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TWO SOLUTIONS FOR FRACTIONAL p-LAPLACIAN INCLUSIONS UNDER NONRESONANCE

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Abstract. We study a pseudo-differential inclusion driven by the fractional p-Laplacian operator and involving a nonsmooth potential, which satisfies nonresonance conditions both at the origin and at infinity. Using variational methods based on nonsmooth critical point theory (Clarke's subdifferential), we establish existence of at least two constant sign solutions (one positive, the other negative), enjoying Hölder regularity

1. Introduction and main result

In this article we study the problem

$$\begin{gathered} (-\Delta)_p^s u \in \partial j(x,u) \quad \text{in } \Omega \\ u = 0 \quad \text{in } \Omega^c. \end{gathered} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^N$ ($N > 1$) is a bounded domain with a C^2 boundary $\partial\Omega$, $p > 1$ and $s \in (0,1)$ are real numbers such that $ps < N$, and $(-\Delta)_p^s$ denotes the fractional p-Laplacian, namely the nonlinear, nonlocal operator defined for all $u : \mathbb{R}^N \to \mathbb{R}$ smooth enough and all $x \in \mathbb{R}^N$ by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{B_\varepsilon^c(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} \, dy \tag{1.2}$$

(which in the linear case $p = 2$ reduces to the fractional Laplacian up to a dimensional constant $C(N,p,s) > 0$, see [6, 14]). Moreover, $\partial j(x,\cdot)$ denotes the generalized subdifferential (in the sense of Clarke [10]) of a potential $j : \Omega \times \mathbb{R} \to \mathbb{R}$ which is assumed to be measurable in Ω and locally Lipschitz continuous in $\mathbb{R}$. Thus, problem (1.1) can be referred to as a pseudo-differential inclusion in Ω, coupled with a Dirichlet-type condition in $\Omega^c = \mathbb{R}^N \setminus \Omega$ (due to the nonlocal nature of the operator $(-\Delta)_p^s$).

Nonsmooth problems driven by linear and nonlinear operators, such as the p-Laplacian, have been extensively studied in a variational perspective, since the pioneering work [9]. The use of variational methods based on nonsmooth critical point theory allows to establish several existence and multiplicity results for problems related to locally Lipschitz potentials, which can be equivalently formulated

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as either differential inclusions or hemivariational inequalities, see [2, 11, 18, 20, 23, 27, 31, 34, 35, 36] and the monographs [16, 32, 33].

The study of nonlocal problems driven by fractional-type operators (both linear and nonlinear) is more recent but rapidly expanding, because of both the important applications of nonlocal diffusion in several disciplines (for instance, in mechanics, game theory, population dynamics, and probability) and to the intrinsic mathematical interest: indeed, fractional operators induce a class of integral equations, exhibiting many common features with partial differential equations. Out of a vast literature, let us mention the results of [1, 4, 6, 7, 17, 21, 24, 40, 42, 43] (linear case) [3, 5, 13, 15, 19, 22, 25, 26, 28, 30, 38, 39, 41] (p-case), as well as [8, 14, 29] for a general introduction to fractional operators.

Our work stands at the conjunction of these two branches of research. By applying nonsmooth critical point theory, we shall prove the existence of two constant sign, smooth weak solutions for problem (1.1). Precisely, on the nonsmooth potential j we will assume the following:

- (H1) $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(\cdot, 0) = 0$, $j(\cdot, t)$ is measurable in Ω for all $t \in \mathbb{R}$, $j(x, \cdot)$ is locally Lipschitz continuous in $\mathbb{R}$ for a.e. $x \in \Omega$. Moreover,
- (H2) for all $\rho > 0$ there exists $a_\rho \in L^\infty(\Omega)_+$ such that for a.e. $x \in \Omega$, all $|t| \leq \rho$, and all $\xi \in \partial j(x,t)$, $|\xi| \leq a_\rho(x)$;
- (H3) there exists $\theta \in L^\infty(\Omega)_+$ such that $\theta \leq \lambda_1$, $\theta \not\equiv \lambda_1$, and uniformly for a.e. $x \in \Omega$,
$$\limsup_{|t|\to\infty} \max_{\xi\in\partial j(x,t)} \frac{\xi}{|t|^{p-2}t} \leq \theta(x);$$
- (H4) there exist $\eta_1, \eta_2 \in L^\infty(\Omega)_+$ such that $\lambda_1 \leq \eta_1 \leq \eta_2$, $\eta_1 \not\equiv \lambda_1$, and uniformly for a.e. $x \in \Omega$
$$\eta_1(x) \leq \liminf_{t\to 0} \min_{\xi\in\partial j(x,t)} \frac{\xi}{|t|^{p-2}t} \leq \limsup_{t\to 0} \max_{\xi\in\partial j(x,t)} \frac{\xi}{|t|^{p-2}t} \leq \eta_2(x);$$
- (H5) for a.e. $x \in \Omega$, all $t \in \mathbb{R}$, and all $\xi \in \partial j(x,t)$ $\xi t \geq 0$.

