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SYMMETRY AND REGULARITY OF AN OPTIMIZATION PROBLEM RELATED TO A NONLINEAR BVP

CLAUDIA ANEDDA, FABRIZIO CUCCU

ABSTRACT. We consider the functional

$$f \mapsto \int_\Omega \big(\frac{q+1}{2}|Du_f|^2 - u_f|u_f|^q f\big)dx,$$

where u_f is the unique nontrivial weak solution of the boundary-value problem

$$-\Delta u = f|u|^q \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain. We prove a result of Steiner symmetry preservation and, if $n = 2$, we show the regularity of the level sets of minimizers.

1. Introduction

Let Ω be a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. We consider the Dirichlet problem

$$\begin{gathered} -\Delta u = f|u|^q \quad \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \end{gathered} \tag{1.1}$$

where $0 \le q < \min\{1, 4/n\}$ and f is a nonnegative bounded function non identically zero. We consider nontrivial solutions of (1.1) in $H_0^1(\Omega)$. The equation (1.1) is the Euler-Lagrange equation of the integral functional

$$v \mapsto \int_\Omega \Big(\frac{q+1}{2}|Dv|^2 - v|v|^q f\Big)dx, \quad v \in H_0^1(\Omega).$$

By using a standard compactness argument, it can be proved that there exists a nontrivial minimizer of the above functional. This minimizer is a nontrivial solution of (1.1).

From the maximum principle, every nontrivial solution of (1.1) is positive. Then, by [5, Theorem 3.2] the uniqueness of problem (1.1) follows. To underscore the dependence on f of the solution of (1.1), we denote it by u_f. Moreover, $u_f \in W^{2,2}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ for all α, $0 < \alpha < 1$ (see [8, 10]).

Let f_0 be a fixed bounded nonnegative function. We study the problem

$$\inf_{v \in H_0^1(\Omega),\, f \in \mathfrak{F}(f_0)} \int_\Omega \Big(\frac{q+1}{2}|Dv|^2 - v|v|^q f\Big)dx, \tag{1.2}$$

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where, denoting by $|A|$ the Lebesgue measure of a set A,

$$\mathfrak{F}(f_0)=\{f\in L^\infty(\Omega):|\{f\geq c\}|=|\{f_0\geq c\}|\ \forall c\in\mathbb{R}\};\tag{1.3}$$

here $\mathfrak{F}(f_0)$ is called class of rearrangements of f_0 (see [9]).

Problems of this kind are not new; see for example [1, 2, 3, 5]. From the results in [5] it follows that (1.2) has a minimum and a representation formula. Let

$$E(f)=\inf_{v\in H_0^1(\Omega)}\int_\Omega\Big(\frac{q+1}{2}|Dv|^2-v|v|^qf\Big)dx.\tag{1.4}$$

Renaming q' the constant q and putting $p=2$ and $q'=q+1$ in [5] we have

$$E(f)=\frac{q'-2}{2}\,I(f),$$

where

$$I(f)=\sup_{H_0^1(\Omega)}\frac{q'}{2-q'}\int_\Omega\Big(\frac{2}{q'}f|v|^{q'}-|Dv|^2\Big)dx$$

is defined in the same paper. By [5, Theorem 2.2] it follows that there exist minimizers of $E(f)$ and that, if $\overline{f}$ is a minimizer, there exists an increasing function ϕ such that

$$\overline{f}=\phi(u_{\overline{f}}).\tag{1.5}$$

We denote by $\operatorname{supp} f$ the support of f, and we call a level set of f the set $\{x\in\Omega: f(x)>c\}$, for some constant c.

In Section 2, we consider a Steiner symmetric domain Ω and f_0 bounded and nonnegative, such that $|\operatorname{supp} f_0|<|\Omega|$. Under these assumptions, we prove that the level sets of the minimizer $\overline{f}$ are Steiner symmetric with respect to the same hyperplane of Ω. As a consequence, we have exactly one optimizer when Ω is a ball.

Chanillo, Kenig and To [4] studied the regularity of the minimizers to the problem

$$\lambda(\alpha,A)=\inf_{u\in H_0^1(\Omega),\,\|u\|_2,|D|=A}\int_\Omega|Du|^2\,dx+\alpha\int_D u^2\,dx,$$

where $\Omega\subset\mathbb{R}^2$ is a bounded domain, $0<A<|\Omega|$ and $\alpha>0$. In particular they prove that, if D is a minimizer, then ∂D is analytic.

