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# ON THE BLOW-UP TIME OF A PARABOLIC SYSTEM WITH DAMPING TERMS

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**Abstract**

This paper deals with a nonlinear and weakly coupled parabolic system, containing damping terms, under Dirichlet boundary conditions. Precisely, for the solutions which blow up in finite time, the blow-up time is determined by means of an appropriate resolution method; moreover, the same algorithm is also implemented to discuss some properties of these solutions.

**Keywords:** blow-up, parabolic system, lower bound, blow-up time, numerical solution

**2010 Mathematics Subject Classification:** 35B30, 35K15, 35K55, 65M22

**1. Introduction.** In this work we analyze the following weakly coupled system

$$(1)\qquad \begin{cases} u_t &= \Delta u + v^p - |\nabla u|^q, \;\; \mathbf{x} \in \Omega, \;\; t \in (0, t^\star), & \text{(a)}\\ v_t &= \Delta v + u^p - |\nabla v|^q, \;\; \mathbf{x} \in \Omega, \;\; t \in (0, t^\star), & \text{(b)}\\ u &= 0 \;\text{ and }\; v = 0, \;\; \mathbf{x} \in \partial\Omega, \;\; t \in (0, t^\star), & \text{(c)}\\ u &= u_0(\mathbf{x}) \geq 0 \;\text{ and }\; v = v_0(\mathbf{x}) \geq 0, \;\; \mathbf{x} \in \Omega, & \text{(d)} \end{cases}$$

where $t^\star$ is the blow-up time, $\Omega$ is a bounded and convex domain of $\mathbb{R}^3$, with the origin inside and whose boundary $\partial\Omega$ is sufficiently smooth, $p > q > 1$, and $u_0(\mathbf{x})$

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and $v_0(\mathbf{x})$ are nonnegative functions in $\Omega$, satisfying the compatibility conditions on $\partial\Omega$. It follows by the maximum principle that in the interval of existence $u(\mathbf{x},t) \geq 0$ and $v(\mathbf{x},t) \geq 0$. Furthermore, let us remark that the gradient terms in (1) have a damping effect, contrasting the power source terms and working against blow-up.

If only an unknown is considered, system (1) is reduced to a single equation, strongly studied in [1–3], and for which distinct results about blow-up have been obtained under more general assumptions on the equation and the boundary conditions (see [4]). In the same way, [5–7] provide good references about upper and lower bounds of blow-up time for solutions of nonlinear parabolic problems under various boundary conditions. Furthermore, we refer to [8, 9] for some results concerning the elliptic case.

On the other hand, in [10] an explicit lower bound of the blow-up time for a classical solution of system (1) is directly derived. This estimate depends on the geometry of $\Omega$, and on the data of the problem $p$, $q$, $u_0(\mathbf{x})$ and $v_0(\mathbf{x})$ (see Section 2). In this parallel work, we want to calculate the real value of the blow-up time. In this sense, we remark that there exist numerous papers devoted to the computational solutions of blow-up problems on bounded or unbounded domains of $\mathbb{R}$ and $\mathbb{R}^2$ (see for example [11, 12]).

Herein (see Section 3), starting from the weak formulation of (1), and using a semidiscrete finite element method (see [13]), system (1) is solved and, consequently, the value of $t^\star$ is computed. Moreover, some numerical examples that confirm the theoretical result obtained in [10] and allow to observe other interesting phenomena connected to the behaviour of the solution are shown. Finally, some concluding remarks are drawn in Section 4.

