Bounds for blow-up time in nonlinear parabolic systems under various boundary conditions.

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Abstract. We consider blow-up solutions to parabolic systems, coupled through their non linearities under various boundary conditions with nonlinearities depending on the gradient solution. To obtain a lower bound to blow up time t^* for the vector solution, Sobolev-type inequalities are introduced to make use of a differential inequality technique. In addition for Dirichlet systems sufficient conditions are introduced to derive an upper bound for t^* and to have a criterion for the global existence of the vector solution.

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1 Introduction

For the qualitative study of solutions to parabolic equations and systems which solutions present the blow-up phenomena, we refer to the book of Quittner-Souplet [8], where one can find also various reaction-diffusion systems arising as mathematical models in chemistry, physics and population dynamics.
For results about these systems with nonlinearities depending on the solution there is a list of recent papers, we cite only [5], [6] and [7] and the references therein. Interesting results for the blow-up time for parabolic problems with nonlinearities depending on gradient solution are present in [1] and in [2].
When the solution blows up at some finite time t^*, t^* cannot in general be determined explicitly. In this paper we consider nonlinear parabolic systems

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with different conditions on the boundary, coupled through the nonlinearities depending on the gradient of the solutions. Our interest is mostly addressed to obtain explicit lower bounds for t^*, since in practical situations upper bounds are less interesting. Among the papers firstly appeared on lower bounds, we cite [10] and [11]. In deriving such lower bounds we introduce Sobolev-type inequalities which produce the restriction $\Omega \in \mathbb{R}^3$.
In sec.2 a Dirichlet system is considered and conditions on data are derived to obtain a lower bound for t^* , a criterion to exclude the blow up phenomena and an upper bound for t^*. A lower bound for Neumann and Robin systems are also derived in sec.3 and sec.4 if the spatial domain Ω is convex.

2 Bounds for blow-up time in Dirichlet systems

We consider the following initial-boundary value problem

$$
(2.1) \qquad \begin{cases} \Delta u + f_1(|\nabla v|) = u_t & \text{in } \Omega \times (0, t^*), \\ \Delta v + f_2(|\nabla u|) = v_t & \text{in } \Omega \times (0, t^*), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0, t^*), \\ v(x,t) = 0 & \text{on } \partial\Omega \times (0, t^*), \\ u(x,0) = u_0(x) & \text{on } \Omega, \\ v(x,0) = v_0(x) & \text{on } \Omega, \end{cases}
$$

where Ω is a bounded domain in $\mathbb{R}^3$, with smooth boundary $\partial\Omega$, t^* is the blow-up time and f_1, f_2 are non negative functions. Through the paper we will refer to system (2.1) with these assumptions on Ω, $\partial\Omega$, f_1, f_2, changing only the boundary and the compatibility conditions. For Dirichlet system we assume that the functions u_0, v_0 are non negative in Ω with $u_0(x) = v_0(x) = 0$ on $\partial\Omega$.
In this section we derive upper and lower bounds for the blow-up time under restrictions on nonlinearities and initial data.
Moreover we prove that if the initial data are "small enough" the blow-up phenomena cannot occur.

Lower bound

For the vector solution of system (2.1) which becomes unbounded at some finite time t^* , we define the auxiliary function

$$\Phi(t) := \int_\Omega (u^2+v^2)^{2p}dx, \quad p>1. \tag{2.2}$$

We prove the following result:

Theorem 2.1 *If the vector solution (u,v) of (2.1) becomes unbounded in the norm Φ at some finite time t^*, and if*

$$uf_1(|\nabla v|)+vf_2(|\nabla u|) \le K(u^2+v^2)^{(1-\beta)(p+1)}(|\nabla u|^2+|\nabla v|^2)^\beta, \tag{2.3}$$

for positive constants K,β, with $0<\beta<1$ and

$$1-K\beta \ge 0, \tag{2.4}$$

then t^* is bounded from below by

$$t^* \ge \frac{1}{2C_1\Phi^2(0)}, \tag{2.5}$$

with C_1 a positive constant and

$$\Phi(0) := \int_\Omega (u_0^2(x)+v_0^2(x))^{2p}dx. \tag{2.6}$$

Proof.

