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Aspects of Snyder geometry

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Abstract

The Snyder spacetime represents the first proposal of noncommutative geometry. It still retains a significant role because of its property of preserving Lorentz invariance. In the thesis, several different aspects of the model are investigated. In particular, the results include: a calculation of the orbits of a particle in Schwarzschild spacetime in the setting of the relativistic Snyder geometry; the definition of the path integral in one- and two-dimensional Snyder space, in the traditional setting and using the Faddeev-Jackiw formalism; a study of the representations of the three-dimensional Euclidean Snyder-de Sitter algebra (namely, the extension of the Snyder model to a spacetime background of constant curvature) with the calculation of the spectrum of the operators of position and momentum squared; a generalisation of the Snyder model, which includes all possible deformations compatible with Lorentz invariance, and an investigation of it within the Hopf algebroid setting, including the discussion of the twist operator, the R-matrix and the deformed addition of momenta.

Contents

Chapter 1

Introduction

The structure of spacetime at Planck scale lengths, where quantum gravitational effects cannot be ignored, is still unknown since this is an area of physics where it is practically impossible to obtain physical data. Already in 1854 Riemann suggested that, at these lengths, the known concept of spacetime should be changed. Heisenberg proposed the idea of noncommutative spaces as a solution to the problem of ultraviolet divergences in his letter to Peierls and through Pauli this idea reached Oppenheimer. It was Oppenheimer's student Snyder who in 1947 for the first time formulated the idea mathematically [56].

In his paper, Snyder has shown that the introduction of a minimal unit of length necessarily leads to a noncommutative algebra of spacetime coordinates, but also that the assumption of Lorentz covariance does not impose a requirement for the spacetime to be continuous.

Because of the success of the renormalisation theory, the idea of noncommutative spaces had been disregarded until the 80's, when mathematicians Connes and Woronowicz introduced the notion of a differential structure within the noncommutative framework. At that time motives for studying noncommutative spaces came also from string theory, which gave similar predictions for the structure of spacetime at small length scales. Furthermore, different approaches to the unification of all physical interactions into one renormalisable quantum field theory also pointed to the need of introducing a natural unit of length. The motivation for studying them came from several different approaches to quantum gravity as well, like for example, loop quantum gravity.

However, regardless of the renewed interest, most work carried out so far has been related to the so called Moyal plane, the most simple type of noncommutative space where the commutator of the spacetime coordinates is given by a constant tensor or, for example, the κ-Minkowski spacetime, a Lie-algebra type deformation of the Minkowski spacetime. Less interest has been devoted to Snyder's original proposal, even though, contrary to the before mentioned examples, it has the property that it preserves Lorentz invariance.

In his paper, Snyder has shown that if one assumes that the spectra of the spacetime coordinate operators are invariant under Lorentz transformations, there exists a solution that admits a natural unit of length. Lorentz invariant commutation relations for the spacetime coordinates $\hat{x}_\mu$ and momenta $\hat{p}_\mu$ are then given by

$$[\hat{x}_\mu, \hat{x}_\nu] = i\beta^2 \hat{J}_{\mu\nu}, \quad [\hat{p}_\mu, \hat{p}_\nu] = 0, \quad [\hat{x}_\mu, \hat{p}_\nu] = i\left(\eta_{\mu\nu} + \beta^2 \hat{p}_\mu \hat{p}_\nu\right), \tag{1.1}$$

where β^2 is a coupling constant, usually assumed to be of the scale of the square of the Planck length and $\hat{J}_{\mu\nu}$ are the generators of the Lorentz algebra, which satisfy the usual commutation relations

$$[\hat{J}_{\mu\nu}, \hat{J}_{\rho\sigma}] = i\left(\eta_{\nu\rho}\hat{J}_{\mu\sigma} - \eta_{\mu\rho}\hat{J}_{\nu\sigma} - \eta_{\sigma\mu}\hat{J}_{\rho\nu} + \eta_{\sigma\nu}\hat{J}_{\rho\mu}\right), \tag{1.2}$$

$$[\hat{J}_{\mu\nu}, \hat{p}_\mu] = i\left(\eta_{\nu\lambda}\hat{p}_\mu - \eta_{\mu\lambda}\hat{p}_\nu\right), \quad [\hat{J}_{\mu\nu}, \hat{x}_\mu] = i\left(\eta_{\nu\lambda}\hat{x}_\mu - \eta_{\mu\lambda}\hat{x}_\nu\right). \tag{1.3}$$

The Snyder algebra is defined by the commutation relations (1.1) - (1.3) and is nonlinear. Since it can be obtained by constraining the momenta to lie on a hypersphere in a (4+1)-dimensional space, the Snyder model can be viewed as the equivalent of de Sitter spacetime for momentum space. In the limit $\beta^2 \to 0$, one retrieves the usual commutation relations of quantum mechanics.

Restricting the model to three-dimensional Euclidean space, i.e. to its spatial section, gives its nonrelativistic limit, which still contains the main features of the model, but for example allows the implementation of quantum mechanics. In its classical limit, the model is described by a noncanonical symplectic structure and its study requires the methods for treating constrained Hamiltonian systems [12]. The model implies a generalisation of the uncertainty relations [29, 8], which gives rise to a lower bound in the uncertainty of the position, and the discreteness of the spectra of area and volume [52].

The new results of the thesis come from [48, 49, 50, 25] and [37] and are presented in sections 2.3, 3.3, 4.2, 5.2 and 5.3 respectively. The thesis is structured as follows: The second chapter deals with the classical limit of the Snyder model and hence starts with a review of the Dirac method for constrained Hamiltonian systems [12]. After that, some known results [47] on the classical dynamics of the Snyder model, which display the formalism, are surveyed. The concluding section gives the results of [48], where the orbit of a particle in Schwarzschild spacetime was calculated, assuming that the dynamics is governed by the Snyder symplectic structure.

The third chapter is devoted to the problem of the construction of the path integral for the Snyder space, which was discussed in [49], following the traditional procedure and in the first-order formalism of Faddeev and Jackiw. Before presenting the results a review of the methods used [53, 14, 23] is given.

The fourth chapter concerns the Snyder-de Sitter model, a generalisation of the Snyder model to a spacetime of constant curvature. First, some of the known properties of the model [45] are revised and following that new results concerning the spectrum of the operators of position and momentum squared from [50] are given.

In the fifth chapter a generalisation of the Snyder model is considered using Hopf algebroid techniques. Before discussing the results on the generalised Snyder model [37], a short survey of Hopf algebras and of the twist operator [1] is given and the Hopf algebroid structure of noncommutative space, which was elaborated in [25] for the case of the κ-Minkowski spacetime, is explained.

Throughout the thesis, we have set $\hbar = c = 1$, and the usual convention of summation over repeated indices is understood. The scalar product between a and b is denoted as

$a \cdot b$, a^2 stands for $a^2 = a \cdot a$, and the flat metric $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$. Greek indices $\mu, \nu, ...$ take the value in $\{0, 1, 2, 3\}$, while the Latin ones, i, $j, k, ...$, run from 1 to 3.

Chapter 2

The classical limit of the Snyder model

A useful starting point for studying the Snyder spacetime is to consider its classical limit. Since it is described by a phase space with noncanonical symplectic structure, its dynamics need to be investigated using Hamiltonian methods.

Studying the classical motion of a nonrelativistic particle in Snyder space within this framework, it is possible to find exact solutions of the equations of motion for a free particle and for the case of a harmonic potential [44]. It results that the free motion is trivial, but in the presence of external forces the classical dynamics is modified. E.g., in the case of the harmonic oscillator, while the motion is periodic, it is not given by a simple trigonometric function like in classical mechanics and the frequency of oscillation depends on the energy, as in special relativity.

Whether these features can be extended to the relativistic dynamics is a nontrivial question, since Hamiltonian dynamics of a particle in the relativistic domain is constrained and it is necessary to use the Dirac formalism. Another point to make is that, because the relativistic Snyder model has nontrivial Poisson brackets between time and spatial coordinates, its nonrelativistic limit doesn't necessarily coincide with the nonrelativistic theory.

The chapter begins with a short review of the Dirac formalism [12] for treating constrained Hamiltonian systems. It then presents the main results of [47], where these methods were used to study the classical dynamics of the relativistic Snyder model. It concludes with presenting the work of [48], which follows the previous approach when calculating the orbits of a particle in Schwarzschild spacetime, under the assumption that the dynamics is governed by a Snyder symplectic structure.

2.1 Dirac's method for constrained Hamiltonian systems

Starting from an action functional

$$S[x_i(t)] = \int dt L(x_i, \dot{x}_i), \tag{2.1}$$

where $x_i(t)$ are canonical coordinates, $\dot{x}_i = dx_i/dt$ are canonical momenta, the Lagrangian $L(x_i, \dot{x}_i)$ has no explicit t-dependence, the canonical momenta are defined as $p^i = \partial L/\partial \dot{x}_i$, and requiring that the variation of the action is stationary, gives the Euler equations

$$\frac{dp^i}{dt} - \frac{\partial L}{\partial x_i} = 0. \tag{2.2}$$

Choosing the Poisson brackets as

$$\{A, B\} = \frac{\partial A}{\partial x_i}\frac{\partial B}{\partial p^i} - \frac{\partial A}{\partial p^i}\frac{\partial B}{\partial x_i}, \tag{2.3}$$

gives $\{p^i, x_j\} = -\delta^i_j$, with δ^i_j the Kronecker delta, and then the canonical Hamiltonian

$$H_c(p^i, x_i) = p^i \dot{x}_i - L(x_i, p^i), \tag{2.4}$$

generates the Hamiltonian equations of motion

$$\dot{x}_i = \{x_i, H_c\} = \frac{\partial H_c}{\partial p^i}, \quad \dot{p^i} = \{p^i, H_c\} = -\frac{\partial H_c}{\partial x_i}. \tag{2.5}$$

A Lagrangian $L(x_i, \dot{x}_i)$ is defined as being singular if the velocities $\dot{x}_i$ cannot be expressed uniquely in terms of the canonical coordinates and momenta. A necessary and sufficient condition that L be singular is

$$Det\frac{\partial^2 L}{\partial\dot{x}_i\partial\dot{x}_j} = 0. \tag{2.6}$$

The above is a consequence of the existence of certain primary constrains

$$\vartheta_m(x,p) \approx 0, \quad m = 1,..,M \tag{2.7}$$

following from the form of the Lagrangian alone. $\approx$, read as weakly zero, means that ϑ_m may have nonvanishing canonical Poisson brackets with some canonical variables.

In this case the canonical Hamiltonian is not unique and can be replaced by the effective Hamiltonian

$$\tilde{H} = H_c + n_m\vartheta_m(x,p) \approx H_c, \tag{2.8}$$

which generates new equations of motion

$$\dot{x}_i = \{x_i, \tilde{H}\} \approx \frac{\partial H_c}{\partial p^i} + n_m\frac{\partial\vartheta_m}{\partial p^i}, \tag{2.9}$$

$$\dot{p}^i = \{p^i, \tilde{H}\} \approx -\frac{\partial H_c}{\partial x_i} - n_m\frac{\partial\vartheta_m}{\partial x_i}. \tag{2.10}$$

To have a consistent system, it must be required that the t-derivatives of the constraints (2.7) are zero or weakly zero, which is the case if they are linear combinations of the constraints

$$\dot{\vartheta}_l = \{\vartheta_l, \tilde{H}\} \approx \{\vartheta_l, H_c\} + n_m\{\vartheta_l, \vartheta_m\} \approx 0. \tag{2.11}$$

The above equation can be true as a consequence of the original primary constraints. If it is not, it can either impose conditions on the form of n_m, or it can imply new relations among the canonical coordinates and momenta, independent of n_m. These relations are then secondary constraints, which must be added to the original ones. The process is repeated until all independent constraints and conditions on n_m are found. For K additional constraints, the complete set is

$$\vartheta_m(x,p) \approx 0, \quad m = 1,..,M+K = T, \tag{2.12}$$

and the consistency of all the constraints with the equations of motion requires that there exists a solution for n_m as a function of the canonical coordinates and momenta. Hence, $\tilde{H}$ can also be expressed in terms of x and p, $\tilde{H} = \tilde{H}(x,p)$.

Functions of x and p are defined as first class quantities if their Poisson bracket with all the constraints (2.12) is weakly equal to 0 and as second class if at least one of them is not. From here, all the constraints (2.12) can be divided into a set of all the linearly independent first class constraints

$$\vartheta_{1i}(x,p) \approx 0, \quad i = 1,..,I, \tag{2.13}$$

and the set of the remaining $N = T - I$ second class constraints

$$\vartheta_{2i}(x,p) \approx 0, \quad i = 1,...,N. \tag{2.14}$$

The second class constraints give rise to a nonsingular N x N matrix of Poisson brackets

$$C_{\alpha\beta} = \{\vartheta_{2\alpha}, \vartheta_{2\beta}\}. \tag{2.15}$$

Its inverse $C^{-1}_{\alpha\beta}$ exists and satisfies $C_{\alpha\beta}C^{-1}_{\beta\gamma} = \delta_{\alpha\gamma}$.

For a given dynamical variable A, it can be shown that a new variable A' defined by

$$A' = A - \{A, \vartheta_{2\alpha}\}C^{-1}_{\alpha\beta}\vartheta_{2\beta} \tag{2.16}$$

has vanishing brackets with all second class constraints. Defining the Dirac bracket as

$$\{A,B\}^* = \{A,B\} - \{A, \vartheta_{2\alpha}\}C^{-1}_{\alpha\beta}\{\vartheta_{2\beta}, B\}, \tag{2.17}$$

it follows that

$$\{A, B\}^* \approx \{A', B'\} \approx \{A', B\} \approx \{A, B'\}. \tag{2.18}$$

Since the Dirac bracket of anything with a second class constraint vanishes, if all Poisson brackets are replaced with Dirac brackets, all second class constraints can be be set strongly to zero. It can also be shown that the Dirac brackets will satisfy the Jacobi identity.

Setting the effective Hamiltonian $\tilde{H}$ to

$$\tilde{H} = H' = H_c - \{H_c, \vartheta_{2\alpha}\}C^{-1}_{\alpha\beta}\vartheta_{2\beta}, \tag{2.19}$$

so that $n_m(x, p) = -\{H_c, \vartheta_{2\alpha}\}C^{-1}_{\alpha\beta}$, $\tilde{H}$ will be first class. Since the equations of motion aren't altered by adding any linear combination of the I first class constraints to this choice of $\tilde{H}$, the Hamiltonian is still not completely determined, so the total Hamiltonian is taken to be

$$H = H' + l_i\vartheta_{1i}(x, p). \tag{2.20}$$

H will have vanishing brackets with all the constraints and will lead to new equations of motion which will explicitly involve the functions l_i. These functions appear in the Hamiltonian because the original Lagrangian possessed I gauge degrees of freedom associated with the first class constraints ϑ_{1i}. The values of the l_i can be fixed by choosing explicit forms for each gauge and imposing them as constraints that do not follow from the Lagrangian.

2.2 Classical dynamics on Snyder spacetime

The formalism of the previous section can be applied to study the classical dynamics of a relativistic particle in Snyder spacetime, both for a free particle and for the case of a particle subject to an external force generated by a harmonic potential [47].

2.2.1 Free particle

In (1+1)-dimensional spacetime, which is considered for simplicity, the Snyder fundamental Poisson brackets are defined by

$$\{x_\mu, p_\nu\} = \eta_{\mu\nu} + \beta^2 p_\mu p_\nu, \quad \{x_\mu, x_\nu\} = \beta^2 J_{\mu\nu}, \quad \{p_\mu, p_\nu\} = 0, \quad \mu, \nu \in \{0, 1\}, \tag{2.21}$$

where $\eta_{\mu\nu}$ is the flat metric, $J_{\mu\nu} = x_\mu p_\nu - p_\nu x_\mu$ is the generator of the Lorentz transformations.

Since Lorentz invariance is preserved, the Hamiltonian can be chosen as in special relativity

$$H = \frac{\lambda}{2}(p^2 - m^2), \tag{2.22}$$

with $p^2 = p_0^2 - p_1^2$ and λ a Lagrange multiplier enforcing the mass shell constraint $\chi_1 = p^2 - m^2 = 0$. The Hamiltonian equations that follow are

$$\dot{x}_\mu = \{x_\mu, H\} = \lambda(1 + \beta^2 p^2)p_\mu = \lambda(1 + \beta^2 m^2)p_\mu, \quad \dot{p}_\mu = \{p_\mu, H\} = 0, \tag{2.23}$$

with the dot denoting the derivative with respect to the evolution parameter. Since the constraint $\chi_1 = 0$ is first class, a further constraint must be imposed to eliminate the redundant degrees of freedom x_0 and p_0 and reduce the system to motion in one spatial dimension with external time.

The standard choice is to identify the evolution parameter with the time coordinate, $\chi_2 = x_0 - t = 0$, and it gives

$$C \equiv \{\chi_2, \chi_1\} = (1 + \beta^2 m^2)p_0. \tag{2.24}$$

From the requirement that $\dot{\chi_2} = 0$, it follows that $\lambda = 1/C$.

The Dirac brackets are defined as

$$\{A, B\}^* = \{A, B\} + \{A, \chi_2\}C^{-1}\{\chi_1, B\} - \{A, \chi_1\}C^{-1}\{\chi_2, B\}, \tag{2.25}$$

and for the independent variables x_1, p_1, they read

$$\Delta \equiv \{x_1, p_1\}^* = -1, \tag{2.26}$$

as in special relativity.

The reduced Hamiltonian K reads

$$K = p_0 = \sqrt{p_1^2 + m^2}, \tag{2.27}$$

and results in the Hamiltonian equations

$$\frac{dx_1}{dt} = \frac{p_1}{\sqrt{p_1^2 + m^2}}, \qquad \frac{dp_1}{dt} = 0, \tag{2.28}$$

which coincide with the equations of motion of a free particle in special relativity, showing that for the case of a free particle motion in Snyder spacetime is trivial.

Another possible gauge could be given by a constant rescaling of time, $t = \sqrt{1 + \beta^2 m^2} x_0 = \sqrt{1 + \beta^2 p^2} x_0$, motivated by the fact that the natural metric of spacetime, invariant under Snyder transformations is $ds^2 = (1 + \beta^2 p^2) dx^2$, with dx^2 the Minkowski metric.

For this choice of gauge, $\{\chi_2, \chi_1\} = (1 + \beta^2 m^2)^{3/2} p_0$, with the Dirac brackets still given by (2.26). The reduced Hamiltonian is given by

$$K = \frac{p_0}{\sqrt{1 + \beta^2 m^2}} = \sqrt{\frac{p_1^2 + m^2}{1 + \beta^2 m^2}}, \tag{2.29}$$

and the Hamiltonian equations are

$$\frac{dx_1}{dt} = \frac{p_1}{\sqrt{(1 + \beta^2 m^2)(p_1^2 + m^2)}}, \qquad \frac{dp_1}{dt} = 0. \tag{2.30}$$

2.2.2 Harmonic oscillator

For the case of a harmonic potential, that depends only on the spatial position of the particle, $V = V(x_1)$, the Hamiltonian considered [46]

$$H = \frac{\lambda}{2}\left(p^2 - (m+V)^2\right) = 0, \tag{2.31}$$

enforces the constraint $\chi_1 = p^2 - (m+V)^2 = 0$. The equations of motion that follow are given by

$$\begin{aligned} \dot{x}_0 &= \lambda\left((1+\beta^2 p^2)p_0 + \beta^2 J(m+V)V'\right), & \dot{p}_0 &= \lambda p_0 p_1 (m+V)V', \\ \dot{x}_1 &= \lambda(1+\beta^2 p^2)p_1, & \dot{p}_1 &= -\lambda(1-\beta^2 p_1^2)(m+V)V', \end{aligned} \tag{2.32}$$

with the prime denoting a derivative with respect to x_1 and $J \equiv J_{10}$ the generator of Lorentz transformations.

One finds that in the interacting case finding a gauge compatible with the nontrivial symplectic structure is not easy and it is necessary to chose time that depends on the dynamics of the model

$$\chi = S x_0 - t = 0, \tag{2.33}$$

with

$$S = \sqrt{1+\beta^2(m+V)^2} = \sqrt{1+\beta^2 p^2}. \tag{2.34}$$

It follows that

$$\begin{aligned} C = \{\chi_2, \chi_1\} &= S(1+\beta^2 p^2)p_0 + \beta^2 S J(m+V)V' + \beta^2 S^{-1}(1+\beta^2 p^2)J(m+V)V' p_1 x_0 \\ &= S(S^2 + (m+V)V' x_1)p_0, \end{aligned} \tag{2.35}$$

the Lagrange multiplier is given by $\lambda = 1/C$ and the Dirac bracket of x_1 and p_1 reads

$$\{x_1, p_1\}^* = -\frac{S^2}{S^2 + (m+V)V' x_1}. \tag{2.36}$$

The form of the reduced Hamiltonian K that generates motion on the reduced phase space induced by these Dirac brackets is given by

$$K = \frac{p_0}{S} = \sqrt{\frac{p_1^2 + (m+V)^2}{1+\beta^2(m+V)^2}}, \tag{2.37}$$

and the equations of motion that follow are

$$\frac{dx_1}{dt} = \frac{1}{B}\frac{p_1}{K}, \quad \frac{dp_1}{dt} = -\frac{1-\beta^2 p_1^2}{(1-\beta^2 m^2)B}\frac{(m+V)V'}{K}, \tag{2.38}$$

with $B = S^2 + (m+V)V'x_1$. The system of equations can be solved by defining an auxiliary time variable τ, such that $dt = Bd\tau$ and using the conservation of the reduced Hamiltonian K, which gives

$$p_1^2 = K^2 - (1-\beta^2 K^2)(m+V)^2. \tag{2.39}$$

After redefining the energy, for the case of the harmonic oscillator with potential $V = \frac{\kappa}{2}x_1^2$, it follows that

$$x_1 = \sqrt{\frac{E^2 - m^2}{\kappa E}} sd(\omega\tau, q), \tag{2.40}$$

where

$$\omega^2 = \frac{\kappa}{E} = \frac{\kappa\sqrt{1-\beta^2 K^2}}{K}, \quad q = \frac{E-m}{2E} = \frac{K - m\sqrt{1-\beta^2 K^2}}{2K}, \tag{2.41}$$

and $sd(\omega\tau, q)$ is a Jacobian elliptic function. The period of oscillation T_0 can be written in terms of the complete elliptic integral $\mathbf{K}(q)$

$$T_0 = \frac{4}{\omega}\mathbf{K}(q) \sim \frac{2\pi}{\omega_0}\left(1 - \frac{3}{8}\frac{E-m}{m}\right), \tag{2.42}$$

where $\omega_0 = \sqrt{\frac{\kappa}{m}}$ is the frequency of the nonrelativistic oscillator.

The solution for the momentum is given by

$$p_1 = \frac{E^2 - m^2}{1+\beta^2 E^2} cd(\omega\tau, q) nd(\omega\tau, q), \tag{2.43}$$

where $cd(\omega\tau, q)$ and $nd(\omega\tau, q)$ are Jacobian elliptic functions.

In terms of the physical time variable t, the solution reads

$$t = (1 + \beta^2 E^2)\tau - \frac{\beta^2(E^2 - m^2)}{\omega} sd(\omega\tau, q)cd(\omega\tau, q)nd(\omega\tau, q), \tag{2.44}$$

and the period of oscillation, given by $T = t(T_0)$,

$$T = \frac{4(1 + \beta^2 E^2)}{\omega}\mathbf{K}(q) = \frac{4}{(1 - \beta^2 K^2)\omega}\mathbf{K}(q), \tag{2.45}$$

contains energy-dependent contributions coming from both special relativity and Snyder dynamics.

2.3 Snyder dynamics in a Schwarzschild spacetime

The validity of noncommutative geometry is presumably limited to Planck-scale physics, so it would be reasonable to assume that its effects cannot be extended to macroscopic systems, where the classical limit holds, e.g. the solar system.

This was confirmed by previous studies of planetary motion based on Snyder dynamics [10], which, when confronted with observations, predict for the coupling constant of the model a scale well bellow the Planck scale, which would be expected on dimensional grounds.

