69th Conference of the Italian Thermal Machines Engineering Association, ATI2014

A Multi-Dimensional Heat Conduction Analysis: Analytical Solutions Versus F.E. Methods in Simple and Complex Geometries with Experimental Results Comparison

Francesco Floris[a,*], Bulut Ilemin[a], Pier Francesco Orrù[a]

[a]Department of Meccanical, Chemical and Material Engineering, University of Cagliari, Piazza d'Armi, Cagliari 09123, Italy

Abstract

Computer codes are widely used to predict heat transfer fields. Modeling is accomplished in multidimensional media with homogenous or not homogenous thermal conductivity, with or without volume heat sources and enthalpy flux.

This paper compares the analytical solution of temperature fields in a few physical cases such as aliment cakes, capacitors, gas turbine blades, tanks with infinite element computer results and experimental results.

The analytical solution of heat transfer partial differential equations presented in this paper appears in the form of the sum of effects. One of them is an infinite series in term of eigenvalues that is easily managed through mathematical commercial codes available even for palmar calculations. A comparison with experimental results shows that the concept of analytical solution has application in many physical phenomena without going to the complexity of computer code modeling.

Nomenclature

a_t	[m²/s]	Thermal diffusivity
a	[m]	Radius diameter of capacitor

* Corresponding author. Tel.: +39-070-6755714; fax: +39-070-6755717.
  E-mail address: francesco.floris@dimcm.unica.it

Bi_a	[-]	Biot number (ha/k)
Bi_c	[-]	Biot number (hc/k)
c_p	[j/kgK]	Specific heat at constant pressure
c	[m]	Radius diameter of beef cake
C_n	[-]	Coefficients (constants)
Fo_a	[-]	Fourier number
h	[W/m^{2}K]	Convective heat transfer coefficient
J_0	[-]	Zeroth order Bessel function of the first kind
J_1	[-]	First order Bessel function of the first kind
k	[W/mK]	Thermal conductivity
l	[m]	Height diameter
L	[m]	Half height diameter
r	[m]	Radial coordinate
t	[s]	Time
T	[K]	Temperature
V	[m/s]	Velocity
W_i	[W/m^3]	Volumetric power
Y_0	[-]	Zeroth order Bessel function of the second kind
z	[m]	Axial coordinate

Greek Symbols

α_n	[m^{-1}]	Eigenvalues
β_m	[m^{-1}]	Eigenvalues
λ_n	[-]	Roots
ρ	[kg/m^3]	Density
θ	[K]	Temperature difference
τ	[-]	Function of time only
∇^2	[-]	Laplacian

Subscripts

∞	[K]	Ambient temperature

1. Introduction

The common practice to infer the temperature field-time dependent or pseudo-steady condition, with or without internal heat generation- in a homogeneous object of simple or complex geometry subject to heat transfer to the surface, is to apply finite-element or finite differences methods as are now available commercially in computer codes as ANSYS or others. The analytical solutions are available for steady/unsteady state problems of stationary cylinders or plates with given boundary conditions both by radiation and convection at constant thermal conductivity [1,2,3].

More complex geometries, as gas turbine blades under radiation, convection quantitatively different in suction and pressure side and leading edge were studied and temperature field solved via analytical solution in cases of constant thermal conductivity [4]. Exact solutions were also found in the case of moving heat source in stir welding where the material cools by means of heat conduction and convection [5].

In cases where conductivity may be different in the three dimensions in space (anisotropy) and strongly dependent by temperature, or the homogeneity of the material is not granted, the exact solution of the energy conservation in the body is not possible.

Despite the complexity of many engineering structures or tinplate cans filled with aliments we undertake the work to demonstrate that the reduction to simpler version of more complex heat conduction equations is possible and the exact analytical solution is comparable with the approximate finite-element solution at variable k –anisotropy included- even when the number of spatial variables is reduced.

The motivation of the present paper is to demonstrate that it is often possible, with a palmar calculator and math codes loaded, to infer solutions without moving to a more complex thermal modelling and to sophisticated computer solvers.

