

## A Quantum Computational Semantics for Epistemic Logical Operators. Part I: Epistemic Structures

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**Abstract** Some critical open problems of epistemic logics can be investigated in the framework of a quantum computational approach. The basic idea is to interpret sentences like “Alice knows that Bob does not understand that  $\pi$  is irrational” as pieces of quantum information (generally represented by density operators of convenient Hilbert spaces). Logical epistemic operators (*to understand, to know...*) are dealt with as (generally irreversible) quantum operations, which are, in a sense, similar to measurement-procedures. This approach permits us to model some characteristic epistemic processes, that concern both human and artificial intelligence. For instance, the operation of “memorizing and retrieving information” can be formally represented, in this framework, by using a quantum teleportation phenomenon.

**Keywords** Truth perspectives · Epistemic structure · Quantum channel

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## 1 Introduction

Logical theories of epistemic operators (*to know, to believe, . . .*) have given rise to a number of interesting open questions. Most standard approaches (based on extensions of classical logic) succeed in modelling a general notion of “potential knowledge”. In this framework, a sentence like “Alice knows that  $\pi$  is irrational” turns out to have the meaning “Alice *could know* that  $\pi$  is irrational”, rather than “Alice *actually knows* that  $\pi$  is irrational”. A consequence of such a strong characterization of knowledge is the unrealistic phenomenon of *logical omniscience*, according to which knowing a sentence implies knowing all its logical consequences.

A weaker approach to epistemic logics can be developed in the framework of a *quantum computational semantics*. The aim is trying to describe forms of “actual knowledge”, which should somehow reflect the real limitations both of human and of artificial intelligence.

In quantum computational semantics meanings of sentences are represented as pieces of quantum information (mathematically described as density operators living in convenient Hilbert spaces), while the logical connectives correspond to special examples of quantum logical gates. How to interpret, in this framework, epistemic sentences like “Alice knows that Bob does not understand that  $\pi$  is irrational”? The leading idea can be sketched as follows. The semantics is based on abstract structures that contain finite sets of *epistemic agents* evolving in time. Each agent (say, Alice at a particular time) is characterized by two fundamental epistemic parameters:

- A set of density operators, representing the information that is accessible to our agent.
- A “truth-conception” (called the *truth-perspective* of the agent in question), which is technically determined by the choice of an orthonormal basis of the two-dimensional Hilbert space  $\mathbb{C}^2$ . In this way, any pair of qubits, corresponding to the elements of the basis that has been chosen, can be regarded as a particular idea about the truth-values *Truth* and *Falsity*. From a physical point of view, we can imagine that a truth-perspective is associated to a physical apparatus that permits one to measure a given observable.

The knowledge operations, described in this semantics, turn out to be deeply different from quantum logical gates, since they cannot be, generally, represented by unitary quantum operations. The “act of knowing” seems to involve some intrinsic irreversibility, which is, in a sense, similar to what happens in the case of measurement-procedures.

The first part of this article is devoted to a mathematical description of the notion of *epistemic structure* in a Hilbert-space environment, while the semantics for an epistemic quantum computational language is developed in the second part. We will analyze, in this framework, some epistemic situations that seem to characterize “real” processes of acquiring and transmitting information.

## 2 Quantum Information and Truth-Perspectives

We will first recall some basic notions of quantum computation that will be used in our semantics. The general mathematical environment is the  $n$ -fold tensor product of the Hilbert space  $\mathbb{C}^2$ :

$$\mathcal{H}^{(n)} := \underbrace{\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_{n\text{-times}},$$

where all pieces of quantum information live. The elements  $|1\rangle = (0, 1)$  and  $|0\rangle = (1, 0)$  of the canonical orthonormal basis  $B^{(1)}$  of  $\mathbb{C}^2$  represent, in this framework, the two classical

bits, which can be also regarded as the canonical truth-values *Truth* and *Falsity*, respectively. The canonical basis of  $\mathcal{H}^{(n)}$  is the set

$$B^{(n)} = \{|x_1\rangle \otimes \cdots \otimes |x_n\rangle : |x_1\rangle, \dots, |x_n\rangle \in B^{(1)}\}.$$

As usual, we will briefly write  $|x_1, \dots, x_n\rangle$  instead of  $|x_1\rangle \otimes \cdots \otimes |x_n\rangle$ . By definition, a *quregister* is a unit vector of  $\mathcal{H}^{(n)}$ : *qubits* are special cases of quregisters, living in the space  $\mathcal{H}^{(1)}$ . Quregisters thus correspond to pure states, namely to maximal pieces of information about the quantum systems that are supposed to store a given amount of quantum information. We shall also make reference to *mixtures* of quregisters, to be called *qumixes*, associated to density operators  $\rho$  of  $\mathcal{H}^{(n)}$ . We will denote by  $\mathfrak{D}(\mathcal{H}^{(n)})$  the set of all qumixes of  $\mathcal{H}^{(n)}$ , while  $\mathfrak{D} = \bigcup_n \{\mathfrak{D}(\mathcal{H}^{(n)})\}$  will represent the set of all possible qumixes. Of course, quregisters can be represented as special cases of qumixes, having the form  $P_{|\psi\rangle}$  (the projection over the one-dimensional closed subspace determined by the quregister  $|\psi\rangle$ ).

From an intuitive point of view, a basis-change in  $\mathbb{C}^2$  can be regarded as a change of our *truth-perspective*. While in the canonical case, the truth-values *Truth* and *Falsity* are identified with the two classical bits  $|1\rangle$  and  $|0\rangle$ , assuming a different basis corresponds to a different idea of *Truth* and *Falsity*. Since any basis-change in  $\mathbb{C}^2$  is determined by a unitary operator, we can identify a *truth-perspective* with a unitary operator  $\mathfrak{T}$  of  $\mathbb{C}^2$ . We will write:

$$|1_{\mathfrak{T}}\rangle = \mathfrak{T}|1\rangle; \quad |0_{\mathfrak{T}}\rangle = \mathfrak{T}|0\rangle,$$

and we will assume that  $|1_{\mathfrak{T}}\rangle$  and  $|0_{\mathfrak{T}}\rangle$  represent respectively the truth-values *Truth* and *Falsity* of the truth-perspective  $\mathfrak{T}$ . The *canonical truth-perspective* is, of course, determined by the identity operator  $I$  of  $\mathbb{C}^2$ . We will indicate by  $B_{\mathfrak{T}}^{(1)}$  the orthonormal basis determined by  $\mathfrak{T}$ ; while  $B_1^{(1)}$  will represent the canonical basis.