We compute

$$\Phi'(t) = 4p\int_\Omega (u^2+v^2)^{2p-1}(uu_t+vv_t)dx = 4p\int_\Omega (u^2+v^2)^{2p-1}(u\Delta u+v\Delta v)dx+$$
$$4p\int_\Omega (u^2+v^2)^{2p-1}(u\ f_1(|\nabla v|)+v\ f_2(|\nabla u|))dx.$$

By the divergence theorem and the boundary conditions in (2.1) we obtain

$$\Phi'(t) = -8p(2p-1)\int_\Omega (u^2+v^2)^{2p-2}(u\nabla u+v\nabla v)(u\nabla u+v\nabla v)dx- \tag{2.7}$$

$$4p\int_\Omega (u^2+v^2)^{2p-1}(|\nabla u|^2+|\nabla v|^2)dx + 4p\int_\Omega (u^2+v^2)^{2p-1}(uf_1+vf_2)dx.$$

Now, by inserting (2.3) in (2.7) we obtain

$$\Phi'(t) \le -8p(2p-1)\int_\Omega (u^2+v^2)^{2p-2}(u\nabla u+v\nabla v)(u\nabla u+v\nabla v)dx \tag{2.8}$$
$$-4p\int_\Omega (u^2+v^2)^{2p-1}(|\nabla u|^2+|\nabla v|^2)dx$$
$$+4pK\int_\Omega (u^2+v^2)^{2p-1}\Big[(u^2+v^2)^{p+1}\Big]^{1-\beta}\Big[|\nabla u|^2+|\nabla v|^2\Big]^{\beta}dx.$$

Following Theo.1 in [7], we use in the last term of (2.8) the basic inequality

$$a^r\, b^s \le ra+sb, \quad r+s=1, \quad a,b>0. \tag{2.9}$$

We obtain

$$\int_\Omega (u^2+v^2)^{2p-1}\Big[\Big(1-\beta\Big)(u^2+v^2)^{p+1}+\beta\Big(|\nabla u|^2+|\nabla v|^2\Big)\Big]dx = \tag{2.10}$$
$$\Big(1-\beta\Big)\int_\Omega (u^2+v^2)^{3p}dx+\beta\int_\Omega (u^2+v^2)^{2p-1}\Big(|\nabla u|^2+|\nabla v|^2\Big)\Big]dx.$$

By replacing (2.10)in (2.8) we have

$$\Phi'(t) \le -8p(2p-1)\int_\Omega (u^2+v^2)^{2p-2}(u\nabla u+v\nabla v)(u\nabla u+v\nabla v)dx+ \tag{2.11}$$

$$4p(K\beta-1)\int_\Omega (u^2+v^2)^{2p-1}(|\nabla u|^2+|\nabla v|^2)dx+4pK(1-\beta)\int_\Omega (u^2+v^2)^{3p}dx.$$

The inequality

$$(u^2+v^2)(|\nabla u|^2+|\nabla v|^2) \ge (u\nabla u+v\nabla v)(u\nabla u+v\nabla v) \tag{2.12}$$

can be used in (2.11), since (2.4) holds. It follows that

(2.13)

$$\Phi'(t) \le -4p^2\Big(4-\frac{1}{p}-\beta\frac{K}{p}\Big)\int_\Omega (u^2+v^2)^{2p-2}(u\nabla u+v\nabla v)(u\nabla u+v\nabla v)dx$$

$$+4pK(1-\beta)\int_\Omega (u^2+v^2)^{3p}dx.$$

For simplicity we now put

$$w=(u^2+v^2)^p \tag{2.14}$$

and compute

$$|\nabla w|^2 = 4p^2(u^2+v^2)^{2p-2}(u\nabla u+v\nabla v)(u\nabla u+v\nabla v). \tag{2.15}$$

We now replace (2.14) and (2.15) into (2.13) to write

$$\Phi'(t)\le -\Big(4-\frac{1}{p}-\beta\frac{K}{p}\Big)\int_\Omega |\nabla w|^2dx+4pK(1-\beta)\int_\Omega w^3dx. \tag{2.16}$$

In the last term we use Schwarz's inequality:

$$\int_\Omega w^3dx\le\Big(\int_\Omega w^2dx\Big)^{\frac12}\Big(\int_\Omega w^4dx\Big)^{\frac12}\le \tag{2.17}$$

$$\Big(\int_\Omega w^2dx\Big)^{\frac12}\Big(\int_\Omega w^2dx\Big)^{\frac14}\Big(\int_\Omega w^6dx\Big)^{\frac14}$$

