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SELECTED ESSAYS ON PLATO'S *REPUBLIC*

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Chapter 11

Mathematics and its reform in Plato's time

1. The state of affairs

Differently from many other occasions in which Glaucon passionately expresses his approval to examine new questions (6.510c2, 6.510d3-5, 6.511a1, 6.511b1-3, 7.525a9-11, 7.525d8-9, 7.527c6-9), when Socrates invites him to reflect on mathematics we have the feeling that the two characters are talking about a state of affairs that is well-known to both of them: they do not seem to be 'inventing' the mathematics they examine but, rather, they seem to take it from the cultural environment that they fundamentally represent. They seem very interested in this subject, even though this kind of knowledge seems to be in a rather unsatisfactory, if not ridiculous shape (7.525e1, 7.527a6, 7.529e4, 7.530a2, 7.531a3-4). Mathematical knowledge needs to be reformed, and Socrates and Glaucon do not seem afraid of undertaking this task. But what does the mathematics they want to improve consist in? What does it entail? And what is its current state?

1.1 Mathemata *without a family name*

Mathematics is presented in the *Republic* as a group of disciplines that are 'sisters' (6.510c5, 6.511b2, 7.530d8; cf. below, §1.4), i.e. related and akin to each other. But they do not have a proper family name: only after Aristotle these disciplines will be called 'mathematical sciences' or 'mathematics'.[500] Socrates simply lists them one by one: geometry and astronomy, which are mentioned more often, are preceded by arithmetic and *logistike*, i.e. the 'science of calculation', and are followed by harmonics, according to the list that Socrates explicitly says to have created too quickly (7.528d7-8).[501]

All these disciplines are labelled *mathemata* and are included in the educational curriculum designed for the future philosophers. But their status fluctuates between that of a technique and that of a science: in general they are called *technai* in keeping with a widespread convention (6.511b2, 6.511c6, 7.532c4, 7.533b3-6, 7.533d4) even though, according to a different convention, they are also called *epistemai* (7.533d4-5); giving them a more precise name would be problematic, and this question is sidestepped by using the term *dianoia* in-

[500] Cf. Brandwood (1976) and Bonitz (1955), where are reported two occurrences of *mathematikon* in Plato, who uses the term in relation to the process of learning and acquiring knowledge (*Tim.* 88c1; *Soph.* 219c2); differently from Aristotle, in Plato there is no occurrence of *mathematike, mathematikai, mahtematikoi* or *mathematika*. All the Pythagorean sources in which these terms appear are either Aristotelian or post-Aristotelian, as results in the general index of Diels-Kranz (1951), vol.3 264 a40-b18, under the heading '*mathematikos*'.

[501] Cf. Mueller (1992), who underlines that in the Academy more types of mathematics were studied than the ones mentioned in *Resp.* 7. In particular, the absence of *metretike* is striking, a discipline that instead is mentioned in the *Laws* (7.817e6-7): this discipline, which belongs to the field of geometrics, must have had a similar function in this field to that performed by *logistike* in the field of arithmetic and can be related to the metrology mentioned by Philodemus (cf. Gaiser 1988, 76-77, 88-91).

stead (7.533d5-7; cf. 6.511d4-5 and §1.4 below). This is not a serious problem (cf. 7.533d7-e2), but it is still interesting since it probably reflects the fact that, in the Athenian cultural environment of the late 5th-cent and of the first half of the 4th century, mathematics was practiced both at an elementary level, i.e. as a set of techniques that are useful in everyday life, and at an advanced one, as types of intellectual and scientific knowledge.

1.2 A kind of 'gymnastics'?

The dialogue between Socrates and Glaucon, in the course of which they outline the general curriculum of the mathematical disciplines that the future philosophers/politicians will have to study (7.521c1-531c8), consists in a progressive examination which aims at defining in an increasingly precise way a set of *mathemata* that possess a certain type of 'power' (7.521c1-d12, esp.7.521c10-d1; cf. §2 below). Since its earliest stages, this examination proceeds by rejecting some commonplaces and by discussing contemporary convictions and practices (e.g., on arithmetic and *logistike*, cf. 7.532a1-3, 7.525b11-e5; on plane geometry, see 7.527a1-b2; on solid geometry, cf. 7.528b6-c8 and 7.528d8-e5). So it is not surprising to note that the first 'candidates' for the type of *mathema* they are in search of are the basic disciplines of music and gymnastics, mentioned explicitly in 7.521d13-522b3, with the traditional addition of grammar (cf. *Prot.* 312b1-3).

In Book 3 – to which Socrates refers in 7.521c13 and 7.522a2, 7.522a4, 7.522b2 – the educational training designed for the Guardians consisted in basic physical and musical training, which seems to share with the *mathema* sought in Book 7 a strict relation with the rational part of the soul (*logistikon*) and the discipline of philosophy, as well as with the irascible soul and the art of war.[502] Similarly to the *mathema* Socrates and Glaucon are in search of, both gymnastics and music aim at developing harmony, equilibrium and good proportions (7.522a2-4; cf. 3.410b-412a). However, since gymnastics deals with the growth and decay of the body, it 'depends on things that grow and perish' (7.522e2-5); both gymnastics and music, which is its 'counterpart' (7.522a3), lack the ability to 'draw the soul away from the world of becoming to the world of being' (7.521d3-4) and therefore their candidacy as the *mathema* sought by Socrates and Glaucon is rejected.

But this does not mean that the practice of this *mathema* cannot be described as a sort of 'gymnastics': the educational curriculum of mathematics must clarify how the process of *gymnazein en mathemasi pollois* must take place (6.503e3, 6.504d1), since the future philosopher and Guardians will have to follow it (6.503b5) in order to acquire the ability to reach the *megista mathemata* (6.503e3-4, 6.504a3, cf. 6.504d4, 6.504e1) and, especially, the 'Idea of the Good' (6.505a2). In this context, the term 'gymnastics' implies a series of endurance tests, similar to taking physical exercises or controlling pains and pleasures (6.503a; cf. 3.413c-414a). Any person who is able to practice this kind of gymnastics in the field of mathematics, especially with regard to *logistike*, increases his/her intellectual capabilities, which could be naturally more or less sharp (7.526b5-9; cf. also 7.527d5-528a3 and 7.535b7-8).

These arguments seem to have been fairly common in the pedagogic discourse of 4th century Athens, as is shown for instance by Isocrates (*Ant.* 262-267). But the *Republic* warns against the danger of interpreting contemporary pedagogy in a narrow sense: the kind of mathematics which represents, to use Isocrates' terms, a sort of 'gymnastics for the soul'

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502 Cf. *Resp.* 3.403c-404e, on gymnastics intended both as lifestyle and as diet, accompanied by 'austere' types of harmony and rhythm; cf. 3.397b-403c on the musical characteristics of songs and melodies, mentioned in 7.522a4-6, and 4.486a-397b on the part of music which deals with discourses and tales, mentioned in 7.522a6-8; cf. 3.410b-412b on the relation between gymnastics and music, rational and irascible soul, philosophy and the art of war.

(266) is not that which is interested in the world of becoming; on the contrary, this type of mathematics is an activity that is proper to the rational faculty of the soul and examines 'what is always'; therefore, its power is not only that of making us smarter and quicker in our reasoning since childhood, as Isocrates correctly maintains.[503]

1.3 'Banausic' techniques

The connection between knowledge and power, which characterises the *mathema* sought by Socrates and Glaucon, represents an excellent reason to consider many *technai* as possible candidates for this role, but this option is discarded quickly: in fact Socrates, referring back to Book 1, observes that 'all the arts were in our opinion base and mechanical (*banausoi*)' (7.522b4-5).

The adjective *hapasai* does not admit any exceptions. So even the *logistike techne* mentioned in Book 1 (1.337a-b; 1.340d-e) must be regarded as unsuitable together with elementary knowledge of numbers, which both Xenophon (*Mem.* 4.4.7; *Oec.* 8.14) and Plato (*Leg.* 5.747b) associate with craft practice, house management, commercial trade and the administration of the city.

In 5th-cent. Athens, it seems that elementary training in basic numeracy was regularly provided for the practical advantages they produce; starting from the 4th century, also basic elements of geometry and astronomy were ordinarily taught (cf. Isocr. *Panat* 26-28, *Antid.* 261-268) and their usefulness in relation to war was praised.[504]

However, the new educational curriculum outlined in the *Republic* does not seem to exclude traditional studies in elementary mathematics, with a technical character. Socrates and Glaucon take for granted the inclusion of simple numerical abilities in the new programme (7.522c5-6) and, even more significantly, they emphasise the importance of *logistike* and elementary geometry for the education of the Guardians, which will need to be also 'athletes of war' (7.521d5; cf. §2.4 below). In the *Philebus*, (55d5-56c11), Socrates maintains that the mathematical knowledge that is entailed in *cheirotechnai* purify them, in that they make them more precise.

The problem is that these disciplines contain an element that makes them inadequate in comparison to the *mathema* pursued in this passage: their 'banausic' element, their connection with manual labour and everyday life, which draws them 'downward' (*Resp.* 7.527b10-11). From the point of view of the 'multitude' (7.527d5), this is a valuable aspect, in that it determines the immediate and great usefulness of elementary mathematics which, for this reason as well as for its simple nature, is presented as the mathematics 'of the majority' (*Phil.* 56d5). Socrates and Glaucon, instead, are interested in a *mathema* which, due to its complexity, will be reserved to the 'few best' (*Resp.* 7.526c6, on which see §2.1 below) and must lead the people who practise it to 'look at what is' (7.521d4) and not – as *technai* do – to 'base' activities such as commercial trade (7.525c3-4). Even more importantly, they must not lead to focus one's attention on 'the world of becoming' (7.526e6-7), where objects either 'come into being and perish' (7.527b5-6) or are 'built and produced' according to 'opinions' and 'desires' (7.533b3-5).

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503 However Isocrates and Plato disagree on one point: the former does not recognise to mathematics any kind of educational power in relation to politics (Isocr. *Panat.* 28-29, *Antid.* 268-269).

504 Cf. Heath (1981), 1.18-25; the fact that this was a peculiarity of the Athenian environment, as opposed to the Spartan one, is underlined by Morrow (1960), 350-351, who analyses the mathematical curriculum of the *Laws* (7.817e-823c).

1.4 Types of intellectual knowledge

The way in which the mathematical curriculum is outlined in the *Republic* is characterised by a contraposition that is explicitly established in the *Philebus* between mass mathematics and 'philosophically-oriented' one (on arithmetic, cf. 56d4-6, 56e8; on *logistike* and geometry, cf. 56e7-57a4). The power of the sought *mathema* to help the soul to ascend toward 'what is always' corresponds to the ability of converting it to 'true philosophy' (7.521c7-8; cf. 7.523a1-3 on numeracy; 7.524d3, 7.525d1-2, 7.525c1-6 on arithmetic and *logistike*; 7.527b9-10, 7.527a6-b1, 7.527b5 on geometry).

'Pythagorean' mathematics

Within the macro-dialogue developed with Glaucon, Socrates starts a sort of micro-dialogue with some experts in this kind of 'philosophical mathematics' (7.525d9; 7.5261), i.e. the kind of intellectual knowledge that is pursued in order to achieve theoretical ends (7.525d8-526a7, on which cf.§2.6). The position maintained by these experts is characterised by the same type of argument that defines 'philosophical arithmetic' in the *Philebus* (56d9-e6): numbers are made of undifferentiated and indivisible units. This conception is attributed by Aristotle to the Pythagoreans (DK 58 B 5, DK 58 B 9, DK 58 B 10) and these are the only philosophers mentioned by Socrates in this passage because he states to agree with them, as Glaucon does too, about the Pythagorean conception of the 'sisterly' relationship existing between astronomy and harmonics (7.530d6-10; cf. 6.510c5, 511b2).

Perhaps this is an allusion to Archytas' theories, as can be seen in an extant fragment that is generally considered authentic (DK 47 B 1), where geometry and arithmetic, astronomy and music are defined as *mathemata adelphea*.[505] Other scholars thought instead of Philolaus, who defined geometry as 'the mother of all other sciences' (DK 44 a 7a), granting it a higher importance, while Archytas reserved this higher status to astronomy and harmonics. Other sources attribute the same *quadrivium* to the 'Pythagoreans' in general, even though they mention them following an order that is still different.[506]

The reason why arithmetic and music, as well as geometry and astronomy are regarded as 'sisters' is quite clear – and the Platonic Socrates demonstrates to be familiar with it, as well as with the 'counterpart' admitted by the Pythagoreans (*Resp.* 7.530d4), who likened astronomy and harmonics respectively to the eyes and to the ears (7.530d6-7).