**2. Main result: an explicit lower bound.** In [10], for any solution $(u, v)$ of (1), the authors introduce the following auxiliary function

$$W(t) = \int_\Omega (u^{2p} + v^{2p}) d\mathbf{x}, \tag{2}$$

and give this

**Definition 2.1.** *The solution of problem* (1) *blows up at time* $t^\star$ *in* $W$*-norm if*

$$\lim_{t\to t^\star} W(t) = +\infty.$$

In order to derive a lower bound for the blow-up time, they prove the following fundamental

**Theorem 2.1.** *Let* $(u, v)$ *be a classical solution of* (1); *then a lower bound for the blow-up time* $t^\star$ *of any blowing up solution in* $W$*-norm* (2) *is*

$$t^\star \geq \frac{1}{2\mathcal{A}W_0^2}. \tag{3}$$

*In* (3), $W_0 = W(0) = \int_\Omega (u_0^{2p} + v_0^{2p}) d\mathbf{x}$ *and* $\mathcal{A} = \mathcal{A}(p, q, |\Omega|, d, \rho_0, \lambda_1)$, *being*

$$\rho_0 = \min_{\partial\Omega}(\mathbf{x} \cdot \boldsymbol{\nu}) > 0 \quad \textit{and} \quad d = \max_{\overline{\Omega}} |\mathbf{x}|,$$

*with* $\boldsymbol{\nu}$ *the normal unit vector to* $\partial\Omega$, *and* $\lambda_1$ *the first eigenvalue of the problem*

$$\begin{cases} \Delta w + \lambda w = 0 \quad \mathbf{x} \in \Omega, \quad w > 0, \\ w = 0, \quad \mathbf{x} \in \partial\Omega. \end{cases}$$

The proof of this theorem is structured as follows: by supposing $W(t)$ blowing up at time $t^\star$, there exists a time $t_1$ (that might also be 0) such that $W(t_1) = W(0)$ and $W(t) > W(t_1)$, $t \in (t_1, t^\star)$ (see Fig. 1). Therefore, by using both functional and algebraic inequalities (all the details are in [10]), this relation

$$W'(t) \le \mathcal{A} W^3(t), \ \forall\, t \in [t_1, t^\star), \tag{4}$$

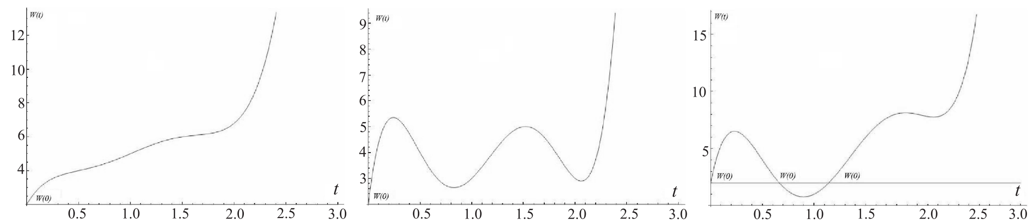
is proven. Lastly, (3) is directly obtained by integrating (4) from $t_1$ to $t^\star$:

$$\frac{1}{2\mathcal{A}W_0^2} = \int_{W(t_1)=W_0}^{W(t^\star)=\infty} \frac{\mathrm{d}W}{\mathcal{A}W^3} \le \int_{t_1}^{t^\star} d\tau \le \int_0^{t^\star} d\tau = t^\star. \tag{5}$$

**Remark 2.1.** We want to underline that estimate (3) can be obtained in different way, so that it represents ***one*** of the possible lower bounds of $t^\star$ in $W$-norm; anyway, once the data are given, by arranging the proof of Theorem 2.1 (see, again, [10]) it is possible to consider a constrained minimization problem on $\mathcal{A}$, whose resolution leads to ***the*** optimal lower bound, i.e. the greatest value of $T = \dfrac{1}{2\mathcal{A}W_0^2}$ verifying (3). Our interest to this optimal value is connected with the maximal interval of existence of the solution $(u, v)$ of (1).

