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Analysis and explicit solvability of degenerate tensorial problems

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Abstract

This paper studies a two dimensional boundary value problem described by a tensorial equation in a bounded domain. Once its more general definition is given, we conclude that its analysis is linked to the resolution of an overdetermined hyperbolic problem; hence some discussions and considerations are presented. Secondly, for a simplified version of the original formulation, which leads to a degenerate problem on a rectangle, we prove the existence and uniqueness of a solution under proper assumptions on the data.

Keywords: Degenerate hyperbolic equations; Boundary value problems

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1 Introduction, motivation and structure of the paper

It is well known that the Dirichlet problem associated to a hyperbolic equation is often employed as an example of an ill posed problem in the theory of hyperbolic partial differential equations ([1]). Nevertheless, a wide range of real problems arising in nature (gas dynamics, torsion theory of shells with alternating sign curvature, mechanical behaviors of bending structures, etc.), are in fact mathematically described through hyperbolic equations; thereafter, a deserving undertaking is developing a casuistry for which such problems are, indeed, well posed.

In this sense, matter of this investigation are the existence and uniqueness of a solution to a tensorial boundary value problem whose analysis requires the study of a hyperbolic equation. Precisely, the corresponding formulation models the equilibrium of membrane structures, used in civil engineering applications. It is worth to underline that these equilibrium equations are not new, but linked to those dealing with shell structures (see the fundamental monograph [2] and also [3])[1] and are given by:

$$\begin{cases} \sigma_{xx,x} + \sigma_{xy,y} = 0 & \text{in } \Omega, \\ \sigma_{xy,x} + \sigma_{yy,y} = 0 & \text{in } \Omega, \\ z_{,xx}\sigma_{xx} + 2z_{,xy}\sigma_{xy} + z_{,yy}\sigma_{yy} = 0 & \text{in } \Omega, \\ \text{boundary conditions} & \text{on } \Gamma = \partial\Omega. \end{cases} \tag{1}$$

[1]In this paper the partial derivative of a function f with respect to a certain variable w is indicated with $f_{,w}$; similar symbols concerning higher order derivatives (double or mixed) are introduced in a natural way.

In system (1), $z = z(x, y)$ is a regular function defined in a bounded domain $\Omega \subset \mathbb{R}^2$, with piecewise smooth boundary $\Gamma = \partial\Omega$, and its graph represents the shape of the shell; similarly,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(x, y) = \begin{pmatrix} \sigma_{xx}(x, y) & \sigma_{xy}(x, y) \\ \sigma_{yx}(x, y) & \sigma_{yy}(x, y) \end{pmatrix}$$

is a symmetric second order stress tensor which determines the state (compression or tension) of the same shell.

The main difference between the equilibrium approach for shell structures and that for membrane ones are discussed, for instance, in [4, 5, 6] and references therein. Even though this paper goes beyond the technical and physical aspects addressed in the previous three references, it must be specified that, from the mathematical point of view, if on the one hand no assumption on the function z nor the tensor $\boldsymbol{\sigma}$ appearing in system (1) is required for the analysis of the equilibrium of shell structures, on the other hand considering membrane elements implies the following restrictions:

$(H1)$ the graph of z represents an almost everywhere (a.e.) negative Gaussian curvature (alternating sign curvature) surface[2] i.e.

$$z_{,xx}z_{,yy} - z_{,xy}^2 < 0 \quad \text{a.e. in} \quad \bar{\Omega};$$

$(H2)$ tensor $\boldsymbol{\sigma}$ is almost everywhere positive definite i.e.

$$\sigma_{xx}\xi_1^2 + 2\sigma_{xy}\xi_1\xi_2 + \sigma_{yy}\xi_2^2 > 0 \quad \forall\,(\xi_1, \xi_2) \neq (0, 0) \quad \text{and} \quad \text{a.e. in} \quad \bar{\Omega}^{[3]}.$$

In other words, *any* pair $(z, \boldsymbol{\sigma})$ verifying system (1) models the equilibrium of a shell whose shape has not necessarily a constant sign curvature and with very general stress state (compression, tension or both). Of course, for a given z with a.e. positive Gaussian curvature, an a.e. negative, a.e. positive or alternating sign definite $\boldsymbol{\sigma}$ balancing z might be derived; nevertheless, *no one* of these cases would represent the equilibrium of a membrane structure. Indeed, a balanced pair $(z, \boldsymbol{\sigma})$ for a fixed z with a.e. negative Gaussian curvature and a.e. negative, a.e. positive or alternating sign definite $\boldsymbol{\sigma}$, idealizes in *only one case* the equilibrium of a membrane structure.

Coming back to the framework of the equilibrium for membrane structures (which, as already commented, justifies this investigation), the discussion presented above naturally allows us to define two complementary approaches resulting from system (1):

$(\mathcal{HP})$ a problem of hyperbolic type where the tensor $\boldsymbol{\sigma}$ is the unknown: given a function z with a.e. negative Gaussian curvature find an a.e. positive definite tensor $\boldsymbol{\sigma}$ fulfilling (1);

$(\mathcal{EP})$ a problem of elliptic type where the function z is the unknown: given an a.e. positive definite tensor $\boldsymbol{\sigma}$ verifying the first two PDEs of (1), find a function z with a.e. negative Gaussian curvature fulfilling (1).

[2] With some abuse of language we also use sentences as *z has an a.e. negative Gaussian curvature* or *z is a function with a.e. negative Gaussian curvature* or similar; in any case, no misunderstanding will be possible from the context.

