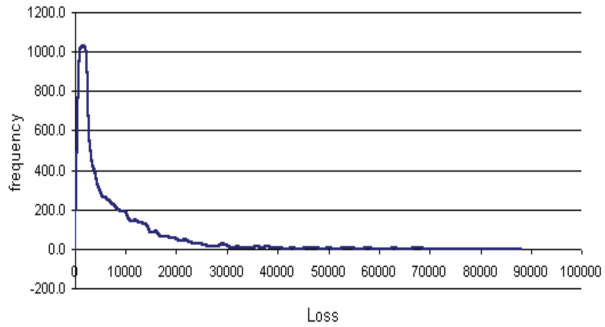

Fig. 1. Frequency distribution of Loss

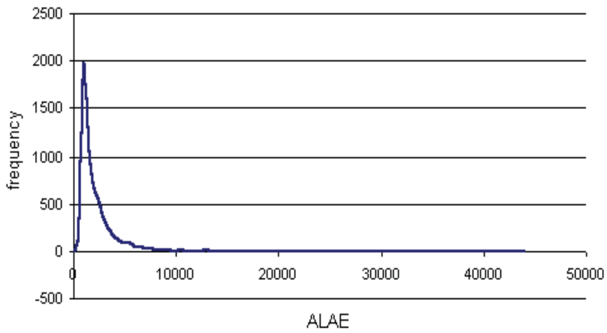

Fig. 2. Frequency distribution of ALAE

The next figure is a scatter plot of Loss versus ALAE on a logarithmic scale; the correlation coefficient ρ between the two variables is 0,3471 while the Kendall τ is 0,3750.

[1] ALAE are company expenses specifically attributable to the settlement of an individual claim (lawyers fees, claim investigation expenses, etc.).

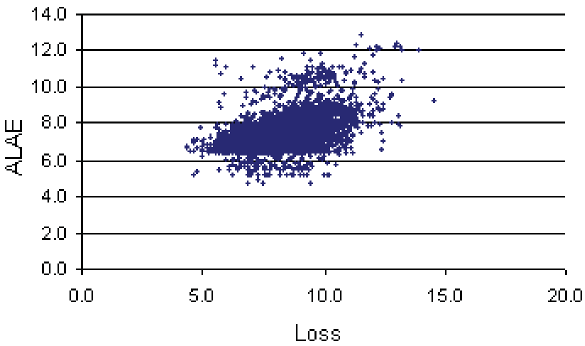

Fig. 3. Plot of ALAE vs Loss on a logarithmic scale

The figure and the correlation coefficients suggest a strong relationship between X and Y.

2.2. The fitting of the marginal distributions. To apply a copula function we need to identify the appropriate marginals for X and Y.

We can fit different distributions to our empirical data[1] and choose the best fit by the use of classical statistics.

In this step maximum likelihood estimation (MLE[2]) was used with 21 different theoretical distributions checked; only a few of them have satisfied both the Kolmogorov-Smirnov (KS) test and the Chi Square test.

For loss empirical data the best estimate has been given from the Weibull distribution with a Chi square test value of 0,00227 (and a confidence level greater than 99%) and a KS test value of 0,055430; for ALAE empirical data the best estimate has been given from the Lognormal distribution with a Chi square test value of 0,00834 (corresponding to a confidence level greater than 99%) and a KS test value of 0,077154.

The following figures 4, 5 show the empirical and the fitted cumulative distribution function for Loss and ALAE.

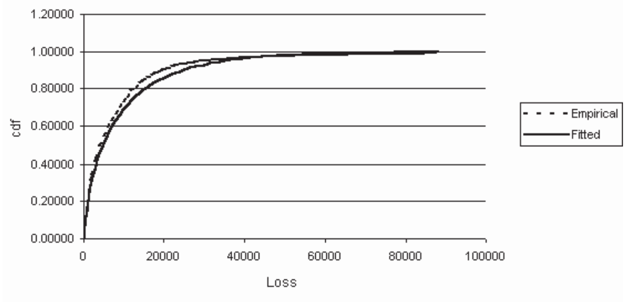

Fig. 4. Empirical distribution and fitted cdf of Loss

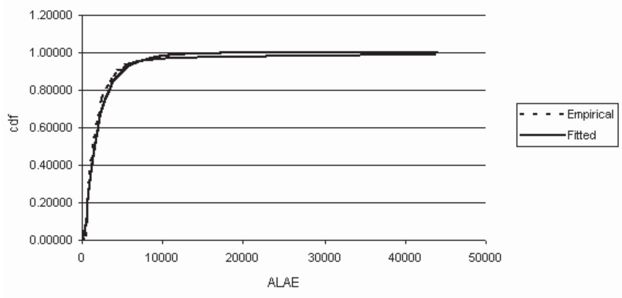

Fig. 5. Empirical distribution and fitted cdf of ALAE

Estimated parameters of the two distributions are 0,0013 and 0,7387 for the Weibull; 7,3753 and 0,8918 for the Lognormal.