Now to estimate $\int_\Omega w^6dx$ we use the Sobolev-Talenti inequality (see [9] with $q=6, p=2, m=3$) and we get for $\Omega\in\mathbb{R}^3$

$$\Big(\int_\Omega w^6dx\Big)^{\frac14}\le C_T\Big(\int_\Omega |\nabla w|^2dx\Big)^{\frac34} \tag{2.18}$$

where the best Sobolev constant $C_T=2\pi^{-1}3^{-\frac34}$.
Since $0<\beta<1$, we can replace (2.17) ,(2.18) in (2.16) and write with $k_1:=\Big(4-\frac{1}{p}-\beta\frac{K}{p}\Big)$ and $k_2:=pK(1-\beta)$

$$\Phi'(t)\le -k_1\int_\Omega |\nabla w|^2dx+4k_2C_T\Big(\int_\Omega w^2dx\Big)^{\frac34}\Big(\int_\Omega |\nabla w|^2dx\Big)^{\frac34}= \tag{2.19}$$

$$-k_1\int_\Omega |\nabla w|^2dx+4k_2C_T\Big[\Big(\int_\Omega w^2dx\Big)^3\Big]^{\frac14}\Big(\int_\Omega |\nabla w|^2dx\Big)^{\frac34}.$$

If τ is an arbitrary positive constant , we write

$$(2.20)\quad \Phi'(t) \le -k_1 \int_\Omega |\nabla w|^2 dx + 4k_2 C_T \Big[\frac{1}{\tau^3}\Big(\int_\Omega w^2 dx\Big)^3\Big]^{\frac{1}{4}} \Big(\tau \int_\Omega |\nabla w|^2 dx\Big)^{\frac{3}{4}}$$

and then use the basic inequality (2.9). It follows that

$$(2.21)\quad \Phi'(t) \le -k_1 \int_\Omega |\nabla w|^2 dx + \frac{k_2}{\tau^3} C_T \Big(\int_\Omega w^2 dx\Big)^3 + 3k_2 \tau C_T \int_\Omega |\nabla w|^2 dx$$

$$= (3k_2 \tau C_T - k_1) \int_\Omega |\nabla w|^2 dx + \frac{k_2}{\tau^3} C_T \Big(\int_\Omega w^2 dx\Big)^3$$

and choosing in (2.21)

$$\tau = \frac{k_1}{3k_2 C_T}$$

we obtain

$$(2.22)\qquad \Phi'(t) \le C_1 \Phi(t)^3,$$

with

$$C_1 = \Big(\frac{3}{k_1}\Big)^3 (k_2 C_T)^4.$$

An integration of (2.22) from 0 to t results in

$$\frac{1}{\Phi(0)^2} - \frac{1}{\Phi(t)^2} \le 2C_1 t, \quad t \in (0, t^*)$$

and implies that if the solution blows up at time t^*, then

$$t^* \ge \frac{1}{2C_1[\Phi(0)]^2},$$

with $\Phi(0)$ in (2.6).

Criterion for Global Existence.

In this subsection we introduce restrictions on initial data which, together with (2.3), imply that blow up cannot occur.

Theorem 2.2 *Let (u, v) be the vector solution of problem (2.1). Suppose (2.3) holds. If the initial data $u_0(x), v_0(x)$ satisfy the condition*

$$\int_\Omega [u_0^2 + v_0^2]^{2p} dx \leq \left[\frac{k_1}{4k_2 C_T}\right]^2 \sqrt{\lambda_1}, \tag{2.23}$$

with k_1, k_2, C_T as in Theo.2.1 and λ_1 the first positive eigenvalue of the fixed membrane problem

$$\Delta\psi + \lambda_1\psi = 0, \quad \psi > 0, \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial\Omega, \tag{2.24}$$

then the vector solution (u, v), assumed to exist, cannot blow up in finite time.

Proof.
Following the proof of Theo 2.1, with Φ defined in (2.2) and the function w defined in (2.14), (2.19) can be rewritten as

$$\Phi'(t) \leq \Big(\int_\Omega |\nabla w|^2 dx\Big)^{\frac{3}{4}} \Big\{-k_1\Big(\int_\Omega |\nabla w|^2 dx\Big)^{\frac{1}{4}} + 4k_2 C_T\Big(\int_\Omega w^2 dx\Big)^{\frac{3}{4}}\Big\}. \tag{2.25}$$

By Rayleigh inequality for λ_1

$$\lambda_1 \int_\Omega w^2 dx \leq \int_\Omega |\nabla w|^2 dx, \tag{2.26}$$

replaced in (2.25) we have

$$\Phi'(t) \leq \Big(\int_\Omega |\nabla w|^2 dx\Big)^{\frac{3}{4}} \Big(\int_\Omega w^2 dx\Big)^{\frac{1}{4}} \Big\{-k_1\lambda_1^{\frac{1}{4}} + 4k_2 C_T\Big(\int_\Omega w^2 dx\Big)^{\frac{1}{2}}\Big\} \tag{2.27}$$

By (2.23) and (2.27) we deduce that

$$\int_\Omega w^2 dx \leq \left[\frac{k_1}{4k_2 C_T}\right]^2 \sqrt{\lambda_1}.$$

Then Φ is nonincreasing and is bounded for all time. The vector solution (u, v) of (2.1) is then bounded in Φ norm for all $t > 0$.

