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Research Article

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Three nontrivial solutions for nonlinear fractional Laplacian equations

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Abstract: We study a Dirichlet-type boundary value problem for a pseudodifferential equation driven by the fractional Laplacian, proving the existence of three non-zero solutions. When the reaction term is sublinear at infinity, we apply the second deformation theorem and spectral theory. When the reaction term is superlinear at infinity, we apply the mountain pass theorem and Morse theory.

Keywords: Fractional Laplacian, eigenvalue problems, Morse theory

MSC 2010: 35R11, 35P30, 49F15

1 Introduction

The present paper deals with the following Dirichlet-type boundary value problem for a nonlinear equation driven by the fractional Laplacian:

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ ($N > 1$) is a bounded domain with a C^2 boundary, $\Omega^c = \mathbb{R}^N \setminus \Omega$, $s \in (0,1)$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. The fractional Laplacian operator is defined for any sufficiently smooth function $u : \mathbb{R}^N \to \mathbb{R}$ and all $x \in \mathbb{R}^N$ by

$$(-\Delta)^s u(x) = C_{N,s} \lim_{\varepsilon \to 0^+} \int\limits_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy, \tag{1.2}$$

where $C_{N,s} > 0$ is a suitable normalization constant. Throughout the paper we will always assume $C_{N,s} = 1$ (for a precise evaluation of $C_{N,s}$, consistent with alternative definitions of the fractional Laplacian, see [10, Remark 3.11]).

Fractional operators have gained increasing popularity in recent years. This is both due to the intrinsic mathematical interest of such subject and to the various applications that they allow. Indeed, nonlocal pseudodifferential operators such as $(-\Delta)^s$ are naturally involved in continuum mechanics, population dynamics, game theory and other phenomena, as the infinitesimal generators of Lévy-type stochastical processes (see [12]).

Roughly speaking, the outstanding feature of operators like $(-\Delta)^s$ is *nonlocality*, i.e., the dependence of $(-\Delta)^s u(x)$ on the values of $u(y)$ not only for y conveniently near to x, but for all $y \in \mathbb{R}^N$. While such nonlocality

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makes our operator particularly suitable to describe phenomena allowing "jumps", it makes things delicate in dealing with regularity, sign, and other typically *local* attributes of solutions. This is one reason why the study of nonlinear equations involving $(-\Delta)^s$ (or closely related operators) started with the case in which the domain is $\mathbb{R}^N$, providing existence of solutions, regularity, a priori bounds and maximum principles (see [10, 11], and [3] for some existence results). The natural functional setting for such study is provided by fractional Sobolev spaces (see [17]).

On the other hand, nonlocality obviously produces some difficulties in finding an analogous to Dirichlet-type boundary conditions on bounded domains. The standard formulation of the Dirichlet problem for fractional equations in a bounded domain Ω was set in the series of papers [35–37], simply by requiring that the solution u vanishes a.e. outside Ω. Our problem (1.1) follows such a standard. While interior regularity of solutions of (1.1) can be handled just as in the unbounded case, boundary regularity and behavior of solutions (e.g., the Hopf property) came forth as a serious difficulty, which was mostly overcome by means of weighted Hölder-type function spaces (see [4, 22, 25, 34]).

Once provided with the appropriate functional formulation, problem (1.1) becomes variational, in the sense that its weak solutions can be detected as critical points of a C^1 energy functional φ, defined on a fractional Sobolev space. So we can prove existence and multiplicity of such solutions by applying to φ several abstract results of critical point theory, such as minimax principles (see [33]) and Morse theory (see [13]). Some results of this type can be found, for instance, in [6, 14, 18, 26, 28, 30, 39].

In the present paper, we will employ much of the research accomplished so far in order to prove the existence of three non-zero solutions for problem (1.1) (one positive, one negative, and the third with indefinite sign), when $f(x,\cdot)$ has a subcritical growth and satisfies convenient conditions at zero and at infinity. Precisely, we will consider the following two cases:

(a) If $f(x,\cdot)$ is *sublinear* at infinity, and at most linear at zero, then we apply the second deformation theorem and some spectral properties of $(-\Delta)^s$ (namely, a characterization of the second eigenvalue which, for the local case, goes back to [16]).

(b) If $f(x,\cdot)$ is *superlinear* at infinity, and satisfies a mild version of the Ambrosetti–Rabinowitz condition, then we apply the mountain pass theorem and the Poincaré–Hopf identity based on the computation of critical groups (thus proving a nonlocal analogous of the result of [38]).

In both cases, truncations of the energy functional φ will be an essential tool, so we will make use of a topological result established in [25], which relates local minimizers of the truncated and uncut functionals, respectively.

Our work strongly relies on the joint application of mutually independent results, and we decided to privilege simplicity rather than generality. One possible generalization of our results is towards linear nonlocal operators of the type

$$\mathcal{L}_K u(x) = \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{K(x,y)}\,dy,$$

where $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ is a weight function exhibiting an asymptotic behavior similar to that of the standard weight $|x-y|^{N+2s}$ (see [35]). Another possible extension may deal with the fractional p-Laplacian, namely the nonlinear nonlocal operator defined by

$$(-\Delta)_p^s u(x) = 2\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+2s}}\,dy,$$

where $p \in (1,\infty)$. Some existence and multiplicity results for fractional p-Laplacian problems, obtained through critical point theory and Morse theory, can be found in [23]. Nevertheless, the methods used in the present paper cannot be easily extended to $(-\Delta)_p^s$ due to the lack of a complete boundary regularity theory like that developed in [34] for $(-\Delta)^s$ (some results in this direction are proved in [24]).

The paper has the following structure: In Section 2 we recall the variational formulation of our problem and some basic properties of solutions, together with some results from critical point theory. In Section 3 we prove our multiplicity result for the sublinear case. And in Section 4 we deal with the superlinear case.

2 Preliminary results

In this section we recall some results that will be used in our arguments.

2.1 Variational formulation and some properties of problem (1.1)

For all measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ we set

$$[u]_{s,2}^2 = \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy.$$

Then we define the fractional Sobolev space

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : [u]_{s,2} < \infty\}$$

(see [17]). We restrict ourselves to the subspace

$$H_0^s(\Omega) = \{u \in H^s(\mathbb{R}^N) : u(x) = 0 \text{ for a.e. } x \in \Omega^c\},$$

which is a separable Hilbert space under the norm $\|u\| = [u]_{s,2}$ (see [35]). We denote by $H^{-s}(\Omega)$ the topological dual of $H_0^s(\Omega)$ and by $\langle\cdot,\cdot\rangle$ the scalar product of $H_0^s(\Omega)$ (or the duality pairing between $H^{-s}(\Omega)$ and $H_0^s(\Omega)$). In this connection we mention the following useful inequality, holding for all $u \in H_0^s(\Omega)$:

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+2s}}\,dx\,dy \leqslant -\|u^-\|^2, \tag{2.1}$$

where u^- stands for the negative part of u (see [25]). The critical exponent is defined as $2_s^* = \frac{2N}{N-2s}$, and the embedding $H_0^s(\Omega) \hookrightarrow L^p(\Omega)$ is continuous and compact for all $p \in [1, 2_s^*)$ (see [17, Lemma 8]). Moreover, we introduce the positive order cone

$$H_0^s(\Omega)_+ = \{u \in H_0^s(\Omega) : u(x) \geqslant 0 \text{ for a.e. } x \in \Omega\},$$

which has an empty interior with respect to the $H_0^s(\Omega)$-topology. The space $H_0^s(\Omega)$ provides the natural framework for the study of problem (1.1).