2.3. Estimating Archimedean copulas. Schweizer & Wolff (1981) established that the value of the parameter α characterizing each family of Archimedean copulas can be related to the Kendall's measure of concordance τ. The relationships are shown in the table below.

Table 2. Relationship between α and τ

Family	τ
Gumbel (1990)	$1-\frac{1}{\alpha}$
Clayton (1978)	$\frac{\alpha}{\alpha+2}$
Frank (1979)	$1-\frac{4}{\alpha}\cdot\left(D_1\left(-\alpha\right)-1\right)$

From the calculation of the Kendall's measure of concordance of our bivariate data, we obtain τ equal to 0,375.

[1] Many statistical packages are able to realize the necessary calculations.

[2] MLE method is one of the fundamental general methods for constructing estimators of unknown parameters in statistical estimation theory. Suppose one has, for an observation X with distribution P_ϑ depending on an unknown parameter $\vartheta \in \Theta \subseteq R^k$, the task to estimate ϑ. Assuming that all measures P_ϑ are absolutely continuous relative to a common measure *v*, the likelihood function is defined by

$$L(\vartheta) = \frac{dP_\vartheta}{d\nu}(X).$$

The maximum likelihood method recommends taking as an estimator for the statistic defined by $L(\hat{\vartheta}) = \max_{\vartheta \in \Theta} L(\vartheta)$.

$\hat{\vartheta}$ is called the maximum likelihood estimator. In a broad class of cases the maximum-likelihood estimator is the solution of a likelihood equation

$$\frac{\partial}{\partial \vartheta_i} \log \ L(\vartheta) = 0 \qquad (*)$$

$i = 1,2,\dots, k,$

$\vartheta = (\vartheta_1, \vartheta_2, \dots, \vartheta_k).$

Example. Let $X = (X_1, X_2, \dots, X_n)$ be a sequence of independent random variables (observations) with common distribution P_ϑ, $\vartheta \in \Theta$. If there is a density $f(x,\vartheta) = \frac{dP_\vartheta}{dm}(x)$ relative to some measure m, then $L(\vartheta) = \prod_{j=1}^{n} f(X_j, \vartheta)$ and the equations (*) take the form $\sum_{j=1}^{n} \frac{\partial}{\partial \vartheta_i} \log f(X_j, \vartheta) = 0$

$i = 1, 2, \dots, k.$

This value gives $\alpha = 1{,}6$ for the Gumbel, $\alpha = 1{,}2$ for the Clayton and $\alpha = 3{,}826$ for the Frank.

Now for each of these different copulas we must verify how close it fits the data by comparison with the empirical sample.

This fit test can be made using a procedure developed by Genest & Rivest (1993) whose algorithm is well described by Frees & Valdez (1998).

The procedure has the following steps:

- identify an intermediate variable $Z_i = F(X_i, Y_i)$ that has distribution function $K(z)$;
- for Archimedean copulas this function is $K(z) = z - \left(\frac{d \ln \phi_\alpha(z)}{dz} \right)^{-1}$;
- define $Z_i = \frac{card\left\{ (X_j, Y_j) : X_j < X_i, Y_j < Y_i \right\}}{N-1}$ and calculate the empirical version of $K(z)$, $K_N(z)$;
- reply the procedure for each copula under examination and compare the parametric estimate with the non parametric one;
- choose the "best" copula by using an adequate criterion (like a graphical test and/or a minimum square error analysis).

From our data we obtain the following forms of $K(z)$ for the copulas under examination:

Table 3. The function *K(z)*

Family	τ
Gumbel (1990)	$\frac{z \cdot (\alpha - \ln z)}{\alpha}$
Clayton (1978)	$\frac{z \cdot (1 + \alpha - z^{\alpha})}{\alpha}$
Frank (1979)	$\frac{\alpha \cdot z - (exp(\alpha \cdot z) - 1) \cdot \ln\left(\frac{exp(-\alpha \cdot z) - 1}{exp(-\alpha) - 1} \right)}{\alpha}$

The empirical version of $K_N(z)$ and the three $K(z)$ coming from the fitted Archimedean copulas are presented in Figure 6 below.

The corresponding mean square errors for the three copulas are 0,03473 for the Frank, 0,22405 for the Clayton and 0,02235 for the Gumbel.