In the rest of this work we focus on investigating for which value of $t^\star$

$$\lim_{t \to t^\star} W(t) = +\infty.$$




(*a*) $W(t)$ increasing; $t_1 = 0$ (*b*) $W(t)$ oscillating; $t_1 = 0$ (*c*) $W(t)$ oscillating; $t_1 > 0$

Fig. 1. Possible behaviour of the $W$-norm in terms of time, once $W(t)$ is supposed to blow up at finite time $t^\star$

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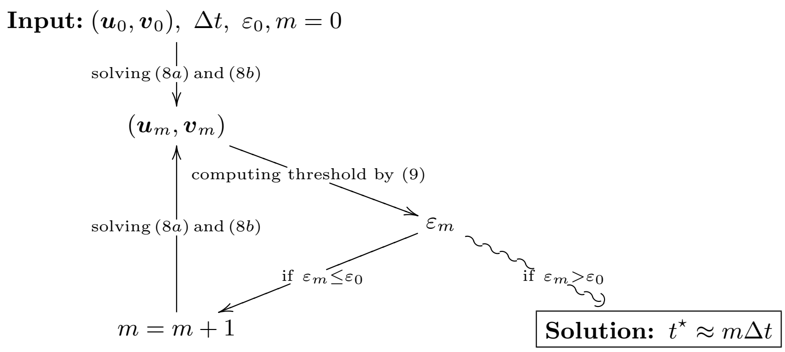
Computation of the blow-up time $t^\star$. The necessary input data are the threshold $\varepsilon_0$, the time step $\Delta t$ and the initial datum $(\boldsymbol{u}_0, \boldsymbol{v}_0)$; successively, it is possible to calculate the sequences $(\boldsymbol{u}_m, \boldsymbol{v}_m)$ and $\varepsilon_m$ and, consequently, to compute $t^\star$




**3. Numerical discretization and examples.** In this section a resolution procedure for system (1), based on a mixed semidiscrete in space and a single-step method in time, is presented.

***3.1. Finite element method: semi-discretization in space.*** If a mesh of $\Omega$ is fixed, and $N$ represents the total number of nodes of $\Omega$, let $(\mathcal{U}, \mathcal{V})$ be the numerical approximation of the solution $(u, v)$ of (1): therefore,

$$
\begin{cases}
\mathcal{U}(\mathbf{x}, t) = \displaystyle\sum_{i=1}^{N} u^i(t)\varphi^i(\mathbf{x}), \\
\mathcal{V}(\mathbf{x}, t) = \displaystyle\sum_{i=1}^{N} v^i(t)\varphi^i(\mathbf{x}),
\end{cases}
\tag{6}
$$

where $\varphi^i(\mathbf{x}) \in H_0^1(\Omega)$ is the standard hat basis at the vertex $\mathbf{x}^i$, for $i = 1, \ldots, N$.

Thanks to the divergence theorem and the homogeneous boundary conditions (1c), by multiplying both (1a) and (1b) by a generic test function $\varphi^j(\mathbf{x})$, the following variational form in space is achieved:

$$
\begin{cases}
(\mathcal{U}_t, \varphi^j) + (\nabla\mathcal{U}, \nabla\varphi^j) = (\mathcal{V}^p, \varphi^j) - (|\nabla\mathcal{U}|^q, \varphi^j), \\
(\mathcal{V}_t, \varphi^j) + (\nabla\mathcal{V}, \nabla\varphi^j) = (\mathcal{U}^p, \varphi^j) - (|\nabla\mathcal{V}|^q, \varphi^j),
\end{cases}
\quad j = 1, \ldots, N,\ t \geq 0,
\tag{7}
$$

where $(f, g) = \int_\Omega f(\mathbf{x})g(\mathbf{x})d\mathbf{x}$ denotes the $L^2$ inner product.

To compute the evolutions in time of both coefficients $u^i$ and $v^i$ appearing in (6), let $\Delta t = t_{m+1} - t_m$ be a given time step, with $m = 0, 1, 2, \ldots$ ($t_0 = 0$), and $(\mathcal{U}_m, \mathcal{V}_m)$ the approximation of $(\mathcal{U}(\mathbf{x}, t), \mathcal{V}(\mathbf{x}, t))$ at time $t_m$. By applying an implicit Euler finite difference approximation to system (7), it is seen that