The family relation between arithmetic and geometry is more complicated. Proclus relates it to the fact that both deal with 'quantities',[507] but this argument is not earlier than Aristotle, who finds a confirmation of his conviction in the theory of proportions attributed to Eudoxus (cf. *An. Post.* 1.5). In the course of the experiment with the slave in the *Meno* (82d8-e1), the keyword of the 'Eudoxian' theory of proportions (*pelike*) occurs several times and indicates indifferently a numerical or geometrical 'magnitude'; therefore, it seems that also in Plato's dialogue there is a concept of 'magnitude' that includes both numbers and images, as

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505 The authenticity of this fragment is controversial due to its similarities to *Republic* 7, as explained by Centrone (1996), 71-72; cf. as well Huffman (2005), 103-107.

506 For instance arithmetic, which studies number in themselves, is followed by harmonics, which studies relations between numbers, and after them come geometry and astronomy, which respectively examine static and dynamic magnitudes. Cf. Heath (1981), 1.12 and Adam (1902), *Appendix* 2 to *Resp.* 7.

507 Cf. Friedlein (1967), 35-36.

is the case of Aristotle's 'quantity'. However, the types of proportions that appear more frequently in Plato are pre-Eudoxian ones, discovered by the Pythagoreans.[508]

The closest connection of arithmetic and geometry that is attested in the Pythagoreanism of the 5th and 4th centuries and is related to the 'theory of figurate numbers', which was well-known in the Platonic Academy probably thanks to Philolaus' discussion of it (DK 44 A 13 = Speusippo 122 Isnardi-Parente).[509] Numbers were represented by drawing one dot per each unit and the position of the dots was organised so as to generate different types of plane or solid shapes; therefore, numbers were classified in keeping with the shape they corresponded to, so they could be 'triangular', 'square', 'polygonal', 'cube' and so on.[510] The theory of figurate numbers revealed a family resemblance between arithmetic and geometry, even though there are many reasons why they differ from each other, such as for instance the existence of incommensurable geometric magnitudes, i.e. entities that cannot not be expressed by positive integer numbers. But in the field of Pythagorean mathematics many tools were developed to solve these problems, among which a prominent role was detained by *logistike*, a science that Archytas conceived as a 'bearer of harmony' (DK 47 B 3) and, therefore, as being superior to geometry (DK 47 B 4, esp. 9-10).

So arithmetic, 'logistic', geometry, astronomy and harmonics represent a family of affiliated disciplines, further united by the manner in which they are practiced, as is emphasised in *Republic* 7: Pythagoras devised a 'philosophical' approach to mathematics, 'investigating mathematical problems conceptually and theoretically' (DK 14 A 6a).

In addition, in *Republic* 10 Pythagoras is remembered in relation to the personal and selective character of his teaching, which allowed his successors to lead a lifestyle that made them 'distinguished' (10.600b2-5). Despite the necessary caution,[511] it is possible to assume that Plato sympathised with the idea to avoid communicating these doctrines (including mathematical ones) to everybody and preferred instead reserving them to the few best that are destined to govern. And actually the diffusion of mathematical theories that are more or less directly related to Pythagorean circles does not seem a very widespread phenomenon in the Greek culture of the 5th and 4th centuries.[512]

Mathematics taught by the Sophists

It is necessary to introduce another means to transmit high scientific knowledge in order to get Glaucon – the prototype of the Athenian man with a superior, up-to-date education – to accept Socrates' remarks in which mathematics is regarded as a type of intellectual knowl-

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[508] It consists in the distinction between arithmetic, geometric and harmonic means (on which cf. *Gorg*. 508a, *Tim*. 31b-36d, *Leg*. 6.757a-d), reconstructed by Heath (1981), 1.84-97. Cf. Cattanei (2007).

[509] On Philolaus as the main source of the Aristotelian testimonies on the Pythagoreans, cf. Huffman (1993), where fragments DK 11 B1-7 are regarded as probably authentic. Speusippus' reliance on Philolaus on this point is discussed by Isnardi Parente (1980), 368-377.

[510] This theory did not apply only to the first four positive numbers but was developed in the form of theorems that comprised all possible combinations of numbers: cf. Heath (1981), 1.76-84. In addition, one should not think only of the numerological aspect of this theory: it represents 'a link between wild numeric speculation and scientific mathematics' (von Fritz 1971b, 50-51).

[511] I refer to the distinction between 'acousmatics' and 'mathematics', on which see Zhmud (1992); cf. also Centrone (1996), 81-83 and Burkert (1972), 192-208.

[512] Cf. DK 44 A 1, as well as Burkert (1972), 218-238, on the anecdote according to which Plato managed to acquire a copy of Philolaus' book after many struggles during one of his trips to Italy. Other exceptional cases of the end of the 5th century are represented by Hippocrates of Chios (cf. below, §1.6), who went to Athens in order to teach geometry in exchange for money (DK 42 A 2), and by the diffusion of the discovery of incommensurability made by Hippasus of Metapontum, a contemporary of Philolaus (cf. Centrone 1996, 84-85, Burkert 1972, 447-465, together with von Fritz 1965 and Heller 1965).

edge: in fact, the average educated Athenian citizen represented by Glaucon was likely to have had some advanced mathematical training as this subject was taught by numerous sophists. Let us look at Protagoras' words, in the homonymous dialogue: 'the other sophists damage the young because, when they have escaped from the arts, they bring them back and force them to study these subjects again against their will, teaching them arithmetic and astronomy and geometry and music – and here he glanced at Hippias' (*Prot.* 318d9-e4). The person who is accused here is Hippias of Elis who, in *Hippias Major* (285b) and *Hippias Minor* (366c-368e), is characterised as being a teacher of *logistike*, geometry, astronomy, 'rhythms' and *harmoniai*; he seems to have been the same Hippias to whom Proclus attributes the invention of a 'squaring' curve, useful to solve the problem of the trisection of any given angle and employed in relation to the problem of the squaring of a circle.[513] According to Plato's dialogues, Hippias and other sophists – except Protagoras – taught the disciplines of the Pythagorean *quadrivium* at a higher level to people older than 14 years who completed the elementary level, comprising also a basic training in technical arithmetic and geometry. And, still according to the *Hippias Major* (285b-e), this curriculum was rejected in some occasions, for instance in Sparta, while it was readily and enthusiastically accepted in Athens. It is likely that it contents derived from the earliest mathematical treatises that at this stage started to circulate in the Greek world: Hippocrates of Chios, who is told to have earned a living in Athens by giving lessons in exchange for money, is attributed the composition of the first *Elements of geometry* of our civilisation and Protagoras, a fellow-citizen of Democritus, devoted more than one work to arithmetical, geometrical and astronomical questions, giving shape to intellectual interests that had already been shown by Anaxagoras and, perhaps, by Thales.[514]

Mathematical practices in the Academy

Situations similar to that described in the *Theaetetus* must have been rather common in high-culture Athenian circles of Plato's time and in the Academy. The central character of this scene is Theodore, friend and pupil of Protagoras but not a defender of his, because he refuses to accept the identity of science and sensations (164d8-165a3; cf. 170c5-9) since he 'moved on to geometry too early' (165a1-2). Facing a small and selected group of boys (168d8-e5), he takes the role of 'the expert' in geometry, astronomy, *logistike* and harmonics (168e) and gives a lecture on irrational numbers: with the aid of some images, he demonstrates that the lengths corresponding to the square root of the numbers comprised between 3 and 17 that are not perfect squares are incommensurable to a foot-long line, which was taken as the basic unit of measurement (147d2-6). His young pupils, some of which are very talented in mathematics (such as Theaetetus), discuss the content of the lecture and ask themselves whether it would not be possible to include all these irrational numbers in a unitary definition (147c8-d1, 147d7-e1). They actually manage to do so, 'as far as it is humanly possible' (148b3): thanks to the distinction between figurate number that are square and equilateral on the one hand, rectangular on the other, they are able to define two classes of lines, respectively denominated 'lengths' and 'powers', which are incommensurable to each other 'with regard to their length but not to the surfaces they can generate' (147e5-148b2, esp. 148a6-b2).[515]

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[513] Cf. DK 86 A 11, DK 86 A 12, DK 86 B 12, DK 86 B 21, together with Procl. *In pr. Eucl.* 326, 1.10 and Heath (1981), 182-183.

[514] Cf. *Hipp. Maj.* 281c, with Heath (1981), 118-140 (on Thales and Egypt), 170-174 (on Anaxagoras), 176-183 (on Democritus). About the composition of the first mathematical treatises, cf. Cambiano (1992).

[515] On this passage of the *Theaetetus*, cf. Knorr (1975), who analyses also the mathematical research undertaken by Theodore of Cyrene, on which cf. DK 43 A 1-5.

Historians believe that, after Eudemus of Rhodes, some mathematicians and astronomers such as Theodore of Cyrene and Theaetetus, Eudoxus of Cnidus, Menaechmus and Dinostratus, Laodamas of Tasus and Philip of Opus followed this kind of methods in order to find solutions to various problems; they tried to formulate theories with the highest possible degree of universality and soundness and then systematised this newly acquired knowledge by compositing written *Elements*, which will reach its canonical form in Euclid's *Elementa*.[516]

A 'laboratory'

The habit of calling these disciplines 'sciences' is explained by the fact that they are regarded as types of intellectual knowledge which are pursued in order to achieve theoretical results and by the fact that they are characterised by the very rigorous nature of their demonstrations (7.533d4-5). So what would be the reason behind the choice to abandon this term and choose *dianoia* instead?

In the simile of the Line (6.509d-511e), as well as in some passages of the simile of the Cave which are indicated as being related to this question at the end of the examination of the mathematical curriculum (7.514a-521b), two aspects are particularly emphasised and both concern the limitations of mathematics: first of all, mathematicians regard their 'hypotheses' (i.e. definitions, axioms and postulates) as conventionally valid, without any justifications, and use them to develop their demonstrations (6.510c6-d2); second, these demonstrations are based on visual representations, physical models, shade projections or images that appear on reflecting surfaces, therefore are not entirely free from perception and sensible imagination (6.510b4-5). Moreover mathematical language employs expressions borrowed from the physical realm and from practical processes (such as building or joining/dividing parts), a characteristics that it seems radically inadequate for a theoretical science (7.525e, 7.527a; see also in this volume Franco Repellini's essay, *The Line and the Cave*). The *Republic* draws what we would call the epistemological and ontological consequences of these two limitations and, therefore, highlights the need to conceive mathematics as a kind of *dianoia*, i.e. a type of knowledge that is intermediate between purely intellectual and sensible knowledge (6.511d1-5, 7.533b3-534b2) and has a sort of dream-like attitude toward reality (7.533b6-c5). Socrates laments its 'persistent *adynamia*' to keep the attention focused on the entities toward which they are capable of leading the soul, while they linger on 'shadows' (7.532b9-c2); admittedly, these are 'shadows of what truly is', therefore they are different from the 'shadows of statues' that could be seen in the darkness of the cave and corresponded to sensible objects (7.532c2-3, cf. 7.515c1-2, 7.516a6). However, this does not stop mathematics from 'dreaming about what really is' (7.533b6): a dream which means, in the first place, that it is incapable of giving an account of its own 'principles' (7.533c2) and, therefore, cannot really set itself free from the realm of convention and turn into a real science (7.533c2-5).

Also other sources, the oldest of which are some Aristotelian passages and some sections of Euclid's' *Elements*, suggest that 5th- and 4th-century mathematics had not reached yet a stable identity. We are told that there were 'many debates' in these fields, and this is not to be intended only as a generic expression: these disciplines aim at finding *archai*, which only Aristotle will divide into definitions, hypotheses, axioms and postulates (*An. Post.* 1.2).[517]

Aristotle mentions many cases of controversial geometrical definitions: for instance, the definition of tangent as 'the straight line that touches a circle in one point' triggered Prota-

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[516] Boyer (1968), ch. 6 §3. On 'minor' mathematicians associated with Plato's Academy, cf. Lasserre (1987), Lasserre (1966), Lasserre (1964). On the history of Greek geometry, which is based on the role of 'elements', cf. Tannery (1988), 130-141 (devoted to the members of the ancient Academy).