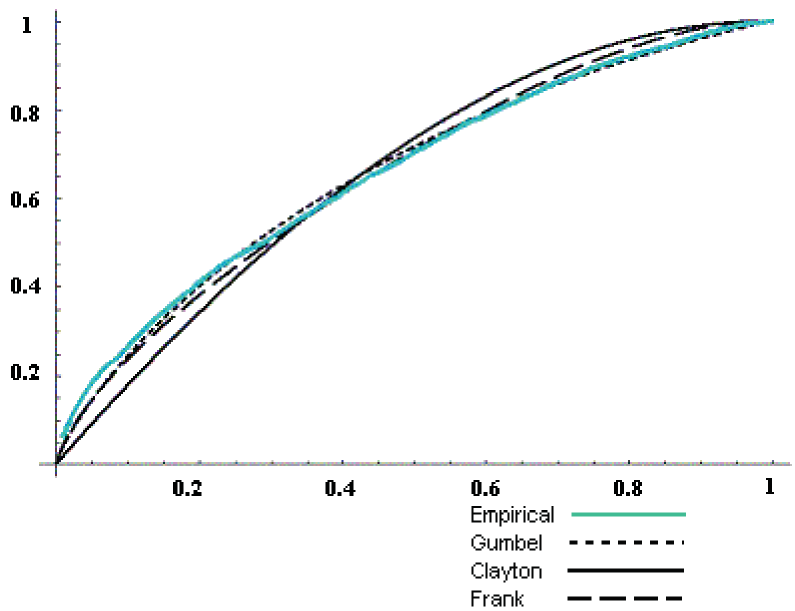

Fig. 6. Empirical and theoretical values of $K(z)$

It is evident both from the figure and from the errors that only the Clayton copula must be rejected while both the Frank and the Gumbel provide a good fit.

2.4. A reinsurance strategy. In this section after having identified the best copula that expresses the joint distribution of (X, Y), we calculate the expected payment on a policy with limit L and insurer's retention R (see Frees & Valdez, 1998).

If we assume a pro-rata sharing of expenses, we have the following expression of the stochastic payment:

$$P(X,Y)=\begin{cases}0 \ \ if \ \ X<R,\\ X-R+\dfrac{X-R}{X}\cdot Y \ \ if \ \ R\le X<L,\\ L-R+\dfrac{L-R}{L}\cdot Y \ \ if \ \ X\ge L.\end{cases}$$

The calculation of the expected payment can be made via Monte Carlo simulation. In this way we must simulate a large number M of bivariate data (x_k, y_k) from the bivariate distribution model.

The reinsurer's estimated payment is given by

$$P(X,Y)=\frac{1}{M}\cdot\sum_{k=1}^{M}P(x_k,y_k)$$

and its standard error is:

$$S[P(X,Y)]=\left[\frac{1}{M}\cdot\left(\frac{\sum_{k=1}^{M}P(x_k,y_k)^2}{M}-P(X,Y)^2\right)\right]^{1/2}$$

The necessary algorithms for these calculations are described in Frees & Valdez (1998); we present here a short version of the algorithm restricted to the bivariate case; the steps are the following:

- generate U_1 and U_2 independent uniform (0,1) random numbers;
- set $X_1 = F_1^{-1}(U_1)$;
- calculate X_2 as the solution of

$$U_2=\frac{\varphi^{-1(1)}(\varphi(F_1(x_1))+\varphi(F_2(x_2)))}{\varphi^{-1(1)}(\varphi(F_1(x_1)))}.$$

Results coming from the application can be compared with those obtained in independence condition that is the most used way of thinking in actuarial field. Tables 4, 5 show the values of the premia the reinsurer would have assessed to cover costs of losses and expenses according to various policy limits L and retentions R (standard errors in parentheses).

Table 4. Reinsurance premia provided by Monte Carlo simulation and their standard errors; Frank case

	0	0.25	0.5	0.75	0.95
5,000	5,804.1 (25.9)	3,781.8 (22.4)	2,320.2 (15.7)	1,083.0 (8.1)	206.6 (1.6)
10,000	7,785.7 (38.1)	4,653.9 (33.4)	2,685.8 (23.1)	1,192.8 (11.7)	219.2 (2.3)
15,000	9,071.8 (49.3)	5,001.3 (42.4)	2,723.7 (28.5)	1,143.0 (14.1)	201.5 (2.8)
20,000	9,937.7 (58.8)	5,058.8 (49.3)	2,594.0 (32.3)	1,039.3 (15.7)	178.3 (3.0)
25,000	10,545.8 (66.6)	4,961.7 (54.4)	2,403.6 (34.9)	926.8 (16.6)	156.2 (3.2)