A sketch of the cases under consideration is shown in Figure 1 and Figure 2. An aluminium electrolyte capacitor of cylindrical winding of aluminium anode and cathode foils separated by papers impregnated with a liquid electrolyte and a cylindrical tincan filled with beef homogenate.
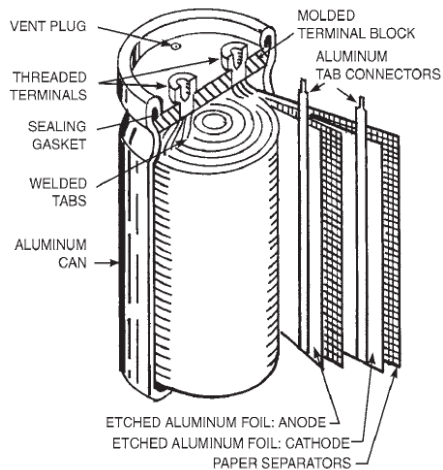
Case 1: The general equation of energy conservation applied to the capacitor should consider the dependency of k by space and temperature and the heat generation due to joule effect. The case is considered independent by time.

Figure 1 Aluminium Electrolytic Capacitor

Case 2: The general equation of heat conduction applied to the beef tincan has no heat generation, is time dependent and k is variable with temperature only (isotrope).

Figure 2 Beef Cake in Tincan

- The case 1 of the electrolytic capacitor was FE-computer-solved 3-D [6], since the thermal conductivity is assumed to be anisotropic, much larger in the axial direction than in the radial direction and a negligible contribution was taken into account in the angular direction. The present work found an analytical solution of the case and compared it to FE-computer-solved results. An electric circuit was set up in order to check superficial temperature of the device under assigned convection heat transfer through ventilation. A few simplifications were made on the value of thermal conduction leading to known physical quantities as dimensionless variables: Biot number in x and r directions. The analytical result, obtained as a series expansion, has the advantage of being correct and usable in a large range of devices, provided an adequate number of eigenvalues is employed.

- The case 2: The finite element method was applied to the calculation of temperature profiles in the sterilization of a real tinplate can filled with beef homogenate cake, under unsteady heating and cooling stages. [7]. In this work the analytical solution is obtained based on the separation of variables in space and time using Bessel functions of order 0 and 1 and a large (up to 30) roots that satisfy boundary conditions. Comparison is then made between FE

results, experimental data and analytical solution. The very close agreement between analytical results and FE solutions and then experimental data, suggests exact solutions, easy to find in literature [1, 2, 8], as a means to infer information in manufacturing process and temperature fields.

1.1. Convection

In both cases 1 and 2, convection is the means to cool the device. It is a surface effect that depends on fluid mechanics, medium temperature, heat-mass transfer properties as density, specific heat and viscosity, geometry of device and direction of flow. The device is supposed to be at a higher temperature than the environment by ΔT. The power dissipated by the medium should be equal to the power generated by the capacitor in steady state, according to the following balance of energy:

$$W_i \pi a^2 L = h 2 \pi a L \Delta T \tag{1.1}$$

At standard atmospheric pressure and temperature controlled, the medium would be air at values ranging from 20 to 30 °C. The approximate value of h, used as known quantity in the analysis is linked to velocity of cross flow air via the correlation:

$$h = 11\sqrt{(V + 0.25) / (0.25)} \ [\text{W/m}^2\text{K}] \tag{1.2}$$

According to [6] equation lumps together the effect of natural convection, forced convection and radiation.

2. Analytical Solution of Case 1

As we want to make a comparison of experimental data with FE computer modelling and exact solutions we set up a simple test stand shown in Figure 3.

Figure 3 Test Setup

The governing equation for conduction is:

$$\nabla^2 \theta + \frac{W_i}{k} = \frac{\rho c_p}{k} \frac{\partial \theta}{\partial t}$$

Where ∇^2 is the laplacian operator, θ is the spatial temperature distribution referred to the known ambient

temperature T_∞.

When the exact steady solution is sought the unsteady term on the r.h.s. (right hand side) is zero. In cylindrical coordinates, with z and r the only spatial coordinates and heat generation, the general equation becomes:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right)+\frac{\partial^2 \theta}{\partial z^2}+\frac{W_i}{k}=0 \tag{2.1}$$

ρ is material density assumed the body is homogeneous, k is the material thermal conductivity, W_i is the volumetric heat generation, c is the specific heat.