In Section 3, following the ideas in [4], we give our main result. We restrict our attention to $\Omega\subset\mathbb{R}^2$. Let $b_1,\dots,b_m>0$ and $0<a_1<\dots<a_m<|\Omega|$, $m\geq 2$, be fixed. Consider $f_0=b_1\chi_{G_1}+\dots+b_m\chi_{G_m}$, where $|G_i|=a_i$ for all i, $G_i\subset G_{i+1}$, $i=1,\dots,m-1$.

We call η the minimum in (1.2); i.e.,

$$\eta=\int_\Omega\Big(\frac{q+1}{2}|Du_{\overline{f}}|^2-u_{\overline{f}}|u_{\overline{f}}|^q\,\overline{f}\Big)dx,$$

where $\overline{f}$ and $u_{\overline{f}}$ are, respectively, the minimizing function and the corresponding solution of (1.1).

In this case (1.5) becomes

$$\overline{f}=\sum_{i=1}^m b_i\chi_{D_i},$$

where

$$D_1=\{u_{\overline{f}}>c_1\},\quad D_2=\{u_{\overline{f}}>c_2\},\quad\dots,\quad D_m=\{u_{\overline{f}}>c_m\},$$

for suitable constants $c_1 > c_2 > \dots > c_m > 0$.

We show regularity of ∂D_i for each i proving that $|Du_{\overline{f}}| > 0$ in ∂D_i. Following the method used in [4], we consider

$$E(s,t) = \int_\Omega \Big(\frac{q+1}{2}|Du_{\overline{f}} + sDv|^2 - (u_{\overline{f}} + sv)|u_{\overline{f}} + sv|^q \overline{f}_t\Big)dx - \eta, \tag{1.6}$$

where $v \in H_0^1(\Omega)$, $\overline{f}_t$ is a family of functions such that $\overline{f}_t \in \mathfrak{F}(f_0)$ with $\overline{f}_0 = \overline{f}$, and $s \in \mathbb{R}$. We have

$$E(s,t) \geq E(0,0) = 0 \quad \forall s,t.$$

Therefore, $(s,t) = (0,0)$ is a minimum point; it follows that

$$\begin{vmatrix} \frac{\partial^2 E}{\partial s^2}(0,0) & \frac{\partial^2 E}{\partial s \partial t}(0,0) \\ \frac{\partial^2 E}{\partial t \partial s}(0,0) & \frac{\partial^2 E}{\partial t^2}(0,0) \end{vmatrix} \geq 0. \tag{1.7}$$

Expanding (1.7) in detail and using some lemmas from [4] we prove that the boundaries of level sets of $\overline{f}$ are regular.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$, $f_0 = \sum_{i=1}^m b_i \chi_{G_i}$ with $m \geq 2$, and $\overline{f} = \sum_{i=1}^m b_i \chi_{D_i}$ a minimizer of (1.2). Then $|Du_{\overline{f}}| > 0$ on ∂D_i, $i = 1, \dots, m$.*

2. Symmetry

In this section we consider Steiner symmetric domains. We prove that, under suitable conditions on f_0 in (1.3), minimizers inherit Steiner symmetry.

Definition 2.1. Let $P \subset \mathbb{R}^n$ be a hyperplane. We say that a set $A \subset \mathbb{R}^n$ is *Steiner symmetric* relative to the hyperplane P if for every straight line L perpendicular to P, the set $A \cap P$ is either empty or a symmetric segment with respect to P.

To prove the symmetry, we need [6, Theorem 3.6 and Corollary 3.9], that, for more convenience for the reader, we state here. These results are related to the classical paper [7].

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^n$ be bounded, connected and Steiner symmetric relative to the hyperplane P. Assume that $u : \overline{\Omega} \to \mathbb{R}$ has the following properties:*

- *$u \in C(\overline{\Omega}) \cap C^1(\Omega)$, $u > 0$ in Ω, $u|_{\partial\Omega} = 0$;*
- *for all $\phi \in C_0^\infty(\Omega)$,*

$$\int_\Omega Du \cdot D\phi \, dx = \int_\Omega \phi F(u) \, dx,$$

where F has a decomposition $F = F_1 + F_2$ such that $F_1 : [0,\infty) \to \mathbb{R}$ is locally Lipschitz continuous, while $F_2 : [0,\infty) \to \mathbb{R}$ is non-decreasing and identically 0 on $[0,\epsilon]$ for some $\epsilon > 0$.

Then u is symmetric with respect to P and $\frac{\partial u}{\partial \mathbf{v}}(x) < 0$, where $\mathbf{v}$ is a unit vector orthogonal to P and x belongs to the part of Ω that lies in the halfspace (with origin in P) in which $\mathbf{v}$ points.