Table 5. Reinsurance premia provided by Monte Carlo simulation and their standard errors; Gumble case

	0	0.25	0.5	0.75	0.95
5000	5,784.6 (36.7)	3,835.5 (31.2)	2,345.5 (21.9)	1,089.7 (11.3)	207.1 (2.3)
10000	7,752.7 (54.0)	4,666.4 (46.7)	2,673.7 (32.3)	1,184.6 (16.3)	216.7 (3.3)
15000	9,025.7 (69.6)	4,978.6 (59.3)	2,694.8 (39.8)	1,124.9 (19.6)	197.8 (3.9)
20000	9,883.4 (82.7)	5,017.1 (68.9)	2,556.3 (45.1)	1,021.6 (21.8)	175.2 (4.2)
25000	10,490.5 (93.8)	4,916.8 (76.2)	2,371.4 (48.9)	916.5 (23.3)	155.9 (4.5)

Table 6. Reinsurance premia provided by Monte Carlo simulation and their standard errors; independence case

	0	0.25	0.5	0.75	0.95
5.000	5,773.8 (32.5)	3,515.0 (28.2)	2,115.1 (19.7)	977.2 (10.1)	184.9 (2.0)
10.000	7,768.7 (47.1)	4,357.5 (42.8)	2,489.8 (29.5)	1,098.3 (14.9)	201.5 (3.0)
15.000	9,063.8 (62.0)	4,717.3 (55.2)	2,551.4 (37.1)	1,065.4 (18.3)	186.5 (3.6)
20.000	9,934.3 (75.0)	4,795.8 (64.9)	2,446.4 (42.6)	975.9 (20.6)	168.0 (4.0)
25.000	10,544.0 (86.0)	4,718.9 (72.3)	2,275.5 (46.4)	876.9 (22.0)	146.5 (4.2)

Table 6 shows premia obtained in the common independence hypothesis.

The comparison between Tables 4/5 and Table 6 highlights some results:

- from a practical point of view, there is only a slight difference between results coming from the Gumbel and the Frank case;
- premia vary considerably passing from dependence to independence hypothesis and those obtained in independence conditions are generally undervalued;
- the level of undervaluation grows with the retention to policy limit ratio L/R; on the average, it passes from about 0 to 7,75% when P_ϑ passes from 0 to X.

Conclusions

The work is essentially devoted to describe the main mathematical features and properties of copula functions and their possible role in some non-life actuarial valuations.

In particular, following the approach of Frees & Valdez (1998) and Klugman & Parsa (1999) we develop a practical scheme for the application of copula functions to the valuation of the premia of a reinsurance strategy in the presence of a policy limit and insurer's retention.

The valuations of the premia are made via Monte Carlo simulation and the results obtained are compared to those deriving from the traditional independence hypothesis.

Empirical results show that the correct way to model dependence permits to avoid the undervaluation of reinsurance premia.

The relative simplicity in estimating the right copula from empirical data and the use of algorithms able to be programmed also on a common PC makes this probabilistic instrument easy to be used by insurers and reinsurers to improve their valuation "ability" and to realize more efficient and precise estimation of their assets and liabilities.

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Appendix

In this section, we present additional features about copula functions.

We deduce the following remarks from the copula definition.

- The canonical representation can be written equivalently. Consider two continuous distributions, G_1 and G_2, and let $Y_i = G_i^{-1}(U_i)$. The distribution G of $Y = (Y_1, Y_2)$ will be:

  $G(y_1, y_2) = C_{(X_1,X_2)}\left(G_1(y_1), G_2(y_2)\right)$ so that $C(x_1, x_2) = F\left(F_1^{-1}\left(G_1(x_1)\right), F_2^{-1}\left(G_2(x_2)\right)\right)$

with $U_i = F_i(X_i)$, F distribution of (X_1, X_2). This construction is called translation method.

- The definition of 2-*copula* can be generalized analogously to the *n-dimensional* case. The canonical form of the *n-dimensional* distribution takes the following form, according to Sklar's theorem:

  $F(x_1,\ldots,x_n) = C\left(F_1(x_1),\ldots,F_n(x_n)\right)$

  where $F_1(x_1),\ldots,F_n(x_n)$ are the *n* marginal distributions and C represents an *n-copula*.