Any unitary operator  $\mathfrak{T}$  of  $\mathcal{H}^{(1)}$  can be naturally extended to a unitary operator  $\mathfrak{T}^{(n)}$  of  $\mathcal{H}^{(n)}$  (for any  $n \geq 1$ ):

$$\mathfrak{T}^{(n)}|x_1, \dots, x_n\rangle = \mathfrak{T}|x_1\rangle \otimes \cdots \otimes \mathfrak{T}|x_n\rangle.$$

Accordingly, any choice of a unitary operator  $\mathfrak{T}$  of  $\mathcal{H}^{(1)}$  determines an orthonormal basis  $B_{\mathfrak{T}}^{(n)}$  for  $\mathcal{H}^{(n)}$  such that:

$$B_{\mathfrak{T}}^{(n)} = \{\mathfrak{T}^{(n)}|x_1, \dots, x_n\rangle : |x_1, \dots, x_n\rangle \in B_1^{(n)}\}.$$

Instead of  $\mathfrak{T}^{(n)}|x_1, \dots, x_n\rangle$  we will also write  $|x_{1_{\mathfrak{T}}}, \dots, x_{n_{\mathfrak{T}}}\rangle$ .

The elements of  $B_{\mathfrak{T}}^{(1)}$  will be called the  *$\mathfrak{T}$ -bits* of  $\mathcal{H}^{(1)}$ ; while the elements of  $B_{\mathfrak{T}}^{(n)}$  will represent the  *$\mathfrak{T}$ -registers* of  $\mathcal{H}^{(n)}$ .

On this ground the notions of *truth*, *falsity* and *probability* with respect to any truth-perspective  $\mathfrak{T}$  can be defined in a natural way.

**Definition 2.1** ( $\mathfrak{T}$ -true and  $\mathfrak{T}$ -false registers)

- $|x_{1_{\mathfrak{T}}}, \dots, x_{n_{\mathfrak{T}}}\rangle$  is a  *$\mathfrak{T}$ -true register* iff  $|x_{n_{\mathfrak{T}}}\rangle = |1_{\mathfrak{T}}\rangle$ ;
- $|x_{1_{\mathfrak{T}}}, \dots, x_{n_{\mathfrak{T}}}\rangle$  is a  *$\mathfrak{T}$ -false register* iff  $|x_{n_{\mathfrak{T}}}\rangle = |0_{\mathfrak{T}}\rangle$ .

In other words, the  $\mathfrak{T}$ -truth-value of a  $\mathfrak{T}$ -register (which corresponds to a sequence of  $\mathfrak{T}$ -bits) is determined by its last element.<sup>1</sup>

**Definition 2.2** ( $\mathfrak{T}$ -truth and  $\mathfrak{T}$ -falsity)

- The  $\mathfrak{T}$ -truth of  $\mathcal{H}^{(n)}$  is the projection operator  ${}^{\mathfrak{T}}P_1^{(n)}$  that projects over the closed subspace spanned by the set of all  $\mathfrak{T}$ - true registers.
- The  $\mathfrak{T}$ -falsity of  $\mathcal{H}^{(n)}$  is the projection operator  ${}^{\mathfrak{T}}P_0^{(n)}$  that projects over the closed subspace spanned by the set of all  $\mathfrak{T}$ - false registers.

In this way, truth and falsity are dealt with as mathematical representatives of possible physical properties. Accordingly, by applying the Born-rule, one can naturally define the probability-value of any qumix with respect to the truth-perspective  $\mathfrak{T}$ .

**Definition 2.3** ( $\mathfrak{T}$ -Probability) For any  $\rho \in \mathfrak{D}(\mathcal{H}^{(n)})$ ,

$$p_{\mathfrak{T}}(\rho) := \text{Tr}({}^{\mathfrak{T}}P_1^{(n)}\rho),$$

where  $\text{Tr}$  is the trace-functional.

We interpret  $p_{\mathfrak{T}}(\rho)$  as the probability that the information  $\rho$  satisfies the  $\mathfrak{T}$ -Truth. In the particular case of qubits, we will obviously obtain:

$$p_{\mathfrak{T}}(a_0|0_{\mathfrak{T}} + a_1|1_{\mathfrak{T}}) = |a_1|^2.$$

For any choice of a truth-perspective  $\mathfrak{T}$ , the set  $\mathfrak{D}$  of all qumixes can be pre-ordered by a relation that is defined in terms of the probability-function  $p_{\mathfrak{T}}$ .

**Definition 2.4** (Preorder)

$$\rho \leq_{\mathfrak{T}} \sigma \quad \text{iff} \quad p_{\mathfrak{T}}(\rho) \leq p_{\mathfrak{T}}(\sigma).$$

As is well known, quantum information is processed by *quantum logical gates* (briefly, *gates*): unitary operators that transform quregisters into quregisters in a reversible way. Let us recall the definition of some gates that play a special role both from the computational and from the logical point of view.

**Definition 2.5** (The negation) For any  $n \geq 1$ , the *negation* on  $\mathcal{H}^{(n)}$  is the linear operator  $\text{NOT}^{(n)}$  such that, for every element  $|x_1, \dots, x_n\rangle$  of the canonical basis,

$$\text{NOT}^{(n)}|x_1, \dots, x_n\rangle = |x_1, \dots, x_{n-1}\rangle \otimes |1 - x_n\rangle.$$

In particular, we obtain:

$$\text{NOT}^{(1)}|0\rangle = |1\rangle; \quad \text{NOT}^{(1)}|1\rangle = |0\rangle,$$

according to the classical truth-table of negation.

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<sup>1</sup>As we will see, the application of a classical reversible gate to a register  $|x_1, \dots, x_n\rangle$  transforms the (canonical) bit  $|x_n\rangle$  into the target-bit  $|x'_n\rangle$ , which behaves as the final truth-value. This justifies our choice in Definition 2.1.

**Definition 2.6** (The Toffoli gate) For any  $n, m, p \geq 1$ , the *Toffoli gate* is the linear operator  $\mathbb{T}^{(n,m,p)}$  defined on  $\mathcal{H}^{(n+m+p)}$  such that, for every element  $|x_1, \dots, x_n\rangle \otimes |y_1, \dots, y_m\rangle \otimes |z_1, \dots, z_p\rangle$  of the canonical basis,

$$\begin{aligned} &\mathbb{T}^{(n,m,p)}|x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_p\rangle \\ &= |x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_{p-1}\rangle \otimes |x_n y_m \hat{+} z_p\rangle, \end{aligned}$$

where  $\hat{+}$  represents the addition modulo 2.

