

**AN EXACT SOLUTION FOR THE MACROSCOPIC
APPROACH TO EXTENDED THERMODYNAMICS OF
DENSE GASES WITH MANY MOMENTS**

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Abstract: Extended Thermodynamics of Dense Gases with an arbitrary but fixed number of moments has been recently studied in literature; the arbitrariness of the number of moments is linked to a number N and the resulting model is called an $(N) - Model$. As usual in Extended Thermodynamics, in the field equations some unknown functions appear; restriction on their generalities are obtained by imposing the entropy principle, the Galilean relativity principle and some symmetry conditions.

The solution of these conditions is called the "closure problem" and it has not been written explicitly because an hard notation is necessary, but it has been shown how the theory is self-generating in the sense that, if we know the closure of the $(N) - Model$, than we will be able to find that of the $(N + 1) - Model$. Instead of this, we find here an exact solution which holds for every number N .

AMS Subject Classification: 74A15, 76N15, 80A17

Key Words: thermodynamics, gas dynamics, thermodynamics of continua

1. Introduction

Extended Thermodynamics of Dense Gases (ETDG) with many moments proposes for every number N the following balance equations

$$\begin{aligned} \partial_t F^{i_1 \dots i_n} + \partial_k F^{ki_1 \dots i_n} &= P^{i_1 \dots i_n} \quad \text{for } n = 0, \dots, N + 2. \\ \partial_t G^{i_1 \dots i_m} + \partial_k G^{ki_1 \dots i_m} &= Q^{i_1 \dots i_m} \quad \text{for } m = 0, \dots, N. \end{aligned} \quad (1)$$

and the resulting model is called an $N - Model$.

We recall that the earlier versions of Extended Thermodynamics (ET) used only eqs. (1)₁ obtaining the important result to have a symmetric hyperbolic system of partial differential equations, with finite speed of propagation of shock waves and other important properties (See refs. [1]-[6]). These properties were not satisfied by the models of Ordinary Thermodynamics except for some simple cases. But the structure of ET implied a restriction on the state function relating the pressure p to the mass density ρ and energy density ϵ , that is $p = \frac{2}{3}\rho\epsilon$. This problem have been overcome in ETDG by considering all the balance equations (1) of which the first one is called the "Mass-Block" of equations and the second is called the "Energy-Block" of equations (See refs. [7]-[14] as examples).

In (1) the independent variables are $F^{i_1 \dots i_n}$ and $G^{i_1 \dots i_m}$; the quantities $F^{ki_1 \dots i_n}$ and $G^{ki_1 \dots i_m}$ are their corresponding fluxes. We see that each flux is equal to the independent variable of the subsequent equation, except for the flux in the last equation of the Mass-Block and for that in the last equation of the Energy-Block; for these last fluxes we know only that they are symmetric tensors. We will refer to this property as "the symmetry conditions". The problem will be closed when we know the expressions of $F^{ki_1 \dots i_{N+2}}$ and $G^{ki_1 \dots i_N}$ as functions of the independent variables. Restrictions on their generality are obtained by imposing the entropy principle, the Galilean relativity principle and the symmetry conditions.

The Entropy Principle can be exploited through Liu's Theorem [15] and by using a bright idea conceived by Ruggeri [16]; so it becomes equivalent to assuming the existence of Lagrange Multipliers μ_A and λ_B which can be taken as independent variables and, after that, we have

$$\begin{aligned} F^{i_1 \dots i_n} &= \frac{\partial h'}{\partial \mu_{i_1 \dots i_n}} \quad , \quad G^{i_1 \dots i_m} = \frac{\partial h'}{\partial \lambda_{i_1 \dots i_m}} \\ F^{ki_1 \dots i_n} &= \frac{\partial h'^k}{\partial \mu_{i_1 \dots i_n}} \quad , \quad G^{ki_1 \dots i_m} = \frac{\partial h'^k}{\partial \lambda_{i_1 \dots i_m}}. \end{aligned} \quad (2)$$

which expresses all the moments in terms of only two unknown functions, the 4-potentials h' , h'^k . A nice consequence of eqs. (2) is that the field equations assume the symmetric form. Another consequence of (2) is that the above mentioned symmetry conditions and the Galilean Relativity Principle can be expressed as

$$\begin{aligned} \frac{\partial h'^k}{\partial \mu_{i_1 \dots i_n}} &= \frac{\partial h'}{\partial \mu_{k i_1 \dots i_n}} \quad \text{for } n = 0, \dots, N + 1; \quad \frac{\partial h'^{[k}}{\partial \mu_{i_1] i_2 \dots i_{N+2}}} = 0 \\ \frac{\partial h'^k}{\partial \lambda_{i_1 \dots i_m}} &= \frac{\partial h'}{\partial \lambda_{k i_1 \dots i_m}} \quad \text{for } m = 0, \dots, N - 1; \quad \frac{\partial h'^{[k}}{\partial \lambda_{i_1] \dots i_N}} = 0. \\ \frac{\partial h'}{\partial \mu_k} \mu_i &+ \sum_{n=1}^{N+1} \frac{\partial h'}{\partial \mu_{k j_1 \dots j_n}} [(n + 1) \mu_{i j_1 \dots j_n} + 2 \lambda_{j_1 \dots j_{n-1}} \delta_{j_n i}] + \\ &+ \sum_{s=0}^{N-1} \frac{\partial h'}{\partial \lambda_{k h_1 \dots h_s}} (s + 1) \lambda_{i h_1 \dots h_s} + h' \delta^{ki} = 0. \end{aligned} \tag{3}$$

We don't go in the details on how the the Galilean Relativity Principle, thanks also to the other conditions, is equivalent to (3)₅; this equivalence can be already found in literature as [17]-[18] and others. In the next section, eqs. (4) and (5), we will exhibit a particular but significative solution of the conditions (3), which is expressed through a Taylor's expansion around equilibrium; this is defined as the state where $\mu_{i_1 \dots i_n} = 0$ for $n = 1, \dots, N + 2$, $\lambda_{i_1 \dots i_m} = 0$ for $m = 1, \dots, N$, so that the only variables which are not zero at equilibrium are μ and λ . The first of these is the chemical potential, while $\lambda = \frac{1}{2T}$ with T absolute temperature.

In sect. 3 we will report a part of the proof that eqs. (4) and (5) give a solution of (3). The second and final part of the proof is described in sect. 4.