The boundary conditions are the following:

z any value	r=0	$\dfrac{\partial \theta}{\partial r}=0$	b.c.1
z any value	r=a	$\dfrac{\partial \theta}{\partial r}=-\dfrac{h}{k}\theta(a,z)$	b.c.2
r any value	z=L/2	$\dfrac{\partial \theta}{\partial z}=0$	b.c.3
r any value	z=L	$\dfrac{\partial \theta}{\partial z}=-\dfrac{h}{k}\theta(r,L)$	b.c.4

If there were no heat generation, the equation would be solved through the separation of variables, since boundary conditions are homogeneous in terms of temperature difference with ambient medium.

Since the volumetric power density W_i is the known term in the equation, we moved to investigate a solution given by superposition as the sum of two effects θ_1 and θ_2, that obey to two different differential equations: a steady state one-dimensional heat equation with power density and a partial second order differential equation, homogeneous in θ_2 with no heat generation:

$$\theta(r,z)=\theta_1(r)+\theta_2(r,z) \tag{2.2}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta_1}{dr}\right)+\frac{W_i}{k}=0 \tag{2.3}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta_2}{\partial r}\right)+\frac{\partial^2 \theta_2}{\partial z^2}=0 \tag{2.4}$$

Then the new boundary conditions are, taking into account symmetry at center r=0, z=L:

For eq.(2.3)

for r=0	$\dfrac{d\theta_1}{dr}=0$	b.c.5
for r=a	$\dfrac{d\theta_1}{dr}=-\dfrac{h}{k}\theta_1(a)$	b.c.6

For eq.(2.4)

for z, r=0	$\dfrac{\partial \theta_2}{\partial r}=0$	b.c.7

for z, r=a $$\frac{\partial \theta_2}{\partial r} = -\frac{h}{k}\theta_2(a,z)$$ b.c.8

for r, z=0 $$\frac{\partial \theta_2}{\partial z} = 0$$ b.c.9

for r, z=L $$\frac{\partial \theta_2}{\partial z} = -\frac{h}{k}\left[\theta_1(r)+\theta_2(r,L)\right]$$ b.c.10

The solution of equation (2.3) is straight forward, coupled with heat balance:

Volumetric heat generation * Volume of the body = Convection heat coefficient * External surface * θ_{surf} that brings to a simplified expression:

$$\frac{W_i a}{h(2+a/L)} = \theta_{surf} \tag{2.5}$$

$$\theta_1(r) = \frac{W_i a^2}{4k}(1-\frac{r^2}{a^2}) + \frac{W_i a}{h(2+a/L)} \tag{2.6}$$

$$\theta_2(r,z) = R(r)Z(z) \tag{2.7}$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{Z''}{Z} = 0 \tag{2.8}$$

$$Z - \alpha^2 Z = 0 \tag{2.9}$$

$$r^2 R'' + r R' + \alpha^2 r^2 R = 0 \tag{2.10}$$

The general solution of the above set of equations is:

$$\theta_2 = \left[C_1 \sinh(\alpha z) + C_2 \cosh(\alpha z)\right]\left[C_3 J_0(\alpha r) + C_4 Y_0(\alpha r)\right] \tag{2.11}$$

The above set of ordinary differential equations (ODE) in conjunction with the boundary conditions gives:

$$\frac{\alpha a}{Bi} = \frac{J_0(\alpha a)}{J_1(\alpha a)} \tag{2.12}$$

$$(Bi = ha/k)$$

$$\lambda_n = \alpha_n a \tag{2.13}$$

$$\theta_{2,n} = C_n \cosh(\alpha_n z) J_0(\alpha_n r) \tag{2.14}$$

We made a use of the properties of Bessel and trigonometric functions to cancel part of the 4 unknown constants and solve the B.C. equations (first order ODE) for which the roots (eigenvalues) are obtained.

We now are left with infinite functions $\theta_{2,n}$ unknown by a constant C_n, because they satisfy b.c. 7,8 and 9. Therefore we will look for a series expansion that satisfies the remaining b.c. 10.