Upper bound for the blow up time.

Under suitable conditions on nonlinearities, we derive now an upper bound for t^*.

Theorem 2.3 *Let (u, v) be the vector solution of problem (2.1). Assume that for some positive constant c*

$$f_1(|\nabla v|) + f_2(|\nabla u|) \geq c\Big(|\nabla u| + |\nabla v|\Big)^4, \tag{2.28}$$

and assume that the initial data u_0, v_0 are large enough in the following sense:

$$\int_\Omega (u_0 + v_0)^3 dx > \frac{9|\Omega|}{2c\lambda_1}, \tag{2.29}$$

$|\Omega|$ is the volume of the domain and λ_1 is the first eigenvalue in problem (2.24). Then

$$\max_{x\in\Omega}(u, v) \to +\infty$$

as $t \to t^*$ with

$$t^* \leq T^* := -\frac{3}{8\lambda_1} ln\Big[1 - \Big(\frac{9|\Omega|}{2c\lambda_1\Psi_0}\Big)\Big]$$

and $\Psi_0 = \int_\Omega (u_0 + v_0)dx$.

Proof.

We add the two equations in (2.1) and by (2.28) we have $(\Delta u - u_t) + (\Delta v - v_t) = -f_1(|\nabla v|) - f_2(|\nabla u|) \leq -c\Big(|\nabla u| + |\nabla v|\Big)^4 \leq -c|\nabla(u + v)|^4$.

We define $\tilde{w} = u + v$.

The function $\tilde{w}$ satisfies

$$\Delta\tilde{w} - \tilde{w}_t + c|\nabla\tilde{w}|^4 \leq 0.$$

If $\bar{w}$ is the solution of the problem

$$\begin{cases} \Delta\bar{w} + c|\nabla\bar{w}|^4 = \bar{w}_t \quad \text{in } \Omega \times (0, t^*), \\ \bar{w}(x, t) = 0 \quad \text{on } \partial\Omega \times (0, t^*), \\ \bar{w}(x, 0) = u_0(x) + v_0(x) \geq 0 \quad \text{on } \Omega, \end{cases} \tag{2.30}$$

we have by comparison principle

$$\tilde{w} \geq \bar{w}.$$

We now consider the auxiliary function:

$$\Psi(t) = \int_\Omega \bar{w}^3 dx. \tag{2.31}$$

Following Theo.1 in [4] we compute

$$\Psi'(t) = 3\int_\Omega \bar{w}^2 \bar{w}_t dx = 3\int_\Omega \bar{w}^2 \Delta \bar{w} + 3c\int_\Omega \bar{w}^2 |\nabla \bar{w}|^4 dx.$$

By divergence theorem we obtain

$$\Psi'(t) = -6\int_\Omega \bar{w}|\nabla \bar{w}|^2 dx + 3c\int_\Omega \bar{w}^2 |\nabla \bar{w}|^4 dx. \tag{2.32}$$

In the first term in (2.32) by Schwarz inequality we write

$$\int_\Omega \bar{w}|\nabla \bar{w}|^2 dx \leq \Big(\int_\Omega \bar{w}^2 |\nabla \bar{w}|^4 dx\Big)^{\frac{1}{2}} |\Omega|^{\frac{1}{2}}. \tag{2.33}$$

By Rayleigh's principle we have

$$\int_\Omega \bar{w}|\nabla \bar{w}|^2 dx = \frac{4}{9}\int_\Omega |\nabla \bar{w}^{\frac{3}{2}}|^2 dx \geq \frac{4\lambda_1}{9}\int_\Omega \bar{w}^3 dx \tag{2.34}$$

Combining (2.32), (2.33) and (2.34) we obtain

$$\begin{aligned} \Psi'(t) &\geq 3\int_\Omega \bar{w}|\nabla \bar{w}|^2 dx\Big[-2 + \frac{c}{|\Omega|}\Big(\int_\Omega \bar{w}|\nabla \bar{w}|^2 dx\Big)\Big] \\ &\geq 6\int_\Omega \bar{w}|\nabla \bar{w}|^2 dx\Big[-1 + \frac{2c\lambda_1}{9|\Omega|}\int_\Omega \bar{w}^3 dx\Big]. \end{aligned} \tag{2.35}$$