Theorem 2.3. *Let Ω be Steiner symmetric and f_0 a bounded nonnegative function. If $|\operatorname{supp} f_0| < |\Omega|$ and $\overline{f} \in \mathfrak{F}(f_0)$ is a minimizer of (1.2), then the level sets of $\overline{f}$ are Steiner symmetric with respect to the same hyperplane of Ω.*

Proof. Let $u = u_{\overline{f}}$ be the solution of (1.2). Then $u \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ and satisfies

$$\int_\Omega Du \cdot D\psi \, dx = \int_\Omega \psi f u^q \, dx \quad \forall \psi \in C_0^\infty(\Omega).$$

Since $u > 0$ and since (from (1.5)) $\overline{f} = \phi(u)$ with ϕ increasing function, it follows that $\phi(u) \equiv 0$ on $\{x \in \Omega : u(x) < d\}$ for some positive constant d. Then we have $u^q \overline{f} = F_1(u) + F_2(u)$ with $F_1(u) \equiv 0$ and $F_2(u) = \phi(u)u^q$. From Theorem 2.1 and $\overline{f} = \phi(u)$ we have the assertion. □

Remark 2.4. By this theorem, if Ω is an open ball and $|\operatorname{supp} f_0| < |\Omega|$, then $\overline{f}$ is radially symmetric and decreasing.

3. Regularity of the free boundaries

In this section we prove the following result.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$, $f_0 = \sum_{i=1}^m b_i \chi_{G_i}$ with $m \geq 2$, and $\overline{f} = \sum_{i=1}^m b_i \chi_{D_i}$ a minimizer of* (1.2)*. Then $|Du_{\overline{f}}| > 0$ on ∂D_i, $i = 1, \dots, m$.*

We use the notation introduced in Section 1. Without loss of generality we can assume $m = 3$; the general case easily follows. Let $\overline{f} = b_1\chi_{D_1} + b_2\chi_{D_2} + b_3\chi_{D_3}$. We will prove $|Du_{\overline{f}}| > 0$ in ∂D_2; we omit the proof for D_1 and D_3 because it is similar. We define the family $\overline{f}_t$ by replacing only the set D_2 by a family of domains $D_2(t)$.

First of all, we explain how to define the family $D_2(t)$.
In the sequel we use the notation introduced in [4], reorganized according to our needs.

We call a curve $\gamma : [a, b] \to \mathbb{R}$, $-\infty < a < b < \infty$, regular if:

(i) it is simple, that is: if $a \leq x < y \leq b$ and $x \neq a$ or $y \neq b$, then $\gamma(x) \neq \gamma(y)$;
(ii) $\|\gamma\|_{C^2(a,b)}$ is finite;
(iii) $|\gamma'|$ is uniformly bounded away from zero.

If, in addiction, $\gamma(a) = \gamma(b)$, we say that the curve is closed and regular. If the domain of γ is (a, b) we say that γ is regular (respectively, closed and regular) if the continuous extension of γ to $[a, b]$ is regular (respectively, closed and regular).

Now, we introduce the notation

$$\mathcal{F} := \partial D_2; \quad \mathcal{F}^* := \mathcal{F} \cap \{|Du_{\overline{f}}| > 0\}.$$

Let $J = \cup_{k=1}^p J_k$ be a finite union of open bounded intervals $J_k \subset \mathbb{R}$, $\gamma = (\gamma_1, \gamma_2) : J \to \mathcal{F}^*$ a simple curve which is regular on each interval J_k and $\overline{\gamma(J)} \subset \mathcal{F}^*$. We suppose that $\operatorname{dist}\big(\gamma(J_k), \gamma(J_h)\big) > 0$ for $1 \leq h \neq k \leq p$. Assume also that $|\gamma'| \geq \theta$ on J. For each $\xi \in J$, we denote by $\mathbf{N}(\xi) = \big(N_1(\xi), N_2(\xi)\big)$ the outward unit normal with respect to D_2 at $\gamma(\xi)$. We also define the tangent vector to γ $\mathbf{N}^\perp(\xi) = \big(-N_2(\xi), N_1(\xi)\big)$, and $\mathbf{N}'$ the first derivative of $\mathbf{N}$.

Reversing the direction of γ if necessary, we will assume, without loss of generality, that γ' and $\mathbf{N}^\perp$ have the same direction; i.e., $\angle \gamma', \mathbf{N}^\perp\rangle = |\gamma'|$. We observe that, because γ is C^2 and simple on $\overline{J_k}$, for each k there exists $\beta_k > 0$ such that the function

$$\phi_k : J_k \times [-\beta_k, \beta_k] \to \mathbb{R}^2, \ (\xi, \beta) \mapsto (x_1, x_2) = \phi_k(\xi, \beta) = \gamma(\xi) + \beta\mathbf{N}(\xi)$$

is injective.

