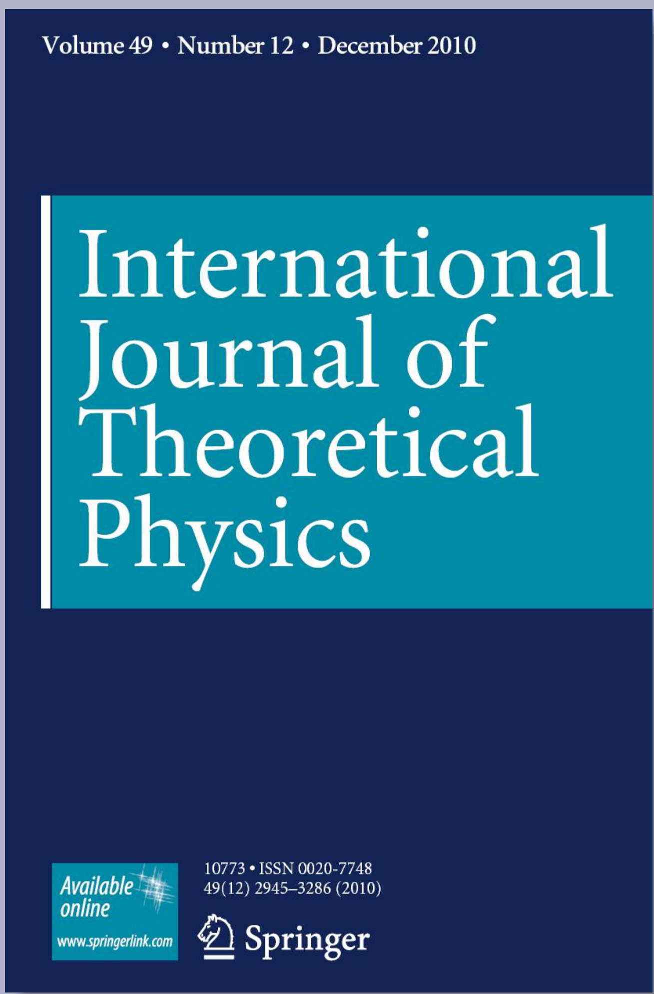
Towards Quantum Computational Logics

International Journal of Theoretical Physics

ISSN 0020-7748
Volume 49
Number 12

Int J Theor Phys (2010)
49:3158-3165
DOI 10.1007/
s10773-010-0368-4



Springer

Int J Theor Phys (2010) 49: 3158–3165
DOI 10.1007/s10773-010-0368-4

Towards Quantum Computational Logics

Antonio Ledda · Giuseppe Sergioli

Received: 29 October 2008 / Accepted: 21 April 2010 / Published online: 26 May 2010

Abstract Quantum computational logics have recently stirred increasing attention (Cattaneo et al. in Math. Slovaca 54:87–108, 2004; Ledda et al. in Stud. Log. 82(2):245–270, 2006; Giuntini et al. in Stud. Log. 87(1):99–128, 2007). In this paper we outline their motivations and report on the state of the art of the approach to the logic of quantum computation that has been recently taken up and developed by our research group.

1 Introduction

Recently, logical and algebraic structures essentially related to quantum computation have been introduced and investigated; let us recall, among the others, [4, 6, 9]. In particular, in [6, 9] the varieties of *quasi-MV algebras* and of $\sqrt{'}$ *quasi-MV algebras* have been introduced. These structures represent a convenient algebraic generalization of the algebra of density operators equipped with the operations "Łukasiewicz sum" $\oplus$ and square root of the negation $\sqrt{\mathtt{NOT}}$ [5].

In Sect. 2 we will present the foundational motivations of the above mentioned structures, and in Sect. 3 we will summarize the main results in the theory of $\sqrt{'}$ quasi-MV algebras.

2 From Quantum Computational Structures to Algebras from Quantum Computation

The purpose of this section is to recall some basic notions from the theory of quantum computation. In particular we will compare two different models: the first one based on

Authors has been funded by the Regione Autonoma della Sardegna, PO Sardegna FSE 2007-2013, L.R.7/2007.