Definition 2.1. A function $u \in H_0^s(\Omega)$ is a (weak) solution of (1.1) if for all $v \in H_0^s(\Omega)$, we have

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy = \int_\Omega f(x,u)v\,dx.$$

In all the forthcoming results we will assume the following subcritical growth condition on the nonlinearity f:
(H0) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping, satisfying

$$|f(x,t)| \leqslant a_0(1+|t|^{p-1}) \quad \text{for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R}\ (a_0 > 0,\ p \in (1, 2_s^*)).$$

Under such assumption, we are able to extend to problem (1.1) some basic results holding for elliptic boundary value problems, starting with a simple a priori bound.

Proposition 2.2 ([25, Theorem 3.2]). *Let* (H0) *hold. Then there exists a continuous, nondecreasing function* $M : \mathbb{R}_+ \to \mathbb{R}_+$ *such that for every weak solution* $u \in H_0^s(\Omega)$ *of* (1.1) *one has* $u \in L^\infty(\Omega)$ *and*

$$\|u\|_\infty \leqslant M(\|u\|_{2_s^*}).$$

While solutions of fractional equations exhibit good interior regularity properties, they may have a singular behavior on the boundary. So, instead of the usual space $C^1(\overline{\Omega})$, they are better embedded in the following weighted Hölder-type spaces: Set $\delta(x) = \operatorname{dist}(x, \Omega^c)$ for all $x \in \mathbb{R}^N$ and define

$$C_\delta^0(\overline{\Omega}) = \Big\{u \in C^0(\overline{\Omega}) : \frac{u}{\delta^s} \in C^0(\overline{\Omega})\Big\}, \quad C_\delta^\alpha(\overline{\Omega}) = \Big\{u \in C^0(\overline{\Omega}) : \frac{u}{\delta^s} \in C^\alpha(\overline{\Omega})\Big\} \quad (\alpha \in (0,1)),$$

endowed with the norms

$$\|u\|_{0,\delta} = \left\|\frac{u}{\delta^s}\right\|_\infty, \quad \|u\|_{\alpha,\delta} = \|u\|_{0,\delta} + \sup_{x\neq y} \frac{|u(x)/\delta^s(x) - u(y)/\delta^s(y)|}{|x-y|^\alpha},$$

respectively. For all $0 \leqslant \alpha < \beta < 1$ the embedding $C^\beta_\delta(\overline{\Omega}) \hookrightarrow C^\alpha_\delta(\overline{\Omega})$ is continuous and compact. In this case, the positive cone $C^0_\delta(\overline{\Omega})_+$ has a nonempty interior given by

$$\operatorname{int}(C^0_\delta(\overline{\Omega})_+) = \Big\{u \in C^0_\delta(\overline{\Omega}) : \frac{u(x)}{\delta^s(x)} > 0 \text{ for all } x \in \overline{\Omega}\Big\}.$$

From Proposition 2.2 and [34, Theorem 1.2] we have the following global regularity result.

Proposition 2.3. *Let* (H0) *hold. Then there exist* $\alpha \in (0, \min\{s, 1-s\})$ *and* $C > 0$ *such that for all solutions* $u \in H^s_0(\Omega)$ *of* (1.1) *one has* $u \in C^\alpha_\delta(\overline{\Omega})$ *and*

$$\|u\|_{\alpha,\delta} \leqslant C(1 + \|u\|_{2^*_s}).$$

We now turn to sign properties of solutions of (1.1). We begin with a weak maximum principle.

Proposition 2.4 ([25, Theorem 2.4]). *Let* (H0) *hold and* $f(x,t) \geqslant 0$ *for a.e.* $x \in \Omega$ *and all* $t \in \mathbb{R}$. *If* $u \in H^s_0(\Omega)$ *is a solution of* (1.1), *then* u *is lower semicontinuous and* $u(x) \geqslant 0$ *for all* $x \in \Omega$.

Moreover, we have the following fractional Hopf lemma.

Proposition 2.5 ([22, Lemma 1.2]). *Let* (H0) *hold and* $f(x,t) \geqslant -ct$ *for a.e.* $x \in \Omega$ *and all* $t \in \mathbb{R}$ $(c > 0)$. *If* $u \in H^s_0(\Omega)_+$ *is a solution of* (1.1), *where* u *is lower semicontinuous, then either* $u(x) = 0$ *for all* $x \in \Omega$ *or* $u \in \operatorname{int}(C^0_\delta(\overline{\Omega})_+)$.

Remark 2.6. In its original version from [22], the above Hopf lemma requires that u satisfies $(-\Delta)^s u = f(x,u)$ *pointwisely* in Ω, while we deal with *weak* solutions. In fact, any weak solution u of (1.1) has a higher interior regularity than that displayed in Proposition 2.3, as $u \in C^{1,\beta}(\Omega)$ for any $\beta \in (\max\{0, 2s-1\}, 2s)$ (see [34, Corollary 5.6]). Hence, also recalling that $u = 0$ in Ω^c, one can see that the limit in (1.2) exists in $\mathbb{R}$ and the equation is satisfied pointwisely (see [24, Proposition 2.12]).

Now we introduce an energy functional for problem (1.1). For all $(x,t) \in \Omega \times \mathbb{R}$ set

$$F(x,t) = \int_0^t f(x,\tau)\,d\tau,$$

and for all $u \in H^s_0(\Omega)$ set

$$\varphi(u) = \frac{\|u\|^2}{2} - \int_\Omega F(x,u)\,dx. \tag{2.2}$$

By the continuous embedding $H^s_0(\Omega) \hookrightarrow L^p(\Omega)$, we have $\varphi \in C^1(H^s_0(\Omega))$, and for all $u, v \in H^s_0(\Omega)$ we have

$$\varphi'(u)(v) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy - \int_\Omega f(x,u)v\,dx.$$

So recalling Definition 2.1, we have that u is a solution of (1.1) if and only if $\varphi'(u) = 0$ in $H^{-s}(\Omega)$. Among critical points of φ, *local minimizers* play a preeminent role. We recall, in this connection, a useful topological result relating such minimizers in the $H^s_0(\Omega)$-topology and in $C^0_\delta(\overline{\Omega})$-topology, respectively (a fractional version of the classical result of [9]).

Proposition 2.7 ([25, Theorem 1.1]). *Let* (H0) *hold,* φ *be defined as above, and* $u \in H^s_0(\Omega)$. *Then the following conditions are equivalent:*

(i) *There exists* $r > 0$ *such that* $\varphi(u+v) \geqslant \varphi(u)$ *for all* $v \in H^s_0(\Omega)$, $\|v\| \leqslant r$.
(ii) *There exists* $\rho > 0$ *such that* $\varphi(u+v) \geqslant \varphi(u)$ *for all* $v \in H^s_0(\Omega) \cap C^0_\delta(\overline{\Omega})$, $\|v\|_{0,\delta} \leqslant \rho$.

In the proof of our result we will need some spectral properties of $(-\Delta)^s$. Let us consider the following eigenvalue problem:

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{2.3}$$

Just as in the local case, we say that $\lambda > 0$ is an eigenvalue of $(-\Delta)^s$ if problem (2.3) has a non-zero solution $u \in H_0^s(\Omega)$, which is called a λ-eigenfunction. From the current literature we have rather complete information about the first two eigenvalues of $(-\Delta)^s$.