[517] Toth (1998a), 71.

goras' criticism since, empirically, a tangent line touches a circle in two points (DK 80 B 7 = Arist. *Met.* 3.2, 997b35-998a4); another example concerns a 'foot-long line', which is taken as the standard unit of measurement even though its physical representation drawn in the sand is not breadthless nor exactly one foot long, aspects that render false all demonstrations based on it (*An. Pr.* 1.41 49b33-37; *An. Post.* 1.10, 76b39-77a3; *Met.* 13.3, 1078a19-21; *Met.* 14.2, 1089a21-22). These are episodes of a 'battle against geometrical definitions' that started in the 5th century with the sophists and then moved on to some Socratic circles; this battle, which ultimately will be reflected in Sextus Empiricus' arguments, will concern also the arithmetic definition of unity as the indivisible principle of numbers, a fundamental axiom of mathematics.[518]

At the same time, it is possible to detect some traces of the mathematicians' effort to make their language more precise and give shape to their argumentative strategies. So many technical terms are coined, such as the aforementioned *pelike* or even the very term *monas*, which is not attested before the *Philebus*, even though it probably derives from Philolaus and Archytas (DK 44 b 10). But mathematicians also strived to distinguish terms that were normally employed in everyday life from their mathematical acceptations, as for instance in the case of numbering terms, 'circle', 'sphere', etc.[519] So far as argumentative strategies are concerned, the auxiliary role of visual representations, whether graphical or mechanical, becomes increasingly clear, a role that is well-established already in Aristotle and which is significantly reflected by the mathematical meaning of the verb *graphesthai*.[520] At any rate, mathematics start to become a field in which different types of reasoning can be tested and ultimately can acquire canonical forms: for instance, a strict relation is recognised between the Aristotelian 'apagogic' reasoning and the 'hypothesis-based process' which is mainly followed by mathematicians in Plato's dialogues. More generally, it is clear that in this phase mathematics is adopting an increasingly rigorous axiomatic-deductive approach, which will reach its fully developed form in Euclid's work.[521]

So these are 'experimental' disciplines, with regard to their foundation, language and method: this is the status of mathematics between 5th and 4th centuries BC. This interlocutory stage, however, is far from primitive and it is not necessarily true that they did not deal with complex and technical mathematical questions. The 'axiomatic sensibility' shown by them, i.e. the tendency to ground one's reasoning on a small number of basic propositions that are universally recognised as being true, led some interpreters to think that Euclid's first geometry ran into a crisis that regarded its fundamental aspects, similarly to what happened in the 19th century after the birth of non-Euclidean geometries.[522] According to its more recent reconstruction, *logistike* reveals numerous connections with the theory of continuous fractions developed by Fermat, Legendre, Gauss and others.[523]

Independently of these kinds of interpretative approaches, the reader of these passages is left with a very strong impression that the historical and cultural reason of Socrates' ambiva-

518 Cf. Cattanei (1996), 95-97 with the corresponding footnotes. On the axiomatic nature of the Pythagorean principle of the indivisibility of the unity – which corresponds to Peano's 'fourth axiom' – cf. Toth (1998a), 72.

519 Cf. *Phil.* 56d-3, with Annas (1976), 6-7, 14-15, and Cattanei (1996), 38-40.

520 Cf. Toth (1967), chapter 1.

521 On the relation between apagogic method and hypothesis-based procedure, cf. Caveing (1990); on different types of mathematical reasoning in Plato and Euclid, cf. Mueller (1992), 175-194.

522 Cf. Toth (1998a), esp. 407-641, with Mugler (1941). Knorr (1975), 306-313, and Lloyd (1979), esp. 117, are very critical of the possibility of a non-Euclidean development of pre-Euclidean geometry.

523 Cf. Fowler (1990), 111-117.

lent attitude towards mathematics is related to the historical situation in which they find themselves: the promising, though not entirely stable condition of 'laboratory sciences'.[524]

1.5 Arithmetic and logistike

As mentioned above, two of these sciences – arithmetic and *logistike* – are the first to be identified by Socrates as parts of the *mathema* he is looking for. Socrates and Glaucon mention them always together, but it is not casual that *logistike* is the discipline that invariably captures their attention.

'Number and calculation' (*arithmon te kai logismon*) enter the scene of the dialogue in a curious manner: Socrates regards the elementary skill of being able to 'count and calculate' (7.522d1-2) and 'distinguish one and two and three' as a 'trifling matter' (*phaulon*, 7.522c5-6), a skill that 'is among the first that everyone must acquire' (7.522c2).

Socrates considers this discipline as a possible candidate for the role of the sought *mathema* because it 'applies to all things alike' (7.522b9), i.e. it is a kind of 'common knowledge' that must be used by 'all other arts, intellectual activities and sciences' (7.522c1-2), including 'even the art of war' (7.522c10). But in relation to the art of war 'even a partial training in counting' would be sufficient, while in order to coincide with the sought *mathema* 'number and calculation' should acquire the 'more refined state' proper to the 'science of calculation and arithmetic' (7.525a9). In this form, the 'science of calculation' in particular is regarded as useful to 'all other disciplines' (7.526b6), since it produces an effect that is widely recognised in the pedagogy of Plato's time: it trains the mind (7.526b5-9).

So what do *logistike* and arithmetic deal with and what is the relation between them? In 7.525a9-11 we read that 'the science of calculation and arithmetic is wholly concerned with number'. On the basis of *Gorgias* 451b and *Charmides* 166a, as well as *Theaetetus* 198a and *Statesman* 259e, we know that arithmetic studies numbers in themselves and their properties, while *logistike* studies the relations established between numbers and their properties.[525] Between the two, arithmetic has a more basic function, given that no relations among numbers can be established without numbers; however, *logistike* tends to assume a dominant position since the core of the Greek theoretical analysis of numbers 'consists in the notion of ratio (*logos*), as in the case of geometry'.[526]

Numbers and units, even and odd.

It has been noted that the Greek numbering system was so complex that Euclid's 'arithmetic Books' (7-9) do not mention any number at all, while in the graphical illustrations of ratios, numbers are represented as segments – a system that is completely unrelated to the representation of numbers used in calculations.[527] This remark can be applied to the whole of arithmetic that is discussed by Socrates and Glaucon: it is far from casual that many traces of this method can be identified in Books 7-9 of Euclid's *Elements* and this aspect, combined with other testimonies, led some scholars to regard it as a 'Pythagorean' one.

The first definitions provided in Book 7 of Euclid's *Elements* are those of unit and number: 'a unit (*monas*) is that by virtue of which each of the things that exist is called one (*hen*);

[524] I borrow this expression from Caveing (1990), interpreting it in a wider sense than that of the original text.

[525] On the basis of a scholium to the *Charmides*, some scholars maintained that *logistike* is a sort of 'applied mathematics' (cf. Heath 1981, 14-15). This interpretative mistake has been rectified since the appearance of Klein (1968).

[526] Cf. Netz (2001), 776.

[527] Netz (2001), 776.

'a number is a multitude composed of units (*arithmos ... to ek monadon sunkeimenon plethos*). Euclid's definition represents an elaboration of other contemporary formulations, which also qualify the unit as absolutely indivisible and conceive numbers as sets of indivisible units.[528] So arithmetic becomes the 'study of unity' (7.525a2: *mathesis peri to hen*), in which the characters and properties of the unity are strictly related to that of 'any other number' (7.515a6-7). Socrates sides with the Pythagoreans in opposing the mathematics of merchants and is sympathetic to one of the most important numerical consequences of the theory of the indivisibility of the unit: fractions, intended as *parts* of units, cannot possibly belong to the realm of numbers (7.525d5-526a7, with §2.6 below). More generally, the definitions of some fundamental properties of numbers and the subsequent division of numbers into classes, which are useful in order to demonstrate groups of theorems, depend on the 'Pythagorean' definitions of number and unity.[529]

According to Socrates and Glaucon (6.510c4), the distinction between 'even' and 'odd', which is included in the set of definitions provided in *Elem.* 7 and presupposes *Elem.* 7.1-2, is regarded as an actual synonym of 'arithemetic'.[530] Actually, definitions 6 and 7 of Book 7 say respectively that 'an even number is that which is divisible into two equal parts' and 'an odd number is that which is not divisible into two equal parts, or that which differs by an unit from an even number', where the term 'part' indicates a group of units contained in the number, a submultiple of it (cf. *Elem.* 7, def. 3). But the *Elements* contain a whole theory based on the definitions of 'even' and 'odd': this theory is illustrated in propositions 21-34 of *Elements* 9.[531] For instance, the first group of propositions (21-30) establish that the sum of even numbers gives an even number, the sum of an even and an odd number gives an even number, while the sum of odd numbers gives an even number, and so on with regard to subtractions, multiplications and divisions.

Some aspects of the theory of 'even' and 'odd' numbers seem to suggest that it lays on the background of Socrates and Glaucon's reflections on mathematics. First of all, the universe of numbers is evidently regarded as being coextensive to that of 'evens' and 'odds'; at the same time, the rules according to which something belongs to this universe are established. The basic definitions of this theory perform implicitly the role of axioms.[532] More precisely, if 'evens' and 'odds' are regarded as a set, this set is going to be 'characterised by the fact that its elements can be connected by means of four types of binary compositions, and the results of these compositions correspond to elements of the same set. These binary compositions are the four operations, addition, subtraction, multiplication and division [...]. This means especially that there is an element that belongs to this set which is neutral with regard to multiplications, such as our number 1, which corresponds also to the Pythagorean *monad.* On the basis of the theorems presented in *Elem.* 9.28-29, it is clear that this element is to be

[528] Heath (1956), 2.280; further specifications are provided by Fowler (1999).

[529] This was the case also with regard to the theory of 'figurate numbers', mentioned above in §1.4: the theorems comprised in this theory are mostly based on the distinction between 'plain' and 'solid' numbers and, more specifically, between square and cubic numbers, which coincide with definitions 16-19 elaborated in Book 7 of the *Elements* on the basis of definitions 1 and 2 (unity and number), together with definition 15.

[530] This is attested also in *Pol.* 262d-e, *Eutiphr.* 12d-e, on which cf. Annas (1976), 8-10.

[531] Cf. Becker (1938); for a different perspective, cf. Knorr (1975), 134-135. Van der Waerden (1957) insists on the ancientness of this theory, underlining its relations with the incommensurability of the diagonal and the side of a square, remembered in Aristotle, *An. Pr.* 1.23, 41a23-30 (cf. note 48 below); the same position is presented in van der Waerden (1979). Zhmud (1997), 158-165, agrees as well on the fact that this theory belonged to ancient Pythagoreanism.

[532] Toth (1998a), 167.

identified with odd numbers. Even numbers, instead, perform the role of our zero (0), the neutral element of additions'.[533]

So is revealed the 'absolute superfluous character' of the theory of 'odd and even',[534] as its rules (such as 0+0=0, 1×0=0, 1×1=1, 0-0=0) are not applicable to mathematical studies or in everyday life. There are no practical applications of these theories – an aspect that in Plato's view is doubtless positive, though certainly was not so for the average man. On the contrary, it is not surprising that the only attested application of these theories in antiquity is to be found in Epicharmus' comedies and is used as a comical expedient.[535]

In keeping with this theory, also the behaviour of the indivisible monad reveals a paradoxical trait: it is a full and perfect unity, given that it is indivisible, but at the same time it can be infinitely multiplied, as it can be repeated infinitely in the series of natural numbers; even more importantly, the monad can lead to surprising formulas which seem clear only if, in keeping with what has been clarified above, the sign 1 is used to indicate the monad as well as all odd numbers, while 0 represents all even numbers. Following this convention, the theorem that establishes that a sum of odd numbers gives an even number as its result would be represented as 1+1=0.[536]

Socrates and Glaucon focus on the paradoxical aspect of a conception of unity in which 'it appears at the same time as one and as infinitely multiple' (7.525a4-5) – a paradox that is extended to 'any other number' (7.525a6). However, they do not provide an explanation of the way in which arithmetic and *logistike* deal with units and numbers that are at the same time unitary and multiple. They simply exclude that the unity's multiplicity corresponds to a multiplicity of parts in which it can be divided (7.525d5-526a7) and instead they highlight that it is a very hard exercise in the science of calculation (7.526c2; cf. below §§ 1.6, 2.5 and 2.6). This last indication leads us to consider a purely arithmetical explanation insufficient, such as that according to which each unity is one but can be infinitely repeated, and each number is one in itself but can be infinitely repeated.[537] Probably it is necessary to identify some complex examples of theoretical *logistike*, as we will do later on, in order to see how the contemporary presence of unity and multiplicity is organised in the case of units and numbers, which were mentioned by Socrates. And still the fact that Socrates does not provide the reader with precise considerations on this question may lead us to think that this is an implicit acknowledgement of a general attitude that can be detected in the fields of arithmetic and science of calculation: the attitude to contemplate paradoxical and contrasting conception of the unity, without being surprised by some unusual aspects of its behaviour.

Logoi *and* antanairesis.