These estimates have however been obtained from a Newtonian theory, while the effect of general relativity cannot be neglected at these scales. Because of this, in [48] we have repeated the calculations in a relativistic setting. The results have partially confirmed those of previous works [10], since the corrections to relativistic dynamics due to Snyder mechanics turn out to be of the same order of magnitude as the ones obtained in the Newtonian approximation, although numerically different.

Our starting point is the classical limit of the Snyder model, which is based on the noncanonical Poisson brackets

$$\{x_\mu, p_\nu\} = \eta_{\mu\nu} + \beta^2 p_\mu p_\nu, \quad \{x_\mu, x_\nu\} = \beta^2 J_{\mu\nu}, \quad \{p_\mu, p_\nu\} = 0, \tag{2.46}$$

where $J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$, and $\eta_{\mu\nu}$ is the flat metric with signature (-1, 1, 1, 1).

The approach to the problem of planetary orbits is rather conservative: the Hamiltonian equation of a free particle in a Schwarzschild background is written down and it is assumed that the only changes in the dynamics are due to the Snyder noncanonical symplectic structure (2.46). In particular, the same Hamiltonian as in general relativity is chosen, even though the Snyder symmetries may allow for more general choices.

2.3.1 Particle motion in flat spacetime

The formalism is set by considering first the free motion of a particle in three-dimensional flat Snyder spacetime. The spatial sections are parametrized with polar coordinates, defined in terms of cartesian coordinates as

$$t = x_0 = -x^0, \quad \rho = \sqrt{(x^1)^2 + (x^2)^2}, \quad \theta = arctan\frac{x^2}{x^1}. \tag{2.47}$$

The corresponding momentum components read

$$p_t = p_0, \quad p_\rho = \frac{x^1 p_1 + x^2 p_2}{\sqrt{(x^1)^2 + (x^2)^2}}, \quad p_\theta \equiv J_{12} = x_1 p_2 - x_2 p_1. \tag{2.48}$$

With these definitions, the Poisson brackets for polar coordinates in Snyder space following from (2.46) are

$$\{t, p_t\} = -1 + \beta^2 p_t^2, \quad \{\rho, p_\rho\} = 1 + \beta^2 \left(p_\rho^2 + \frac{p_\theta^2}{\rho^2}\right), \quad \{\theta, p_\theta\} = 1, \tag{2.49}$$
$$\{\rho, \theta\} = \beta^2 \frac{p_\theta}{\rho}, \quad \{t, \rho\} = \beta^2 \left(t p_\rho - \rho p_t\right), \quad \{t, \theta\} = \beta^2 \frac{t p_\theta}{\rho^2},$$
$$\{p_t, p_\rho\} = -\beta^2 \frac{p_t p_\theta^2}{\rho^3}, \quad \{p_t, p_\theta\} = \{p_\rho, p_\theta\} = \{t, p_\theta\} = \{\rho, p_\theta\} = 0,$$
$$\{t, p_\rho\} = \beta^2 \left(p_t, p_\rho + \frac{t p_\theta^2}{\rho^3}\right), \quad \{\rho, p_t\} = \beta^2 p_t p_\rho, \quad \{\theta, p_t\} = \beta^2 \frac{p_t p_\theta}{\rho^2}, \quad \{\theta, p_\rho\} = \beta^2 \frac{p_\rho p_\theta}{\rho^2}.$$

Here, contrary to the canonical case, the choice of polar coordinates changes the symplectic structure.

The Hamiltonian is chosen as in special relativity,

$$H = \frac{\lambda}{2}\left(-p_t^2 + p_\rho^2 + \frac{p_\theta^2}{\rho^2} + m^2\right) = 0, \tag{2.50}$$

with λ a Lagrange multiplier enforcing the mass shell constraint. The choice of the Hamiltonian is not unique, but (2.50) seems to be the most reasonable in this context.

The Hamiltonian equations derived from (2.49) and (2.50) are

$$\dot{t} = \lambda\Delta p_t, \quad \dot{\rho} = \lambda\Delta p_\rho, \quad \dot{\theta} = \lambda\Delta \frac{p_\theta}{\rho^2},$$
$$\dot{p}_t = 0, \quad \dot{p}_\rho = \lambda\Delta \frac{p_\theta^2}{\rho^3}, \quad \dot{p}_\theta = 0, \tag{2.51}$$

with $\Delta = 1 - \beta^2 m^2$. Hence, as in special relativity, the momenta p_θ and p_t are constants of motion, that according to the standard notations, are denoted as ml and E respectively. They can be identified with the angular momentum and energy of the particle. As in the case of 1+1 dimensions [47], which was discussed in Sec. 2.2.1, all the equations are identical to those of classical relativity, except that they are multiplied by the common factor Δ. Their solution can therefore be obtained as in special relativity, after a redefinition of proper time.

In particular one should choose a gauge by fixing the time variable, in order to eliminate the Hamiltonian constraint (2.50) by means of the Dirac formalism. However, if one is only interested in the equation of the orbits, it is not necessary to fix the gauge since, like in special relativity,

$$\frac{d\rho}{d\theta} = \frac{\dot{\rho}}{\dot{\theta}} = \rho^2 \frac{p_\rho}{p_\theta}, \tag{2.52}$$

does not depend on λ. From the Hamiltonian constraint (2.50), it follows that in flat space

$$p_\rho = \sqrt{E^2 - m^2 \left(1 + \frac{l^2}{\rho^2}\right)}, \tag{2.53}$$

and hence

$$\rho' \equiv \frac{d\rho}{d\theta} = \frac{\rho}{l} \sqrt{\left(\frac{E^2}{m^2} - 1\right)\rho^2 - l^2}, \tag{2.54}$$

which is solved by

$$\rho = \frac{l}{\sqrt{E^2/m^2 - 1}} \frac{1}{cos(\theta - \theta_0)}, \tag{2.55}$$

which describes a straight line in polar coordinates and coincides with the solution of classical special relativity.

2.3.2 Particle motion in Schwarzschild spacetime

We consider next the motion of a planet in the Schwarzschild spacetime with metric

$$ds^2 = -A(\rho)dt^2 + A^{-1}(\rho)d\rho^2 + \rho^2 d\Omega^2, \tag{2.56}$$

where

$$A(\rho) = 1 - \frac{2M}{\rho}, \tag{2.57}$$

and M is the mass of the sun. As in special relativity, due to the conservation of the angular momentum, the problem can be reduced to 2+1 dimensions.

The Hamiltonian is chosen as in standard relativity,

$$H = \frac{\lambda}{2}\left(-\frac{p_t^2}{A} + Ap_\rho^2 + \frac{p_\theta^2}{\rho^2} + m^2\right) = 0, \tag{2.58}$$

where m is the mass of the planet.

The field equations derived from (2.49) and (2.58) are

$$\begin{aligned}
\dot{t} &= \lambda\left[p_t\left(A^{-1} - \beta^2 m^2 - \beta^2\frac{M}{\rho}\left(p_\rho^2 + \frac{p_t^2}{A^2}\right)\right) + \beta^2\frac{Mtp_\rho}{\rho}\left(p_\rho^2 + \frac{p_t^2}{A^2} - 2\frac{p_\theta^2}{\rho^2}\right)\right],\\
\dot{\rho} &= \lambda\left[A - \beta^2 m^2 - \frac{2\beta^2 M p_\theta^2}{\rho^3}\right]p_\rho, \quad \dot{\theta} = \lambda\frac{p_\theta}{\rho^2}\left[1 - \beta^2 m^2 - \frac{\beta^2 M}{\rho}\left(p_\rho^2 + \frac{p_t^2}{A^2}\right)\right],\\
\dot{p}_t &= -\lambda\left[\frac{\beta^2 M p_t p_\rho}{\rho^2}\left(p_\rho^2 - \frac{2p_\theta^2}{\rho^2} + \frac{p_t^2}{A^2}\right)\right], \quad \dot{p}_\theta = 0,\\
\dot{p}_\rho &= \lambda\left[(1-\beta^2 m^2)\frac{p_\theta^2}{\rho^3} - \frac{M}{\rho^2}\left[\left(p_\rho^2 + \frac{p_t^2}{A^2}\right)\left(1 + \beta^2\left(p_\rho^2 + \frac{p_\theta^2}{\rho^2}\right)\right) - 2\beta^2\frac{p_\rho^2 p_\theta^2}{\rho}\right]\right].
\end{aligned} \tag{2.59}$$

The equation of the orbit is obtained as in the case of particle motion in flat spacetime. Now, while p_θ is still a constant, p_t is no longer conserved. Instead, it can be checked that the quantity

$$E = \frac{p_t}{\sqrt{1 + \beta^2(-p_t^2 + p_\rho^2 + p_\theta^2/\rho^2)}} \tag{2.60}$$

is conserved and plays the role of the energy. It follows that

$$p_t^2 = \frac{E^2}{1+\beta^2 E^2}\left[1 + \beta^2\left(p_\rho^2 + p_\theta^2/\rho^2\right)\right]. \tag{2.61}$$

Moreover, (2.58) and (2.61) imply that

$$p_\rho^2 = \frac{E^2(1+\beta^2 m^2 l^2/\rho^2) - m^2(1+\beta^2 E^2)(1 + l^2/\rho^2)A}{(1+\beta^2 E^2)A^2 - \beta^2 E^2}, \tag{2.62}$$

where $l = p_\theta/m$ is defined.

The equation of the orbits is conveniently written in terms of the variable $u = 1/\rho$ as

$$\frac{du}{d\theta} = -\frac{1}{\rho^2}\frac{\dot\rho}{\dot\theta} = -\frac{A - \beta^2 m^2(1 + 2Ml^2u^3)}{1 - \beta^2 m^2 - \beta^2 Mu(p_\rho^2 + p_t^2/A^2)ml}. \tag{2.63}$$

Substituting in (2.63) the values of p_ρ and p_t deduced from (2.61) and (2.62), one can write down a differential equation for the single variable $u(\theta)$.

The calculations are very involved and the equation can only be solved perturbatively. One can first expand in the Snyder parameter $\beta^2 m^2$ and then adopt the usual expansion used in standard textbooks on general relativity to solve for the Schwarzschild orbits. To this end, it is useful to define the dimensionless quantities $v = \frac{l^2}{M}u$ and $\epsilon = \frac{M^2}{l^2}$. The parameter ϵ is small for planetary orbits, and can be taken as an expansion parameter. Moreover, it is assumed that $\beta^2 m^2 \ll \epsilon$, since the Snyder corrections are expected to be small with respect to those of general relativity. Also, by the viral theorem, and the definition (2.60) of E, $E^2 - m^2 \sim m^2(\epsilon q + \beta^2 E^2)$, with q a parameter of order unity.

The first-order expansion in both $\beta^2 m^2$ and ϵ gives, after lengthy calculations,

$$v'^2 = q + 2v - v^2 + 2\epsilon v^3 + \beta^2 m^2(2v + 4\epsilon(qv + v^2)). \tag{2.64}$$

It is convenient to take the derivative of this expression, which gives

$$v'' = 1 + \beta^2 m^2 - v + \epsilon(3v^2 + \beta^2 m^2(2q + 4v)). \tag{2.65}$$

Expanding $v = v_0 + \epsilon v_1 + ...$, at zeroth order one obtains a Newtonian approximation of the solution

$$v_0 = 1 + \beta^2 m^2 + e\,cos\theta, \quad e = 1 + \frac{q}{\epsilon} = 1 + \frac{l^2(E^2 - m^2)}{M^2 m^2}, \tag{2.66}$$

while v_1 satisfies

$$v_1'' + v_1 = 3 + (10 + 2q)\beta^2 m^2 + 2(3 + 5\beta^2 m^2)e\,cos\theta + 3e^2 cos^2\theta, \tag{2.67}$$

which is solved by

$$v_1 = 3\left(1 + \frac{e^2}{2}\right) + 2\beta^2 m^2(5 + q^2) + e(3 + 5\beta^2 m^2)\theta\, sin\theta - \frac{e^2}{2}cos2\theta. \tag{2.68}$$

The solution at first order is therefore

$$\begin{aligned} v \quad \approx \quad & (1 + \beta^2 m^2) + \epsilon\left[3\left(1 + \frac{e^2}{2}\right) + 2\beta^2 m^2(5 + q^2)\right] + e\, cos\left[(1 - \epsilon(3 + 5\beta^2 m^2))\theta\right] \\ & -\frac{\epsilon}{2}e^2 cos2\theta. \end{aligned} \tag{2.69}$$

From this expression one can easily obtain the perihelion shift as

$$\delta\theta = 2\pi\epsilon(3 + 5\beta^2 m^2) \approx \frac{6\pi M^2}{l^2}\left(1 + \frac{5}{3}\beta^2 m^2\right). \tag{2.70}$$

The first term is of course the one predicted by general relativity, while the second depends on the mass of the planet. This dependence is of course a consequence of the breaking of the equivalence principle in Snyder mechanics.

It turns out that the Schwarzschild geodesics are slightly deformed. In particular, a shift of the perihelion arises in addition to that predicted by general relativity. In a Newtonian setting, the shift due to Snyder mechanics is given by $\delta\theta = -2\pi\beta^2 m^2 M^2/l^2$ [10]. While the order of magnitude of the Snyder correction is the same as that obtained from the relativistic model, its sign is opposite. Therefore, calculations based on Newtonian mechanics are not very reliable in this context. In any case, it has been shown [10] that for these corrections to be compatible with the observed discrepancy of the perihelion shift of Mercury from the predictions of general relativity, β must be less than 10^{-9} in Planck units. This estimate remains true in the relativistic case.

Another important outcome of the investigation is that the principle of equivalence is broken in Snyder mechanics, since the corrections to the equation of the geodesics depend on a parameter $\beta^2 m^2$, which is a function of the mass m of the particle. This effect is a consequence of the nontrivial dependence of the dynamics on the momenta of the particles, and it also puts strong limits on the value of the coupling constant β if the validity of Snyder mechanics at planetary scale is assumed. Experimental data show that violation of the equivalence principle are less than one part in 10^{12} [58]. It follows that $\beta < 10^{-26}$ in Planck units for planetary masses of order $10^{24}kg = 10^{32}M_{Pl}$. This bound is even stronger than the previous one.

These results seem to indicate that if one assumes that Snyder mechanics holds at scales compatible with the orbit of planets, the coupling constant β must be less than its natural value of order 1 in Planck units by many orders of magnitude.

This is expected, as the limitation of the validity of Snyder mechanics to microscopic physics should be justified. This problem can be related to the soccer-ball problem of doubly special relativity [33]. In fact, in Snyder spacetime the summation rules for the momenta must be nonlinear, since the translation invariance is deformed [7], and, following a reasoning analogous to that of [33], should be arranged in such a way that classical mechanics holds at macroscopic scales. A related argument, that has not been thoroughly investigated yet, is that passing from the quantum-gravity regime to its classical limit some kind of decoherence should occur so that classical mechanics is recovered, as in the classical limit of quantum mechanics. However, the detailed mechanism of this transition from the Plank to the larger scales has not been figured out yet.

Chapter 3

Path integral in Snyder space

The Feynman path integral provides an alternative formulation of quantum mechanics and it rests on the idea that the integral kernel (i.e. the propagator) of the time-evolution operator, which is the amplitude for a particle to get from the point a to the point b, can be expressed as a sum over all possible paths connecting the two points, where each path's contribution is weighed by a factor which is proportional to the action. Hence, the probability $P(b, a)$ to go from a point x_a at the time t_a to the point x_b at t_b is the absolute square $P(b, a) = |K(b, a)|^2$ of an amplitude $K(b, a)$ to go from a to b, and this amplitude is the sum of contributions $\Phi[x(t)]$ from each path

$$K(b, a) = \Sigma_{over\,all\,paths} \Phi[x(t)], \tag{3.1}$$

where the contribution of a path has a phase proportional to the action S (of the corresponding classical system)

$$\Phi[x(t)] = const.\, e^{iS[x(t)]}. \tag{3.2}$$

A characteristic of noncommutative spaces is that the corresponding classical phase space is not canonical, i.e. the Poisson brackets do not have the usual form. However, since the standard definition of the path integral assumes a canonical phase space, it is necessary to extend the standard formalism to include this more general situation.

This problem has been afforded in a variety of ways and several different approaches have been proposed for the definition of the path integral on noncommutative spaces. The first is based on the noncanonical structure of the phase space: Darboux's theorem ensures that it is always possible to find a transformation to canonical (and hence commutative) coordinates, which will deform the measure of the integral, but otherwise allow the use of the standard formulation of the path integral [36, 3, 13]. A different approach uses the standard integration measure, but treats the products in the integrand as star products between functions of noncommutative coordinates [16]. This second framework is more suitable for a generalisation to field theory. Alternatively, some authors propose the adoption of smeared (coherent state) bases for the Hilbert space to avoid the use of noncommutative coordinates in the computation of the path integral [55]. However, most of the work on this subject has been developed for the so-called Moyal plane [51], the most simple type of noncommutative space, whose Poisson brackets are constant tensors.

The chapter begins with a brief survey of the usual procedure for the reduction of Lagrange's equations to the Hamiltonian form, following the presentation of [53], after which the Faddeev-Jackiw formalism for the Hamiltonian reduction of unconstrained and constrained systems is reviewed. It ends with a presentation of the results obtained in [49], where these methods have been used for the Snyder model.

3.1 Reduction of Lagrange's equations to the Hamiltonian form

The usual procedure [53] for the reduction of Lagrange's equations to the Hamiltonian form starts with the prescription for the canonical momenta

$$p_i = \frac{\partial L}{\partial \dot{x}_i}, \quad i = 1, ..., n. \tag{3.3}$$

Assuming that the Lagrangian is not singular

$$det\left(\frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j}\right) \neq 0, \tag{3.4}$$

the velocities $\dot{x}_i$, and hence also the Hamiltonian, can be expressed in terms of the canonical variables, and from the Lagrange's equations, one gets the Hamiltonian equations in the usual form

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}, \quad i = 1, ..., n. \tag{3.5}$$

These can be written in a unified form

$$\dot{\xi}_i - \Omega_{ij}\frac{\partial H}{\partial \xi_j} = 0, \quad i, j = 1, .., 2n \tag{3.6}$$

where

$$\xi_i = \begin{cases} x_i, & i = 1, ..., n \\ p_i, & i = n+1, ..., 2n, \end{cases} \tag{3.7}$$

$$\Omega_{ij} = \begin{pmatrix} 0_{n\times n} & 1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{pmatrix}. \tag{3.8}$$

An equivalent form of these equations is given by

$$(\Omega^{-1})^{ij}\dot{\xi}_j - \frac{\partial H}{\partial \xi_i} = 0, \tag{3.9}$$

where

$$(\Omega^{-1})^{ij} = \begin{pmatrix} 0_{n\times n} & -1_{n\times n} \\ 1_{n\times n} & 0_{n\times n} \end{pmatrix}. \tag{3.10}$$

The action can also be written in a unified way

$$S = \int dt\left(p^i \dot{x}_i - H\right) \tag{3.11}$$

$$\equiv \int dt\left(a^i(\xi)\dot{\xi}_i - H(\xi)\right), \tag{3.12}$$

with

$$a^i = \begin{cases} p^i, & i = 1, ..., n \\ 0, & i = n+1, ..., 2n. \end{cases} \tag{3.13}$$

From the variational principle, one then gets Hamilton's equations in the following form

$$\left(\frac{\partial a^j}{\partial \xi_i} - \frac{\partial a^i}{\partial \xi_j}\right)\dot{\xi}_j - \frac{\partial H}{\partial \xi_i} \equiv (\Omega^{-1})^{ij}\dot{\xi}_j - \frac{\partial H}{\partial \xi_i} = 0, \tag{3.14}$$

where the following identity has been used

$$\frac{\partial a^j}{\partial \xi_i} - \frac{\partial a^i}{\partial \xi_j} \equiv (\Omega^{-1})^{ij}, \quad i, j = 1, ..., 2n. \tag{3.15}$$

3.2 The Faddeev-Jackiw formalism

An alternative approach to the Dirac method for the Hamiltonian reduction of unconstrained and constrained systems, which is based on Darboux's theorem, has been developed by L. Faddeev and R. Jackiw [14, 23].

The starting point of this method is a first-order Lagrangian, i.e., one that is linear in time derivatives, noting that this does not necessarily imply the existence of constraints. In fact, using the same Legendre transform which is used to pass from a Lagrangian to Hamiltonian, a second-order Lagrangian can be transformed into a first-order one. This means that, for a given Hamiltonian description of dynamics, one can always construct a first-order Lagrangian whose configuration space coincides with the Hamiltonian phase space.

Thus, the starting point is a general first-order Lagrangian

$$L = a^i(\xi)\dot{\xi}_i - V(\xi), \tag{3.16}$$

where a^i has the character of a vector potential for an Abelian gauge theory, as modifying $a^i(\xi)$ by a total derivative does not affect the dynamics, because it would change the Lagrangian by a total time derivative. Furthermore, since the Lagrangian is first-order in $\dot{\xi}_i$, velocities don't appear in the combination $\frac{\partial L}{\partial \dot{\xi}_i}\dot{\xi}_i$, so if the Hamiltonian is defined with the usual Legendre transform, it can be identified with V

$$H = \frac{\partial L}{\partial \dot{\xi}_i}\dot{\xi}_i - L = V, \tag{3.17}$$

and the Lagrangian can be written as

$$L = a^i(\xi)\dot{\xi}_i - H(\xi), \tag{3.18}$$

where the first term of the right side defines the canonical one-form $a^i(\xi)d\xi_i \equiv a(\xi)$.

An introduction to the method is given by considering the special case of its simplest realisation, in which $a^i(\xi)$ is linear in ξ_i

$$a^i(\xi) = \frac{1}{2}\xi_j\omega^{ji}, \tag{3.19}$$

where the constant matrix ω^{ij} is antisymmetric, since any symmetric part would only contribute a total time derivative to L. The Euler-Lagrange equations that follow are

$$\omega^{ij}\dot{\xi}_j = \frac{\partial H(\xi)}{\partial \xi_i}. \tag{3.20}$$

If the antisymmetric matrix ω^{ij} possesses an inverse, $(\omega^{-1})_{ij}$, in which case it has to be even-dimensional, i.e. $i, j = 1, ..., N = 2n$, it follows that ξ_i satisfy the evolution equation

$$\dot{\xi}_i = (\omega^{-1})_{ij} \frac{\partial H(\xi)}{\partial \xi_j}, \tag{3.21}$$

and there are no constraints. With the identification $\omega^{ij} \leftrightarrow (\Omega^{-1})^{ij}$, eqs. (3.20) and (3.21) coincide with (3.9) and (3.6) respectively.

The classical Poisson brackets are defined in such a way that they reproduce these equations by commutation with the Hamiltonian

$$\begin{aligned} \dot{\xi}_i &= (\omega^{-1})_{ij} \frac{\partial H(\xi)}{\partial \xi_j} = \{H(\xi), \xi_i\} \\ &= \{\xi_j, \xi_i\} \frac{\partial H(\xi)}{\partial \xi_j}, \end{aligned} \tag{3.22}$$

which implies that

$$\{\xi_i, \xi_j\} = (\omega^{-1})_{ij}, \tag{3.23}$$

and for general functions of ξ

$$\{A(\xi), B(\xi)\} = \frac{\partial A(\xi)}{\partial \xi_i} (\omega^{-1})_{ij} \frac{\partial B(\xi)}{\partial \xi_j}. \tag{3.24}$$

In the more general case, where $a^i(\xi)$ is an arbitrary function of ξ_i, that does not depend on time, the Euler-Lagrange equations of (3.18) are

$$(\Omega^{-1})^{ij}(\xi)\dot{\xi}_j = \frac{\partial H(\xi)}{\partial \xi_i}, \tag{3.25}$$

where

$$(\Omega^{-1})^{ij}(\xi) = \frac{\partial a^j(\xi)}{\partial \xi_i} - \frac{\partial a^i(\xi)}{\partial \xi_j}. \tag{3.26}$$

$(\Omega^{-1})^{ij}(\xi)$ behaves as a gauge invariant field strength and is called the symplectic two-form, $\frac{1}{2}(\Omega^{-1})^{ij}(\xi) d\xi_i d\xi_j = \Omega^{-1}(\xi)$. Since $\Omega^{-1} = da$ it is exact and hence also closed $d\Omega^{-1} = 0$.