2. An Exact Solution of Conditions (3)

Before writing our solution we need to describe an hard but straightforward

Notation: To do the derivatives with respect to $\mu_{i_1 \dots i_n}$ a number p_n of times, we use the compact form $A_{n,1}$ to indicate a set of n indexes, $A_{n,2}$ to indicate another set of n indexes, and so on up to A_{n,p_n} . For example,

$$\frac{\partial^{p_2} h'^k}{\partial \mu_{h_1 k_1} \partial \mu_{h_2 k_2} \dots \partial \mu_{h_{p_2} k_{p_2}}} \quad \text{in the compact notation becomes} \\ \frac{\partial^{p_2} h'^k}{\partial \mu_{A_{2,1}} \partial \mu_{A_{2,2}} \dots \partial \mu_{A_{2,p_2}}}$$

because the first index of $A_{\dots, \dots}$ indicates how many indexes has the corresponding μ_{\dots} , while the second index of $A_{\dots, \dots}$ indicates how many derivatives we are taking with respect to it.

$$\begin{aligned} \text{Similarly, } & \frac{\partial^{p_3} h'^k}{\partial \mu_{h_1 k_1 j_1} \partial \mu_{h_2 k_2 j_2} \cdots \partial \mu_{h_{p_3} k_{p_3} j_{p_3}}} \\ \text{becomes } & \frac{\partial^{p_3} h'^k}{\partial \mu_{A_{3,1}} \partial \mu_{A_{3,2}} \cdots \partial \mu_{A_{3,p_3}}} \end{aligned}$$

and so on. By using this notation, we can now write our solution and it is

$$\begin{aligned} h' = & \sum_{\substack{0 \cdots \infty \\ p_1, \dots, p_{N+2}, r_1, \dots, r_N \\ \sum_{i=1}^{N+2} ip_i + \\ + \sum_{j=1}^N jr_j \text{ even}}} \frac{1}{p_1! p_2!} \cdots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \cdots \frac{1}{r_N!} \cdot \quad (4) \\ & \cdot \frac{[1 + \sum_{i=1}^{N+2} ip_i + \sum_{j=1}^N jr_j]!!}{1 + \sum_{i=1}^{N+2} ip_i + \sum_{j=1}^N jr_j} \\ & \cdot \frac{\partial^{1 + \sum_{i=1}^{N+2} p_i + \sum_{j=1}^N r_j}}{\partial \lambda^{\sum_{j=1}^N r_j} \partial \mu^{1 + \sum_{i=1}^{N+2} p_i}} \left[\left(\frac{-1}{2\lambda} \right)^{\frac{1}{2} [\sum_{i=1}^{N+2} ip_i + \sum_{j=1}^N jr_j]} \right. \\ & \quad \left. \cdot \psi_{\frac{1}{2}} [p_1 + 2p_2 + \cdots + (N+2)p_{N+2} + r_1 + 2r_2 + \cdots + Nr_N] \right] \cdot \\ & \cdot \delta^{(A_{1,1} \cdots A_{1,p_1} \cdots A_{N+2,1} \cdots A_{N+2,p_{N+2}} B_{1,1} \cdots B_{1,r_1} \cdots B_{N,1} \cdots B_{N,r_N})} \\ & \cdot \mu_{A_{1,1}} \cdots \mu_{A_{1,p_1}} \cdots \mu_{A_{N+2,1}} \cdots \mu_{A_{N+2,p_{N+2}}} \cdot \\ & \cdot \lambda_{B_{1,1}} \cdots \lambda_{B_{1,r_1}} \cdots \lambda_{B_{N,1}} \cdots \lambda_{B_{N,r_N}} \cdot \end{aligned}$$

$$\begin{aligned} h'^k = & \sum_{\substack{0 \cdots \infty \\ p_1, \dots, p_{N+2}, r_1, \dots, r_N \\ \sum_{i=1}^{N+2} ip_i + \\ + \sum_{j=1}^N jr_j \text{ odd}}} \frac{1}{p_1! p_2!} \cdots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \cdots \frac{1}{r_N!} \cdot \quad (5) \\ & \cdot \left[\sum_{i=1}^{N+2} ip_i + \sum_{j=1}^N jr_j \right]!! \cdot \end{aligned}$$

$$\frac{\partial^{1+\sum_{i=1}^{N+2} p_i + \sum_{j=1}^N r_j}}{\partial \lambda^{\sum_{j=1}^N r_j} \partial \mu^{1+\sum_{i=1}^{N+2} p_i}} \left[\left(\frac{-1}{2\lambda} \right)^{\frac{1}{2}[1+\sum_{i=1}^{N+2} i p_i + \sum_{j=1}^N j r_j]} \cdot \psi_{\frac{1}{2}[1+p_1+2p_2+\dots+(N+2)p_{N+2}+r_1+2r_2+\dots+Nr_N]} \cdot \delta^{(kA_{1,1}\dots A_{1,p_1}\dots A_{N+2,1}\dots A_{N+2,p_{N+2}} B_{1,1}\dots B_{1,r_1}\dots B_{N,1}\dots B_{N,r_N})} \cdot \mu_{A_{1,1}} \dots \mu_{A_{1,p_1}} \dots \mu_{A_{N+2,1}} \dots \mu_{A_{N+2,p_{N+2}}} \cdot \lambda_{B_{1,1}} \dots \lambda_{B_{1,r_1}} \dots \lambda_{B_{N,1}} \dots \lambda_{B_{N,r_N}} \right]$$

where ψ_n is a family of functions depending only on μ and λ and constrained only by

$$\frac{\partial \psi_{n+1}}{\partial \mu} = \psi_n. \tag{6}$$

Despite the appearance, these expressions are not complicated. In fact, the factorial $\frac{1}{p_1!}$ appears also in the Taylor's expansions of functions depending on a single variable; we have here one of these for every variable, both of the mass block than of the energy block. Moreover, in these expressions appears the sum of these numbers $p_1 + p_2 + \dots + p_{N+2}$ of the mass block and $r_1 + r_2 + \dots + r_N$ of the energy block. Moreover, these numbers appear also through $p_1 + 2p_2 + \dots + (N + 2)p_{N+2} + r_1 + 2r_2 + \dots + Nr_N$ where these numbers are multiplied by the order of the Lagrange multiplier which they represent; after that, their sum is taken. The condition " $p_1 + 2p_2 + \dots + (N + 2)p_{N+2} + r_1 + 2r_2 + \dots + Nr_N$ odd" in the expression of h^k is necessary because the following δ^{\dots} must have an even number of indexes; similarly, for the expression of h' . Here we find only a letter δ^{\dots} but it is understood that it is a shortened symbol denoting the product of some δ^{\dots} each one with 2 indexes, and with a final symmetrization over all these indexes. Finally, in the last line there is the product of the variables with respect to which we have done the Taylor's expansions.

3. Proof of the Solution (4), (5) – Part I

We prove now that (4), (5) is a solution of (3)₁₋₄, while (3)₅ will be considered in the next section.