**Definition 2.7** (The XOR-gate) For any  $n, m \geq 1$ , the *Toffoli gate* is the linear operator  $\text{XOR}^{(n,m)}$  defined on  $\mathcal{H}^{(n+m)}$  such that, for every element  $|x_1, \dots, x_n\rangle \otimes |y_1, \dots, y_m\rangle$  of the canonical basis,

$$\text{XOR}^{(n,m)}|x_1, \dots, x_n, y_1, \dots, y_m\rangle = |x_1, \dots, x_n, y_1, \dots, y_{m-1}\rangle \otimes |x_n \hat{+} y_m\rangle,$$

where  $\hat{+}$  represents the addition modulo 2.

**Definition 2.8** (The SWAP-gate) For any  $n \geq 1$ , for any  $i$  and for any  $j$  (where  $1 \leq i \leq n$  and  $1 \leq j \leq n$ ), the *SWAP gate* is the linear operator  $\text{SWAP}_{(i,j)}^{(n)}$  defined on  $\mathcal{H}^{(n)}$  such that, for every element  $|x_1, \dots, x_i, \dots, x_j, \dots, x_n\rangle$  of the canonical basis,

$$\text{SWAP}_{(i,j)}^{(n)}|x_1, \dots, x_i, \dots, x_j, \dots, x_n\rangle = |x_1, \dots, x_j, \dots, x_i, \dots, x_n\rangle.$$

In other words,  $\text{SWAP}_{(i,j)}^{(n)}$  exchanges the  $i$ -th with the  $j$ -th element in any element of the basis.

**Definition 2.9** (The Hadamard-gate) For any  $n \geq 1$ , the *Hadamard-gate* on  $\mathcal{H}^{(n)}$  is the linear operator  $\sqrt{\mathbb{I}}^{(n)}$  such that for every element  $|x_1, \dots, x_n\rangle$  of the canonical basis:

$$\sqrt{\mathbb{I}}^{(n)}|x_1, \dots, x_n\rangle = |x_1, \dots, x_{n-1}\rangle \otimes \frac{1}{\sqrt{2}}((-1)^{x_n}|x_n\rangle + |1 - x_n\rangle).$$

In particular we obtain:

$$\sqrt{\mathbb{I}}^{(1)}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle); \quad \sqrt{\mathbb{I}}^{(1)}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Hence,  $\sqrt{\mathbb{I}}^{(1)}$  transforms bits into genuine qubits.

**Definition 2.10** (The square root of NOT) For any  $n \geq 1$ , the *square root of NOT* on  $\mathcal{H}^{(n)}$  is the linear operator  $\sqrt{\text{NOT}}^{(n)}$  such that for every element  $|x_1, \dots, x_n\rangle$  of the canonical basis:

$$\sqrt{\text{NOT}}^{(n)}|x_1, \dots, x_n\rangle = |x_1, \dots, x_{n-1}\rangle \otimes \left( \frac{1-i}{2}|x_n\rangle + \frac{1+i}{2}|1-x_n\rangle \right),$$

where  $i = \sqrt{-1}$ .

All gates can be naturally transposed from the canonical truth-perspective to any truth-perspective  $\mathfrak{T}$ . Let  $G^{(n)}$  be any gate defined with respect to the canonical truth-perspective. The *twin-gate*  $G_{\mathfrak{T}}^{(n)}$ , defined with respect to the truth-perspective  $\mathfrak{T}$ , is determined as follows:

$$G_{\mathfrak{T}}^{(n)} := \mathfrak{T}^{(n)} G^{(n)} \mathfrak{T}^{(n)\dagger},$$

where  $\mathfrak{T}^{(n)\dagger}$  is the adjoint of  $\mathfrak{T}$ .

All  $\mathfrak{T}$ -gates can be canonically extended to the set  $\mathfrak{D}$  of all qumixes. Let  $G_{\mathfrak{T}}$  be any gate defined on  $\mathcal{H}^{(n)}$ . The corresponding *qumix gate* (also called *unitary quantum operation*)  ${}^{\mathfrak{D}}G_{\mathfrak{T}}$  is defined as follows for any  $\rho \in \mathfrak{D}(\mathcal{H}^{(n)})$ :

$${}^{\mathfrak{D}}G_{\mathfrak{T}}\rho = G_{\mathfrak{T}}\rho G_{\mathfrak{T}}^{\dagger}.$$

It is interesting to consider a convenient notion of *distance* between truth-perspectives. As is well known, different definitions of distance between vectors can be found in the literature. For our aims it is convenient to adopt the Fubini-Study definition of distance between two qubits.

**Definition 2.11** (The Fubini-Study distance) Let  $|\psi\rangle$  and  $|\varphi\rangle$  be two qubits.

$$d(|\psi\rangle, |\varphi\rangle) = \frac{2}{\pi} \arccos |\langle \psi | \varphi \rangle|.$$

This notion of distance satisfies the following conditions:

- (1)  $d(|\psi\rangle, |\varphi\rangle)$  is a metric distance;
- (2)  $|\psi\rangle \perp |\varphi\rangle \Rightarrow d(|\psi\rangle, |\varphi\rangle) = 1$ ;
- (3)  $d(|1\rangle, |1_{Bell}\rangle) = \frac{1}{2}$ , where  $|1\rangle$  is the canonical truth, while  $|1_{Bell}\rangle = \sqrt{I}^{(1)}|1\rangle = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$  represents the *Bell-truth* (which corresponds to a maximal uncertainty with respect to the canonical truth).

On this ground, one can naturally define the *epistemic distance* between two truth-perspectives.

**Definition 2.12** (Epistemic distance) Let  $\mathfrak{T}_1$  and  $\mathfrak{T}_2$  be two truth-perspectives.

$$d^{Ep}(\mathfrak{T}_1, \mathfrak{T}_2) = d(|1_{\mathfrak{T}_1}\rangle, |1_{\mathfrak{T}_2}\rangle).$$

In other words, the epistemic distance between the truth-perspectives  $\mathfrak{T}_1$  and  $\mathfrak{T}_2$  is identified with the distance between the two qubits that represent the truth-value *Truth* in  $\mathfrak{T}_1$  and in  $\mathfrak{T}_2$ , respectively.

