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Multi-state models for evaluating conversion options in life insurance☆

Guglielmo D'Amico[a,*], **Montserrat Guillen**[b], **Raimondo Manca**[c], **Filippo Petroni**[d]

[a]*Department of Pharmacy, University "G. d'Annunzio" of Chieti-Pescara, Chieti, Italy*

[b]*Department of Econometrics, Statistics and Economics, University of Barcelona, Barcelona, Spain*

[c]*MEMOTEF Department, University "La Sapienza", Rome, Italy*

[d]*Department of Business, University of Cagliari, Cagliari, Italy*

g.damico@unich.it (G. D'Amico), mguillen@ub.edu (M. Guillen),
raimondo.manca@uniroma1.it (R. Manca), fpetroni@unica.it (F. Petroni)

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Abstract In this paper we propose a multi-state model for the evaluation of the conversion option contract. The multi-state model is based on age-indexed semi-Markov chains that are able to reproduce many important aspects that influence the valuation of the option such as the duration problem, the time non-homogeneity and the ageing effect. The value of the conversion option is evaluated after the formal description of this contract.

Keywords Semi-Markov chain, temporary insurance policy, permanent insurance policy
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☆This work is dedicated to Prof. Dmitrii Silvestrov in recognition of his contribution to actuarial mathematics.

*Corresponding author.

1 Introduction

The conversion option is an option that allows the policyholder to convert his original temporary insurance policy (TIP) to permanent insurance policy (PIP) before the initial policy is due.

Insurance companies may find convenient this kind of contract because it may be much less expensive to convert the initial policy instead of issuing a new one. On the other side the policyholder may be interested in converting the contract because, at the time of conversion, insurance companies do not require any evidence of insurability and calculate the new premium according to the age at the issue of the original contract. However, at the time of conversion the insured individual has to pay the difference of cash value between the original TIP and converted PIP.

The literature on conversion option is not large and the main reference is represented by the article [17] where a valuation model was constructed based on mortality tables and then extended to a Lee–Carter model of mortality. A related article is [15] where the author considered an exchange option that is available in Norway.

In general, insurance companies collect data in form of sequences of events concerning the health status of the policyholders. Therefore they can evaluate survival probabilities taking into account for the health evolution of the insured person. This means that the adoption of a multi-state model can improve the evaluation process of policy-linked contracts like the conversion option when compared with information extracted from simple mortality tables. Indeed, as argued in [11], mortality rates are limited to accurately predict the dynamics of mortality. Moreover recent literature includes contributions where multi-state models, based on Markov chains, have been advanced as a valuable alternative to traditional mortality models see, e.g., [12, 13, 18, 10].

A general approach based on semi-Markov processes has been applied to problems of disability insurance also in recent years, see [16, 4, 5, 14]. Their appropriateness is due to the rejection of the geometric (exponential in continuous time model) distribution hypothesis for modeling the waiting times in a health status before making a transition in another state. Indeed, the geometric (exponential) hypothesis results in the lack of memory property that is very convenient from a mathematical point of view but is rarely supported by empirical evidence.

In this paper we focus on the evaluation of the conversion options when an age-indexed semi-Markov multi-state model describes the evolution of the health status of the policyholder. To this end we first derive transition probabilities for the model and then we develop the evaluation procedure by analyzing the TIP and PIP contracts and the conversion option. The obtained results represent the generalization of the results of [17] in a more general framework. Particularly, we show that the value of the conversion option depends on many parameters that are contemporary managed by our model such as the health status evolution of the policyholder, the age of the policyholder and the chronological time effect due to medical-scientific progress.

We start in Section 2 by describing the age indexed semi-Markov model. In Section 3, we explain the valuation procedure of the conversion option and we calculate its value. The paper ends with some conclusions and suggestions for further research.

2 Age-indexed semi-Markov model

Following the approach of [9] it is possible to give a tractable extension of discrete time non-homogeneous semi-Markov chains useful to consider different aspects that are relevant for the evaluation of the conversion option like the duration problem, the non-homogeneity and the ageing effect. This approach has been further generalized in [1–3] where general indexed semi-Markov processes were investigated and applied to different problems.

On a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we consider two sequences of random variables that evolve jointly:

$$J_n : \Omega \to E = \{1, 2, \dots, D\},$$

$$T_n : \Omega \to \mathbb{N}.$$

J_n represents the state at the n-th transition which can be identified with one of the mutually exclusive elements of the set E. In our framework, the set E contains all possible values of the health-status of the policyholder, included the death state denoted by D. The quantity T_n denotes the time of the n-th transition, i.e. the time when the policyholder enters in the health-status J_n.