Because dist $\big(\gamma(J_k), \gamma(J_h)\big) > 0$ for all $h \neq k$, we can find a number $\beta_0 > 0$ and we can paste together the functions ϕ_k to obtain a function ϕ injective on $J \times [-\beta_0, \beta_0]$. Choose β_0 such that dist $\big(\phi(J \times [-\beta_0, \beta_0]), \partial D_1\big) > 0$ and dist $\big(\phi(J \times [-\beta_0, \beta_0]), \partial D_3\big) > 0$.

Now, we define

$$K = D_2 \setminus \phi\big(J \times (-\beta_0, 0]\big);$$

for $t \in (-t_0, t_0)$ we define

$$D_2(t) = K \cup \{\phi(\xi, \beta) : \xi \in J,\, \beta < g(\xi, t)\}, \tag{3.1}$$

where $g : J \times (-t_0, t_0) \to \mathbb{R}$, $t_0 > 0$, is a function such that

$$g(\xi, t),\ g_t(\xi, t),\ g_{tt}(\xi, t) \in C(\overline{J}) \quad \forall t \in (-t_0, t_0) \tag{3.2}$$

and

$$g(\xi, 0) \equiv 0 \quad \forall \xi \in J. \tag{3.3}$$

We observe that $D_2(0) = D_2$. Next we compute the measure of $D_2(t)$. Put $A(t) = |D_2(t)|$ and $A = |D_2(0)| = |D_2|$; we have

$$A(t) = |D_2| + \int_J \int_0^{g(\xi,t)} J(\xi, \beta)\, d\beta d\xi,$$

where

$$\begin{aligned} J(\xi, \beta) = \frac{\partial(x_1, x_2)}{\partial(\xi, \beta)} &= \begin{vmatrix} \gamma_1' + \beta N_1' & N_1 \\ \gamma_2' + \beta N_2' & N_2 \end{vmatrix} \\ &= \big| - \langle \gamma', \mathbf{N}^{\perp} \rangle - \beta \langle \mathbf{N}', \mathbf{N}^{\perp} \rangle \big| = \big| |\gamma'| + \beta \langle \mathbf{N}', \mathbf{N}^{\perp} \rangle \big|. \end{aligned}$$

We show that $|\gamma'| + \beta \langle \mathbf{N}', \mathbf{N}^{\perp} \rangle \geq 0$. Indeed, from the fact that $\|\gamma\|_{C^2(J)} < \infty$, we have $\|\langle \mathbf{N}', \mathbf{N}^{\perp} \rangle\|_{L^\infty(J)} < \infty$. Substituting t_0 by a smaller positive number if necessary, we can assume that

$$\|g\|_{L^\infty(J \times (-t_0, t_0))} < \beta_0$$

and

$$\|\langle \mathbf{N}', \mathbf{N}^{\perp} \rangle\|_{L^\infty(J)}\ \|g\|_{L^\infty(J \times (-t_0, t_0))} < \theta.$$

Note that the first of these assumptions guarantees that $\partial D_2(t)$ has positive distance from ∂D_1 and ∂D_3. We have

$$|\beta|\, \big|\langle \mathbf{N}', \mathbf{N}^{\perp} \rangle\big| \leq \|g\|_{L^\infty(J \times (-t_0, t_0))} \|\langle \mathbf{N}', \mathbf{N}^{\perp} \rangle\|_{L^\infty(J)} \leq \theta \leq |\gamma'|$$

for all $\xi \in J$ and $|\beta| \leq \|g\|_{L^\infty(J \times (-t_0, t_0))}$. Thus, $J(\xi, \beta) = |\gamma'| + \beta \langle \mathbf{N}', \mathbf{N}^{\perp} \rangle$. Substituting into the formula for $A(t)$ we have

$$\begin{aligned} A(t) &= A + \int_J \int_0^{g(\xi,t)} (|\gamma'| + \beta \langle \mathbf{N}', \mathbf{N}^{\perp} \rangle) d\beta d\xi \\ &= A + \int_J \Big(g(\xi, t)|\gamma'| + \frac{1}{2} (g(\xi, t))^2 \langle \mathbf{N}', \mathbf{N}^{\perp} \rangle \Big) d\xi. \end{aligned}$$