$$\begin{cases} (\frac{\mathcal{U}_{m+1}-\mathcal{U}_m}{\Delta t}, \varphi^j) + (\nabla \mathcal{U}_{m+1}, \nabla \varphi^j) = (\mathcal{V}_m^p, \varphi^j) - (|\nabla \mathcal{U}_m|^q, \varphi^j), \\ (\frac{\mathcal{V}_{m+1}-\mathcal{V}_m}{\Delta t}, \varphi^j) + (\nabla \mathcal{V}_{m+1}, \nabla \varphi^j) = (\mathcal{U}_m^p, \varphi^j) - (|\nabla \mathcal{V}_m|^q, \varphi^j), \end{cases}$$

i.e., taking into account (6),

$$(8) \qquad \begin{cases} \boldsymbol{M}\dfrac{\boldsymbol{u}_{m+1} - \boldsymbol{u}_m}{\Delta t} + \boldsymbol{K}\boldsymbol{u}_{m+1} = \mathcal{F}_p(\boldsymbol{v}_m) - \mathcal{G}_q(\boldsymbol{u}_m), & \text{(a)} \\ \boldsymbol{M}\dfrac{\boldsymbol{v}_{m+1} - \boldsymbol{v}_m}{\Delta t} + \boldsymbol{K}\boldsymbol{v}_{m+1} = \mathcal{F}_p(\boldsymbol{u}_m) - \mathcal{G}_q(\boldsymbol{v}_m), & \text{(b)} \end{cases}$$

with

$$\begin{cases} \boldsymbol{M} \in \mathbb{R}^{N\times N} \text{ (mass matrix)}: \ M_{ij} = \int_\Omega \varphi^i(\mathbf{x})\varphi^j(\mathbf{x})d\mathbf{x}, \\ \boldsymbol{K} \in \mathbb{R}^{N\times N} \text{ (stiffness matrix)}: \ K_{ij} = \int_\Omega \nabla\varphi^i(\mathbf{x}) \cdot \nabla\varphi^j(\mathbf{x})d\mathbf{x}, \\ \mathcal{F}_p(\boldsymbol{v}_m) \in \mathbb{R}^N : \mathcal{F}_p(\boldsymbol{v}_m)_j = \int_\Omega (\sum_{i=1}^N v_m^i \varphi^i(\mathbf{x}))^p \varphi^j(\mathbf{x})d\mathbf{x} \text{ (similarly } \mathcal{F}_p(\boldsymbol{u}_m)_j), \\ \mathcal{G}_q(\boldsymbol{u}_m) \in \mathbb{R}^N : \mathcal{G}_q(\boldsymbol{u}_m)_j = \int_\Omega (\sum_{i=1}^N u_m^i |\nabla\varphi^i(\mathbf{x})|)^q \varphi^j(\mathbf{x})d\mathbf{x} \text{ (similarly } \mathcal{G}_q(\boldsymbol{v}_m)_j), \end{cases}$$

being $\boldsymbol{u}_m = (u_m^1, \ldots, u_m^N)^T$ and $\boldsymbol{v}_m = (v_m^1, \ldots, v_m^N)^T$, where $T$ represents, in this case, the transposition operator. In these circumstances, $(u_m^i, v_m^i)$ is the approximation of the solution $(u, v)$ of problem (1) at time $t_m$, for $m = 0, 1, 2, \ldots$, and at space point $\mathbf{x}^i$, for $i = 1, 2, \ldots, N$.

With regards to the estimate of the blow-up time $t^\star$, the following numerical resolution algorithm is proposed. Let $\varepsilon_0$ be a fixed threshold: once the initial datum $(\boldsymbol{u}_0, \boldsymbol{v}_0)$ is given, $\boldsymbol{u}_1$ and $\boldsymbol{v}_1$ are computed from (8a) and (8b), respectively. Successively, $(\boldsymbol{u}_1, \boldsymbol{v}_1)$ is used to actualize $(\boldsymbol{u}_2, \boldsymbol{v}_2)$, and so on. Moreover, according to (2), we exit the loop when the $W$-norm at step $m$

$$(9) \qquad \varepsilon_m = \int_\Omega \Big[\Big(\sum_{i=1}^n u_m^i \varphi^i(\mathbf{x})\Big)^{2p} + \Big(\sum_{i=1}^n v_m^i \varphi^i(\mathbf{x})\Big)^{2p}\Big]d\mathbf{x},$$

is greater than the initial threshold $\varepsilon_0$ (*Stopping Criterion*); consequently, $t^\star \approx m\Delta t$ (see the scheme in Table 1).