As is well known, a crucial notion of quantum theory and of quantum information is the concept of *entanglement*. Consider a composite quantum system  $S = S_1 + \dots + S_n$ . According to the quantum theoretic formalism, the *reduced state function* determines for any state  $\rho$  of  $S$  the *reduced state*  $Red^{i_1, \dots, i_m}(\rho)$  of any subsystem  $S_{i_1} + \dots + S_{i_m}$  (where  $1 \leq i_1 \leq n, \dots, 1 \leq i_m \leq n$ ). A characteristic case that arises in entanglement-phenomena is the following: while  $\rho$  (the state of the global system) is pure (a maximal information), the reduced state  $Red^{i_1, \dots, i_m}(\rho)$  is generally a mixture (a non-maximal information). Hence our information about the *whole* cannot be reconstructed as a function of our pieces of information about the *parts*.

In the second part of this article we will see how these characteristic *holistic* features of the quantum theoretic formalism will play an important role in the development of the epistemic semantics [4].

**Definition 2.13** (*n*-partite entangled quregister) A quregister  $|\psi\rangle$  of  $\mathcal{H}^{(n)}$  is called an *n-partite entangled* iff all reduced states  $Red^1(|\psi\rangle), \dots, Red^n(|\psi\rangle)$  are proper mixtures.

As a consequence an *n*-partite entangled quregister cannot be represented as a tensor product of the reduced states of its parts.

When all reduced states  $Red^1(|\psi\rangle), \dots, Red^n(|\psi\rangle)$  are the qumix  $\frac{1}{2}I$  (which represents a perfect ambiguous information) one says that  $|\psi\rangle$  is *maximally entangled*.

**Definition 2.14** (Entangled quregister with respect to some parts) A quregister  $|\psi\rangle$  of  $\mathcal{H}^{(n)}$  is called *entangled* with respect to its parts labelled by the indices  $i_1, \dots, i_h$  (with  $1 \leq i_1, \dots, i_h \leq n$ ) iff the reduced states  $Red^{i_1}(|\psi\rangle), \dots, Red^{i_h}(|\psi\rangle)$  are proper mixtures.

Since the notion of reduced state is independent of the choice of a particular basis, it turns out that the status of *n-partite entangled quregisters*, *maximally entangled quregisters* and *entangled quregisters with respect to some parts* is invariant under changes of truth-perspective.

*Example 2.1*

- The quregister

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0, 0, 0\rangle + |1, 1, 1\rangle)$$

- is a 3-partite maximally entangled quregister of  $\mathcal{H}^{(3)}$ ;
- the quregister

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0, 0, 0\rangle + |1, 1, 0\rangle)$$

is an entangled quregister of  $\mathcal{H}^{(3)}$  with respect to its first and second part.

### 3 Epistemic Situations and Epistemic Structures

Any logical analysis of epistemic phenomena naturally refers to a set of agents (say, Alice, Bob, ...), possibly evolving in time. Let  $T = (t_1, \dots, t_n)$  be a sequence of times (which can be thought of as “short” time-intervals) and let  $Ag$  be a finite set of epistemic agents, described as functions of the times in  $T$ . For any  $a \in Ag$  and any  $t$  of  $T$ , we write  $a(t) = a_t$ . Each  $a_t$  is associated with a characteristic *epistemic situation*, which consists of the following elements:

1. A truth-perspective  $\mathfrak{T}_{a_t}$ , representing the truth-conception of  $a$  at time  $t$ .
2. A set  $EpD_{a_t}$  of qumixes, representing the information that is virtually accessible to  $a_t$  (a kind of *virtual memory*).
3. Two epistemic maps  $U_{a_t}$  and  $K_{a_t}$ , that permit us to transform any qumix living in a space  $\mathcal{H}^{(n)}$  into a qumix living in the same space. From an intuitive point of view,  $U_{a_t}\rho$  is to be interpreted as:  $a_t$  understands  $\rho$  (or,  $a_t$  has information about  $\rho$ ); while  $K_{a_t}\rho$  is to be interpreted as:  $a_t$  knows  $\rho$ .

**Definition 3.1** (Epistemic situation) An *epistemic situation* for an agent  $a_t$  is a system

$$EpSit_{a_t} = (\mathfrak{T}_{a_t}, EpD_{a_t}, U_{a_t}, K_{a_t}),$$

where:

1.  $\mathfrak{T}_{a_t}$  is a truth-perspective, representing the truth-conception of  $a_t$ .
2.  $EpD_{a_t}$  is a set of qumixes, representing the virtual memory of  $a_t$ . We indicate by  $EpD_{a_t}^{(n)}$  the set  $EpD_{a_t} \cap \mathfrak{D}(\mathcal{H}^{(n)})$ .
3.  $U_{a_t}$  is a map that assigns to any  $n \geq 1$  a map, called (*logical*) *understanding operation*:

$$U_{a_t}^{(n)} : \mathcal{B}(\mathcal{H}^{(n)}) \mapsto \mathcal{B}(\mathcal{H}^{(n)}),$$

where  $\mathcal{B}(\mathcal{H}^{(n)})$  is the set of all bounded operators of  $\mathcal{H}^{(n)}$ . The following conditions are required:

- 3.1.  $\rho \in \mathfrak{D}(\mathcal{H}^{(n)}) \implies U_{a_t}^{(n)}\rho \in \mathfrak{D}(\mathcal{H}^{(n)})$ .
- 3.2.  $\rho \notin EpD_{a_t}^{(n)} \implies U_{a_t}^{(n)}\rho = \overline{\rho_0}$  (where  $\overline{\rho_0}$  is a fixed element of  $\mathfrak{D}(\mathcal{H}^{(n)})$ ).
4.  $K_{a_t}$  is a map that assigns to any  $n \geq 1$  a map, called (*logical*) *knowledge operation*:

$$K_{a_t}^{(n)} : \mathcal{B}(\mathcal{H}^{(n)}) \mapsto \mathcal{B}(\mathcal{H}^{(n)}).$$

The following conditions are required:

- 4.1.  $\rho \in \mathfrak{D}(\mathcal{H}^{(n)}) \implies K_{a_t}^{(n)}\rho \in \mathfrak{D}(\mathcal{H}^{(n)})$ .
- 4.2.  $\rho \notin EpD_{a_t}^{(n)} \implies K_{a_t}^{(n)}\rho = \overline{\rho_0}$  (where  $\overline{\rho_0}$  is a fixed element of  $\mathfrak{D}(\mathcal{H}^{(n)})$ ).
- 4.3.  $K_{a_t}^{(n)}\rho \preceq_{\mathfrak{T}_{a_t}} \rho$ , for any  $\rho \in EpD_{\mathfrak{T}_{a_t}}^{(n)}$  (where  $\preceq_{\mathfrak{T}_{a_t}}$  is the preorder relation defined by Definition 2.4).
- 4.4.  $K_{a_t}^{(n)}\rho \preceq_{\mathfrak{T}_{a_t}} U_{a_t}^{(n)}\rho$ , for any  $\rho \in EpD_{a_t}^{(n)}$ .