We define the age-index process by the relation:

$$A_n = A_{n-1} + T_n - T_{n-1}, \quad n \in \mathbb{N}, \tag{1}$$

where A_0 is known. From now on we will set $A_0 = a$ and as usually $T_0 = 0$. This implies that by recursive substitution $A_n = a + T_n$, that is the age at the time of the n-th transition is given by the initial age ($A_0 = a$) plus the time of occurrence of the n-th transition (T_n).

The key assumption is to consider the triple (J_n, T_n, A_n) like a non-homogeneous Markov Renewal Process with index:

$$\begin{aligned}&\mathbb{P}\big[J_{n+1} = j, T_{n+1} \le t \;\big|\; \sigma(J_h, T_h, A_h,\ h \le t), J_n = i, T_n = s, A_n = a + s\big]\\ &\quad = \mathbb{P}[J_{n+1} = j, T_{n+1} \le t \mid J_n = i, T_n = s, A_n = a + s] = {}^{a}Q_{ij}(s;t),\end{aligned} \tag{2}$$

where $\sigma(J_h, T_h, A_h,\ h \le t)$ is the natural filtration of the three-variate process $(J_h, T_h, A_h)_{h\in\mathbb{N}}$.

Relation (2) affirms that the knowledge of the values J_n, T_n, A_n is sufficient to give the conditional distribution of the couple J_{n+1}, T_{n+1} whatever the values of the past variables might be. Let us denote by ${}^{a}p_{ij}(s)$ transition probabilities of the embedded non-homogeneous age indexed Markov chain:

$${}^{a}p_{ij}(s) := \mathbb{P}[J_{n+1} = j \mid J_n = i, T_n = s, A_n = a + s] = \lim_{t\to\infty} {}^{a}Q_{ij}(s;t).$$

Furthermore, it is necessary to introduce the probability that the process will remain in the state i up to the time t given the entrance in i at time s:

$${}^{a}\overline{H}_i(s;t) = \mathbb{P}[T_{n+1} > t \mid J_n = i, T_n = s, A_n = a + s] = 1 - \sum_{j\in E} {}^{a}Q_{ij}(s;t).$$

Now it is possible to define the distribution function of the waiting time in each state i, given that the state successively occupied is known

$$
\begin{aligned}
{}^{a}G_{ij}(s;t) &:= \mathbb{P}[T_{n+1} \leq t \mid J_{n+1} = j, J_n = i, T_n = s, A_n = a+s] \\
&= \begin{cases} \frac{{}^{a}Q_{ij}(s;t)}{{}^{a}p_{ij}(s)} & \text{if } {}^{a}p_{ij}(s) \neq 0, \\ 1 & \text{if } {}^{a}p_{ij}(s) = 0. \end{cases}
\end{aligned}
$$

The main advantage of semi-Markov models as compared to Markovian models is that in a semi-Markovian environment the probability distribution functions ${}^{a}G_{ij}(s;\cdot)$ can be of any type. On the contrary, in a Markovian model they should be geometrically distributed. Since disability data have shown rejection of the geometricity of the waiting time distributions (see, e.g. [8, 16, 4, 7]), semi-Markovian models are more appropriate to describe the dynamics of health-status evolution in time.

Let us denote by ${}^{a}N(t) = \sup\{n \in \mathbb{N} : T_n \leq t \mid A_0 = a\}$ the process counting the number of transitions up to time t and define consequently the age-indexed semi-Markov chain by

$$
{}^{a}Z(t) = J_{{}^{a}N(t)}.
$$

In the valuation procedure it will be useful to introduce the backward recurrence time process $B(t) = t - T_{{}^{a}N(t)}$. It denotes the time elapsed from the last transition of the system. The relevance of this process in the disability insurance modeling has been described in [4].

To characterize the probabilistic evolution of the system we introduce the following transition probability function:

Definition 1. The age-indexed semi-Markov transition probability function with initial and final backward is the matrix-valued function

$$
{}^{a+s-u}\boldsymbol{\Phi}\big(u,s;u',t\big) = \big({}^{a+s-u}\phi_{ij}\big(u,s;u',t\big)\big), \quad i,j \in E, \ u,s,u',t \in \mathbb{N},
$$

whose generic element ${}^{a+s-u}\phi_{ij}(u,s;u',t)$ expresses the probability

$$
\mathbb{P}\big[{}^{a}Z(t)=j, B(t)=u' \mid {}^{a}Z(s)=i, B(s)=u, A_{{}^{a}N(s)}=a+T_{{}^{a}N(s)}\big]. \tag{3}
$$

In disability insurance the probability (3) can be interpreted as the probability that an insured will be at time t in a disability of degree j and duration u' given that at time s she/he was in a disability of degree i and duration u and of age $a+s$.