In this work we will mainly dedicate to problem $(\mathcal{HP})$: furthermore, due to the high complementarity between $(\mathcal{HP})$ and $(\mathcal{EP})$, we might make mention to this latter approach, for which partial results are available in the literature.

The remaining structure of the paper is drawn as follows: In §2 we formulate the so called **General Problem** associated to $(\mathcal{HP})$, which is a very broad (tensorial) boundary value problem modeling an optimal mechanical scenario appearing in membrane structures. As detailed in [4, 5, 7], the boundary of the domain is split in two parts; on a portion, mechanically corresponding to the boundary of the membrane tensioned by rigid elements (which admit any geometrical shape), a Dirichlet boundary condition is assumed, whilst on the remaining part, associated to the complementary boundary of the membrane tensioned by cables (which cannot be straight lines, nor changing curvature curves), an unusual boundary relation is given. We discuss the main mathematical properties of this formulation, also in terms of other well known results, and we conclude that this is an overdetermined, generally ill posed, problem, for which the part of the domain with the singular boundary condition (free boundary) plays the role of a further unknown. In addition, §3 deals with the analytical resolution of the **Reduced Problem**, a simplified version of the **General Problem**, linked to a more restrictive physical situation, where the membrane is only tensioned by rigid elements: we examine a specific case in a rectangle for which the resulting Dirichlet boundary problem admits an explicit unique solution. Specifically, once a polynomial for the function z is fixed in such a way that its graph identifies a surface with a.e. negative Gaussian curvature, by manipulating the tensorial expressions of the problem, the main equation reads $cy^{2(n-1)}\sigma_{yy,xx} - \sigma_{yy,yy} = 0$ in $(0,a)\times(-b,0)$, with some $a, b, c > 0$ and n an integer greater than 1, exactly degenerating for $y = 0$. Connected to the last partial differential equation (PDE), the question of well posedness of boundary value problems for linear second order PDEs of the form $\psi(y)u_{,xx} - u_{,yy} = 0$, where ψ is a sufficiently regular function with specific properties, has been studied in several works: contributions as [8, 9, 10, 11] (and references therein) include discussions concerning the notorious special case of the mixed elliptic-hyperbolic Tricomi equation, obtained for $\psi(y) = y$, and provide a general comprehensive picture of the whole analysis. Also in line with these works, we cite paper [12], employed in this present investigation to prove the main result asserted in Theorem 3.1 and, in particular, to construct the claimed explicit solution $\boldsymbol{\sigma}$ to system (1). Finally, in order to mathematically point out the different physical behaviors between shells and membranes, in §4 we also solve the same **Reduced Problem** presented in §3 but in the case where no restriction on the sign of $\boldsymbol{\sigma}$ is required (Theorem 4.1); besides, we give a graphical representation of the derived solutions corresponding to the two mechanical situations.

2 The General Problem

The following section includes some necessary tools used to our main purposes.

2.1 Definition of the domain and the boundary data

In order to formulate the General Problem associated to system (1), we need to properly define its domain and boundary data. The items below address these questions and are graphically represented in the left side of Figure 1.

Assumptions 2.1 *We consider a function $z = z(x, y)$ with a.e. negative Gaussian curvature in $\bar{\Omega}$, in the sense of (H1), Ω being a bounded subset of $\mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$, obtained by the union of two portions; precisely $\Gamma = \partial\Omega = \Gamma^r \cup \Gamma^c$ and has the following properties*[4]*:*

(i) *Γ^c is represented by a regular curve in $\mathbb{R}^2$, with no vanishing curvature, whose parametrization is given by $\gamma(t) = (x(t), y(t))$, with $t \in [t_0, t_1]$, and obtained by solving this ordinary differential equation:*

$$z_{,xx}(x')^2 + 2z_{,xy}x'y' + z_{,yy}(y')^2 = 0. \tag{2}$$

(ii) *Γ^r is arbitrarily fixed, but in such a way that $\Gamma^r \cap \Gamma^c = \{P_0, P_1\}$, where $P_0 = \gamma(t_0)$ and $P_1 = \gamma(t_1)$.*

(iii) *$\boldsymbol{n}$ is the outward unit vector to Γ.*

(iv) *$\boldsymbol{f}^r = (f_1^r, f_2^r)$ and $\boldsymbol{f}^c = (f_1^c, f_2^c)$ are two regular vectorial fields, per unit length, defined on Γ^r and Γ^c, respectively; in addition the continuity conditions $\boldsymbol{f}^r(P_0) = \boldsymbol{f}^c(P_0)$ and $\boldsymbol{f}^r(P_1) = \boldsymbol{f}^c(P_1)$ have to be satisfied.*

2.2 Mathematical formulation of the General Problem

Let us now describe the details of the General Problem we are interested in.

General Problem 1 *Under the hypothesis of Assumptions 2.1, find a symmetric and a.e. positive definite second order tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}(x, y)$ in $\bar{\Omega}$ and a real function g defined in Γ^c such that*

$$\begin{cases} \sigma_{xx,x} + \sigma_{xy,y} = 0 \quad \text{in } \Omega, & \text{(3a)} \\ \sigma_{xy,x} + \sigma_{yy,y} = 0 \quad \text{in } \Omega, & \text{(3b)} \\ z_{,xx}\sigma_{xx} + 2z_{,xy}\sigma_{xy} + z_{,yy}\sigma_{yy} = 0 \quad \text{in } \bar{\Omega}, & \text{(3c)} \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{f}^r \quad \text{on } \Gamma^r, & \text{(3d)} \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{f}^c \quad \text{on } \Gamma^c, & \text{(3e)} \end{cases}$$

and

$$\begin{cases} (gx')' = f_1^c \quad \text{on } \Gamma^c \ (t \in [t_0, t_1]), \\ (gy')' = f_2^c \quad \text{on } \Gamma^c \ (t \in [t_0, t_1]), \\ g(t_0) = g_0, \end{cases} \tag{4}$$

where g_0 is a given real number.