A. Ledda · G. Sergioli (✉)
Department of Education, University of Cagliari, Via Is Mirrionis 1, 09123 Cagliari, Italy
e-mail: giuseppe.sergioli@gmail.com

A. Ledda
e-mail: antonio.ledda78@gmail.com

unitary operators (quantum gates) acting on unit vectors (quantum registers), and the second one based on quantum operations acting on density operators.

In quantum mechanics every physical system S is associated to an appropriate Hilbert space $\mathcal{H}^S$. A state of the physical system S is said to be *pure* if and only if it represents a maximal information quantity, that is an information which cannot be increased by any further observation. A pure state is mathematically represented by a unit vector in the Hilbert space $\mathcal{H}^S$.

Let us first consider a physical system S whose associated Hilbert space is $\mathbb{C}^2$ and let $\mathcal{B} = \{|0\rangle, |1\rangle\}$ be its (canonical) computational basis, where $|0\rangle = \binom{1}{0}$ and $|1\rangle = \binom{0}{1}$. In this case, the general form of a vector is the following:

$$|\varphi\rangle = a_0 |0\rangle + a_1 |1\rangle,$$

where a_0 and a_1 are complex numbers such that $|a_0|^2 + |a_1|^2 = 1$.

A pure state in $\mathbb{C}^2$ is usually called *qubit*; it represents the quantum computational counterpart of the classical bit. As dictated by the Born rule, $|a_0|^2$ ($|a_1|^2$, respectively) yields the probability of the information described by the pure state $|0\rangle$ ($|1\rangle$, respectively), which, from a logical point of view, corresponds to the falsity (truth).

As regards the mathematical representation of a number of certain physical systems interacting with one another, we resort the notion of tensor product Hilbert spaces. Suppose we have to deal with a physical system S composed by n subsystems, say $S_1, \ldots, S_n$. Let $\mathcal{H}^{S_i}$ be the Hilbert spaces associated to S_i, for $1 \leq i \leq n$. The Hilbert space $\mathcal{H}$ associated to S will be the tensor product $\mathcal{H}^{S_1} \otimes \cdots \otimes \mathcal{H}^{S_n}$ of the spaces associated to $S_1, \ldots, S_n$. If, for every i, j, $S_i = S_j$, we use the notation $\bigotimes^n \mathcal{H}^{S_i}$ in place of $\mathcal{H}^{S_i} \otimes \cdots \otimes \mathcal{H}^{S_i}$.

As we have seen, qubits "live" in the space $\mathbb{C}^2$. *Quregisters* are the tensor product analogues of qubits: by quregister, in fact, we mean any unit vector in $\bigotimes^n \mathbb{C}^2$. Quregisters are the quantum counterpart of classical registers—i.e. finite strings of bits.

Let $\mathfrak{R}(\bigotimes^n \mathbb{C}^2)$ be the set of all quregisters of $\bigotimes^n \mathbb{C}^2$. We denote by

$$\mathfrak{R} := \bigcup_{n=1}^{\infty} \left(\mathfrak{R} \left(\bigotimes^n \mathbb{C}^2 \right) \right)$$

the set of all quregisters in $\mathbb{C}^2$ or in a tensor power of $\mathbb{C}^2$. The (canonical) computational basis of $\bigotimes^n \mathbb{C}^2$ is defined accordingly and will be denoted by $\mathcal{B}^{(\mathrm{n})}$.

Like in the classical case, also in quantum computation the evolution of a state is obtained by the application of a (quantum) gate to a (quantum) register.

In classical computation a gate is a *generally irreversible* function $f : \{0, 1\}^n \to \{0, 1\}$. On the contrary, in quantum computation a quantum gate is a *reversible* operator U (transforming quregisters into quregisters), since U is unitary.

By looking at the output quregister, we can always trace back the corresponding input quregister.

Typical examples of quantum gates are the quantum $\mathtt{Not}$ and $\sqrt{\mathtt{Not}}$ gates:

Example 1 For any $n \geq 1$ and for every element $|x_1, \ldots, x_n\rangle$ of the computational basis $\mathcal{B}^{(\mathrm{n})}$,

$$\mathtt{Not}^{(\mathrm{n})}(|x_1, \ldots, x_n\rangle) = |x_1, \ldots, x_{n-1}\rangle \otimes |1 - x_n\rangle;$$

$$\sqrt{\mathtt{Not}}^{(\mathrm{n})}(|x_1, \ldots, x_n\rangle) = |x_1, \ldots, x_{n-1}\rangle \otimes \frac{1}{2}\left((1+i)|x_n\rangle + (1-i)|1 - x_n\rangle\right).$$

The basic property of $\sqrt{\mathtt{Not}}^{(\mathrm{n})}$ is the following: for any $|\psi\rangle \in \mathfrak{R}(\bigotimes^n \mathbb{C}^2)$, $\sqrt{\mathtt{Not}}^{(\mathrm{n})}(\sqrt{\mathtt{Not}}^{(\mathrm{n})}(|\psi\rangle)) = \mathtt{Not}^{(\mathrm{n})}(|\psi\rangle)$.