Proposition 1. *The age-indexed semi-Markov transition probability function with initial and final backward satisfy the following recursive system of equations*

$$
\begin{aligned}
{}^{a+s-u}\phi_{ij}\big(u,s;u',t\big) &= 1_{\{i=j\}} 1_{\{u'=t-s+u\}} \frac{{}^{a+s-u}\overline{H}_i(s-u;t)}{{}^{a+s-u}\overline{H}_i(s-u;s)} \\
&+ \sum_{k\in E} \sum_{\theta=s+1}^{t-u'} \frac{{}^{a+s-u}q_{ik}(s-u;\theta)}{{}^{a+s-u}\overline{H}_i(s-u;s)} \cdot {}^{a+\theta}\phi_{kj}\big(0,\theta;u',t\big),
\end{aligned} \tag{4}
$$

where

$$
\begin{aligned}
{}^{a+s}q_{ij}(s;t) &= \mathbb{P}[J_{n+1}=j, T_{n+1}=t \mid J_n=i, T_n=s, A_n=a+s] \\
&= \begin{cases} {}^{a+s}Q_{ij}(s;t) - {}^{a+s}Q_{ij}(s;t-1) & \textit{if } t > s, \\ 0 & \textit{if } t = s. \end{cases}
\end{aligned} \tag{5}
$$

Proof. Let us denote by $\mathbb{P}_{(i,s-u,a+s-u)}(\cdot)$ the probability measure

$$\mathbb{P}\big(\cdot \mid {}^{a}Z(s)=i,\, T_{{}^{a}N(s)} = s-u,\, A_{{}^{a}N(s)}=a+s-u\big),$$

and by $\mathbb{P}_{(i,s-u,a+s-u,>s)}(\cdot)$ the probability measure

$$\mathbb{P}\big(\cdot \mid {}^{a}Z(s)=i,\, T_{{}^{a}N(s)} = s-u,\, A_{{}^{a}N(s)}=a+s-u,\, T_{{}^{a}N(s)+1} > s\big).$$

Observe that the information set $\{{}^{a}Z(s)=i,\, B(s)=u,\, A_{{}^{a}N(s)}=a+T_{{}^{a}N(s)}\}$ is equivalent to $\{{}^{a}Z(s)=i,\, T_{{}^{a}N(s)} = s-u,\, T_{{}^{a}N(s)+1} > s,\, A_{{}^{a}N(s)}=a+s-u\}$, so that the age-indexed semi-Markov transition probability function can be denoted by

$$\begin{aligned}{}^{a+s-u}\phi_{ij}\big(u, s; u', t\big) &= \mathbb{P}_{(i,s-u,a+s-u,>s)}\big[{}^{a}Z(t)=j,\, B(t)=u'\big]\\ &= \mathbb{P}_{(i,s-u,a+s-u,>s)}\big[{}^{a}Z(t)=j,\, T_{{}^{a}N(t)} = t-u',\, T_{{}^{a}N(s)+1} > t\big]\\ &\quad + \mathbb{P}_{(i,s-u,a+s-u,>s)}\big[{}^{a}Z(t)=j, T_{{}^{a}N(t)} = t-u',\, T_{{}^{a}N(s)+1} \le t\big].\end{aligned} \tag{6}$$

The first summand of (6) can be represented as follows:

$$\begin{aligned}&\frac{\mathbb{P}_{(i,s-u,a+s-u,>s)}[\, T_{{}^{a}N(s)+1} > t, {}^{a}Z(t)=j,\, T_{{}^{a}N(t)} = t-u']}{\mathbb{P}_{(i,s-u,a+s-u,>s)}[T_{{}^{a}N(s)+1} > s]}\\ &= \frac{1}{\mathbb{P}_{(i,s-u,a+s-u,>s)}[T_{{}^{a}N(s)+1} > s]}\\ &\quad \cdot \big(\mathbb{P}_{(i,s-u,a+s-u,>s)}\big[\, T_{{}^{a}N(s)+1} > t, {}^{a}Z(t)=j,\, T_{{}^{a}N(t)} = t-u'\big]\\ &\quad \cdot \mathbb{P}_{(i,s-u,a+s-u,>s)}\big[\, T_{{}^{a}N(t)} = t-u'\big]\\ &\quad \cdot \mathbb{P}\big[T_{{}^{a}N(s)+1} > t \mid {}^{a}Z(s)=i,\, T_{{}^{a}N(s)} = s-u,\, A_{{}^{a}N(s)}=a+s-u\big]\big)\\ &= \frac{1}{{}^{a+s-u}\overline{H}_i(s-u; s)} \cdot \big(1_{\{i=j\}} \cdot 1_{\{u'=t-s+u\}} \cdot {}^{a+s-u}\,\overline{H}_i(s-u; t)\big).\end{aligned}$$