For the non-singular, unconstrained case, the antisymmetric $N \times N$, $(N = 2n)$, matrix

$(\Omega^{-1})^{ij}$ has an inverse and the Euler-Lagrange equations imply

$$\dot{\xi}_i = \Omega_{ij}(\xi)\frac{\partial H(\xi)}{\partial \xi^j}. \tag{3.27}$$

In order to reproduce this evolution equation by commutation with H, the bracket needs to be chosen as

$$\{\xi_i, \xi_j\} = \Omega_{ij}(\xi). \tag{3.28}$$

Even though the case of an arbitrary $a^i(\xi)$ and unconstrained dynamics seems more general, it is actually included in the special case where $a^i(\xi)$ is linear in ξ. Using Darboux's theorem, it can be shown that an arbitrary vector potential (one-form $a^i d\xi_i$) whose associated field strength (two-form $d(a^i d\xi_i) = \frac{1}{2}(\Omega^{-1})^{ij} d\xi_i d\xi_j$ is non-singular (the matrix $(\Omega^{-1})^{ij}$ possesses an inverse), can be mapped by a coordinate transformation onto (3.19) with ω^{ij} non-singular. Because of this, apart from a gauge term, $a_i(\xi)$ can always be expressed as

$$a^i(\xi) = \frac{1}{2}\zeta_k(\xi)\omega^{kl}\frac{\partial \zeta_l(\xi)}{\partial \xi_i}, \tag{3.29}$$

and $(\Omega^{-1})^{ij}(\xi)$ as

$$(\Omega^{-1})^{ij}(\xi) = \frac{\partial \zeta_k(\xi)}{\partial \xi_i}\omega^{kl}\frac{\partial \zeta_l(\xi)}{\partial \xi_j}. \tag{3.30}$$

In terms of the new coordinates ζ_i the field strength is ω^{ij}, a constant and non-singular matrix. Furthermore, modifying the Gram-Schmidt argument, it is possible to construct a basis in which the antisymmetric $N \times N$ matrix ω_{ij} takes the block-off-diagonal form

$$\omega^{ij} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{3.31}$$

where I is the n-dimensional unit matrix ($N = 2n$).

When writing down the path integral for the non-singular, first-order dynamics, the action corresponding to the Lagrangian (3.18) is simply given by

$$S = \int (a^i(\xi)d\xi_i - H(\xi))dt, \tag{3.32}$$

but the measure is not minimal, instead the path integral is given by

$$A = \int \mathcal{D}\xi_i \left|det\,\Omega_{jk}\right|^{-\frac{1}{2}} e^{iS}. \tag{3.33}$$

The reason for the factor $\left|det\,\Omega_{jk}\right|^{-\frac{1}{2}}$ is that when going from the canonical variables ζ_i to the coordinates ξ_i, the Jacobian of the transformation appears in the path integral measure,

$$\prod \mathcal{D}\zeta_i = det\left|\frac{\partial\zeta_i}{\partial\xi_j}\right| \prod \mathcal{D}\xi_j = \sqrt{det(\Omega^{-1})^{ij}} \prod \mathcal{D}\xi_i. \tag{3.34}$$

3.3 Construction of the path integral for the Snyder space

We discuss the definition of path integrals, both in the traditional setting, which was recalled in Sec. 3.1, and in the first-order formalism of Faddeev and Jackiw, described in Sec. 3.2, for one- and two-dimensional Snyder space [49]. In its nonrelativistic version the Snyder space commutation relations are given by

$$[\hat{x}_i, \hat{p}_j] = i(\delta_{ij} + \beta^2 \hat{p}_i \hat{p}_j), \quad [\hat{x}_i, \hat{x}_j] = i\beta^2 \hat{J}_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0. \tag{3.35}$$

Following the Hamiltonian formalism of the two previous sections, we consider non-canonical Poisson brackets of the type

$$\{\xi_i, \xi_j\} = \Omega_{ij}(\xi), \tag{3.36}$$

with ξ_i denoting the phase space variables x_i and p_i, and Ω_{ij} an invertible matrix. The

Hamiltonian equations for the Hamiltonian $H(\xi)$ are then still given by (3.6) or equivalently (3.9), but for the Snyder space $\Omega_{ij}(\xi)$ is no longer given by (3.8).

If one wants to obtain these equations from the variation of a first-order action of the form (3.12), the condition (3.15) still needs to hold, however, a^i will now not be given by (3.13). Going in the other direction, knowing the form of the matrix Ω_{ij}, one can solve (3.15) for the a^i and thus write down the action that generates the Hamiltonian equations (3.6).

3.3.1 One-dimensional Snyder path integral

Even though noncommutativity is absent in the case of the one-dimensional Snyder model, the symplectic structure is still noncanonical, and thus an investigation of the one-dimensional Snyder model is useful for understanding the higher-dimensional case.

When investigating the Snyder model, it is necessary to use the phase space formulation of the path integral. For a particle satisfying canonical Poisson brackets, moving in a one-dimensional space, the path integral is defined as

$$A = \int \mathcal{D}p\mathcal{D}x\, e^{iS}, \tag{3.37}$$

where

$$S = \int_{t_i}^{t_f} L dt = \int_{t_i}^{t_f} (p\dot{x} - H(x,p))dt \tag{3.38}$$

is the action and $\mathcal{D}p\mathcal{D}x$ is a measure on the space of paths in phase space that will be defined bellow.

It can be shown that in a momentum basis the transition amplitude $A(p_f, p_i)$ from an initial state of momentum p_i at time t_i to a final state of momentum p_f at time t_f is given by

$$A(p_f, p_i) = \left\langle p_f \left| e^{-i\hat{H}(t_f - t_i)} \right| p_i \right\rangle. \tag{3.39}$$

A momentum basis is chosen because, when considering Snyder space, the standard position variables do not commute and hence do not form a complete set of observables.

The aim is to generalise this formula to the one-dimensional Snyder phase space, whose only nontrivial Poisson bracket is

$$\{x, p\} = 1 + \beta^2 p^2. \tag{3.40}$$

Given the Hamiltonian $H = p^2/2 + V(x)$, the Hamiltonian equations in Snyder space read

$$\dot{x} = (1 + \beta^2 p^2)p, \quad \dot{p} = -(1 + \beta^2 p^2)\frac{\partial V}{\partial x}. \tag{3.41}$$

These equations can be obtained from an action principle following the procedure of the previous section. Defining $\xi_1 = x$, $\xi_2 = p$, the inverse of the symplectic matrix associated to (3.40) is

$$(\Omega^{-1})^{ij} = \frac{1}{1 + \beta^2 p^2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.42}$$

Inserting into (3.15), one can get the particular solution

$$a_1 = 0, \quad a_2 = \frac{-x}{1 + \beta^2 p^2}, \tag{3.43}$$

from where follows the action

$$S = \int \left(-\frac{x\dot{p}}{1 + \beta^2 p^2} - H \right) dt = \int \left(\frac{arctan\,\beta p}{\beta}\dot{x} - H \right) dt, \tag{3.44}$$

where the two expressions are related by an integration by parts. It is next shown that inserting (3.44) into (3.37) gives the correct expression for the path integral.

When considering the quantum mechanics of the one-dimensional Snyder model [26], the Poisson bracket (3.40) goes into the commutator

$$[\hat{x}, \hat{p}] = i(1 + \beta^2 \hat{p}^2). \tag{3.45}$$

The operators $\hat{x}$ and $\hat{p}$ satisfying (3.45) can be represented in a momentum basis by [22]

$$\hat{p} = p, \quad \hat{x} = i(1 + \beta^2 p^2)\frac{\partial}{\partial p}. \tag{3.46}$$

These operators are hermitian with respect to the scalar product

$$\langle\psi|\phi\rangle = \int_{-\infty}^{+\infty} \frac{dp}{1 + \beta^2 p^2}\psi^*(p)\phi(p). \tag{3.47}$$

The identity operator can therefore be expanded in terms of momentum eigenstates $|p\rangle$ as [26]

$$1 = \int_{-\infty}^{+\infty} \frac{dp}{1 + \beta^2 p^2}|p\rangle\langle p|, \tag{3.48}$$

with $\langle p|p'\rangle = (1 + \beta^2 p^2)\delta(p - p')$.

The eigenvalue equation for the position operator, $\hat{x}\,|x\rangle = x\,|x\rangle$, has formal solutions (which are not physical, because they have infinite energy [26], but are sufficiently regular to adopt them in this setting)

$$\langle p|x\rangle \propto e^{-ix\frac{arctan\beta p}{\beta}}. \tag{3.49}$$

These states form an overcomplete set. However, one can choose a discrete basis with $x = 2\beta n$, n integer, which satisfies the completeness relation [26]

$$1 = \int dx\,|x\rangle\,\langle x|\,, \tag{3.50}$$

with $\langle x|x'\rangle = \delta_{xx'}$ and, for simplicity of notation, an integral sign is used for the infinite sum over n.

Going back to the path integral, splitting the interval $t_f - t_i$ into N intervals of equal length $\epsilon = t_k - t_{k-1}$, the Trotter product formula gives

$$A = \lim_{\epsilon\to 0}\left\langle p_f \left|\left(e^{-i\epsilon\frac{\hat{p}^2}{2}}\, e^{-i\epsilon V(\hat{x})}\right)^{1/\epsilon}\right| p_i\right\rangle. \tag{3.51}$$

This follows from the Baker-Campbell-Hausdorff formula and the fact that the commutator terms are higher order in $1/N$ and then vanish for $N \to \infty$, independently from the deformation of the symplectic structure [54].

Inserting the completeness relations (3.48) and (3.50) between each pair of operators, this reduces to

$$A = \lim_{N\to\infty} \int \prod_{k=1}^{N-1} \frac{dp_{(k)}}{1+\beta^2 p_{(k)}^2} \int \prod_{k=1}^{N} dx_{(k)} \prod_{k=1}^{N} \langle p_{(k)}|x_{(k)}\rangle \left\langle x_{(k)} \left| e^{-i\epsilon\hat{H}} \right| p_{(k-1)} \right\rangle . \tag{3.52}$$

Recalling that

$$\begin{aligned} \left\langle x_{(k)} \left| e^{-i\epsilon\frac{\hat{p}^2}{2}} \right| p_{(k-1)} \right\rangle &= e^{-i\epsilon\frac{p^2}{2}} \langle x_{(k)}|p_{(k-1)}\rangle , \\ \left\langle p_{(k)} \left| e^{-i\epsilon V(\hat{x})} \right| x_{(k)} \right\rangle &= e^{-i\epsilon V(x)} \langle p_{(k)}|x_{(k)}\rangle , \end{aligned} \tag{3.53}$$

and taking into account (3.49),one then obtains

$$A = \lim_{N\to\infty} \int \prod_{k=1}^{N-1} \frac{dp_{(k)}}{1+\beta^2 p_{(k)}^2} \int \prod_{k=1}^{N} dx_{(k)}\, e^{-i\sum_{k=1}^{N}\left[\frac{1}{2}\epsilon p_{(k-1)}^2 + \epsilon V(x_{(k)}) + \frac{1}{\beta} x_{(k)}(arctan\,\beta p_{(k)} - arctan\,\beta p_{(k-1)})\right]}. \tag{3.54}$$

Finally, in the limit $\epsilon \to \infty$,

$$\frac{arctan\,\beta p_{(k)}}{\beta} - \frac{arctan\,\beta p_{(k-1)}}{\beta} \approx \frac{\dot{p}_{(k)}\epsilon}{1+\beta^2 p_{(k)}^2}, \tag{3.55}$$

and hence one recovers for A the form (3.37) with S given by (3.44) and

$$\mathcal{D}p = \lim_{N\to\infty} \prod_{k=1}^{N-1} \frac{dp_{(k)}}{1+\beta^2 p_{(k)}^2}, \quad \mathcal{D}x = \lim_{N\to\infty} \prod_{k=1}^{N} dx_{(k)}, \tag{3.56}$$

which proves the claim. This result was previously obtained in [57], but with an incorrect measure for the path integral.

3.3.2 Two-dimensional Snyder path integral

In higher dimensions the problem is more difficult since the position operators $\hat{x}_i$ do not commute and their eigenfunctions cannot be taken as a basis for the Hilbert space. The Poisson brackets in D dimensions are

$$\{x_i, p_j\} = \delta_{ij} + \beta^2 p_i p_j, \quad \{x_i, x_j\} = \beta^2 J_{ij}, \quad \{p_i, p_j\} = 0, \tag{3.57}$$

where $J_{ij} = p_j x_i - p_i x_j$, $i = 1, ..., D$.

However, most problems in higher dimensions are most conveniently addressed using (hyper-)spherical coordinates. In particular, the $D = 2$ is discussed, for which polar coordinates are chosen to parametrise the momentum space, and their canonically conjugate variables for the position space. More precisely, it is defined

$$p_\rho = \sqrt{p_1^2 + p_2^2}, \quad p_\theta = arctan\,\frac{p_1}{p_2}, \tag{3.58}$$

and

$$\rho = \frac{p_1 x_1 + p_2 x_2}{\sqrt{p_1^2 + p_2^2}}, \quad J = J_{12} = p_2 x_1 - p_1 x_2. \tag{3.59}$$

The position coordinates ρ and J essentially correspond to the parallel and orthogonal components of the position vector with respect to the momentum of the particle. The phase space polar coordinates defined above obey the Poisson brackets

$$\begin{aligned} &\{p_\rho, p_\theta\} = 0, \quad \{\rho, J\} = 0, \quad \{p_\rho, J\} = 0, \\ &\{\rho, p_\rho\} = 1 + \beta^2 p_\rho^2, \quad \{J, p_\theta\} = 1, \quad \{p_\theta, \rho\} = 0. \end{aligned} \tag{3.60}$$

Since the Poisson bracket of ρ and J vanishes, the corresponding quantum operators commute and form a basis for the position space. An alternative basis would be constituted by the operators $\hat{r}^2$ and $\hat{J}$, where $\hat{r}^2 = \hat{x}_1^2 + \hat{x}_2^2$ [32]. These coordinates however give rise to more complicated formulas.

Defining $\xi_1 = \rho$, $\xi_2 = J$, $\xi_3 = p_\rho$ and $\xi_4 = p_\theta$, the symplectic matrix associated with the Poisson brackets (3.60) takes the simple form

$$\Omega_{ij} = \begin{pmatrix} 0 & 0 & 1+\beta^2 p_\rho^2 & 0 \\ 0 & 0 & 0 & 1 \\ -(1+\beta^2 p_\rho^2) & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{3.61}$$

with inverse

$$(\Omega^{-1})^{ij} = \frac{1}{1+\beta^2 p_\rho^2} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -(1+\beta^2 p_\rho^2) \\ 1 & 0 & 0 & 0 \\ 0 & 1+\beta^2 p_\rho^2 & 0 & 0 \end{pmatrix}. \tag{3.62}$$

A solution of eq. (3.15) for this symplectic structure is then

$$a_1 = a_2 = 0, \quad a_3 = \frac{-\rho}{1+\beta^2 p_\rho^2}, \quad a_4 = -J, \tag{3.63}$$

from which one can write the action that generates the classical Hamilton equations. This is

$$S = -\int \left(\frac{\rho \dot{p}_\rho}{1+\beta^2 p_\rho^2} + J \dot{p}_\theta + H \right) dt = \int \left(\frac{\arctan \beta p_\rho}{\beta} \dot{\rho} + p_\theta \dot{J} - H \right) dt, \tag{3.64}$$

where the two expressions are related by an integration by part, and $H = \frac{p_\rho^2}{2} + V$.

When considering the quantum theory, the quantum operators must be defined carefully, because of ordering ambiguities. The ordering that is adopted is the one with $\hat{p}_i$ always on the left. The commutation relations are

$$[\hat{x}_i, \hat{p}_j] = i(\delta_{ij} + \beta^2 \hat{p}_i \hat{p}_j), \quad [\hat{x}_i, \hat{x}_j] = i\beta^2 \hat{J}_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0. \tag{3.65}$$

A representation of (3.65) is given by [22]

$$\hat{p}_i = p_i, \quad \hat{x}_i = i\left(\frac{\partial}{\partial p_i} + \beta^2 p_i p_j \frac{\partial}{\partial p_j}\right). \tag{3.66}$$

The hermitian operators corresponding to the classical polar coordinates are

$$\hat{p}_\rho = \sqrt{p_i^2} \equiv p_\rho, \quad \hat{p}_\theta = arctan\,\frac{p_1}{p_2} \equiv p_\theta, \tag{3.67}$$

and

$$\hat{\rho} = i(1+\beta^2 p_\rho^2)\left(\frac{\partial}{\partial p_\rho} + \frac{1}{2p_\rho}\right), \quad \hat{J} = i\frac{\partial}{\partial p_\theta}, \tag{3.68}$$

where the scalar product

$$\langle\psi|\phi\rangle = \int_{-\infty}^{+\infty} \frac{p_\rho dp_\rho dp_\theta}{1+\beta^2 p_\rho^2}\psi^*(p)\phi(p) \tag{3.69}$$

is understood. The completeness relations for momentum eigenstates are therefore

$$\int_{-\infty}^{+\infty} \frac{p_\rho dp_\rho}{1+\beta^2 p_\rho^2}\int_0^{2\pi} dp_\theta \left|p_\rho, p_\theta\right\rangle\left\langle p_\rho, p_\theta\right| = 1. \tag{3.70}$$

The eigenvalue equations for the position operators read (in this case as well, the eigenfunctions are not physical because their energy diverges)

$$\hat{\rho}\,|\rho\rangle = \rho\,|\rho\rangle\,, \quad \left\langle p_\rho|\rho\right\rangle \propto \frac{1}{\sqrt{p_\rho}}e^{-i\rho\frac{arctan\beta p_\rho}{\beta}}, \tag{3.71}$$

and

$$\hat{J}\,|J\rangle = J\,|J\rangle\,, \quad \langle p_\theta|J\rangle \propto e^{-iJp_\theta}, \tag{3.72}$$

with integer J.

Defining a basis $|\rho, J\rangle = |\rho\rangle\,|J\rangle$, one has

$$\left\langle p_\rho, p_\theta|\rho, J\right\rangle = \frac{1}{\sqrt{p_\rho}}e^{-i\left(\rho\frac{acrtan\beta p_\rho}{\beta}+Jp_\theta\right)}, \tag{3.73}$$

with

$$\int_0^\infty d\rho \int_{-\infty}^\infty dJ \, |\rho, J\rangle \langle \rho, J| = 1. \tag{3.74}$$

Proceeding as in the one-dimensional case, one can show that for infinitesimal ϵ

$$\left\langle \rho, J \left| e^{-i\epsilon H} \right| p_\rho, p_\theta \right\rangle \sim \frac{1}{\sqrt{p_\rho}} e^{i\left(\rho \frac{\arctan \beta p_\rho}{\beta} + J p_\theta - \epsilon H\right)}, \tag{3.75}$$

and hence one obtains in the limit $\epsilon \to 0$ the formula (3.37) with action (3.64), and the measure adapted to two dimensions

$$\mathcal{D}p = \lim_{N\to\infty} \prod_{k=1}^{N-1} \int \frac{d^2 p_{(k)}}{1 + \beta^2 p_{(k)}^2}, \quad \mathcal{D}x = \lim_{N\to\infty} \prod_{k=1}^{N} \int d^2 x_{(k)}, \tag{3.76}$$

where $d^2 p_{(k)} = p_{\rho(k)} dp_{\rho(k)} dp_{\theta(k)}$ and $d^2 x_{(k)} = d\rho_{(k)} dJ_{(k)} / p_{\rho(k)}$.

In terms of the previous coordinates, the classical Hamiltonian $H = \frac{p_\rho^2}{2} + V(r^2)$ takes the form $H = \frac{p_\rho^2}{2} + V\left(\rho^2 + \frac{J^2}{p_\rho^2}\right)$. However, it is known that in order to obtain the correct result for the path integral in polar coordinates, that takes into account the hermitian nature of the operator $\hat{\rho}^2$, an additional term $-1/2p_\rho^2$ must be added to the classical two-dimensional action [15]. Hence, the correct effective potential will be $V = V\left(\rho^2 + \frac{J^2 - 1/2}{p_\rho^2}\right)$.

3.3.3 Faddeev-Jackiw formalism

The previous results can also be obtained in an easier way just assuming the validity of the canonical path integral framework and employing the first-order formalism introduced by Faddeev and Jackiw. As it was shown in Sec. 3.2, using a Darboux transformation from the original variables ξ_i to new canonical variables, that for noncanonical variables the path integral can be written in the form of (3.33), and S is given by (3.32) [23].

For the Poisson brackets (3.57), it can be shown by induction that for the Snyder model in any dimension

$$\left| det \Omega_{ij} \right| = (1 + \beta^2 p_i^2)^2, \tag{3.77}$$

from where the measures (3.56) and (3.76) follow.

In this framework the explicit form of the Darboux's transformation is not needed. However, several possibilities are known for the Snyder model, e.g., in terms of canonical variables P_i and X_i, one may choose $p_i = P_i$, $x_i = X + \beta^2 X_k P_k P_i$ [56]. A different transformation was proposed in [44, 45], with $p_i = P_i/\sqrt{1-\beta^2P^2}$, $x_i = \sqrt{1-\beta^2P^2}X_i$. All these choices give equivalent results, as it also follows from the study of the Schrödinger equation [44, 45, 32].

3.3.4 Two-dimensional examples

If the formalism is applied to the case of a free particle, integration over the angular variables p_θ and J simply yields a delta function $\delta\left(p_\theta^{(i)} - p_\theta^{(f)}\right)$, and one is left with an integral over the radial coordinates,

$$\int \mathcal{D}\rho \int \mathcal{D}p_\rho \exp\left[i\int\left(\frac{\rho\dot{p}_\rho}{1+\beta^2p_\rho^2} + \frac{p_\rho^2}{2}\right)dt\right], \tag{3.78}$$

where

$$\mathcal{D}p_\rho = \lim_{N\to\infty}\prod_{k=1}^{N-1}\int\frac{dp_{\rho(k)}}{1+\beta^2p_{\rho(k)}^2}. \tag{3.79}$$

Performing a change of variables$P_\rho = \beta^{-1}arctan\,\beta p_\rho$, one gets

$$\int \mathcal{D}\rho \int \mathcal{D}P_\rho \exp\left[i\int\left(\rho\dot{P}_\rho + \frac{tan^2P_\rho}{2}\right)dt\right], \tag{3.80}$$

with

$$\mathcal{D}P_\rho = \lim_{N\to\infty}\prod_{k=1}^{N-1}\int dP_\rho^{(k)}. \tag{3.81}$$

The integration on ρ gives in turn a delta function $\delta\left(P_\rho^{(i)} - P_\rho^{(f)}\right)$ and one is left with

$$\int \mathcal{D}P_\rho \exp\left[\frac{i}{2}\int tan^2P_\rho dt\right]. \tag{3.82}$$

In the harmonic oscillator case, the classical potential is $V = \omega^2r^2 \to \omega^2\left(\rho^2 + \frac{J^2-1/4}{p_\rho^2}\right)$.