- Let us begin with $(3)_1$ in the case $n \geq 1$.

To verify it we have to take into account that $\mu_{i_1 \dots i_n}$ here is denoted with $\mu_{A_n,1}$, or with $\mu_{A_n,2}$, and so on up to μ_{A_n,p_n} . Similarly, $\mu_{ki_1 \dots i_n}$ is denoted with $\mu_{A_{n+1},1}$, or with $\mu_{A_{n+1},2}$, and so on up to $\mu_{A_{n+1},p_{n+1}}$. Consequently, the left hand side of $(3)_1$ with $n \geq 1$ causes, with respect to the expression (5), a rise of one unity of the index p_n ; similarly, the right hand side of $(3)_1$ with $n \geq 1$ causes, with respect to the expression (4), a rise of one unity of the index p_{n+1} . More precisely, we obtain for both sides the following expression

$$\begin{aligned} & \sum_{p_1, \dots, p_{N+2}, r_1, \dots, r_N}^{0 \dots \infty} \frac{1}{p_1!} \frac{1}{p_2!} \dots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \dots \frac{1}{r_N!} \cdot \\ & \quad n + \sum_{i=1}^{N+2} ip_i + \\ & \quad + \sum_{j=1}^N jr_j \text{ odd} \\ & \quad \cdot \left[n + \sum_{i=1}^{N+2} ip_i + \sum_{j=1}^N jr_j \right] !! \cdot \\ & \quad \cdot \frac{\partial^{2+\sum_{i=1}^{N+2} p_i + \sum_{j=1}^N r_j}}{\partial \lambda^{\sum_{j=1}^N r_j} \partial \mu^{2+\sum_{i=1}^{N+2} p_i}} \left[\left(\frac{-1}{2\lambda} \right)^{\frac{1}{2} [n+1+\sum_{i=1}^{N+2} ip_i + \sum_{j=1}^N jr_j]} \right] \cdot \\ & \quad \cdot \psi_{\frac{1}{2}} [n+1+p_1+2p_2+\dots+(N+2)p_{N+2}+r_1+2r_2+\dots+Nr_N] \cdot \\ & \quad \cdot \delta^{(ki_1 \dots i_n A_{1,1} \dots A_{1,p_1} \dots A_{N+2,1} \dots A_{N+2,p_{N+2}} B_{1,1} \dots B_{1,r_1} \dots B_{N,1} \dots B_{N,r_N})} \cdot \\ & \quad \cdot \mu_{A_{1,1}} \dots \mu_{A_{1,p_1}} \dots \mu_{A_{N+2,1}} \dots \mu_{A_{N+2,p_{N+2}}} \cdot \\ & \quad \lambda_{B_{1,1}} \dots \lambda_{B_{1,r_1}} \dots \lambda_{B_{N,1}} \dots \lambda_{B_{N,r_N}} \cdot \end{aligned}$$

- Regarding $(3)_1$ in the case $n = 0$, we have the same situation of the above case for its right hand side, while for its left hand side we have to take simply the derivative of (5) with respect to μ . More precisely, we obtain

for both sides the following expression

$$\begin{aligned}
 & \sum_{p_1, \dots, p_{N+2}, r_1, \dots, r_N}^{0 \dots \infty} \frac{1}{p_1!} \frac{1}{p_2!} \dots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \dots \frac{1}{r_N!} \cdot \\
 & \quad \sum_{i=1}^{N+2} i p_i + \\
 & \quad + \sum_{j=1}^N j r_j \text{ odd} \\
 & \quad \cdot \left[\sum_{i=1}^{N+2} i p_i + \sum_{j=1}^N j r_j \right] !! \cdot \\
 & \quad \cdot \frac{\partial^{2 + \sum_{i=1}^{N+2} p_i + \sum_{j=1}^N r_j}}{\partial \lambda^{\sum_{j=1}^N r_j} \partial \mu^{2 + \sum_{i=1}^{N+2} p_i}} \left[\left(\frac{-1}{2\lambda} \right)^{\frac{1}{2} [1 + \sum_{i=1}^{N+2} i p_i + \sum_{j=1}^N j r_j]} \right. \\
 & \quad \cdot \psi^{\frac{1}{2} [1 + p_1 + 2p_2 + \dots + (N+2)p_{N+2} + r_1 + 2r_2 + \dots + N r_N]} \left. \right] \cdot \\
 & \quad \cdot \delta^{(k A_{1,1} \dots A_{1,p_1} \dots A_{N+2,1} \dots A_{N+2,p_{N+2}} B_{1,1} \dots B_{1,r_1} \dots B_{N,1} \dots B_{N,r_N})} \cdot \\
 & \quad \cdot \mu_{A_{1,1}} \dots \mu_{A_{1,p_1}} \dots \mu_{A_{N+2,1}} \dots \mu_{A_{N+2,p_{N+2}}} \cdot \\
 & \quad \lambda_{B_{1,1}} \dots \lambda_{B_{1,r_1}} \dots \lambda_{B_{N,1}} \dots \lambda_{B_{N,r_N}} \cdot
 \end{aligned}$$

- It is easy to verify (3)_{2,4}.
- Let us continue verifying (3)₃ in the case $m \geq 1$.

To verify it we have to take into account that $\lambda_{i_1 \dots i_m}$ here is denoted with $\lambda_{B_{m,1}}$, or with $\lambda_{B_{m,2}}$, and so on up to $\lambda_{B_{m,r_m}}$. Similarly, $\lambda_{k i_1 \dots i_m}$ is denoted with $\lambda_{B_{m+1,1}}$, or with $\lambda_{B_{m+1,2}}$, and so on up to $\lambda_{B_{m+1,r_{m+1}}}$. Consequently, the left hand side of (3)₃ with $m \geq 1$ causes, with respect to the expression (5), a rise of one unity of the index r_m ; similarly, the right hand side of (3)₃ with $m \geq 1$ causes, with respect to the expression (4), a rise of one unity of the index r_{m+1} . More precisely, we obtain for both sides the following expression