It seems that the main theoretical difficulties posed by the conception of the indivisibility of the unity and that of numbers as sets of indivisible units had been examined early on by means of a theory of *ratios* or *logoi*. This is what *logistike* is supposed to be, which in the *Charmides* is defined as the science that 'is concerned with the even and the odd in their numerical relations to themselves and *to one another*' (166a5-7). The terms *logismos, logizesthai* and the activity of the *logistikon* – which had already been introduced as key terms at the beginning of the simile of the Line (6.510c3) and are employed often in the mathemati-

533 Toth (1998a), 167-168.

534 Toth (1998a), 169.

535 DK 23 B 2 quotes *Elem.* 7.7; for more information related to Epicharmus, who lived at the beginning of the 5th century BC, cf. below §2.6.

536 Cf. Toth (1998a), 169.

537 Cf. *Resp.* 7.526a3 with *Phil.* 56e2. On the ontological implications of this characteristics of units and numbers, cf. Szlezák (2000) *ad* 525d-e with Gaiser (1986), esp. 91 note 10.

cal *curriculum* outlined in *Republic* 7 – are concerned with *ratios* and *relations* established between positive integers.

In order to understand what kind of *logos* is examined by *logistike* it is necessary to go back to a stage that preceded the theory of proportions presented in Euclid's *Elements* 5, a theory that is traditionally attributed to Eudoxus and regarded by Aristotle as a sort of 'general mathematics';[538] it is necessary to find a definition of *logos* 'that is closer to the concept of *algorithm* or a process of numerical computation' of which we have 'fragmentary and uncertain testimonies', although 'Aristotle provides a clear and explicit formulation of it'.[539] Book 8 of the *Topica*, esp. 158b29-35, provides us with the best definition of the kind of *logos* examined by *logistike*, which is denominated *antanairesis* or *antiphairesis*:[540] Aristotle states that, in the case of a parallelogram divided into two by a straight line that is parallel to one of its sides, both the bases and the surface are divided in 'similar ways, since the surface and the bases have the same *antanairesis* and this is actually the definition of "the same ratio" ' (158b31-35). The term *antanairesis* hints at a 'sort of "antagonism" (*anti*), a mutual confrontation between magnitudes, as well as at the resolution of this conflict', i.e. 'the idea of finding a "balance" between opposed parts', ultimately outlining a process that consists in the 'repetition of the same elementary operation of subtraction on different magnitudes that are related to each other by a recursive formula'.[541]

This is what is generally indicated in mathematical texts as an 'Euclidean algorithm'. Its basic example consists in the relation between two line segments. Given two line segments a and b, where $a > b$ (fig.1), it is possible to establish the ratio they have to each other by taking the smallest line segment b as a unit of measurement and then proceed by means of 'repeated subtractions' (fig. 1): the smallest line segment b is subtracted from the biggest one a certain amount of times, until the latter is completely measured, i.e. it corresponds to a certain number of units. If, for instance, $a = 6$ cm and $b = 2$ cm, $a = 3b$.

a [————————][————————][————————]

b[————————]

(fig. 1: *Repeated subtractions*)

However, it is likely that repeated subtractions of the smaller line segment from the bigger one at some point produce a *remainder* between the segment that must be measured and a certain number of units, as would happen for instance if $c = 7$ cm and $b = 2$ cm. In this case, there would be a remainder $r = 1$ cm in excess of $3b$ but in defect to $4b$ (fig. 2). At this point, it is necessary to proceed by 'reciprocal subtraction': the remainder is taken as a unit of measurement and is subtracted from the initial smaller line segment. In our case, r must be subtracted from b. The procedure stops when the remainder measures comprehensively, i.e. without additional remainders, the segment from which it is subtracted. In our example, this happens immediately since the remainder $r = 1$ cm measures without remainder the segment $b = 2$ cm after the first subtraction. The line segment r, therefore, results to be the highest common measure between the initial line segments, c and b, in that it can measure them precisely,

[538] Cf. Cattanei (1996), 74-81.

[539] Cf. Zellini (1999), 131.

[540] Wallies (1891), 1.545.15-19.

[541] Zellini (1999), 179.

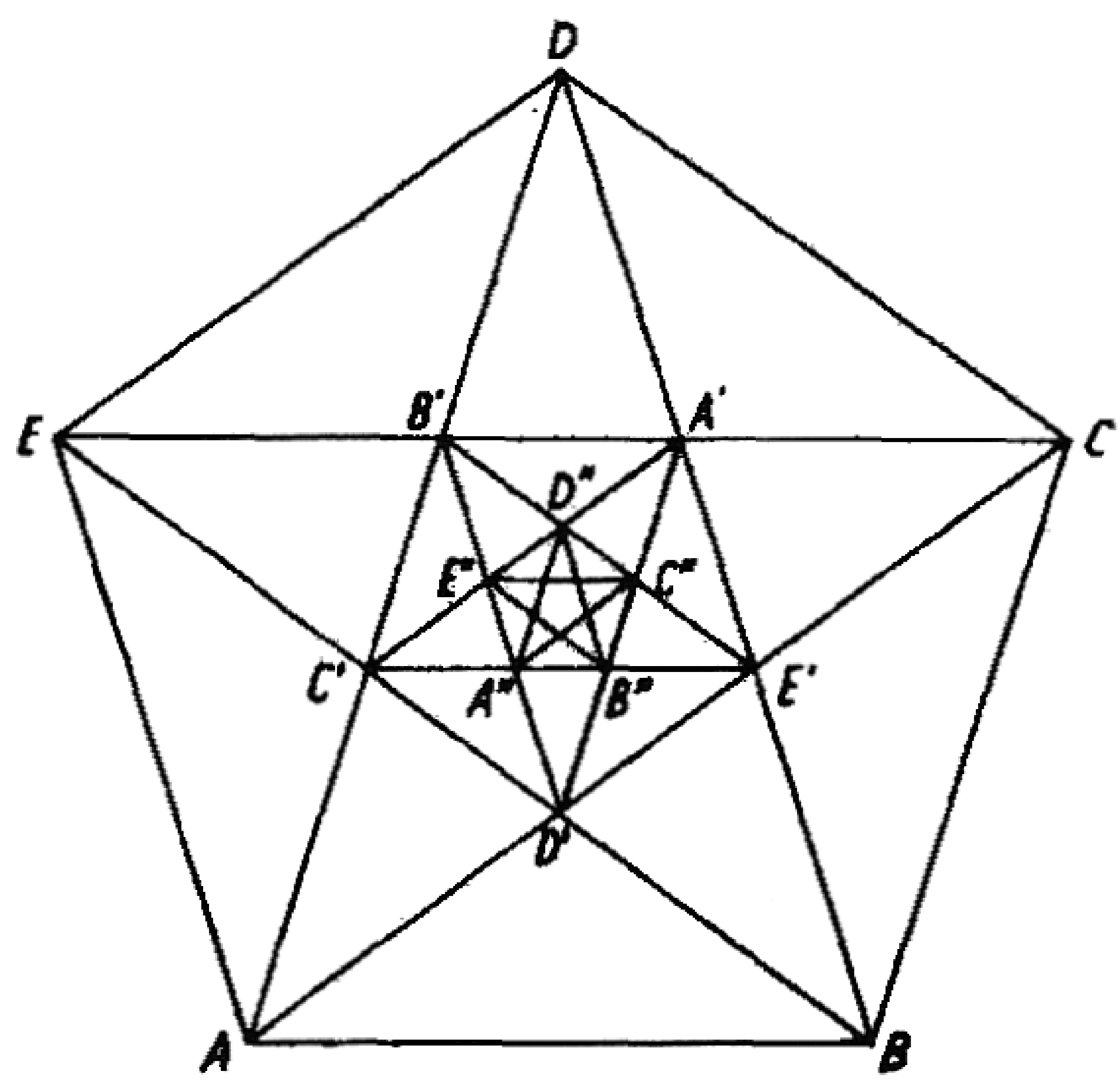
with no remainders. The overall ratio established between *c* and *b* can be expressed as an ordered couple of positive integer numbers that corresponds to the result of the two phases of the 'reciprocal subtraction', [3, 2].

c [————][————][————][——]
(*r*)
b [————]

r [——]

(fig. 2: *Reciprocal subtraction*)

In this case, the initial line segments are commensurable. Instead in the case of incommensurable magnitudes – e.g. the diagonal and the side of a regular pentagon or those of a square – it is not possible to define any kind of remainder that can measure them both, so the 'antanairetic' process is infinite, even though the remainder becomes increasingly smaller and increasingly closer to being identical, a result that however is never reached. Any attempt at constructing geometrically the 'reciprocal subtraction' of incommensurable magnitudes leads necessarily to a cyclical repetition of the same, though increasingly smaller image. For instance the *antanairesis* of the side and the diagonal of a regular pentagon leads to the construction of the so-called 'pentagram' (fig. 3).[542]



[542] More precisely: the side AB, which belongs to the bigger pentagon ABCDE, is subtracted once from its diagonal AC; given that AB=AE' and E'B=E'B', AC-AB=E'B', i.e. corresponds to the side of the pentagon A'B'C'D'E'. The diagonal and the side of A'B'C'D'E' are examined following the same procedure in a direction that is analogous to the previous one, leading to an even smaller pentagon A''B''C''D''E''. Following the same procedure, the analysis of each pentagon obtained by means of 'reciprocal subtraction' will invariably lead to another smaller pentagon. On this construction, see von Fritz (1965), esp. 295-298, and Heller (1965), esp. 328-330. A similar image composed of increasingly smaller squares results from the *antanairesis* of the diagonal and side of a square, as showed for instance by Netz (2001), 762 and Toth (1998b), 25, 50, 55, 61, 64, 74.

(fig. 3: *Reciprocal subtractions between incommensurable magnitudes*)

Even in the case of incommensurable magnitudes, the amount of subtractions undertaken in the course of the antanairetic procedure can be expressed by means of a series of couples of positive integers; clearly this series is infinite and open, differently from the finite and close ones that indicate the ratios between commensurable magnitudes.[543]

Therefore, in a context that deals with a theory of calculation based on *antanairesis*, the word *logos* indicates a ratio, which can be expressed by an *ordered couple of integer numbers* and corresponds to one of the stages of a recursive process of measurement of two magnitudes, which can be expressed in an ordered, either open or close series (*taxis*) of *logoi*.

The presence of antanairetic calculations in many passages of ancient mathematical literature led scholars to identify an authentic '*antanairetic enlightenment*' in pre-Euclidean mathematics, while the obscuration of this procedure in subsequent mathematics would derive from specific historical and theoretical causes.[544] At any rate, it is clear how the development and study of antanairetic *logoi* can overcome some of the most restrictive limitations of 'Pythagorean' arithmetic. The concept of *logos* as *ordered couple of integers*, and not as a number itself, makes it possible to overcome the contradiction between a type of mathematics that does not accept the division of the unit and one that does, as was the case of fractional mathematics:[545] 'the statement that '1', as a single and *individual* object, is indivisible, while the *logos* [1, 1], which consists in a *couple* of numbers, is divisible does not present any contradictions nor any kind of logical difficulty whatsoever, even if the couple represents the relation of the one with itself'.[546]

However, the arithmetic-algebraic field does not seem to be the most promising one for the development of antanairetic *logistike*: the best results are achieved with regard to the geometrical problem of incommensurable magnitudes. Among the different possible methodologies by means of which the existence of incommensurability might have been discovered, a particularly relevant status is usually attributed to the one reported by Aristotle (*An. Pr.* 1.23, 41a23-30) and reconstructed on the basis of Alexander of Aphrodisias' comments, according to which the commensurability of a diagonal would imply the existence of a number that is neither even nor odd.[547] But all pre-Euclidean demonstrations of the existence of incommensurability, even the one which presents a possible geometrical variant of the Aristotelian procedure, 'seem regularly interested in establishing unlimited sequences of similar

[543] For instance, on the basis of the combined evidence provided in *Men.* 82e-84a, *Resp.* 8.546d and *Theaet.* 196a-199b, Toth (1998b), 42-45, was able to identify the first four members of the infinite series of couples of numbers determined by the *antanairesis* of the side and diagonal of a square which, according to him, is identified in the maieutic experiment of the *Meno*: [1, 1]; [3, 2]; [7, 5]; [17, 12]; etc.

[544] This is the fundamental belief maintained by Fowler (1999), which is presented more briefly in Fowler (1990).

[545] Cf. below §2.6. A valid examination of these aspects of Pythagorean mathematics is provided in Klein (1968).

[546] Toth (1998a), 77.