All the notions previously mentioned are formulated in the framework of the usual approach to quantum computation, based on unitary transformations of pure states.

According to [2], however, such a representation is unduly restrictive insomuch as it does not encompass open systems where, for example, coupling with environment and measurement processes may occur. In these cases, the evolution of a state is no longer reversible. In [2], the authors formulate a more general model of quantum computational processes, where quregisters and unitary operators are replaced by density operators[1] (*qumixes*, mixtures of pure states) and quantum operations [7].

Let $\mathfrak{D}(\bigotimes^n \mathbb{C}^2)$ be the set of all density operators on $\bigotimes^n \mathbb{C}^2$. We denote by

$$\mathfrak{D} := \bigcup_{n=1}^{\infty} \left(\mathfrak{D} \left(\bigotimes^n \mathbb{C}^2 \right) \right)$$

the set of all density operators in $\mathbb{C}^2$ or in a tensor product of $\mathbb{C}^2$. This set is a convenient representation of the set of all qumixes.

Any quregister can be regarded as a limit case of a qumix: a quregister is a density operator which is also a *projection operator*.

If $\rho \in \mathfrak{D}(\bigotimes^n \mathbb{C}^2)$ is a qumix, *the probability of truth* (denoted by) $\mathtt{p}(\rho)$ of a density operator ρ is given by $\mathtt{tr}(P_1^{(\mathrm{n})}\rho)$, where $\mathtt{tr}$ is the trace functional, $P_1^{(\mathrm{n})} = \mathbb{I}^{(\mathrm{n}-1)} \otimes (|1\rangle\langle 1|)$ and $\mathbb{I}$ is the 2×2 identity matrix; analogously, the *probability of falsity of* ρ is $\mathtt{tr}(P_0^{(\mathrm{n})}\rho)$, where $P_0^{(\mathrm{n})} = \mathbb{I}^{(\mathrm{n}-1)} \otimes (|0\rangle\langle 0|)$. Intuitively, $\mathtt{p}(\rho)$ $(1 - \mathtt{p}(\rho)$, respectively) represents the probability that the information stored by the qumix ρ is true (false).

Interestingly enough, qumixes are connected with the real closed unit interval $[0, 1]$. For, given a real number $\lambda \in [0, 1]$ and $n \in \mathbb{N}^+$, we can define a qumix $\rho_\lambda^{(\mathrm{n})}$ in the following way: $\rho_\lambda^{(\mathrm{n})} = (1 - \lambda)k_n P_0^{(\mathrm{n})} + \lambda k_n P_1^{(\mathrm{n})}$.[2] This observation will play a key role in what follows. Moreover, one can verify that $\mathtt{p}(\rho_\lambda^{(\mathrm{n})}) = \lambda$.

From a physical point of view, using qumixes instead of pure states has plenty of advantages. First of all, every physical system is not completely isolated from the rest of the universe, but it always interacts with it, and by this reason a state of a physical system is better represented by a qumix (mixed state) instead of a quregister (pure state). Moreover, as Aharonov, Kitaev and Nisan have shown [2], taking into account quantum circuits with qumixes allows us to treat some critical problems (such as measurements in the middle of computation, decoherence and noise, etc. . .) that are difficult or impossible to be dealt with the usual approach. It should be noticed, however, that the Aharonov-Kitaev-Nisan model and the standard model are polynomially equivalent in computational power [2].

We have seen that evolutions of pure states are unitary transformations. If we are concerned with mixed states, evolutions are mathematically represented by *quantum operations*.

Definition 2 [8] A *quantum operation* is a trace preserving, completely positive, linear map from density operators to density operators.

Let us stress the fact that the notion of quantum operation includes both reversible and irreversible transformations [10].