For the sake of simplicity, we will generally write  $U_{a_t}\rho$  and  $K_{a_t}\rho$ , instead of  $U_{a_t}^{(n)}\rho$  and  $K_{a_t}^{(n)}\rho$ .

According to Definition 3.1, whenever an information  $\rho$  does not belong to the epistemic domain of  $a_t$ , then both  $U_{a_t}\rho$  and  $K_{a_t}\rho$  collapse into a fixed element (which may be identified, for instance, with the maximally uncertain information  $\frac{1}{2}I^{(n)}$  or with the  $\mathfrak{T}_{a_t}$ -Falsity  $\mathfrak{T}_{a_t}P_0^{(n)}$  of the space  $\mathcal{H}^{(n)}$  where  $\rho$  lives). At the same time, whenever  $\rho$  belongs to the epistemic domain of  $a_t$ , it seems reasonable to assume that the probability-values of  $\rho$  and  $K_{a_t}\rho$  are correlated: the probability of the quantum information asserting that “ $\rho$  is known by  $a_t$ ” should always be less than or equal to the probability of  $\rho$  (with respect to the truth-perspective of  $a_t$ ) (condition 4.3.). Hence, in particular, we have:

$$p_{\mathfrak{T}_{a_t}}(K_{a_t}\rho) = 1 \implies p_{\mathfrak{T}_{a_t}}(\rho) = 1.$$

But generally, not the other way around! In other words, pieces of quantum information that are known are true (with respect to the truth-perspective of the agent in question). Also condition 4.4. appears quite natural: knowing implies understanding.

A knowledge operation  $K_{a_t}$  is called *non-trivial* iff for at least one qumix  $\rho$ ,  $p_{\mathfrak{T}_{a_t}}(K_{a_t}\rho) < p_{\mathfrak{T}_{a_t}}(\rho)$ . Notice that knowledge operations do not generally preserve pure states [1].

For any agent  $a_t$  whose epistemic situation is  $(\mathfrak{T}_{a_t}, EpD_{a_t}, U_{a_t}, K_{a_t})$ , two special sets play an important intuitive role. The first set represents a kind of *active memory* of  $a_t$ , and



can be defined as follows:

$$ActMem(a_t) := \{\rho \in EpD_{a_t} : \mathcal{P}_{\mathfrak{I}_{a_t}}(\mathbf{U}_{a_t}\rho) = 1\}.$$

While the epistemic domain of  $a_t$  represents the virtual memory of  $a_t$ ,  $ActMem(a_t)$  can be regarded as the set containing all pieces of information that are actually understood by agent  $a$  at time  $t$ . Another important set, representing the *actual knowledge* of  $a$  at time  $t$ , is defined as follows:

$$ActKnowl(a_t) := \{\rho \in EpD_{a_t} : \mathcal{P}_{\mathfrak{I}_{a_t}}(\mathbf{K}_{a_t}\rho) = 1\}.$$

By definition of *epistemic situation* one immediately obtains:

$$ActKnowl(a_t) \subseteq ActMem(a_t) \subseteq EpD(a_t).$$

Using the concepts defined above, we can now introduce the notion of *epistemic quantum computational structure* (which will play an important role in the development of the epistemic semantics).

**Definition 3.2** (Epistemic quantum computational structure) An *epistemic quantum computational structure* is a system

$$S = (T, Ag, \mathbf{EpSit})$$

where:

1.  $T$  is a time-sequence  $(t_1, \dots, t_n)$ .
2.  $Ag$  is a finite set of epistemic agents  $a$  represented as functions of the times  $t$  in  $T$ .
3.  $\mathbf{EpSit}$  is a map that assigns to any agent  $a$  at time  $t$  an epistemic situation

$$EpSit_{a_t} = (\mathfrak{I}_{a_t}, EpD_{a_t}, \mathbf{I}_{a_t}, \mathbf{K}_{a_t}).$$

It may happen that, at any time, all agents of an epistemic quantum computational structure  $S$  share one and the same truth-perspective. In other words, for any agents  $a, b$  and for any times  $t_i, t_j$ :  $\mathfrak{I}_{a_{t_i}} = \mathfrak{I}_{b_{t_j}}$ . In such a case we will say that  $S$  is (*epistemically*) *harmonic*.

It is interesting to isolate some characteristic properties that may be satisfied by the agents of an epistemic quantum computational structure.

**Definition 3.3** Let  $S = (T, Ag, \mathbf{EpSit})$  be an epistemic quantum computational structure and let  $a$  be an agent of  $S$ .

- $a$  has a *sound epistemic capacity* iff for any time  $t$ , the qumixes  $\mathfrak{I}_{a_t} P_1^{(1)}$  and  $\mathfrak{I}_{a_t} P_0^{(1)}$  belong to the epistemic domain of  $a_t$ . Furthermore,  $\mathbf{K}_{a_t}^{\mathfrak{I}_{a_t} P_1^{(1)}} = \mathfrak{I}_{a_t} P_1^{(1)}$  and  $\mathbf{K}_{a_t}^{\mathfrak{I}_{a_t} P_0^{(1)}} = \mathfrak{I}_{a_t} P_0^{(1)}$ . In other words, at any time, agent  $a$  has access to the truth-values of his/her truth-perspective, assigning to them the “right” probability-values.
- $a$  has a *perfect epistemic capacity* iff for any time  $t$  and any qumix  $\rho$  belonging to the epistemic domain of  $a_t$ ,  $\mathbf{K}_{a_t}\rho = \rho$ . Hence, at any time  $a$  assigns the “right” probability-values to all pieces of information that belong to his/her epistemic domain.
- $a$  has a *maximal epistemic capacity* iff, at any time  $t$ ,  $a$  has a perfect epistemic capacity and his/her epistemic domain coincides with the set  $\mathcal{D}$  of all possible qumixes.

Notice that a maximal epistemic capacity does not imply *omniscience* (i.e. the capacity of *deciding* any piece of information). For, in quantum computational logics the *excluded-middle principle*

$$\forall \rho \in \mathfrak{D}(\mathcal{H}^{(n)}): \text{ either } \mathfrak{p}_{\mathfrak{T}}(\rho) = 1 \quad \text{or} \quad \mathfrak{p}_{\mathfrak{T}}(\mathfrak{D}\text{NOT}_{\mathfrak{T}}^{(n)} \rho) = 1$$

is, generally, violated.