In (H3) and (H4), $\lambda_1 > 0$ denotes the principal eigenvalue of $(-\Delta)_p^s$ with Dirichlet conditions in Ω (see Section 2 below), so these conditions conjure a nonresonance phenomenon both at infinity and at the origin. Here we present an example of a potential satisfying (H1):

Example 1.1. Let $\theta, \eta \in L^\infty(\Omega)_+$ be such that $\theta < \lambda_1 < \eta$, and $j : \Omega \times \mathbb{R} \to \mathbb{R}$ be defined for all $(x,t) \in \Omega \times \mathbb{R}$ by
$$j(x,t) = \begin{cases} \frac{\eta(x)}{p}|t|^p & \text{if } |t| \leq 1 \\ \frac{\theta(x)}{p}|t|^p + \ln(|t|^p) + \frac{\eta(x)-\theta(x)}{p} & \text{if } |t| > 1. \end{cases}$$
Then j satisfies (H1)–(H5).

To the best of our knowledge, this is the first existence/multiplicity result for a nonlocal problem involving fractional operators and set-valued reactions in higher dimension, while we should mention [44, 45] for the ordinary case (the first based on fixed point methods, the second on nonsmooth variational methods). We also recall a nice application of nonsmooth analysis to a single-valued nonlocal equation in [12]. Our main result is as follows:

Theorem 1.2. *If hypotheses* (H1)–(H5) *hold, then problem* (1.1) *admits at least two solutions* $u_+, u_- \in C^\alpha(\overline{\Omega})$ *(*$\alpha \in (0,1)$*) such that* $u_-(x) < 0 < u_+(x)$ *for all* $x \in \Omega$*.*

This article has the following structure: in Section 2 we recall some basic notions from nonsmooth critical point theory, as well as some useful results on the fractional p-Laplacian; and in Section 3 we prove our main result.
Notation: Throughout the paper, for any $A \subset \mathbb{R}^N$ we shall set $A^c = \mathbb{R}^N \setminus A$. For any two measurable functions $f, g : \Omega \to \mathbb{R}$, $f \le g$ will mean that $f(x) \le g(x)$ for a.e. $x \in \Omega$ (and similar expressions). The positive (resp., negative) part of f is denoted f^+ (resp., f^-). If X is an ordered Banach space, X_+ will denote its non-negative order cone. For all $q \in [1, \infty]$, $\|\cdot\|_q$ denotes the standard norm of $L^q(\Omega)$ (or $L^q(\mathbb{R}^N)$, which will be clear from the context). Every function u defined in Ω will be identified with its 0-extension to $\mathbb{R}^N$.

2. Preliminaries

We begin this section by recalling some basic definitions and results of nonsmooth critical point theory. For the details, we refer to [10, 16, 32]. Let $(X, \|\cdot\|)$ be a real Banach space and $(X^*, \|\cdot\|_*)$ its topological dual. A functional $\varphi : X \to \mathbb{R}$ is said to be *locally Lipschitz continuous* if for every $u \in X$ there exist a neighborhood U of u and $L > 0$ such that

$$|\varphi(v) - \varphi(w)| \le L\|v - w\| \text{ for all } v, w \in U.$$

From now on we assume φ to be locally Lipschitz continuous. The *generalized directional derivative* of φ at u along $v \in X$ is

$$\varphi^\circ(u; v) = \limsup_{w \to u,\, t \to 0^+} \frac{\varphi(w + tv) - \varphi(w)}{t}.$$

The *generalized subdifferential* of φ at u is the set

$$\partial\varphi(u) = \big\{u^* \in X^* : \langle u^*, v\rangle \le \varphi^\circ(u; v) \text{ for all } v \in X\big\}.$$