2.3 Analysis and discussion of the General Problem

Since the vectorial field $\boldsymbol{f}^c$ has to verify both expressions (3e) and (4), it cannot be uniquely and arbitrarily assigned (see the Counterexample 1 below). Essentially this singularity is tied to the fact that the unknowns $\boldsymbol{\sigma}$ and g are coupled through $\boldsymbol{f}^c$ and that, even more, they are especially linked to the domain Γ^c; subsequently,

[4]Let us remark that as indicated in the paragraphs of §1 dealing with the **General Problem**, the superscripts r and c stand for *rigid* and *cable*.

the General Problem 1 represents an overdetermined system which, commonly, does not admit solutions.

Moreover, so far we did not manage to derive a nontrivial analytical solution to the same problem; indeed the question how to fix $(z, \boldsymbol{\sigma}, \gamma, \boldsymbol{f}^c, \boldsymbol{f}^r, g, g_0)$ such that all the relations (2), (3) and (4) hold seems rather challenging. In line with this, the case where z is a linear function has no mathematical interest (and even less physical), since relation (3c) is automatically verified and, in addition, any $\gamma(t) = (x(t), y(t))$ is compatible with condition (2); hence the problem merely loses its intrinsic nature. The same mathematical and physical reasons make that functions behaving as $z(x, y) = \alpha^2x^2 - \beta^2y^2$, with $\alpha, \beta \in \mathbb{R}_0$, do not lead to stimulating issues, since (2) would infer straight lines parametrized as $\gamma(t) = \big(t, \mp(\alpha/\beta)t + \text{constant}\big)$ for Γ^c, and essentially the General Problem would "degenerate" to the forthcoming Reduced Problem (see page 6).

Returning to overdetermined boundary value problems, there exists a large amount of literature dealing with the subject; in general these problems are prescribed by a classical partial differential equation where both Dirichlet and Neumann boundary conditions are imposed on the boundary of the domain. Some meriting questions about the analysis are the proof of the existence of a solution, possibly uniqueness and the study of its properties. The main characteristic of the overdetermined problems is that such an over-determination makes the domain itself unknown (free boundary problems), or in general it cannot be arbitrarily assigned, resulting solvable only in precise domains; beyond the landmark result by Serrin [13], we refer also to [14, 15, 16, 17] for contributions regarding both elliptic and hyperbolic equations.

Remark 1 *As to the specific problem we are focusing here, let us quote that the elliptic version of the General Problem 1, herein indicated with $(\mathcal{EP})$ and briefly defined in §1, has been deeply discussed by the same author of this paper in recent investigations. We mention that the complete formulation of problem $(\mathcal{EP})$ corresponds to a boundary value problem, in the unknown z, described by an elliptic differential equation in Ω. The portion Γ^c of Γ is indeed constructed by means of $\boldsymbol{\sigma}$ (which in this case is fixed) and $\boldsymbol{f}^c$. Finally, the whole Γ is endowed with Dirichlet boundary conditions but, in accordance to overdetermined problems, on Γ^c another relation involving $z_{,y}$ and replacing expression (2) has to be satisfied as well. The technical aspects for the construction of Γ^c and the definition of the complete boundary value problem are available in [18] and [7]; in particular, as for the General Problem, the questions of the existence and the derivation of an explicit solution are still open. Conversely in the last two aforementioned contributions an equivalent number of numerical procedures exactly tied to free boundary approaches are proposed and employed as resolution methods.*

Counterexample 1 *(Ill posedness of the General Problem). Let us fix $z(x, y) = -A^2x^4 + 6B^2y^2$ (with $A, B \in \mathbb{R}_0$). From equation (2), we can choose as Γ^c the curve $\gamma(t) = (t, (A/2B)t^2)$, with $t \in [t_0, t_1]$. In addition, $\sigma_{xx} = 1$, $\sigma_{xy} = \sigma_{yx} = 0$ and $\sigma_{yy} = (A^2/B^2)x^2$ is a symmetric and positive definite tensor a.e. in $\mathbb{R}^2$ which solves equations (3a), (3b) and (3c). As to the expression of $\boldsymbol{f}^c$, since $\boldsymbol{n} = 1/||\gamma'(t)||(-(A/B)t, 1)$, relation (3e) infers $\boldsymbol{f}^c = (-(A/B)t, (A^2/B^2)t^2)$ on Γ^c; thereafter from the first and last conditions of (4) we arrive at $g(t) =$*

$-(A/2B)t^2 + g_0 + (A/2B)t_0^2$ which, in view of the second relation of (4), leads to the following incongruence: $-(3A^2/2B^2)t^2 + g_0(A/B) + (A^2/2B^2)t_0^2 = (A^2/B^2)t^2$ for all $t \in [t_0, t_1]$.