To obtain $|D_2(t)| = |D_2|$ for all $t \in (-t_0, t_0)$, we find the further constraint on g:

$$\int_J \Big(g(\xi, t)|\gamma'| + \frac{1}{2}\ \big(g(\xi, t)\big)^2 \langle \mathbf{N}, \mathbf{N}^{\perp} \rangle \Big) d\xi = 0 \quad \forall t \in (-t_0, t_0). \tag{3.4}$$

Moreover, we calculate the derivatives of $A(t)$, that we will use later.

$$\begin{aligned} A'(t) &= \int_J \big(g_t(\xi,t)|\gamma'(\xi)| + g(\xi,t)g_t(\xi,t)\langle \mathbf{N}', \mathbf{N}^\perp\rangle\big)d\xi = 0; \\ A''(t) &= \int_J \big(g_{tt}(\xi,t)|\gamma'(\xi)| + \big(g(\xi,t)g_{tt}(\xi,t) + g_t^2(\xi,t)\big)\langle \mathbf{N}', \mathbf{N}^\perp\rangle\big)d\xi = 0. \end{aligned} \tag{3.5}$$

Once we have defined the family $D_2(t)$, we can go back to the functional (1.6). The following lemma describes (1.7) with $\overline{f}_t = b_1\chi_{D_1} + b_2\chi_{D_2(t)} + b_3\chi_{D_3}$. We find an inequality corresponding to [4, (2.3) of Lemma 2.1].

Lemma 3.2. *Let $\overline{f}_t = b_1\chi_{D_1} + b_2\chi_{D_2(t)} + b_3\chi_{D_3}$, where the variation of domain $D_2(t)$ is described by* (3.1) *and $g : J \times (-t_0, t_0) \to \mathbb{R}$, $t_0 > 0$, satisfies* (3.2), (3.3) *and* (3.4). *Then, for all $v \in H_0^1(\Omega)$, the conditions* (1.7) *becomes*

$$\begin{aligned} &\int_\Omega \big(|Dv|^2 - qu_{\overline{f}}\ v^2|u_{\overline{f}}|^{q-2}\overline{f}\big)dx \cdot \int_\gamma g_t^2(\gamma^{-1},0)|Du_{\overline{f}}|d\sigma \\ &\geq b_2c_2^q\Big(\int_\gamma g_t(\gamma^{-1},0)\ vd\sigma\Big)^2. \end{aligned} \tag{3.6}$$

Proof. We calculate the second derivative of the functional (1.6), with respect to s. We have

$$\frac{\partial E}{\partial s} = (q+1)\int_\Omega \big(\langle Du_{\overline{f}} + sDv, Dv\rangle - v|u_{\overline{f}} + sv|^q\overline{f}_t\big)dx$$

and

$$\frac{\partial^2 E}{\partial s^2}(0,0) = (q+1)\int_\Omega \big(|Dv|^2 - qu_{\overline{f}}\ v^2|u_{\overline{f}}|^{q-2}\overline{f}\big)dx. \tag{3.7}$$

Before calculating the second derivative of E with respect to t, we rewrite (1.6) in the form

$$\begin{aligned} E(s,t) = &\int_\Omega \frac{q+1}{2}\ |Du_{\overline{f}} + sDv|^2dx - b_1\int_{D_1}(u_{\overline{f}} + sv)|u_{\overline{f}} + sv|^qdx \\ &- b_2\int_{D_2(t)}(u_{\overline{f}} + sv)|u_{\overline{f}} + sv|^qdx - b_3\int_{D_3}(u_{\overline{f}} + sv)|u_{\overline{f}} + sv|^qdx - \eta. \end{aligned}$$

We observe that, if $F : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, then

$$\int_{D_2(t)} F - \int_{D_2} F = \int_J\int_0^{g(\xi,t)} F\big(\phi(\xi, g(\xi,\beta))\big)J(\xi,\beta)d\beta d\xi;$$

whence, from the Fundamental Theorem of Calculus,

$$\frac{\partial}{\partial t}\int_{D_2(t)} F = \int_J g_t(\xi,t)F\big(\phi(\xi,g(\xi,t))\big)J(\xi,g(\xi,t))d\xi.$$