We say that u is a *critical point* of φ if $0 \in \partial\varphi(u)$. The following Lemmas display some useful properties of the notions introduced above, see [16, Propositions 1.3.7-1.3.12]:

Lemma 2.1. *If* φ, $\psi : X \to \mathbb{R}$ *are locally Lipschitz continuous, then*

(i) $\varphi^\circ(u; \cdot)$ *is positively homogeneous, sub-additive and continuous for all* $u \in X$*;*
(ii) $\varphi^\circ(u; -v) = (-\varphi)^\circ(u; v)$ *for all* $u, v \in X$*;*
(iii) *if* $\varphi \in C^1(X)$*, then* $\varphi^\circ(u; v) = \langle \varphi'(u), v\rangle$ *for all* $u, v \in X$*;*
(iv) $(\varphi + \psi)^\circ(u; v) \le \varphi^\circ(u; v) + \psi^\circ(u; v)$ *for all* $u, v \in X$*.*

Lemma 2.2. *If* $\varphi, \psi : X \to \mathbb{R}$ *are locally Lipschitz continuous, then*

(i) $\partial\varphi(u)$ *is convex, closed and weakly* compact for all* $u \in X$*;*
(ii) *the multifunction* $\partial\varphi : X \to 2^{X^*}$ *is upper semicontinuous with respect to the weak* topology on* X^**;*
(iii) *if* $\varphi \in C^1(X)$*, then* $\partial\varphi(u) = \{\varphi'(u)\}$ *for all* $u \in X$*;*
(iv) $\partial(\lambda\varphi)(u) = \lambda\partial\varphi(u)$ *for all* $\lambda \in \mathbb{R}$*,* $u \in X$*;*
(v) $\partial(\varphi + \psi)(u) \subseteq \partial\varphi(u) + \partial\psi(u)$ *for all* $u \in X$*;*
(vi) *if* u *is a local minimizer (or maximizer) of* φ*, then* $0 \in \partial\varphi(u)$*.*

Now we deal with integral functionals defined on L^p-spaces by means of locally Lipschitz continuous potentials. Let $\Omega \subset \mathbb{R}^N$ be as in the Introduction and j_0 be a potential satisfying

(H6) $j_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $j_0(\cdot, t)$ is measurable in Ω for all $t \in \mathbb{R}$, $j_0(x, \cdot)$ is locally Lipschitz continuous in $\mathbb{R}$ for a.e. $x \in \Omega$. Moreover, there exists $a_0 > 0$ such that for a.e. $x \in \Omega$, all $t \in \mathbb{R}$, and all $\xi \in \partial j_0(x, t)$
$$|\xi| \le a_0 |t|^{p-1}.$$

For $u \in L^p(\Omega)$, we set
$$J_0(u) = \int_\Omega j_0(x, u)\, dx,$$
and define the set-valued Nemytzkij operator
$$N_0(u) = \{w \in L^{p'}(\Omega) : \, w(x) \in \partial j_0(x, u(x)) \text{ for a.e. } x \in \Omega\}.$$
From [10, Theorem 2.7.5] we have the following statement.

Lemma 2.3. *If j_0 satisfies* (H6)*, then $J_0 : L^p(\Omega) \to \mathbb{R}$ is Lipschitz continuous on any bounded subset of $L^p(\Omega)$. Moreover, for all $u \in L^p(\Omega)$, $w \in \partial J_0(u)$ one has $w \in N_0(u)$.*

Now we collect some useful results related to the fractional p-Laplacian defined in (1.2). First we fix a functional-analytical framework, following [14, 19]. For all measurable $u : \mathbb{R}^N \to \mathbb{R}$ we define the Gagliardo seminorm $[u]_{s,p}$ by setting
$$[u]_{s,p}^p = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}}\, dx\, dy,$$
then we introduce the fractional Sobolev space
$$W^{s,p}(\mathbb{R}^N) = \big\{u \in L^p(\mathbb{R}^N) : \, [u]_{s,p} < \infty\big\},$$
endowed with the norm $\|u\|_{s,p}^p = \|u\|_p^p + [u]_{s,p}^p$. Letting Ω be as in the Introduction, and taking into account the Dirichlet-type condition, we restrict ourselves to the space
$$W_0^{s,p}(\Omega) = \big\{u \in W^{s,p}(\mathbb{R}^N) : \, u(x) = 0 \text{ for a.e. } x \in \Omega^c\big\}.$$
Because of the fractional Poincaré inequality (see [14, Theorem 7.1]), $W_0^{s,p}(\Omega)$ can be normed by means of $\|u\| = [u]_{s,p}$. With such a norm, $(W_0^{s,p}(\Omega), \|\cdot\|)$ is a separable, uniformly convex (hence, reflexive) Banach space. We set
$$p_s^* = \frac{Np}{N - ps}.$$
Then the embedding $W_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for all $q \in [1, p_s^*]$ and compact for all $q \in [1, p_s^*)$ (in particular, for $q = p$), see [14, Corollary 7.2]. Moreover, we denote by $(W_0^{s,p}(\Omega)^*, \|\cdot\|_*)$ the topological dual of $(W_0^{s,p}(\Omega), \|\cdot\|)$.