Proposition 2.8. *The spectrum of $(-\Delta)^s$ consists of a nondecreasing sequence $0 < \lambda_s^1(\Omega) < \lambda_s^2(\Omega) \leqslant \cdots$ of positive numbers, in particular:*

(i) (See [36, Proposition 9]) *The eigenvalue $\lambda_s^1(\Omega)$ is simple and the unique $L^2(\Omega)$-normalized eigenfunction is $\hat{u}_1 \in \operatorname{int}(C_\delta^0(\overline{\Omega})_+)$ such that $\|\hat{u}_1\|_2 = 1$, moreover $\lambda_s^1(\Omega)$ admits the variational characterization*

$$\lambda_s^1(\Omega) = \inf_{u \in H_0^s(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\|u\|_2^2}.$$

(ii) (See [21, Proposition 2.8]) *The eigenvalue $\lambda_s^2(\Omega)$ is the smallest eigenvalue in the interval $(\lambda_s^1(\Omega), \infty)$, the $\lambda_s^2(\Omega)$-eigenfunctions are nodal, moreover $\lambda_s^2(\Omega)$ admits the variational characterization*

$$\lambda_s^2(\Omega) = \inf_{\gamma \in \Gamma_1} \max_{t \in [0,1]} \|\gamma(t)\|^2,$$

where

$$\Gamma_1 = \{\gamma \in C([0,1], H_0^s(\Omega)) : \gamma(0) = \hat{u}_1,\ \gamma(1) = -\hat{u}_1,\ \|\gamma(t)\|_2 = 1 \textit{ for all } t \in [0,1]\}.$$

Note that (ii) above is a fractional version of a classical result of [16], and that Proposition 2.8 holds as well for $(-\Delta)_p^s$ (see [7, 20]). For further information about the spectra of $(-\Delta)^s$ and $(-\Delta)_p^s$ see also [27, 31, 37].

2.2 Some recalls of critical point theory

Variational methods are based on abstract critical point theory, and the latter includes many results, depicting the rich topology that nonlinear and nonconvex functionals may exhibit. We recall here some well-known results which will be our major tools, mainly following [29] (see also [33]).

Let $(X, \|\cdot\|)$ be a reflexive Banach space, $(X^*, \|\cdot\|_*)$ be its topological dual, and $\varphi \in C^1(X)$ be a functional. By $K(\varphi)$ we denote the set of all critical points of φ, i.e., those points $u \in X$ such that $\varphi'(u) = 0$ in X^*, while for all $c \in \mathbb{R}$ we set

$$K_c(\varphi) = \{u \in K(\varphi) : \varphi(u) = c\},$$

besides we set

$$\overline{\varphi}^c = \{u \in X : \varphi(u) \leqslant c\}.$$

Most results require the following Cerami compactness condition (a weaker version of the Palais–Smale condition):

$$\begin{cases} \text{Any sequence } (u_n) \text{ in } X, \text{ such that } (\varphi(u_n)) \text{ is bounded in } \mathbb{R} \text{ and } (1 + \|u_n\|)\varphi(u_n) \to 0 \text{ in } X^* \\ \text{admits a (strongly) convergent subsequence.} \end{cases} \tag{C}$$

We recall a version of the mountain pass theorem (see [2, 32] for the original result).

Theorem 2.9 ([29, Theorem 5.40]). *Let $\varphi \in C^1(X)$ satisfy (C), $u_0, u_1 \in X$, $r \in (0, \|u_1 - u_0\|)$ be such that*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \eta_r := \inf_{\|u-u_0\|=r} \varphi(u),$$

moreover, let

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0,\ \gamma(1) = u_1\}, \quad c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)).$$

Then $c \geqslant \eta_r$ and $K_c(\varphi) \neq \emptyset$.

We will also use the second deformation theorem.

Theorem 2.10 ([29, Theorem 5.34]). *Let $\varphi \in C^1(X)$ satisfy* (C)*, let $a < b$ be real numbers such that $K_c(\varphi) = \emptyset$ for all $c \in (a, b)$ and $K_a(\varphi)$ is a finite set. Then, there exists a continuous deformation*

$$h : [0,1] \times (\overline{\varphi}^b \setminus K_b(\varphi)) \to (\overline{\varphi}^b \setminus K_b(\varphi))$$

such that the following hold:
(i) $h(0, u) = u$, $h(1, u) \in \overline{\varphi}^a$ *for all* $u \in (\overline{\varphi}^b \setminus K_b(\varphi))$,
(ii) $h(t, u) = u$ *for all* $(t, u) \in [0, 1] \times \overline{\varphi}^a$,
(iii) $t \mapsto \varphi(h(t, u))$ *is decreasing in* $[0, 1]$ *for all* $u \in (\overline{\varphi}^b \setminus K_b(\varphi))$.

In particular, (i)–(ii) above mean that $\overline{\varphi}^a$ is a *strong deformation retract* of $\overline{\varphi}^b$ (see [29, Definition 5.33 (b)]). Note that, if a is the global minimum of φ and is attained at a unique point $u_0 \in X$, and there are no critical levels of φ in (a, b), then by Theorem 2.10 the set $\overline{\varphi}^b \setminus K_b(\varphi)$ is *contractible* (see [29, Definition 6.22]).

We conclude this section by recalling some basic notions from Morse theory (see [5, 13] for details). Let $\varphi \in C^1(X)$ satisfy (C) and $u \in K_c(\varphi)$ ($c \in \mathbb{R}$) be an *isolated* critical point of φ, i.e., there exists a neighborhood $U \subset X$ of u such that $K(\varphi) \cap U = \{u\}$. Then, for all integers $k \geqslant 0$ the *k-th critical group of φ at u* is defined as

$$C_k(\varphi, u) = H_k(\overline{\varphi}^c \cap U, \overline{\varphi}^c \cap U \setminus \{u\}), \tag{2.4}$$

where $H_k(\cdot\,,\cdot)$ is the k-th (singular) homology group of a topological pair (see [29, Definition 6.9]). All these groups are real linear spaces. Note that, by the excision property of homology groups, (2.4) is invariant with respect to U. In particular, if $u \in K(\varphi)$ is a strict local minimizer and an isolated critical point, then for all $k \geqslant 0$ we have

$$C_k(\varphi, u) = \delta_{k,0}\mathbb{R}, \tag{2.5}$$

where $\delta_{k,h}$ is the Kronecker symbol (see [29, Example 6.45 (a)]). Critical groups describe the homology of sublevel sets.

Proposition 2.11 ([29, Lemma 6.55]). *Let $\varphi \in C^1(X)$ satisfy* (C)*, let $a < c < b$ be real numbers such that c is the only critical value of φ in $[a, b]$ and $K_c(\varphi)$ is a finite set. Then for all $k \in \mathbb{N}$ we have*

$$H_k(\overline{\varphi}^b, \overline{\varphi}^a) = \bigoplus_{u \in K_c(\varphi)} C_k(\varphi, u).$$

Now assume that

$$\inf_{u \in K(\varphi)} \varphi(u) =: \bar{c} > -\infty.$$

Then we can as well define the *k-th critical group of φ at infinity* as

$$C_k(\varphi, \infty) = H_k(X, \overline{\varphi}^c), \tag{2.6}$$

with $c < \bar{c}$ (this definition is also invariant with respect to c). Critical groups at critical points and at infinity are related by the Poincaré–Hopf formula (one of the Morse relations).

Theorem 2.12 ([29, Remark 6.58]). *Let $\varphi \in C^1(X)$ satisfy* (C)*, let $a < b$ be real numbers such that the set*

$$K_a^b(\varphi) = \{u \in K(\varphi) : a \leqslant \varphi(u) \leqslant b\}$$

is finite. Then

$$\sum_{k=0}^{\infty} \sum_{u \in K_a^b(\varphi)} (-1)^k \dim(C_k(\varphi, u)) = \sum_{k=0}^{\infty} (-1)^k \dim(C_k(\varphi, \infty)).$$

Notation

Throughout the paper, $B_r(x)$ will denote the open ball of radius $r > 0$ centered at $x \in \mathbb{R}^N$ and $C > 0$ will be a constant whose value may change from line to line.

3 The sublinear case

In this section we prove the existence of three non-zero solutions of problem (1.1) when $f(x,\cdot)$ is sublinear at infinity, by means of the second deformation theorem and spectral theory. Precisely, we make on the nonlinearity of f the following assumptions:

(H1) $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory mapping, satisfying
- (i) $|f(x,t)|\leqslant a_0(1+|t|^{p-1})$ for a.e. $x\in\Omega$ and all $t\in\mathbb{R}$ ($a_0>0$, $p\in(2,2_s^*)$),
- (ii) $f(x,t)t\geqslant 0$ for a.e. $x\in\Omega$ and all $t\in\mathbb{R}$,
- (iii) $\limsup_{|t|\to\infty}F(x,t)/t^2\leq 0$ uniformly for a.e. $x\in\Omega$,
- (iv) $\liminf_{t\to 0}F(x,t)/t^2\geq\beta$ uniformly for a.e. $x\in\Omega$ ($\beta>0$).