It is convenient to integrate first in p_θ, getting the conservation of angular momentum, $\delta\left(J^{(i)} - J^{(f)}\right)$. The integral then reduces to a sum on different J sectors

$$\int \mathcal{D}\rho \int \frac{\mathcal{D}p_\rho}{1+\beta^2 p_\rho^2} \exp\left[i \int \left(\frac{\rho \dot{p}_\rho}{1+\beta^2 p_\rho^2} + \frac{p_\rho^2}{2} + \omega^2\rho^2 + \omega^2 \frac{J^2 - 1/4}{p_\rho^2}\right) dt\right]. \tag{3.83}$$

Defining a new variable P_ρ as before, one gets

$$\int \mathcal{D}\rho \int \mathcal{D}P_\rho \exp\left[i \int \left(\rho \dot{P}_\rho + \omega^2\rho^2 + \frac{tan^2 P_\rho}{2} + \omega^2 \frac{J^2 - 1/4}{tan^2 P_\rho}\right) dt\right], \tag{3.84}$$

and the gaussian integration over ρ yields

$$\int \mathcal{D}P_\rho \exp\left[-i \int \left(\frac{\dot{P}_\rho^2}{4\omega^2} - \frac{tan^2 P_\rho}{2} - \omega^2 \frac{J^2 - 1/4}{tan^2 P_\rho}\right) dt\right]. \tag{3.85}$$

This path integral can be evaluated at least perturbatively by standard methods and is similar to the one obtained for the one-dimensional case in [57].

Adopting the approach of [3], which is based on the noncanonical structure of phase space, we have discussed the formulation of one-particle nonrelativistic quantum mechanics through path integral techniques for the Snyder model. A detailed derivation was given starting from the definition of the path integral and using a representation of the operators in terms of canonical coordinates. We have also shown that the same results can be obtained in a more formal way using the techniques introduced in [14] for the study of first-order systems, taking for granted the definition of the path integral for canonical variables.

Chapter 4

The Snyder-de Sitter model

The Snyder model can be considered as an example of doubly special relativity (DSR), a theory where a new fundamental scale is introduced by deforming special relativity [4, 5]. Models of this sort can be described in terms of a curved momentum space [29], which in the Snyder case is a 3-sphere S^3. In general, the non-trivial geometry of the momentum space entails some remarkable consequences on the definition of locality, that loses its absolute meaning and becomes observer dependent, suggesting the possibility that locality is a relative property [6].

The Snyder-de Sitter (SdS) model is a generalisation of the Snyder model to a spacetime background of constant curvature. It is an example of a noncommutative spacetime admitting two fundamental scales besides the speed of light, and is invariant under the action of the de Sitter group.

The extension of the Snyder model to a spacetime background of constant curvature was proposed in [30], motivated by the necessity to include the cosmological constant $\Lambda \sim \alpha^2$ among the bare parameters of a theory of quantum gravity. Because it includes three fundamental constants, the theory was originally called triply special relativity (TSR). The most relevant feature of this generalisation is its duality for the interchange between position and momenta, that realises the Born reciprocity principle [11].

In the first section, a review of the classical and quantum mechanics of the nonrelativistic Snyder model in curved space, which was investigated in [45] is given. Following that, the results of [50], where the representations of the 3-dimensional Euclidean Snyder-de Sitter algebra and the spectrum of the position and momentum operators were studied, are presented.

4.1 Dynamics of the non-relativistic Snyder model in curved space

The non-relativistic version of the SdS model, i.e. the Snyder model restricted to a three-dimensional sphere, can be studied by means of a nonlinear transformation relating the SdS phase space variables to canonical ones, allowing one to investigate both the dynamics of a free particle and the harmonic oscillator case [45].

The starting point is the TSR algebra, which is generated by the position $\hat{\mathcal{X}}_\mu$, momenta $\hat{\mathcal{P}}_\mu$ and Lorentz generators $\hat{J}_{\mu\nu}$, which satisfy

$$\begin{aligned}
&[\hat{\mathcal{X}}_\mu, \hat{\mathcal{X}}_\nu] = i\beta^2 \hat{J}_{\mu\nu}, \quad [\hat{\mathcal{P}}_\mu, \hat{\mathcal{P}}_\nu] = i\alpha^2 \hat{J}_{\mu\nu},\\
&[\hat{\mathcal{X}}_\mu, \hat{\mathcal{P}}_\nu] = i\left(\eta_{\mu\nu} + \alpha^2 \hat{\mathcal{X}}_\mu \hat{\mathcal{X}}_\nu + \beta^2 \hat{\mathcal{P}}_\mu \hat{\mathcal{P}}_\nu + \alpha\beta(\hat{\mathcal{X}}_\mu \hat{\mathcal{P}}_\nu + \hat{\mathcal{P}}_\mu \hat{\mathcal{X}}_\nu - \hat{J}_{\mu\nu})\right),\\
&[\hat{J}_{\mu\nu}, \hat{\mathcal{X}}_\lambda] = i(\eta_{\mu\lambda}\hat{\mathcal{X}}_\nu - \eta_{\nu\lambda}\hat{\mathcal{X}}_\mu), \quad [\hat{J}_{\mu\nu}, \hat{\mathcal{P}}_\lambda] = i(\eta_{\mu\lambda}\hat{\mathcal{P}}_\nu - \eta_{\nu\lambda}\hat{\mathcal{P}}_\mu),
\end{aligned} \tag{4.1}$$

and the usual Lorentz algebra commutation relations satisfied by the $\hat{J}_{\mu\nu}$. The $\hat{J}_{\mu\nu}$ and $\hat{\mathcal{P}}_\mu$ generate a de Sitter or anti-de Sitter subalgebra (depending on the sign of α^2) that describes the spacetime symmetries of the model. As in DSR, the action of the translations on the spatial coordinates is nonlinear. The coupling constants α and β have dimensions of inverse length and inverse mass respectively and are usually identified with the square root of the cosmological constant (α) and the inverse of the Planck length (β). In the following it is required that $\alpha\beta \ll 1$. The limit $\alpha \to 0$ gives the flat Snyder model, while the limit $\beta \to 0$ yields the Heisenberg algebra of quantum mechanics in a de Sitter background endowed with projective coordinates.

From (4.1) it follows that in TSR both position and momentum components do not commute among themselves. While the noncommutativity of position coordinates in (4.1) characterises Snyder spaces, the noncommutativity of momenta is typical of curved space-times. What can also be noted is the duality of the model under position and momentum interchange through $\hat{\mathcal{X}}_\mu \longleftrightarrow \hat{\mathcal{P}}_\mu$, $\alpha \longleftrightarrow \beta$.

4.1.1 Classical mechanics

In classical mechanics, motion in the non-relativistic SdS model can be described by postulating a non-canonical symplectic structure of the phase space, with fundamental Poisson brackets given by [42]

$$
\begin{aligned}
&\{\mathcal{X}_i, \mathcal{X}_j\} = \beta^2 J_{ij}, \quad \{\mathcal{P}_i, \mathcal{P}_j\} = \alpha^2 J_{ij}, \\
&\{\mathcal{X}_i, \mathcal{P}_j\} = \delta_{ij} + \alpha^2 \mathcal{X}_i \mathcal{X}_j + \beta^2 \mathcal{P}_i \mathcal{P}_j + \alpha\beta \mathcal{P}_i \mathcal{X}_j,
\end{aligned} \tag{4.2}
$$

where $i, j = 1, 2, 3$. These Poisson brackets are obtained from (4.1) by introducing the standard expression $J_{ij} = \mathcal{X}_i \mathcal{P}_j - \mathcal{X}_j \mathcal{P}_i$ for the generators of rotation.

In the notation that is adopted, α^2 and β^2 are allowed to be negative, but, in order for the Jacobi identities to hold, they must have the same sign. When they are both negative, $\alpha\beta$ is also taken to be negative, while for expressions linear in α and β it is defined $\alpha = \sqrt{|\alpha^2|}$, $\beta = \sqrt{|\beta^2|}$. The case of positive coupling constants corresponds to the Snyder model on a spherical background, while the model with α^2 and β^2 negative gives rise to the anti-Snyder model on a pseudosphere, the so-called anti-Snyder-de Sitter (aSdS) model.

The Poisson brackets (4.2) can be obtained from those of the flat Snyder model, with position variables x_i and momentum variables p_i obeying

$$
\{x_i, x_j\} = \beta^2 (x_i p_j - x_j p_i), \quad \{p_i, p_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij} + \beta^2 p_i p_j, \tag{4.3}
$$

by performing a linear unimodular, but non-symplectic transformation of the phase space coordinates

$$\mathcal{X}_i = x_i + \frac{\beta}{\alpha}\lambda p_i, \quad \mathcal{P}_i = (1-\lambda)p_i - \frac{\alpha}{\beta}x_i, \tag{4.4}$$

where λ is a free parameter, that can be chosen arbitrarily.

It can be shown that the phase space variables x_i and p_i can be written in terms of the coordinates X_i and P_i, that satisfy canonical Poisson brackets, through the nonlinear transformation [44]

$$p_i = \frac{P_i}{\sqrt{1-\beta^2 P^2}}, \quad x_i = \sqrt{1-\beta^2 P^2}X_i. \tag{4.5}$$

Combining (4.4) and (4.5), the coordinates $\mathcal{X}_i$ and $\mathcal{P}_i$, that satisfy the SdS Poisson brackets can be written in terms of the canonical coordinates X_i,P_i, as

$$\begin{aligned}
\mathcal{X}_i &= \sqrt{1-\beta^2 P^2}X_i + \lambda\frac{\beta}{\alpha}\frac{P_i}{\sqrt{1-\beta^2 P^2}}, \\
\mathcal{P}_i &= -\frac{\alpha}{\beta}\sqrt{1-\beta^2 P^2}X_i + (1-\lambda)\frac{P_i}{\sqrt{1-\beta^2 P^2}},
\end{aligned} \tag{4.6}$$

with the inverse transformations

$$\begin{aligned}
X_i &= \frac{(1-\lambda)\alpha\mathcal{X}_i - \lambda\beta\mathcal{P}_i}{\alpha\sqrt{1+(\alpha\mathcal{X}_k+\beta\mathcal{P}_k)(\alpha\mathcal{X}_k+\beta\mathcal{P}_k)}}, \\
P_i &= \frac{\beta\mathcal{P}_i + \alpha\mathcal{X}_i}{\beta\sqrt{1+(\alpha\mathcal{X}_k+\beta\mathcal{P}_k)(\alpha\mathcal{X}_k+\beta\mathcal{P}_k)}}.
\end{aligned} \tag{4.7}$$

If $\alpha^2, \beta^2 < 0$, it is necessary to impose $|\alpha\mathcal{X}_k + \beta\mathcal{P}_k| < 1$, and hence the range of definition of the coordinates depends on the value of the momentum. A possible interpretation for this is that the radius of the pseudosphere is a function of the momentum of the particle, which is a common situation in DSR theories defined on curved spaces, where the metric properties are momentum dependent [43, 34].

While the full phase space is invariant under the $SO(5)$ or $SO(4,1)$ group, depending on the sign of the coupling constant, the symmetries of the configuration space are described by the $SO(4)$ or $SO(3,1)$ groups, which leave invariant a three-dimensional space of constant, positive or negative curvature, respectively, and which are generated by the angular momentum $J_{ij} = \mathcal{X}_i\mathcal{P}_j - \mathcal{X}_j\mathcal{P}_i$ and momentum $\mathcal{P}_i$.

Since the phase space coordinates transform as vectors under the action of the generators of rotation J_{ij},

$$\{J_{ij}, \mathcal{X}_k\} = \delta_{ik}\mathcal{X}_j - \delta_{jk}\mathcal{X}_i, \quad \{J_{ij}, \mathcal{P}_k\} = \delta_{ik}\mathcal{P}_j - \delta_{jk}\mathcal{P}_i, \tag{4.8}$$

the symmetry under rotations is realised in the usual way.

The action of the translation generators $\mathcal{P}_i$ on phase space variables can be read from (4.2): the momenta transform according to the standard law for a constant curvature space, but the action on the position coordinates is deformed and is nonlinear.

Because of the Jacobi identities, the fundamental Poisson brackets (4.2) transform covariantly under the action of the symmetry group.

For the Hamiltonian of a free particle defined in the usual way, $H = \frac{\mathcal{P}^2}{2m}$, the deformed Poisson structure leads to the following Hamiltonian equations

$$\begin{aligned} \dot{\mathcal{X}}_i &= \left(1 + \beta^2\mathcal{P}^2 + 2\alpha\beta\mathcal{X}_k\mathcal{P}_k\right)\mathcal{P}_i + \alpha^2\mathcal{X}_k\mathcal{P}_k\mathcal{X}_i, \\ \dot{\mathcal{P}}_i &= -\alpha^2\left(\mathcal{P}^2\mathcal{X}_i - \mathcal{X}_k\mathcal{P}_k\mathcal{P}_i\right). \end{aligned} \tag{4.9}$$

Since the relation between velocity and momentum is no longer linear, the equations are in general difficult to solve. However, in one dimension the Poisson brackets reduce to

$$\{\mathcal{X}, \mathcal{P}\} = 1 + (\alpha\mathcal{X} + \beta\mathcal{P})^2, \tag{4.10}$$

leading to the Hamilton equations

$$\dot{\mathcal{X}} = \left(1 + (\alpha\mathcal{X} + \beta\mathcal{P})^2\right)\frac{\mathcal{P}}{2m}, \quad \dot{\mathcal{P}} = 0. \tag{4.11}$$

In this case, the momentum is constant $\mathcal{P} = \mathcal{P}_0$, and integrating the first equation gives

$$\begin{aligned} \mathcal{X} &= \tfrac{1}{\alpha} tan\, \tfrac{\alpha \mathcal{P}_0}{m} t - \tfrac{\beta}{\alpha}\mathcal{P}_0, \quad \alpha^2 > 0, \\ \mathcal{X} &= \tfrac{1}{\alpha} tanh\, \tfrac{\alpha \mathcal{P}_0}{m} t - \tfrac{\beta}{\alpha}\mathcal{P}_0, \quad \alpha^2 < 0. \end{aligned} \tag{4.12}$$

It can also be seen that in the second case, the condition $|\alpha\mathcal{X} + \beta\mathcal{P}| < 1$, is fulfilled as required.

When considering the example of the one-dimensional harmonic oscillator, the Hamiltonian is given by

$$H = \frac{\mathcal{P}^2}{2m} + \frac{m\omega_0^2 \mathcal{X}^2}{2}. \tag{4.13}$$

In the case of $\alpha^2, \beta^2 > 0$, and for unit mass, the Hamilton equations are given by

$$\dot{\mathcal{X}} = \left(1 + (\alpha\mathcal{X} + \beta\mathcal{P})^2\right)\mathcal{P}, \quad \dot{\mathcal{P}} = -\omega_0^2\left(1 + (\alpha\mathcal{X} + \beta\mathcal{P})^2\right)\mathcal{X}. \tag{4.14}$$

From here, it follows that the Hamiltonian is conserved,

$$\frac{\mathcal{P}^2}{2} + \frac{\omega_0^2 \mathcal{X}^2}{2} = E, \tag{4.15}$$

with E the total energy of the oscillator.

After some work, the equations can be solved to find

$$\begin{aligned} \mathcal{X} &= \sqrt{\frac{2E}{\gamma}} \frac{\frac{\alpha}{\omega_0} sin\, \omega t - \beta\sqrt{1 + 2\gamma E}\, cos\, \omega t}{\omega_0 \sqrt{1 + 2\gamma E\, cos^2 \omega t}}, \\ \mathcal{P} &= \sqrt{\frac{2E}{\gamma}} \frac{\beta\, sin\, \omega t + \frac{\alpha}{\omega_0}\sqrt{1 + 2\gamma E}\, cos\, \omega t}{\sqrt{1 + 2\gamma E\, cos^2 \omega t}}, \end{aligned} \tag{4.16}$$

where

$$\gamma = \beta^2 + \frac{\alpha^2}{\omega_0^2}, \quad \omega = \sqrt{1 + 2\gamma E}\,\omega_0. \tag{4.17}$$

The solutions are periodic, but not sinusoidal, and the frequency now depends on the energy of the oscillator. In order for the frequency to be real, the energy must be such that $1 + 2\gamma E \geq 0$, which is always true if the energy is positive. In the limit $\alpha \to 0$, one recovers the flat Snyder oscillator.

In the aSdS case, the same relation between the frequency and the energy is recovered, but with with negative γ, and the solutions of the Hamilton equations are given by

$$\begin{aligned} \mathcal{X} &= \sqrt{\frac{2E}{|\gamma|}}\, \frac{\frac{\alpha}{\omega_0} sin\,\omega t - \beta\sqrt{1+2\gamma E}\, cos\,\omega t}{\omega_0\sqrt{1+2\gamma E\, cos^2\omega t}}, \\ \mathcal{P} &= \sqrt{\frac{2E}{|\gamma|}}\, \frac{\beta\, sin\,\omega t + \frac{\alpha}{\omega_0}\sqrt{1+2\gamma E}\, cos\,\omega t}{\sqrt{1+2\gamma E\, cos^2\omega t}}. \end{aligned} \tag{4.18}$$

The condition imposed on the energy in order for the solutions to be real now gives rise to an upper bound for the energy, $2E < \omega_0^2/(|\beta^2|\omega_0^2 + |\alpha^2|)$, consistently with the bound $|\alpha\mathcal{X} + \beta\mathcal{P}| < 1$.

4.1.2 Quantum mechanics

An equivalent form of the (a)SdS commutation relations (4.1), between the position and momentum operators, $\hat{\mathcal{X}}_i$ and $\hat{\mathcal{P}}_i$, is given by

$$\begin{aligned} &\left[\hat{\mathcal{X}}_i, \hat{\mathcal{X}}_j\right] = i\beta^2 \hat{J}_{ij}, \quad \left[\hat{\mathcal{P}}_i, \hat{\mathcal{P}}_j\right] = i\alpha^2 \hat{J}_{ij}, \\ &\left[\hat{\mathcal{X}}_i, \hat{\mathcal{P}}_j\right] = i\left(\delta_{ij} + \alpha^2\hat{\mathcal{X}}_i\hat{\mathcal{X}}_j + \beta^2\hat{\mathcal{P}}_i\hat{\mathcal{P}}_j + \frac{\alpha\beta}{2}\left(3\hat{\mathcal{P}}_i\hat{\mathcal{X}}_j + \hat{\mathcal{X}}_j\hat{\mathcal{P}}_i - \hat{\mathcal{P}}_j\hat{\mathcal{X}}_i + \hat{\mathcal{X}}_i\hat{\mathcal{P}}_j\right)\right), \end{aligned} \tag{4.19}$$

where the last equation is obtained from (4.1) after substituting the representation $\hat{J}_{ij} = \frac{1}{2}\left(\hat{\mathcal{P}}_i\hat{\mathcal{X}}_j + \hat{\mathcal{X}}_j\hat{\mathcal{P}}_i - \hat{\mathcal{P}}_j\hat{\mathcal{X}}_i - \hat{\mathcal{X}}_i\hat{\mathcal{P}}_j\right)$ for the angular momentum. The lower degree of symmetry of (4.19) with respect to the classical expression (4.1) is due to operator ordering problems.

The deformed commutation relations imply a modification of the Heisenberg uncertainty relations and give rise to minimal uncertainty of both position and momentum. In the simple case in which $\langle\hat{\mathcal{P}}_i\rangle = \langle\hat{\mathcal{X}}_i\rangle = 0$, the uncertainty relations that follow from (4.1) for states with vanishing angular momentum are

$$\begin{aligned}
\Delta\mathcal{X}_i\Delta\mathcal{P}_j &\geq \frac{1}{2}\left|\left\langle\left[\hat{\mathcal{X}}_i,\hat{\mathcal{P}}_j\right]\right\rangle\right| = \frac{1}{2}\left|\delta_{ij} + \left\langle\left(\alpha\hat{\mathcal{X}}_i + \beta\hat{\mathcal{P}}_i\right)\left(\alpha\hat{\mathcal{X}}_j + \beta\hat{\mathcal{P}}_j\right)\right\rangle\right| \\
&\geq \frac{1}{2}\left|\delta_{ij} + \alpha^2\Delta\mathcal{X}_i\Delta\mathcal{X}_j\beta^2\Delta\mathcal{P}_i\Delta\mathcal{P}_j - \alpha\beta\left(\Delta\mathcal{X}_i\Delta\mathcal{P}_j + \Delta\mathcal{X}_j\Delta\mathcal{P}_i\right)\right|, \qquad (4.20)
\end{aligned}$$

where the last step follows after using the Schwartz inequality.

In the one-dimensional case, the algebra (4.19) simplifies

$$\left[\hat{\mathcal{X}},\hat{\mathcal{P}}\right] = i\left(1 + \alpha^2\hat{\mathcal{X}}^2 + \beta^2\hat{\mathcal{P}}^2 + \alpha\beta\left(\hat{\mathcal{X}}\hat{\mathcal{P}} + \hat{\mathcal{P}}\hat{\mathcal{X}}\right)\right), \qquad (4.21)$$

and the uncertainty relations (4.20) reduce to

$$\Delta\mathcal{X}\Delta\mathcal{P} \geq \frac{1}{2}\frac{\left|1 + \alpha^2(\Delta\mathcal{X})^2 + \beta^2(\Delta\mathcal{P})^2\right|}{1 + \alpha\beta}. \qquad (4.22)$$

If $\alpha^2, \beta^2 > 0$, they imply the existence of both minimal position and momentum uncertainties, given by

$$\Delta\mathcal{X}_M = \frac{\beta}{\sqrt{1 + 2\alpha\beta}} \sim \beta(1 - \alpha\beta), \quad \Delta\mathcal{P}_M = \frac{\alpha}{\sqrt{1 + 2\alpha\beta}} \sim \alpha(1 - \alpha\beta). \qquad (4.23)$$

For $\alpha^2, \beta^2 < 0$, no minimal uncertainties emerge.

Using the relation between the SdS coordinates and the canonical ones (4.6), it is possible to define a realisation of the position and momentum operators $\hat{\mathcal{X}}_i$ and $\hat{\mathcal{P}}_i$ that satisfies the algebra (4.19). The action of the operators on a Hilbert space of functions of a variable P_i are then given by

$$\begin{aligned}
\hat{\mathcal{X}}_i &= \hat{x}_i + \lambda\frac{\beta}{\alpha}\hat{p}_i = i\sqrt{1 - \beta^2P^2}\frac{\partial}{\partial P_i} + \lambda\frac{\beta}{\alpha}\frac{P_i}{\sqrt{1 - \beta^2P^2}}, \\
\hat{\mathcal{P}}_i &= -\frac{\alpha}{\beta}\hat{x}_i + (1 - \lambda)\hat{p}_i = -i\frac{\alpha}{\beta}\sqrt{1 - \beta^2P^2}\frac{\partial}{\partial P_i} + (1 - \lambda)\frac{P_i}{\sqrt{1 - \beta^2P^2}}. \qquad (4.24)
\end{aligned}$$

For $\alpha^2, \beta^2 > 0$, the range of allowed values of P_i is bounded by $P^2 < 1/\beta^2$. Otherwise, if

$\alpha^2, \beta^2 < 0$, all real values of P_i are allowed, but the upper bound of the eigenstates of the operator $\left(\alpha\hat{\mathcal{X}}_k + \beta\hat{\mathcal{P}}_k\right)\left(\alpha\hat{\mathcal{X}}_k + \beta\hat{\mathcal{P}}_k\right)$ is 1, as in the classical limit.

A simple example of the action of the operators can be given by considering a particle in one dimension. For this case, it is convenient to set $\lambda = 0$ in (4.24). If $\alpha^2, \beta^2 > 0$, in order to get symmetric operators, i.e.

$$\left(\hat{\mathcal{P}}\psi, \phi\right) = \left(\psi, \hat{\mathcal{P}}\phi\right), \quad \left(\hat{\mathcal{X}}\psi, \phi\right) = \left(\psi, \hat{\mathcal{X}}\phi\right), \tag{4.25}$$

the scalar product must be defined as

$$(\psi, \phi) = \int_{-1/\beta}^{1/\beta} \frac{dP}{\sqrt{1 - \beta^2 P^2}} \psi^*(P)\phi(P), \tag{4.26}$$

and the wave functions have to satisfy periodic boundary conditions, $\psi(-1/\beta) = \psi(1/\beta)$.

For $\alpha^2, \beta^2 < 0$, the scalar product is defined by

$$(\psi, \phi) = \int_{-\infty}^{\infty} \frac{dP}{\sqrt{1 - \beta^2 P^2}} \psi^*(P)\phi(P), \tag{4.27}$$

and only the convergence of the integral is required.