The operator $(-\Delta)_p^s$ can be represented by a duality mapping $A : W_0^{s,p}(\Omega) \to W_0^{s,p}(\Omega)^*$ defined for all $u, v \in W_0^{s,p}(\Omega)$ by
$$\langle A(u), v\rangle = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+ps}}\, dx\, dy,$$
which satisfies the (S)-property, namely, whenever $u_n \rightharpoonup u$ in $W_0^{s,p}(\Omega)$ and
$$\lim_n \langle A(u_n), u_n - u\rangle = 0,$$

then we have $u_n \to u$ in $W_0^{s,p}(\Omega)$ (see [19, 37]). Now we consider the (1.1)-type problem

$$\begin{gathered}(-\Delta)_p^s u \in \partial j_0(x,u) \quad \text{in } \Omega \\ u = 0 \quad \text{in } \Omega^c,\end{gathered} \tag{2.1}$$

where j_0 satisfies (H6). We introduce the following notion of weak (or variational) solution.

Definition 2.4. A function $u \in W_0^{s,p}(\Omega)$ is a (weak) solution of (2.1) if there exists $w \in N_0(u)$ such that for all $v \in W_0^{s,p}(\Omega)$

$$\langle A(u), v\rangle = \int_\Omega wv\,dx.$$

Recalling that $W_0^{s,p}(\Omega) \hookrightarrow L^p(\Omega)$, conversely we have $L^{p'}(\Omega) \hookrightarrow W_0^{s,p}(\Omega)^*$, so Definition 2.4 can be rephrased by

$$A(u) = w \quad \text{in } W_0^{s,p}(\Omega)^*. \tag{2.2}$$

By means of (2.2), problem (1.1) is somewhat reduced to a pseudodifferential equation (with single-valued right hand side), to which we can apply most recent results from fractional calculus of variations. We begin with uniform L^∞-bounds, whose proof closely follows that of [19, Theorem 3.1]:

Lemma 2.5. *If j_0 satisfies* (H6)*, then there exists $C_0 > 0$ such that for all solution $u \in W_0^{s,p}(\Omega)$ of* (2.1) *one has $u \in L^\infty(\Omega)$ and*

$$\|u\|_\infty \le C_0(1 + \|u\|).$$

Proof. Without loss of generality we may assume $u^+ \not\equiv 0$ (the case $u^- \not\equiv 0$ is analogous). By Definition 2.4, there exists $w \in N_0(u)$ such that (2.2) holds. By (H6) we have for a.e. $x \in \Omega$

$$|w(x)| \le a_0|u(x)|^{p-1}. \tag{2.3}$$

Choose $\rho \ge \max\{1, \|u\|_p^{-1}\}$ (to be determined later) and set $v = (\rho\|u\|_p)^{-1}u \in W_0^{s,p}(\Omega)$, so that $\|v\|_p = \rho^{-1} \le 1$ and $A(v) = (\rho\|u\|_p^{-1})^{p-1}A(u)$ by $(p-1)$-homogeneity of the fractional p-Laplacian. For all $n \in \mathbb{N}$ set

$$v_n = \Big(v - 1 + \frac{1}{2^n}\Big)^+ \in W_0^{s,p}(\Omega),$$

in particular $v_0 = v^+$. The sequence (v_n) is pointwise nonincreasing as for all $n \in \mathbb{N}$, a.e. $x \in \Omega$ we have $0 \le v_{n+1}(x) \le v_n(x)$, and $v_n(x) \to (v(x)-1)^+$ as $n \to \infty$ for a.e. $x \in \Omega$. Moreover we have for all $n \in \mathbb{N}$,

$$\big\{v_{n+1} > 0\big\} \subseteq \big\{0 < v < (2^{n+1}-1)v_n\big\} \cap \big\{v_n > \frac{1}{2^{n+1}}\big\}. \tag{2.4}$$