$$\begin{aligned}
 & \sum_{p_1, \dots, p_{N+2}, r_1, \dots, r_N}^{0 \dots \infty} \frac{1}{p_1!} \frac{1}{p_2!} \dots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \dots \frac{1}{r_N!} \cdot \\
 & \quad m + \sum_{i=1}^{N+2} ip_i + \\
 & \quad + \sum_{j=1}^N jr_j \text{ odd} \\
 & \quad \cdot \left[m + \sum_{i=1}^{N+2} ip_i + \sum_{j=1}^N jr_j \right] !! \cdot \\
 & \quad \cdot \frac{\partial^{2+\sum_{i=1}^{N+2} p_i + \sum_{j=1}^N r_j}}{\partial \lambda^{1+\sum_{j=1}^N r_j} \partial \mu^{1+\sum_{i=1}^{N+2} p_i}} \left[\left(\frac{-1}{2\lambda} \right)^{\frac{1}{2} [m+1+\sum_{i=1}^{N+2} ip_i + \sum_{j=1}^N jr_j]} \right. \\
 & \quad \cdot \psi_{\frac{1}{2} [m+1+p_1+2p_2+\dots+(N+2)p_{N+2}+r_1+2r_2+\dots+Nr_N]} \\
 & \quad \cdot \delta^{(ki_1 \dots i_m A_{1,1} \dots A_{1,p_1} \dots A_{N+2,1} \dots A_{N+2,p_{N+2}} B_{1,1} \dots B_{1,r_1} \dots B_{N,1} \dots B_{N,r_N})} \cdot \\
 & \quad \cdot \mu_{A_{1,1}} \dots \mu_{A_{1,p_1}} \dots \mu_{A_{N+2,1}} \dots \mu_{A_{N+2,p_{N+2}}} \cdot \\
 & \quad \lambda_{B_{1,1}} \dots \lambda_{B_{1,r_1}} \dots \lambda_{B_{N,1}} \dots \lambda_{B_{N,r_N}} \cdot
 \end{aligned}$$

- Let us conclude verifying (3)₃ in the case $m = 0$.

We have the same situation of the above case for its right hand side, while for its left hand side we have to take simply the derivative of (5) with respect to λ . More precisely, we obtain for both sides the following expression

$$\begin{aligned}
 & \sum_{p_1, \dots, p_{N+2}, r_1, \dots, r_N}^{0 \dots \infty} \frac{1}{p_1!} \frac{1}{p_2!} \dots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \dots \frac{1}{r_N!} \cdot \\
 & \quad \sum_{i=1}^{N+2} ip_i + \\
 & \quad + \sum_{j=1}^N jr_j \text{ odd} \\
 & \quad \cdot \left[\sum_{i=1}^{N+2} ip_i + \sum_{j=1}^N jr_j \right] !! \cdot
 \end{aligned}$$

$$\begin{aligned} & \frac{\partial^{2+\sum_{i=1}^{N+2} p_i + \sum_{j=1}^N r_j}}{\partial \lambda^{1+\sum_{j=1}^N r_j} \partial \mu^{1+\sum_{i=1}^{N+2} p_i}} \left[\left(\frac{-1}{2\lambda} \right)^{\frac{1}{2} [1+\sum_{i=1}^{N+2} ip_i + \sum_{j=1}^N jr_j]} \right. \\ & \cdot \psi^{\frac{1}{2} [1+p_1+2p_2+\dots+(N+2)p_{N+2}+r_1+2r_2+\dots+Nr_N]} \cdot \\ & \cdot \delta^{(kA_{1,1} \dots A_{1,p_1} \dots A_{N+2,1} \dots A_{N+2,p_{N+2}} B_{1,1} \dots B_{1,r_1} \dots B_{N,1} \dots B_{N,r_N})} \cdot \\ & \cdot \mu_{A_{1,1}} \dots \mu_{A_{1,p_1}} \dots \mu_{A_{N+2,1}} \dots \mu_{A_{N+2,p_{N+2}}} \cdot \\ & \lambda_{B_{1,1}} \dots \lambda_{B_{1,r_1}} \dots \lambda_{B_{N,1}} \dots \lambda_{B_{N,r_N}} \cdot \end{aligned}$$

So we have finished to verify that (4), (5) is a solution of (3)₁₋₄ and there remains to prove that it satisfies also (3)₅.

4. Proof of the Solution (4), (5) – Part II

We prove now that (4), (5) is a solution of (3)₅. To this regard we note firstly that in this condition we can put under an unique summation the coefficients of $\mu\dots$, from the first coefficient of a $\lambda\dots$ we can isolate the term with $n = 1$ and in the other ones we can change the index of the summation according to the law $n = 2 + s$. In this way (3)₅ can be rewritten as

$$\begin{aligned} & \sum_{n=0}^{N+1} \frac{\partial h'}{\partial \mu_{kj_1 \dots j_n}} (n+1) \mu_{ij_1 \dots j_n} + \sum_{s=0}^{N-1} 2\lambda_{j_1 \dots j_{s+1}} \frac{\partial h'}{\partial \mu_{kj_1 \dots j_{s+1}i}} + \quad (7) \\ & + \frac{\partial h'}{\partial \mu_{ki}} 2\lambda + \sum_{s=0}^{N-1} \frac{\partial h'}{\partial \lambda_{kh_1 \dots h_s}} (s+1) \lambda_{ih_1 \dots h_s} + h' \delta^{ki} = 0. \end{aligned}$$

- Now, for the first term of this relation we can use (4) and the derivation causes a presence of the factor p_{n+1} , the substitution of $\mu_{A_{n+1,1}} \dots \mu_{A_{n+1,p_{n+1}}}$ with $\mu_{A_{n+1,1}} \dots \mu_{A_{n+1,p_{n+1}-1}}$ and the new free indexes $kj_1 \dots j_n$ in the expression of δ^{ki} ; more precisely, that term becomes

$$\sum_{n=0}^{N+1} (n+1) \mu_{ij_1 \dots j_n} \frac{\partial h'}{\partial \mu_{kj_1 \dots j_n}} = \sum_{n=0}^{N+1} (n+1) p_{n+1} \cdot \quad (8)$$