Using the above relation with $F = (u+sv)\big|u+sv\big|^q$, we have

$$\frac{\partial E}{\partial t} = -b_2\int_J g_t(\xi,t)\big(u_{\overline{f}} + sv\big)\big|u_{\overline{f}} + sv\big|^q J(\xi,g(\xi,t))\,d\xi,$$

where, for simplicity of notation, we set $u_{\overline{f}}\big(\phi(\xi,g(\xi,t))\big) = u_{\overline{f}}$ and $v\big(\phi(\xi,g(\xi,t))\big) = v$. Moreover

$$\begin{aligned} \frac{\partial^2 E}{\partial t^2} = -b_2\int_J |u_{\overline{f}} + sv|^q\Big\{&\big[g_{tt}(\xi,t)(u_{\overline{f}} + sv) + (q+1)g_t^2(\xi,t)\langle Du_{\overline{f}} + sDv, \mathbf{N}\rangle\big] \\ &\times J(\xi,g(\xi,t)) + g_t^2(\xi,t)(u_{\overline{f}} + sv)\langle \mathbf{N}', \mathbf{N}^\perp\rangle\Big\}d\xi, \end{aligned}$$

where we have used that

$$\frac{\partial}{\partial t}\ u_{\overline{f}}\big(\phi(\xi,g(\xi,t))\big)=\langle Du_{\overline{f}}\big(\phi(\xi,g(\xi,t))\big),\mathbf{N}\rangle g_t(\xi,t),$$

$$\frac{\partial}{\partial t}J(\xi,g(\xi,t))=g_t\langle\mathbf{N}',\mathbf{N}^{\perp}\rangle.$$

We note that, when $t=0$,

$$u_{\overline{f}}\big(\phi(\xi,g(\xi,t))\big)=u_{\overline{f}}(\gamma(\xi))=c_2,$$

$Du_{\overline{f}}\big(\phi(\xi,g(\xi,0))\big)=-|Du_{\overline{f}}\big(\gamma(\xi)\big)|\mathbf{N}(\xi)$ and $J(\xi,g(\xi,0))=J(\xi,0)=|\gamma'(\xi)|$. Evaluating the above expression in $(0,0)$, we find

$$\begin{aligned}\frac{\partial^2E}{\partial t^2}(0,0)=&-b_2c_2^{q+1}\int_J\Big[g_{tt}(\xi,0)|\gamma'(\xi)|+g_t^2(\xi,0)\langle\mathbf{N}',\mathbf{N}^{\perp}\rangle\Big]d\xi\\&+b_2c_2^q(q+1)\int_J g_t^2(\xi,0)|Du_{\overline{f}}(\gamma(\xi))||\gamma'(\xi)|d\xi.\end{aligned}$$

By using (3.5) with $t=0$ we find

$$\begin{aligned}\frac{\partial^2E}{\partial t^2}(0,0)&=b_2c_2^q(q+1)\int_J g_t^2(\xi,0)|Du_{\overline{f}}(\gamma(\xi))|\,|\gamma'(\xi)|d\xi\\&=b_2c_2^q(q+1)\int_\gamma g_t^2(\gamma^{-1},0)|Du_{\overline{f}}|\,d\sigma.\end{aligned}\tag{3.8}$$

We also have

$$\frac{\partial^2E}{\partial s\partial t}=-b_2(q+1)\int_J g_t(\xi,t)v|u_{\overline{f}}+sv|^qJ(\xi,g(\xi,t))\,d\xi;$$

that is,

$$\begin{aligned}\frac{\partial^2E}{\partial s\partial t}(0,0)&=-b_2c_2^q(q+1)\int_J g_t(\xi,0)\ v\big(\gamma(\xi)\big)|\gamma'(\xi)|\,d\xi\\&=-b_2c_2^q(q+1)\int_\gamma g_t(\gamma^{-1},0)\ v\,d\sigma.\end{aligned}\tag{3.9}$$

Using (1.7) in the form

$$\frac{\partial^2E}{\partial s^2}(0,0)\frac{\partial^2E}{\partial t^2}(0,0)\geq\Big(\frac{\partial^2E}{\partial s\partial t}(0,0)\Big)^2,$$

and using (3.7), (3.8) and (3.9) in this inequality, we obtain (3.6). □

Note that in inequality (3.6) only $g(\gamma^{-1},0)$ appears. Moreover, $g(\gamma^{-1},0)$ has null integral on γ. Indeed, differentiating (3.4) with respect to t and putting $t=0$, we obtain

$$\int_J g(\xi,0)|\gamma'|d\xi=0.$$

Now a natural question arises: does inequality (3.6) hold for any function h with null integral on γ? The answer is contained in the following result.

Lemma 3.3. *Let J and γ be the same as described. Let $h:\gamma\to\mathbb{R}$ bounded, continuous and such that $\int_\gamma h\,d\sigma=0$. Then, for all $v\in H_0^1(\Omega)$ and for all $a\in\mathbb{R}$ we have*

$$\int_\Omega\Big(|Dv|^2-qu_{\overline{f}}\ v^2|u_{\overline{f}}|^{q-2}\overline{f}\Big)dx\cdot\int_\gamma h^2|Du_{\overline{f}}|\,d\sigma\geq b_2c_2^q\Big(\int_\gamma h(v-a)\,d\sigma\Big)^2.\tag{3.10}$$

The proof of the above lemma is similar to that of [4, Lemma 2.2]; we omit it. The following lemma is an analogue to [4, Lemma 3.1].