3 The Reduced Problem

3.1 Mathematical formulation of the Reduced Problem

Let us now introduce the Reduced Problem; essentially, its definition corresponds to set $\Gamma^c = \emptyset$ in Assumptions 2.1. Therefore, Γ^c, $\boldsymbol{f}^c$ and g do not take part in the formulation and, subsequently, we have $\Gamma = \Gamma^r = \partial\Omega$; moreover, for convenience, we avoid the superscript r for $\boldsymbol{f}^r$, and we directly consider $\boldsymbol{f}$ as a given vectorial field, per unit length, on $\Gamma = \partial\Omega$.

Reduced Problem 1 *Under the hypothesis of Assumptions 2.1, let us set $\Gamma^c = \emptyset$. Find a symmetric and a.e. positive definite second order tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}(x, y)$ in $\bar{\Omega}$ such that*

$$\begin{cases} \sigma_{xx,x} + \sigma_{xy,y} = 0 \quad \text{in } \Omega, & \text{(5a)} \\ \sigma_{xy,x} + \sigma_{yy,y} = 0 \quad \text{in } \Omega, & \text{(5b)} \\ z_{,xx}\sigma_{xx} + 2z_{,xy}\sigma_{xy} + z_{,yy}\sigma_{yy} = 0 \quad \text{in } \bar{\Omega}, & \text{(5c)} \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{f} \quad \text{on } \Gamma. & \text{(5d)} \end{cases}$$

In the rest of this section we show the existence and uniqueness of a solution to the Reduced Problem 1 defined in a rectangle.

3.2 A case of explicit resolution in a rectangle

For any $a, c_1 > 0$ and $n \in \mathbb{N}$ with $n > 1$, let us consider the rectangle $\Omega = (0, a) \times (-b, 0)$, with $b = n^{1/n}$, and the function $z(x, y) = c_1x^2 - c_2y^{2n}$, where $c_2 = a^2c_1/(n(2n-1))$, which satisfies $z_{,xx}z_{,yy} - z_{,xy}^2 < 0$ a.e. in $\bar{\Omega}$. Differentiating (5a) with respect to x and (5b) to y, and subtracting the results each other, give

$$\sigma_{xx,xx} - \sigma_{yy,yy} = 0 \quad \text{in } \Omega. \tag{6}$$

On the other hand, in view of the expression of z, relation (5c) infers

$$\sigma_{xx} = \frac{c_2n(2n-1)}{c_1}y^{2(n-1)}\sigma_{yy} \quad \text{in } \bar{\Omega}. \tag{7}$$

Hence, (6) and (7) lead to

$$\frac{c_2n(2n-1)}{c_1}y^{2(n-1)}\sigma_{yy,xx} - \sigma_{yy,yy} = 0 \quad \text{in } \Omega, \tag{8}$$

which degenerates for $y = 0$. In order to endow this equation with the desired Dirichlet conditions for the unknown σ_{yy}, let us treat the vectorial field $\boldsymbol{f}$ on Γ and let us write

$$\boldsymbol{f}=\begin{cases}(\gamma_1(y),\gamma_2(y)) & \text{on } x=a,\\ (-\gamma_1(y),\gamma_3(y)) & \text{on } x=0,\\ (\theta_1(x),\theta_2(x)) & \text{on } \ y=0,\\ (\theta_3(x),-\theta_2(x)) & \text{on } \ y=-b.\end{cases}$$

In the previous definition, $\gamma_i(y)$ and $\theta_i(x)$ (for $i=1,2,3$) are continuous functions for $-b\le y\le 0$ and $0\le x\le a$, respectively, which will be chosen later on; subsequently, being $\boldsymbol{n}=(\mp 1,0)$, respectively on $x=0$ and $x=a$, and $\boldsymbol{n}=(0,\mp 1)$, respectively on $y=-b$ and $y=0$, the boundary conditions (5d) read

$$\begin{cases}\sigma_{xx}(a,y)=\sigma_{xx}(0,y)=\gamma_1(y),\\ \sigma_{xy}(a,y)=\gamma_2(y) \quad \text{and} \quad \sigma_{xy}(0,y)=-\gamma_3(y),\\ \sigma_{xy}(x,0)=\theta_1(x) \quad \text{and} \quad \sigma_{xy}(x,-b)=-\theta_3(x),\\ \sigma_{yy}(x,0)=\sigma_{yy}(x,-b)=\theta_2(x).\end{cases} \tag{9}$$

Additionally, if $\gamma_1=\gamma_1(y)$ is such that the function

$$\gamma(y)=\frac{\gamma_1(y)c_1}{c_2n(2n-1)y^{2(n-1)}}$$

is itself continuous in $-b\le y\le 0$, in light of (7), (8) and (9) we arrive to this boundary value problem

$$\begin{cases}\frac{c_2n(2n-1)}{c_1}y^{2(n-1)}\sigma_{yy,xx}-\sigma_{yy,yy}=0 \quad \text{in } \Omega,\\ \sigma_{yy}(x,0)=\sigma_{yy}(x,-b)=\theta_2(x),\\ \sigma_{yy}(0,y)=\sigma_{yy}(a,y)=\gamma(y),\end{cases} \tag{10}$$

for which we fix these proper assumptions:

$$\gamma(0)=\gamma(-b)=\theta_2(0)=\theta_2(a)=K\in\mathbb{R} \quad \text{and} \quad \gamma_2(0)=H\in\mathbb{R}. \tag{11}$$