$$\begin{aligned}
 & \sum_{p_1, \dots, p_{N+2}, r_1, \dots, r_N}^{0 \dots \infty} \frac{1}{p_1!} \frac{1}{p_2!} \dots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \dots \frac{1}{r_N!} \cdot \\
 & \sum_{i=1}^{N+2} i p_i + \\
 & + \sum_{j=1}^N j r_j \text{ even} \\
 & \cdot \frac{\left[1 + \sum_{i=1}^{N+2} i p_i + \sum_{j=1}^N j r_j \right] !!}{1 + \sum_{i=1}^{N+2} i p_i + \sum_{j=1}^N j r_j} \cdot \\
 & \cdot \frac{\partial^{1 + \sum_{i=1}^{N+2} p_i + \sum_{j=1}^N r_j}}{\partial \lambda^{\sum_{j=1}^N r_j} \partial \mu^{1 + \sum_{i=1}^{N+2} p_i}} \left[\left(\frac{-1}{2\lambda} \right)^{\frac{1}{2} [\sum_{i=1}^{N+2} i p_i + \sum_{j=1}^N j r_j]} \cdot \right. \\
 & \cdot \psi_{\frac{1}{2} [p_1 + 2p_2 + \dots + (N+2)p_{N+2} + r_1 + 2r_2 + \dots + N r_N]} \left. \right] \cdot \\
 & \cdot \delta^{j_{n+1} i} \delta^{(k j_1 \dots j_n A_{1,1} \dots A_{1,p_1} \dots A_{n+1,1} \dots A_{n+1,p_{n+1}-1} \dots A_{N+2,1} \dots A_{N+2,p_{N+2}})} \\
 & \cdot B_{1,1} \dots B_{1,r_1} \dots B_{N,1} \dots B_{N,r_N} \mu_{A_{1,1}} \dots \mu_{A_{1,p_1}} \dots \mu_{A_{n+1,1}} \dots \mu_{A_{n+1,p_{n+1}-1}} \\
 & \mu_{j_1 \dots j_{n+1}} \dots \mu_{A_{N+2,1}} \dots \mu_{A_{N+2,p_{N+2}}} \cdot \\
 & \cdot \lambda_{B_{1,1}} \dots \lambda_{B_{1,r_1}} \dots \lambda_{B_{N,1}} \dots \lambda_{B_{N,r_N}},
 \end{aligned}$$

where we have also substituted $\mu_{i j_1 \dots j_n}$ with $\mu_{j_1 \dots j_{n+1}} \delta^{j_{n+1} i}$. Now in this expression we can insert a symmetrization over all the indexes of the set

$$j_1 \dots j_{n+1} A_{n+1,1} \dots A_{n+1,p_{n+1}-1}$$

because that expression remains the same if we exchange two of these indexes. This fact is evident if the two indexes are taken between $j_1 \dots j_{n+1}$; for the proof in the other cases, let us consider the shortened expression

$$\delta^{j_{n+1} i} \delta^{(k j_1 \dots j_n k_1 \dots k_n k_{n+1} \dots)} \mu_{j_1 \dots j_{n+1}} \mu_{k_1 \dots k_{n+1}};$$

here we can exchange the nomes of the indexes j . with those of the k ., so that the above shortened expression becomes

$$\delta^{k_{n+1} i} \delta^{(k k_1 \dots k_n j_1 \dots j_n j_{n+1} \dots)} \mu_{k_1 \dots k_{n+1}} \mu_{j_1 \dots j_{n+1}}.$$

Now we can exchange the indexes $k_1 \dots k_n$ with the indexes $j_1 \dots j_n$ in the expression of $\delta^{(k k_1 \dots k_n j_1 \dots j_n j_{n+1} \dots)}$ because this is a symmetric tensor.

We obtain $\delta^{k_{n+1}i} \delta^{(k_{j_1} \dots j_n k_1 \dots k_n j_{n+1} \dots)} \mu_{k_1 \dots k_{n+1}} \mu_{j_1 \dots j_{n+1}}$. By comparing this result with the expression which we started from, it is the same if he had exchanged the indexes j_{n+1} and k_{n+1} . This completes the proof of the fact that the expression (8) remains the same if we exchange two indexes of the set $j_1 \dots j_{n+1} A_{n+1,1} \dots A_{n+1,p_{n+1}-1}$; so we can insert there a symmetrization over those indexes and (8) becomes

$$\begin{aligned}
 \sum_{n=0}^{N+1} (n+1) \mu_{i_{j_1} \dots j_n} \frac{\partial h'}{\partial \mu_{k_{j_1} \dots j_n}} &= \sum_{n=0}^{N+1} (n+1) p_{n+1} \cdot \quad (9) \\
 \sum_{0 \dots \infty} & \frac{1}{p_1!} \frac{1}{p_2!} \dots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \dots \frac{1}{r_N!} \cdot \\
 p_1, \dots, p_{N+2}, r_1, \dots, r_N & \\
 \sum_{i=1}^{N+2} i p_i + & \\
 + \sum_{j=1}^N j r_j \text{ even} & \\
 \frac{\left[1 + \sum_{i=1}^{N+2} i p_i + \sum_{j=1}^N j r_j \right]!!}{1 + \sum_{i=1}^{N+2} i p_i + \sum_{j=1}^N j r_j} \cdot & \\
 \frac{\partial^{1 + \sum_{i=1}^{N+2} p_i + \sum_{j=1}^N r_j}}{\partial \lambda^{\sum_{j=1}^N r_j} \partial \mu^{1 + \sum_{i=1}^{N+2} p_i}} \left[\left(\frac{-1}{2\lambda} \right)^{\frac{1}{2} [\sum_{i=1}^{N+2} i p_i + \sum_{j=1}^N j r_j]} \right. & \\
 \cdot \psi_{\frac{1}{2} [p_1 + 2p_2 + \dots + (N+2)p_{N+2} + r_1 + 2r_2 + \dots + N r_N]} & \\
 \cdot \underline{\delta^{j_{n+1}i}} \delta^{(\underline{k_{j_1} \dots j_n A_{1,1} \dots A_{1,p_1} \dots A_{n+1,1} \dots A_{n+1,p_{n+1}-1} \dots A_{N+2,1} \dots A_{N+2,p_{N+2}}} & \\
 \underline{B_{1,1} \dots B_{1,r_1} \dots B_{N,1} \dots B_{N,r_N}})} \mu_{A_{1,1}} \dots \mu_{A_{1,p_1}} \dots \mu_{A_{n+1,1}} \dots \mu_{A_{n+1,p_{n+1}-1}} & \\
 \mu_{j_1 \dots j_{n+1}} \dots \mu_{A_{N+2,1}} \dots \mu_{A_{N+2,p_{N+2}}} \cdot & \\
 \cdot \lambda_{B_{1,1}} \dots \lambda_{B_{1,r_1}} \dots \lambda_{B_{N,1}} \dots \lambda_{B_{N,r_N}}, &
 \end{aligned}$$

where underlined indexes denote symmetrization over these indexes. Now we observe that $(n+1)p_{n+1}$ is exactly the number of the indexes of the set $j_1 \dots j_{n+1} A_{n+1,1} \dots A_{n+1,p_{n+1}-1}$ and that, thanks to the summation $\sum_{n=0}^{N+1}$ the index near i in $\delta^{j_{n+1}i}$ can be every index of the set

$$A_{1,1} \dots A_{1,p_1} \dots A_{n+1,1} \dots A_{n+1,p_{n+1}} \dots A_{N+2,1} \dots A_{N+2,p_{N+2}}.$$