The second summand of (6) can be represented as follows:

$$\begin{aligned}&\frac{\mathbb{P}_{(i,s-u,a+s-u)}[\,{}^{a}Z(t)=j,\, T_{{}^{a}N(t)} = t-u',\, s < T_{{}^{a}N(s)+1} \le t]}{\mathbb{P}_{(i,s-u,a+s-u,>s)}[T_{{}^{a}N(s)+1} > s]}\\ &= \frac{1}{{}^{a+s-u}\overline{H}_i(s-u; s)} \sum_{k\in E}\sum_{\theta=s+1}^{t-u'} \mathbb{P}_{(i,s-u,a+s-u)}\big[\,{}^{a}Z(t)=j,\, T_{{}^{a}N(t)} = t-u',\\ &\qquad J_{{}^{a}N(s)+1} = k,\, T_{{}^{a}N(s)+1} = \theta\big]\\ &= \frac{1}{{}^{a+s-u}\overline{H}_i(s-u; s)}\\ &\quad \cdot \sum_{k\in E}\sum_{\theta=s+1}^{t-u'} \mathbb{P}_{(i,s-u,a+s-u)}\big[\,{}^{a}Z(t)=j, T_{{}^{a}N(t)} = t-u' \mid J_{{}^{a}N(s)+1}=k,\, T_{{}^{a}N(s)+1}=\theta\big]\\ &\quad \cdot \mathbb{P}_{(i,s-u,a+s-u)}[\, J_{{}^{a}N(s)+1} = k,\, T_{{}^{a}N(s)+1} = \theta]\\ &= \sum_{k\in E}\sum_{\theta=s+1}^{t-u'} \frac{{}^{a+s-u}q_{ik}(s-u;\theta)}{{}^{a+s-u}\overline{H}_i(s-u; s)} \cdot {}^{a+\theta}\phi_{kj}\big(0, \theta; u', t\big).\end{aligned}$$

The last equality is obtained using the assumption (2) on the Markovianity of the triple (J_n, T_n, A_n) with respect to transition times T_n and the definition of the age-indexed semi-Markov kernel given in formula (5). □

The above-presented transition probabilities generalize the corresponding transition probabilities with initial backward derived in [6] by including the dependence on the final backward. Moreover they generalize the transition probabilities with initial and final backward given in [4] by including the dependence on the age-index process.

In the sequel of the paper we need to consider survival functions for our age-indexed model. To this end we introduce the hitting time of state D (death of the policyholder) given the occupancy of state i at time s with age $a+s$ and duration in the state equal to u:

$$^{a+s-u}T_{i,D}(u,s) := \inf\{t > s : {}^aZ(t) = D \mid {}^aZ(s) = i, B(s) = u\}.$$

Definition 2. The survival function of the age-indexed semi-Markov chain is the vector valued function $^{a+s-u}\mathbf{S}(u,s;t) = (^{a+s-u}S_i(u,s;t))$, $i \in E$, $u, s, t \in \mathbb{N}$ with generic element given by:

$$^{a+s-u}S_i(u,s;t) := \mathbb{P}\big[{}^aT_{i,D}(u,s) > t\big]. \tag{7}$$

It denotes the probability to not enter state D in the time interval $(s,t]$ given the occupancy of state i at time s being aged $a+s$ with entrance in this state with last transition u periods before. This function can be calculated using the following relation:

$$^{a+s-u}S_i(u,s;t) = \sum_{j \neq D} \sum_{u'=0}^{t-s+u} {}^{a+s-u}\phi_{ij}\big(u,s;u',t\big).$$

It is simple to note that

$$\begin{aligned} \mathbb{P}\big[{}^{a+s-u}T_{i,D}(u,s) = t\big] &= {}^{a+s-u}S_i(u,s;t-1) - {}^{a+s-u}S_i(u,s;t) \\ &=: \Delta^{a+s-u}S_i(u,s;t-1). \end{aligned} \tag{8}$$

3 The conversion option in life insurance

Let us consider the general situation where a female insured aged x at the initial time 0 with a health state $i \in E$ buys an n-year term insurance policy (TIP). When the policy is almost due, if she is still alive she decides to extend the policy for the rest of her life. The extension can be done by converting the initial TIP into a PIP or buying a new PIP. In Figure 1 we report a diagram that summarizes the time schedule of a conversion option contract. It should be remarked that at time n, the decision to convert the TIP into a PIP or to purchase a new PIP should be taken considering the new health state of the policyholder ($^aZ(n)$), the duration in this state ($B(n)$) and the age $(x+n)$.

The valuation of the conversion option needs the study of two kinds of contracts involved here: the TIP and the PIP contracts.