When studying the spectrum of the position and momentum operators, certain difficulties arise. Starting for example from the eigenvalue equation

$$\hat{\mathcal{X}}\psi_{\mathcal{X}} = \mathcal{X}\psi_{\mathcal{X}}, \tag{4.28}$$

the solutions are given by

$$\psi_{\mathcal{X}} = C\, e^{-\frac{i\mathcal{X}}{\beta}\, arcsin \beta P}, \tag{4.29}$$

with C a normalisation constant and $\mathcal{X} = 2n\beta$. While the eigenstates (4.29) have vanishing position uncertainty, their energy diverges, and hence they cannot be accepted as physical. Here however, unlike in ordinary quantum mechanics, position eigenstates cannot be obtained as a limit of states with finite energy. This is because the existence of a minimal indetermination of the positions implies that no exactly localised states can exist. One can then, considering that $\left[\hat{X}, \hat{\mathcal{P}}\right] = [\hat{x}, \hat{p}]$, guess that the functions that minimise the commutator expectation value are the same as the ones that give rise to minimal uncertainty in flat Snyder space, i.e.

$$\psi_X = C\sqrt{1-\beta^2 P^2}\, e^{-\frac{iX}{\beta}\, arcsin\beta P}. \tag{4.30}$$

The functions (4.30) satisfy the boundary conditions without the need of quantising the parameter X. For these functions,

$$\Delta X = \beta, \quad \Delta\mathcal{P} = \sqrt{\frac{1}{\beta^2} + \alpha^2}, \quad \Delta X \Delta\mathcal{P} = \sqrt{1+\alpha^2\beta^2}, \tag{4.31}$$

and ΔX is close to its minimal value (4.23), while $\Delta\mathcal{P}$ is extremely large due to the smallness of β.

The properties of the momentum operator $\hat{\mathcal{P}}$ are similar to those of $\hat{X}$. The eigenvalue equation

$$\hat{\mathcal{P}}\psi_{\mathcal{P}} = \mathcal{P}\psi_{\mathcal{P}}, \tag{4.32}$$

has the solution

$$\psi_{\mathcal{P}} = C(1-\beta^2P^2)^{\frac{i}{2\alpha\beta}}\, e^{\frac{i\mathcal{P}}{\alpha}\, arcsin\beta P}, \tag{4.33}$$

with $\mathcal{P} = 2n\alpha$, and it exhibits problems analogous to those affecting the position operator eigenstates. Here, $\Delta\mathcal{P} = 0$, but the expectation value of $\hat{X}^2$ diverges. As before, it is possible to define a basis,

$$\psi_{\mathcal{P}} = C(1-\beta^2P^2)^{\frac{1}{2}+\frac{i}{2\alpha\beta}}\, e^{\frac{i\mathcal{P}}{\alpha}\, arcsin\beta P}, \tag{4.34}$$

in which $\mathcal{P}$ is not quantised, and the expectation value of $\hat{X}^2$ is finite. The uncertainties in this basis are

$$\Delta X = \sqrt{\frac{1}{\alpha^2} + \beta^2}, \quad \Delta\mathcal{P} = \alpha, \quad \Delta X \Delta\mathcal{P} = \sqrt{1+\alpha^2\beta^2}. \tag{4.35}$$

Hence, a basis of states more physical then the formal eigenstates (4.29) and (4.33), is given by smeared functions like (4.30) and (4.34), which describe a fuzzy phase space, with no sharply defined values of positions and momenta.

In the aSdS case, since no minimal indetermination arises, the situation is analogous to ordinary quantum mechanics and the formal eigenfunctions are limits of states with finite $\Delta\mathcal{X}$ and $\Delta\mathcal{P}$. For the position operator, the eigenfunctions are

$$\psi_{\mathcal{X}} = C\, e^{-\frac{i\mathcal{X}}{\beta}\, arcsinh\beta P}, \tag{4.36}$$

and for the momentum

$$\psi_{\mathcal{P}} = C(1-\beta^2P^2)^{\frac{i}{2\alpha\beta}}\, e^{\frac{i\mathcal{P}}{\alpha}\, arcsinh\beta P}. \tag{4.37}$$

The invariance of the configuration space of the classical model under the $SO(4)$ or $SO(3,1)$ group can be extended to the quantum case.

The rotations are generated by

$$\hat{J}_{ij} = \frac{1}{2}\left(\hat{\mathcal{X}}_i\hat{\mathcal{P}}_j + \hat{\mathcal{P}}_j\hat{\mathcal{X}}_i - \hat{\mathcal{X}}_j\hat{\mathcal{P}}_i - \hat{\mathcal{P}}_i\hat{\mathcal{X}}_j\right) = \hat{p}_j\hat{x}_i - \hat{p}_i\hat{x}_j = i\left(P_j\frac{\partial}{\partial P_i} - P_i\frac{\partial}{\partial P_j}\right), \tag{4.38}$$

and act in the standard way. The spectrum of $\hat{J}_{ij}$ is the same as in ordinary quantum mechanics, and defining $\hat{J}_i = \epsilon_{ijk}\hat{J}_{jk}$, the eigenfunctions in spherical coordinates in the momentum representation are given by the standard spherical harmonics

$$\hat{J}^2 Y_{lm}(P_\theta, P_\phi) = l(l+1)Y_{lm}(P_\theta, P_\phi), \quad \hat{J}_3 Y_{lm}(P_\theta, P_\phi) = mY_{lm}(P_\theta, P_\phi). \tag{4.39}$$

The translations are generated by the momentum operators $\hat{\mathcal{P}}_i$, that act according to (4.19). Their action on momenta is the usual one for a space of constant curvature, but the one on position variables is deformed and takes a nonlinear form. The commutation relations (4.19) transform covariantly under these symmetries.

As a simple example of one-dimensional dynamics, one can consider the Schrödinger equation for a free particle. In the representation (4.24), with $\lambda = 0$ and for unit mass, it reads

$$\frac{d^2\psi}{dP^2} - \left(\beta - \frac{2i}{\alpha}\right)\frac{\beta P}{1-\beta^2P^2}\frac{d\psi}{dP} - \frac{\beta^2}{\alpha^2}\left(\frac{P^2 - i\alpha/\beta}{(1-\beta^2P^2)^2} - \frac{2E}{1-\beta^2P^2}\right)\psi = 0. \tag{4.40}$$

In the SdS case, the solutions of this equation, that vanish at $P = \pm 1/\beta$ are

$$\psi = const. \times (1-\beta^2P^2)^{\frac{i}{2\alpha\beta}}\, cos\left(\frac{\sqrt{2E}}{\alpha}\, arcsin \beta P\right), \tag{4.41}$$

with $E = \frac{1}{2}\alpha^2 n^2$, for odd integer n. These solutions have finite values of $\Delta\mathcal{X}$.

In the case of aSdS, the solutions are given by the momentum eigenfunctions (4.37) and the energy is not quantised.

If one wants to consider the case of the one-dimensional harmonic oscillator, with the Hamiltonian given by

$$H = \frac{\hat{\mathcal{P}}^2}{2m} + \frac{m\omega_0^2\hat{\mathcal{X}}^2}{2}, \tag{4.42}$$

it is convenient to choose the coefficient λ in (4.24) such that the cross term $\hat{\mathcal{P}}\hat{\mathcal{X}} + \hat{\mathcal{X}}\hat{\mathcal{P}}$ in the Hamiltonian vanishes. This is achieved with the choice of the gauge

$$\lambda = \frac{\alpha^2}{\beta^2\omega_0^2 + \alpha^2}. \tag{4.43}$$

The Schrödinger equation for unit mass now reads

$$\frac{1}{2}\frac{\beta^2\omega_0^2}{\beta^2\omega_0^2+\alpha^2}\left(\hat{\mathcal{P}}^2 + \frac{(\beta^2\omega_0^2+\alpha^2)^2}{\beta^4\omega_0^2}\hat{\mathcal{X}}^2\right)\psi = E\psi. \tag{4.44}$$

Using the realisation (4.24), this equation becomes

$$\frac{d^2\psi}{dP^2} - \frac{\beta^2 P}{1-\beta^2P^2}\frac{d\psi}{dP} - \frac{1}{\omega^2}\left(\frac{P^2}{(1-\beta^2P^2)^2} - \frac{2\varepsilon}{1-\beta^2P^2}\right)\psi = 0, \tag{4.45}$$

where $\omega = \left(1 + \frac{\alpha^2}{\beta^2\omega_0^2}\right)\omega_0$ and $\varepsilon = \left(1 + \frac{\alpha^2}{\beta^2\omega_0^2}\right)E$.

After certain redefinitions of variables and requiring that ψ vanish at $P = \pm 1/\beta$, one finds the solution in terms of Gegenbauer polynomials C_n^α [2]

$$\psi = const. \times (1 - \beta^2 P^2)^{\alpha/2} \, C_n^\alpha(\beta P), \tag{4.46}$$

with $\alpha = \frac{1}{2}(1 + \sqrt{1 + 4\mu})$, and

$$\mu = \frac{\omega_0^2}{(\beta^2 \omega_0^2 + \alpha^2)^2}. \tag{4.47}$$

The spectrum of energy is given by

$$E = \left(n + \frac{1}{2}\right)\omega_0 \sqrt{1 + \frac{(\beta^2\omega_0^2 + \alpha^2)^2}{4\omega_0^2}} + \left(n^2 + n + \frac{1}{2}\right)\frac{\beta^2\omega_0^2 + \alpha^2}{2}, \tag{4.48}$$

and it exhibits corrections of order $(\beta^2\omega_0 + \alpha^2/\omega_0)$ with respect to the standard case and a duality for $\beta^2\omega_0 \leftrightarrow \alpha^2/\omega_0$.

For the case of $\alpha^2, \beta^2 < 0$, the energy spectrum is the analytic continuation of (4.48) to negative values of α^2 and β^2. Since the energy becomes negative for great n, it is necessary to impose an upper bound on the allowed values of n.

The previous considerations can also be generalised to three dimensions.

4.2 Spectrum of position and momentum squared

The representations of the three-dimensional Euclidean SdS algebra can be studied starting from those of the Snyder algebra and exploiting the geometrical properties of the phase space that can be identified with a Grassmannian manifold [50].

For the given problem, it is more convenient to consider as generators of rotation $\hat{J}_k = \frac{1}{2}\epsilon_{ijk}\hat{J}_{ij}$, which satisfy

$$\left[\hat{J}_i, \hat{J}_j\right] = i\epsilon_{ijk}\hat{J}_k, \quad \left[\hat{J}_i, \hat{X}_j\right] = -i\epsilon_{ijk}\hat{X}_k, \quad \left[\hat{J}_i, \hat{\mathcal{P}}_j\right] = -i\epsilon_{ijk}\hat{\mathcal{P}}_k. \tag{4.49}$$

The rest of the commutation relations of the SdS algebra are then given by

$$\left[\hat{\mathcal{X}}_i, \hat{\mathcal{X}}_j\right] = i\beta^2 \epsilon_{ijk} \hat{J}_k, \quad \left[\hat{\mathcal{P}}_i, \hat{\mathcal{P}}_j\right] = i\alpha^2 \epsilon_{ijk} \hat{J}_k,$$
$$\left[\hat{\mathcal{X}}_i, \hat{\mathcal{P}}_j\right] = i\left(\delta_{ij} + \alpha^2 \hat{\mathcal{X}}_i \hat{\mathcal{X}}_j + \beta^2 \hat{\mathcal{P}}_j \hat{\mathcal{P}}_i + \alpha\beta\left(\hat{\mathcal{X}}_j \hat{\mathcal{P}}_i + \hat{\mathcal{P}}_i \hat{\mathcal{X}}_j\right)\right), \tag{4.50}$$

where $\hat{J}_{ij} = \frac{1}{2}\left(\hat{\mathcal{X}}_i \hat{\mathcal{P}}_j + \hat{\mathcal{P}}_j \hat{\mathcal{X}}_i - \hat{\mathcal{X}}_j \hat{\mathcal{P}}_j - \hat{\mathcal{P}}_i \hat{\mathcal{X}}_j\right)$.

The SdS algebra can be considered as a nonlinear realisation of a model proposed by Yang [59], which differs from SdS only in the assumption of a standard Heisenberg algebra for positions and momenta, $\left[\hat{\mathcal{X}}_i, \hat{\mathcal{P}}_j\right] = i\hat{K}\delta_{ij}$, $\hat{K}$ being a central charge for the rotation group, satisfying $\left[\hat{K}, \hat{\mathcal{X}}_i\right] = i\alpha^2 \hat{\mathcal{P}}_i$, $\left[\hat{K}, \hat{\mathcal{P}}_i\right] = -i\beta^2 \hat{\mathcal{X}}_i$. With the identifications $\hat{J}_{ij} = \hat{J}_{ij}$, $\alpha\hat{\mathcal{X}}_i = \hat{J}_{4i}$, $\beta\hat{\mathcal{P}}_i = \hat{J}_{5i}$, $\hat{K} = \hat{J}_{45}$, the Yang model reproduces an $SO(5)$ algebra with generators $\hat{J}_{\mu\nu}$, $\mu, \nu = 1, ..., 5$.

The three-dimensional nonrelativistic SdS model also enjoys an $SO(5)$ symmetry. Its phase space can be realised on the six-dimensional Grassmanian coset space $Gr(3,5) = SO(5)/SO(3) \times SO(2)$, with $SO(3)$ generated by the $\hat{J}_{ij}$ and $SO(2)$ by $\hat{J}_{45}$.

The space $Gr(3,5)$ can be parametrised by homogeneous coordinates $\mathcal{X}_\mu$ and $\mathcal{P}_\mu$, that satisfy the constraints [17]

$$\alpha^2 \mathcal{X}_\mu^2 = 1, \quad \beta^2 \mathcal{P}_\mu^2 = 1, \quad \mathcal{X}_\mu \mathcal{P}_\mu = 0. \tag{4.51}$$

This parametrisation associates a one-parameter set of matrices to each coset. One can then identify the variables $\mathcal{X}_\mu$ and $\mathcal{P}_\mu$ with canonical coordinates of a ten-dimensional phase space and hence reduce it to a six-dimensional phase space parametrised by $\mathcal{X}_i$ and $\mathcal{P}_i$ by eliminating the constraints (4.51), using the Dirac formalism [12], which was recalled in Sec. 2.1. However, in order to obtain a one-to-one parametrisation, it is necessary to impose a further constraint on the parameters $\mathcal{X}_4$, $\mathcal{X}_5$, $\mathcal{P}_4$, $\mathcal{P}_5$. This is also required by the Dirac formalism since the constraints (4.51) split into one first class and two second class constraints [42]. Unfortunately, not every constraint leads to the SdS algebra and it is therefore necessary to choose a suitable gauge [42]. In particular, the choice $\alpha\mathcal{X}_5 + \beta\mathcal{P}_5 = 0$ yields the algebra (4.50).

4.2.1 The nonrelativistic Snyder algebra

Before considering the SdS case, it is useful to review some results on the representation of the Snyder algebra from [44, 32].

The Snyder algebra, as the limit of the SdS algebra for $\alpha \to 0$, contains an $SO(4)$ subalgebra of $SO(5)$ generated by $\hat{x}_i = \hat{J}_{4i}$ and $\hat{J}_i = \frac{1}{2}\epsilon_{ijk}\hat{J}_{jk}$, while the momentum space is realised as the coset space $S^3 = SO(4)/SO(3)$.

The representations of the $SO(4)$ algebra can be labelled by the eigenvalues of $\hat{A}^2$, $\hat{J}^2$ and $\hat{J}_3$, where $\hat{A}_i$ is the operator defined as $\hat{A}_i = \frac{1}{2}\left(\hat{J}_i + \beta^{-1}\hat{x}_i\right)$, so that $\hat{x}^2 = \beta^2\left(4\hat{A}^2 - \hat{J}^2\right)$,

$$\begin{aligned}\hat{J}^2|j,l,m\rangle &= l(l+1)|j,l,m\rangle,\\ \hat{J}_3|j,l,m\rangle &= m|j,l,m\rangle,\\ \hat{x}^2|j,l,m\rangle &= \beta^2(4j(j+1) - l(l+1))|j,l,m\rangle,\end{aligned} \tag{4.52}$$

with $0 \le l \le 2j$, and $j(j+1)$ being the eigenvalue of $\hat{A}^2$. The eigenvalues of $\hat{x}^2$ have degeneration $2l+1$.

The momentum space can be realised as a 3-sphere, obtained by imposing the constraint $p_i^2 + p_4^2 = 1/\beta^2$ on a four-vector p. This can be shown algebraically [32], or from a Dirac reduction of phase space [18].

The representation I

It is possible to define different representations of the Snyder algebra on a Hilbert space and they are usually given in a momentum representation. One possibility is [32, 22]

$$\hat{p}_i = p_i, \quad \hat{x}_i = i\frac{\partial}{\partial p_i} + i\beta^2 p_i\left(p_j\frac{\partial}{\partial p_j} + \mu\right), \quad \hat{J}_i = -i\epsilon_{ijk}p_j\frac{\partial}{\partial p_k}, \tag{4.53}$$

with the operators $\hat{x}_i$, $\hat{p}_i$, $\hat{J}_i$ acting on functions $\psi(p_i)$ of the Hilbert space, μ an arbitrary real parameter and $-\infty < p_i < \infty$. The operators are symmetric for the measure

$$\frac{d^3p}{(1+\beta^2p^2)^{2-\mu}}, \tag{4.54}$$

provided that the functions $\psi(p_i)$ go to infinity like $\left(\frac{1}{\sqrt{p^2}}\right)^{\mu-1/2}$.

In this representation the operator $\hat{x}^2$ takes the form

$$\hat{x}^2 = -(1+\beta^2p_\rho^2)^2\left(\frac{\partial^2}{\partial p_\rho^2}+\frac{2}{p_\rho}\frac{\partial}{\partial p_\rho}\right)-\mu\beta^2\left(2(1+\beta^2p_\rho^2)p_\rho\frac{\partial}{\partial p_\rho}+(1+\mu)\beta^2p_\rho^2+3\right)+\frac{\hat{J}_i^2}{p_\rho^2}. \tag{4.55}$$

For $\mu = 0$, the equation $\hat{x}^2\phi = x^2\phi$ has eigenfunctions [32]

$$\phi_{nlm} = const. \times sin^l\chi C_n^{(l+1)}(cos\chi)Y_m^l(p_\theta, p_\varphi), \tag{4.56}$$

where the polar coordinates $p_\rho,\ p_\theta,\ p_\varphi$ have been used, and χ is defined as $\chi = arctan\beta p_\rho$. The functions $C_n^{(a)}$ are Gegenbauer polynomials with n a nonnegative integer parameter and $Y_m^l(p_\theta, p_\varphi)$ are spherical harmonics.

For $\mu \neq 0$, the eigenfunctions (4.56) are simply multiplied by $cos^\mu\chi$. The eigenvalues are independent of μ and read

$$x^2 = \beta^2(n^2+2nl+2n+l), \tag{4.57}$$

with $0 \le l \le n$. They can be identified with (4.52) by setting $n = 2j - l$.

The operator $\hat{p}^2 = \hat{p}_\rho^2$ is trivial and its spectrum extends to the positive real line.

The representation II

An alternative representation is given by [44]

$$\hat{p}_i = \frac{p_i}{\sqrt{1-\beta^2p^2}}, \quad \hat{x}_i = i\sqrt{1-\beta^2p^2}\frac{\partial}{\partial p_i}, \tag{4.58}$$

with $p^2 < 1/\beta^2$. In this representation the operators are symmetric for the measure

$$\frac{d^3p}{\sqrt{1-\beta^2p^2}}, \tag{4.59}$$

and the operator $\hat{x}^2$ takes the form

$$\hat{x}^2 = -(1-\beta^2p_\rho^2)^2\frac{\partial^2}{\partial p_\rho^2} - \frac{2-3\beta^2p_\rho^2}{p_\rho}\frac{\partial}{\partial p_\rho} + \frac{(1-\beta^2p_\rho^2)\hat{J}^2}{p_\rho^2}. \tag{4.60}$$

Its eigenfunctions are given by

$$\phi_{qlm} = const. \times\ sin^l\eta\, cos\,\eta P_q^{(1/2,l+1/2)}(cos\,2\eta)Y_m^l(p_\theta, p_\varphi), \tag{4.61}$$

where $\eta = arcsin\beta p_\rho$ and $P_q^{a,b)}$ are Jacobi polynomials with q a nonnegative integer. The eigenvalues are given by $\beta^2((2q+l+2)^2 - l(l+1) - 1)$. Taking $q = (n-1)/2$, one recovers the eigenvalues (4.57).

4.2.2 The nonrelativistic SdS algebra

As in the previous section, the representations of the operators $\hat{\mathcal{X}}_i$ and $\hat{\mathcal{P}}_i$ that satisfy the SdS algebra can be obtained from the operators $\hat{x}_i$ and $\hat{p}_i$ of the Snyder algebra by taking the linear combinations [45]

$$\hat{\mathcal{X}}_i = \hat{x}_i + \lambda\frac{\beta}{\alpha}\hat{p}_i, \quad \hat{\mathcal{P}}_i = (1-\lambda)\hat{p}_i - \frac{\alpha}{\beta}\hat{x}_i, \tag{4.62}$$

with the inverse

$$\hat{x}_i = (1-\lambda)\hat{\mathcal{X}}_i - \lambda\frac{\beta}{\alpha}\hat{\mathcal{P}}_i, \quad \hat{p}_i = \hat{\mathcal{P}}_i + \frac{\alpha}{\beta}\hat{\mathcal{X}}_i, \tag{4.63}$$

where λ is a free parameter. Since representations with different values of λ are related by unitary transformations [45], for simplicity only the case $\lambda = 0$ is considered in the following.

The relation between Snyder and SdS representations can be understood by considering the embedding of S^3 into $Gr(3,5)$, corresponding to the branching $SO(5) \to SO(4)$. The vectors of $Gr(3,5)$ satisfy the constraints (4.51), while those of S^3 satisfy $\beta^2(p_i^2 + p_4^2) = 1$. Taking into account the SdS gauge constraint $\alpha\mathcal{X}_5 + \beta\mathcal{P}_5 = 0$, one can see that the combination $p_\mu = \mathcal{P}_\mu + \frac{\alpha}{\beta}\mathcal{X}$ defined as in (4.2.2) satisfies the same constraints as the vectors of $SO(4)/SO(3)$ and then transforms as the Snyder momentum.

The momentum representation I

Setting $\lambda = 0$, from (4.62) and (4.53), one obtains the representation

$$\begin{aligned} \hat{\mathcal{X}}_i &= i\frac{\partial}{\partial P_i} + i\beta^2 P_i\left(P_j\frac{\partial}{\partial P_j} + \mu\right), \\ \hat{\mathcal{P}}_i &= P_i - i\frac{\alpha}{\beta}\left(\frac{\partial}{\partial P_i} + \beta^2 P_i\left(P_j\frac{\partial}{\partial P_j} + \mu\right)\right). \end{aligned} \tag{4.64}$$

As for the Snyder model, the eigenfunctions can be written in terms of $\hat{\mathcal{X}}^2$, $\hat{J}^2$ and $\hat{J}_3$.

Since $\hat{\mathcal{X}}^2 = \hat{x}^2$, the equation

$$\hat{\mathcal{X}}^2\psi = \mathcal{X}^2\psi \tag{4.65}$$

has the same eigenfunctions (4.56) and eigenvalues (4.57) as in the Snyder model.