These facts allow to rewrite (9) as

$$\begin{aligned}
 \sum_{n=0}^{N+1} (n+1) \mu_{ij_1 \dots j_n} \frac{\partial h'}{\partial \mu_{kj_1 \dots j_n}} &= \left[\sum_{n=0}^{N+1} (n+1) p_{n+1} \right] \cdot \tag{10} \\
 &\sum_{p_1, \dots, p_{N+2}, r_1, \dots, r_N}^{0 \dots \infty} \frac{1}{p_1!} \frac{1}{p_2!} \dots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \dots \frac{1}{r_N!} \cdot \\
 &\frac{\sum_{i=1}^{N+2} i p_i + \sum_{j=1}^N j r_j \text{ even}}{\left[1 + \sum_{i=1}^{N+2} i p_i + \sum_{j=1}^N j r_j \right]!!} \cdot \\
 &\frac{\partial^{1+\sum_{i=1}^{N+2} p_i + \sum_{j=1}^N r_j}}{\partial \lambda^{\sum_{j=1}^N r_j} \partial \mu^{1+\sum_{i=1}^{N+2} p_i}} \left[\left(\frac{-1}{2\lambda} \right)^{\frac{1}{2} [\sum_{i=1}^{N+2} i p_i + \sum_{j=1}^N j r_j]} \right. \\
 &\cdot \psi_{\frac{1}{2} [p_1 + 2p_2 + \dots + (N+2)p_{N+2} + r_1 + 2r_2 + \dots + N r_N]} \\
 &\cdot \underline{\delta^{j_{n+1} i} \delta^{(kj_1 \dots j_n A_{1,1} \dots A_{1,p_1} \dots A_{n+1,1} \dots A_{n+1,p_{n+1}-1} \dots A_{N+2,1} \dots A_{N+2,p_{N+2}})}} \\
 &B_{1,1} \dots B_{1,r_1} \dots B_{N,1} \dots B_{N,r_N} \mu_{A_{1,1}} \dots \mu_{A_{1,p_1}} \dots \mu_{A_{n+1,1}} \dots \mu_{A_{n+1,p_{n+1}-1}} \\
 &\mu_{j_1} \dots \mu_{j_{n+1}} \dots \mu_{A_{N+2,1}} \dots \mu_{A_{N+2,p_{N+2}}} \cdot \\
 &\cdot \lambda_{B_{1,1}} \dots \lambda_{B_{1,r_1}} \dots \lambda_{B_{N,1}} \dots \lambda_{B_{N,r_N}},
 \end{aligned}$$

- For the fourth term of (7) we can do similar passages (the difference is that we have the $\lambda \dots$ instead of the $\mu \dots$, $N - 2$ instead of N and s instead of n); in this way that term becomes

$$\sum_{s=0}^{N-1} (s+1) \lambda_{ih_1 \dots h_s} \frac{\partial h'}{\partial \lambda_{kh_1 \dots h_s}} = \left[\sum_{s=0}^{N-1} (s+1) r_{s+1} \right] \cdot \tag{11}$$

$$\begin{aligned}
 & \sum_{p_1, \dots, p_{N+2}, r_1, \dots, r_N}^{0 \dots \infty} \frac{1}{p_1!} \frac{1}{p_2!} \dots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \dots \frac{1}{r_N!} \cdot \\
 & \sum_{i=1}^{N+2} ip_i + \\
 & + \sum_{j=1}^N jr_j \text{ even} \\
 & \cdot \frac{\left[1 + \sum_{i=1}^{N+2} ip_i + \sum_{j=1}^N jr_j \right]!!}{1 + \sum_{i=1}^{N+2} ip_i + \sum_{j=1}^N jr_j} \cdot \\
 & \cdot \frac{\partial^{1 + \sum_{i=1}^{N+2} p_i + \sum_{j=1}^N r_j}}{\partial \lambda^{\sum_{j=1}^N r_j} \partial \mu^{1 + \sum_{i=1}^{N+2} p_i}} \left[\left(\frac{-1}{2\lambda} \right)^{\frac{1}{2} [\sum_{i=1}^{N+2} ip_i + \sum_{j=1}^N jr_j]} \right. \\
 & \cdot \psi_{\frac{1}{2} [p_1 + 2p_2 + \dots + (N+2)p_{N+2} + r_1 + 2r_2 + \dots + Nr_N]} \left. \right] \cdot \\
 & \cdot \underline{\delta^{h_{s+1} i}} \delta^{(kh_1 \dots h_s A_{1,1} \dots A_{1,p_1} \dots A_{N+2,1} \dots A_{N+2,p_{N+2}})} \\
 & \cdot \underline{B_{1,1} \dots B_{1,r_1} \dots B_{s+1,1} \dots B_{s+1,r_{s+1}-1} \dots B_{N,1} \dots B_{N,r_N}} \cdot \\
 & \cdot \mu_{A_{1,1}} \dots \mu_{A_{1,p_1}} \dots \mu_{A_{N+2,1}} \dots \mu_{A_{N+2,p_{N+2}}} \cdot \\
 & \cdot \lambda_{B_{1,1}} \dots \lambda_{B_{1,r_1}} \dots \lambda_{B_{s+1,1}} \dots \lambda_{B_{s+1,r_{s+1}-1}} \dots \lambda_{B_{N,1}} \dots \lambda_{B_{N,r_N}} \cdot
 \end{aligned}$$

- If we look at the last term of (7), we see that it can be written together with (10) and (11) and they become

$$\begin{aligned}
 & \sum_{n=0}^{N+1} \frac{\partial h'}{\partial \mu_{kj_1 \dots j_n}} (n+1) \mu_{ij_1 \dots j_n} + \sum_{s=0}^{N-1} \frac{\partial h'}{\partial \lambda_{kh_1 \dots h_s}} (s+1) \lambda_{ih_1 \dots h_s} + \\
 & + h' \delta^{ki} = \left[1 + \sum_{n=0}^{N+1} (n+1) p_{n+1} + \sum_{s=0}^{N-1} (s+1) r_{s+1} \right] \cdot
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{p_1, \dots, p_{N+2}, r_1, \dots, r_N}^{0 \dots \infty} \frac{1}{p_1!} \frac{1}{p_2!} \dots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \dots \frac{1}{r_N!} \cdot \\
 & \sum_{i=1}^{N+2} ip_i + \\
 & + \sum_{j=1}^N jr_j \text{ even}
 \end{aligned}$$