**Remark 3.1.** It is well known that the Euler method presents only a linear accuracy with respect to the step size, i.e. it is a first order method. On the other hand, some modifications of this same method can return a better accuracy of the result; for instance, a very common approach is the so called Crank–Nicolson method (see [13]), that is a quadratic (i.e. second order) method. In many situations this method is unconditionally stable; however, the approximated solutions could contain some kind of oscillations. Contrary, the less accurate Euler method is stable and immune to oscillations; moreover, it is also very simple to implement, very intuitive, and the computed solution can be as close to the exact one as desired. As a consequence, since it is appropriate to the aims of this research, it is the one we will use.

***3.2. Numerical tests.*** Let us solve system (1) and discuss the following examples:

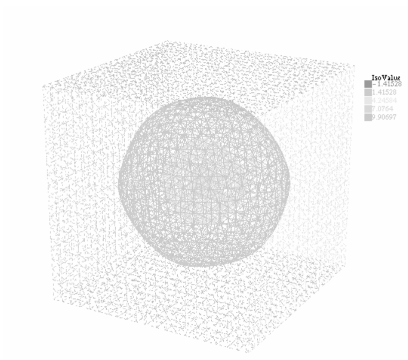
- Test 1 deals with the computation of the blow-up time $t^\star$ in terms of the integration step $\Delta t$ and the comparison with the lower bound $T = \dfrac{1}{2\mathcal{A}W_0^2}$ estimated in Theorem 2.1.
- Test 2 focuses on the analysis of the value of $t^\star$ with $p$ and $q$ varying.
- Test 3 shows the influence of the initial data on $t^\star$.

All these cases are computed in the domain $Q = \Omega \times \mathbb{R}_0^+$, being $\Omega = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^3$ the cube with centre in the origin and length $\pi$. Therefore, it is checked that $|\Omega| = \pi^3$, $d = \frac{\sqrt{3}}{2}\pi$, $\rho_0 = \frac{\pi}{2}$, $\lambda_1 = 3$ (see Theorem 2.1).

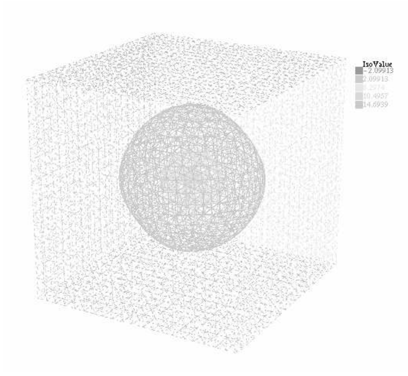
**Test 1.** Let us fix $p = 3$, $q = 1.1$, $u_0(\mathbf{x}) = 2(\pi/2 - |x|)(\pi/2 - |y|)(\pi/2 - |z|)$ and $v_0(\mathbf{x}) = 1.5(\pi/2 - |x|)(\pi/2 - |y|)(\pi/2 - |z|)$; moreover let $\varepsilon_0 = 10^7$. Table 2 shows the values of the blow-up time $t^\star$. The approximated value of $t^\star$ approaches to 0.021 and $\varepsilon_m$ gets closer to the fixed threshold $\varepsilon_0$ with $\Delta t$ decreasing. On the other hand, as explained in Remark 2.1, it is possible to show that these data return this optimal lower bound of $T = 7.1 \times 10^{-15}$; as a consequence, the lower bound $T \ll t^\star$. Lastly, the qualitative solution $(u, v)$ is represented in Fig. 2 in three different instants of time, $t_1 = 0.01, t_2 = 0.018$ and $t_3 = 0.025$.