From (2.29) we deduce that

$$\int_\Omega \bar{w}^3 dx > \frac{9|\Omega|}{2c\lambda_1}, \tag{2.36}$$

so that $\Psi(t)$ is a nondecreasing function.
In view of (2.34), we reformulate (2.35), as a differential inequality for $\Psi(t)$:

$$\Psi'(t) \geq \frac{8\lambda_1}{3}\Psi(t)\Big[-1+\frac{2c\lambda_1}{9|\Omega|}\Psi(t)\Big]. \tag{2.37}$$

We now see that for the reciprocal

$$\Big[\Psi(t)\Big]^{-1} := z(t) \tag{2.38}$$

we have $z'(t) = -\Psi^{-2}\Psi'$.
Moreover, by using (2.37), $z(t)$ satisfies the following differential inequality

$$\frac{d}{dt}\Big[z(t)e^{-\frac{8\lambda_1}{3}t}\Big] \leq -\frac{16\ c\lambda_1^2}{27|\Omega|}\ e^{-\frac{8\lambda_1}{3}t}. \tag{2.39}$$

Integrating (2.39) from 0 to t, we obtain

$$z(t) \leq \Big\{z(0) - \frac{2c\lambda_1}{9|\Omega|}\Big[1-e^{-\frac{8}{3}\lambda_1 t}\Big]\Big\}e^{\frac{8}{3}\lambda_1 t} \tag{2.40}$$

It follows from (2.38) and (2.40) that $\Psi(t)$ blows up at some positive time $t_0 < T^*$, with T^* defined in

$$z(0) = \frac{2c\lambda_1}{9|\Omega|}\Big[1-e^{-\frac{8}{3}\lambda_1 T^*}\Big].$$

Since $z(0) = \Psi_0^{-1}$ we obtain

$$t_0 \leq -\frac{3}{8\lambda_1}\ ln\Big[1-\frac{9|\Omega|}{2c\lambda_1\Psi_0}\Big]. \tag{2.41}$$

As a consequence, T^* is also an upper bound for the blow up time of $\bar{w}$ and then of $\tilde{w} = u + v$.

3 Lower bound for the blow up time in Neumann systems.

In this section we consider the system (2.1) with Dirichlet boundary condition replaced by Neumann ones:

$$\frac{\partial u}{\partial n}(x,t) = \frac{\partial v}{\partial n}(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,t^*), \tag{3.1}$$

with compatibility conditions $\frac{\partial u_0}{\partial n} = \frac{\partial v_0}{\partial n} = 0$ on $\partial\Omega$. In order to obtain a lower bound for t^*, we observe that Theo 2.1 cannot be extended to this case, since the Talenti-Soblolev inequality used there to estimate $\int_\Omega w^6 dx$, is valid only under Dirichlet condition on the boundary.
Under the hypothesis of convexity of the domain Ω, we introduce now another inequality of Sobolev type, which can be used with different boundary conditions.

Lemma 3.1.(Sobolev type inequality).
Let Ω be a bounded convex domain in $\mathbb{R}^3$. Let $\rho = \min_{\partial\Omega}(x_i\nu_i)$, where ν_i denotes the i-th component of the unit normal vector, and $d^2 = \max_\Omega(x_i x_i), i = 1, 2, 3$, with summation convention over repeated indexes. For an arbitrary positive constant γ, for any nonnegative C^1- function $U(x)$ defined in Ω, the following inequality holds

$$\int_\Omega U^3 dx \leq a_1\Big(\int_\Omega U^2 dx\Big)^{\frac{3}{2}} + a_2\Big(\int_\Omega U^2 dx\Big)^3 + a_3\int_\Omega |\nabla U|^2 dx, \tag{3.2}$$

with $a_1 := \sqrt{2}\Big(\frac{3}{2\rho}\Big)^{\frac{3}{2}}$, $a_2 := \frac{\sqrt{2}}{4\gamma^3}\Big(1+\frac{d}{\rho}\Big)^{\frac{3}{2}}$ $a_3 := \frac{3}{4}\sqrt{2}\gamma\Big(1+\frac{d}{\rho}\Big)^{\frac{3}{2}}$.