[547] Aristotle emphasises the impulse given by the discovery of incommensurability to the development of theoretical knowledge (*Met.* 1.2, 983a11-21) and employs the image of a 'commensurable diagonal' as one of his favourite examples of absurdity (cf. Bonitz, 1955, s.vv. *diametros*, *symmetros*). The inclination to regard the discovery of incommensurability as a crucial turn in archaic Greek mathematics dates back to him and is generally accepted by the majority of contemporary interpreters; an exception to this is represented by Knorr (1975), esp. 298-302, who shows that there were *many* examples of incommensurability in the earliest stages of ancient Greek mathematical thought. For a reconstruction of the proof of *An. Pr.* 1.23, 41a23-30, cf. Fowler (1999), 294-308.

geometrical shapes', sequences that were to be translated into figures in the *taxeis* of the antanairetic *logoi*.[548]

The antanairetic *logoi*, therefore, are capable to express in an open, though ordered and 'rational' manner the geometrical phenomenon that was indicated as 'irrational' (*alogon*) and 'inexpressible' (*arrheton*): the ratio between magnitudes that lack a common measure, which could not be expressed by means of positive integer numbers.

Perhaps it is exactly due to this 'cohesive force'[549] that *logistike* deserves the praise it receives in the aforementioned fragments attributed to Archytas, where is underlined its ability to 'make discordance stop and enhance concordance', in that it is able to 'demonstrate constructions that geometry is not able to' (DK 47 B 3-4). And perhaps it was exactly thanks to Archytas that *logistike* acquired such a prominent role in Plato's writings.

In fact, there are many traces of antanairetic *logoi* in the dialogues. In *Republic* 8, when the *logismos* of the 'nuptial number' is outlined, Socrates mentions the 'rational' or 'effable' diagonal and the 'irrational' or 'ineffable' diagonal of 5, referring to the infinite series of lateral and diagonal numbers, which corresponds to the *antanairesis* of the diagonal and the side of a square (8.546c4-6). According to some interpreters, the first passages of the same *antanairesis* are presented in *Meno* (82b-86c), other sections are quoted in the *Theaetetus* (196a, 199b), while the section of the *Parmenides* that examines the dialectics of old and new (154b-d) provides us with one of the most complete examples of *antanairesis* that have been preserved.[550]

Socrates and Glaucon do not give the reader any explicit examples of the *logistike* they are talking about, even though both mention 'symmetry' (7.530a1) and 'irrational lines' (7.534d5). However, the curriculum of mathematical studies presented in the *Laws* basically completes that of the *Republic* by adding that 'problems concerning the essential nature of the commensurable and the incommensurable' should be 'endlessly examined and solved' (*Leg.* 7.820b3-c9, with 7.817e5-820e7). The only real example of mathematical demonstration mentioned in the simile of the Line regards the square and its diagonal (*Resp.* 6.510d7-8). A demanding example of theoretical *logistike* that is familiar to Socrates and Glaucon could be exactly that of identifying the numerical series relative to the *antanairesis* of two incommensurable lines, such as the diagonal and the side of a square. So, in relation to the discovery of incommensurability, the mathematics of *antanairesis* represents 'a way to put back together the separate pieces in which numbers and magnitudes had been divided', presenting itself as a 'type of reflection that entails a contraposition between 'same' and 'other' and ends up in a calculation (*logismos*) inspired to a principle of similarity and balance of opposites'[551] In this way *logistike*, i.e. the link between arithmetic and geometry, would be characterised by a compresence of opposites, in particular by the simultaneous presence of unity and infinite multiplicity to which Socrates relates the *dynamis* of the *mathema* he is looking for, as if it were a kind of illness that ultimately produces a healthy state of the soul (cf. §2.5).

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548 Cf. Zellini (1999), 168-171, esp. 167.

549 Cf. Zellini (1999), 165.

550 These passages are discussed by Fowler (1999), 393; Toth (1998a), 204-220 presents a detailed analysis of the antanairetic 'chase' of the *Parmenides*, while in Toth (1998b) the experiment of the *Meno* is reconstructed in similar terms to those proposed by Fowler. Both show the familiarity of many Platonic dialogues with conceptual constellations related to antanairetic mathematical calculations (limit vs. unlimited, excess vs. defect, etc.); many examples of this can be found in the *Philebus*.

551 Zellini (1999), 161, 133.

1.6 Plain and solid geometry

After accepting arithmetic and *logistike*, Socrates invites Glaucon to see whether 'a second discipline, which derives from the former one, is fitting to our purposes'; 'What is that? Do you mean geometry?' (7.526c8-10) replies Glaucon assuredly, given that he showed to be acquainted with this discipline since the last sections of Book 6. Socrates agrees with him, but immediately hastens to distinguish between knowledge of the basic elements of geometry (7.526d7-8), which is useful to war (7.526d1, on which see §2.4) and a 'more advanced and useful part of it' (7.526d8-9), which it much more interesting to him not only for its well-known ability to make the mind sharper (7.527c6-9) but also in order to 'examine whether it tends to that aim of ours' (7.526d9-e1), that is the conversion of the soul toward 'what truly is'.

This examination will produce two positive results: in the end both the most common type of geometry, the plain one, and the lesser-known and new one, i.e. solid geometry, will prove to be suitable to Socrates' end. But one particular aspect of their characterisation is striking: they are explicitly depicted by means of terms that are related to vision (cf. 7.526e1-5, 7.527b10-11, 7.527e2-3). The close relation between geometry and the 'visible' nature of the images it studies was already underlined in the simile of the Line (7.510d5-511a1). According to Socrates and Glaucon, a crucial question is that of defining the role of *graphein*, 'drawing' or 'modelling' visible representations in the field of theoretical geometry: these activities are indicated by means of terms such as 'squaring, applying and adding' (7.527a8-9), in order to establish 'equals or doubles or any other ratio' (7.529d6-530a1, on which see §2.6 below).[552] The centrality of the visual aspect of geometry seems to be far from casual, and the same goes for Socrates' specification that geometry must be examined 'by means of problems' that can stimulate one's intelligence (7.530b6) and for the emphasis he puts on the 'necessitating' power of geometry (7.525d6, 7.526b1-2, 7.526e2, 7.526e7, on which cf. § 2.3 below). In the 5th century, the emphasis is put on the 'new visual type of geometry, as well as the role of surprising results and rigorous demonstrations'.[553]

The 'visual geometry' of Hippocrates of Chios

The hero of this type of geometry is Hippocrates of Chios, to whom is attributed the 'squaring of the lunula' that is illustrated by Simplicius in a passage of his commentary to Aristotle's *Physics* which dates back to Eudemus (DK 42 B 3). This demonstration is a part of the problem of the 'squaring of the circle', which is an actual enigma, a paradox that is actually comically exploited by Aristophanes and other comic playwrights. Such as Zeno turns the idea of motion into its opposite, Hippocrates demonstrates that a curve shape, the lunula, turns into a rectilinear one, or rather into a square, despite the failure of all attempts to apply to the circle the method of triangulation that was used in order to 'square' rectilinear shapes. And he

552 On the translation of *parateinein* as 'apply' in 7.527a8, cf. Mugler (1958), 333 and Mugler (1941), 75-77.

553 Netz (2001), 762. The very analysis of numbers and ratios seems to have entailed some 'visual' aspects in 5th- and 4th-century BC Greece: the arithmetic Books of Euclid's *Elements* – as I observed above (§1.5) – used line segments in order to demonstrate theorems on numbers; but even more importantly we have seen how there was an arithmetic of 'figurate numbers' and that the stable structure of antanairetic demonstrations of incommensurability consisted in the construction of an unlimited sequence of similar geometrical shapes, which are expressed by means of a series of *logoi* but can also be represented by concentric figures nowadays known as fractals, such as the one reproduced in fig. 3. The visual aspect of Greek mathematics – as underlined by Netz (2001), 757 – has been particularly emphasised by scholars who criticise the idea of a purely intellectual genesis and development of the discipline, which is vigorously defended in Szabó (1969) and in the more recent Szabó (1994).

demonstrates this rigorously, although 'the necessity of his conclusion follows from observing an image'.[554]

So Hippocrates 'squares, applies and adds' – as Socrates would say with disappointment (7.527a8-9, on which cf. §2.6) – even though he does not do that *praxeos heneka* (7.527a7); on the contrary it is clear that, in addition to facing a problem that stimulates intelligence, his work entails a set of definitions and argumentative chains which make plausible the testimony according to which he wrote the first *Elements* of geometry of our civilisation, even though the age in which he lived was too early for this kind of work.

*The Academic 'proto-*Elements*' and Eudoxus*

The composition of 'proto-*Elements*' probably dates back to the Platonic Academy of the first half of the 4th century, and this work will ultimately become part of Euclid's ones. The geometrical 'hypotheses' mentioned in the simile of the Line are all comprised in the definitions and postulates of Book 1 of Euclid's *Elements*.[555] Starting from these 'hypotheses', which were universally accepted by all mathematicians, deductive demonstrations were developed in keeping with rigorous criteria of exactness and clarity, but still entailed visual components.

In relation to this, we can examine the case of the theory of proportions attributed to Eudoxus, which is presented in Book 5 of Euclid's *Elements* and, even though is not mentioned in Book 7 of the *Republic* (which deals with an earlier theory of *logoi*), is hinted at in other Platonic texts.[556] Even though the attribution to Eudoxus is uncertain, the fact that Aristotle mentions it means that its formulation cannot be later than mid-fourth century BC. This is a general, complex and rigorous theory: it can be applied in line of principle to all possible types of magnitude – indicated by the neologism *pelike* – and is based on a complex definition of *logos* (*Elem.* 5, deff. 3-5) which does not entail any reference to integer numbers but is expressed only 'in terms of magnitudes that are bigger or smaller than others' and employs the general relations of 'equality', 'excess' and 'defect'.[557] So some interpreters thought that this discipline was developed in order to deal with the phenomenon of incommensurability, since its definition of *logos* covers both commensurable and incommensurable magnitudes; however, its theoretical core consists in the truly philosophical effort to 'grasp the essence' of proportionality, together with the truly mathematical effort to resolve problems in keeping with a different theoretical framework (cf. e.g. *Elem.* 5.7). On the other hand, the actual theorems entailed in this theory focus on geometrical magnitudes and, in particular, on line segments: so, 'in looking for the most general theory, Eudoxus ended up establishing a "geometrical" theory', yielding to the 'central role of geometry' – a position that, being a 'consequence of the role of demonstrations based on visual representations', determines the strongly visual character of Greek mathematics.[558]

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554 Netz (2001), 760. For a representation of the image used in this demonstration, cf. Netz (2001), 761. Additional explanations are provided in Boyer (1968), ch. 5 § 4-5, and Heath (1981), 183-200.

555 The 'shapes' (6.510c4: *schemata*), and the 'square with its diagonal' (6.510d6-7), correspond to *Elem.* 1 deff. 14-15, 18-22 and postt. 1-3, i.e. the Euclidean definitions of 'shape', 'circle', 'centre', 'diameter', 'half circle', 'rectilinear shapes', equilateral, isosceles and scalene 'triangle' and 'quadrilateral shape' and the postulates concerning straight lines and the circumference. The 'three types of angles' (6.510c5) correspond to *Elem.* 1 deff. 10-12, which define right, obtuse and acute angles, as well as *Elem.* 1, post. 4, which establishes the equivalence of all right angles.

556 Cf. Heath (1956), 2.112-90

557 Netz (2001), 766.

558 Cf. Netz (2001), 762.

Stereometry and pessimism

Socrates and Glaucon examine as well a science that was newly developed in the 4th century, a science called 'stereometry', to use the term provided in *Epinomis* 990d8, which consists in the theoretical study of solid shapes. Both characters seem aware of the innovative nature of this discipline to such a degree that they initially forget to mention it as a third *mathema* that precedes astronomy, which instead is the discipline that studies solids in movement (*Resp.* 7.527d1-528b5, 7.528d2-e5; cf. 7.528a9-b3 and 7.528d8). Glaucon justifies this omission by observing that 'this subject does not appear to have been discovered yet' (7.528b4-5). Socrates specifies that it actually has been discovered but it has neither been examined with the necessary dedication nor in the right manner, and it is 'in a ridiculous state' (7.528d9). Socrates depicts a pessimistic picture: the 'crowd' puts it among 'useless types of knowledge' (7.527d5), so that 'these inquiries, which are difficult in themselves, are languidly pursued as no city holds them in honour' (7.528b6-7); on the other hand the researchers, who do not 'entirely understand the usefulness' of their studies (7.528c5-6), 'need a director, in the absence of whom they cannot reach any discovery': a director like this 'is hard to find' and, even if there was one, he would not be accepted by 'those who currently do research in this field, due to their arrogance' (7.528c1). But who are these oblivious and arrogant investigators? And who could supervise them?