The calculation of the operator $\hat{\mathcal{P}}^2$ is more involved. From (4.62), setting $\lambda = 0$, one gets

$$\hat{\mathcal{P}}^2 = \hat{p}^2 - \frac{\alpha}{\beta}(\hat{x}_i\hat{p}_i + \hat{p}_i\hat{x}_i) + \frac{\alpha^2}{\beta^2}\hat{x}^2. \tag{4.66}$$

The second term of the right hand side can be written as

$$\hat{x}_i\hat{p}_i + \hat{p}_i\hat{x}_i = 2i(1 + \beta^2 P_\rho^2)P_\rho\frac{\partial}{\partial P_\rho} + 3i + i\beta^2(1 + 2\mu)P_\rho^2. \tag{4.67}$$

From (4.55) and (4.67) it then follows

$$\begin{aligned}\hat{\mathcal{P}}^2 &= -\frac{\alpha^2}{\beta^2}\left((1+\beta^2P_\rho^2)^2\frac{\partial^2}{\partial P_\rho^2} + (1+\beta^2P_\rho^2)\left(1+\beta^2P_\rho^2 + i\frac{\beta}{\alpha}P_\rho^2\right)\frac{2}{P_\rho}\frac{\partial}{\partial P_\rho} - \frac{\hat{J}^2}{P_\rho^2}\right)\\ &\quad -\frac{\alpha}{\beta}\left(3i + \left(i\beta^2 - \frac{\beta}{\alpha}\right)P_\rho^2\right)\\ &\quad +\mu\alpha^2\left(2(1+\beta^2P_\rho^2)P_\rho\frac{\partial}{\partial P_\rho} + (1+\mu)\beta^2P_\rho^2 + 3 - 2i\frac{\beta}{\alpha}P_\rho^2\right).\end{aligned} \tag{4.68}$$

As for the Snyder model, it is straightforward to check that the solutions with $\mu \neq 0$ can be obtained by multiplying those with vanishing μ by $cos^\mu\chi$, so only the $\mu = 0$ case is considered. Then the solutions of the eigenvalue equation $\hat{\mathcal{P}}^2\phi = \mathcal{P}^2\phi$ can be deduced from those of (4.65) by noting that the substitution $\phi = (1+\beta^2P_\rho^2)^{-i/2\alpha\beta}\psi$ brings the equation to the same form as (4.55), with $x^2 \to (\beta^2/\alpha^2)\mathcal{P}^2$, and hence its eigenfunctions differ only by a phase from those of $\hat{\mathcal{X}}^2$, and are given by

$$\phi_{nlm} = const. \times sin^l\chi\, cos^{i/\alpha\beta}\chi C_n^{(l+1)}(cos\chi)Y_m^l(P_\theta, P_\varphi). \tag{4.69}$$

The operators $\hat{\mathcal{X}}^2$ and $\hat{\mathcal{P}}^2$ are therefore related by a unitary transformation, and the eigenvalues of $\hat{\mathcal{P}}^2$ are the same as those of $\hat{\mathcal{X}}^2$, except for a multiplicative constant

$$\mathcal{P}^2 = \alpha^2(n^2 + 2nl + 2n + l). \tag{4.70}$$

This could have been predicted on the ground of the duality between $\hat{\mathcal{X}}_i$ and $\hat{\mathcal{P}}_i$. It follows that in the SdS model the eigenvalues of the momentum squared (and hence of the energy) are also quantised and they do not depend on β.

The momentum representation II

The prescription (4.58) along with the relation between the SdS and Snyder algebras gives an alternative representation of the SdS algebra. For $\lambda = 0$, one obtains

$$\begin{aligned} \hat{\mathcal{X}}_i &= i\sqrt{1-\beta^2P^2}\frac{\partial}{\partial P_i}, \\ \hat{\mathcal{P}}_i &= \frac{P_i}{\sqrt{1-\beta^2P^2}} - i\frac{\alpha}{\beta}\sqrt{1-\beta^2P^2}\frac{\partial}{\partial P_i}. \end{aligned} \tag{4.71}$$

As before, the operator $\hat{\mathcal{X}}^2$ coincides with $\hat{x}^2$ and its eigenfunctions and eigenvalues are given by (4.61) and (4.57) respectively.

On the other hand, the operator $\hat{\mathcal{P}}^2$ is given by

$$\begin{aligned} \hat{\mathcal{P}}^2 &= -\frac{\alpha^2}{\beta^2}\left((1-\beta^2P_\rho^2)\frac{\partial^2}{\partial P_\rho^2} + \frac{2-(3\beta^2+2i(\beta/\alpha))P_\rho^2}{P_\rho}\frac{\partial}{\partial P_\rho}\right) \\ &\quad -\frac{(1+2i\alpha\beta)P_\rho^2+3i(\alpha/\beta)}{1-\beta^2P_\rho^2} + \frac{\alpha^2\hat{J}^2}{\beta^2P_\rho^2}. \end{aligned} \tag{4.72}$$

This result has been obtained in [45] for a sightly different operator.

In analogy to the previous calculations, the eigenvalue equation for $\hat{\mathcal{P}}^2$ can be reduced to the form (4.60) by introducing a function ψ such that $\phi = (1-\beta^2P_\rho^2)^{i/2\alpha\beta}\psi$. The solution is therefore

$$\phi_{qml} = const.\times\, sin^l\eta\, cos^{1+i/2\alpha\beta}\eta P_q^{(l+1/2,\,1/2)}(cos\,2\eta)Y_m^l(P_\theta,P_\varphi), \tag{4.73}$$

with $\eta = arcsin\beta P_\rho$. As in the Snyder case, taking $q = (n+1)/2$, one recovers the eigenvalues (4.70).

The position representation

The duality of the SdS algebra for the interchange of $\hat{\mathcal{X}}_i$ and $\hat{\mathcal{P}}_i$ permits to define position representations by simply exchanging the roles of the phase space coordinates. Alternatively, such representations can be obtained starting from those of the symmetries of S^3

in Beltrami coordinates and using transformations analogous to (4.62).

From (4.64) and (4.71) one can in this way obtain the action of the momentum and position operators on the Hilbert space of functions of X_i. For $\lambda = 0$, they take the form

$$\begin{aligned} \hat{\mathcal{P}}_i &= i\frac{\partial}{\partial X_i} + i\alpha^2 X_i\left(X_k\frac{\partial}{\partial X_k} + \mu\right), \\ \hat{\mathcal{X}}_i &= X_i - i\frac{\beta}{\alpha}\left(\frac{\partial}{\partial X_i} + \alpha^2 X_i\left(X_k\frac{\partial}{\partial X_k} + \mu\right)\right), \end{aligned} \tag{4.74}$$

and

$$\begin{aligned} \hat{\mathcal{P}}_i &= i\sqrt{1-\alpha^2X^2}\frac{\partial}{\partial X_i}, \\ \hat{\mathcal{X}}_i &= \frac{X_i}{\sqrt{1-\alpha^2X^2}} - i\frac{\beta}{\alpha}\sqrt{1-\alpha^2X^2}\frac{\partial}{\partial X_i}, \end{aligned} \tag{4.75}$$

respectively. Position representations can be useful in some problems, like the hydrogen atom, where the potential is a nontrivial function of X_i.

All the results could also be extended to the case of negative coupling constants. In this case the algebra would no longer be compact and it is expected that the spectra of position and momentum be continuous [45].

With the aim of generalising the results on the algebraic structure of the three-dimensional nonrelativistic Snyder model investigated in [32], to the case of a curved background, we have studied the representations of the three-dimensional Euclidean SdS algebra. In the SdS case, the algebraic structure is less useful than in the Snyder case, because for SdS the spectrum of the position operator cannot be derived directly from the algebraic structure of the theory. However, since it is possible to find a relation between the representations of the Snyder and the SdS algebra, one can find the spectrum of the square of the position and momentum operators analytically.

Chapter 5

Snyder-type spaces from the Hopf algebroid point of view

From the noncommutative geometry point of view, relatively few investigations have been performed on the Snyder model. Except for a series of papers [8, 9, 39], where the model was generalised and the star product, coproduct and antipodes were calculated, the model was also investigated in [19], where it was considered from a geometrical point of view as a coset in momentum space, with results equivalent to those of [8, 9]. The construction of a quantum field theory on Snyder spacetime was also started in these papers.

The chapter begins with a short review of Hopf algebras and the twist operator [1] and follows with explaining the Hopf algebroid structure of (noncommutative) spacetime [25]. In the last section, the results on the Snyder space from [37] are presented.

5.1 Hopf algebras and the twist operator

5.1.1 Hopf algebras

An associative algebra with unit is a vector space A over a field K equipped with two linear maps

$$m : A \otimes A \to A, \quad \textit{the product} \tag{5.1}$$

$$\eta : K \to A, \quad \textit{the unit}, \tag{5.2}$$

such that $\forall a,\ b \in A$

$$m(m \otimes 1) = m(1 \otimes m), \tag{5.3}$$

$$m(\eta \otimes 1) = m(1 \otimes \eta), \tag{5.4}$$

where $\otimes$ is the tensor product, $m(a \otimes b) = ab$ is the product of a and b, the mapping η is determined by its value $\eta(1_K) \in A$, which is the unit element in A, for 1_K the unit element in K. The identity map $1 :\ A \to A$, is such that $1(a) = a$, $\forall a \in A$.

A coalgebra is a vector space over a field K equipped with two linear mappings

$$\Delta : A \to A \otimes A, \quad \textit{the coproduct}, \tag{5.5}$$

$$\epsilon : A \to K, \quad \textit{the counit}, \tag{5.6}$$

such that

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta, \tag{5.7}$$

$$(\epsilon \otimes 1)\Delta = 1 = (1 \otimes \epsilon)\Delta. \tag{5.8}$$

Eq. (5.3) is the associativity condition and (5.8) is referred to as the coassociativity of the comultiplication Δ. In eqs. (5.4) and (5.8) the isomorphism of algebras $K \otimes A$ and A is understood, so that the objects $\alpha \otimes a$ and a, $\forall \alpha \in K$, $\forall a \in A$, can be identified.

A bialgebra A is a vector space which is an algebra and a coalgebra such that $\forall a,\ b \in A$ the following conditions hold

$$\begin{aligned}\Delta(ab) &= \Delta(a)\Delta(b),\\ \epsilon(ab) &= \epsilon(a)\epsilon(b),\\ \Delta(1) &= 1 \otimes 1,\\ \epsilon(1) &= 1_K.\end{aligned} \tag{5.9}$$

A bialgebra A is a Hopf algebra if there exists an additional mapping

$$S : A \to A, \tag{5.10}$$

called the antipode, such that

$$m(S \otimes 1)\Delta = \eta\epsilon = m(1 \otimes S)\Delta. \tag{5.11}$$

If A is an algebra with generators g_i and relations $\mathcal{R}$: $g_i g_j - g_j g_i - ic^k_{ij} g_k = 0$, the universal enveloping algebra $U(A)$ of A is the free algebra generated by the elements g_i and divided by the ideal generated by the relations $\mathcal{R}$.

The universal enveloping algebra of the Poincaré algebra, which is generated by the momenta P_μ and the Lorentz generators $J_{\mu\nu}$, provides an example of a Hopf algebra if the coproducts, counits and antipodes are given by the following

$$\Delta(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu, \quad \epsilon(P_\mu) = 0, \quad S(P_\mu) = -P_\mu, \tag{5.12}$$

$$\Delta(J_{\mu\nu}) = J_{\mu\nu} \otimes 1 + 1 \otimes J_{\mu\nu}, \quad \epsilon(J_{\mu\nu}) = 0, \quad S(J_{\mu\nu}) = -J_{\mu\nu}. \tag{5.13}$$

5.1.2 The twist operator

The twist operator $\mathcal{F}$ is in invertible element $\mathcal{F} \in U(A) \otimes U(A)$ that satisfies

1. the cocycle condition

$$(\mathcal{F} \otimes 1)(\Delta \otimes 1)\mathcal{F} = (1 \otimes \mathcal{F})(1 \otimes \Delta)\mathcal{F}, \tag{5.14}$$

$$(\mathcal{F}^{-1} \otimes 1)(\Delta \otimes 1)\mathcal{F}^{-1} = (1 \otimes \mathcal{F}^{-1})(1 \otimes \Delta)\mathcal{F}^{-1}, \tag{5.15}$$

2. the normalisation condition

$$m(\epsilon \otimes 1)\mathcal{F} = 1 = m(1 \otimes \epsilon)\mathcal{F}. \tag{5.16}$$

For the primitive coproduct Δ_0 defined as

$$\Delta_0(g_i) = g_i \otimes 1 + 1 \otimes g_i, \quad g_i \in A, \tag{5.17}$$

one can define the twisted coproduct Δ_t as

$$\Delta_t(g_i) = \mathcal{F}\Delta_0(g_i)\mathcal{F}^{-1}. \tag{5.18}$$

If $U(A)$ is a Hopf algebra with the coproduct (5.17), then $U_t(A)$, defined such that the algebraic sector is the same as that of $U(A)$, but the coproduct is replaced with (5.18), will also have all the properties of a Hopf algebra.

5.2 Hopf algebroid structure of (deformed) phase space

The Hopf algebra formalism is a useful tool for dealing with the symmetries of noncommutative spaces, but it cannot be used for describing the phase space, as the universal enveloping algebra of the Heisenberg algebra does not satisfy the properties of a Hopf algebra. Considering a different type of noncommutative spacetime, the κ-Minkowski spacetime, it was shown that the proper framework for describing the (deformed) phase space is that of Hopf algebroids [25].

5.2.1 The undeformed Hopf algebroid

The Heisenberg algebra $\mathcal{H}$ (i.e. quantum phase space) is generated by the commutative coordinates x_μ and momenta p_μ that satisfy the following relations

$$\begin{aligned}
&[x_\mu, x_\nu] \equiv x_\mu x_\nu - x_\nu x_\mu = 0,\\
&[p_\mu, p_\nu] \equiv p_\mu p_\nu - p_\nu p_\mu = 0,\\
&[p_\mu, x_\nu] \equiv p_\mu x_\nu - x_\nu p_\mu = -i\eta_{\mu\nu} 1.
\end{aligned} \tag{5.19}$$

The quantum phase space $\mathcal{H}$ is defined as the free unital algebra generated by x_μ and p_μ, divided by the ideal generated by the relations (5.19). The bases elements in $\mathcal{H}$ are chosen to be normally ordered monomials, i.e. the coordinates x_μ are left from the momenta p_μ. $\mathcal{H}$ can symbolically be written as $\mathcal{H} = \mathcal{A}\mathcal{T}$, where $\mathcal{A}$ is the unital commutative algebra generated by x_μ and $\mathcal{T}$ is the unital commutative algebra generated by p_μ, $\mathcal{T} = \mathcal{C}[[p]]$.

The coproducts of the generators are given by

$$\Delta_0 x_\mu = x_\mu \otimes 1, \quad \Delta_0 p_\mu = p_\mu \otimes 1 + 1 \otimes p_\mu, \tag{5.20}$$

where an equivalence class in $\mathcal{H} \otimes \mathcal{H}$ is generated by the ideal $\mathcal{I}_0 = U_+(\mathcal{R}_0)(\mathcal{A} \otimes 1)\Delta_0\mathcal{T}$. Here, $U_+(\mathcal{R}_0)$ is the universal enveloping algebra generated by $(\mathcal{R}_0)_\mu \equiv x_\mu \otimes 1 - 1 \otimes x_\mu$, but without the unit element [40], and $\Delta_0\mathcal{T} = \mathcal{C}[[\Delta_0 p]] \subset \mathcal{T} \otimes \mathcal{T}$ is the image of $\mathcal{T}$ in $\mathcal{T} \otimes \mathcal{T}$. The elements $\mathcal{R}_0$ satisfy the following properties

$$[\Delta_0 x_\mu, \mathcal{R}_0] = 0, \quad [\Delta_0 p_\mu, \mathcal{R}_0] = 0, \quad [(\mathcal{R}_0)_\mu, (\mathcal{R}_0)_\nu] = 0. \tag{5.21}$$

Since the Heisenberg algebra $\mathcal{H}$ can be written as $\mathcal{H} = \mathcal{A}\mathcal{T}$, it can be shown that $\Delta_0\mathcal{H} = U(\mathcal{R}_0)(\mathcal{A} \otimes 1)\Delta_0\mathcal{T}/\mathcal{I}_0 = [(\mathcal{A} \otimes 1)\Delta_0\mathcal{T} + \mathcal{I}_0]/\mathcal{I}_0$ is an algebra isomorphic to $\mathcal{H}$, and that

$\Delta_0\mathcal{A} = (\mathcal{A}\otimes 1 + \mathcal{I}_0)/\mathcal{I}_0 = (\mathcal{A}\otimes\mathcal{A} + \mathcal{I}_0)/\mathcal{I}_0$ is an algebra isomorphic to $\mathcal{A}$.

For every generator $h \in \mathcal{H}$, the counit ϵ_0 can be defined by the action of h on 1. The coproduct Δ_0 and the counit lead to a bialgebroid structure of the quantum phase space.

The undeformed Hopf algebroid is defined by the total algebra $\mathcal{H}$ (quantum phase space), base algebra $\mathcal{A}$, multiplication m, coproduct Δ_0, antipode S_0, counit ϵ_0, source map α_0 and target map β_0.

The coproduct is a mapping

$$\Delta_0 : \mathcal{H} \mapsto U(\mathcal{R}_0)(\mathcal{A}\otimes\mathcal{A})\Delta_0\mathcal{T}/\mathcal{I}_0, \tag{5.22}$$

defined by (5.20). It is a homomorphism and satisfies the coassociativity condition

$$(\Delta_0 \otimes 1)\Delta_0 = (1 \otimes \Delta_0)\Delta_0. \tag{5.23}$$

The antipode S_0 is a mapping $S_0 : \mathcal{H} \mapsto \mathcal{H}$ and an antihomomorphism $S_0(h_1h_2) = S_0(h_2)S_0(h_1)$, $\forall h_1, h_2 \in \mathcal{H}$. For the generators of the Heisenberg algebra it is given by

$$S_0(x_\mu) = x_\mu, \quad S_0(p_\mu) = -p_\mu. \tag{5.24}$$

The counit $\epsilon_0 : \mathcal{H} \mapsto \mathcal{A}$ is defined by $\epsilon_0(h) = h \triangleright 1 \in \mathcal{A} \subset \mathcal{H}$, $\forall h \in \mathcal{H}$, where the action $\triangleright$ is defined by

$$x_\mu \triangleright f(x) = x_\mu f(x), \quad p_\mu \triangleright f(x) = -i\frac{\partial f(x)}{\partial x^\mu}. \tag{5.25}$$

It can be seen that $\epsilon_0(\mathcal{H}) = \mathcal{A}$. The target map $\alpha_0 : \mathcal{A} \mapsto \mathcal{H}$ and the source map $\beta_0 : \mathcal{A} \mapsto \mathcal{H}$ are equal and for $f(x) \in \mathcal{A}$ are given by $\alpha_0(f(x)) = \beta_0(f(x)) = f(x)$.

The coproduct Δ_0, antipode S_0 and counit ϵ_0 satisfy the following relations

$$\begin{aligned} m(\epsilon_0 \otimes 1)\Delta_0 &= m(1 \otimes \epsilon_0)\Delta_0 = 1, \\ m(S_0 \otimes 1)\Delta_0 &= m(1 \otimes S_0)\Delta_0 = \epsilon_0, \end{aligned} \tag{5.26}$$

and hence are highly related through (5.23)-(5.26). For example, in the momentum representation

$$\epsilon_0^{mom}(p_\mu) = p_\mu, \quad \epsilon_0^{mom}(x_\mu) = 0, \tag{5.27}$$

and then

$$\Delta_0^{mom}(p_\mu) = p_\mu \otimes 1, \quad \Delta_0^{mom}(x_\mu) = x_\mu \otimes 1 + 1 \otimes x_\mu, \tag{5.28}$$

and

$$S_0^{mom}(p_\mu) = p_\mu, \quad S_0^{mom}(x_\mu) = -x_\mu. \tag{5.29}$$

5.2.2 The twisted Hopf algebroid

When dealing with noncommutative spaces, the spacetime is generated by a set of coordinates $\hat{x}_\mu$, which no longer commute, i.e. the relations ($\mathcal{R}_0$) no longer hold, but are replaced by a different set which determines the type of noncommutative spacetime.

For a large class of spacetimes it is possible to find an isomorphism between the noncommutative algebra $\hat{\mathcal{A}}$, generated by the $\hat{x}_\mu$ and the algebra $\mathcal{A}_*$, which is generated by the commutative coordinates x_μ, but the multiplication is replaced with the star product. The star product between two elements $f(x)$, $g(x) \in \mathcal{A}_*$ is defined by

$$f(x) * g(x) = \hat{f}(\hat{x})\hat{g}(\hat{x}) \triangleright 1, \tag{5.30}$$

where $\hat{f}(\hat{x})$ and $\hat{g}(\hat{x})$ are elements of $\hat{A}$, the action $\triangleright$ is defined by (5.25) and in order to calculate the right hand side of (5.30) one needs to choose a specific realisation of the noncommutative coordinates $\hat{x}_\mu$ in terms of the commutative phase space variables x_μ, p_μ.

The deformed phase space, generated by $\hat{x}_\mu$ and p_μ, can be denoted as the deformed Heisenberg algebra $\hat{\mathcal{H}}$. It is convenient to define the action [27] $\blacktriangleright$: $\hat{\mathcal{H}} \otimes \hat{\mathcal{A}} \mapsto \hat{\mathcal{A}}$, where symbolically $\hat{\mathcal{H}} = \hat{\mathcal{A}}\mathcal{T}$, $\hat{\mathcal{A}}$ being the subalgebra of $\hat{\mathcal{H}}$ generated by the $\hat{x}_\mu$ and $\mathcal{T}$ the subalgebra of $\hat{\mathcal{H}}$ generated by p_μ

$$\hat{x}_\mu \blacktriangleright \hat{g}(\hat{x}) = \hat{x}_\mu \hat{g}(\hat{x}), \quad p_\mu \blacktriangleright 1 = 0, \quad p_\mu \blacktriangleright \hat{x}_\nu = -i\eta_{\mu\nu}. \tag{5.31}$$

From the Leibniz rule

$$\hat{x}_\mu \blacktriangleright \hat{f}(\hat{x})\hat{g}(\hat{x}) = \hat{x}_\mu \hat{f}(\hat{x})\hat{g}(\hat{x}), \tag{5.32}$$

follows the coproduct of $\Delta\hat{x}_\mu$

$$\Delta\hat{x}_\mu = \hat{x}_\mu \otimes 1. \tag{5.33}$$

For any choice of realisation, one way of constructing the corresponding coproducts Δ for x_μ and p_μ is using the Leibniz rules for $p_\mu \triangleright (f * g)$ and $x_\mu \triangleright (f * g)$ obtained from the property $h_1 h_2 \triangleright f(x) = h_1 \triangleright (h_2 \triangleright f(x))$, where $h_1,\ h_2 \in \mathcal{H}$ [40].