Example 3.1. Let $a\in L^\infty(\Omega)$ be a function such that $a(x)\geqslant 2\beta>0$ for a.e. $x\in\Omega$. For all $(x,t)\in\Omega\times\mathbb{R}$ set

$$f(x,t)=a(x)\,\mathrm{sign}(t)\ln(1+|t|).$$

Then f satisfies hypotheses (H1).

Clearly, by hypothesis (H1) (ii), problem (1.1) always has the zero solution. First we prove that, for $\beta>0$ big enough, problem (1.1) has two constant sign solutions.

Proposition 3.2. *Let* (H1) *hold with* $\beta>\lambda_s^1(\Omega)/2$. *Then problem* (1.1) *admits at least two non-zero solutions* $u_\pm\in\pm\mathrm{int}\,(C_\delta^0(\overline{\Omega})_+)$.

Proof. We define φ as in (2.2). Besides, we introduce two truncated energy functionals by setting

$$\varphi_\pm(u)=\frac{\|u\|^2}{2}-\int_\Omega F_\pm(x,u)\,dx \tag{3.1}$$

for all $u\in H_0^s(\Omega)$, where for all $(x,t)\in\Omega\times\mathbb{R}$ we have set

$$f_\pm(x,t)=f(x,\pm t^\pm),\qquad F_\pm(x,t)=\int_0^t f_\pm(x,\tau)\,d\tau.$$

We focus on the functional φ_+. Clearly, $\varphi_+\in C^1(H_0^s(\Omega))$. We now prove that φ_+ is coercive in $H_0^s(\Omega)$, i.e.,

$$\lim_{\|u\|\to\infty}\varphi_+(u)=\infty. \tag{3.2}$$

Indeed, by hypotheses (H1) (i)–(iii), for all $\varepsilon>0$ we can find $C_\varepsilon>0$ such that for a.e. $x\in\Omega$ and all $t\in\mathbb{R}$ we have

$$0\leqslant F_+(x,t)\leqslant C_\varepsilon+\varepsilon t^2. \tag{3.3}$$

By Proposition 2.8 (i) and (3.3), we have

$$\varphi_+(u)\geqslant\frac{\|u\|^2}{2}-\int_\Omega(C_\varepsilon+\varepsilon u^2)\,dx\geqslant\Big(\frac{1}{2}-\frac{\varepsilon}{\lambda_s^1(\Omega)}\Big)\|u\|^2-C_\varepsilon|\Omega|$$

for all $u\in H_0^s(\Omega)$. If we choose $\varepsilon<\lambda_s^1(\Omega)/2$, the latter tends to ∞ as $\|u\|\to\infty$, so (3.2) follows. Moreover, φ_+ is sequentially weakly lower semicontinuous in $H_0^s(\Omega)$. Indeed, let $u_n\rightharpoonup u$ in $H_0^s(\Omega)$. Passing if necessary to a subsequence, we may assume $u_n\to u$ in $L^p(\Omega)$ and $u_n(x)\to u(x)$ for a.e. $x\in\Omega$, moreover, there exists $g\in L^p(\Omega)$ such that $|u_n(x)|\leqslant g(x)$ for a.e. $x\in\Omega$ and all $n\in\mathbb{N}$ (see [8, Theorem 4.9]). Hence,

$$\lim_n\int_\Omega F_+(x,u_n)\,dx=\int_\Omega F_+(x,u)\,dx.$$

Besides, by convexity we have

$$\liminf_n\frac{\|u_n\|^2}{2}\geqslant\frac{\|u\|^2}{2},$$

so

$$\liminf_n \varphi_+(u_n) \geqslant \varphi_+(u).$$

Then we easily go back to the original sequence. Thus, there exists $u_+ \in H^s_0(\Omega)$ such that

$$\varphi_+(u_+) = \inf_{u \in H^s_0(\Omega)} \varphi_+(u). \tag{3.4}$$

In particular, $u_+ \in K(\varphi_+)$. We note that u_+ is a solution of the (1.1)-type problem

$$\begin{cases} (-\Delta)^s u_+ = f_+(x, u_+) & \text{in } \Omega, \\ u_+ = 0 & \text{in } \Omega^c. \end{cases}$$

By (H1) (ii) and Proposition 2.4 we have $u_+ \in H^s_0(\Omega)_+$. It remains to prove that $u_+ \neq 0$. Here we use our assumption on β: let $\beta' \in (0, \beta)$ be such that $\beta' > \lambda^1_s(\Omega)/2$. By (H1) (iv), we can find $\sigma > 0$ such that $F_+(x, t) > \beta' t^2$ for a.e. $x \in \Omega$ and all $|t| \leqslant \sigma$. Let $\hat{u}_1 \in \operatorname{int}(C^0_\delta(\overline{\Omega})_+)$ be defined as in Proposition 2.8 (i), then for $\mu > 0$ small enough we have $\|\mu\hat{u}_1\|_\infty \leqslant \sigma$, hence

$$\varphi_+(\mu\hat{u}_1) \leqslant \frac{\|\mu\hat{u}_1\|^2}{2} - \int_\Omega \beta'(\mu\hat{u}_1)^2\,dx = \mu^2\Big(\frac{1}{2} - \frac{\beta'}{\lambda^1_s(\Omega)}\Big)\|\hat{u}_1\|^2 < 0.$$

By (3.4) we have $\varphi_+(u_+) < 0$, hence $u_+ \neq 0$. By Proposition 2.5 we deduce $u_+ \in \operatorname{int}(C^0_\delta(\overline{\Omega})_+)$. Noting that $\varphi(u) = \varphi_+(u)$ for all $u \geqslant 0$, we see that u_+ is a local minimizer of φ in $C^0_\delta(\overline{\Omega})$, hence by Proposition 2.7 a local minimizer of φ in $H^s_0(\Omega)$. In particular, $u_+ \in K(\varphi)$, hence u_+ is a positive solution of (1.1).

Similarly, we find another local minimizer $u_- \in -\operatorname{int}(C^0_\delta(\overline{\Omega})_+)$ of φ, which turns out to be a negative solution of (1.1). □

Now, taking $\beta > 0$ even bigger, we achieve a third non-zero solution.

Theorem 3.3. *Let* (H1) *hold with* $\beta > \lambda^2_s(\Omega)/2$. *Then problem* (1.1) *admits at least three non-zero solutions* $u_\pm \in \pm\operatorname{int}(C^0_\delta(\overline{\Omega})_+)$, $\tilde{u} \in C^0_\delta(\overline{\Omega}) \setminus \{0\}$.