When all agents of an epistemic quantum computational structure  $\mathcal{S}$  have a sound (perfect, maximal) capacity, we will say that  $\mathcal{S}$  is *sound (perfect, maximal)*.

In many concrete epistemic situations agents use to *interact*. In order to describe such phenomenon from an abstract point of view, we introduce the notion of *epistemic quantum computational structure with interacting agents*.

**Definition 3.4** (Epistemic quantum computational structure with interacting agents) An *epistemic quantum computational structure with interacting agents* is a system

$$\mathcal{S} = (T, Ag, \mathbf{EpSit}, Int),$$

where:

1.  $(T, Ag, \mathbf{EpSit})$  is an epistemic quantum computational structure;
2.  $Int$  is a map that associates to any time  $t \in T$  a set of pairs  $(a_t, b_t)$  (where  $a, b \in Ag$ ). The intuitive interpretation of  $(a_t, b_t) \in Int(t)$  is: the agents  $a$  and  $b$  interact at time  $t$ ;
3.  $(a_t, b_t) \in Int(t) \Rightarrow \exists t' \geq t \exists \rho [(\rho \in ActMem(a_t) \text{ and } \rho \in ActMem(b_{t'})) \text{ or } (\rho \in ActMem(b_t) \text{ and } \rho \in ActMem(a_{t'}))]$ . In other words, as a consequence of the interaction, there is at least one piece of information  $\rho$  such that at time  $t$  agent  $a$  certainly understands  $\rho$ , while at a later time  $t'$  agent  $b$  certainly understands  $\rho$ ; or viceversa.

What can be said about the characteristic mathematical properties of epistemic operations? Is it possible to represent the knowledge operations  $\mathbf{K}_{a_t}^{(n)}$  occurring in an epistemic quantum computational structure as special cases of qumix gates? This question has a negative answer. One can prove that non-trivial knowledge operations cannot be represented by unitary quantum operations [1].

At the same time, some interesting knowledge operations can be represented by the more general notion of *quantum channel* (which represents a special case of the concept of *quantum operation*<sup>2</sup>).

**Definition 3.5** (Quantum channel) A *quantum channel* on  $\mathcal{H}^{(n)}$  is a linear map  $\mathcal{E}$  from  $\mathcal{B}(\mathcal{H}^{(n)})$  to  $\mathcal{B}(\mathcal{H}^{(n)})$  that satisfies the following properties:

- for any  $A \in \mathcal{B}(\mathcal{H}^{(n)})$ ,  $\text{Tr}(\mathcal{E}(A)) = \text{Tr}(A)$ ;
- $\mathcal{E}$  is completely positive.

From the definition one immediately obtains that any quantum channel maps qumixes into qumixes.

A useful characterization of quantum channels is stated by *Kraus first representation theorem* [6].

<sup>2</sup>See for instance [3] and [5].

**Theorem 3.1** *A map*

$$\mathcal{E} : \mathcal{B}(\mathcal{H}^{(n)}) \mapsto \mathcal{B}(\mathcal{H}^{(n)})$$

is a quantum channel on  $\mathcal{H}^{(n)}$  iff for some set  $I$  of indices there exists a set  $\{E_i\}_{i \in I}$  of elements of  $\mathcal{B}(\mathcal{H}^{(n)})$  satisfying the following conditions:

- (1)  $\sum_i E_i^\dagger E_i = \mathbf{I}^{(n)}$ ;
- (2)  $\forall A \in \mathcal{B}(\mathcal{H}^{(n)}) : \mathcal{E}(A) = \sum_i E_i A E_i^\dagger$ .

Of course, qumix gates  ${}^{\mathfrak{D}}G^{(n)}$  are special cases of quantum channels, for which  $\{E_i\}_{i \in I} = \{G^{(n)}\}$ .

One can prove that there exist uncountably many quantum channels that are non-trivial knowledge operations of the space  $\mathcal{H}^{(n)}$  with respect to any truth-perspective [1].

An interesting example of a quantum channel that gives rise to a knowledge operation is the *depolarizing channel*. Let us refer to the space  $\mathcal{H}^{(1)}$  and let  $p \in [0, 1]$ . Consider the following system of operators:

$$E_0 = \frac{\sqrt{4-3p}}{2} \mathbf{I}^{(1)}; \quad E_1 = \frac{\sqrt{p}}{2} X; \quad E_2 = \frac{\sqrt{p}}{2} Y; \quad E_3 = \frac{\sqrt{p}}{2} Z$$

(where  $X, Y, Z$  are the three Pauli-matrices). Define  ${}^p\mathcal{D}_{\mathfrak{X}}^{(1)}$  as follows for any  $\rho \in \mathfrak{D}(\mathbb{C}^2)$ :

$${}^p\mathcal{D}_{\mathfrak{X}}^{(1)} \rho = \sum_{i=0}^3 \mathfrak{X} E_i \mathfrak{X}^\dagger \rho \mathfrak{X} E_i^\dagger \mathfrak{X}^\dagger.$$

It turns out that  ${}^p\mathcal{D}_{\mathfrak{X}}^{(1)}$  is a quantum channel, called *depolarizing channel*. Notice that for any truth-perspective  $\mathfrak{X}$ ,  ${}^p\mathcal{D}_{\mathfrak{X}}^{(1)} = {}^p\mathcal{D}_{\mathbf{I}}^{(1)}$ .

The channel  ${}^p\mathcal{D}_{\mathbf{I}}^{(1)}$  gives rise to a corresponding knowledge operation  ${}^p\mathbf{KD}_{\alpha_t}^{(1)}$  for an agent  $\alpha_t$  (who is supposed to belong to an epistemic quantum computational structure  $\mathcal{S}$ ).

**Definition 3.6** (A depolarizing knowledge operation  ${}^p\mathbf{KD}_{\alpha_t}^{(1)}$ ) Define  ${}^p\mathbf{KD}_{\alpha_t}^{(1)}$  as follows:

- (1)  $EpD_{\alpha_t} \subseteq \{\rho \in \mathfrak{D}(\mathcal{H}^{(1)}) : \mathfrak{P}_{\mathfrak{X}_{\alpha_t}}(\rho) \geq \frac{1}{2}\}$ .
- (2)  $\rho \in EpD_{\alpha_t} \Rightarrow {}^p\mathbf{KD}_{\alpha_t}^{(1)} \rho = {}^p\mathcal{D}_{\mathbf{I}}^{(1)} \rho$ .