$$\frac{\partial \theta_{2,n}}{\partial z}\bigg|_{z=L} = -\frac{h}{k}\Big[\theta_1(r) + \sum{}_n C_n \cos(\alpha_n L) J_0(\alpha_n r)\Big] \tag{2.15}$$

$$C_n\left\{\alpha_n \sinh(\alpha_n L)\int_0^a rJ_0{}^2(\alpha_n r)dr + \frac{h}{k}\cosh(\alpha_n L)\int_0^L rJ_0{}^2(\alpha_n r)dr\right\}$$
$$= -\frac{hW_i a^2}{4k^2}\int_0^a rJ_0(\alpha_n r)dr + \frac{hW_i}{4k^2}\int_0^a r^3 J_0(\alpha_n r)dr - \frac{h}{k}\frac{W_i a}{h(2+a/L)}\int_0^a rJ_0(\alpha_n r)dr \tag{2.16}$$

$$\frac{a^2}{2}C_n = \left[\alpha_n \sinh(\alpha_n L) + \frac{h}{k}\cosh(\alpha_n L)\right]\left[J_1{}^2(\alpha_n a) + J_0{}^2(\alpha_n a)\right]$$
$$= \frac{hW_i a^2}{2k^2 \alpha_n{}^2}J_0(\alpha_n a) + \frac{a}{\alpha_n}J_1(\alpha_n a)\left[-\frac{h}{k}B - \frac{hW_i}{k^2 \alpha_n{}^2}\right] \tag{2.17}$$

$$\left(B = \frac{W_i a}{h(2+a/L)}\right)$$

$$C_n = \frac{\dfrac{hW_i aBi}{k^2 \alpha_n J_1(\lambda_n)} - \dfrac{2Bi^2 W_i}{\alpha_n J_1(\lambda_n)k(2+a/L)} - \dfrac{2hW_i Bi^2}{\alpha_n{}^3 ak^2 J_1(\lambda_n)}}{\left[\alpha_n \sinh(\alpha_n L) + \dfrac{h}{k}\cosh(\alpha_n L)\right]\left[Bi^2 + \lambda_n{}^2\right]} \tag{2.18}$$

Final Solution: $\theta(r,z) = \theta_1(z) + \theta_2(r,z)$

$$\theta = \frac{W_i a^2}{4k}\left(1 - \frac{r^2}{a^2}\right) + \frac{W_i a}{h(2+a/L)} + \sum_{n=1}^{\infty} C_n \cosh(\alpha_n z)J_0(\alpha_n r) \tag{2.19}$$

Since the analytical solution is the summation of an infinite series, only a finite number of terms are taken for obtaining the analytical solution after checking the effect of number of terms on the result. It is found that with more than 30 terms the solution converges, it does not change appreciably when an additional number of terms is used.

The above expression may easily be inserted in a palmar calculator that has "math" solvers. How far is the analytical solution from more sophisticated FEM solvers that allow anisotropy of conductivity and 3-D analysis with finer grids is shown in the following Figures.
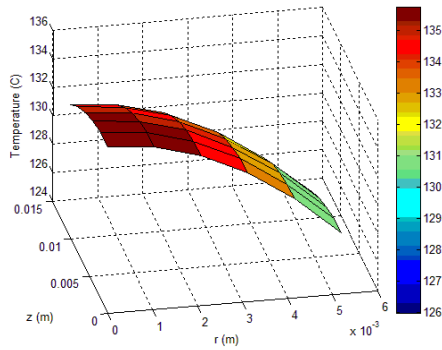
Figure 4 Temperature distribution of capacitor
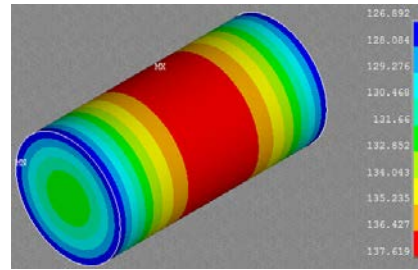with analytical solution.

Figure 5 Temperature distribution of capacitor
with FEM analysis.

2.1. FEM analysis and comparison

The discrepancy between the numerical solution and the analytical solution, at least in this case, is low and do not suggest the need to move to a complete FEM study with commercial solvers.

The FEM software package we used –widely popular in stress and thermal simulation- allows the data input of anisotropic material and the temperature dependency of thermal conductivity. It allows also a full three dimension analysis. We then used it to simulate the conduction in the capacitor of identical drawing, properties and b.c.
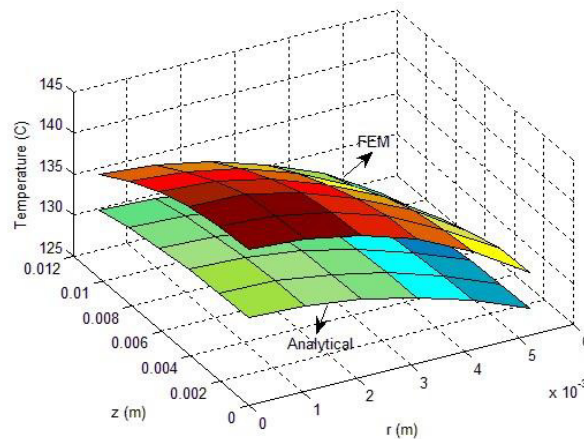

Figure 6 Comparison of Analytical and FEM Analysis.