Proof. First we note, arguing as in the proof of (3.2), that

$$\lim_{\|u\|\to\infty} \varphi(u) = \infty. \tag{3.5}$$

Now we prove that φ satisfies (C) (which in this case is equivalent to the Palais–Smale condition). Let (u_n) be a sequence in $H^s_0(\Omega)$ such that $|\varphi(u_n)| \leqslant C$ for all $n \in \mathbb{N}$ and $(1 + \|u_n\|)\varphi'(u_n) \to 0$ in $H^{-s}(\Omega)$. By (3.5), the sequence (u_n) is bounded in $H^s_0(\Omega)$. Hence, passing if necessary to a subsequence, we may assume $u_n \rightharpoonup u$ in $H^s_0(\Omega)$, $u_n \to u$ in $L^p(\Omega)$ and $L^1(\Omega)$, and $u_n(x) \to u(x)$ for a.e. $x \in \Omega$, with some $u \in H^s_0(\Omega)$. Moreover, by [8, Theorem 4.9] there exists $g \in L^p(\Omega)$ such that $|u_n(x)| \leqslant g(x)$ for all $n \in \mathbb{N}$ and a.e. $x \in \Omega$. Using such relations along with (H1) (i), we have

$$\begin{aligned} \|u_n - u\|^2 &= \langle u_n, u_n - u\rangle - \langle u, u_n - u\rangle \\ &= \varphi'(u_n)(u_n - u) + \int_\Omega f(x, u_n)(u_n - u)\,dx - \langle u, u_n - u\rangle \\ &\leqslant \|\varphi'(u_n)\|_*\|u_n - u\| + \int_\Omega a_0(1 + |u_n|^{p-1})|u_n - u|\,dx - \langle u, u_n - u\rangle \\ &\leqslant \|\varphi'(u_n)\|_*\|u_n - u\| + a_0\big(\|u_n - u\|_1 + \|u_n\|_p^{p-1}\|u_n - u\|_p\big) - \langle u, u_n - u\rangle \end{aligned}$$

for all $n \in \mathbb{N}$ and the latter tends to 0 as $n \to \infty$. Thus, $u_n \to u$ in $H^s_0(\Omega)$.

By (H1) (ii) we have $0 \in K(\varphi)$, while from Proposition 3.2 we know that $u_\pm \in K(\varphi) \setminus \{0\}$. We aim at proving the existence of a further critical point $\tilde{u} \in H^s_0(\Omega)$. We argue by contradiction, assuming

$$K(\varphi) = \{0, u_+, u_-\}. \tag{3.6}$$

It is not restrictive to assume that $\varphi(u_+) \geqslant \varphi(u_-)$ and that u_+ is a strict local minimizer of φ, so we can find $r \in (0, \|u_+ - u_-\|)$ such that $\varphi(u) > \varphi(u_+)$ for all $u \in H^s_0(\Omega)$, $0 < \|u - u_+\| \leqslant r$. Moreover, we have

$$\eta_r := \inf_{\|u-u_+\|=r} \varphi(u) > \varphi(u_+). \tag{3.7}$$

Otherwise, we could find a sequence (u_n) in $H^s_0(\Omega)$ such that $\|u_n - u_+\| = r$ for all $n \in \mathbb{N}$, $\varphi(u_n) \to \varphi(u_+)$ and $\varphi'(u_n) \to 0$ in $H^{-s}(\Omega)$ (see [29, Corollary 5.12]). Then by (C) we would have $u_n \to \bar{u}$ in $H^s_0(\Omega)$ for some $\bar{u} \in H^s_0(\Omega)$, $\|\bar{u} - u_+\| = r$, hence in turn $\varphi(\bar{u}) = \varphi(u_+)$, which is a contradiction.

Now set

$$\Gamma = \{\gamma \in C([0,1], H^s_0(\Omega)) : \gamma(0) = u_+,\ \gamma(1) = u_-\}, \qquad c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t)).$$

By Theorem 2.9 we have $c \geqslant \eta_r$ and there exists $\tilde{u} \in K_c(\varphi)$. By (3.7) we have $\tilde{u} \neq u_\pm$. So, (3.6) implies $\tilde{u} = 0$, hence $c = 0$. To reach a contradiction, we will construct a path $\gamma \in \Gamma$ such that

$$\max_{t\in[0,1]} \varphi(\gamma(t)) < 0, \tag{3.8}$$

so that $c < 0$. Let $\beta' \in (0, \beta)$, $\theta > 0$ be such that

$$\beta' > \frac{\lambda^2_s(\Omega) + \theta}{2}. \tag{3.9}$$

By (H1) (iv) there exists $\sigma > 0$ such that $F(x,t) > \beta' t^2$ for a.e. $x \in \Omega$ and all $|t| \leqslant \sigma$. Besides, by Proposition 2.8 (ii) there exists $\gamma_1 \in \Gamma_1$ such that

$$\max_{t\in[0,1]} \|\gamma_1(t)\|^2 < \lambda^2_s(\Omega) + \theta. \tag{3.10}$$

Since $C^\infty_0(\Omega)$ is dense in $H^s_0(\Omega)$ (see [19, Theorem 2]), we can choose $\gamma_1(t) \in L^\infty(\Omega)$ for all $t \in [0,1]$ and γ_1 continuous with respect to the $L^\infty(\Omega)$-topology. So, by choosing $\varepsilon > 0$ small enough, we have $\|\varepsilon\gamma_1(t)\|_\infty \leqslant \sigma$ for all $t \in [0,1]$. Thus, by (3.10) and recalling that $\|\gamma_1(t)\|_2 = 1$, we have for all $t \in [0,1]$ that

$$\varphi(\varepsilon\gamma_1(t)) \leqslant \frac{\varepsilon^2\|\gamma_1(t)\|^2}{2} - \beta'\varepsilon^2\|\gamma_1(t)\|^2_2 < \varepsilon^2\Big(\frac{\lambda^2_s(\Omega) + \theta}{2} - \beta'\Big),$$

and the latter is negative by (3.9). Then $\varepsilon\gamma_1$ is a continuous path joining $\varepsilon\hat{u}_1$ and $-\varepsilon\hat{u}_1$ such that

$$\max_{t\in[0,1]} \varphi(\varepsilon\gamma_1(t)) < 0. \tag{3.11}$$

By (H1) (ii) and Proposition 2.4, it is easily seen that $K(\varphi_+) \subseteq K(\varphi)$. More precisely, by (3.6), we have $K(\varphi_+) = \{0, u_+\}$. Set $a = \varphi_+(u_+)$, $b = 0$. Then $\overline{\varphi}^a_+ = \{u_+\}$ and φ_+ satisfies all assumptions of Theorem 2.10, so there exists a continuous deformation $h_+ : [0,1] \times (\overline{\varphi}^0_+ \setminus \{0\}) \to (\overline{\varphi}^0_+ \setminus \{0\})$ such that

$$\begin{cases} h_+(0,u) = u, \quad h_+(1,u) = u_+ & \text{for all } u \in (\overline{\varphi}^0_+ \setminus \{0\}),\\ h_+(t,u_+) = u_+ & \text{for all } t \in [0,1],\\ \qquad t \mapsto \varphi_+(h_+(t,u)) & \text{is decreasing for all } u \in (\overline{\varphi}^0_+ \setminus \{0\}). \end{cases}$$

In particular, the set $\overline{\varphi}^0_+ \setminus \{0\}$ turns out to be contractible. Set

$$\gamma_+(t) = h_+(t, \varepsilon\hat{u}_1)$$

for all $t \in [0,1]$. Then $\gamma_+ \in C([0,1], H^s_0(\Omega))$ is a path joining $\varepsilon\hat{u}_1$ and u_+, such that $\varphi_+(\gamma_+(t)) < 0$ for all $t \in [0,1]$. Note that $\varphi(u) \leqslant \varphi_+(u)$ for all $u \in H^s_0(\Omega)$, indeed we have

$$\varphi_+(u) - \varphi(u) = \int_\Omega (F(x,u) - F_+(x,u))\,dx = \int_{\{u<0\}} F(x,u)\,dx,$$

and the latter is non-negative by (H1) (ii). So we have

$$\max_{t\in[0,1]} \varphi(\gamma_+(t)) < 0. \tag{3.12}$$

Similarly, we construct a path $\gamma_- \in C([0,1], H^s_0(\Omega))$ joining $-\varepsilon\hat{u}_1$ and u_-, such that

$$\max_{t\in[0,1]} \varphi(\gamma_-(t)) < 0. \tag{3.13}$$

Concatenating γ_+, $\varepsilon\gamma_1$, and γ_- (with convenient changes of parameter) and considering (3.11)–(3.13), we construct a path $\gamma \in \Gamma$ satisfying (3.8), against (3.6) and the definition of the mountain pass level c.