Probability density. Suppose that the bivariate $X = (X_1, X_2)$ possesses a density function. We can then express it by means of the marginal density functions and the copula in the following manner:

$$f(x_1,x_2) = c\left(F_1(x_1),F_2(x_2)\right)\cdot f_1(x_1)\cdot f_2(x_2) \text{ with } c(u_1,u_2) = \frac{\partial C(u_1,u_2)}{\partial u_1 \partial u_2}.$$

The condition $C(v_1,v_2) - C(v_1,u_2) - C(u_1,v_2) + C(u_1,u_2) \geq 0$ leads to the positivity of the density $c(u_1,u_2) \geq 0$.

In the case of *n-dimensional* distributions, if the density function exists we will get analogously:

$$f(x_1,\ldots,x_n) = c\left(F_1(x_1),\ldots,F_n(x_n)\right)\cdot \prod_{i=1}^{n} f_i(x_i) \text{ with: } c(u_1,\ldots,u_n) = \frac{\partial^n C(u_1,\ldots,u_n)}{\partial u_1 \cdots \partial u_n}.$$

The density of a copula can then be written as $c(u_1,u_2) = \dfrac{f\left(F_1^{-1}(u_1),F_2^{-1}(u_2)\right)}{f_1\left(F_1^{-1}(u_1)\right)\cdot f_2\left(F_2^{-1}(u_2)\right)}$.

Other copulas examples

- **The product copula.** The product copula is $C^{\perp}(u_1,u_2) = u_1 \cdot u_2$ which density is $c^{\perp}(u_1,u_2) = 1$.

  We deduce that a distribution constructed with this copula satisfies: $f(x_1,x_2) = f_1(x_1)\cdot f_2(x_2)$ which characterizes independence between X_1 and X_2.

- **Gumbel logistic copula.** The Gumbel Logistic copula is:

  $$C(u_1,u_2) = \frac{u_1\cdot u_2}{u_1 + u_2 - u_1\cdot u_2} = F(F_1^{-1}(u_1),F_2^{-1}(u_2)),$$

  where $F(x_1,x_2) = (1+e^{-x_1}+e^{-x_2})^{-1}$ is the Gumbel logistic 2-distribution having marginals $F_1(x_1) = (1+e^{-x_1})^{-1}$ and $F_2(x_2) = (1+e^{-x_2})^{-1}$; moreover, quantiles have the expression $F_1^{-1}(u_1) = \log u_1 - \log(1-u_1)$ and $F_2^{-1}(u_2) = \log u_2 - \log(1-u_2)$.

  The density function is:

  $$c(u_1,u_2) = \frac{2u_1\cdot u_2}{(u_1+u_2-u_1\cdot u_2)^3}.$$

- **Gumble-Barnett copula.** Gumble-Barnett copula is:

  $$C(u_1,u_2,\vartheta) = u_1\cdot u_2\cdot e^{-\vartheta\cdot\log u_1\cdot\log u_2}.$$

  One easily verifies that $C(0,u,\vartheta) = C(u,0,\vartheta) = 0$ and $C(1,u,\vartheta) = C(u,1,\vartheta) = u$.

  Density is given by:

  $$c(u_1,u_2,\vartheta) = (1-\vartheta-\vartheta\cdot(\log u_1+\log u_2)+\vartheta^2\cdot\log u_1\cdot\log u_2)\cdot e^{-\vartheta\cdot\log u_1\cdot\log u_2}.$$

- **Normal copula.** The normal copula is given by:

  $$C_\rho^{Ga} = \Phi_\rho^n(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_n)),$$

  where we supposed that $Z = (Z_1,\ldots,Z_n)$ has normal distribution $N_n(\mu,\Sigma)$ with marginals $F(Z_i)$, where $Z_i \sim N(\mu_i,\Sigma_{ii})$ and ρ represents the linear correlation matrix corresponding to the covariance matrix Σ.

We denote Φ_ρ^n the multivariate normal distribution function with correlation matrix ρ and Φ^{-1} is the inverse of the standard univariate normal distribution.

The density of the normal copula is:

$$c(u,\rho) = |\rho|^{-\frac{1}{2}} \cdot exp\left(-\frac{1}{2}\zeta^T \cdot (\rho^{-1} - I) \cdot \zeta \right),$$

where $\zeta = \left(\Phi^{-1}(u_1),...,\Phi^{-1}(u_n)\right)$.

- **FGM copula (Farlie-Gumbel-Morgenstern).** The FGM copula is given by:

  $C_\vartheta(u_1,u_2) = u_1 \cdot u_2 \cdot (1 + \vartheta(1-u_1)\cdot(1-u_2)).$

- **The *t* Student copula.** Let the variate $Z=(Z_1,...,Z_n) \sim N_n(0,\Sigma)$ with non-degenerate marginals and let

$$X = \mu + \frac{\sqrt{\nu}}{\sqrt{S}} \cdot Z \ ,$$

where Z and $S \sim \chi_\nu^2$ are independent. We will say that X has a *t Student* distribution with degrees of freedom ν, mean μ (if $\nu > 1$) and covariance matrix $\frac{\nu}{\nu-2}\cdot\Sigma$ (if $\nu > 2$).