$$\begin{aligned}
 & \frac{\left[1 + \sum_{i=1}^{N+2} ip_i + \sum_{j=1}^N jr_j\right]!!}{1 + \sum_{i=1}^{N+2} ip_i + \sum_{j=1}^N jr_j} \cdot \\
 & \frac{\partial^{1+\sum_{i=1}^{N+2} p_i + \sum_{j=1}^N r_j}}{\partial \lambda^{\sum_{j=1}^N r_j} \partial \mu^{1+\sum_{i=1}^{N+2} p_i}} \left[\left(\frac{-1}{2\lambda}\right)^{\frac{1}{2}[\sum_{i=1}^{N+2} ip_i + \sum_{j=1}^N jr_j]} \cdot \right. \\
 & \left. \cdot \psi_{\frac{1}{2}[p_1+2p_2+\dots+(N+2)p_{N+2}+r_1+2r_2+\dots+Nr_N]} \right] \cdot \\
 & \cdot \delta^{ki} \delta^{\underline{(A_{1,1} \dots A_{1,p_1} \dots A_{N+2,1} \dots A_{N+2,p_{N+2}} B_{1,1} \dots B_{1,r_1} \dots B_{N,1} \dots B_{N,r_N})}} \cdot \\
 & \cdot \mu_{A_{1,1}} \dots \mu_{A_{1,p_1}} \dots \mu_{A_{N+2,1}} \dots \mu_{A_{N+2,p_{N+2}}} \cdot \\
 & \lambda_{B_{1,1}} \dots \lambda_{B_{1,r_1}} \dots \lambda_{B_{N,1}} \dots \lambda_{B_{N,r_N}}, \tag{12}
 \end{aligned}$$

where also the index k has been put under the symmetrization and we recall for the sequel is that a property of symmetrization is $\delta^{ik} \delta^{\underline{(\dots)}} = \delta^{ik} \delta^{\dots} = \delta^{(ik \dots)}$. It is interesting to see that the coefficient in square bracket at the beginning of the right hand side of eq. (12) has become equal to the denominator of the half factorial of that right hand side!

- For the second term of (7), we can use (4) and the derivation causes someway the rising of one unity of the index p_2 ; more precisely we have

$$\begin{aligned}
 \frac{\partial h'}{\partial \mu_{ki}} 2\lambda &= 2\lambda \sum_{\substack{0 \dots \infty \\ p_1, \dots, p_{N+2}, r_1, \dots, r_N}} \frac{1}{p_1!} \dots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \dots \frac{1}{r_N!} \cdot \\
 & \sum_{i=1}^{N+2} ip_i + \\
 & + \sum_{j=1}^N jr_j \text{ even} \\
 & \cdot \left[1 + \sum_{i=1}^{N+2} ip_i + \sum_{j=1}^N jr_j \right] !! \cdot \\
 & \frac{\partial^{2+\sum_{i=1}^{N+2} p_i + \sum_{j=1}^N r_j}}{\partial \lambda^{\sum_{j=1}^N r_j} \partial \mu^{2+\sum_{i=1}^{N+2} p_i}} \left[\left(\frac{-1}{2\lambda}\right)^{\frac{1}{2}[2+\sum_{i=1}^{N+2} ip_i + \sum_{j=1}^N jr_j]} \right] \cdot
 \end{aligned}$$

$$\begin{aligned}
 & \cdot \psi_{\frac{1}{2}} [2+p_1+2p_2+\dots+(N+2)p_{N+2}+r_1+2r_2+\dots+Nr_N] \Big] \cdot \\
 & \cdot \delta^{(kiA_{1,1}\dots A_{1,p_1}\dots A_{N+2,1}\dots A_{N+2,p_{N+2}} B_{1,1}\dots B_{1,r_1}\dots B_{N,1}\dots B_{N,r_N})} \cdot \\
 & \cdot \mu_{A_{1,1}} \dots \mu_{A_{1,p_1}} \dots \mu_{A_{N+2,1}} \dots \mu_{A_{N+2,p_{N+2}}} \cdot \\
 & \lambda_{B_{1,1}} \dots \lambda_{B_{1,r_1}} \dots \lambda_{B_{N,1}} \dots \lambda_{B_{N,r_N}} \cdot
 \end{aligned} \tag{13}$$

- There remains to consider the third term of (7); for it we can use (4) and the derivation causes somehow the rising of one unity of the index p_{s+3} ; more precisely we have

$$\begin{aligned}
 \sum_{s=0}^{N-1} 2\lambda_{j_1\dots j_{s+1}} \frac{\partial h'}{\partial \mu_{kj_1\dots j_{s+1}i}} &= \sum_{s=0}^{N-1} 2\lambda_{j_1\dots j_{s+1}} \cdot \\
 \sum_{p_1, \dots, p_{N+2}, r_1, \dots, r_N}^{0\dots\infty} & \frac{1}{p_1!} \frac{1}{p_2!} \dots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \dots \frac{1}{r_N!} \cdot \\
 s+3+\sum_{i=1}^{N+2} ip_i+ & + \sum_{j=1}^N jr_j \text{ even} \\
 \frac{[s+4+\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^N jr_j]!!}{s+4+\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^N jr_j} & \cdot \\
 \frac{\partial^{2+\sum_{i=1}^{N+2} p_i+\sum_{j=1}^N r_j}}{\partial \lambda^{\sum_{j=1}^N r_j} \partial \mu^{2+\sum_{i=1}^{N+2} p_i}} \left[\left(\frac{-1}{2\lambda} \right)^{\frac{1}{2}[s+3+\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^N jr_j]} \right] & \cdot \\
 \cdot \psi_{\frac{1}{2}} [s+3+p_1+2p_2+\dots+(N+2)p_{N+2}+r_1+2r_2+\dots+Nr_N] \Big] \cdot & \\
 \cdot \delta^{(kj_1\dots j_{s+1}iA_{1,1}\dots A_{1,p_1}\dots A_{N+2,1}\dots A_{N+2,p_{N+2}} B_{1,1}\dots B_{1,r_1}\dots B_{N,1}\dots B_{N,r_N})} \cdot & \\
 \cdot \mu_{A_{1,1}} \dots \mu_{A_{1,p_1}} \dots \mu_{A_{N+2,1}} \dots \mu_{A_{N+2,p_{N+2}}} \cdot & \\
 \lambda_{B_{1,1}} \dots \lambda_{B_{1,r_1}} \dots \lambda_{B_{N,1}} \dots \lambda_{B_{N,r_N}} \cdot &
 \end{aligned} \tag{14}$$