So, we conclude that there exists a fourth critical point $\tilde{u} \in K(\varphi) \setminus \{0, u_+, u_-\}$, which turns out to be a non-zero solution of (1.1), concluding the proof. □

4 The superlinear case

In this section we prove the existence of three non-zero solutions of problem (1.1) when $f(x,\cdot)$ is superlinear at infinity. Following an idea first appeared in [38], we will apply the mountain pass theorem and Morse theory. Precisely, we make the following assumptions on the nonlinearity f:

(H2) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping, satisfying
- (i) $|f(x,t)| \leqslant a_0(1 + |t|^{p-1})$ for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$ ($a_0 > 0$, $p \in (2, 2^*_s)$),
- (ii) $f(x,t)t \leqslant 0$ for a.e. $x \in \Omega$ and all $t \in [-\sigma, \sigma]$ ($\sigma > 0$),
- (iii) $f(x,t)t \geqslant -c_0 t^2$ for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$ ($c_0 > 0$),
- (iv) $\lim_{|t|\to\infty} F(x,t)/t^2 = \infty$ uniformly for a.e. $x \in \Omega$,
- (v) $\liminf_{|t|\to\infty} (f(x,t)t - 2F(x,t))/|t|^q > 0$ uniformly for a.e. $x \in \Omega$ ($q \in (\frac{(p-2)N}{2s}, 2^*_s)$).

Condition (H2) (v) is a mild version of the classical Ambrosetti–Rabinowitz condition (see [33]), and an easy computation shows that we can always assume $q < p$ in it. Such a condition was first introduced in [15].

Example 4.1. Let $a, b \in L^\infty(\Omega)$ be such that $a(x) \geqslant \alpha$, $b(x) \geqslant \beta$ for a.e. $x \in \Omega$ ($\alpha, \beta > 0$), and set

$$f(x,t) = -a(x)t + b(x)|t|^{p-2}t$$

for all $(x,t) \in \Omega \times \mathbb{R}$. Then f satisfies hypotheses (H2) with convenient a_0, c_0, σ, and q. This choice of f belongs in the class of concave-convex nonlinearities, whose study (in the classical case $s = 1$) started with [1].

By hypothesis (H2) (ii), problem (1.1) admits the zero solution. We focus now on constant sign solutions.

Proposition 4.2. *Let* (H2) *hold. Then* (1.1) *admits at least two non-zero solutions* $u_\pm \in \pm\mathrm{int}\,(C^0_\delta(\overline{\Omega})_+)$.

Proof. We define φ, $\varphi_\pm$ as in (2.2), (3.1). We focus mainly on φ_+.

First we prove that φ_+ satisfies (C). Let (u_n) be a sequence in $H^s_0(\Omega)$ such that $|\varphi_+(u_n)| \leqslant C$ for all $n \in \mathbb{N}$ and $(1 + \|u_n\|)\varphi'_+(u_n) \to 0$ in $H^{-s}(\Omega)$. Then we have

$$-\|u_n\|^2 + \int_\Omega f_+(x, u_n)u_n \, dx \leqslant C,$$

$$\|u_n\|^2 - 2\int_\Omega F_+(x, u_n)\, dx \leqslant C$$

for all $n \in \mathbb{N}$, which imply

$$\int_\Omega (f_+(x, u_n)u_n - 2F_+(x, u_n))\, dx \leqslant C. \tag{4.1}$$

Clearly, (H2) (v) yields

$$\lim_{t\to\infty} \frac{f_+(x,t)t - 2F_+(x,t)}{t^q} > 0$$

uniformly for a.e. $x \in \Omega$. So we can find $\beta, M > 0$ such that $f_+(x,t)t - 2F_+(x,t) \geqslant \beta t^q$ for a.e. $x \in \Omega$ and all $t > M$. We claim that (u_n) is bounded in $L^q(\Omega)$. Indeed, for all $n \in \mathbb{N}$ we have

$$\|u_n\|_q^q = \|u_n^+\|_q^q + \|u_n^-\|_q^q.$$

By the previous inequality we have

$$\begin{aligned}\beta\|u_n^+\|_q^q &= \int\limits_{\{0<u_n\leqslant M\}} \beta u_n^q\,dx + \int\limits_{\{u_n>M\}} \beta u_n^q\,dx \\ &\leqslant \beta M^q|\Omega| + \int\limits_{\{u_n>M\}} (f_+(x,u_n)u_n - 2F_+(x,u_n))\,dx \\ &\leqslant C + \int\limits_{\Omega} (f_+(x,u_n)u_n - 2F_+(x,u_n))\,dx,\end{aligned}$$

and the latter is bounded by (4.1). Besides, using (2.1) and recalling that $f_+(x,t)t^- = 0$ for all $(x,t) \in \Omega\times\mathbb{R}$, we get

$$\begin{aligned}\|u_n^-\|^2 &\leqslant -\iint\limits_{\mathbb{R}^N\times\mathbb{R}^n} \frac{(u_n(x)-u_n(y))(u_n^-(x)-u_n^-(y))}{|x-y|^{N+2s}}\,dx\,dy \\ &= -\varphi_+'(u_n)(u_n^-) \\ &\leqslant \|\varphi_+'(u_n)\|_*\|u_n^-\|,\end{aligned}$$

which implies that $\|u_n^-\|$ is bounded in $\mathbb{R}$. By the continuous embedding $H_0^s(\Omega)\hookrightarrow L^q(\Omega)$, this yields $\|u_n^-\|_q \to 0$ as $n\to\infty$. So we deduce that $\|u_n\|_q$ is bounded in $\mathbb{R}$.

Using this fact, we want to show that (u_n) is bounded in $H_0^s(\Omega)$ as well. Since $q<p<2_s^*$ in our assumptions, we can find $\tau\in(0,1)$ such that

$$\frac{1}{p} = \frac{1-\tau}{q} + \frac{\tau}{2_s^*}.$$

By the interpolation inequality (see [8, p. 93]) and the continuous embedding $H_0^s(\Omega)\hookrightarrow L^{2_s^*}(\Omega)$, we have

$$\|u_n\|_p \leqslant \|u_n\|_q^{1-\tau}\|u_n\|_{2_s^*}^{\tau} \leqslant C\|u_n\|^{\tau} \tag{4.2}$$

for all $n\in\mathbb{N}$. Again by $(1+\|u_n\|)\varphi_+'(u_n)\to 0$ in $H^{-s}(u_n)$ and (H2) (i) we have

$$\begin{aligned}\|u_n\|^2 &\leqslant \int\limits_{\Omega} f_+(x,u_n)u_n\,dx + C \\ &\leqslant \int\limits_{\Omega} a_0(1+|u_n|^{p-1})|u_n|\,dx + C \\ &\leqslant C(1+\|u_n\|_1+\|u_n\|_p^p)\end{aligned}$$

for all $n\in\mathbb{N}$. By (4.2) and the continuous embeddings $H_0^s(\Omega)\hookrightarrow L^1(\Omega), L^p(\Omega)$ we see that

$$\|u_n\|^2 \leqslant C(1+\|u_n\|+\|u_n\|^{p\tau}).$$

Since $p\tau<2$ we deduce that (u_n) is bounded in $H_0^s(\Omega)$. Now we conclude as in the proof of Theorem 3.3.