Hence, let us translate the unknown σ_{yy} through

$$u(x,y)=\sigma_{yy}(x,y)-[\gamma(y)+\theta_2(x)-K], \tag{12}$$

and successively let us rescale x by the homogeneous dilation mapping $(0,1)$ onto $(0,a)$ and given by $x(X)=aX$; the new variable $U(X,y)=u(aX,y)$ and data $\Theta(X)=\theta_2(ax)$ are so obtained. These two transformations, in conjunction with the relation $c_2=a^2c_1/(n(2n-1))$, reduce (10) to

$$\begin{cases}y^{2(n-1)}U_{,XX}-U_{,yy}+\Theta''(X)y^{2(n-1)}-\gamma''(y)=0 \quad \text{in } (0,1)\times(-b,0),\\ U(X,0)=U(X,-b)=0,\\ U(0,y)=U(1,y)=0.\end{cases} \tag{13}$$

Now, let us impose $\Theta''(X)y^{2(n-1)} - \gamma''(y) = 0$, i.e. $\Theta''(X) = \gamma''(y)/y^{2(n-1)} = \lambda$, for some $\lambda \in \mathbb{R}$. This provides, thanks to the first continuity conditions (11) which for $-b \le y \le 0$ and $0 \le x \le 1$ are $\gamma(0) = \gamma(-b) = \Theta(0) = \Theta(1) = K$,

$$\Theta(X) = \frac{\lambda}{2}X^2 - \frac{\lambda}{2}X + K \quad \text{and} \quad \gamma(y) = \frac{\lambda}{2n(2n-1)}y^{2n} + \frac{\lambda b^{2n-1}}{2n(2n-1)}y + K; \tag{14}$$

as a consequence, problem (13) is equivalent to

$$\begin{cases} (-y)^{2(n-1)}U_{,XX} - U_{,yy} = 0 \quad \text{in } (0,1) \times (-b,0), \\ U(X,0) = U(X,-b) = 0, \\ U(0,y) = U(1,y) = 0. \end{cases} \tag{15}$$

According to the theory of second order linear PDE's, and by virtue of the fact that $b = n^{1/n}$, the characteristic curves associated to the equation $(-y)^{2(n-1)}U_{,XX} - U_{,yy} = 0$ and exactly passing through the vertexes of the rectangle $(0,1) \times (-b,0)$ are (see the right side of Figure 1)

$$X = \frac{1}{n}(-y)^n \quad \text{and} \quad 1 - X = \frac{1}{n}(-y)^n.$$

Thereafter we can rely on the main statement given in [12] and apply its result to the boundary value problem (15); hence we conclude that it admits a unique solution in $(0,1) \times (-b,0)$ which is continuously differentiable everywhere in its closure, possibly except along the mentioned characteristics. As to our specific case, in view of the homogeneous boundary conditions, $U(X,y) \equiv 0$ (and hence also $u(x,y) \equiv 0$) is the unique function with such properties solving problem (15).

Coming back to the tensorial unknown $\boldsymbol{\sigma}$ in $\bar{\Omega}$, expression (14) produces through the relations $X = x/a$, (12) and (7)

$$\begin{cases} \sigma_{yy}(x,y) = \frac{\lambda}{2a^2}x^2 - \frac{\lambda}{2a}x + \frac{\lambda}{2n(2n-1)}y^{2n} + \frac{\lambda b^{2n-1}}{2n(2n-1)}y + K & \text{in } \bar{\Omega}, \\ \sigma_{xx}(x,y) = a^2 y^{2(n-1)}\sigma_{yy}(x,y) & \text{in } \bar{\Omega}. \end{cases} \tag{16}$$

As to $\sigma_{xy} = \sigma_{yx}$, from (5b) we deduce

$$\sigma_{xy}(x,y) = -\int \sigma_{yy,y}dx = -\frac{2n\lambda y^{2n-1} + \lambda b^{2n-1}}{2n(2n-1)}x + h(y),$$

so that imposing (5a) we get

$$h'(y) = \frac{a}{2}\lambda y^{2(n-1)} \Leftrightarrow h(y) = \frac{a\lambda y^{2n-1}}{2(2n-1)} + h_0, \; h_0 \in \mathbb{R}.$$

Now, taking into account the second position in (11) and the boundary conditions (9), the last two expressions yield to

$$\begin{aligned} \sigma_{xy}(x,y) &= \sigma_{yx}(x,y) \\ &= -\frac{2n\lambda y^{2n-1} + \lambda b^{2n-1}}{2n(2n-1)}x + \frac{a\lambda y^{2n-1}}{2(2n-1)} + H + \frac{a\lambda b^{2n-1}}{2n(2n-1)} \quad \text{in } \bar{\Omega}. \end{aligned} \tag{17}$$