If X_i has distribution G_i, then the distribution function of $G_1(X_1),\ldots,G_n(X_n)$ is the t_ν copula $C_{\nu,\rho}{}^t$ where ρ is the linear correlation matrix associated to Σ. The density of the *t copula* is:

$$c(u_1,\ldots,u_n) = |\rho|^{-1/2} \frac{\Gamma\left(\frac{\nu+n}{2}\right)\left[\Gamma\left(\frac{\nu}{2}\right)\right]^n \left(1+\frac{1}{\nu}\xi^T\rho^{-1}\xi\right)^{-\frac{\nu+n}{2}}}{\left[\Gamma\left(\frac{\nu+1}{2}\right)\right]^n \Gamma\left(\frac{\nu}{2}\right)\prod_{i=1}^n\left(1+\frac{\xi_i^2}{\nu}\right)^{-\frac{\nu+1}{2}}},$$

where Γ is the gamma function and $\zeta_i = t_\nu^{-1}(u_i)$. For $\nu \to \infty$ we obtain the normal copula.

- **The cubic copula.** The cubic copula has the form:

$$C(u_1,u_2) = u_1 \cdot u_2 + \alpha \cdot \left(u_1\cdot(u_1-1)\cdot(2u_1-1)\right)\cdot\left(u_2\cdot(u_2-1)\cdot(2u_2-1)\right)$$

  with $\alpha \in [-1,2]$.

- **Two parameters copulas.** We can define copulas with two parameters, for example, an Archimedean copula whose generator is defined by composing two generators of other Archimedean copulas: $\phi = \phi_1 \cdot \phi_2$.
- **Alternative method.** Another method for constructing copulas is the following:

  Let $f,g : [0,1] \to R$ with $\int_0^1 f(x)dx = \int_0^1 g(y)dy = 0$ and $f(x)\cdot g(y) \geq -1$ for every $x,y \in [0,1]$. Then $h(x,y) = 1 + f(x)\cdot g(y)$ is a bivariate density in $[0,1]\times[0,1]$.

  Consequently, we can define the copula

$$C(x,y) = x\cdot y + \left(\int_0^x f(u)du\right)\cdot\left(\int_0^y g(v)dv\right).$$

Copulas properties. Let's start with some properties, which need conditional probabilities.

- $Pr\{U_1 \leq u_1, U_2 > u_2\} = u_1 - C(u_1,u_2);$
- $Pr\{U_1 \leq u_1 | U_2 \leq u_2\} = \frac{C(u_1,u_2)}{u_2};$
- $Pr\{U_1 \leq u_1 | U_2 > u_2\} = \frac{u_1 - C(u_1,u_2)}{1-u_2};$

- $Pr\{U_1 > u_1, U_2 > u_2\} = 1 - u_1 - u_2 + C(u_1, u_2);$
- $Pr\{U_1 > u_1 | U_2 > u_2\} = \frac{1 - u_1 - u_2 + C(u_1, u_2)}{1 - u_2};$
- $Pr\{U_1 \le u_1 | U_2 = u_2\} = C_{1|2}(u_1, u_2) = \partial_2 C(u_1, u_2).$

Remark. Starting with a copula C, we can construct some other copulas in the following way:

- $\hat{C}(u_1, u_2) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2)$ is the "survival" copula;
- $\tilde{C}(u_1, u_2) = u_1 + u_2 - C(u_1, u_2)$ is the dual copula;
- $C^*(u_1, u_2) = 1 - C(1 - u_1, 1 - u_2)$ is the co-copula,

which satisfy the following properties:

- $Pr\{U_1 \le u_1, U_2 \le u_2\} = C(F_1(u_1), F_2(u_2));$
- $Pr\{U_1 > u_1, U_2 > u_2\} = \hat{C}(S_1(u_1), S_2(u_2));$
- $Pr\{U_1 \le u_1, U_2 \le u_2\} = \tilde{C}(F_1(u_1), F_2(u_2));$
- $Pr\{U_1 > u_1, U_2 > u_2\} = C^*(S_1(u_1), S_2(u_2)),$

where $S(x) = 1 - F(x)$.