[1] A *density operator* is a positive, self adjoint, trace class operator ρ such that $\mathtt{tr}(\rho) = 1$.

[2] Where k_n is a normalization coefficient equal to $\frac{1}{2^{n-1}}$.

Examples of reversible quantum operations are NOT and $\sqrt{\mathtt{NOT}}$:

Example 3 Let $\rho \in \mathfrak{D}(\bigotimes^n \mathbb{C}^2)$, then:

$$\mathtt{NOT}\rho = \mathtt{Not}^{(n)}\rho\mathtt{Not}^{(n)},$$
$$\sqrt{\mathtt{NOT}}\rho = \sqrt{\mathtt{Not}}^{(n)}\rho\sqrt{\mathtt{Not}}^{(n)*},$$

where * is the adjoint of $\sqrt{\mathtt{Not}}^{(n)}$.

It can be seen that: $\sqrt{\mathtt{NOT}}^{(n)}\sqrt{\mathtt{NOT}}^{(n)}\rho = \mathtt{NOT}^{(n)}\rho$.

As an example of an irreversible quantum operation we can consider the *partial trace* operation.

There are interesting transformations of density operators into density operators that are not even quantum operations but that can be approximated via quantum operations [5], for example *the Łukasiewicz disjunction*:

Example 4 Let $\sigma \in \mathfrak{D}(\bigotimes^n \mathbb{C}^2)$ and $\tau \in \mathfrak{D}(\bigotimes^m \mathbb{C}^2)$, then:

$$\sigma \oplus \tau = \rho^{(1)}_{\mathtt{p}(\sigma)\oplus\mathtt{p}(\tau)}$$

where $\oplus$ is the Łukasiewicz "truncated sum", i.e. $\min(x+y,1)$, for $x, y \in [0,1]$.

We have now all the required ingredients [4] to define a preorder relation $\preceq$ on $\mathfrak{D}$ as follows: for any $\rho, \sigma \in \mathfrak{D}$, $\rho \preceq \sigma$ if and only if $\mathtt{p}(\rho) \leq \mathtt{p}(\sigma)$ and $\mathtt{p}(\sqrt{\mathtt{NOT}}^{(n)}\rho) \leq \mathtt{p}(\sqrt{\mathtt{NOT}}^{(n)}\sigma)$. In virtue of the definition above we can introduce an equivalence relation $\approxeq$ on $\mathfrak{D}$ in the following way: for any $\rho, \sigma \in \mathfrak{D}$, $\rho \approxeq \sigma$ if and only if $\rho \preceq \sigma$ and $\sigma \preceq \rho$. It turns out that $\approxeq$ is a congruence relation with respect to $\oplus$ and $\sqrt{\mathtt{NOT}}$ [4]. Therefore we end up with the following quantum computational structure:

$$\mathbf{D}/_{\approxeq} = \left\langle \mathfrak{D}/_{\approxeq}, \oplus, \sqrt{\mathtt{NOT}}, P_0^{(1)}/_{\approxeq}, P_1^{(1)}/_{\approxeq}, \rho_{\frac{1}{2}}/_{\approxeq} \right\rangle. \tag{1}$$

Let us consider the relation $\approxeq\restriction_{\mathbb{C}^2}$, the restriction of $\approxeq$ to $\mathbb{C}^2$; it can be straightforwardly verified that $\approxeq\restriction_{\mathbb{C}^2}$ is a congruence relation on $\mathfrak{D}(\mathbb{C}^2)$ with respect to $\oplus$ and $\sqrt{\mathtt{NOT}}$. Therefore we obtain the algebra

$$\mathbf{D}\left(\mathbb{C}^2\right)/_{\approxeq\restriction_{\mathbb{C}^2}} = \left\langle \mathfrak{D}\left(\mathbb{C}^2\right)/_{\approxeq\restriction_{\mathbb{C}^2}}, \oplus, \sqrt{\mathtt{NOT}}, P_0^{(1)}/_{\approxeq\restriction_{\mathbb{C}^2}}, P_1^{(1)}/_{\approxeq\restriction_{\mathbb{C}^2}}, \rho_{\frac{1}{2}}/_{\approxeq\restriction_{\mathbb{C}^2}} \right\rangle. \tag{2}$$

Surprisingly enough, as a consequence of Theorem 7.1 in [4], the algebra $\mathbf{D}(\mathbb{C}^2)/_{\approxeq\restriction_{\mathbb{C}^2}}$ in (2) is isomorphic to the algebra $\mathbf{D}/_{\approxeq}$ in (1).