It is possible to hypothesise that the 'bad stereometers' are to be identified with 'current astronomers' (7.530c2-3), who are criticised in another section (7.528e3-530c3) but are also mentioned in a negative tone by Socrates at the beginning of his discourse on solid geometry (7.527d1-528a3). It is possible that contemporary stereometrical practices included at least some basic concepts about solid shapes, especially considering that – as I have already underlined –geometry and astronomy were strictly related to each other in 4th-century Athenian higher education. But the astronomers who, as Glaucon himself (7.527d2-4), praise the usefulness of astronomy while they underestimate solid geometry, though having some basic training in this discipline too, practice astronomy for the wrong reasons. And the general consent that they certainly obtain makes them arrogant toward anyone who takes a different perspective.[559]

The 'director' of research: Archytas?

The alternative perspective, which should be promoted also by the 'director' invoked by Socrates (7.528b7), consists in accepting the non-trivial idea that 'in every soul there is an instrument that is purified and rekindled by such subjects' (7.527e; cf. 10.602e): this is the rational part of the soul, *logistikon*, which is 'more important to preserve than ten thousand

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559 Burnyeat (1987), esp. 218, suggests that in 7.528a-e Plato observes the earlier chronological setting of the dialogue and alludes to the *status* of solid geometry that preceded the 'systematic work' undertaken by Theaetetus which at the time of the composition of the *Republic* would have been completed. However the allusion made in same context to the 'elegance' of stereometrical studies (7.528d1), an allusion that probably hints at the theory of the regular solids developed precisely by Theaetetus, seems to me rather problematic: if this were the case Plato would both observe and disregard the historically fictional setting in just a few lines, and both cases would be connected with Theaetetus. In general, the fact that the composition of the *Republic* required at least approximately ten years (cf. Vegetti 1998b, 18-22), together with the 'experimental' character of mathematics in Plato's time, makes it extremely difficult to establish the presence or the absence of Theaetetus, or even Eudoxus, from different parts of the dialogue; for these reasons, the approach followed in this essay is rather that of trying to relate these Platonic passages with a *period* of the development of mathematics which, among other things, comprised the advent of the contributions of Theaetetus and Eudoxus, though this did not happen in a vacuum or in a primitive phase.

eyes, since only with it can the truth be seen' (7.527d8, cf. 7.527e3). So the 'director' undertakes and promotes studies in solid geometry for purely theoretical reasons, aiming at the rational element that inhabits the human soul and the research of the truth. The 'director' should also cooperate with the city, i.e. take political actions, in order to stimulate these studies (7.528c1), and is an 'uncommon person' (7.528b8-9). These characteristics led scholars to identify him with two historical figures: Eudoxus or, more likely, Archytas, who were both geometricians and astronomers.[560] In particular, one of the most significant contributions to stereometry in the first half of the 4th century is attributed to Archytas and regards the cube, i.e. the only shape expressly mentioned by Socrates in this passage (7.528b3) in relation to a ratio that experts in geometry – again according to Socrates – want to establish precisely: the double ratio (7.530a1). The problem examined by Archytas is that of the duplication of the cube which, together with the squaring of the circle and the trisection of a given angle, represented one of the great challenges of ancient mathematics.[561] The solution, which is attributed to him by an unquestionable tradition, is rather complex: it seems that it was based on two given straight lines and two proportional means that look for each other and involved 'complex spatial movements' of plain and solid figures, such as straight lines, circumferences, chords, tangents, parallels, triangles, cylinders and cones.[562] It is precisely its complexity, which probably aimed at impressing people, which indicates that it probably was the first solution to this problem, from which we can derive two considerations on the beginning of stereometry: it is a discipline that was still conceived in physical terms, so far as it involved movements of figures in its demonstrations, but at the same time it had a complex and precise terminological apparatus, 'which permitted to describe unambiguously the most complicated configurations'.[563]

Stereometry and optimism: Theaetetus' 'charm'

The type of research undertaken by Archytas or Eudoxus may have elicited the development of a more general positive trend, such as that to which Socrates refers with a touch of optimism when he says that 'even now, lightly esteemed as they are by the multitude and hampered by the ignorance of their students as to the true reasons for pursuing them, in the face of all these obstacles the studies on solid geometry nevertheless force their way by their inherent elegance and it would not surprise us if the truth about them were made apparent' (7.528c4-8). Even though he was not able to follow Socrates' previous observations on stereometry, Glaucon immediately approves this specific remark of his (7.528d1). It seems that some members of the Academy, which are remembered as pupils of Plato and Eudoxus,

[560] Cf. Heath (1981), 1.12-13, the contributions of these two historical figures, also with regard to astronomy, are examined in Heath (1981), 1.213-216, 1.246-251, 1.322-329. As is well-known, Eudoxus' progress in astronomy regarded the system of celestial spheres, on which see Heath (1981), 1.329-335. An alternative tradition maintains that Plato would have appointed himself as director of stereometrical studies, a tradition that has been strongly emphasised by Szlezák (2000), who refers to Gaiser (1988), 152 and, therefore, to Philodemus and to Eudemus fr. 133 Wehrli. But some doubts are raised by the dubious tradition that attributes to Plato some original contributions to stereometry (cf. note 62), while it seems rather plausible that Archytas developed the *first* solutions to problems of solid geometry in the Academy, solutions that would have led to many subsequent developments.

[561] As mentioned in the previous note, an extremely dubious tradition attributes to Plato an attempt to solve the question of the duplication of the cube: according to Heath (1981), 1.245-246, 255, this attempt must be interpreted as a solution developed in the Academy, in response to those proposed by Archytas and Eudoxus. Cf. Knorr (1986), esp. 49-66 on the academic solutions to the problem of the duplication of the cube.

[562] Cf. Netz (2001), 765.

[563] Cf. Netz (2001), 765.

actually reached significant and subtle results in the field of stereometry such as, for instance, the theory of the conical section attributed to Menaecmus, which comprises among other things also a solution to the problem of the duplication of the cube that is much more linear than that attributed to Archytas.[564] But why do they talk of the 'elegance' or 'charm' (*charis*) of this discipline?

As Aristotle will argue on the basis of an Academic argument, all kinds of mathematics have a certain type of beauty, as they are characterised by order, symmetry and definition.[565] However, the beauty of solid geometry is praised repeatedly in one of Plato's dialogues, the *Timaeus*, precisely every time that the discourse deals with the five regular solid shapes and the proportional ratios that exist between them (cf. esp. 31.b-32c, 53b-56b). And, in relation to these passages, Book 13 of Euclid's *Elements* is also relevant, in that it contains the theory of regular solids that is attributed to Plato's young friend Theaetetus. It is generally believed that also the theory of irrationals, presented in Book 10 of the *Elements*, is to be ascribed to him, as also an aforementioned passage of the *Theaetetus* seems to suggest (147c8-148b3).[566] In both cases, Theaetetus shows his mastery on the subject he deals with by formulating careful definitions and classifications: so he manages to create some order in the chaotic dimension of incommensurable magnitudes, although he does not do so exhaustively, while Book 13 of the *Elements* provides 'a complete and definitive systematisation of a specific field of research' by means of 'a clear and linear account of a complete and beautiful theory'.[567] With regard to solids whose sides consist in equal regular polygons, two problems are posed: that of their inclusions in a sphere, that of comparing the edges to each other and that of comparing the edges to the radius of the sphere which contains the solid in question. This last problem highlights again the need to establish ratios between lines that are incommensurable, such as for instance in the case of the dodecahedron and icosahedron; however, this case is included in the classes of incommensurable ratios that are defined in Book 10 of the *Elements* and, therefore, has a specific place in the universe of geometry.

2. A reform of power

Arithmetic, *logistike*, geometry and stereometry are thoroughly reformed as the curriculum of mathematical studies is outlined in *Republic* 7. Some aspects of this reform emerged already in the course of the previous description of the disciplines included, but Socrates and Glaucon clarify that it is crucial to give these discipline a new power (*dynamis*). It is only due to this ultimate aim that it is necessary to insist on those traits of mathematical power that are still known as 'Platonistic', i.e. on the intellectual and theoretical, non-empirical nature of the objects examined by these disciplines, nature that is independent of our thoughts about them.[568]

Plato's reform, though, is not only 'Platonistic', because it delves deeply into contemporary trends in mathematical studies and deals with fashionable contrasting approaches, which

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[564] Cf. Heath (1981), 1.251-255, Boyer (1968), ch. 6 § 10, and Netz (2001), 770-772.

[565] Cf. *Met.* 13.3, 1078a31-b6. The Academic origin of this argument, which Aristotle uses against Aristippus, is demonstrated by the connection between this passage and *Eth. Eud.* 1.8, 1218a15-32, on which see Brunschwig (1971).

[566] Cf. Heath (1981), 1.209-212, 1.294-297, on the theory of regular solids in Plato, together with Heath (1956), 3.10-13 for an introduction to Eucl. *Elem.* 10 and Heath (1956), 3.438-511 for an analytical commentary to Eucl. *Elem.* 13. On the beauty of the proportional ratios mentioned in the *Timaeus*, cf. Périllié (2001).

[567] Cf. Netz (2001), 767 ff.

[568] Cf. Bernays (1964).

included a low technico-practical type of knowledge and a highly scientific and theoretical one; Plato exploits the flexible identity of this 'laboratory discipline' and accepts its tendency to deal with contradictions that sometimes are rather paradoxical (commensurable/incommensurable, unity/infinite multiplicity, visible/intellectual), undertaking an infinite effort to organise them harmonically. This whole energetic and dynamic cultural heritage – which includes also didactic habits, scientific procedures, styles of reasoning, linguistic conventions, established or developing theories – is transformed into a *dynamis* that is capable of acting on a psychological, scientific, philosophical and political level in a 'advantageous', 'useful' and 'appropriate' manner.[569]

2.1. A type of power reserved to the aristoi

Who is going to manage the new power of mathematics? Clearly not the mass, as no forms of 'democratic' control or fruition of it are envisaged. Socrates underlines again that the teaching of 'reformed' mathematics – i.e. higher scientific studies and not simply technical abilities – will be enforced by the city's laws and will be limited to the future Guardians (7.525b11-c1, on arithmetic and *logistike*, 7.527c1-2 on geometry, 7.528e4-5 on stereometry). At any rate, it is clear that 'only the best natures' (7.526c5-6) should be trained in the new mathematics and not only because these disciplines are hard. The 'best natures' are the few *aristoi* that are destined to become the true 'Guardians': the whole mathematical *curriculum* is devoted to their education (7.521c1-2). And the fact that they are not 'a crowd' but actually are opposed to the 'multitude' is clarified both in the passages dealing with the *curriculum* itself (7.527d5) and in the aforementioned passage of the *Philebus* (56d-e, cf. §1.4).

2.2 An 'auroral' power

In the initial stages of their analysis, Socrates and Glaucon describe the *dynamis* of the *mathema* they are seeking (7.521c10-d1) by means of two light metaphors.[570]

This *dynamis* consists in being able to draw people up (*anagein*) to the light, bringing them back (*anerchomai*) from Hades, the realm of shades, to the gods (7.521c2-3): these divinities actually *are* lights, since they coincide with the stars that are mentioned in the simile of the Sun (6.508a4). At the same time, this *dynamis* is able to turn the soul around (*periagogein)* 'from a day whose light is darkness to the veritable day' (7.521c6-7), i.e. from a starry night to full daylight.

The sought *mathema* should also be able to bring about a change from a given situation to the opposite one. The initial situation – Hades, and nocturnal light – can be interpreted as a parallel to the situation in which vision is dimmed and therefore it is possible to see only 'shadows' – a situation that is mentioned in all the three great 'similes' (6.508c4-10, the Sun; 7.510a1, the Line; 7.515b, 7.516b, the Cave) and coincides with examining sensible objects in

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[569] Friedländer (1979), 123, writes the following: 'Numbers and visual representations have always had for the Greeks some peculiar trait that brought them to an higher level than the 'colourless' one, so to speak, of mathematics, something that was beautiful and magic. According to Plato (...) mathematical sciences had a meaning that went beyond that of their individual character of science and led to the higher type of being [...]. But this is precisely what the *Platonic turn* consists in: individual sciences lead the soul up to the truth, aim at the science of what always is, purify the organ of the soul and are useful to the research of what is good and beautiful. Therefore, they cannot be intended as autonomous, in the current meaning of the term [...]. However, the opposite position that claims that they consist only in abstract speculations is not less undermined by the facts'.