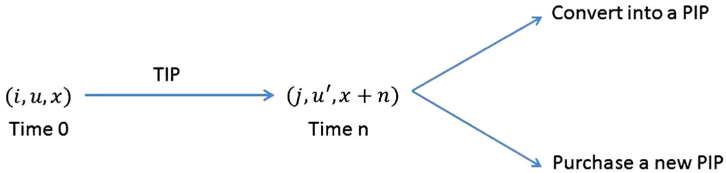

Fig. 1. A conversion option diagram

3.1 Temporary insurance policy contract

Term insurance policies provide coverage for a limited time (n years) and gives to the policyholder a benefit in case of death. In this paper without loss of generality we assume that the benefit is set to 1 Euro. The possession of this coverage is subordinated to the payment, by the policyholder, of an yearly premium until the occurrence of the death event or the expiry of the contract whichever occur before.

For the TIP contract, let us introduce the random variable (r.v.) *conditional Present Value of Death Benefit* denoted by $(PVDB)_{i,u,x}$. It takes value δ^s when the death of the policyholder occurs at any time $s \leq n$. Given the initial conditions $\{{}^aZ(0) = i, B(0) = u, A(0) = x\}$, the death event may occur at time s with probability ${}^xS_i(u,0;s-1) - {}^xS_i(u,0;s)$, then it results in

$$\begin{aligned}\mathcal{A}_{i,u}(x,0,n) :=& \mathbb{E}\big[(PVDB)_{i,u,x}\big] = \sum_{s=1}^{n} \mathbb{P}\big[{}^xT_{i,D}(u,0) = s\big] \cdot 1 \cdot \delta^s \\ =& \sum_{s=1}^{n} \Delta\, {}^xS_i(u,0;s-1)\delta^s.\end{aligned}$$

Let us introduce the r.v. *conditional Present Value of Unitary Premiums* denoted by $(PVUP)_{i,u,x}$. Since premiums are paid in the due case, the r.v. $(PVUP)_{i,u,x}$ takes value $\sum_{r=0}^{s-1} \delta^r$ when the death of the policyholder occurs at time $s \leq n-1$ and value $\sum_{r=0}^{n} \delta^r$ if she will survive time n.

Let us denote by $p_{i,u}(x,0)$ the annual premium for an n-TIP with 1 Euro payable at the year of death of an insured of age x, in health state i obtained u years before. Then the r.v. *conditional Present Value of Premiums* denoted by $(PVP)_{i,u,x}$ is simply defined by

$$\begin{aligned}&(PVP)_{i,u,x} := p_{i,u}(x,0) \cdot (PVUP)_{i,u,x}, \quad \text{for } i \neq D, \\ &(PVP)_{i,u,x} := 0, \quad \text{for } i = D,\end{aligned} \tag{9}$$

and then it results in

$$\begin{aligned}\mathcal{P}_{i,u}(x,0,n) :=& \mathbb{E}\big[(PVP)_{i,u,x}\big] \\ =& \sum_{s=1}^{n-1}\left(p_{i,u}(x,0)\sum_{r=0}^{s-1}\delta^r\right)\Delta\, {}^xS_i(u,0;s-1) \\ &+ \left(p_{i,u}(x,0)\sum_{r=1}^{n}\delta^r\right){}^xS_i(u,0;n).\end{aligned}$$

Furthermore if we assume that premiums are fixed according to the equivalence principle, i.e. in a way such that the actuarial present value of premiums should be equal to the actuarial present value of benefits (see e.g. [8]), then we have that:

$$\mathcal{A}_{i,u}(x,0,n)=\mathcal{P}_{i,u}(x,0,n),$$

from which we recover the fair premium

$$p_{i,u}(x,0)=\frac{\sum_{s=1}^{n}\Delta\,{}^{x}S_i(u,0;s-1)\delta^s}{\sum_{s=1}^{n-1}\sum_{r=1}^{s-1}\delta^r\Delta\,{}^{x}S_i(u,0;s-1)+\sum_{r=1}^{n}\delta^r\,{}^{x}S_i(u,0;n)}. \tag{10}$$

3.2 Permanent insurance policy

Permanent insurance policies provide coverage for an unlimited time horizon and gives to the policyholder a benefit of 1 Euro in case of death. The possession of this coverage is subordinated to the payment, by the policyholder, of an yearly premium until the occurrence of the death event.