We give now some fundamental properties of copulas:

Proposition 1. A copula C is uniformly continuous in its domain. Besides, it can be shown that $|C(v_1, v_2) - C(u_1, u_2)| \le |v_1 - u_1| + |v_2 - u_2|$.

Proposition 2. Partial derivatives $\partial_1 C$ and $\partial_2 C$ exist for every $(u_1, u_2) \in [0,1] \times [0,1]$ and they satisfy the following properties: $0 \le \partial_1 C(u_1, u_2) \le 1$ and $0 \le \partial_2 C(u_1, u_2) \le 1$.

Proposition 3. Let X_1, X_2 be two continuous variates with marginals F_1 and F_2 and copula $C(X_1, X_2)$. If h_1 and h_2 are two strictly increasing functions on $\mathrm{Im}\, X_1$ and $\mathrm{Im}\, X_2$ then $C(h_1(X_1), h_2(X_2)) = C(X_1, X_2)$, in other words the copula function is invariant under strictly increasing transformations of the variates.

Linear correlation. Remember that given two variates X, Y with finite variance we define the linear correlation coefficient between X and Y as

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sigma(X) \cdot \sigma(Y)},$$

where $Cov(X, Y) = E(X \cdot Y) - E(X) \cdot E(Y)$ is the covariance and $\sigma^2(X), \sigma^2(Y)$ are the variances of X and Y, respectively. Moreover, we have that $\rho(X, Y) \in [-1, 1]$.

Correlation is a linear dependence measure. It means that in case of perfect linear correlation $Y = a \cdot X + b$, one gets $\rho(X, Y) = \pm 1$ and in case of independence we obtain $\rho(X, Y) = 0$ (as $Cov(X, Y) = 0$).

In financial theory, correlation plays a central role. For example, in the Capital Asset Pricing Model (CAPM) or in Arbitrage Pricing Theory (APT) correlation is used as a dependence measure between financial instruments. These theories rely on the hypothesis that assets follow a multivariate normal distribution. In actuarial fields, the increasing complexity of problems involved leads to develop great interest in modeling dependent risks. The concept of correlation represents closely the idea of dependence between variates in the framework of normal distributions (or more generally, for spherical or elliptical distributions).

Nevertheless, empirical observations show that, in actuarial and financial fields, normal distributions rarely occur. The notion of dependence between variates requires more suitable tools.

The advantages of linear correlation are mainly due to calculus facilities and to the following property: given two applications $A,B : R^n \to R^m$ with $A(x) = A \cdot x + a$ and $B(x) = B \cdot x + b$ where $A,B \in R^{m\times n}; a,b \in R^m$, we have: $Cov(A\cdot X + a, B\cdot Y + b) = A\cdot Cov(X,Y)\cdot B^T$.

Moreover, correlation turns out to be a natural risk measure for normal distributions.

Let see now some inconvenient:

- variance of X and Y must be finite; this restriction causes problems when we use fat tailed distributions;
- independence between variates implies they are uncorrelated (i.e. $\rho = 0$); the converse is not true (it holds only in the normal case);
- correlation is not invariant under a strictly increasing transformation: let $T : R \to R$, with $T' > 0$, so $\rho(T(X),T(Y)) \neq \rho(X,Y)$;
- in general, the knowledge of marginals and correlation does not allow to determine the joint distribution;
- given marginals of X and Y, not all values between -1 and $+1$ can be reached by a suitable determination of the joint distribution.

Rank correlation

Definition 1. Consider the variates X, Y with marginals F_1 and F_2 and joint distribution F. The Spearman's rank correlation ("Spearman's ρ") is defined as $\rho_S(X,Y) = \rho\left(F_1(X), F_2(Y)\right)$ where ρ is the usual linear correlation. Let $(X_1, X_2), (Y_1, Y_2)$ two independent couples of variates from F, then Kendall's rank correlation ("Kendall's τ") is given by

$$\rho_\tau(X,Y) = \Pr\left\{(X_1 - X_2)\cdot(Y_1 - Y_2) > 0\right\} - \Pr\left\{(X_1 - X_2)\cdot(Y_1 - Y_2) < 0\right\}.$$

We can assume that ρ_S is the correlation of the copula C associated to (X,Y); both ρ_S and ρ_τ measure the monotonic dependence degree between X and Y (whereas linear correlation only measures the linear dependence degree).

We list some fundamental properties of ρ_S and ρ_τ.