[570] In general on light metaphors in Plato, cf. Napolitano Valditara (1994); however this study does not examine the metaphors I have just mentioned.

order to obtain sensible knowledge. The passage from this condition to the opposite one is described as an 'ascent towards what really is' (7.521c7), or as a transition 'from the world of becoming to that of being' (7.5121d2-3) and, therefore, as 'true philosophy' (7.521c8).

This passage does not happen easily: 'so this, it seems, would not be the whirling (*peristrophe*) of the shell in the children's game'. So it is not just a matter of turning a shell, or a piece of shard as in the game of *ostraka,* which entails throwing potshards or shells which are black on one side, called *nux*, and white on the other, called *hemera*.[571] Turning these shards around means moving from the pitch of the night to full daylight, and this quick shift happens by pure chance, given that the shards turn around after having been thrown. The passage that the sought *mathema* must be able to cause, instead, must be neither immediate nor casual. It seems to be gradual and recursive, similar to the *periagoge* from a starry night to full daylight that takes place every day at dawn: when the stars set and the sun rises. Also the words *anagein* and *anerchomai*, as well as *epanodos* and *periagoge* can be used to describe the periodical rising of the sun.[572] Socrates and Glaucon look for a *mathema* that has this kind of 'auroral' power: a power that is able to bring regularly close to each other opposite conditions of light and shadows, trying to make them more similar and balanced; a power that is exerted on the soul and assimilates the opposite conditions of the sensible and the intelligible realms, of becoming and being, constantly trying to balance them more accurately. This is the most relevant aspect of the 'conversion' and 'ascent' of the soul to 'true philosophy'. And even though the *mathema* that is capable of doing this still lacks a name, the fact that it is so similar to dawn generates some expectations: among the gods who in Greek mythology are related to the crack of dawn there is Prometheus who, as Aeschylus remembers (*Pr.* 456-458), was the first man who was able to show others how 'to discern the risings of the stars and their settings' and invented for them 'numbers, too'.

2.3 A power that 'draws upward'

Another image that Socrates and Glaucon employ in order to describe the *dynamis* of the *mathema* they are looking for as well as, afterwards, the power of arithmetic, *logistike* and geometry, is that of a 'drawing force' (*holkon*) that brings the soul 'away from the world of becoming to the world of being' (7.521d3), 'toward the essence' (7.524e1; cf. 7.523a2-3) and 'toward the truth' (7.527b9: *holkon pros aletheian*).[573]

This force draws the soul, i.e. is in the soul, affects it, and leads it toward a target (*agein* and similar terms: 7.522b1, 7.523a1, 7.532a6, 7.525a1, 7.525b1), while in the field of geometry it acts by making the soul 'tend' toward a specific aim (*teinein*, 7.526e1-2); most importantly, however, it causes a sort of 'conversion' (*metastrophe*, 7.525a1, 7.525c5, 7.526e3) of the soul, forcing it to turn in a specific direction (*anankazein* and similar terms: cf. 7.526d6, 7.526b1-2, 7.526e2, 7.526e6).[574]

From 'the world of becoming', the soul turns to and is forcefully pushed towards the opposite 'world of being and *ousia*', of 'truth' and 'pure thought (*noesis*)' (7.521d4, 7.523a3, 7.523d8, 7.524b4, 7.525d5, 7.524e11, 7.525a1, 7.525b5, 7.526b1-2, 7.526e4, 7.526e6, 7.527b5-6), and even toward the 'Idea of the Good' (7.526e3). The tension between these opposite poles, from which the soul cannot escape, moves from the bottom to the top: *lo-*

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571 Cf. *LSJ*, s.v. *ostrakon*.

572 Cf. *LSJ*, s.v. *periago, anago, anerchomai, epanodos*.

573 Cf. *LSJ*, s.vv. *holkos* – among its possible meanings there is also the attractive force of a magnet – and *helktikos*, which occurs in 7.523a2-3.

574 The 'necessitating' power of mathematics is emphasised in the *Laws*, 7.818b, on which see Morrow (1960), 344-345.

gistike 'directs the soul upward and compels it to discourse about pure numbers' (7.525d5) and geometry, being 'knowledge of what always is' (7.527b7-8), and not of 'what comes into being and passes away' (7.527b5-6), is able to 'draw the soul to truth, [...], directing upward the faculties that now wrongly are turned earthward' (7.527b10-11).

As we have seen, the rational part of the soul (*logistikon*) – the 'organ of the soul' that is more precious than 'a thousand eyes' in that it is capable 'to see the truth' – is affected by the power of attraction that is exerted by mathematics on the *psyche,* which draws it upwards and in this way heals it from the 'blindness' it caught by engaging in 'other activities' (7.527d7-e3).[575] The 'study of the one' – provided that the one presents itself also as infinitely multiple – can already lead the soul to look to the opposite world, forcing it to 'contemplate' (*thea*) the realm of being (7.525a1). Above all geometry, 'whose entire purpose consists in acquiring knowledge' (7.527b1), despite the fact that its images, procedures and language are far too tightly bound to the realm of practice (7.527a6-9), has the power to attract the attention of the soul and enable it to get a better vision of the 'Idea of the Good' (7.526e1, cf. 7.526e2-4, 7.526e6-7).

The image of the soul 'being drawn' specifies further the nature of the 'auroral' power that mathematics needs: it suggests that the progressive and balanced juxtaposition of sensible and intelligible dimensions, of becoming and being, that they are capable of determining entails also a kind of tension between two opposite pole and turns into an ascending attraction exerted by the second pole on the first – a sort of 'longitudinal magnetic current' that goes through the soul and enhances its intellectual ability to 'see' the truth.

2.4 A type of power that is useful in war

Socrates and Glaucon will describe in detail this enhancement of intellectual abilities by comparing it to a kind of 'illness' that awakes and alerts the soul. But they highlight first another aspect, which instead is often overlooked by contemporary interpreters since it is regarded as irrelevant, if not disquieting, in relation to contemporary mathematics and physics.[576] The type of knowledge, Socrates says, must not be 'useless to soldiers', given that the future Guardians will have to be 'athletes of war' in their youth. (7.521d4-11). Arithmetic and *logistike* are included in the sought *mathema* because 'our Guardian is soldier and philosopher in one' (7.525b8-9) and 'a soldier must learn these disciplines in order to marshal his troops, and a philosopher because he must rise out of the region of generation and lay hold on essence or he can never become a true reckoner' (7.525b3-6). Similarly geometry, 'as applies to the conduct of war, is obviously suitable. For in dealing with encampments, the occupation of strong places, the bringing of troops into column and line and all the other formations of an army in actual battle and on the march, an officer who had studied geometry would be a very different person from what he would be if he had not' (7.526d1-6; cf. 527d4 on astronomy).

[575] Szlezák (2000), 977-978, connects this passage with 4.439d, 9.589c-e, 9.590d, and 10.611b-d (where the *logistikon* is also defined as 'divine') and underlines that the mathematical *curriculum* is also presented as a way to fortify the divine element of the soul; on the other hand, given that the education of the intellectual faculties is accompanied by a general training of the soul toward virtue (cf. Szlezák 2000, 977, *ad* 7.518c), the educational power of mathematics does not only affect the soul by drawing it 'upwards' but, though the role it plays in the art of war, contributes as well to the overall development of the soul.

[576] Cf. Heisenberg (1962), who among other things undertakes a re-evaluation of the Platonic theory of the five regular solids of the *Timaeus* (67-68) and opens his essay by quoting atomic weapons as an example of the exceptional influence of modern mathematics and physics on the 'political structure of the contemporary world' (27).

As we have said before, it is very plausible that here the Platonic reform includes one of the reasons why basic mathematical training was customary in Athens in the first half of the 4th century. Socrates and Glaucon agree on the fact that an expert general would need only 'a slight modicum of geometry and calculation' (7.526e7-8) and that, in relation to war, the only useful part of mathematics (which however is very much useful and crucial) is the 'trifling matter' of 'distinguishing one, two and three – in sum, number and calculation' (7.522c4-7).

However, the usefulness of elementary mathematics to the art of war must not be regarded as a vile or ridiculous aspect, differently from the ability of commercial dealers to keep track of their sells and purchases (cf. 7.525c2-6 on *logistike*, 7.527a6-8, 7.527c2-6 on geometry).

The meaning of the comic scene that regards Agamemnon and Palamedes (7.522d1-e4 with §2.6 below) must be interpreted in keeping with this assessment of the art of war as being noble, useful and therefore closer to a philosophical kind of activity than to a commercial or 'banausic' one. Palamedes is one of the mythical 'inventors' of the number, together with Prometheus, Hermes, the Egyptian god Theuth and others, and one of his main activities is 'creating *taxeis*': in addition to organising the army in *taxeis*, Palamedes classifies and arranges ships as well as all other objects, as if they had never been counted before (7.522d3-5). These are very simple operations, but they all involve the ability to '*logizesthai kai arithmein*' (7.522d3-4, 7.522e3). Without this skill, one would be a truly 'ridiculous' (7.522d1) and 'odd general' (7.522d9), such as Agamemnon who did not even know how many feet he had since he was not able to count (7.522d6-7). Differently from Palamedes, Agamemnon did not even have the numerical skills that Socrates regarded as 'trivial'. But Palamedes seems to have gone further. According to Socrates' specification, this is attested in tragedies (7.522d2): a fragment of Sophocles states that Palamedes' model of earthy *taxeis* corresponded to the order of stars in the sky, and in turn the system of 'signs' that is present in the sky corresponds to numerical *taxeis*.[577] Sky and earth are connected and brought closer to each other by means of numerical *taxeis*. So Palamedes tried to assimilate sky and earth, what is above and what is below, by means of an arithmetical order. His art of war employs number and *logismos*, albeit at a basic level, in a way that is analogous to that required for the soul to turn away from the world of becoming and look at the truth and being. And it is far from casual that, in order to reach these results, it is necessary to go through contrasts that are similar to a war.

2.5 A 'vital' disease

The 'antagonism of opposites' represents the origin of the auroral *dynamis* that draws the soul and is useful to war; this *dynamis* was regarded since the beginning as the fundamental trait of the sought *mathema,* which was finally identified in arithmetic, *logistike* and geometry. Socrates and Glaucon discuss this aspect extensively, while they develop a new 'simile' (7.522e5-525d7, 7.524d9: *analogizou*). Their discourse is centred on the idea that arithmetic and *logistike* trigger a sort of psychological 'disease' that is characterised by a confused and simultaneous perception of opposite conditions: one and infinitely multiple, big and small, sensible and intelligible. This illness is far from deadly: it 'provokes the soul to reconsideration' (7.523b1, 7.523b9, 7.523c1, 7.523e1, 7.524b4, 7.524d3, 7.524d4: *parakalein*; cf. *Phaed.* 89c7; Thuc. 5.31.2) and to 'awaken *noesis*' (7.523e1, 7.524d5), i.e. its purely intellectual faculty, in order to pose itself some doubts and start a complex and demanding process of examination and discussion that aims at distinguishing the true nature of things. Also in the

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577 Pearson (1963), 2.86-87. For other references to Palamedes as the inventor of numbers in Aeschylus, Sophocles and Euripides, cf. Adam *ad loc.* and Szlezák (2000), 978.

case of mathematics, the power of 'dawn' is that of waking up human beings, and the passage from Hades to daylight truly awakens the person who sleeps; in addition, the drawing force associated to this process evokes again war echoes, with the ideas of contrast and opposition they entail.

Characteristics and power of the confusional state: the 'three fingers'

Socrates proposes a parallel with perceptions and gradually focuses on those related to 'seeing'. When someone perceives some kind of feeling – Socrates tells Glaucon – there are some objects that 'do not provoke thought to reconsideration' while others 'always invite the intellect to reflection' (7.523a10-b2). The objects that do not stimulate the intellect are those in relation to which 'judgement by sensation seems adequate' (7.523b1-2), while the objects that lead the intellect to further examinations are those whose 'sensation yields nothing that can be trusted' (7.523b3-4).