Relatively to the PIP contract let us introduce the r.v. *conditional Present Value of Death Benefits* denoted by $(\widetilde{PVDB})_{i,u,x}$. It takes value δ^s when the death of the policyholder occurs at time $s\in\mathbb{N}$. In analogy with the TIP case it results in

$$\begin{aligned}\tilde{\mathcal{A}}_{i,u}(x,0)&:=\mathbb{E}\big[(\widetilde{PVDB})_{i,u,x}\big]=\sum_{s=1}^{\infty}\mathbb{P}\big[{}^{x}T_{i,D}(u,0)=s\big]\cdot 1\cdot\delta^s\\&=\sum_{s=1}^{\infty}\Delta^{x}S_i(u,0;s-1)\delta^s.\end{aligned}$$

Let us introduce the r.v. *conditional Present Value of Unitary Premiums* denoted by $(\widetilde{PVUP})_{i,u,x}$. Premiums are paid until the occurrence of the death of the policyholder, formally the r.v. $(\widetilde{PVUP})_{i,u,x}$ assumes value $\sum_{r=1}^{s-1}\delta^r$ when the death of the policyholder occurs at time $s\in\mathbb{N}$.

Let us denote by $\tilde{p}_{i,u}(x,0)$ the annual premium for a PIP with 1 Euro payable at the year of death of an insured of age x, in health state i obtained u years before. Then the r.v. *conditional Present Value of Premiums* denoted by $(\widetilde{PVP})_{i,u,x}$ is simply defined by

$$\begin{aligned}&(\widetilde{PVP})_{i,u,x}:=\tilde{p}_{i,u}(x,0)\cdot(\widetilde{PVUP})_{i,u,x},\quad \text{for } i\neq D,\\&(\widetilde{PVP})_{i,u,x}:=0,\quad \text{for } i=D,\end{aligned} \tag{11}$$

and then it results in

$$\tilde{\mathcal{P}}_{i,u}(x,0):=\mathbb{E}\big[(\widetilde{PVP})_{i,u,x}\big]=\sum_{s=1}^{\infty}\tilde{p}_{i,u}(x,0)\sum_{r=1}^{s-1}\delta^r\Delta\,{}^{x}S_i(u,0;s-1).$$

Furthermore if we assume that premiums are fixed according to the equivalence principle we have that:

$$\tilde{\mathcal{A}}_{i,u}(x,0)=\tilde{\mathcal{P}}_{i,u}(x,0),$$

from which we recover the fair premium

$$\tilde{p}_{i,u}(x,0)=\frac{\sum_{s=1}^{\infty}\delta^s\Delta\,{}^{x}S_i(u,0;s-1)}{\sum_{s=1}^{\infty}\sum_{r=1}^{s-1}\delta^r\Delta\,{}^{x}S_i(u,0;s-1)}. \tag{12}$$

3.3 Valuation of the conversion option

In this subsection we develop the valuation procedure for conversion options when survival probability functions are derived from a multi-state model of the policyholder's health. The valuation makes use of the random variables introduced for describing the TIP and PIP contracts and what we called *exercise set* of the option. The introduction of the exercise set is a prerogative of our model and was not present in earlier studies on conversion options. We remember that the policyholder possesses a TIP issued at time zero with maturity n and at time n should decide to prolong the insurance coverage either by means of converting the TIP into a PIP or purchasing a new PIP.

We define the r.v. *conditional Conversion Gain* as

$$(CG)_{i,u,x} = \big[(PVDB)_{i,u,x}\big] - \big[(PVP)_{i,u,x} \;\big|\; \text{conversion}\big], \tag{13}$$

where $[(PVDB)_{i,u,x}]$ is the r.v. denoting the present value of death benefits and $[(PVP)_{i,u,x} \mid \text{conversion}]$ is the r.v. describing the present value of premiums when the policyholder possesses an option to convert the original TIP into a PIP before the expiry of the TIP.

They are both conditional on the information set $\{{}^{a}Z(0) = i, B(0) = u, A(0) = x\}$ describing the initial health conditions of the policyholder at the inception time zero. The formal definition of the r.v. $[(PVP)_{i,u,x} \mid \text{conversion}]$ is given in Definition 4 below.

Similarly it is possible to define the r.v. *conditional No Conversion Gain* as

$$(NCG)_{i,u,x} = \big[(PVDB)_{i,u,x}\big] - \big[(PVP)_{i,u,x} \;\big|\; \text{no conversion}\big], \tag{14}$$

where $[(PVP)_{i,u,x} \mid \text{no conversion}]$ is the r.v. denoting the present value of premiums when the policyholder does not possess an option to convert the original TIP into a PIP and then must purchase a new PIP at time n if she wants to extend the insurance protection. The formal definition of the r.v. $[(PVP)_{i,u,x} \mid \text{no conversion}]$ is given in Definition 3 below.