Indeed, for a.e. $x \in \Omega$ such that $v_{n+1}(x) > 0$ we have $v(x) > 1 - 2^{-n-1} > 1 - 2^{-n}$, hence $v(x) > 0$ and $v_n(x) > 2^{-n-1}$. Further, we have

$$\begin{aligned}(2^{n+1}-1)v_n(x) - v(x) &= (2^{n+1}-2)v(x) + (2^{n+1}-1)\Big(\frac{1}{2^n} - 1\Big) \\ &= (2^{n+1}-2)\Big(v(x) - 1 + \frac{1}{2^{n+1}}\Big) > 0,\end{aligned}$$

which proves (2.4). Set $R_n = \|v_n\|_p^p$, so (R_n) is a nonincreasing sequence in $(0,1)$. We claim that

$$\lim_n R_n = 0. \tag{2.5}$$

Indeed, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
R_{n+1} &= \int_{\{v_{n+1}>0\}} v_{n+1}^p \, dx \\
&\le \|v_{n+1}\|_{p_s^*}^p \Big|\Big\{v_n > \frac{1}{2^{n+1}}\Big\}\Big|^{\frac{p_s^*-p}{p_s^*}} \quad \text{(by Hölder's inequality and (2.4))} \\
&\le \|v_{n+1}\|_{p_s^*}^p \Big(2^{p(n+1)} \int_{\{v_n>2^{-n-1}\}} v_n^p \, dx\Big)^{\frac{p_s^*-p}{p_s^*}} \quad \text{(by Chebyshev's inequality)} \\
&\le c_1 \|v_{n+1}\|^p \Big(2^{p(n+1)} R_n\Big)^{\frac{p_s^*-p}{p_s^*}} \quad \text{(by the fractional Sobolev inequality, } c_1>0\text{)}.
\end{aligned}$$

Besides, testing (2.2) with $v_{n+1} \in W_0^{s,p}(\Omega)$ we have

$$\begin{aligned}
\|v_{n+1}\|^p &\le \langle A(v), v_{n+1}\rangle \\
&= (\rho\|u\|_p)^{1-p} \int_\Omega w v_{n+1} \, dx \\
&\le (\rho\|u\|_p)^{1-p} \int_{\{v_{n+1}>0\}} a_0 |u|^{p-1} v_{n+1} \, dx \quad \text{(by (2.3))} \\
&= a_0 \int_{\{v_{n+1}>0\}} |v|^{p-1} v_{n+1} \, dx \\
&\le a_0 \int_{\{v_{n+1}>0\}} \big((2^{n+1}-1)v_n\big)^{p-1} v_{n+1} \, dx \quad \text{(by (2.4))} \\
&\le a_0 (2^{n+1}-1)^{p-1} \int_{\{v_n>0\}} v_n^p \, dx \quad \text{(by monotonicity of } (v_n)\text{)} \\
&= a_0 (2^{n+1}-1)^{p-1} R_n.
\end{aligned}$$

Concatenating the above inequalities, we obtain the recursive formula

$$R_{n+1} \le H^{n+1} R_n^{1+\beta}, \tag{2.6}$$

where the constants $H > 1$, $\beta \in (0,1)$ (independent of u) are given by

$$H = \max\{1, a_0 c_1\} 2^{2p-1}, \ \beta = \frac{p_s^* - p}{p_s^*}.$$

Now we fix

$$\rho = \max\big\{1, \|u\|_p^{-1}, H^{\frac{1+\beta}{p\beta^2}}\big\}, \quad \eta = H^{-\frac{1}{\beta}} \in (0,1).$$

We have for that all $n \in \mathbb{N}$,

$$R_n \le \frac{\eta^n}{\rho^p}. \tag{2.7}$$

We argue by induction on $n \in \mathbb{N}$. Clearly $R_0 \le \rho^{-p}$. If (2.7) holds for some $n \ge 1$, then by (2.6) we have

$$R_{n+1} \le H^{n+1} \Big(\frac{\eta^n}{\rho^p}\Big)^{1+\beta} \le \frac{\eta^{n+1}}{\rho^p}.$$

Recalling that $\eta \in (0,1)$, from (2.7) we deduce (2.5). Thus we have $v_n \to 0$ in $L^p(\Omega)$. Passing if necessary to a subsequence we have $v_n(x) \to 0$ for a.e. $x \in \Omega$, which, along with $v_n(x) \to (v(x)-1)^+$, implies $v(x) \le 1$ for a.e. $x \in \Omega$. An analogous argument applies to $-v$, therefore we have $v \in L^\infty(\Omega)$ with $\|v\|_\infty \le 1$.