Theorem 1. Let X, Y be continuous variates with continuous distributions F_1, F_2; joint distribution F and copula C. We then have:

$$\rho_S(X,Y) = \rho_S(Y,X); \rho_\tau(X,Y) = \rho_\tau(Y,X), \tag{1}$$

if X and Y are independent, then $\rho_S(X,Y) = \rho_\tau(X,Y) = 0$, (2)

$$\rho_S, \rho_\tau \in [-1,1], \tag{3}$$

$$\rho_S(X,Y) = 12\int_0^1\int_0^1 \left(C(u,v) - u\cdot v\right) dudv, \tag{4}$$

$$\rho_\tau(X,Y) = 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1, \tag{5}$$

given the strictly monotonous application $T : R \to R$, ρ_S and ρ_τ satisfy property (8) of the last section, (6)

$\rho_S(X,Y) = \rho_\tau(X,Y) = 1$ if and only if $C = C^+$ if and only if $Y = T(X)$ with T increasing, (7)

$\rho_S(X,Y) = \rho_\tau(X,Y) = -1$ if and only if $C = C^-$ if and only if $Y = T(X)$ with T decreasing, (8)

The rank correlation satisfies properties (4), (5), (6) and (8) given in section 1 (definition 1.3.2).

Remarks:

- for the normal copula we have:

$$\rho_s = 6\pi^{-1}\cdot arcsin\left(\frac{\rho}{2}\right),$$

$\rho_\tau = 2\pi^{-1} \cdot arcsin\,\rho;$

- for Gumbel copula we have:

$$\rho_\tau = \frac{\vartheta - 1}{\vartheta};$$

- for the copula FGM:

$$\rho_s = \vartheta / 3,$$

$$\rho_\tau = 2\vartheta / 9;$$

- for Frank copula:

$$\rho_s = 1 - 12\vartheta^{-1} \cdot (D_1(\vartheta) - D_2(\vartheta)),$$

$$\rho_\tau = 1 - 4\vartheta^{-1} \cdot (1 - D_1(\vartheta)),$$

where $D_k(x)$ is the Debye function defined in the following way:

$D_i(x) = \frac{i}{x^i} \int_0^x t^i (e^t - 1)^{-1} dt$, which satisfies $D_1(-x) = D_1(x) + \frac{x}{2};$

- for Archimedean copula we have:

$$\rho_\tau = 1 + 4\int_0^1 \frac{\phi(u)}{\phi'(u)} du \; ;$$

- finally, for Clayton copula:

$$\rho_\tau = \frac{\vartheta}{\vartheta + 2}.$$

Tail dependence. The notion of tail dependence for a couple of continuous variates (X, Y) is linked to the probability of simultaneous extremal values occurring for X and Y (this is essential for applications). Let F and G be the marginals of X and Y, respectively. We give the following:

Definition 2. The upper tail dependence coefficient for (X, Y) is given by the following limit (if it exists):

$\lambda_u = \lim_{u \to 1^-} \Pr\left\{Y > G^{-1}(u) \mid X > F^{-1}(u)\right\}$ with $\lambda_u \in [0,1]$.

Remarks:

- If $\lambda_u \in (0,1]$ we will say that X and Y are asymptotically dependent in the upper tail.
- If $\lambda_u = 0$ we will say that X and Y are asymptotically independent in the upper tail.

If we denote C the copula of the joint distribution, one can prove that:

$$\lambda_u = \lim_{u \to 1^-} \frac{1 - 2u + C(u,u)}{1 - u}.$$

An analogous definition can be given for the lower tail:

$$\lambda_l = \lim_{u \to 0^*} Pr\left\{Y < G^{-1}(u) \middle| X < F^{-1}(u)\right\},$$

or in terms of copulas:

$$\lambda_l = \lim_{u \to 0^+} \frac{C(u,u)}{u}.$$

Examples.

- Copula $C^\perp : \lambda_u = 0; \lambda_l = 0$;
- Copula $C^+ : \lambda_u = 1; \lambda_l = 1$;
- Copula $C^- : \lambda_u = 0; \lambda_l = 0$;
- Gumbel copula $\lambda_u = 2 - 2^{1/\vartheta}; \lambda_l = 0$;

- Clayton copula $\lambda_u = 0; \lambda_l = 2^{-1/\vartheta}$;
- Normal copula: $\lambda_u = \lambda_l = 0$ if $\rho > 1$ whereas $\lambda_u = \lambda_l = 1$ if $\rho = 1$;
- *t* Student copula:

$$\lambda_u = 2 - 2s_{\nu+1} \cdot \left(\frac{(\nu+1)\cdot(1-\rho)}{1+\rho} \right)^{1/2};$$

so that $\lambda_u > 0$ if $\rho > -1$ and $\lambda_u = 0$ if $\rho = -1$.

The results presented in this appendix can also be found in Roncalli (2000) and in Embrechts et al. (2001, 2002).