It is well known that every density operator ρ in $\mathfrak{D}(\mathbb{C}^2)$ can be represented via Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$, i.e.

$$\rho = \frac{1}{2}\left(\mathbb{I} + r_1\sigma_x + r_2\sigma_y + r_3\sigma_z\right),$$

where r_1, r_2, r_3 are real numbers such that $r_1^2 + r_2^2 + r_3^2 \leq 1$. Thus, it is possible to uniquely identify any density operator ρ in $\mathfrak{D}(\mathbb{C}^2)$ with a triple of real numbers $\langle r_1, r_2, r_3\rangle$. Moreover, notice that in the definition of $\approxeq\restriction_{\mathbb{C}^2}$, for any density operator ρ in $\mathfrak{D}(\mathbb{C}^2)/_{\approxeq\restriction_{\mathbb{C}^2}}$, just $\mathtt{p}(\rho)$ and

$\mathtt{p}(\sqrt{\mathtt{NOT}}\rho)$ come into play, since a routine calculation shows that, for any $\rho \in \mathfrak{D}(\mathbb{C}^2)/_{\approxeq\restriction_{\mathbb{C}^2}}$, $\mathtt{p}(\rho) = \frac{1-r_2}{2}$ and $\mathtt{p}(\sqrt{\mathtt{NOT}}\rho) = \frac{1-r_3}{2}$. Therefore, we can uniquely associate to every density operator ρ in $\mathfrak{D}(\mathbb{C}^2)/_{\approxeq\restriction_{\mathbb{C}^2}}$ a pair of real numbers $\langle r_2, r_3\rangle$.

Thus, we finally end up with the set

$$D = \left\{\langle a, b\rangle : (1-2a)^2 + (1-2b)^2 \leq 1\right\} \tag{3}$$

where, for any ρ in $\mathfrak{D}(\mathbb{C}^2)/\approxeq\restriction_{\mathbb{C}^2}$, $a = \mathtt{p}(\rho)$ and $b = \mathtt{p}(\sqrt{\mathtt{NOT}}\rho)$.

The geometrical representation of D amounts to the closed disc with radius $\frac{1}{2}$ and center $\langle\frac{1}{2}, \frac{1}{2}\rangle$.

As a last step, it is straightforward to verify that the algebra $\mathbf{D}(\mathbb{C}^2)/_{\approxeq\restriction_{\mathbb{C}^2}}$ is isomorphic to the algebra $\mathbf{D}_r = \langle D, \oplus, \sqrt{\ }', 0, \frac{1}{2}, 1\rangle$, where:[3]

$$\langle a, b\rangle \oplus \langle a, b\rangle = \left\langle a \oplus b, \frac{1}{2}\right\rangle; \qquad \sqrt{\ }'\langle a, b\rangle = \langle b, 1-a\rangle;$$
$$0 = \left\langle 0, \frac{1}{2}\right\rangle; \qquad \frac{1}{2} = \left\langle \frac{1}{2}, \frac{1}{2}\right\rangle;$$
$$1 = \left\langle 1, \frac{1}{2}\right\rangle.$$

3 Square Root Quasi-MV Algebras

In the previous section we have seen that the set of all qumixes of $\mathfrak{D}(\mathbb{C}^2)/_{\approxeq\restriction_{\mathbb{C}^2}}$ is in bijective correspondence with a subset of the unit complex interval $[\langle 0, 0\rangle, \langle 1, 1\rangle]$, i.e. with the lattice ordered set $D = \{\langle a, b\rangle : a, b \in \mathbb{R}$ and $(1-2a)^2 + (1-2b)^2 \leq 1\}$. Moreover, we have seen how the operations $\oplus$ and $\sqrt{\mathtt{NOT}}$ are realized in this setting.