**Test 2.** The data of the problem are $\Delta t = 10^{-3}$ and $\varepsilon_0 = 10^7$; moreover $u_0(\mathbf{x}) = 5\log(1 + \cos(y))|\cos(x)\cos(z)|$ and $v_0(\mathbf{x}) = 1.5|\cos(x)\cos(y)|(\frac{\pi}{2} - |z|)$. Table 3 illustrates how the blow-up time $t^\star$ depends on the values of $p$ and $q$. Furthermore, let us observe that not any choice of $p$ and $q$ (with $p > q > 1$) returns a blow-up phenomenon; in fact, the last row of the same table reflects that for $p = 3$ and $q = 1.8$ the damping terms *break* the source ones, producing bounded solutions, and therefore a decreasing behaviour of $\varepsilon_m$ through the time.

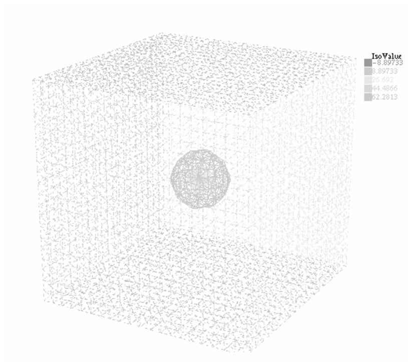
**Test 3.** In this test we study the blow-up time $t^\star$ in terms of the initial data $u_0(\mathbf{x})$ and $v_0(\mathbf{x})$, more exactly depending on $W_0$-norm. Let $p = 3$ and $q = 1.1$; moreover $\varepsilon_0 = 10^7$. Four cases, corresponding to the following initial conditions



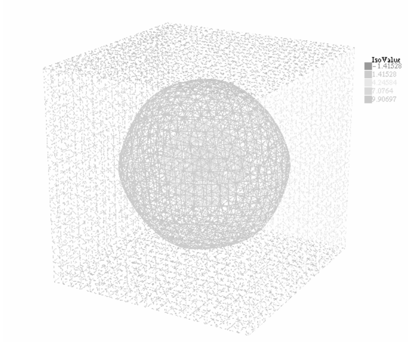
($a$) Solution $u$ at time $t_1 = 0.01$. The value of $u$ at $O$ is approximately 8.49



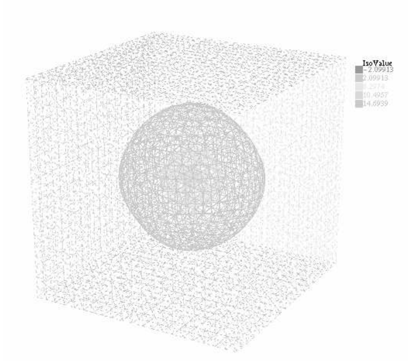
($b$) Solution $u$ at time $t_2 = 0.018$. The value of $u$ at $O$ is approximately 12.59



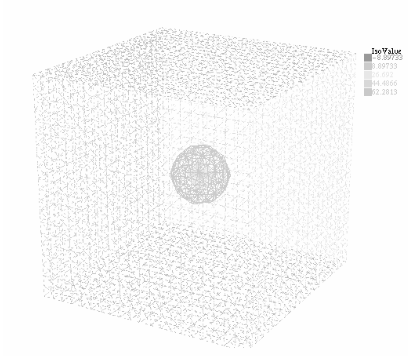
($c$) Solution $u$ at time $t_3 = 0.025$. The value of $u$ at $O$ is approximately 53.38



($d$) Solution $v$ at time $t_1 = 0.01$. The value of $v$ at $O$ is approximately 8.1



($e$) Solution $v$ at time $t_2 = 0.018$. The value of $v$ at $O$ is approximately 12.54



($f$) Solution $v$ at time $t_3 = 0.025$. The value of $v$ at $O$ is approximately 53.38

Fig. 2. Numerical solution. Evolution of the solution $(u, v)$ and its graphical representations; the darker the shade of gray the highest the value of the solution. The values of $u$ and $v$ at the center of the cube (the origin $O$), that is the blow-up point, increase with time, but slower than the $W$-norm (9) (see Table 2)