Lemma 3.4. *Let P be a point on $\mathcal{F} = \partial\{u_{\overline{f}} > c_2\}$. Suppose that for all $k \in \mathbb{Z}^+$ there exist a positive number r_k, a bounded open interval J_k and a regular curve $\gamma_k : J_k \to \mathcal{F}^*$ such that $r_1 > r_2 > \dots \to 0$, $\overline{\gamma_k(J_k)} \subset \mathcal{F}^* \cap B_{r_k}(P) \setminus \overline{B_{r_{k+1}}(P)}$. Then we must have*

$$\sum_{k=1}^{\infty} \int_{\gamma(J_k)} \frac{1}{|Du_{\overline{f}}|}\, d\sigma < \infty.$$

Proof. Without loss of generality, we assume that P is the origin. We suppose also that $J_k \cap J_h = \emptyset$ for all $k \neq h$, and denote all γ_k with γ. We define

$$J_{k,m} = \begin{cases} J_k \cup J_{k+1} \cup \dots \cup J_m & \text{if } m \geq k,\\ \emptyset & \text{otherwise.} \end{cases}$$

We suppose by contradiction that

$$\sum_{k=1}^{\infty} \int_{\gamma(J_k)} \frac{1}{|Du_{\overline{f}}|}\, d\sigma = \infty. \tag{3.11}$$

Let V be a smooth radial function in $\mathbb{R}^2$, decreasing in $|x|$, defined by

$$\begin{cases} V(x) = 2, & |x| = 0\\ 1 < V(x) < 2, & 0 < |x| < 1/2\\ 0 < V(x) < 1, & 1/2 < |x| < 1\\ V(x) = 0, & |x| \geq 1. \end{cases}$$

For all $k \in \mathbb{Z}^+$ we define $v_k(x) = V(\frac{x}{r_k})$. Consider k large enough such that $\operatorname{supp} v_k \subset \Omega$. Now we fix k; we have

$$\begin{cases} v_k(x) - 1 = 1, & |x| = 0\\ 0 < v_k(x) - 1 < 1, & 0 < |x| < r_k/2\\ -1 < v_k(x) - 1 < 0, & r_k/2 < |x| < r_k\\ v_k(x) - 1 = -1, & |x| \geq r_k. \end{cases}$$

Since J_k and $|\gamma'|$ are bounded, $\gamma(J_k)$ is of finite length. Moreover, $|Du_{\overline{f}}|$ is uniformly bounded away from 0 on $\gamma(J_k)$ since $\overline{\gamma(J_k)} \subset \mathcal{F}^*$. Together with the fact that $\gamma(J_{1,k-1}) \subset (B_{r_k})^C$, we have

$$-\infty < \int_{\gamma(J_{1,k-1})} \frac{v_k - 1}{|Du_{\overline{f}}|}\, d\sigma = -\int_{\gamma(J_{1,k-1})} \frac{1}{|Du_{\overline{f}}|}\, d\sigma < 0.$$

Choose m such that $r_m < r_k/2$. From the facts that $v_k(x) - 1 > 0$ in B_{r_m}, $\gamma(J_l) \subset B_{r_m}$ for all $l \geq m$ and $v_k(x) - 1 \to 1$ as $x \to 0$ and (3.11), we have

$$\int_{\gamma(J_{m,l})} \frac{v_k - 1}{|Du_{\overline{f}}|}\, d\sigma \to \infty \quad \text{for } l \to \infty.$$

Consequently, there must be a number $l \geq m$ such that

$$\int_{\gamma(J_{m,l-1})} \frac{v_k - 1}{|Du_{\overline{f}}|}\, d\sigma \leq -\int_{\gamma(J_{1,k-1})} \frac{v_k - 1}{|Du_{\overline{f}}|}\, d\sigma < \int_{\gamma(J_{m,l})} \frac{v_k - 1}{|Du_{\overline{f}}|}\, d\sigma.$$

Choose a subinterval $J_l' \subset J_l$ such that

$$\int_{\gamma(J_{m,l-1})} \frac{v_k - 1}{|Du_{\overline{f}}|}\, d\sigma + \int_{\gamma(J_l')} \frac{v_k - 1}{|Du_{\overline{f}}|}\, d\sigma = - \int_{\gamma(J_{1,k-1})} \frac{v_k - 1}{|Du_{\overline{f}}|}\, d\sigma.$$

Then we have

$$\int_{\gamma(J^k)} \frac{v_k - 1}{|Du_{\overline{f}}|}\, d\sigma = 0,$$

where $J^k = J_{1,k-1} \cup J_{m,l-1} \cup J_l'$.