Lately, in order to guarantee the positive definiteness of $\boldsymbol{\sigma}$ a.e. in $\bar{\Omega}$ in the sense of $(H2)$, we have to impose, inter alia, that $\sigma_{xx}\sigma_{yy} - (\sigma_{xy})^2 > 0$ a.e. in $\bar{\Omega}$. From (16) and (17) we obtained that $\sigma_{xx}(x,0) = 0$ for all $x \in [0,a]$, while that $\sigma_{xy}(x,0) = -\lambda b^{2n-1}x/(2n(2n-1)) + H + a\lambda b^{2n-1}/(2n(2n-1))$ for all $x \in [0,a]$; therefore, without specific assumptions and relations on a, b, n, λ and H, generally $\sigma_{xx}(x,0)\sigma_{yy}(x,0) - (\sigma_{xy}(x,0))^2 < 0$ holds in $[0,a]$. In these conditions, since for continuity arguments there would exist $\varepsilon > 0$ such that $\sigma_{xx}\sigma_{yy} - (\sigma_{xy})^2 < 0$ in $(0,a) \times (-\varepsilon, 0)$, in (17) we have to impose $H = \lambda = 0$ obtaining $\sigma_{xy}(x,y) = \sigma_{yx}(x,y) \equiv 0$ in $\bar{\Omega}$. In addition, in order to avoid the nil solution $\boldsymbol{\sigma} \equiv \mathbf{0}$, we choose a strictly positive value for K and from (16) we explicitly write $\sigma_{yy}(x,y) = K$, $\sigma_{xx}(x,y) = Ka^2y^{2(n-1)}$ in $\bar{\Omega}$ and also obtain the following formulas for the functions $\gamma_i(y)$ and $\theta_i(x)$ defining the boundary conditions (9):

$$\gamma_1(y) = a^2Ky^{2(n-1)}, \quad \theta_2(x) = K, \quad \theta_1(x) = \theta_3(x) = \gamma_2(y) = \gamma_3(y) = 0.$$

We have so proved our main result:

Theorem 3.1 *Let be $a, c_1 > 0$ and $n \in \mathbb{N}$, with $n > 1$. Moreover, for $b = n^{1/n}$ and $c_2 = a^2c_1/(n(2n-1))$, the rectangle $\Omega = (0,a) \times (-b,0)$ and the function $z(x,y) = c_1x^2 - c_2y^{2n}$ are given. Then for any fixed $K > 0$ and vectorial field (per unit length) on $\Gamma = \partial\Omega$*

$$\boldsymbol{f} = \begin{cases} (a^2Ky^{2(n-1)}, 0) & \text{on } x = a, \\ (-a^2Ky^{2(n-1)}, 0) & \text{on } x = 0, \\ (0, K) & \text{on } y = 0, \\ (0, -K) & \text{on } y = -b, \end{cases}$$

the symmetric and a.e. positive definite tensor

$$\boldsymbol{\sigma}(x,y) = \begin{pmatrix} a^2Ky^{2(n-1)} & 0 \\ 0 & K \end{pmatrix} \quad \text{in } \bar{\Omega}, \tag{18}$$

is the unique classical solution of the Reduced Problem 1.

4 The case of no restriction on the sign definiteness of $\boldsymbol{\sigma}$

By retracing the proof of Theorem 3.1 we observe that, behind other technical reasons, the final expression of the solution $\boldsymbol{\sigma}$ derived in (18) is deeply tied to the requirement of the a.e. positivity definiteness of such a tensor; conversely, as announced in the introductory comments of §1, if this restriction is omitted, for the same function $z(x,y) = c_1x^2 - c_2y^{2n}$ the unique solution in $\bar{\Omega}$ exhibits a more general representation, precisely given by (16) and (17). Subsequently, we have this other result which we state without further comments.

Theorem 4.1 *Let be $a, c_1 > 0$ and $n \in \mathbb{N}$, with $n > 1$. Moreover, for $b = n^{1/n}$ and $c_2 = a^2c_1/(n(2n-1))$, the rectangle $\Omega = (0,a) \times (-b,0)$ and the function $z(x,y) = c_1x^2 - c_2y^{2n}$ are given. Then for any fixed $H, K, \lambda \in \mathbb{R}$ and vectorial field* *(per unit length) on* $\Gamma = \partial\Omega$

$$\boldsymbol{f} = \begin{cases} (\gamma_1(y), \gamma_2(y)) & \text{on } x = a, \\ (-\gamma_1(y), \gamma_3(y)) & \text{on } x = 0, \\ (\theta_1(x), \theta_2(x)) & \text{on } y = 0, \\ (\theta_3(x), -\theta_2(x)) & \text{on } y = -b, \end{cases}$$

where

$$\begin{cases} \gamma_1(y) = a^2 y^{2(n-1)}\left(\frac{\lambda}{2n(2n-1)} y^{2n} + \frac{\lambda b^{2n-1}}{2n(2n-1)} y + K\right), \\ \gamma_2(y) = H + a\lambda \frac{n y^{2n-1} - b^{2n-1}(n-1)}{2n(2n-1)}, \\ \gamma_3(y) = -H - a\lambda \frac{b^{2n-1} + y^{2n-1}}{2(2n-1)}, \\ \theta_1(x) = \frac{\lambda b^{2n-1}}{2n(2n-1)}(an - x) + H, \\ \theta_2(x) = \frac{\lambda}{2a^2} x^2 - \frac{\lambda}{2a} x + K, \\ \theta_3(x) = -\frac{\lambda b^{2n-1}}{2n} x - H, \end{cases}$$

the symmetric tensor

$$\begin{cases} \sigma_{yy}(x,y) = \frac{\lambda}{2a^2} x^2 - \frac{\lambda}{2a} x + \frac{\lambda}{2n(2n-1)} y^{2n} + \frac{\lambda b^{2n-1}}{2n(2n-1)} y + K & \text{in } \bar{\Omega}, \\ \sigma_{xx}(x,y) = a^2 y^{2(n-1)} \sigma_{yy}(x,y) & \text{in } \bar{\Omega}, \\ \sigma_{xy}(x,y) = \sigma_{yx}(x,y) = \frac{\lambda y^{2n-1}}{2n-1}\left(\frac{a}{2} - x\right) + \frac{\lambda b^{2n-1}}{2n(2n-1)}(an - x) + H & \text{in } \bar{\Omega}. \end{cases} \quad (19)$$

is the unique classical solution of the Reduced Problem 1.