The relation between the deformed coproducts Δ and the undeformed ones Δ_0, defines the corresponding twist operator

$$\Delta h = \mathcal{F}\Delta_0 h \mathcal{F}^{-1}, \tag{5.34}$$

for every $h \in \mathcal{H}$. Hence,

$$\Delta x_\mu = \mathcal{F}\Delta_0 x_\mu \mathcal{F}^{-1}, \quad \Delta p_\mu = \mathcal{F}\Delta_0 p_\mu \mathcal{F}^{-1}. \tag{5.35}$$

The star product can also be written in terms of the twist operator

$$f(x) * g(x) = m_* \left(f(x) \otimes g(x)\right) = m\left(\mathcal{F}^{-1} \triangleright (f \otimes g)\right), \tag{5.36}$$

with m the multiplication map $m(h_1 \otimes h_2) = h_1 h_2$, and m_* the multiplication map defined by $m_*(h_1 \otimes h_2) = h_1 * h_2,\ \forall h_1,\ h_2 \in \mathcal{H}$. The star product does not change if $\mathcal{F}^{-1} \to \mathcal{F}^{-1} + \mathcal{J}_0$, where $\mathcal{J}_0 = U_+(\mathcal{R}_0)\mathcal{H} \otimes \mathcal{H}$ is the right ideal with the property $m\left(\mathcal{J}_0 \triangleright (f \otimes g)\right) = 0$. On the other hand, $f(x)g(x) = m(f \otimes g) = m_*\left(\mathcal{F} \triangleright (f \otimes g)\right)$ does not change if $\mathcal{F} \to \mathcal{F} + \mathcal{J}$, where $\mathcal{J}$ is also a right ideal defined by $\mathcal{J} = U_+(\mathcal{R})\mathcal{H} \otimes \mathcal{H} = \mathcal{F}\mathcal{J}_0$, with $\mathcal{R}_\mu = \mathcal{F}(\mathcal{R}_0)\mathcal{F}^{-1}$. The right ideal $\mathcal{J}$ has the property $m_*\left(\mathcal{J} \triangleright (f \otimes g)\right) = 0$. The right ideals $\mathcal{J}_0$ and $\mathcal{J}$ also satisfy

$$
\begin{aligned}
&\mathcal{J}_0(\mathcal{H}\otimes\mathcal{H}) = \mathcal{J}_0,\\
&\mathcal{J}(\mathcal{H}\otimes\mathcal{H}) = \mathcal{J},\\
&\mathcal{J} = \mathcal{F}\mathcal{J}_0\mathcal{F}^{-1} = \mathcal{F}\mathcal{J}_0\\
&\mathcal{J}\mathcal{J}_0 \subset \mathcal{J}, \quad \mathcal{J}_0\mathcal{J} \subset \mathcal{J}_0.
\end{aligned} \tag{5.37}
$$

It can be shown that $\Delta\mathcal{H} = U(\mathcal{R})(\mathcal{A}\otimes 1)_{\mathcal{F}}\Delta\mathcal{T}/\mathcal{I}$ is an algebra isomorphic to $\mathcal{H}$, where $(\mathcal{A}\otimes 1)_{\mathcal{F}} = \mathcal{F}(\mathcal{A}\otimes 1)\mathcal{F}^{-1}$ and $\mathcal{I} = \mathcal{F}(\mathcal{I}_0)\mathcal{F}^{-1} = U_+(\mathcal{R})(\mathcal{A}\otimes 1)_{\mathcal{F}}\Delta\mathcal{T}$. It can also be shown that $\Delta\mathcal{A} = \mathcal{F}(\Delta_0\mathcal{A})\mathcal{F}^{-1} = ((\mathcal{A}\otimes 1)_{\mathcal{F}} + \mathcal{I})/\mathcal{I}$ is an algebra isomorphic to $\mathcal{A}$. The elements $\mathcal{R}_\mu$ satisfy the following properties

$$
[\Delta x_\mu, \mathcal{R}] = 0, \quad [\Delta p_\mu, \mathcal{R}] = 0, \quad [\mathcal{R}_\mu, \mathcal{R}_\nu] = 0. \tag{5.38}
$$

The ideals $\mathcal{I}_0$ and $\mathcal{I}$ satisfy

$$
\begin{aligned}
&\mathcal{I}_0\Delta_0\mathcal{H} = \Delta_0\mathcal{H}\mathcal{I}_0 = \mathcal{I}_0,\\
&\mathcal{I}\Delta\mathcal{H} = \Delta\mathcal{H}\mathcal{I} = \mathcal{I},\\
&\mathcal{I}\mathcal{I}_0 \subset \mathcal{J}, \quad \mathcal{I}_0\mathcal{I} \subset \mathcal{J}_0.
\end{aligned} \tag{5.39}
$$

The twist $\mathcal{F} \in (\mathcal{H}\otimes\mathcal{H})/\mathcal{J}$ is a mapping $\mathcal{F} : \Delta_0\mathcal{H} \mapsto \Delta\mathcal{H}$, while its inverse $\mathcal{F}^{-1} \in (\mathcal{H}\otimes\mathcal{H})/\mathcal{J}_0$ is a mapping $\mathcal{F}^{-1} : \Delta\mathcal{H} \mapsto \Delta_0\mathcal{H}$.

The twisted Hopf algebroid is defined by the total algebra $\mathcal{H}$ (quantum phase space), the base algebra $\hat{\mathcal{A}}$ (where the elements of $\hat{\mathcal{A}}$ are taken in a particular realisation), multiplication m, twisted coproduct $\Delta_{\mathcal{F}} \equiv \Delta$, antipode $S_{\mathcal{F}} \equiv S$, counit $\epsilon_{\mathcal{F}} \equiv \hat{\epsilon}$, source map $\hat{\alpha}$ and target map $\hat{\beta}$. The twisted structure also satisfies the axioms of a Hopf algebroid.

Using the twist $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$, that satisfy the cocycle condition (5.14) and the normalisation condition (5.16), the twisted coproduct defined as $\Delta h = \mathcal{F}\Delta_0 h\mathcal{F}^{-1}$, $\forall h \in \mathcal{H}$

will satisfy the coassociativity condition

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta. \tag{5.40}$$

The antipode $S : \mathcal{H} \mapsto \mathcal{H}$ is an antihomomorphism defined by

$$S(h) = \chi S_0(h)\chi^{-1}, \tag{5.41}$$

where $\chi^{-1} = m\left((S_0 \otimes 1)\mathcal{F}^{-1}\right)$.

The counit $\hat{\epsilon} : \mathcal{H} \mapsto \hat{\mathcal{A}} \subset \mathcal{H}$ is defined by

$$\epsilon(h) = m\left(\mathcal{F}^{-1}(\triangleright \otimes 1)(\epsilon_0(h) \otimes 1)\right). \tag{5.42}$$

So it follows that

$$\begin{aligned} \hat{\epsilon}(f) &= \hat{f}, \quad \epsilon_0(\hat{f}) = f, \\ \hat{\epsilon}(f * g) &= \hat{f}\hat{g}, \quad \epsilon_0(\hat{f}\hat{g}) = f * g, \\ \hat{\epsilon}(\epsilon_0(\hat{f})) &= \hat{f}, \quad \epsilon_0(\hat{\epsilon}(f)) = f, \end{aligned} \tag{5.43}$$

$\forall f, g \in \mathcal{A}$, and $\forall \hat{f}, \hat{g} \in \hat{\mathcal{A}}$.

The source and the target map can also be reconstructed from the twist. One first defines the maps $\alpha : \mathcal{A}_* \to \hat{\mathcal{A}} \subset \mathcal{H}$ and $\beta : \mathcal{A}_* \to \mathcal{H}$ by

$$\begin{aligned} \alpha(f(x)) &= m\left(\mathcal{F}^{-1}(\triangleright \otimes 1)(\alpha_0(f(x)) \otimes 1)\right), \quad \alpha_0(f(x)) = f(x), \\ \beta(f(x)) &= m\left(\tilde{\mathcal{F}}^{-1}(\triangleright \otimes 1)(\beta_0(f(x)) \otimes 1)\right), \quad \beta_0(f(x)) = f(x), \end{aligned} \tag{5.44}$$

where $\tilde{\mathcal{F}} = \tau_0 \mathcal{F} \tau_0$ and τ_0 is the flip operator $\tau_0(h_1 \otimes h_2) = h_2 \otimes h_1$, $\forall h_1, h_2 \in \mathcal{H}$. The source and target map are then given by

$$\hat{\alpha} = \alpha\epsilon_0 \mid_{\hat{\mathcal{A}}}, \quad \hat{\beta} = \beta\epsilon_0 \mid_{\hat{\mathcal{A}}}. \tag{5.45}$$

The coproduct Δ, antipode S and counit ϵ satisfy the following relations

$$
\begin{aligned}
&m(\hat{\epsilon}\otimes 1)\Delta = m\left(1\otimes S^{-1}\hat{\epsilon}S\right)\Delta = 1,\\
&m(S\otimes 1)\Delta = S^{-1}\hat{\epsilon}S,\\
&m(1\otimes S)\Delta = \hat{\epsilon},
\end{aligned}
\tag{5.46}
$$

which are compatible with the Hopf algebroid structure in [31]. The relation between the presented approach and Lu's paper [31] is $\alpha\epsilon \mapsto \hat{\epsilon}$ and $\beta\epsilon \mapsto S^{-1}\hat{\epsilon}$, since in the present case $\hat{\epsilon}: \mathcal{H} \mapsto \hat{\mathcal{A}} \subset \mathcal{H}$. In the undeformed case the relation is $\alpha\epsilon \mapsto \epsilon_0$ and $\beta\epsilon \mapsto S_0^{-1}\epsilon_0 = \epsilon_0$.

5.2.3 The Hopf algebroid structure of $\hat{\mathcal{H}}$

The deformed phase space $\hat{\mathcal{H}}$, generated by the noncommutative coordinates $\hat{x}_\mu$ and the momenta p_μ also has a Hopf algebroid structure which is defined by the total algebra $\hat{\mathcal{H}}$, base algebra $\hat{\mathcal{A}} \subset \hat{\mathcal{H}}$, multiplication map m, coproduct Δ (which is for the coordinates $\hat{x}_\mu$ given by (5.33), and for a given realisation needs to be calculated for the momenta), antipode S, counit $\hat{\epsilon}$, source map $\hat{\alpha}$ and target map $\hat{\beta}$. The counit is defined by

$$\hat{\epsilon}(\hat{h}) = \hat{h} \blacktriangleright 1, \quad \forall \hat{h} \in \hat{\mathcal{H}}. \tag{5.47}$$

It is useful to introduce $\hat{y}_\mu$ as the right multiplication by $\hat{x}_\mu$

$$\hat{y}_\mu \blacktriangleright \hat{f}(\hat{x}) = \hat{f}(\hat{x})\hat{x}_\mu, \tag{5.48}$$

from where it follows

$$\Delta\hat{y}_\mu = 1\otimes\hat{y}_\mu. \tag{5.49}$$

The relation $\hat{Q}_\mu = \hat{y}_\mu \otimes 1 - 1\otimes\hat{x}_\mu$, which has the property $\hat{Q}_\mu \blacktriangleright \hat{\mathcal{A}}\otimes\hat{\mathcal{A}} = 0$, generates the right ideal $\mathcal{J} = U_+(\hat{Q})\mathcal{H}\otimes\hat{\mathcal{H}}$, which satisfies $\mathcal{J} \blacktriangleright \hat{\mathcal{A}}\otimes\hat{\mathcal{A}} = 0$.

The antipode S is then defined by $S(\hat{y}_\mu) = \hat{x}_\mu$ and satisfies

$$
\begin{aligned}
&m(\hat{\epsilon} \otimes 1)\Delta = m(1 \otimes S^{-1}\hat{\epsilon}S)\Delta = 1, \\
&m(S \otimes 1)\Delta = S^{-1}\hat{\epsilon}S \\
&m(1 \otimes S)\Delta = \hat{\epsilon}.
\end{aligned} \tag{5.50}
$$

The antipode for p_μ, $S(p_\mu)$ follows from $m(S \otimes 1)\Delta(p_\mu) = m(1 \otimes S)\Delta(p_\mu) = 0$. The source map $\hat{\alpha}$: $\hat{\mathcal{A}} \mapsto \hat{\mathcal{H}}$ is a homomorphism and the target map $\hat{\beta}$: $\hat{\mathcal{A}} \mapsto \hat{\mathcal{H}}$ is an antihomomorphism defined by $\hat{\beta} = S^{-1}\hat{\alpha}$.

5.3 Snyder-type spaces, twisted Poincaré algebra and addition of momenta

5.3.1 Snyder-type spaces

We define generalised Snyder spaces as deformations of ordinary phase space, generated by noncommutative coordinates $\hat{x}_\mu$ and momenta $\hat{p}_\mu$ that span a deformed Heisenberg algebra $\hat{\mathcal{H}}(\hat{x}, \hat{p})$ of the type

$$
[\hat{x}_\mu, \hat{x}_\nu] = i\beta^2 \hat{J}_{\mu\nu}\psi(\beta^2\hat{p}^2), \quad [\hat{p}_\mu, \hat{p}_\nu] = 0, \quad [\hat{p}_\mu, \hat{x}_\nu] = -i\varphi_{\mu\nu}(\beta^2\hat{p}^2). \tag{5.51}
$$

The algebra also includes the Lorentz generators $\hat{J}_{\mu\nu}$ that satisfy the standard relations

$$
\begin{aligned}
&[\hat{J}_{\mu\nu}, \hat{J}_{\rho,\sigma}] = i(\eta_{\mu\nu}\hat{J}_{\nu\sigma} - \eta_{\mu\sigma}\hat{J}_{\nu\rho} + \eta_{\nu\rho}\hat{J}_{\mu\sigma} - \eta_{\nu\sigma}\hat{J}_{\mu\rho}), \\
&[\hat{J}_{\mu\nu}, \hat{p}_\lambda] = i(\eta_{\mu\nu} - \eta_{\lambda\nu}\hat{x}_\mu), \quad [\hat{J}_{\mu\nu}, \hat{x}_\lambda] = i(\eta_{\mu\nu} - \eta_{\lambda\nu}\hat{x}_\mu).
\end{aligned} \tag{5.52}
$$

The functions $\psi(\beta^2\hat{p}^2)$ and $\varphi_{\mu\nu}(\beta^2\hat{p}^2)$ are constrained by the requirement that the Jacobi identities hold and $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$. The commutation relations (5.51)-(5.52) generalise those originally investigated in [56], that are recovered for ψ =const.

The commutative coordinates x_μ and momenta p_μ, that generate the undeformed Heisenberg algebra $\mathcal{H}(x,p)$, satisfy

$$[x_\mu, x_\nu] = 0, \quad [p_\mu, p_\nu] = 0, \quad [p_\mu, x_\nu] = -i\eta_{\mu\nu}. \tag{5.53}$$

The action of x_μ and p_μ on functions $f(x)$ belonging to the enveloping algebra $\mathcal{A}$ generated by the x_μ is defined by

$$x_\mu \triangleright f(x) = x_\mu f(x), \quad p_\mu \triangleright f(x) = -i\frac{\partial f(x)}{\partial x^\mu}. \tag{5.54}$$

The noncommutative coordinates $\hat{x}_\mu$, the momenta $\hat{p}_\mu$ and the Lorentz generators $\hat{J}_{\mu\nu}$ in (5.51)-(5.52) can be expressed in terms of commutative coordinates x_μ and p_μ as [8, 9]

$$\hat{x}_\mu = x_\mu\varphi_1(\beta^2p^2) + \beta^2 x\cdot p p_\mu\varphi_2(\beta^2p^2) + \beta^2 p_\mu\chi(\beta^2p^2), \tag{5.55}$$

$$\hat{p}_\mu = p_\mu, \quad \hat{J}_{\mu\nu} \equiv J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu. \tag{5.56}$$

The function χ does not appear in the defining relations (5.51)-(5.52), but takes into account ambiguities arising from operator ordering of x_μ and p_μ in eq.(5.55).

In terms of the realisation (5.55), the functions $\varphi_{\mu\nu}$ in (5.51) read

$$\varphi_{\mu\nu} = \eta_{\mu\nu}\varphi_1 + \beta^2 p_\mu p_\nu\varphi_2, \tag{5.57}$$

while the Jacobi identities are satisfied if

$$\psi = -2\varphi_1\varphi_1' + \varphi_1\varphi_2 - 2\beta^2p^2\varphi_1'\varphi_2, \tag{5.58}$$

where the prime denotes a derivative with respect to β^2p^2. In particular, the function ψ does not depend on the function χ.

From eq.(5.58), it follows that the coordinates $\hat{x}_\mu$ are commutative for $\varphi_2 = \frac{2\varphi_1'\varphi_1}{\varphi_1 - 2\beta^2p^2\varphi_1'}$, and correspond to the Snyder space for $\varphi_2 = \frac{1+2\varphi_1'\varphi_1}{\varphi_1-2\beta^2p^2\varphi_1'}$. In particular, the Snyder realisation of the Snyder space is recovered for $\varphi_1 = \varphi_2 = 1$, and the Maggiore realisation for $\varphi_1 = \sqrt{1-\beta^2p^2}$, $\varphi_2 = 0$ [8, 9]. There is also another interesting realisation of the Snyder space for $\psi = s$ =const., given by

$$\hat{x}_\mu = x_\mu + \frac{\beta^2}{4}C_\mu, \tag{5.59}$$

where $C_\mu = x_\mu p^2 - 2x\cdot pp_\mu$ are the generators of conformal transformations in momentum space, with $[C_\mu, C_\nu] = 0$.

The algebra (5.51) unifies commutative space $\psi = 0$, and Snyder space $\psi = 1$. Since the Lorentz transformations are not deformed, the Casimir operator of the algebra (5.51)-(5.52) is $C = p^2$.

It can be shown that [27, 41, 28]

$$e^{ik\cdot\hat{x}} \triangleright 1 = e^{iK(k)\cdot x + ig(k)}, \tag{5.60}$$

and

$$e^{ik\cdot\hat{x}} \triangleright e^{iq\cdot x} = e^{i\mathcal{P}(k,q)\cdot x + i\mathcal{Q}(k,q)}, \tag{5.61}$$

where eqs. (5.60) and (5.61) can be seen as the defining relations for the functions $K(k)$, $g(k)$, $\mathcal{P}(k,q)$ and $\mathcal{Q}(k,q)$. It is easily seen that

$$\mathcal{P}_\mu(k,0) = K_\mu(k), \quad \mathcal{P}_\mu(0,q) = q_\mu, \tag{5.62}$$

and it can be checked that

$$e^{-i\lambda k\cdot\hat{x}} p_\mu e^{i\lambda k\cdot\hat{x}} \triangleright e^{iq\cdot x} = \mathcal{P}_\mu(\lambda k, q)e^{iq\cdot x}, \tag{5.63}$$

with λ a real parameter. Differentiating both sides of the last equation by λ, it follows that the function $\mathcal{P}_\mu(\lambda k, q)$ satisfies the differential equation

$$\frac{d\mathcal{P}_\mu(\lambda k, q)}{d\lambda} = k_\alpha \varphi_\mu^\alpha \left(\mathcal{P}(\lambda k, q)\right), \tag{5.64}$$

where the relation (5.57) was used to write the realisation (5.55) in terms of $\varphi_{\mu\nu}$.

The generalised addition of momenta k_μ and q_μ is then defined as [9, 39, 24]

$$k_\mu \oplus q_\mu = \mathcal{D}_\mu(k, q), \tag{5.65}$$

where $\mathcal{D}_\mu(k, 0) = k_\mu$, $\mathcal{D}_\mu(0, q) = q_\mu$, and the function $\mathcal{D}_\mu(k, q)$ can be calculated in terms of $\varphi_{\mu\nu}$ as

$$\mathcal{D}_\mu(k, q) = \mathcal{P}_\mu(K^{-1}(k), q), \tag{5.66}$$

with $K_\mu^{-1}(k)$ the inverse map of $K_\mu(k)$, i.e. $K_\mu^{-1}(K(k)) = k_\mu$. From (5.64) and (5.57) it follows that $\mathcal{P}_\mu(k, q)$ and hence also $\mathcal{D}_\mu(k, q)$ do not depend on the function χ in (5.55).

Writing the inverse of eq.(5.60),

$$e^{ik\cdot x} = e^{iK^{-1}(k)\cdot\hat{x} - ig(K^{-1}(k))} \triangleright 1, \tag{5.67}$$

it follows that the star product of two plane waves is given by

$$\begin{aligned} e^{ik\cdot x} * e^{iq\cdot x} &= e^{iK^{-1}(k)\cdot\hat{x} - ig(K^{-1}(k))} \triangleright e^{iq\cdot x} \\ &= e^{i\mathcal{P}(K^{-1}(k), q)\cdot x + i\mathcal{Q}(K^{-1}(k), q) - ig(K^{-1}(k))}. \end{aligned} \tag{5.68}$$

Using (5.66), noting that $g(k) = \mathcal{Q}(k, 0)$, and defining the function $\mathcal{G}(k, q)$ by

$$\mathcal{G}(k, q) = \mathcal{Q}(K^{-1}(k), q) - \mathcal{Q}(K^{-1}(k), 0), \tag{5.69}$$

it follows that the star product can be written as

$$e^{ik\cdot x} * e^{iq\cdot x} = e^{i\mathcal{D}(k,q)\cdot x + i\mathcal{G}(k,q)}. \tag{5.70}$$

It should be noted that $\mathcal{G}$ vanishes if $\chi(k) = 0$. Defining $\hat{x}^\mu_{(0)} = x^\alpha \varphi^\mu_\alpha$ and using (5.57) when writing down the realisation (5.55) for $\hat{x}_\mu$, one can check that the following equality holds

$$e^{-i\lambda k\cdot \hat{x}_{(0)}} p_\mu e^{i\lambda k\cdot \hat{x}} \triangleright e^{iq\cdot x} = \mathcal{P}_\mu(\lambda k, q) e^{iq\cdot x + i\mathcal{Q}(\lambda k, q)}, \tag{5.71}$$

with λ a real parameter. Differentiating both sides of (5.71) by λ and using (5.64), it follows that $\mathcal{Q}(k, q)$ satisfies the differential equation

$$\frac{d\mathcal{Q}(\lambda k, q)}{d\lambda} = k_\alpha \chi^\alpha \left(\mathcal{P}(\lambda k, q)\right), \tag{5.72}$$

with $\mathcal{Q}(0, q) = 0$ and $\chi^\alpha \equiv p^\alpha \chi(\beta^2 p^2)$.

We next follow the approach of Sec. 5.2. Since the coproduct for the Snyder space does not satisfy the coassociativity condition, the structure corresponding to the phase space will be that of a quasi-Hopf algebroid. Hence,the algebra $\mathcal{A}$, generated by the commutative coordinates x_μ, can be extended to the quasi-Hopf algebroid $\mathcal{H}$ generated by the x_μ and the p_μ, symbolically indicated as $\mathcal{H} = \mathcal{A}\mathcal{T}$, where $\mathcal{T}$ is the algebra generated by the p_μ [25]. In this approach, the coproduct for the momenta Δp_μ is obtained from $\mathcal{D}_\mu(k, q)$ as

$$\Delta p_\mu = \mathcal{D}_\mu(p \otimes 1, 1 \otimes p). \tag{5.73}$$

From this definition it follows that the addition of momenta and the coproduct do not depend on $\chi(\beta^2 p^2)$.

From the coproduct one can then define the twist $\mathcal{F}$, such that $\Delta h = \mathcal{F}\Delta_0 h \mathcal{F}^{-1}$ for any $h \in \mathcal{H}$, as [25, 20]

$$\mathcal{F}^{-1} =: \exp\left\{i(1 \otimes x^\alpha)(\Delta - \Delta_0)p_\alpha + \mathcal{G}(p \otimes 1, 1 \otimes p)\right\} :, \tag{5.74}$$

where $\Delta_0 p_\mu = p_\mu \otimes 1 + 1 \otimes p_\mu$, and : : denotes normal ordering in which the coordinates x_α stand on the left of the momenta p_α.

In this approach, the star product $f * g$ can be defined as

$$(f * g)(x) = m\left(\mathcal{F}^{-1}(\triangleright \otimes \triangleright)(f \otimes g)\right), \quad f, g \in \mathcal{A}, \tag{5.75}$$

with $m: \mathcal{A} \otimes \mathcal{H} \to \mathcal{H}$ the multiplication map of $\mathcal{A}$.

The relation (5.55) between $\hat{x}_\mu$ and x_μ can also be written in terms of the twist as

$$\hat{x}_\mu = m\left(\mathcal{F}^{-1}(\triangleright \otimes 1)(x_\mu \otimes 1)\right) = x_\alpha \varphi^\alpha_\mu(p) + \beta^2 p_\mu \chi(p). \tag{5.76}$$

It follows for consistency

$$\Delta p_\mu = \mathcal{F}(\Delta_0 p_\mu)\mathcal{F}^{-1}, \quad \Delta_0 p_\mu = p_\mu \otimes 1 + 1 \otimes p_\mu, \tag{5.77}$$

in accordance with (5.73).

The coproducts of the momenta were found for special cases in [9]. For the Snyder realisation, it reads

$$\Delta p_\mu = \frac{1}{1 - \beta^2 p_\alpha \otimes p^\alpha}\left(p_\mu \otimes 1 - \frac{\beta^2}{1 + \sqrt{1 + \beta^2 p^2}} p_\mu p_\alpha \otimes p^\alpha + \sqrt{1 + \beta^2 p^2} \otimes p_\mu\right), \tag{5.78}$$

and for the Maggiore realisation

$$\Delta p_\mu = p_\mu \otimes \sqrt{1 - \beta^2 p^2} - \frac{\beta^2}{1 + \sqrt{1 - \beta^2 p^2}} p_\mu p_\alpha \otimes p^\alpha + 1 \otimes p_\mu. \tag{5.79}$$

The coproducts of the Lorentz generators are given by

$$\Delta J_{\mu\nu} = \mathcal{F}(\Delta_0 J_{\mu\nu})\mathcal{F}^{-1}, \quad \Delta_0 J_{\mu\nu} = J_{\mu\nu} \otimes 1 + 1 \otimes J_{\mu\nu}. \tag{5.80}$$

Because of the commutation relations (5.52), the coproduct of $J_{\mu\nu}$ will be trivial, i.e.