The algebraic structure $\mathbf{D}_r$ is based on the "concrete" universe $\mathfrak{D}(\mathbb{C}^2)/_{\approxeq\restriction_{\mathbb{C}^2}}$. We will show that it is possible to "distill" an algebraic abstraction that captures, so to say, all the logical properties of $\mathfrak{D}(\mathbb{C}^2)/_{\approxeq\restriction_{\mathbb{C}^2}}$. To this aim we need to introduce the notion of $\sqrt{\ }'$ *quasi-MV algebra.*

Definition 5 [6] A $\sqrt{\ }'$ *quasi-MV algebra* (for short, $\sqrt{\ }'$*qMV algebra)* is an algebra $\mathbf{A} = \langle A, \oplus, \sqrt{\ }', 0, 1, k\rangle$ of type $\langle 2, 1, 0, 0, 0\rangle$ such that, upon defining $a' = \sqrt{\ }'\sqrt{\ }'a$ for all $a \in A$, the following conditions are satisfied:

A1. $x \oplus (y \oplus z) \approx (x \oplus z) \oplus y$	A5. $(x \oplus 0)' \approx x' \oplus 0$
A2. $x'' \approx x$	A6. $(x \oplus y) \oplus 0 \approx x \oplus y$
A3. $x \oplus 1 \approx 1$	A7. $0' \approx 1$
A4. $(x' \oplus y)' \oplus y \approx (y' \oplus x)' \oplus x$	A8. $k = \sqrt{\ }'k$
	A9. $\sqrt{\ }'(x \oplus y) \oplus 0 = k$

[3] For sake of notational clarity we will use the notation introduced in [9] and [6].

Clearly, the class of all $\sqrt{\prime}$qMV algebras is a variety[4] in their own similarity type, referred to as $\sqrt{\prime}q\mathbb{MV}$.[5] As one can easily realize, the reduct $\langle A, \oplus, ', 0, 1\rangle$ is a generalization of an MV algebra, in that it satisfies all the MV algebraic equations except for $x \oplus 0 \approx x$.

Let us notice in passing that it is impossible to add a square root of the inverse to a nontrivial MV algebra: letting y be 0 in A.9, for all $x \in A$ we would have $\sqrt{\prime}x = k$, whence by A.8 $x' = \sqrt{\prime}\sqrt{\prime}x = \sqrt{\prime}k = k$ and so $x = x'' = k' = \sqrt{\prime}\sqrt{\prime}k = k$.

In $\sqrt{\prime}$qMV algebras we have not only *regular elements* [9], i.e. elements satisfying the equation $x \oplus 0 = x$, but also *coregular* elements, i.e. elements whose square roots of the inverse are regular. In other words, a is coregular just in case $(\sqrt{\prime}a) \oplus 0 = \sqrt{\prime}a$. We denote by $\mathcal{R}(\mathbf{A})$ and $\mathcal{COR}(\mathbf{A})$ the sets of regular and coregular elements, respectively, of $\mathbf{A}$.

In $\sqrt{\prime}$qMV algebras, a "crucial" role is played by the binary relations λ and μ, which are defined as follows:

Definition 6 Let $\mathbf{A}$ be a $\sqrt{\prime}$qMV algebra and let $a, b \in A$. We set:

$$a\lambda^{\mathbf{A}}b \text{ iff } a \leq^{\mathbf{A}} b, b \leq^{\mathbf{A}} a, \sqrt{\prime}a \leq^{\mathbf{A}} \sqrt{\prime}b \text{ and } \sqrt{\prime}b \leq^{\mathbf{A}} \sqrt{\prime}a,$$

where $x \leq y$ iff $x' \oplus y = 1$ is a preordering relation on $\sqrt{\prime}q\mathbb{MV}$. It turns out that $\lambda^{\mathbf{A}}$ is a congruence on every $\sqrt{\prime}$qMV algebra. Following the literature [6], we refer to the relation $\lambda^{\mathbf{A}}$ as the *Cartesian* congruence on a given $\sqrt{\prime}$qMV algebra, and drop once again the superscripts whenever it is clear which algebra is at issue. Likewise, we introduce a congruence which we call the *flat* congruence on a $\sqrt{\prime}$ qMV algebra. Omitting superscripts from the very beginning, we put:

Definition 7 Let $\mathbf{A}$ be a $\sqrt{\prime}$qMV algebra and let $a, b \in A$. We define:

$$a\mu b \text{ iff } a = b \text{ or } a, b \in \mathcal{R}(\mathbf{A}) \cup \mathcal{COR}(\mathbf{A}).$$

Let us now introduce two special classes of $\sqrt{\prime}$ qMV algebras: *Cartesian* algebras—where λ is the identity, Δ- and *flat* algebras—where λ is the universal relation, ∇.