Now we prove that φ_+ is unbounded from below. Indeed, let $\hat{u}_1$ be defined as in Proposition 2.8 (i), and recall that $\|\hat{u}_1\|^2 = \lambda_s^1(\Omega)$, $\|\hat{u}_1\|_2^2 = 1$. By (H2) (iv), given $\theta > \lambda_s^1(\Omega)/2$ we can find $M>0$ such that $F(x,t)\geqslant\theta t^2$ for a.e. $x\in\Omega$ and all $|t|>M$. For all $\mu>0$ we have

$$\begin{aligned}\varphi_+(\mu\hat{u}_1) &= \frac{\mu^2\|\hat{u}_1\|^2}{2} - \int\limits_{\{\mu\hat{u}_1\leqslant M\}} F_+(x,\mu\hat{u}_1)\,dx - \int\limits_{\{\mu\hat{u}_1>M\}} F_+(x,\mu\hat{u}_1)\,dx \\ &\leqslant \frac{\mu^2\lambda_s^1(\Omega)}{2} - \int\limits_{\{\mu\hat{u}_1>M\}} \theta\mu^2\hat{u}_1^2\,dx + C \\ &\leqslant \mu^2\Big(\frac{\lambda_s^1(\Omega)}{2} - \theta\Big) + \theta M^2|\Omega| + C,\end{aligned}$$

and the latter goes to $-\infty$ as $\mu \to \infty$. Thus,

$$\lim_{\mu\to\infty} \varphi_+(\mu\hat{u}_1) = -\infty. \tag{4.3}$$

We claim that 0 is a local minimizer for φ_+. By (H2) (ii) we have $F_+(x,t) \leqslant 0$ for a.e. $x \in \Omega$ and all $|t| \leqslant \sigma$. For all $u \in C^0_\delta(\overline{\Omega})$ with

$$\|u\|_{0,\delta} \leqslant \frac{\sigma}{\operatorname{diam}(\Omega)^s},$$

we have $\|u\|_\infty \leqslant \sigma$, hence

$$\varphi_+(u) \geqslant \frac{\|u\|^2}{2} \geqslant 0.$$

So, 0 is a local minimizer of φ_+ in $C^0_\delta(\overline{\Omega})$. By Proposition 2.7, 0 is as well a local minimizer of φ_+ in $H^s_0(\Omega)$. As usual, it is not restrictive to assume that 0 is a strict local minimizer for φ_+ and (reasoning as in the proof of (3.7)) there exists $r > 0$ such that

$$\eta^+_r := \inf_{\|u\|=r} \varphi_+(u) > 0. \tag{4.4}$$

By (4.3) we can find $\mu > 0$ such that $\|\mu\hat{u}_1\| > r$ and $\varphi_+(\mu\hat{u}_1) < 0$. Set

$$\Gamma_+ = \{\gamma \in C([0,1], H^s_0(\Omega)) : \gamma(0) = 0,\ \gamma(1) = \mu\hat{u}_1\}, \quad c_+ = \inf_{\gamma\in\Gamma_+} \max_{t\in[0,1]} \varphi_+(\gamma(t)).$$

By Theorem 2.9 we have $c_+ \geqslant \eta^+_r$ and there exists $u_+ \in K_{c_+}(\varphi_+)$. From (4.4) we see that $c_+ > 0$, hence $u_+ \neq 0$. Testing $\varphi'_+(u_+) = 0$ with $(u_+)^- \in H^s_0(\Omega)$ and using (2.1), we get

$$-\|(u_+)^-\|^2 \geqslant \varphi'_+(u_+)((u_+)^-) = 0,$$

i.e., $u_+ \in H^s_0(\Omega)_+$ (note that Proposition 2.4 does not apply here). By (H2) (iii) we can apply Proposition 2.5 and deduce $u_+ \in \operatorname{int}(C^0_\delta(\overline{\Omega})_+)$, in particular, $f_+(x,u_+) = f(x,u_+)$ for a.e. $x \in \Omega$. Thus we conclude that $u_+ \in K(\varphi)$ and it is a positive solution of (1.1).

A similar argument, applied to φ_-, leads to the existence of a negative solution $u_- \in -\operatorname{int}(C^0_\delta(\overline{\Omega})_+)$ of problem (1.1). □

Using the critical groups, we can improve the conclusion of Proposition 4.2 under the same assumptions.

Theorem 4.3. *Let* (H2) *hold. Then problem* (1.1) *admits at least three non-zero solutions* $u_\pm \in \pm\operatorname{int}(C^0_\delta(\overline{\Omega})_+)$, $\tilde{u} \in C^0_\delta(\overline{\Omega}) \setminus \{0\}$.

Proof. Reasoning as in the proof of Proposition 4.2 we see that φ, $\varphi_\pm$ satisfy (C), are unbounded from below and have a strict local minimum at 0. Moreover, we know that $0, u_\pm \in K(\varphi)$. We aim at finding a further critical point for φ. We argue by contradiction, assuming

$$K(\varphi) = \{0, u_+, u_-\}. \tag{4.5}$$

In particular, all critical points of φ are isolated. Taking $a < b$ in $\mathbb{R}$ such that all critical levels of φ lie in (a,b), from Theorem 2.12 we have

$$\sum_{k=0}^\infty (-1)^k (\dim C_k(\varphi,0) + \dim C_k(\varphi,u_+) + \dim C_k(\varphi,u_-)) = \sum_{k=0}^\infty (-1)^k \dim C_k(\varphi,\infty). \tag{4.6}$$

Now we will compute all critical groups of φ both at its critical points and at infinity, then we will plug the results into (4.6) to get a contradiction. In doing so, we will also need to compute some critical groups of $\varphi_\pm$.

We begin with critical groups at infinity. For all integers $k \geqslant 0$ we have

$$C_k(\varphi,\infty) = C_k(\varphi_\pm,\infty) = 0. \tag{4.7}$$

We focus on φ (the argument for $\varphi_\pm$ is analogous). We recall from the proof of Proposition 4.2 that

$$\min\{\varphi(u_+), \varphi(u_-)\} > \varphi(0) = 0.$$

We denote the unit sphere in $H^s_0(\Omega)$ by

$$S = \{u \in H^s_0(\Omega) : \|u\| = 1\}.$$

Reasoning as in the proof of (4.3) we see that for all $u \in S$ we have

$$\lim_{\mu\to\infty} \varphi(\mu u) = -\infty.$$

Moreover, since φ is sequentially weakly lower semicontinuous in $H^s_0(\Omega)$, we have

$$\inf_{\|u\|\leqslant 1} \varphi(u) =: \kappa > -\infty.$$

We claim that there exists $c < \kappa$ such that for all $v \in \varphi^{-1}(c)$ we have

$$\varphi'(v)(v) < 0. \tag{4.8}$$

Indeed, by (H2) (v) there exist $\beta, M > 0$ such that $f(x,t)t - 2F(x,t) \geqslant \beta|t|^q$ for a.e. $x \in \Omega$ and all $|t| > M$. Then, using also (H2) (i), for all $v \in \varphi^{-1}(c)$ we have

$$\begin{aligned}
\varphi'(v)(v) &= \|v\|^2 - \int_\Omega f(x,v)v\,dx \\
&= 2\varphi(v) - \int_\Omega \big(f(x,v)v - 2F(x,v)\big)\,dx \\
&\leqslant 2c - \int_{\{|v|>M\}} \beta|v|^q\,dx + \int_{\{|v|\leqslant M\}} \Big(a_0(|v| + |v|^p) + a_0\Big(|v| + \frac{|v|^p}{p}\Big)\Big)\,dx \\
&\leqslant 2c - \beta\|v\|_q^q + \beta M^q|\Omega| + C(M + M^p)|\Omega| \\
&\leqslant 2c + C_M,
\end{aligned}$$

with a constant $C_M > 0$ only depending on M. So, choosing

$$c < \min\Big\{-\frac{C_M}{2}, \kappa\Big\},$$

we get (4.8). Now we apply the implicit function theorem [29, Theorem 7.3] to the function $(\mu, u) \mapsto \varphi(\mu u)$ defined in $(1,\infty) \times S$. By (4.8) we have for all $(\mu, u) \in (1,\infty) \times S$ with $\varphi(\mu u) = c$ that

$$\frac{\partial}{\partial\mu}\varphi(\mu u) = \frac{\varphi'(\mu u)(\mu u)}{\mu} < 0,$$

hence there exists a continuous mapping $\rho : S \to (1,\infty)$ such that for all $(\mu, u) \in (1,\infty) \times S$ we have

$$\varphi(\mu u) \begin{cases} > c & \text{if } \mu < \rho(u), \\ = c & \text{if } \mu = \rho(u), \\ < c & \text{if } \mu > \rho(u). \end{cases}$$