Consider now  ${}^1\mathbf{KD}_{\alpha_t}^{(1)}$  and suppose that the structure  $\mathcal{S}$  satisfies the condition:

$$\rho \notin EpD_{\alpha_t} \Rightarrow {}^1\mathbf{KD}_{\alpha_t}^{(1)} \rho = \frac{1}{2} \mathbf{I}^{(1)}.$$

We obtain: for any  $\rho \in \mathfrak{D}(\mathcal{H}^{(1)})$ ,  ${}^1\mathbf{KD}_{\alpha_t}^{(1)} \rho = {}^1\mathcal{D}_{\mathbf{I}}^{(1)} \rho = \frac{1}{2} \mathbf{I}^{(1)}$ . In other words,  ${}^1\mathbf{KD}_{\alpha_t}^{(1)}$  seems to behave like a “fuzzification-procedure”, that transforms any (certain or uncertain) knowledge into a kind of maximally unsharp piece of information.

Other examples of quantum channels representing knowledge operations that give rise to interesting physical interpretations have been investigated in [7].

Unlike qumix gates, knowledge operations are not generally reversible. One can guess that the intrinsic irreversibility of the *act of knowing* is somehow connected with a loss of information due to the interaction with an environment.

#### 4 Memorizing and Retrieving Information via Teleportation

In epistemic processes that concern both human and artificial intelligence it is customary to distinguish an *internal* from an *external memory*. In the framework of our approach, the internal memory  $IntMem_{\alpha_t}$  of an agent  $\alpha$  (say, Alice) at time  $t$  can be naturally associated with the set  $ActMem(\alpha_t)$ . Hence, a piece of information  $\rho$  will belong to the internal memory of  $\alpha_t$  iff  $\mathcal{P}_{\mathcal{I}_{\alpha_t}}(\mathbf{U}_{\alpha_t}\rho) = 1$ . This means that at time  $t$  Alice has a kind of “aware understanding” of the information  $\rho$ . At the same time, the external memory  $ExtMem_{\alpha_t}$ , can be identified with a convenient subset of the epistemic domain of  $\alpha_t$ . Owing to the concrete limitations of the internal memory, the possibility of “depositing elsewhere” (in an external memory) some pieces of information turns out to be very useful for Alice. Of course, at a later time, Alice should be able to retrieve her “forgotten” information, storing it again in her internal memory.

We will now try to model examples of this kind in the framework of our abstract quantum computational approach. We will refer to a very simple physical situation. At any time  $t$  (of a given time-sequence) the external memory  $ExtMem_{\alpha_t}$  of Alice is supposed to be physically realized by a two-particle system  $S_1 + S_2$ , while the internal memory  $IntMem_{\alpha_t}$  is realized by a single particle  $S_3$ . For any time  $t$ , the global system  $S(t) = (S_1 + S_2 + S_3)(t)$  will represent Alice’s *physical memory-system*. For the sake of simplicity, we suppose that the state of  $S(t)$ , indicated by  $|\Psi^S(t)\rangle$ , is pure. Accordingly,  $|\Psi^S(t)\rangle$  will determine the states of the subsystems, which will be, generally, mixtures. We write:  $\rho^{(S_1+S_2)}(t) = Red^{(1,2)}(|\Psi^S(t)\rangle)$ ;  $\rho^{(S_i)}(t) = Red^{(i)}(|\Psi^S(t)\rangle)$  (where  $1 \leq i \leq 3$ ).

On this basis, we can put:

- $IntMem_{\alpha_t} = \{\rho^{(S_3)}(t)\}$ ;
- $ExtMem_{\alpha_t} = \{\rho^{(S_1)}(t), \rho^{(S_2)}(t), \rho^{(S_1+S_2)}(t)\}$ .

Now we want to describe a process of “memorizing and retrieving information”, by using a quantum teleportation phenomenon [2]. Physically, this process corresponds to the time-evolution of the global memory-system  $S$  (in a given time-interval). Since during this process Alice’s internal and external memories shall interact, we can imagine that Alice’s external memory is associated with an agent  $\beta$  (say, Bob), who can communicate with Alice via a classical channel (as happens in the standard teleportation-cases). Accordingly, our abstract description will naturally make use of epistemic quantum computational structures with interacting agents (Definition 3.4). For the sake of simplicity, we will refer to harmonic structures, where all agents have, at any time, the canonical truth-perspective  $\mathbb{I}$ .

*At time  $t_1$*  We suppose that at the initial time  $t_1$  the global memory-state is the following:

$$|\Psi^S(t_1)\rangle = \frac{1}{\sqrt{2}}(|0, 0\rangle + |1, 1\rangle) \otimes (a_0|0\rangle + a_1|1\rangle).$$

Hence, the state of the external memory is the entangled *Bell-state*, while the state of the internal memory is a qubit. According to our convention, we obtain:  $IntMem_{\alpha_t} = \{\rho^{(S_3)}(t_1)\}$ , where  $\rho^{(S_3)}(t_1) = P_{a_0|0+a_1|1}$ ;  $ExtMem_{\alpha_t} = \{\rho^{(S_1)}(t_1), \rho^{(S_2)}(t_1), \rho^{(S_1+S_2)}(t_1)\}$ , where  $\rho^{(S_1)}(t_1) = \frac{1}{2}\mathbb{I}^{(1)}$ ;  $\rho^{(S_2)}(t_1) = \frac{1}{2}\mathbb{I}^{(1)}$ ;  $\rho^{(S_1+S_2)}(t_1) = P_{\frac{1}{\sqrt{2}}(|0,0\rangle+|1,1\rangle)}$ .

*At time  $t_2$*  In order to “forget” the information  $a_0|0\rangle + a_1|1\rangle$  (stored by her internal memory) Alice acts on her global memory, by applying the gate  $SWAP_{(1,3)}^{(3)}$ , which exchanges the

states of the first and of the third subsystem of  $S$ . As a consequence, we obtain:

$$|\Psi^S(t_2)\rangle = \text{SWAP}_{(1,3)}^{(3)}|\Psi^S(t_1)\rangle = (a_0|0\rangle + a_1|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0, 0\rangle + |1, 1\rangle).$$

Alice’s internal memory is now changed. We have:

$$\rho^{(S_3)}(t_2) = \frac{1}{2}\mathbb{I}^{(1)},$$

which represents a maximally fuzzy information. Roughly, we might say that at time  $t_2$  Alice has “cleared out” her internal memory. At the same time, we have that  $\rho^{(S_1)}(t_2) = P_{a_0|0\rangle+a_1|1\rangle}$  belongs to Alice’s external memory. The operation of memorizing the information  $a_0|0\rangle + a_1|1\rangle$  in the external memory is now completed. Interestingly enough, the entanglement correlation between  $S^{(3)}(t_2)$  and  $S^{(2)}(t_2)$  guarantees to Alice the possibility of interacting with her external memory. It turns out that the transformation  $\rho^{(S_3)}(t_1) \mapsto \rho^{(S_3)}(t_2)$  is described by the depolarizing knowledge operation (considered in the previous section), which transforms any  $\rho$  of  $\mathcal{H}^{(1)}$  into  $\frac{1}{2}\mathbb{I}^{(1)}$ .