$\Delta J_{\mu\nu} = \Delta_0 J_{\mu\nu}$ [9].

For the coordinates, the coproducts are given by

$$\Delta\hat{x}_\mu = \hat{x}_\mu \otimes 1 + \mathcal{I}, \tag{5.81}$$

where $\mathcal{I}$ is the right ideal of $\mathcal{H}$ with the property $m_*\left(\mathcal{I}(\triangleright \otimes \triangleright)(f \otimes g)\right) = 0$, where $m_*(f \otimes g) = f * g$ for $f, g \in \mathcal{A}$. Hence,

$$\Delta x_\mu = \mathcal{F}\Delta_0 x_\mu \mathcal{F}^{-1} = \mathcal{F}(x_\mu \otimes 1)\mathcal{F}^{-1}. \tag{5.82}$$

The antipodes for the Snyder space are trivial [9]

$$S(p_\mu) = -p_\mu, \quad S(J_{\mu\nu}) = -J_{\mu\nu}. \tag{5.83}$$

5.3.2 First order expansion

The study of the general form of the deformed Heisenberg algebra (5.51) is difficult, however it can be studied perturbatively, by expanding the realisation (5.55) of the noncommutative coordinates in powers of β^2

$$\hat{x}_\mu = x_\mu + \beta^2(s_1 x_\mu p^2 + s_2 x \cdot p p_\mu + c p_\mu) + O(\beta^2), \tag{5.84}$$

with parameters s_1, s_2, c. It can be seen that the commutation relations do not depend on the parameter c and up to the first order are given by

$$[\hat{x}_\mu, \hat{x}_\nu] = i\beta^2 s J_{\mu\nu} + O(\beta^2), \quad [p_\mu, \hat{x}_\nu] = -i\left(\eta_{\mu\nu}(1 + \beta^2 s_1 p^2) + \beta^2 s_2 p_\mu p_\nu\right) + O(\beta^4), \tag{5.85}$$

where $s = s_2 - 2s_1$.

The models of [8, 9] are recovered for $s_2 = 1 + 2s_1$. Furthermore, for $s_1 = 0$, $s_2 = 1$, eqs.(5.84)-(5.85) reproduce the exact Snyder realisation, while for $s_1 = -1/2$, $s_2 = 0$, they give the first order expansion of the Maggiore realisation. For $s_2 = 2s_1$, the spacetime is commutative to first order in β^2, and for $s_1 = -s/4$, $s_2 = s/2$, $c = 0$, one gets the exact realisation (5.59).

The first order expression for the function $\mathcal{P}_\mu(k,q)$ is given by

$$\begin{aligned}\mathcal{P}_\mu(k,q) &= q_\mu + \int_0^1 d\lambda \left\{k_\mu + \beta^2 \left[s_1 k_\mu(\lambda k + q)^2 + s_2(\lambda k^2 + k\cdot q)(\lambda k_\mu + q_\mu)\right]\right\} + O(\beta^4) \\ &= k_\mu + q_\mu + \beta^2 \left[\left(s_1 q^2 + \left(s_1 + \frac{s_2}{2}\right) k\cdot q + \frac{s_1 + s_2}{3} k^2\right) k_\mu + s_2 \left(k\cdot q + \frac{k^2}{2}\right) q_\mu\right] \\ &\quad + O(\beta^4)\end{aligned} \tag{5.86}$$

from where it follows that

$$K_\mu^{-1}(k) = k_\mu - \frac{\beta^2}{3}(s_1 + s_2) k^2 k_\mu + O(\beta^4). \tag{5.87}$$

Using these results, it is possible to write down the generalised addition law of the momenta k_μ and q_μ at first order

$$(k \oplus q)_\mu = \mathcal{D}_\mu(k,q) = k_\mu + q_\mu + \beta^2 \left(s_2 k\cdot q q_\mu + s_1 q^2 k_\mu + \left(s_1 + \frac{s_2}{2}\right) k\cdot q k_\mu + \frac{s_2}{2} k^2 q_\mu\right) + O(\beta^4). \tag{5.88}$$

In particular, for the "conformal" case $s_1 = -s/4$, $s_2 = s/2$,

$$(k \oplus q)_\mu = k_\mu + q_\mu + \frac{\beta^2 s}{4}\left(2k\cdot q q_\mu - q^2 k_\mu + k^2 q_\mu\right) + O(\beta^4). \tag{5.89}$$

Another interesting outcome is that for $s_2 = 2s_1 \neq 0$, $s = 0$, although spacetime is commutative up to the first order in β^2, the addition of momenta is deformed

$$(k \oplus q)_\mu \neq k_\mu + q_\mu. \tag{5.90}$$

The Lorentz transformations of momenta are not deformed and, denoting them by $\Lambda(\xi, p)$, with ξ the rapidity parameter, the law of addition of momenta implies that

$$\Lambda(\xi, k \oplus q) = \Lambda(\xi_1, k) \oplus \Lambda(\xi_2, q) \tag{5.91}$$

is satisfied for $\xi_1 = \xi_2 = \xi$. Hence, there are no backreaction factors in the sense of [21, 35], meaning that in composite systems the boosted momenta of the single particles are independent of the momenta of the other particles in the system.

The coproduct up to the first order can be read from (5.88) and is given by

$$\Delta p_\mu = \Delta_0 p_\mu + \beta^2 \left(s_1 p_\mu \otimes p^2 + s_2 p_\alpha \otimes p^\alpha p_\mu + \left(s_1 + \frac{s_2}{2} \right) p_\mu p_\alpha \otimes p^\alpha + \frac{s}{2} p^2 \otimes p_\mu \right) + O(\beta^4). \tag{5.92}$$

The corresponding twist operator $\mathcal{F}^{-1}$ is

$$\mathcal{F}^{-1} = 1 \otimes 1 + i(1 \otimes x_\alpha)(\Delta - \Delta_0)p^\alpha + ic\beta^2 p_\alpha \otimes p^\alpha + O(\beta^4). \tag{5.93}$$

From this, it is possible to calculate the coproducts $\Delta J_{\mu\nu}$, Δx_μ and the antipodes $S(p_\mu)$, $S(J_{\mu\nu})$, $S(x_\mu)$ in the Hopf algebroid sense.

In general, the exact twist will not satisfy the cocycle condition, the corresponding star product will be non-associative and the coproducts Δp_μ, $\Delta J_{\mu\nu}$ will be non-coassociative [9], hence the corresponding phase space will have a quasi-Hopf algebroid structure. An exception is given in the commutative case $s_2 = 2s_1$, when the star product is associative and the corresponding coproduct Δp_μ is cocommutative and coassociative.

Using the twist (5.93) to calculate the coproduct of p_μ as in (5.77), one gets again (5.92), the same result as when using the function $\mathcal{D}$, while using (5.80) to calculate the coproduct of $J_{\mu\nu}$ gives $\Delta J_{\mu\nu} = \Delta_0 J_{\mu\nu} + O(\beta^4)$.

In the special case $s_2 = 2s_1$, $s = 0$, which corresponds to commutative space, it is easily seen from (5.92) that

$$\tilde{\Delta} p_\mu \equiv \tau_0 \Delta p_\mu \tau_0 = \Delta p_\mu, \tag{5.94}$$

i.e. the coproduct is left-right symmetric, with the flip operator τ_0 defined in the usual way as

$$\tau_0(A \otimes B) = B \otimes A. \tag{5.95}$$

The coproduct will also be cocommutative and the corresponding star product will be commutative, but not local.

The flip operator, $\tau = \mathcal{F}\tau_0\mathcal{F}^{-1}$, is relevant in the discussion of the twisted statistics of particles in quantum field theory on noncommutative spaces [40, 20]. Another important operator in this context is the $R-$matrix, which satisfies the relation $R\Delta_0 p_\mu R^{-1} = \tilde{\Delta} p_\mu$. Defining $\tilde{\mathcal{F}} = \tau_0\mathcal{F}\tau_0$, it can be written as

$$R = \tilde{\mathcal{F}}\mathcal{F}^{-1} = 1 \otimes 1 + R_{cl} + O(\beta^4), \tag{5.96}$$

where the classical $R-$matrix R_{cl} reads

$$R_{cl} = (x_\alpha \otimes 1)(\tilde{\Delta} - \Delta_0)p^\alpha - (1 \otimes x_\alpha)(\Delta - \Delta_0)p^\alpha, \tag{5.97}$$

where Δp_μ is given by (5.92). For commutative spaces, R_{cl} is given by

$$R_{cl} = (x_\alpha \otimes 1 - 1 \otimes x_\alpha)(\Delta - \Delta_0)p^\alpha \in \mathcal{I}_0, \tag{5.98}$$

where $\mathcal{I}_0$ is the right ideal of $\mathcal{H}$ with the property $m\left(\mathcal{I}_0 \triangleright (f \otimes g)\right) = 0$. Its relation with the right ideal $\mathcal{I}$ is given by $\mathcal{I} = \mathcal{F}\mathcal{I}_0\mathcal{F}^{-1}$.

5.3.3 Twist for the Snyder realisation

As it turns out, for the case of the Snyder realisation of the Snyder space, it is possible to find the exact twist operator using the perturbative approach introduced in [25], by expanding the coproduct in powers of β^2. The Snyder realisation corresponds to the case $\varphi_1 = \varphi_2 = 1, \chi = 0$, i.e.

$$\hat{x}_\mu = x_\mu + \beta^2 x \cdot p p_\mu. \tag{5.99}$$

The coproduct of the momenta is given by (5.78). It is expanded with respect to the deformation parameter β^2 as $\Delta p_\mu = \sum_{k=0}^{\infty} \Delta_k p_\mu$, with $\Delta_k p_\mu \propto (\beta^2)^k$

$$
\begin{aligned}
\Delta p_\mu \;=\;& p_\mu \otimes 1 + 1 \otimes p_\mu + \beta^2 \left(\frac{1}{2} p_\mu p_\alpha \otimes p^\alpha + p_\alpha \otimes p^\alpha p_\mu + \frac{1}{2} p^2 \otimes p_\mu \right) \\
&+\beta^4 \left(\frac{1}{2} p_\mu p_\alpha p_\beta \otimes p^\alpha p^\beta + p_\alpha p_\beta \otimes p^\alpha p^\beta p_\mu + \frac{1}{8} p_\mu p_\alpha p^2 \otimes p^\alpha - \frac{1}{8} p^4 \otimes p_\mu \right. \\
&\left. + \frac{1}{2} p_\alpha p^2 \otimes p^\alpha p_\mu \right) + \beta^6 \left(\frac{1}{2} p_\mu p_\alpha p_\beta p_\gamma \otimes p^\alpha p^\beta p^\gamma + p_\alpha p_\beta p_\gamma \otimes p^\alpha p^\beta p^\gamma p_\mu \right. \\
&- \frac{1}{16} p_\mu p_\alpha p^4 \otimes p^\alpha + \frac{1}{8} p_\mu p_\alpha p_\beta p^2 \otimes p^\alpha p^\beta + \frac{1}{16} p^6 \otimes p_\mu - \frac{1}{8} p_\alpha p^4 \otimes p^\alpha p_\mu \\
&\left. + \frac{1}{2} p_\alpha p_\beta p^2 \otimes p^\alpha p^\beta p_\mu \right) + O(\beta^8) \qquad (5.100)
\end{aligned}
$$

and the twist is assumed to take the form

$$
\mathcal{F} = e^{f_1 + f_2 + f_3 + \dots}, \qquad (5.101)
$$

where $f_k \propto (\beta^2)^k$. From (5.77) follow the equations satisfied by the f_k, order by order

$$
\begin{aligned}
\left[f_1, \Delta_0 p_\mu \right] &= \Delta_1 p_\mu, \qquad (5.102) \\
\left[f_2, \Delta_0 p_\mu \right] &= \Delta_2 p_\mu - \frac{1}{2} \left[f_1, \left[f_1, \Delta_0 p_\mu \right] \right], \qquad (5.103) \\
\left[f_3, \Delta_0 p_\mu \right] &= \Delta_3 p_\mu - \frac{1}{2} \left(\left[f_1, \left[f_2, \Delta_0 p_\mu \right] \right] + \left[f_2, \left[f_1, \Delta_0 p_\mu \right] \right] \right) \\
&\quad - \frac{1}{3!} \left[f_1, \left[f_1, \left[f_1, \Delta_0 p_\mu \right] \right] \right], \qquad (5.104)
\end{aligned}
$$

and so on. To calculate f_1 one writes down the ansatz

$$
f_1 = \beta^2 (\alpha_1 p^2 \otimes x \cdot p + \alpha_2 p_\alpha p_\beta \otimes x^\alpha p^\beta + \alpha_3 p_\alpha \otimes x \cdot p\, p^\alpha + \alpha_4 p_\alpha \otimes x^\alpha p^2), \qquad (5.105)
$$

and inserts it into (5.102) to determine the unknown coefficients α_i. The resulting expression for f_1 is

$$
f_1 = -i\beta^2 \left(\frac{1}{2} p^2 \otimes x \cdot p + \frac{1}{2} p_\alpha p_\beta \otimes x^\alpha p^\beta + p_\alpha \otimes x \cdot p\, p^\alpha \right). \qquad (5.106)
$$

Inserting this and the ansatz

$$\begin{aligned} f_2 &= \beta^4(\alpha_1 p^4 \otimes x\cdot p + \alpha_2 p_\alpha p_\beta p^2 \otimes x^\alpha p^\beta + \alpha_3 p_\alpha p^2 \otimes x\cdot p\, p^\alpha \\ &\quad + \alpha_4 p_\alpha p^2 \otimes x^\alpha p^2 + \alpha_5 p_\alpha p_\beta p_\gamma \otimes x^\alpha p^\beta p^\gamma), \end{aligned} \tag{5.107}$$

into (5.103), one finds

$$f_2 = i\frac{\beta^4}{2}\left(\frac{1}{2}p^4 \otimes x\cdot p + \frac{1}{2}p_\alpha p_\beta p^2 \otimes x^\alpha p^\beta + p_\alpha p^2 \otimes x\cdot p\, p^\alpha\right). \tag{5.108}$$

An analogous procedure for the third order gives

$$f_3 = -i\frac{\beta^6}{3}\left(\frac{1}{2}p^6 \otimes x\cdot p + \frac{1}{2}p_\alpha p_\beta p^4 \otimes x^\alpha p^\beta + p_\alpha p^4 \otimes x\cdot p\, p^\alpha\right). \tag{5.109}$$

Inductively, it follows that the closed form for the twist is given by

$$\mathcal{F} = \exp\left\{-i\left(\frac{1}{2}p^2 \otimes x\cdot p + \frac{1}{2}p_\alpha p_\beta \otimes x^\alpha p^\beta + p_\alpha \otimes x\cdot p\, p^\alpha\right)\left(\frac{\log(1+\beta^2 p^2)}{p^2} \otimes 1\right)\right\}. \tag{5.110}$$

One can check that (5.110) gives the correct twist for the Snyder space by calculating

$$m\left(\mathcal{F}^{-1}(\triangleright \otimes 1)(x_\mu \otimes 1)\right) = x_\mu + \beta^2 x\cdot p p_\mu. \tag{5.111}$$

An independent verification is to start from (5.74). One gets

$$\begin{aligned} \mathcal{F}^{-1} &= \,:\exp\left\{\frac{i}{1-\beta^2 p_\alpha \otimes p^\alpha}\left[\frac{\beta^2\sqrt{1+\beta^2p^2}}{1+\sqrt{1+\beta^2p^2}}p^\beta p^\gamma \otimes x_\beta p_\gamma + \left(\sqrt{1+\beta^2p^2}-1\right)\otimes x\cdot p \right.\right. \\ &\quad \left.\left. +\beta^2 p_\beta \otimes x\cdot pp^\alpha\right]\right\}:, \end{aligned} \tag{5.112}$$

which expanded up to the second order gives

$$\begin{aligned}\mathcal{F}^{-1} &= 1\otimes 1 + i\beta^2\left(\frac{1}{2}p^\alpha p^\beta\otimes x_\alpha p_\beta + \frac{1}{2}p^2\otimes x\cdot p + p_\alpha\otimes x\cdot pp^\alpha\right)\\ &-\frac{i\beta^4}{2}\left(\frac{1}{4}p^4\otimes x\cdot p - \frac{1}{4}p_\alpha p_\beta p^2\otimes x^\alpha p^\beta - p_\alpha p^2\otimes x\cdot pp^\alpha - p_\alpha p_\beta p_\gamma\otimes x^\alpha p^\beta p^\gamma\right.\\ &\left.-2p_\alpha p_\beta\otimes x\cdot pp^\alpha p^\beta\right) - \frac{\beta^4}{2}\left(\frac{1}{4}p^4\otimes x^\alpha x\cdot pp_\alpha + \frac{1}{2}p_\alpha p_\beta p^2\otimes x^\alpha x\cdot pp^\beta\right.\\ &+p_\alpha p^2\otimes x_\beta x\cdot pp^\beta p^\alpha + \frac{1}{4}p_\alpha p_\beta p_\gamma p_\delta\otimes x^\alpha x^\beta p^\delta p^\gamma + p_\alpha p_\beta p_\gamma\otimes x^\alpha x\cdot pp^\beta p^\gamma\\ &\left.+p_\alpha p_\beta\otimes x_\gamma x\cdot pp^\gamma p^\alpha p^\beta\right) + O(\beta^6). \end{aligned} \tag{5.113}$$

The expression in eq.(5.113) agrees exactly with what one would get from (5.106) and (5.108) using the fact that $\mathcal{F}^{-1} = 1\otimes 1 - f_1 - f_2 + \frac{1}{2}f_1^2 + O(\beta^6)$.

Using the exact twist (5.110) to calculate the coproduct of $J_{\mu\nu}$ one can verify that the coproduct of the Lorentz generators is undeformed to all orders, i.e.

$$\Delta J_{\mu\nu} = \Delta_0 J_{\mu\nu}. \tag{5.114}$$

5.3.4 Twist for the Maggiore realisation

The same procedure can be performed for the Maggiore realisation (5.79). The coproduct when expanded up to the third order, takes the following form

$$\begin{aligned}\Delta p_\mu &= p_\mu\otimes 1 + 1\otimes p_\mu - \frac{\beta^2}{2}\left(p_\mu p_\alpha\otimes p^\alpha + p_\mu\otimes p^2\right) \qquad (5.115)\\ &-\frac{\beta^4}{8}\left(p_\mu\otimes p^4 + p_\mu p_\alpha p^2\otimes p^\alpha\right) - \frac{\beta^6}{16}\left(p_\mu\otimes p^6 + p_\mu p_\alpha p^4\otimes p^\alpha\right) + O(\beta^8).\end{aligned}$$

Using the same procedure as in the previous subsection, one finds

$$
\begin{aligned}
f_1 &= \frac{i\beta^2}{2}\left(p_\alpha \otimes x^\alpha p^2 + p_\alpha p_\beta \otimes x^\alpha p^\beta\right), \\
f_2 &= \frac{i\beta^4}{8}\Big(p_\alpha \otimes x^\alpha p^4 + p_\alpha p^2 \otimes x^\alpha p^2 + 2p_\alpha p_\beta p_\gamma \otimes x^\alpha p^\beta p^\gamma \\
&\quad +2p_\alpha p_\beta \otimes x^\alpha p^\beta p^2 + 2p_\alpha p_\beta p^2 \otimes x^\alpha p^\beta\Big), \\
f_3 &= \frac{i\beta^6}{8}\Big(\frac{1}{2}p_\alpha \otimes x^\alpha p^6 + \frac{4}{3}p_\alpha p_\beta p^4 \otimes x^\alpha p^\beta + \frac{3}{2}p_\alpha p_\beta \otimes x^\alpha p^\beta p^4 \\
&\quad +\frac{7}{12}p_\alpha p^4 \otimes x^\alpha p^2 + \frac{5}{12}p_\alpha p^2 \otimes x^\alpha p^4 + \frac{7}{3}p_\alpha p_\beta p_\gamma \otimes x^\alpha p^\beta p^\gamma p^2 \\
&\quad +\frac{5}{3}p_\alpha p_\beta p^2 \otimes x^\alpha p^\beta p^2 + \frac{4}{3}p_\alpha p_\beta p_\gamma p_\delta \otimes x^\alpha p^\beta p^\gamma p^\delta + 2p_\alpha p_\beta p_\gamma p^2 \otimes x^\alpha p^\beta p^\gamma\Big).
\end{aligned}
\tag{5.116}
$$

In this case it doesn't seem to be possible to find a closed form for the twist. However, the perturbative result, when used to calculate the coproduct of $J_{\mu\nu}$, gives again the primitive coproduct, as it should.

We have extended the previous investigations on the noncommutative geometry of the Snyder model in several directions. It was first generalised further to include in the defining commutations relations all terms compatible with undeformed Lorentz invariance. The corresponding deformed addition of momenta was also obtained and analysed. Adopting the formalism of Hopf algebroids [25], recalled in Sec. 5.2, the twist and the R-matrix were calculated to first order in the deformation parameter in the general case and the exact expression for the twist for the case of the so-called Snyder realisation was obtained. It should be noted however that because of the non-coassociativity of the coproduct, the twist for the Snyder space will not satisfy the cocycle condition in the sense of Hopf algebras (5.1.2), as can be checked directly order by order, and the structure associated to the phase space will be that of a quasi-Hopf algebroid. The formal definition and elaboration of the quasi-Hopf algebroid corresponding to the Snyder space is non-trivial and will be presented in future work.

Chapter 6

Concluding remarks

In this thesis we have studied the Snyder model from several points of view. As we have seen, this model is an example of noncommutative geometry based on a deformation of the Heisenberg algebra, and can also be considered as an example of doubly special relativity. Its relevance is due to its distinctive property of preserving Lorentz invariance, in contrast with analogous models. We list below the original results discussed in this thesis:

We have calculated the orbits of a particle in Schwarzschild spacetime, assuming that the dynamics is governed by a Snyder symplectic structure in a relativistic setting. We found that the perihelion shift of the planets acquires an additional contribution with respect to that predicted by general relativity and that the equivalence principle is violated. Another outcome of our assumptions is that the coupling constant β has to be less than its natural value of order 1 in Planck units by many orders of magnitude. Therefore, the calculations are in accordance with the natural starting presumption that the validity of Snyder mechanics is limited to Planck-scale physics.

The definition of path integrals in one- and two-dimensional Snyder space was discussed in detail, first using the standard techniques and it was then shown how the same results follow, in a much simpler way, employing the formalism introduced by Faddeev and Jackiw.

We have studied the realisations of the three-dimensional Euclidean Snyder-de Sitter algebra and used them to find the spectra of the position and momentum operators squared. Since position and momentum are related by duality, the obtained spectra turn out to be the same, apart from a multiplicative constant. Due to the compactness of the algebra, the spectra are discrete.

We have analysed the Snyder model in the Hopf algebroid framework. We have also introduced a generalisation of the model that includes all possible deformations compatible with Lorentz invariance. The corresponding deformed addition of momenta was obtained and analysed, as well as the twist and the R-matrix up to the first order in the deformation parameter. For the case of the Snyder realisation, we were able to find a closed form for the twist.

An important development of the present work that we started to investigate is the study of quantum field theory in Snyder spaces. We recall that the original motivation of Snyder for introducing his model was to remove the divergences from quantum field theory. In a recent paper [38] we have shown that the free scalar theory, in analogy with other noncommutative models, is equivalent to the commutative theory. This however changes when one considers the interacting theory. It is likely that the modification of the vertices with respect to the commutative theory improve the ultraviolet behaviour, and we plan to study this possibility in the future

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