In order to give an explicit example to each one of the results claimed in Theorems 3.1 and 4.1, we analyze Figures 2 and 3. They graphically show the behavior of the tensor $\boldsymbol{\sigma}$, which solves the Reduced Problem 1, once in the hypothesis of such theorems the same surface $z = c_1 x^2 - c_2 y^{2n}$ and the rectangle $R = (0,a) \times (-b, 0)$ are fixed by means of these values: $n = 3$, $c_1 = 4$ and $a = 5$ (the surface and the domain are shown at the top-lefts corners of Figures 2 and 3).

More precisely, for $K = 5$ in expression (18), Figure 2 represents the case of the equilibrium between stress and shape of a membrane structure. We can realize that, the component σ_{xx} is positive a.e. in R and it increases for $y \to -b$ and constant values of x (see the below part of the top-right corner of Figure 2); in the limit, it exactly corresponds to a zone on the membrane with major tension, along the x-direction, with respect to others (same Figure 2, above part). As to σ_{yy}, it is constant and positive in R so that the corresponding tension along the y-direction is uniformly distributed on the surface (see the lower-left corner of Figure 2); finally, the last Figure 2, at the lower-right corner, highlights the nil contribution of $\sigma_{yx} = 0$ in R, that is the absence of shear stress on the membrane.

Conversely, if in (19) we set $\lambda = 4$, $K = 0.5$ and $H = 2$, the features of the solution $\boldsymbol{\sigma}$ are summarized in Figure 3 , which models the balance between stress and shape for a shell structure. By relaxing the assumption on the sign definiteness of $\boldsymbol{\sigma}$, we do not obtain only positive expressions for the components σ_{xx} and σ_{yy} on the whole R, but also regions of the rectangle where they are negative (see the below part of the top-right and lower-left corner of Figure 3, respectively); this aspect identifies zones of the shell where are present tensions or compressions along both the x- and y-directions (sames corners of Figure 3, but the above part).

By virtue of all of the above, we stress again that the general solution for the tensor $\boldsymbol{\sigma}$ given by relations (18) represents a very particular and simplified case of solution (19). Such a leap has not to appear surprising since, indeed, it is intimately linked to the different natures of the problems: in particular, when a membrane is considered a strong limitation on the state of its stress tensor which exactly balances its shape is naturally expected and absolutely consistent with the mechanical problem.

5 Conclusions

This paper is devoted to a two dimensional boundary value system described by a tensorial equation in a bounded domain. Its more general definition leads to the resolution of an overdetermined hyperbolic problem, whose analysis is complex and represents a challenging open question in the field. Indeed, for a simplified version, whose formulation is given by a degenerate problem on a rectangle, the existence and uniqueness of a solution under proper assumptions on the data can be proven. Behind its pure mathematical interest, this research is motivated by its natural application to real mechanic problems, linked to the equilibrium of membrane and shell structures. In this sense, the derived solutions achieved throughout the paper are totally consistent with the expected results.

List of abbreviations
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

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Authors' contributions
Both authors contributed equally to this work. They both read and approved the final version of the manuscript.

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References

1. Hörmander, L.: Linear Partial Differential Operators. Springer, (1964)
2. Timoshenko, S., Woinowsky-Krieger, S.: Theory of Plates and Shells. McGraw-Hill, New York, (1959)
3. Ventsel, E., Krauthammer, T.: Thin Plates and Shells. Theory, Analysis, and Applications. New York, Basel, (2001)
4. Viglialoro, G., Murcia, J., Martínez, F.: Equilibrium problems in membrane structures with rigid boundaries. Inf. Constr. **61**(516), 57–66 (2009)

5. Viglialoro, G., Murcia, J.: Equilibrium problems in membrane structures with rigid and cable boundaries. Inf. Constr. **63**(524), 49–57 (2011)
6. Viglialoro, G., Murcia, J., Martínez, F.: The 2-d continuous analysis versus the density force method (discrete) for structural membrane equilibrium. Inf. Constr. **65**(531), 349–358 (2013)
7. Viglialoro, G., González, A., Murcia, J.: A mixed finite-element finite-difference method to solve the equilibrium equations of a prestressed membrane having boundary cables. Int. J. Comput. Math. **94**(5), 933–945 (2017)
8. Agmon, S., Nirenberg, L., Protter, M.H.: A maximum principle for a class of hyperbolic equations and applications to equations of mixed elliptic-hyperbolic type. Comm. Pure Appl. Math. **6**, 455–470 (1953)
9. Khachev, M.M.: The Dirichlet problem for the Tricomi equation in a rectangle. Differ. Uravn. **11**(1), 151–160 (1975)
10. Tricomi, F.G.: Sulle equazioni lineari alle derivate parziali di secondo ordine, di tipo misto. Atti Accad. Naz. Lincei Mem. Cl. Fis. Mat. Nat. **5**(14), 134–2470 (1923)
11. Lupo, D., Morawets, C., Payne, K.R.: On Closed Boundary Value Problems for Equations of Mixed Elliptic-Hyperbolic Type. Comm. Pure Appl. Math. **LX**, 1319–1348 (2007)
12. Khachev, M.M.: The Dirichlet Problem for a Degenerate Hyperbolic Equation in a Rectangle. Differential Equations **37**(4), 603–606 (2001)
13. Serrin, J.: A symmetry problem in potential theory. Arch. Rational Mech. Anal. **43**, 304–318 (1971)
14. Agostiniani, V., Magnanini, R.: Symmetries in an overdetermined problem for the Green's function. Discrete Contin. Dyn. Syst. Ser. S **43**, 791–800 (2011)
15. Henrot, A., Philippin, G.A.: Some overdetermined boundary value problems with elliptical free boundaries. SIAM J. Math. Anal. Vol. **29**(2), 309–320 (1998)
16. Nacinovich, M.: Overdetermined hyperbolic systems on l.e. convex sets. Rendiconti del Seminario Matematico della Università di Padova **83**, 107–132 (1990)
17. John, F.: The Dirichlet problem for a hyperbolic equations. Amer. J. Math. **63**, 141–154 (1941)
18. Viglialoro, G., Murcia, J.: A singular elliptic problem related to the membrane equilibrium equations. Int. J. Comput. Math. **90**(10), 2185–2196 (2013)