For the proof of Lemma 3.1, we apply Lemma A2 in [6], with $n = 2$:

$$\int_\Omega U^3 dx \leq \Big\{\frac{3}{2\rho}\int_\Omega U^2 dx + \Big(1+\frac{d}{\rho}\Big)\int_\Omega U|\nabla U| dx\Big\}^{\frac{3}{2}}. \tag{3.3}$$

Now we introduce in the last term of (3.3) the Schwarz inequality and the following basic inequality

$$(a+b)^{3/2} \leq \sqrt{2}(a^{3/2} + b^{3/2}). \tag{3.4}$$

We obtain

$$\int_\Omega U^3 dx \leq \Big\{\frac{3}{2\rho}\int_\Omega U^2 dx + \Big(1+\frac{d}{\rho}\Big)\Big(\int_\Omega U^2 dx\Big)^{1/2}\Big(\int_\Omega |\nabla U|^2 dx\Big)^{1/2}\Big\}^{\frac{3}{2}} \tag{3.5}$$

$$\leq \sqrt{2}\Big\{\Big(\frac{3}{2\rho}\Big)^{\frac{3}{2}}\Big(\int_\Omega U^2 dx\Big)^{\frac{3}{2}} + \Big(1+\frac{d}{\rho}\Big)^{\frac{3}{2}}\Big(\int_\Omega U^2 dx\Big)^{\frac{3}{4}}\Big(\int_\Omega |\nabla U|^2 dx\Big)^{\frac{3}{4}}\Big\}$$

Now let γ be an arbitrary positive quantity. For any a, b positive, the following inequality holds

$$a^{1/4}\, b^{3/4} \leq \frac{1}{4\gamma^3}\, a + \frac{3}{4}\gamma\, b.$$

By using it, we rewrite (3.5) as

$$\int_\Omega U^3 dx \tag{3.6}$$

$$\leq \Big\{ b_1 \Big(\int_\Omega U^2 dx \Big)^{\frac{3}{2}} + b_2 \Big[\frac{1}{4\gamma^3} \Big(\int_\Omega U^2 dx \Big)^3 + \frac{3\gamma}{4} \int_\Omega |\nabla U|^2 dx \Big] \Big\},$$

with $b_1 := \sqrt{2}\Big(\frac{3}{2\rho}\Big)^{\frac{3}{2}}$ and $b_2 := \sqrt{2}\Big(1 + \frac{d}{\rho}\Big)^{\frac{3}{2}}$ From (3.6) we get (3.2) with $a_1 = b_1$ $a_2 = \frac{1}{4\gamma^3} b_2$ and $a_3 = \frac{3}{4}\gamma b_2$ and Lemma 3.1 is proved.

Let $\Phi(t)$ be defined as in (2.2). We prove the following

Theorem 3.1 *Let (u, v) be the vector solution of Neumann system in $\Omega \times (0, t^*)$ with Ω a bounded convex domain in $\mathbb{R}^3$. If the solution becomes unbounded in the norm Φ at some finite time t^*, and if the nonlinearities satisfy (2.3), then t^* is bounded from below by*

$$t^* \geq \int_{\Phi(0)}^{+\infty} \frac{d\eta}{\chi_1 \eta^{\frac{3}{2}} + \chi_2 \eta^3} \tag{3.8}$$

with

$$\Phi(0) = \int_\Omega (u_0 + v_0)^{2p} dx.$$

$$\chi_1 = 4k_2 a_1 \quad \chi_2 = 4k_2 a_2, \tag{3.9}$$

with a_1, a_2 in Lemma 3.1 and k_1, k_2 in Theo.2.1.

Proof.
We follow the proof of Theo 2.1 up to (2.16). Since Lemma 3.1 holds,we insert (3.2) in (2.16) , to obtain with $U := (u^2 + v^2)^p$

$$\Phi'(t) \leq -k_1 \int_\Omega |\nabla U|^2 dx + 4k_2 \int_\Omega U^3 dx \tag{3.11}$$

$$\leq -k_1 \int_\Omega |\nabla U|^2 dx + 4k_2 \Big\{ a_1 (\int_\Omega U^2 dx)^{\frac{3}{2}} + a_2 (\int_\Omega U^2 dx)^3 + a_3 \int_\Omega |\nabla U|^2 dx \Big\}$$
$$= (-k_1 + 4k_2 a_3) \int_\Omega |\nabla U|^2 dx + 4k_2 a_1 \Phi^{\frac{3}{2}} + 4k_2 a_2 \Phi^3.$$

If we choose $\gamma = \dfrac{k_1}{3b_2k_2}$, the coefficient $(-k_1 + 4k_2a_3) = 0$. Then we get the differential inequality

$$\Phi'(t) \leq \chi_1 \Phi^{\frac{3}{2}} + \chi_2 \Phi^3. \tag{3.12}$$

By an integration of (3.12) we obtain (3.8) and Theo.3.1 is proved.