In this expression, we can substitute $\lambda_{j_1 \dots j_{s+1}}$ with $\lambda_{B_{s+1}, r_{s+1}+1}$ and $j_1 \dots j_{s+1}$ with $B_{s+1}, r_{s+1}+1$; moreover, we substitute $\frac{1}{r_{s+1}!}$ with $\frac{r_{s+1}+1}{(r_{s+1}+1)!}$ and, after that, decrease r_{s+1} of one unity (which is equivalent to a change of index). So this expression becomes

$$\sum_{s=0}^{N-1} 2\lambda_{j_1 \dots j_{s+1}} \frac{\partial h'}{\partial \mu_{k j_1 \dots j_{s+1} i}} = \sum_{s=0}^{N-1} 2r_{s+1} \cdot \tag{15}$$

$$\sum_{p_1, \dots, p_{N+2}, r_1, \dots, r_N}^{0 \dots \infty} \frac{1}{p_1!} \frac{1}{p_2!} \dots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \dots \frac{1}{r_N!} \cdot$$

$$2 + \sum_{i=1}^{N+2} i p_i + \sum_{j=1}^N j r_j \text{ even}$$

$$\cdot \frac{[3 + \sum_{i=1}^{N+2} i p_i + \sum_{j=1}^N j r_j]!!}{3 + \sum_{i=1}^{N+2} i p_i + \sum_{j=1}^N j r_j} \cdot$$

$$\cdot \frac{\partial^{1+\sum_{i=1}^{N+2} p_i + \sum_{j=1}^N r_j}}{\partial \lambda^{-1+\sum_{j=1}^N r_j} \partial \mu^{2+\sum_{i=1}^{N+2} p_i}} \left[\left(\frac{-1}{2\lambda} \right)^{\frac{1}{2}[2+\sum_{i=1}^{N+2} i p_i + \sum_{j=1}^N j r_j]} \right]$$

$$\cdot \psi_{\frac{1}{2}}[2+p_1+2p_2+\dots+(N+2)p_{N+2}+r_1+2r_2+\dots+Nr_N]$$

$$\cdot \delta^{(k i A_{1,1} \dots A_{1,p_1} \dots A_{N+2,1} \dots A_{N+2,p_{N+2}} B_{1,1} \dots B_{1,r_1} \dots B_{N,1} \dots B_{N,r_N})}$$

$$\cdot \mu_{A_{1,1}} \dots \mu_{A_{1,p_1}} \dots \mu_{A_{N+2,1}} \dots \mu_{A_{N+2,p_{N+2}}}$$

$$\lambda_{B_{1,1}} \dots \lambda_{B_{1,r_1}} \dots \lambda_{B_{N,1}} \dots \lambda_{B_{N,r_N}}$$

It is true that with change of index we have in the summation the extra term with $r_{s+1} = 0$; but it doesn't effect the result for the presence of the coefficient r_{s+1} .

If we look at the expressions (12), (13) and (15) we conclude that to prove (7) it is sufficient that the following relation holds

$$\frac{\partial^{1+\sum_{i=1}^{N+2} p_i + \sum_{j=1}^N r_j}}{\partial \lambda^{\sum_{j=1}^N r_j} \partial \mu^{1+\sum_{i=1}^{N+2} p_i}} \left[\left(\frac{-1}{2\lambda} \right)^{\frac{1}{2}[\sum_{i=1}^{N+2} i p_i + \sum_{j=1}^N j r_j]} \right]$$

$$\cdot \psi_{\frac{1}{2}}[p_1+2p_2+\dots+(N+2)p_{N+2}+r_1+2r_2+\dots+Nr_N] +$$

$$\begin{aligned}
 & +2\lambda \frac{\partial^{2+\sum_{i=1}^{N+2} p_i + \sum_{j=1}^N r_j}}{\partial \lambda^{\sum_{j=1}^N r_j} \partial \mu^{2+\sum_{i=1}^{N+2} p_i}} \left[\left(\frac{-1}{2\lambda} \right)^{\frac{1}{2}[2+\sum_{i=1}^{N+2} ip_i + \sum_{j=1}^N jr_j]} \right. \\
 & \qquad \qquad \qquad \cdot \psi_{\frac{1}{2}[2+p_1+2p_2+\dots+(N+2)p_{N+2}+r_1+2r_2+\dots+Nr_N]} \left. \right] + \\
 & + \sum_{s=0}^{N-1} 2r_{s+1} \frac{\partial^{1+\sum_{i=1}^{N+2} p_i + \sum_{j=1}^N r_j}}{\partial \lambda^{-1+\sum_{j=1}^N r_j} \partial \mu^{2+\sum_{i=1}^{N+2} p_i}} \left[\left(\frac{-1}{2\lambda} \right)^{\frac{1}{2}[2+\sum_{i=1}^{N+2} ip_i + \sum_{j=1}^N jr_j]} \right. \\
 & \qquad \qquad \qquad \cdot \psi_{\frac{1}{2}[2+p_1+2p_2+\dots+(N+2)p_{N+2}+r_1+2r_2+\dots+Nr_N]} \left. \right] = 0.
 \end{aligned}$$

Now, for the last two terms of this expression we can use (6), after that only the function

$\psi_{\frac{1}{2}[p_1+2p_2+\dots+(N+2)p_{N+2}+r_1+2r_2+\dots+Nr_N]}$ will be present there and to prove that relation it will be sufficient to prove that

$$\frac{\partial^r}{\partial \lambda^r} \vartheta + 2\lambda \frac{\partial^r}{\partial \lambda^r} \left(\frac{-1}{2\lambda} \vartheta \right) + 2r \frac{\partial^{-1+r}}{\partial \lambda^{-1+r}} \left(\frac{-1}{2\lambda} \vartheta \right) = 0, \tag{16}$$

where we have put $r = r_1 + r_2 + \dots + r_N$ and

$$\vartheta = \left[\left(\frac{-1}{2\lambda} \right)^{\frac{1}{2}[p_1+2p_2+\dots+(N+2)p_{N+2}+r_1+2r_2+\dots+Nr_N]} \cdot \psi_{\frac{1}{2}[p_1+2p_2+\dots+(N+2)p_{N+2}+r_1+2r_2+\dots+Nr_N]} \right].$$

Obviously, eq. (16) is an identity because we have

$$\frac{\partial^r}{\partial \lambda^r} \vartheta = \frac{\partial^r}{\partial \lambda^r} \left(-2\lambda \frac{-1}{2\lambda} \vartheta \right) = -2\lambda \frac{\partial^r}{\partial \lambda^r} \left(\frac{-1}{2\lambda} \vartheta \right) - 2r \frac{\partial^{-1+r}}{\partial \lambda^{-1+r}} \left(\frac{-1}{2\lambda} \vartheta \right)$$

so we have finished all the arguments which we had to prove.

Conclusions

We observe that in the particular case $N = 0$, we obtain the 11 moments model and the solution here indicated is exactly that already obtained in literature with the macroscopic approach. In the case $N = 1$ we have that the present one is a particular solution of those already known in literature. Aim of a future research is to find the explicit expression of the general solution.

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