The illness that forces people to think induces a 'confusional state' in which perceptions appear indistinct. However, the lack of distinction is not general – as would be the case of distant appearances and shadow-painting (7.523b5-6), when there is not a marked perception of the distinction between opposite states: that is, at the same moment the perception of an object switches between contradictory positions, since 'the perception no more manifests one thing than its contrary' (7.523c2-3).[578]

At this point, Socrates introduces the example of the three fingers, 'the little finger, the second and the middle' (7.523c5-6). After excluding the possibility that the position of the fingers, their colour or shape could hinder the perception of each finger as a finger, Socrates observes that 'the faculty of sight never signifies to the soul at the same time that the finger is the opposite of a finger' (7.523d5-6). This is the standard, physiological position, which is unlikely to 'provoke or awaken reflection and thought' (7.523d8-e1). In the opposite condition, instead, the eye is not able to distinguish precisely between small and big objects to such an extent that the reciprocal position of each finger becomes indifferent. The same goes for the sense of touch as well as all other senses, which can perceive opposite feelings, triggering the impression that what is perceived has, at the same time, a certain quality and its opposite: e.g. big/small, light/heavy, and so on. 'In these circumstances' states Socrates 'the soul must necessarily be at a loss and wonder what significance this sensation has for it' (7.524a6-7), wondering about the 'strange communications' that come from the senses and 'invite reconsideration' (7.524b1-2).

The soul is put to the test: it must awaken its *logismon te kai noesin* (7.524b4) and examine whether what it perceives is 'only one object or two' (7.524b4-5). If sight perceives two objects, each of them will appear different and individual (7.524b7-8); if the soul recognises that there are *two* things, it recognises at the same time that each of them is one in itself and different from the other. At this point, the two opposites are distinguished by *noesis*: it separates them and conceives the two objects independently from each other, and it is able to do so because otherwise the two objects would not have been recognised as *two* but rather as *one* (7.524b10-c1).

The same happens also in the case of the opposition between big and small: the eye perceives them together, while *noesis* is capable of seeing them (*idein*) as separate from each other. From here comes the need to examine what the quality of being 'big' and what that of

[578] This example is subtler than that described in *Prot.* 356c-d, quoted by Adam *ad loc.*, where *metretike* is examined and 'optical illusions' that provide false perceptions of the dimension of an object is distinguished from the actual dimension of that object; cf. *Resp.* 10.602c-d, where 'measuring, *logizesthai* and weighing' are presented as activities undertaken by the *logistikon* as an antidote to 'optical illusions'.

being 'small' consists in; 'and this is the origin of the designation "intelligible" for the one, and "visible" for the other' (7.524c13), concludes Socrates, suggesting that the very distinction between intelligible and sensible (cf. 6.509d4) results from a separation achieved by *noesis* between opposite states that appear at once, as is the case of the 'ill perception' of *two* objects or in that of a big and a small object, as well as what happens also in the course of the 'preparatory' and 'dianoetic' disciplines of mathematics and *logistike* (7.525a9-10).

One and infinitely multiple

Socrates develops an analogy between 'the number and the one' on the one hand and the two cases of different sensations described above on the other (7.525d2-525b10). The 'healthy condition' which does not stimulate the mind was the one in which each finger was satisfactorily perceived simply as a finger. Analogously, if the one is perceived satisfactorily in itself, either by means of vision or with another sense, it cannot have the force to 'draw us toward the essence' (7.524e1). In order to have this power, the one must present itself as an object of conflicting feelings: it must always be perceived coincidentally with 'some contradiction, so that it no more appears to be one than the opposite' (7.524e2-3). Only in a similar situation, the soul starts to pose itself some questions and examine the problem, in order to distinguish among these two objects: so the soul starts wondering and examines in the first place what is 'one' in itself. Only in this way, the *mathesis peri to hen* shows to be one the disciplines that are capable of leading the soul and convert it to the contemplation of what always is.

At this point, Socrates could have applied the same examples presented above also to arithmetic and *logistike*, i.e. counting *two* objects or evaluating whether something is *big* or *small*; instead, he prefers to introduce another opposition, that between unity and infinite multiplicity, gradually shifting the discourse from the fact observed by Glaucon that 'we see the same thing at once as one and as an indefinite plurality' (7.525a6-8) to a general consideration of the one and of the number (7.525a6-8) and finally to their theoretical study, i.e. arithmetic and *logistike* (7.525a9-11). Therefore, the disciplines that examine numbers, given that they showed to be able to 'lead the soul toward the truth', will constitute at least a part of the sought *mathema* (7.525a6-b3).

Is silence due to familiarity?

Socrates and Glaucon do not specify exactly *how* mathematics manages to embrace both the balanced presence and the reciprocal overlapping of unity and multiplicity, which are crucial in the establishment of its power. This is something that they take for granted, but they also underline that it is far from being a trivial issue; instead they regard is as a very refined task (7.525d1) which entails – as we have seen several times – significant efforts in order to be learned and put into practice (7.525c1-3). But did not 5th- and 4th-century arithmetic show a significant inclination to study different characteristics of the basic unity, including the most paradoxical ones? Did not *logistike*, intended as the theory of antanairetic calculations, entail finding a balance between unity and multiplicity, small and big, 'same' and 'other', in order to attempt at establishing a progressive and reciprocal fading of the opposites into each other? And did it not present in similar terms also unity and indefinite multiplicity, infinitely big and infinitely small, identical and different, as well as an 'arithmetic' translation of the geometrical phenomenon of incommensurability? In addition, did not geometry seem to be traversed by a constitutive tension between 'visible' and 'noetic'? Perhaps, if we were part of the culture represented by Socrates and Glaucon, we would be so familiar with these aspects of mathematics, which according to them are 'sick', that we would not feel the need for them to be mentioned explicitly; and perhaps we would find rather unusual the emphasis they put on the fact that the 'disease' of mathematics has the exceptional power to heal the soul, as well as the city, by forcing the best individuals to study philosophy and become the future governors. But, even more importantly, if we were part of Socrates and Glaucon's culture, we would laugh much more in reading all these passages.

2.6 The end of the comedy

Ancient comedy, as we have noted above, exploited the paradoxical aspects and the complete material uselessness of theoretical mathematics in order make the audience laugh. In particular it seems that Epicharmus, who is presented in the *Theaetetus* as a 'very wise poet' convinced, together with Protagoras, Heraclitus and Empedocles, that 'nothing ever is, while everything always becomes' (153d-e, cf. DK 23 A 6), was very familiar with the Pythagoreans and parodied it on stage. The public laughed at very complex reflections on absurd and trivial topics, such as the fact that an even or an odd number of votes turns into an odd or an even one respectively if another vote is added, or that the measure of an arm seems bigger or smaller if another measure is added or subtracted (DK 23 B 2). It is possible that the public laughed also at sentences such as the following: 'Human life needs number and calculation. We live by means of number and calculation: these are the things that save mortal beings' (DK 23 B 56).

But Socrates and Glaucon laugh at the laughing audience. And one of the most serious messages of the mathematical *curriculum* they outline, i.e. its 'Platonistic' message', derives precisely from their laughter.

Agamemnon and the pig.

The first 'utterly ridiculous' character (7.522d1) appears already in the prologue of the *curriculum*: Agamemnon, the 'very odd' general (7.522d9) who is not even able to count his own feet. Socrates and Glaucon laugh at him because, differently from Palamedes, he is not at all acquainted with *taxeis* and would not be able to count his soldiers and his ships. However, there is another, more serious reason to laugh: the simple 'ability to reckon and number' is the 'most necessary skill for a soldier', not only in order to organise the troops but rather 'if he is to be a man at all' (7.522e1-4). Without knowing, even at a basic level, number, calculation and measure, men are not different from animals (*Tim.* 39b1-c1, *Epin.* 978b7-c6); in the *Laws*, men who display this 'ridiculous and shameful type of ignorance' (7.819d1-2) are assimilated to 'some sort of pig' (7.819d7). So, what kind of a general is the Argive king, given that he is not able to add a foot to a foot? An animal, in the worse case a pig.

Pythagorean irony against the divided unit.

However, in order to escape Socrates and Glaucon mocking, knowing how to add one and one or having basic arithmetic notions is not enough: it is necessary to use them correctly in order to avoid becoming a target of mockery. The 'ridiculous' use of number and calculation is obviously that of the 'amateurs', such as 'dealers and merchants, who use it to sell and buy'; this is the opposite of the serious use of mathematics proper to the 'reformed mathematics' of the *Republic*, which 'look at the nature of numbers by means of thought' and are exercised 'in order to acquire knowledge', 'for the uses of war and for facilitating the conversion of the soul itself from the world of generation to essence and truth' (7.525c1-3). So what is the ridiculous aspect using number and calculation instrumentally?

Socrates reveals it by introducing the Pythagoreans in the scene in order to show how the quality of individuality and indefinite multiplicity that is attributed to the one and to all other numbers should *not* be conceived; in the course of this polemical explanation, Glaucon understands that the one and all numbers cannot be conceived as physical objects but only as objects of *dianoia* (7.525b11-526c4).

For the *mathema peri tous logismous* (7.525d1) to be useful, it is necessary not to consider the contemporaneous presence of the opposite qualities of unity and multiplicity both in the one and in all other numbers as a kind of unity that is divided into multiple or infinite parts, such as a physical body would be. People who use mathematics for practical aims do so on an empirical basis, accepting the possibility that the basic unity can be divided and the existence of fractional numbers: the basic unity is conceived as a physical body that can be divided into two, three, four parts and so on – therefore also numbers, which are made of units, can be divided in the same way.

But now roars the ironic laughter of the 'experts' (7.525d9-e1), who can be identified with the Pythagorean mathematics (as we have seen above in §1.4). These experts laugh at people who try to divide the basic unity in their reasoning, as if it were a physical object. Actually, the more people try to divide it, the more they multiply the unity, as they are interested in making the one actually appear as one and not as comprising many parts (7.525d6-e4). Those who conceive the unity empirically divide the unity into parts, e.g. in halves, thirds, quarts etc., and therefore produce fractional numbers (1/2, 1/3, 1/4, etc.) that are unacceptable in the Pythagorean universe, where the unity must be indivisible and unitary. However, in order to recreate the unit, the experts can simply multiply the fractions by the number of parts in which the unity has been divided – 1/2 times 2, for instance, or 1/3 times 3 – and obtain once more the basic unity.[579]

However the serious matter is another one. 'Suppose now, Glaucon, someone were to ask them, "My good friends, what numbers are these you are talking about, in which the one is such as you postulate, each unity equal to every other without the slightest difference and admitting no division into parts?" – What do you think would be their answer?' (7.526a1-5). Glaucon immediately replies saying something that, in hindsight, seems crucially important: 'This, I think – that they are speaking of units which can only be conceived by thought, and which it is not possible to deal with in any other way' (7.526a6-7).

This is the serious message of this Pythagorean 'trick' against the fractioned unity, the first formulation of 'mathematical arithmetical Platonism' of our civilisation.

Ridiculous geometricians

Socrates and Glaucon's satire does not spare many people who practice plain or solid geometry. 'Ridiculous' geometricians are those who, bound to the realm of becoming and to contingent needs, oppose the approach of 'reformed' scientists, who respond to a very different kind of need, i.e. that of 'contemplating the essence' (7.526e6) until they reach the 'vision of the Idea of the Good' (7.526e1). In the case of stereometry, the 'ridiculous' state of the discipline (7.528d9) is caused by bad and arrogant astronomers, who are interested exclusively in the practical usefulness of their knowledge; this type of knowledge is very different from the new, promising and graceful but definitely misunderstood manner to practice this discipline.

Ordinary geometry is similar to an upside-down world: this is what Socrates and Glaucon find ridiculous, especially with regard to the language by means of which their inhabitants loudly express themselves (7.527a9). 'This science' says Socrates 'is in direct contradiction with the language employed in it by its adepts' (7.527a1-4), because 'their language is most

579 Cf. Szabó (1969), 348-351, who maintains that the theory of the 'divisibility of the unity' is *intrinsically contradictory*, as can be shown by 'indirect demonstrations', while Toth (1998a), 84-85, argues that the contradiction is established between *two* different theses, or rather two *axioms* that are coherent in themselves, i.e. that of the divisibility and that of the indivisibility of the unity.

ludicrous, though they cannot help it, for they speak as if they were doing something and as if all their words were directed towards action. For all their talk is of squaring and applying and adding and the like, whereas in fact the real object of the entire study is pure knowledge' (7.527a5-b1). And even if their drawings were exceptionally beautiful, such as those 'made by Daedalus', a true expert 'would think it absurd to examine them seriously in the expectation of finding in them the absolute truth with regard to equals or doubles or any other ratio' (7.529e1-530a1).

Again, from this grotesque situation, Socrates and Glaucon draw a crucial and exceedingly serious conclusion. 'Must we not agree on a further point?', says Socrates. 'What?' asks Glaucon, and Socrates: 'That geometry is the knowledge of that which always is, and not of a something which at some time comes into being and passes away.' And Glaucon concludes 'That is readily admitted, for geometry is the knowledge of the eternally existent' (7.527b3-8). This is the serious message behind their irony against common and unrefined geometricians: the first formulation of 'Platonism' and geometrical anti-constructivism of our civilisation.

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