Remark 1.
For Neumann system our method in Theo. 2.3 to derive an upper bound for t^* does not apply, since in the proof for Dirchlet case we use a Rayleigh inequality involving the first positive eigenvalue for the fixed membrane problem. The correspondent inequality for the free membrane problem

$$\Delta\phi + \mu_1 \phi = 0 \ \ in\ \Omega, \quad \phi_n = 0 \ \ on\ \partial\Omega,$$

holds only with function with integral mean value zero.

Remark 2.
In [3] bounds for blow-up time are derived for systems with nonlinearities depending on the vector solution (u, v), with the Laplacian operator replaced by an operator of divergence type. There both lower and upper bounds are obtained thanks to the particular choice of the auxiliary function introduced to obtain a differential inequality.

4 Lower bound for the blow up time in Robin systems

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(4.1)\qquad \begin{cases} \Delta u + f_1(|\nabla v|) = u_t & \text{in } \Omega \times (0,t^*), \\ \Delta v + f_2(|\nabla u|) = v_t & \text{in } \Omega \times (0,t^*), \\ \dfrac{\partial u}{\partial n}(x,t) = \beta_1 u, & \text{on } \partial\Omega \times (0,t^*), \\ \dfrac{\partial v}{\partial n}(x,t) = \beta_2 v, & \text{on } \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) & \text{on } \Omega, \\ v(x,0) = v_0(x) & \text{on } \Omega, \end{cases}
$$

where β_1 and β_2 are two positive constants with

$$
(4.2)\qquad \beta_1 \le \beta_2;
$$

f_1, f_2 are continuous non negative functions. Moreover $u_0(x), v_0(x)$ are non negative functions in Ω with bounded first derivative and satisfy the compatibility conditions $\dfrac{\partial u_0}{\partial n} = \beta_1 u_0$, $\dfrac{\partial v_0}{\partial n} = \beta_2 v_0$ on $\partial\Omega$. We note that (u,v) is non negative in x and $t \in (0,t^*)$ by the parabolic maximum principle. Under suitable conditions on non linearities f_1, f_2, by introducing an auxiliary function and conditions on non linearities, we derive a differential inequality from which a lower bound for t^* is obtained.

As observed in Neumann system, Theo.2.1 cannot be extended to this case. In fact we have to estimate an integral over the boundary of Ω. Firstly we prove an inequality to be used in deriving the lower bound.

Lemma 4.1.
Let Ω be a bounded and convex domain in $\mathbb{R}^3$. Let $\rho = \min_{\partial\Omega}(x_i\nu_i)$ and $d = \max_{\Omega}(x_i x_i), i = 1,2,3$, for any nonnegative C^1- function $V(x)$ defined in $\bar{\Omega}$, the following inequality holds

$$
(4.3)\qquad \int_{\partial\Omega} V^2 ds \le \frac{3+d}{\rho}\int_{\Omega} V^2 dx + \frac{d}{\rho}\int_{\Omega} |\nabla V|^2 dx,
$$

To prove Lemma 4.1, we consider for $n = 2$ Lemma A1 in [5],

$$\int_{\partial\Omega} V^2 ds \leq \frac{3}{\rho}\int_{\Omega} V^2 dx + \frac{2d}{\rho}\int_{\Omega} V\ |\nabla V| dx. \tag{4.4}$$

By applying the Schwarz inequality and the inequality (2.9), we get (4.3).

Now let Φ be defined in (2.1).
We proof the following

Theorem 4.1 *If Ω is a bounded and convex domain in $\mathbb{R}^3$, if the vector solution (u, v) of (4.1) becomes unbounded in the norm Φ at some finite time t^*, and if (2.3) and (2.4) hold, then t^* is bounded below by*

$$t^* \geq \int_{\Phi(0)}^{\infty} \frac{d\eta}{\psi(\eta)} \tag{4.5}$$

with

$$\begin{cases} \psi(\Phi) = \xi_1\ \Phi + \xi_2\ \Phi^{\frac{3}{2}} + \xi_3\ \Phi^3 \\ \xi_1 = 4p\beta_2 \dfrac{(3+d)}{\rho} \\ \xi_2 = 4pK(1-\beta)a_1 \\ \xi_3 = 4pK(1-\beta)a_2, \end{cases} \tag{4.6}$$

a_1, a_2 in (3.2)

Proof.