Notice that the state of the global system  $|\Psi^S(t_2)\rangle = (a_0|0\rangle + a_1|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0, 0\rangle + |1, 1\rangle)$  corresponds to the initial state of the standard teleportation-situation, where Bob (who has physical access to the system  $S_1 + S_2$ ) tries to send the qubit  $a_0|0\rangle + a_1|1\rangle$  to the “far” Alice (who has access to  $S_3$ ), by using the entanglement-correlation between  $S_2$  and  $S_3$ . We can now proceed, by applying the steps that are currently used in a teleportation-process.

*At time  $t_3$*  Bob applies the gate  $\text{XOR}^{(1,1)}$  to the external memory-state. As a consequence, we obtain:  $|\Psi^S(t_3)\rangle = [\text{XOR}^{(1,1)} \otimes \mathbb{I}^{(1)}]|\Psi^S(t_2)\rangle = \frac{1}{\sqrt{2}}(a_0|0\rangle \otimes (|0, 0\rangle + |1, 1\rangle) + \frac{1}{\sqrt{2}}(a_1|1\rangle \otimes (|1, 0\rangle + |0, 1\rangle))$ .

It is worth-while noticing that *theoretically* Bob is acting on the whole system  $S$ , while *materially* he is only acting on the subsystem  $S_1 + S_2$  that is accessible to him.

*At time  $t_4$*  Bob applies the gate Hadamard to the system  $S_1$  (whose state is to be teleported into the internal memory). Hence, we obtain:  $|\Psi^S(t_4)\rangle = [\sqrt{\mathbb{I}}^{(1)} \otimes \mathbb{I}^{(1)} \otimes \mathbb{I}^{(1)}]|\Psi^S(t_3)\rangle = \frac{1}{2}[(|0, 0\rangle \otimes (a_0|0\rangle + a_1|1\rangle)) + (|0, 1\rangle \otimes (a_0|1\rangle + a_1|0\rangle)) + (|1, 0\rangle \otimes (a_0|0\rangle - a_1|1\rangle)) + (|1, 1\rangle \otimes (a_0|1\rangle - a_1|0\rangle))]$ .

*At time  $t_5$*  Bob performs a measurement on the external memory, obtaining as a result one of the following possible registers:  $|0, 0\rangle, |0, 1\rangle, |1, 0\rangle, |1, 1\rangle$ . As a consequence, the state of the global system is transformed, by *collapse of the wave-function*; and such transformation is mathematically described by a (generally irreversible) quantum operation.

Let  $P_{|x,y\rangle}^{(2)}$  represent the projection-operator over the closed subspace determined by the register  $|x, y\rangle$ . We obtain four possible states for the global memory-system:

1.  $|\Psi_{00}^S(t_5)\rangle = 2[P_{|0,0\rangle}^{(2)} \otimes \mathbb{I}^{(1)}]|\Psi^S(t_4)\rangle = |0, 0\rangle \otimes (a_0|0\rangle + a_1|1\rangle)$ ;
2.  $|\Psi_{01}^S(t_5)\rangle = 2[P_{|0,1\rangle}^{(2)} \otimes \mathbb{I}^{(1)}]|\Psi^S(t_4)\rangle = |0, 1\rangle \otimes (a_0|1\rangle + a_1|0\rangle)$ ;
3.  $|\Psi_{10}^S(t_5)\rangle = 2[P_{|1,0\rangle}^{(2)} \otimes \mathbb{I}^{(1)}]|\Psi^S(t_4)\rangle = |1, 0\rangle \otimes (a_0|0\rangle - a_1|1\rangle)$ ;
4.  $|\Psi_{11}^S(t_5)\rangle = 2[P_{|1,1\rangle}^{(2)} \otimes \mathbb{I}^{(1)}]|\Psi^S(t_4)\rangle = |1, 1\rangle \otimes (a_0|1\rangle - a_1|0\rangle)$ .

By quantum non-locality, Bob’s action on the external memory has determined an instantaneous transformation of the state  $\rho^{(S_3)}(t_4)$  of the internal memory, which will have now one of the four possible forms:

$$a_0|0\rangle + a_1|1\rangle; \quad a_1|0\rangle + a_0|1\rangle; \quad a_0|0\rangle - a_1|1\rangle; \quad a_0|1\rangle - a_1|0\rangle.$$

Alice's internal memory is no longer fuzzy, since it is storing again a qubit. However, only in the first case this qubit coincides with the original  $a_0|0\rangle + a_1|1\rangle$  that Alice had stored in her internal memory (at the initial time). In spite of this, Alice has the possibility of retrieving her original information, through the application of a convenient gate. We have:  $a_0|0\rangle + a_1|1\rangle = \mathbb{I}^{(1)}(a_0|0\rangle + a_1|1\rangle) = \text{NOT}^{(1)}(a_1|0\rangle + a_0|1\rangle) = Z(a_0|0\rangle - a_1|1\rangle) = \text{NOT}^{(1)}Z(a_1|0\rangle - a_0|1\rangle)$  (where  $Z$  is the third Pauli-matrix).

In this situation, Bob can give an "order" to Alice, by using a classical communication channel. The order will be:

- "apply  $\mathbb{I}^{(1)}$ !" (i.e. "don't do anything!"), in the first case.
- "apply  $\text{NOT}^{(1)}$ !", in the second case.
- "apply  $Z$ !", in the third case.
- "apply  $\text{NOT}^{(1)}Z$ !" in the fourth case.

At time  $t_6$  Alice follows Bob's order and retrieves her original information.

Notice that the transformation  $\rho^{(S_3)}(t_1) \mapsto \rho^{(S_3)}(t_6)$  (from the initial to the final state of the internal memory) is mathematically described by the identity operator. Transformations of this kind (which concern reduced states and are obtained by neglecting the interaction with an environment) generally determine a loss of information; consequently they are described by irreversible quantum operations. Interestingly enough, this is not the case in the situation we have considered here, where the entanglement-correlation between the internal and the external memory, associated with a classical communication, allows Alice to retrieve *exactly* her initial information.

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