Definition 8

1. A $\sqrt{\prime}$ qMV algebra $\mathbf{A}$ is called *Cartesian* iff $\lambda = \Delta$, that is if and only if it satisfies the quasiequation

$$x \oplus 0 \approx y \oplus 0 \wedge \sqrt{\prime}x \oplus 0 \approx \sqrt{\prime}y \oplus 0 \Rightarrow x \approx y.$$

2. A $\sqrt{\prime}$ qMV algebra $\mathbf{A}$ is called *flat* iff $\lambda = \nabla$.

We denote by $\mathbb{F}$ the class of flat $\sqrt{\prime}$ qMV algebras, and by $\mathbb{CAR}$ the class of Cartesian $\sqrt{\prime}$ qMV algebras.

As a consequence of the definition, it can be easily realized that the unique $\sqrt{\prime}$qMV algebra which is both Cartesian and flat is the trivial one-element algebra. It is worth noticing

[4] A variety of algebras is the class of all algebraic structures of a given signature satisfying a given set of identities.

[5] We remark that a variety of term reducts of algebras in $\sqrt{\prime}q\mathbb{MV}$, whose language includes just $\oplus$, $'$, 0 and 1, namely *quasi-MV algebras*, has been deeply investigated in [3, 9, 11].

that $\mathbb{F}$ is a variety, whose equational basis in $\sqrt{'}q\mathbb{MV}$ is given by the single equation $0 \approx 1$, while $\mathbb{CAR}$ is a quasivariety which is not a variety [6].

Cartesian $\sqrt{'}$qMV algebras are special in that they are amenable to a neat representation in terms of algebras of *pairs*. We first introduce a convenient construction on MV algebras having a fixpoint for the inverse:

Definition 9 Let $\mathbf{A} = \langle A, \oplus^{\mathbf{A}}, {}'^{\mathbf{A}}, 0^{\mathbf{A}}, 1^{\mathbf{A}} \rangle$ be an MV algebra and let $k \in A$ be such that $k = k'$. The *pair algebra* over $\mathbf{A}$ is the algebra

$$\mathcal{P}(\mathbf{A}) = \left\langle A^2, \oplus^{\mathcal{P}(\mathbf{A})}, \sqrt{'}^{\mathcal{P}(\mathbf{A})}, 0^{\mathcal{P}(\mathbf{A})}, 1^{\mathcal{P}(\mathbf{A})}, k^{\mathcal{P}(\mathbf{A})} \right\rangle$$

where:

$$\langle a, b \rangle \oplus^{\mathcal{P}(\mathbf{A})} \langle c, d \rangle = \left\langle a \oplus^{\mathbf{A}} c, k \right\rangle; \qquad \sqrt{'}^{\mathcal{P}(\mathbf{A})} \langle a, b \rangle = \left\langle b, a'^{\mathbf{A}} \right\rangle;$$
$$0^{\mathcal{P}(\mathbf{A})} = \left\langle 0^{\mathbf{A}}, k \right\rangle; \qquad k^{\mathcal{P}(\mathbf{A})} = \langle k, k \rangle;$$
$$1^{\mathcal{P}(\mathbf{A})} = \left\langle 1^{\mathbf{A}}, k \right\rangle.$$

It can be proved that, on the one hand, every pair algebra $\mathcal{P}(\mathbf{A})$ over an MV algebra $\mathbf{A}$ is a Cartesian $\sqrt{'}$qMV algebra; on the other hand, conversely, every Cartesian $\sqrt{'}$qMV algebra is embeddable into a pair algebra via the mapping $f(a) = \langle a \oplus 0, \sqrt{'}a \oplus 0 \rangle$:

Theorem 10 *Every Cartesian $\sqrt{'}$qMV algebra* $\mathbf{A}$ *is embeddable into the pair algebra* $\mathcal{P}(\mathbf{R_A})$ *over its MV polynomial subreduct* $\mathbf{R_A}$ *of regular elements.*

We are now ready to state a direct decomposition theorem [6]: any $\sqrt{'}$qMV algebra can be thought of as composed by a Cartesian component ($\mathcal{P}(\mathbf{R_Q})$, the pair $\sqrt{'}$qMV algebra over the MV algebra $\mathbf{R_Q}$ of regular elements of $\mathbf{Q}$) and a flat component ($\mathbf{A}/\mu$):

Theorem 11 *For every $\sqrt{'}$qMV algebra* $\mathbf{Q}$, *there exist a Cartesian algebra* $\mathbf{C}$ *and a flat algebra* $\mathbf{F}$ *such that* $\mathbf{Q}$ *can be embedded into the direct product* $\mathbf{C} \times \mathbf{F}$.