We compute
(4.7)

$$\Phi'(t) = 4p\int_{\Omega}(u^2+v^2)^{2p-1}(uu_t + vv_t)dx = 4p\int_{\Omega}(u^2+v^2)^{2p-1}(u\Delta u + v\Delta v)dx$$

$$+\ 4p\int_{\Omega}(u^2+v^2)^{2p-1}(u\ f_1(|\nabla v|) + v\ f_2(|\nabla u|))dx$$

$$= 4p \int_{\partial\Omega} (u^2+v^2)^{2p-1}(u\frac{\partial u}{\partial n} + v\frac{\partial v}{\partial n})ds$$

$$-8p(2p-1)\int_{\Omega}(u^2+v^2)^{2p-2}(u\nabla u + v\nabla v)(u\nabla u + v\nabla v)dx$$

$$-4p\int_{\Omega}(u^2+v^2)^{2p-1}(|\nabla u|^2+|\nabla v|^2)dx + 4p\int_{\Omega}(u^2+v^2)^{2p-1}(uf_1+vf_2)dx$$

$$= 4pI_1 - 8p(2p-1)I_2 + 4p\ I_3 + 4p\ I_4$$

Due to Robin boundary conditions the integral on $\partial\Omega$ in (4.7) does not disappear. In fact, by the divergence theorem, the boundary conditions in (4.1) and by inserting (4.2), we obtain

$$I_1 \leq \beta_2 \int_{\partial\Omega}(u^2+v^2)^{2p}ds.$$

In the last integral , we apply Lemma 4.1 to obtain

$$I_1 \leq \beta_2\frac{3+d}{\rho}\int_{\Omega}(u^2+v^2)^{2p}dx + \beta_2\frac{d}{\rho}\int_{\Omega}|\nabla(u^2+v^2)^p|^2dx.$$

By using (2.12), we get

$$I_3 \leq \int_{\Omega}(u^2+v^2)^{2p-2}(u\nabla u + v\nabla v)(u\nabla u + v\nabla v)dx.$$

From (2.3) we derive

$$I_4 \leq K\int_{\Omega}(u^2+v^2)^{2p-1}(u^2+v^2)^{(1-\beta)(p+1)}(|\nabla u|^2+|\nabla v|^2)^{\beta}dx$$

$$\leq K(1-\beta)\int_{\Omega}(u^2+v^2)^{3p}dx + \beta K\int_{\Omega}(u^2+v^2)^{2p-1}(|\nabla u|^2+|\nabla v|^2)dx.$$

By replacing for brevity $(u^2+v^2)^p$ with V and inserting the inequalities for $I_i, i=1,3,4$ in (4.7), after some calculations we obtain

$$\Phi'(t) \leq \frac{4p\beta_2(3+d)}{\rho}\int_{\Omega}V^2dx \tag{4.8}$$

$$+\Big[\frac{4p\beta_2 d}{\rho} - (4-\frac{1}{p}-\frac{\beta K}{p})\Big]\int_{\Omega}|\nabla V|^2dx + 4pK(1-\beta)\int_{\Omega}V^3dx.$$

Since Ω is convex, Lemma 3.1 can be applied to the last term in (4.8). We obtain

$$(4.9)\quad \Phi'(t) \leq \frac{4p\beta_2(3+d)}{\rho}\int_\Omega V^2dx + \Big[\frac{4p\beta_2 d}{\rho} - (4-\frac{1}{p}-\frac{\beta K}{p})\Big]\int_\Omega |\nabla V|^2dx$$
$$+ 4pK(1-\beta)\Big[a_1(\int_\Omega V^2dx)^{\frac{3}{2}} + a_2(\int_\Omega V^2dx)^3 + a_3\int_\Omega |\nabla V|^2dx\Big].$$

We write (4.9) as

$$(4.10)\quad \Phi'(t) \leq \xi_1\int_\Omega V^2dx + \xi_2\Big(\int_\Omega V^2dx\Big)^{\frac{3}{2}} + \xi_3\Big(\int_\Omega V^2dx\Big)^3 + \xi_4\int_\Omega |\nabla V|^2dx.$$

with ξ_1, ξ_2, ξ_3 as in(4.6) and

$$(4.11)\qquad \xi_4 = \Big[\frac{4p\beta_2 d}{\rho} - \Big(4-\frac{1}{p}-\frac{\beta K}{p}\Big)\Big] + 4pK(1-\beta)a_3,$$

with $a_3 = \frac{3}{4}\sqrt{2}\,\gamma\Big(1+\frac{d}{\rho}\Big)^{\frac{3}{2}}$. If in (4.11) we choose γ such that

$$\xi_4 = 0,$$

then we get the differential inequality

$$\Phi'(t) \leq \xi_1\Phi + \xi_2\Phi^{\frac{3}{2}} + \xi_3\Phi^3.$$

i.e.

$$(4.12)\qquad \Phi'(t) \leq \psi(\Phi)$$

Finally with an integration of (4.12) from 0 to t^* we obtain (4.5) and Theo.4.1 is proved.

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