Figures

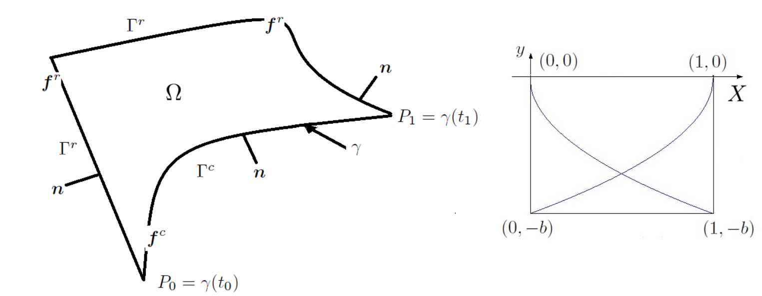

Figure 1 The General Problem. (Left) Representation of the domain and the boundary data. (Right) The Reduced Problem. Representation of the characteristic curves for the equation $(-y)^{2(n-1)}U_{,XX} - U_{,yy} = 0$ in the rectangle $(0,1) \times (-b,0)$.

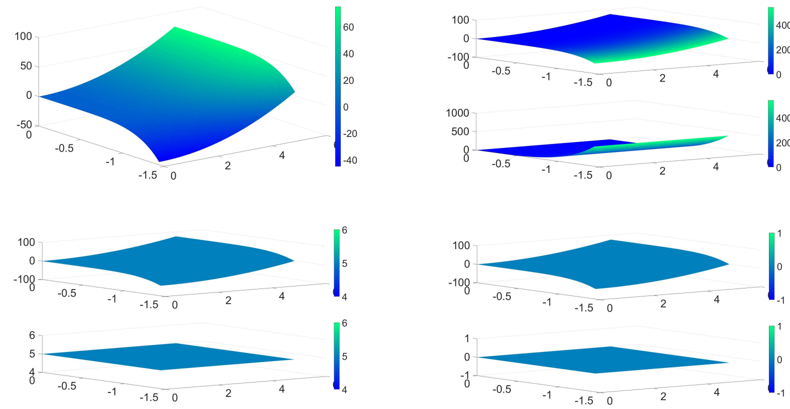

Figure 2 *Membrane structure*: the case of positive definiteness and graphical representation of the solution $\boldsymbol{\sigma}$. (Top-Left) Graph of the function z in R. (Top-Right) Above: distribution of the component σ_{xx} on the surface z. Below: representation of the component σ_{xx} in the rectangle R. (Lower-Left) Above: distribution of the component σ_{yy} on the surface z. Below: representation of the component σ_{yy} in the rectangle R. (Lower-Right) Above: distribution of the component σ_{xy} on the surface z. Below: representation of the component σ_{xy} in the rectangle R.

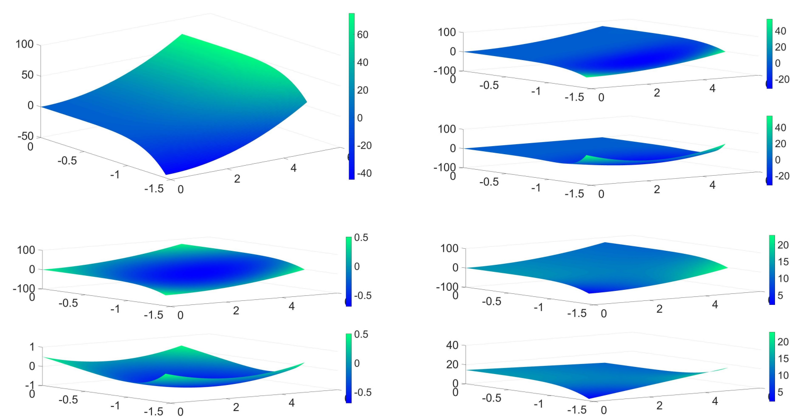

Figure 3 *Shell structure*: the case of no restriction on the sign definiteness and graphical representation of the solution $\boldsymbol{\sigma}$. (Top-Left) Graph of the function z in R. (Top-Right) Above: distribution of the component σ_{xx} on the surface z. Below: representation of the component σ_{xx} in the rectangle R. (Lower-Left) Above: distribution of the component σ_{yy} on the surface z. Below: representation of the component σ_{yy} in the rectangle R. (Lower-Right) Above: distribution of the component σ_{xy} on the surface z. Below: representation of the component σ_{xy} in the rectangle R.