It is shown in [6] that the whole variety $\sqrt{'}q\mathbb{MV}$ is generated by the quasivariety of Cartesian $\sqrt{'}$qMV algebras; in the same paper is also proved a standard completeness result for $\sqrt{'}q\mathbb{MV}$.

Theorem 12 $\mathbf{V}(\mathbb{CAR}) = \sqrt{'}q\mathbb{MV}$.

Theorem 13 *Let* $t, s \in Term(\langle 2, 1, 0, 0, 0 \rangle)$. *Then*

$$\mathbf{D}_r \vDash t \approx s \text{ iff } \sqrt{'}q\mathbb{MV} \vDash t \approx s.$$

Among the additional results proved for $\sqrt{'}$ quasi-MV algebras in [3, 6, 11], we mention the following: finite model property, congruence extension property, amalgamation property, failure of several algebraic properties (including congruence modularity, subtractivity and point regularity), a characterization of free algebras, a characterization of quasi-MV term reducts and subreducts of $\sqrt{'}$quasi-MV algebras.

Conclusion 14 Theorem 13 shows that the logic of quantum computation (whenever qumixes, "Łukasiewicz sum" and square root of the negation are considered) can be formulated in a pure abstract logical way. Identities that hold in the class of all $\sqrt{'}$quasi-MV algebras are exactly the same as those identities that hold in the concrete standard model $\mathbf{D}_r$ based on the class of qumixes of the Hilbert space $\mathbb{C}^2$. Further work is needed to characterize quantum computational logics based on qumixes and quantum gates (operations) that are, in some sense, more representative of the class of all quantum gates. Particularly interesting to this aim is the pair of quantum gates *Toffoli* and *Hadamard* that D. Aharonov has proved to be quantum universal [1].

References

1. Aharonov, D.: A simple proof that Toffoli and Hadamard are quantum universal. arXiv:quant-ph/0301040v1 (2003)
2. Aharonov, D., Kitaev, A., Nisan, N.: Quantum circuits with mixed states. In: STOC '98: Proceedings of the Thirtieth Annual ACM Symposium on Theory of Computing, pp. 20–30. ACM, New York (1998)
3. Bou, F., Paoli, F., Ledda, A., Freytes, H.: On some properties of quasi-MV algebras and $\sqrt{'}$ quasi-MV algebras, part II. Soft Comput. **12**(4), 341–352 (2008)
4. Cattaneo, G., Dalla Chiara, M.L., Giuntini, R., Leporini, R.: Quantum computational structures. Math. Slovaca **54**, 87–108 (2004)
5. Freytes, H., Giuntini, R., Ledda, A., Sergioli, G.: Stone-Weierstrass type theorem for quantum operations. In preparation
6. Giuntini, R., Ledda, A., Paoli, F.: Expanding quasi-MV algebras by a quantum operator. Stud. Log. **87**(1), 99–128 (2007)
7. Gudder, S.: Quantum computational logics. Int. J. Theor. Phys. **42**, 39–47 (2003)
8. Kraus, K.: States, Effects and Operations: Fundamental Notions of Quantum Theory. Lecture Notes in Physics, vol. 190. Springer, Berlin (1983)
9. Ledda, A., Konig, M., Paoli, F., Giuntini, R.: MV algebras and quantum computation. Stud. Log. **82**(2), 245–270 (2006)
10. Nielsen, M., Chuang, I.: Quantum Computation and Quantum Information. Cambridge University Press, Cambridge (2000)
11. Paoli, F., Ledda, A., Giuntini, R., Freytes, H.: On some properties of quasi-MV algebras and $\sqrt{'}$ quasi-MV algebras, part I. Rep. Math. Log. **44**, 53–85 (2008)