So we have

$$\overline{\varphi}^c = \{\mu u : u \in S,\ \mu \in [\rho(u), \infty)\}.$$

Set also

$$E = \{\mu u : u \in S,\ \mu \geqslant 1\}.$$

We can define a continuous deformation $h : [0,1] \times E \to E$ by setting

$$h(t, \mu u) = \begin{cases} (1-t)\mu u + t\rho(u)u & \text{if } \mu < \rho(u), \\ \mu u & \text{if } \mu \geqslant \rho(u) \end{cases}$$

for all $(t, \mu u) \in [0,1] \times E$, so $\overline{\varphi}^c$ is a strong deformation retract of E. Besides, we define another continuous deformation $\tilde{h} : [0,1] \times E \to E$ by setting

$$\tilde{h}(t, \mu u) = (1-t)\mu u + tu$$

for all $(t, \mu u) \in [0,1] \times E$, showing that S is also a strong deformation retract of E. By the choice of c, (2.6), and [29, Corollary 6.15] we have for all $k \geqslant 0$ that

$$C_k(\varphi, \infty) = H_k(H_0^s(\Omega), \overline{\varphi}^c) = H_k(H_0^s(\Omega), E) = H_k(H_0^s(\Omega), S),$$

and the latter is 0 by [29, Propositions 6.24, 6.25] (recall that S is contractible in itself, as $\dim H_0^s(\Omega) = \infty$). Thus we have (4.7).

We compute now the critical points at 0. For all $k \geqslant 0$ we have

$$C_k(\varphi, 0) = C_k(\varphi_\pm, 0) = \delta_{k,0}\mathbb{R}. \tag{4.9}$$

Reasoning as in the proof of Proposition 4.2 and using (4.5), we see that 0 is a strict local minimizer of φ, so (4.9) follows from (2.5) (the argument for $\varphi_\pm$ is analogous).

Finally, we compute the critical groups at $u_\pm$. For all $k \geqslant 0$ we have

$$C_k(\varphi, u_\pm) = \delta_{k,1}\mathbb{R}. \tag{4.10}$$

We consider u_+ (the argument for u_- is analogous). First we note that

$$C_k(\varphi, u_+) = C_k(\varphi_+, u_+). \tag{4.11}$$

Indeed, for all $\tau \in [0,1]$ we define $\psi_\tau \in C^1(H_0^s(\Omega))$ by setting

$$\psi_\tau(u) = (1-\tau)\varphi(u) + \tau\varphi_+(u)$$

for all $u \in H_0^s(\Omega)$. Clearly, we have $u_+ \in K(\psi_\tau)$ for all $\tau \in [0,1]$. Moreover, u_+ is an isolated critical point of ψ_τ uniformly with respect to τ, as we shall prove arguing by contradiction. Assume that there exist sequences (u_n) in $H_0^s(\Omega) \setminus \{u_+\}$, (τ_n) in $(0,1)$ such that $u_n \to u_+$ in $H_0^s(\Omega)$, $\tau_n \to \tau$, and $\psi'_{\tau_n}(u_n) = 0$ in $H^{-s}(\Omega)$ for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$, u_n is a solution of the (1.1)-type problem

$$\begin{cases} (-\Delta)^s u_n = (1-\tau_n)f(x, u_n) + \tau_n f_+(x, u_n) & \text{in } \Omega, \\ u_n = 0 & \text{in } \Omega^c, \end{cases}$$

with a reaction term satisfying (H0) uniformly (i.e., with a_0, p independent of n). By Proposition 2.2 the sequence (u_n) is bounded in $L^\infty(\Omega)$, and by Proposition 2.3 there exist $\alpha \in (0,1)$, $C > 0$ such that for all $n \in \mathbb{N}$ we have $u_n \in C_\delta^\alpha(\overline{\Omega})$ and $\|u_n\|_{\alpha,\delta} \leqslant C$.

By the compact embedding $C_\delta^\alpha(\overline{\Omega}) \hookrightarrow C_\delta^0(\overline{\Omega})$, passing if necessary to a subsequence, we have $u_n \to u_+$ in $C_\delta^0(\overline{\Omega})$, hence $u_n \in \operatorname{int}(C_\delta^0(\overline{\Omega})_+)$ for all $n \in \mathbb{N}$ large enough. This in turn implies that u_n is a solution of (1.1), i.e., a critical point of φ different from 0 and $u_\pm$, against (4.5).

So, by homotopy invariance of critical groups (see [13, Theorem 5.6]), we see that $C_k(\psi_\tau, u_+)$ is independent of $\tau \in [0,1]$. Noting that $\psi_0 = \varphi$ and $\psi_1 = \varphi_+$, we achieve (4.11).

By (4.11), we are reduced to computing $C_k(\varphi_+, u_+)$. Recall that $K(\varphi_+) = \{0, u_+\}$ and fix $a, b \in \mathbb{R}$ such that

$$a < \varphi_+(0) < b < \varphi_+(u_+).$$

Then set $A = \overline{\varphi}_+^a$, $B = \overline{\varphi}_+^b$. We have $A \subset B$ and the following long sequence is exact due to [29, Proposition 6.14]:

$$\cdots \longrightarrow H_k(H_0^s(\Omega), A) \xrightarrow{j_*} H_k(H_0^s(\Omega), B) \xrightarrow{\partial_*} H_{k-1}(B, A) \xrightarrow{i_*} H_{k-1}(H_0^s(\Omega), A) \longrightarrow \cdots.$$

Here j_*, i_* are the group homomorphisms induced by the inclusion mappings $j : (H_0^s(\Omega), A) \to (H_0^s(\Omega), B)$ and $i : (B, A) \to (H_0^s(\Omega), A)$, respectively, and ∂_* is the boundary homomorphism (see [29, Definition 6.9]).

By Proposition 2.11 and (2.6) we have

$$H_k(H_0^s(\Omega), A) = C_k(\varphi_+, \infty), \quad H_k(H_0^s(\Omega), B) = C_k(\varphi_+, u_+), \quad H_{k-1}(B, A) = C_{k-1}(\varphi_+, 0).$$

So, recalling (4.7), the exact sequence rephrases as

$$0 \to C_k(\varphi_+, u_+) \to C_{k-1}(\varphi_+, 0) \to 0,$$

which by (4.9) yields

$$C_k(\varphi_+, u_+) = \delta_{(k-1),0}\mathbb{R} = \delta_{k,1}\mathbb{R}.$$

By (4.11), we get (4.10).

Plugging (4.7), (4.9), and (4.10) into (4.6), we have

$$\sum_{k=0}^{\infty}(-1)^k(\delta_{k,0} + 2\delta_{k,1}) = 0,$$

namely $-1 = 0$, a contradiction. Therefore, (4.5) cannot hold, i.e., there exists a further critical point $\tilde{u} \in K(\varphi) \setminus \{0, u_+, u_-\}$. By Proposition 2.3, we see that $u \in C_\delta^0(\overline{\Omega})$ and it is a solution of (1.1). □

Remark 4.4. A comparison between Theorems 3.3 and 4.3 is now in order. Though formally the statements of such results coincide, the underlying structure of the critical set $K(\varphi)$ changes considerably in the two cases. In the sublinear case we have two local minimizers u_+, u_- and a third non-zero critical point $\tilde{u}$, typically of mountain pass type, while in the superlinear case we have two mountain pass-type points u_+, u_- and a third non-zero critical point of undetermined nature $\tilde{u}$.

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