The results are shown in Figure 4, 5 and 6 as graphical output in the form of a colour plot of the temperature distribution by z and r. It appears to be a good correlation between the simplified 2-D, constant physical properties exact solution and the more close to reality capacitor data as input to FEM.

Finite Element analysis shows that if the local to average properties of conductivity vary in a few percent, the temperature predicted by exact solution with constant properties is quite close with FEM solution with temperature dependency.

We conclude that the analytical method provides results in a short time and may give a hint on the behaviour of the capacitor. The FEM was a tool to check the accuracy of the model. More complex study will rely to finer FEM

grid points and computer code simulation via finite element methods.

3. Analytical Solution of Case 2

The general governing equation for the conduction is the same:

$$\nabla^2 \theta + \frac{W_i}{k} = \frac{\rho c_p}{k} \frac{\partial \theta}{\partial t}$$

(3.1)

Where ∇^2 is the laplacian operator. θ is the spatial temperature distribution referred to the known ambient temperature T_∞.

In the case of beef cake under sterilization there is no volumetric heat generation and the case is unsteady. Therefore the general equation becomes:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} = \frac{1}{a_t} \frac{\partial T}{\partial t}$$

(3.2)

where; $a_t = \frac{k}{\rho c}$

Now the equation is solved with the four spatial boundary conditions as case 1 taking into account symmetry at center z, r=0 and r, z=L. An additional information is needed: initial temperature that is equal to $\theta_i = T_{initial} - T_\infty$ where T_∞ is the heat environment temperature. A product solution is now possible in the form of the separation of variables:

$$\theta(r, z, t) = R(r) \cdot Z(z) \cdot \tau(t)$$

(3.3)

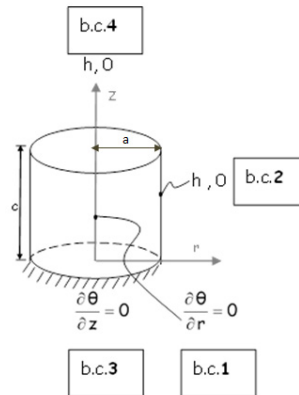Since B.C. are homogenous as shown in the following figure:

Figure 7 Problem description

And we obtain the following set of ODE:

$$\frac{R^{''}}{R} + \frac{1}{r} \frac{R^{'}}{R} = -\alpha^2$$

(3.4)

$$\frac{Z^{"}}{Z} = -\beta^2 \tag{3.5}$$

$$\frac{1}{a_t}\frac{\tau^{'}}{\tau} = -\left(\alpha^2 + \beta^2\right) \tag{3.6}$$

With the B.C. expressed as first order equations that bring to the following solutions:

- The solution of equation in z is:

$$Z = C_3 \, sin\left(\beta z\right) + C_4 \cos\left(\beta z\right) \tag{3.7}$$

And coefficient C_3 is easily eliminated by use of B.C.3.
The solution in r is:

$$R = C_1 \, J_0\left(\alpha r\right) + C_2 \, Y_0\left(\alpha r\right) \tag{3.8}$$

With J and Y the known Bessel functions are [9]. Coefficient C_2 is eliminated by B.C.1.
B.C.2 and 4 give rise to ODE of the first order.

$$tg\left(\beta c\right) = \frac{Bi_c}{\beta c} \tag{3.9}$$

$$\frac{J_1\left(\alpha a\right)}{J_0\left(\alpha a\right)} = \frac{Bi_a}{\alpha a} \tag{3.10}$$

The infinite roots $\alpha_1 a, \alpha_2 a, ..., \alpha_n a,...$ and $\beta_1 c, \beta_2 c, ..., \beta_m c,...$ satisfy the equations.

- The solution of equation in t is:

$$\tau = \exp[-\left(\alpha_n^2 + \beta_m^2\right)a_t \, t] \tag{3.11}$$

We have an infinite series of roots in r and z. Then we apply the initial condition θ_i and assume that is equal to a series of functions with unknown C_{nm}.