The difference between the Conversion Gain and the No Conversion Gain define the r.v. *conditional Net Gain*, i.e.:

$$(G)_{i,u,x} = (CG)_{i,u,x} - (NCG)_{i,u,x}, \tag{15}$$

and its expected value is called conditional *Value of the Conversion Option*, i.e.:

$$(VCO)_{i,u,x} = \mathbb{E}\big[(G)_{i,u,x}\big]. \tag{16}$$

It is simple to realize that

$$(VCO)_{i,u,x} = \mathbb{E}\big[(PVP)_{i,u,x} \;\big|\; \text{no conversion}\big] - \mathbb{E}\big[(PVP)_{i,u,x} \;\big|\; \text{conversion}\big]. \tag{17}$$

Therefore, we need to calculate the expectations on the right hand side of Eq. (17). To do this we proceed first to the formal definition of the two random variables involved in the computation. This requires the introduction of some auxiliary concepts.

Let us consider a time $n \in \mathbb{N}$, then the triple (i, u, x) is called an *n-scenario* if ${}^{a}Z(n) = i$, $B(n) = u$, $A_{N(n)} = x - n + u$.

We say that the 0-scenario (i, u, x) is state-unchanged at time n if the n-scenario will be $(i, u, x+n)$.

Two state-unchanged scenarios share the same health state and duration in this state but are characterized by different ages of the policyholder.

The conditional *cash Value* is defined by

$$V_{i,u}(x+n,n) := \big[\tilde{p}_{i,u}(x+n,n) - p_{i,u}(x+n,0)\big] \cdot \widetilde{PVUP}_{i,u,x} \cdot \delta^n. \tag{18}$$

The expectation of the cash value is the quantity the policyholder has to pay at the time of conversion to the insurance company:

$$\begin{aligned}\mathcal{V}_{i,u}(x+n,n) &:= \mathbb{E}\big[V_{i,u}(x+n,n)\big] \\ &= \big[\tilde{p}_{i,u}(x+n,n) - p_{i,u}(x+n,0)\big] \cdot \sum_{h=n+1}^{\infty} \delta^h \, \Delta^{x+n} S_i(u,n;h-1).\end{aligned} \tag{19}$$

The quantity $\mathcal{V}_{i,u}(x+n,n)$ expresses the gain the policyholder expect to realize buying the conversion option under the hypothesis of an unchanged n-scenario. This quantity is greater or equal than zero because

$$\tilde{p}_{i,u}(x+n,n) \geq p_{i,u}(x+n,0),$$

that is, the premiums for a PIP are greater than the corresponding premium for a TIP given the same n-scenario $(i, u, x+n)$.

In analogy with the financial options, we can define a set where it is convenient to exercise the conversion option. This is a prerogative of the adopted multi-state model because in the paper [17], if the insured person was still alive at the conversion time it was always convenient to prolong the coverage by exercising the option. However, in our more general framework, this is not the case, because given the initial 0-scenario (i, u, x) it is possible after n years that the insured person improves considerably the health state and the prospective expectation of a prolonged life. This has been observed in the evolution of several diseases like HIV infection, see e.g. [7].

Given the 0-scenario (i, u, x), we define the *exercise set* as

$$\begin{aligned}C_{i,u}(x,n) := \Big\{ &(j,u') \in E \times \mathbb{N} : \\ &\mathbb{E}\big[p_{i,u}(x,0) \cdot \widetilde{PVUP}_{i,u,x} + V_{i,u}(x+n,n)\big] \leq \tilde{\mathcal{P}}_{j,u'}(x+n,n)\Big\}.\end{aligned} \tag{20}$$

The set $C_{i,u}(x,n)$ comprehends all couples of health states and durations where it is convenient for the policyholder to exercise the conversion option. Indeed, if the expected payment to face by converting the option $\mathbb{E}[p_{i,u}(x,0) \cdot \widetilde{PVUP}_{i,u,x} + V_{i,u}(x+n,n)]$ is smaller than the expected present value of premiums to be paid for a new PIP in the new n-scenario $(j, u', x+n)$ it is convenient to convert the option because with an inferior cost the policyholder guarantees to herself the same insurance protection. Therefore, if $(j, u') \in C_{i,u}(x,n)$ the policyholder will convert the option; on the contrary, if $(j, u') \in C^c_{i,u}(x,n)$ the policyholder will not convert the option.

Now we are in the position to define the random variables

$$\big[(PVP)_{i,u,x} \bigm| \text{no conversion}\big], \quad \big[(PVP)_{i,u,x} \bigm| \text{conversion}\big].$$

Definition 3. The r.v. $[(PVP)_{i,u,x} \mid \text{no conversion}]$ is defined by the following relation:

$$\big[(PVP)_{i,u,x} \bigm| \text{no conversion}\big] := (PVP)_{i,u,x} + (\widetilde{PVP})_{{}^aZ(n),B(n),A(n)} \cdot \delta^n. \tag{21}$$

Then, the conditional present value of premiums given no conversion is equal to the conditional present value of premiums from the TIP contract plus the conditional present value of premiums of the subsequent PIP calculated under the n-scenario $({}^aZ(n), B(n), A(n))$ and discounted at time zero.