Now we can apply Lemma 3.3 to J^k, γ, v_k, $a = 1$ and $h = \frac{v_k - 1}{|Du_{\overline{f}}|}$ and, after rearranging, obtain

$$\int_\Omega \Big(|Dv_k|^2 - q u_{\overline{f}}\, v_k^2 |u_{\overline{f}}|^{q-2} \overline{f} \Big)\, dx \ge b_2 c_2^q \int_{\gamma(J^k)} \frac{(v_k - 1)^2}{|Du_{\overline{f}}|}\, d\sigma.$$

We find that

$$\int_\Omega \Big(|Dv_k|^2 - q u_{\overline{f}}\, v_k^2 |u_{\overline{f}}|^{q-2} \overline{f} \Big) dx \le \int_{B_1(0)} |DV|^2 dx.$$

By the above estimate, for a suitable constant C, we have

$$\begin{aligned} C \int_{B_1(0)} |DV|^2 dx &\ge \int_{\gamma(J^k)} \frac{(v_k - 1)^2}{|Du_{\overline{f}}|}\, d\sigma \\ &\ge \int_{\gamma(J_{1,k-1})} \frac{(v_k - 1)^2}{|Du_{\overline{f}}|}\, d\sigma \\ &= \sum_{h=1}^{k-1} \int_{\gamma(J_h)} \frac{(v_k - 1)^2}{|Du_{\overline{f}}|}\, d\sigma. \end{aligned}$$

Then, when $k \to \infty$, we have

$$C \int_{B_1(0)} |DV|^2 dx \ge \sum_{h=1}^{\infty} \int_{\gamma(J_h)} \frac{(v_k - 1)^2}{|Du_{\overline{f}}|}\, d\sigma = +\infty,$$

which is a contradiction. So we must have

$$\sum_{k=1}^{\infty} \int_{\gamma(J_k)} \frac{d\sigma}{|Du_{\overline{f}}|} < \infty,$$

as desired. □

Lemma 3.5. *Let P be a point on $\mathcal{F} = \partial\{u_{\overline{f}} > c_2\}$. Suppose that there are numbers $K \in \mathbb{Z}$ and $\overline{\sigma} > 0$ such that, for each $k \ge K$, there exists a regular curve $\gamma_k : J_k \to \mathcal{F}^*$ with the following two properties:*

$$\overline{\gamma(J_k)} \subset \mathcal{F}^* \cap B_{2^{-k}}(P) \setminus \overline{B_{2^{-(k+1)}}(P)},$$
$$\mathcal{H}^1(\gamma_k(J_k)) = \int_{J_k} |\gamma'(\xi)| d\xi > \overline{\sigma}\, 2^{-k}.$$

Then $|Du_{\overline{f}}(P)| > 0$.

For a proof of the above lemma, see [4, Lemma 3.2]. From an intuitive point of view, this lemma says that, if the set $\partial\{u_{\overline{f}} > c_2\} \cap \{|Du_{\overline{f}}| > 0\}$ is big enough around a point of $\partial\{u_{\overline{f}} > c_2\}$, then $|Du_{\overline{f}}| > 0$ at this point.

Now, we are able to prove our main theorem.

Proof of Theorem 3.1. By using the previous Lemmas and superharmonicity of $u_{\overline{f}}$ the Theorem follows from the results of sections 5 and 6 in [4]. $\square$

Open problems. The method used in this paper to prove regularity does not work when the number of level sets of $\overline{f}$ is infinite. Therefore it remains to study the boundaries of level sets of $\overline{f}$ in the case of the rearrangement class $\mathfrak{F}(f_0)$ of a general function f_0.

We can obtain an analogous result to Lemma 3.4 for the p-Laplacian operator, but we cannot go further because we lack a suitable regularity theory for the p-Laplacian operator and its solutions. We think that it is reasonable to guess that a regularity result of the type that we have proven in this work will hold for the situation with the p-Laplacian when $p < 2$.

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Claudia Anedda
Dipartimento di Matematica e Informatica, Universitá di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy
E-mail address: canedda@unica.it

Fabrizio Cuccu
Dipartimento di Matematica e Informatica, Universitá di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy
E-mail address: fcuccu@unica.it