We make a use of the orthogonality of function J and cos with respect to the weighting factor r over the finite interval 0, a and 0, c. We multiply both sides of equation by J*r and cos, integrate the result over the said interval with the assumption that the integral of the infinite sum is equivalent to the sum of integrals. We have:

$$\int_0^a \theta_i \, r \, J_0\left(\alpha_n r\right) dr \int_0^c \cos\left(\beta_m z\right) dz = C_{nm} \int_0^a r \, J_0^{\,2}\left(\alpha_n r\right) dr \int_0^c \cos^2\left(\beta_m z\right) dz \tag{3.12}$$

Hence equation (3.12) gives:

$$C_n = \frac{2 \, Bi_a}{(Bi_a^{\,2} + \lambda_n^{\,2}) J_0\left(\lambda_n\right)} \tag{3.13}$$

$$C_m = \frac{2\sin(\beta_m c)}{\beta_m c + \cos(\beta_m c)\sin(\beta_m c)} \tag{3.14}$$

The final solution in dimensionless form is:

$$\frac{\theta(r,z,t)}{\theta_i} = 2\,Bi_a \sum_{n=1}^{\infty} \frac{J_0\left(\lambda_n \frac{r}{a}\right)e^{-\lambda_n^2 Fo_a}}{(\lambda_n^2 + Bi_a^2)J_0(\lambda_n)} \bullet 2\sum_{m=1}^{\infty} \frac{\sin(\beta_m c)\cos(\beta_m z)e^{-\beta_m^2 a_t t}}{\beta_m c + \sin(\beta_m c)\cos(\beta_m c)} \tag{3.15}$$

Where; $\dfrac{a_t}{a^2}t = Fo_a$ (Fourier Number)   and $\alpha_n\,a = \lambda_n$

3.1. FEM analysis and comparison

In both exact analysis and FEM we assumed the axial thermal symmetry of the material with respect with the abscissa z and r. Dependency of temperature would be then to r, z and time for the FEM solver also.

The solid behaviour of the food is assumed. It is enforced the hypothesis that the heat convective forced flow is uniform all around the can and on both the caps. In the FEM the conductivity of the material is a mild function of temperature and isotropy is given.

The restitution of results is shown in pictures 8, 9, 10 and the comparison with experimental data is offered in the Figure 12.

FEM apparently underestimates the speed of warm up in the material more than it does the exact solution. However the simulation forecasts a time of sterilization larger than in real world, and it is in favour of safety.
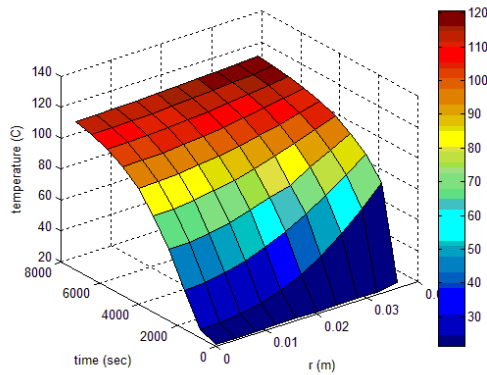

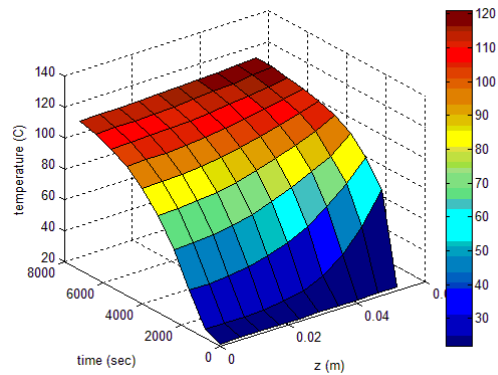Figure 8 Temperature variation at r direction (z=0) during 7200 sec with analytical solution.