It is possible to calculate its expectation that is given here below:

$$\begin{aligned} \mathbb{E}[PVP \mid \text{no conversion}] &= \mathcal{P}_{i,u}(x,0,n) \\ &\quad + \sum_{j\in E}\sum_{u'\geq 0} {}^x\phi_{ij}\big(u,0;u',n\big)\cdot\delta^n\cdot\tilde{\mathcal{P}}_{j,u'}(x+n,n). \end{aligned} \tag{22}$$

Definition 4. The r.v. $[(PVP)_{i,u,x} \mid \text{conversion}]$ is defined by the following relation:

$$\begin{aligned} \big[(PVP)_{i,u,x} \bigm| \text{conversion}\big] &:= (PVP)_{i,u,x} \\ &\quad + \delta^n(\widetilde{PVP})_{{}^aZ(n),B(n),A(n)} \cdot 1_{\{({}^aZ(n),B(n))\in C^c_{i,u}(x,n)\}} \\ &\quad + \delta^n\big[\big(p_{i,u}(x,0)\widetilde{PVUP}\big) + V_{i,u}(x+n,n)\big] \\ &\quad \cdot 1_{\{({}^aZ(n),B(n))\in C_{i,u}(x,n)\}}. \end{aligned} \tag{23}$$

Then, the conditional present value of premiums given the possibility to convert is equal to the conditional present value of premiums from the TIP contract plus the conditional present value of premiums from the PIP calculated under the n-scenario $({}^aZ(n), B(n), A(n))$ and discounted at time zero if this scenario does not belong to the exercise set plus the expected payment to face by converting the option if the n-scenario belongs to the exercise set.

It is possible to calculate the expectation of (23) that is given here below:

$$\begin{aligned} \mathbb{E}[PVP \mid \text{conversion}] &= \mathcal{P}_{i,u}(x,0,n) \\ &\quad + \sum_{(j,u')\in C^c_{i,u}(x,n)} {}^x\phi_{ij}\big(u,0;u',n\big)\cdot\delta^n\cdot\tilde{\mathcal{P}}_{j,u'}(x+n,n) \\ &\quad + \sum_{(j,u')\in C_{i,u}(x,n)} {}^x\phi_{ij}\big(u,0;u',n\big)\cdot\delta^n\cdot\Bigg[V_{i,u}(x+n,n) \\ &\quad + \sum_{h=n+1}^{\infty} p_{i,u}(x,0)\sum_{r=n+1}^{h}\delta^r\,\Delta^{x+n}S_j\big(u',n;h\big)\Bigg]. \end{aligned} \tag{24}$$

Now we are in the position of computing the value of the conversion option by substituting Eqs (23) and (24) in Formula (17). Some algebra gives the following representation:

$$(VCO)_{i,u,x} = \sum_{(j,u')\in C_{i,u}(x,n)} {}^{x}\phi_{ij}\big(u,0;u',n\big)\delta^{n}\cdot\Bigg[\tilde{\mathcal{P}}_{j,u'}(x+n,n)-V_{i,u}(x+n,n) - \sum_{h=n+1}^{\infty} p_{i,u}(x,0)\sum_{r=n+1}^{h}\delta^{r}\big({}^{x+n}S_j\big(u',n;h\big)-{}^{x+n}S_j\big(u',n;h+1\big)\big)\Bigg],$$

from which we realize that $VCO \geq 0$ because on the exercise set $C_{i,u}(x,n)$ the term within square brackets is nonnegative.

We would like to remark that the value of the conversion option is nonnegative unless the exercise set is empty. Moreover the value does depend on the dynamics of the health state of the policyholder and therefore, in our model, it is sensitive to the duration of permanence in the health state, to the chronological time and to the age of the policyholder.

4 Conclusions

The valuation of conversion options in life insurance is an important subject in modern actuarial mathematics.

This study accomplished several goals. First, we proposed a general multistate model that can reproduce important aspects in the modeling of life insurance contracts and we calculated transition probability function for the model. Second, we defined the main variables necessary to the description of the contract and we calculated the value of the conversion option in a very general framework. As particular cases we obtain formulas for the valuation of temporary insurance policy and permanent insurance policy that are embedded in the conversion option contract.

This paper leaves several points opened. First of all the application to real data of the model is by far the most urgent task to be accomplished. This task can be accomplished once a reliable dataset is obtained and adequate computer programmes are built. Then, the possibility to extend the results to more complex models is also relevant, in this light a possible extension to subordinated semi-Markov chains is worth mentioning.

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