(I.C.), are analyzed:

$$\begin{cases} u_1 = 5\log(1+\cos(y))|\cos(x)\cos(z)|, \\ v_1 = 0, \end{cases}$$

$$\begin{cases} u_2 = 5\log(1+\cos(y))|\cos(x)\cos(z)|, \\ v_2 = 0.7(\pi/2-|x|)(\pi/2-|y|)(\pi/2-|z|), \end{cases}$$

$$\begin{cases} u_3 = 5\log(1+\cos(y))|\cos(x)\cos(z)|, \\ v_3 = 0.8(\pi/2-|x|)(\pi/2-|y|)(\pi/2-|z|), \end{cases}$$

$$\begin{cases} u_4 = 5\log(1+\cos(y))|\cos(x)\cos(z)|, \\ v_4 = 0.9(\pi/2-|x|)(\pi/2-|y|)(\pi/2-|z|). \end{cases}$$

The results are shown in Table 4; also in this case, the blow-up time is connected to the magnitude of the initial data. In particular, certain initial conditions can produce global solutions in time, as illustrated in the first row of the same Table 4.

**4. Concluding remarks.** This paper studies the blowing-up solutions of a nonlinear and weakly coupled parabolic system defined in a bounded domain of $\mathbb{R}^3$; the equations contain power source and damping terms, and Dirichlet boundary conditions are fixed. Starting from a theoretical result concerning a

Table 2

Computing the blow-up time $t^\star$ in terms of $\Delta t$; comparison with respect the lower bound $T$

| $\Delta t$ | $t^\star$ | $T$ | $\varepsilon_m$ |
|---|---|---|---|
| $10^{-2}$ | 0.05 | $7.1 \times 10^{-15}$ | $2.8 \times 10^6$ |
| $10^{-3}$ | 0.024 | $7.1 \times 10^{-15}$ | $5.5 \times 10^6$ |
| $10^{-4}$ | 0.0215 | $7.1 \times 10^{-15}$ | $8.3 \times 10^6$ |
| $10^{-5}$ | 0.0211 | $7.1 \times 10^{-15}$ | $9.1 \times 10^6$ |

Table 3

Analyzing the blow-up time $t^\star$ in terms of $p$ and $q$

| $p$ | $q$ | $t^\star$ | $\varepsilon_m$ |
|---|---|---|---|
| 3 | 1.1 | 0.148 | $2.9 \times 10^6$ |
| 3.4 | 1.8 | 0.06 | $4.5 \times 10^6$ |
| 3.3 | 1.8 | 0.08 | $9.3 \times 10^6$ |
| 3 | 1.8 | 1.14 | $7.6 \times 10^{-6}$ |

Table 4

Analyzing the blow-up time in terms of $W_0$

| I.C. | $W_0$ | $t^\star$ | $\varepsilon_m$ |
|---|---|---|---|
| $(u_1, v_1)$ | 1810.75 | 1.6 | $4.8 \times 10^{-10}$ |
| $(u_2, v_2)$ | 1846.81 | 0.288 | $9.3 \times 10^{6}$ |
| $(u_3, v_3)$ | 1891.11 | 0.165 | $5.3 \times 10^{6}$ |
| $(u_4, v_4)$ | 1973.63 | 0.128 | $3.14 \times 10^{6}$ |

lower bound for the blow-up time, we propose and employ a procedure capable to directly calculate its real value. It is achieved by applying a mixed semidiscrete in space and a single-step method in time algorithm to the system. Furthermore, the problem is numerically solved in different cases; the analysis of the results shows that:

- the numerical method is coherent with respect to the theoretical approach, in the sense that in those cases in which the solution blows up the blow-up time verifies the lower bound estimate,
- the optimal lower bound for the blow-up time (in $W$-norm) is significantly smaller with respect to its real value,
- the $W$-norm of the solution can grow up more and more with respect to the value of the solution at its blow-up point,
- the system is sensitive with respect to small variations of its data: in fact, initial conditions or parameters slightly different from each other can return both blowing up or global solutions.

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