Figure 9 Temperature variation at z direction (r=0) during 7200 sec with analytical solution.
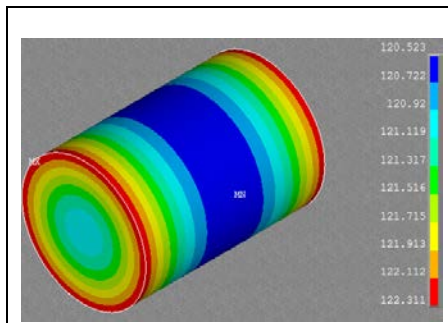
Figure 10 The temperature distribution of ailment cake after 7200 sec with FEM analysis.
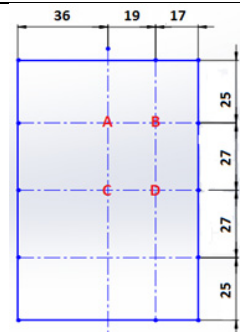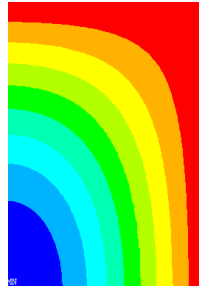
Figure 11 Locations within the box in which the temperature is measured.

In this particular case a better match with experiments is obtained by the simple hypothesis that external tincan temperature is equal to environmental T_∞ since forced ventilation is quite turbulent.

Figure 12 (a), (b), (c) and (d) shows the different situations (analytical, FEM) as compared to measurements.
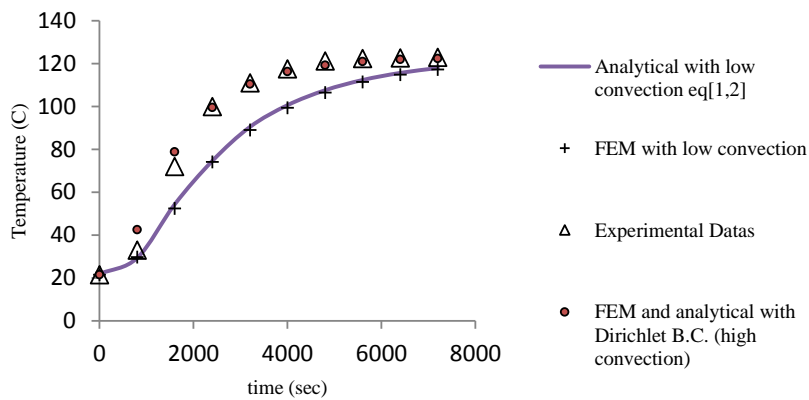

Figure 12 (a) Comparison of Analytical, Numerical and Experimental results of temperature variation at A point.
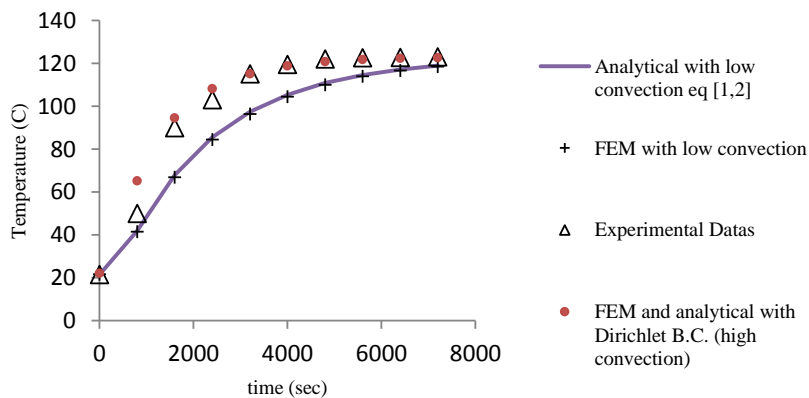

Figure 12 (b) Comparison of Analytical, Numerical and Experimental results of temperature variation at B point.
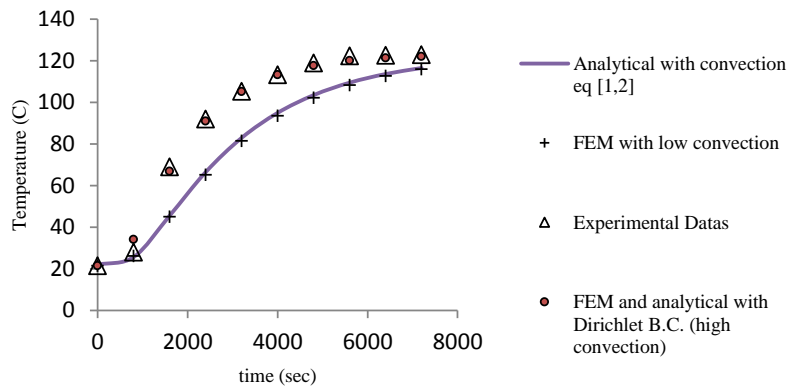
Figure 12 (c) Comparison of Analytical, Numerical and Experimental results of temperature variation at C point.
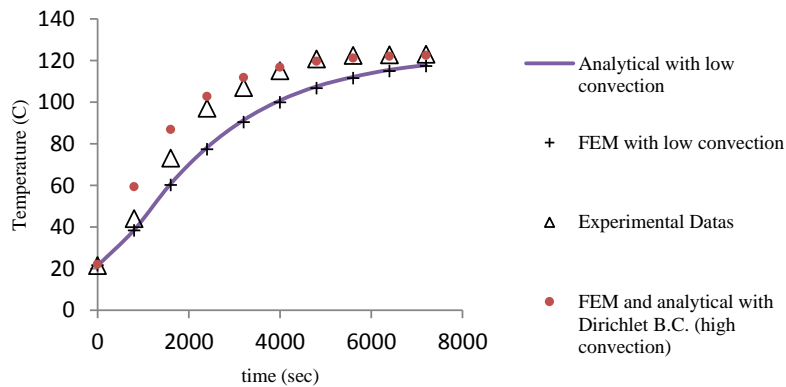

Figure 12 (d) Comparison of Analytical, Numerical and Experimental results of temperature variation at D point.

4. Conclusions

An analytical solution is obtained for the steady state two-D temperature distribution in a capacitor, and for the two-D transient temperature in a food can under sterilization at high temperature.

In both cases are known: internal volumetric heat generation (1st case) and initial temperature (2nd case) with boundary conditions given as assigned convection, environment temperature and surface heat transfer coefficient.

The results are obtained in terms of a series expansion solution involving Bessel functions based on the principle of superposition (1st case) and separation of variables (1st case), and superposition only in three variables (2nd case).

The eigenvalues required by the series expansion are solved by root solving method making an use of orthogonality of Bessel functions and homogeneous b.c. together. A not homogeneous 4th b.c. brings to a Fourier analysis with an unknown coefficient left.

It has been found that a finite number of roots can be used to obtain the analytical solution with reasonable accuracy. In particular 30 terms for the two-D problem were found to be sufficient.

A comparison between the analytical results and the numerical results obtained through a FEM package were in excellent agreement at least in the capacitor analysis.

A larger discrepancy was found in the case of food sterilization. However, analytical results match better the experimental data, in this case.

Therefore the analytical solution is valuable because it is obtained by simple math solvers and palmar calculators and is a means of validating the numerical schemes or vice versa when experimental data of the engineering problem are available.

Also the analytical solution can be used to obtain an accurate temperature distribution in any location of the object as alternative to a large population of temperature values in grid points.

References

[1] Arpaci VS. Conduction Heat Transfer. Reading, Massachusetts: Addison-Wesley; 1966.

[2] Eckert ERG, Drake RM. Analysis of Heat and Mass Transfer. New York: Hemisphere Publishing Co.; 1987.

[3] Floris F, Ilemin B, and Orrù FF. Quasi 1-D analysis of heat equation with exact solutions and comparison with numerical simulation in liquid/vapour pressure tanks, waterwalls and hot drawing machines. Paper presented at: 31st UIT Heat Transfer Conference; 2013 June 25-27; Como, Italy.

[4] Floris F, Ibba G. Determinazione del campo di temperatura in una pala di turbina esposta a flussi termici non simmetrici, e parzialmente refrigerata al contorno. Paper presented at: 61° Congresso Nazionale ATI.; 2006 Settembre 11-15; Perugia, Italy.

[5] Floris F. et al. Validazione di un modello di simulazione del flusso termico in una giunzione Friction Stir Welded. 1° Congresso Nazionale del Coordinamento della Meccanica Italiana, Palermo, Italy, 20-22 Giugno, 2010.

[6] Parler Jr SG. Thermal modelling of aluminium electrolytic capacitors. Paper presented at: IEEE Industry Applications Society Conference; 1999 October.

[7] Mazzola P, Cindio di B, Mignoli L, Massini R, Dipollina G, Pedrelli T, et al. Applicazione del metodo agli elementi finiti alla sterilizzazione di omogeneizzato di manzo in scatola.

[8] Carslaw HS, Jaeger JC. Conduction of Heat in Solids. 2nd Edition. Oxford: Clarendon Press; 1959.

[9] Bowman F. Introduction to Bessel Functions. New York: Dover Publications; 1958.