Going back to u and recalling the definition of ρ, we have $u \in L^\infty(\Omega)$ with

$$\begin{aligned}\|u\|_\infty &\le \rho\|u\|_p \\ &= \max\{\|u\|_p, 1, H^{\frac{1}{p\beta^2}}\|u\|_p\} \\ &\le C_0(1+\|u\|)\end{aligned}$$

for some $C_0 > 0$ which does not depend on u. □

Remark 2.6. If in (H6) the exponent $p-1$ is replaced by $q-1$ for some $q \in (1, p_s^*)$, some uniform L^∞-bounds still hold (see [19]). We kept this special assumption for the sake of simplicity.

Weak solutions exhibit Hölder regularity up to the boundary. For $\alpha \in (0,1)$, we shall use the function space $C^\alpha(\overline{\Omega})$, endowed with the norm

$$\|u\|_{C^\alpha(\overline{\Omega})} = \|u\|_\infty + \sup_{x,y\in\Omega,\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

Lemma 2.7. *If j_0 satisfies* (H6)*, then there exist $\alpha \in (0,s]$, $K_0 > 0$ such that for all solution $u \in W_0^{s,p}(\Omega)$ of* (2.1) *one has $u \in C^\alpha(\overline{\Omega})$ and*

$$\|u\|_{C^\alpha(\overline{\Omega})} \le K_0(1+\|u\|).$$

Proof. By Lemma 2.5 we have $u \in L^\infty(\Omega)$ and $\|u\|_\infty \le C_0(1+\|u\|)$, with $C_0 > 0$ independent of u. Let $w \in N_0(u)$ be as in Definition 2.4, then by (H6) we have

$$\|w\|_\infty \le a_0\|u\|_\infty^{p-1} \le c_2(1+\|u\|^{p-1})$$

for some $c_2 > 0$ independent of u. Now [22, Theorem 1.1] implies $u \in C^\alpha(\overline{\Omega})$ and

$$\|u\|_{C^\alpha(\overline{\Omega})} \le c_3\|w\|_\infty^{\frac{1}{p-1}} \le K_0(1+\|u\|),$$

with $c_3, K_0 > 0$ independent of u. □

No regularity higher than C^s can be expected in the fractional framework, as was pointed out in [40] even for the linear case (fractional Laplacian). In particular, solutions do not, in general, admit an outward normal derivative at the points of $\partial\Omega$ and, as a consequence, the Hopf property is stated in terms of a Hölder-type quotient (see [13] and Lemmas 3.1 and 3.2 below).

Similarly to the case of the p-Laplacian ($s=1$), the spectrum of $(-\Delta)_p^s$ includes a sequence $0 < \lambda_1 < \lambda_2 \le \dots \le \lambda_k \le \dots$ of variational eigenvalues with min-max characterizations (see [5, 15, 25, 28, 38, 41] for a detailed description of such eigenvalues). Here we shall only use the following properties of λ_1:

Lemma 2.8. *The principal eigenvalue λ_1 of $(-\Delta)_p^s$ in $W_0^{s,p}(\Omega)$ is simple and isolated (as en element of the spectrum), with the following variational characterization:*

$$\lambda_1 = \inf_{u\in W_0^{s,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p}{\|u\|_p^p}.$$

The corresponding positive, $L^p(\Omega)$-normalized eigenfunction is $u_1 \in C^\alpha(\overline{\Omega})$.

The following result is crucial in obtaining the constant sign solutions of (1.1), exploiting hypothesis (H1) (H2) (nonresonance at infinity):

Proposition 2.9. *Let $\theta \in L^\infty(\Omega)_+$ be such that $\theta \leq \lambda_1$, $\theta \not\equiv \lambda_1$, and $\psi \in C^1(W_0^{s,p}(\Omega))$ be defined by*

$$\psi(u) = \|u\|^p - \int_\Omega \theta(x)|u|^p \, dx.$$

Then there exists $\theta_0 \in (0, \infty)$ such that for all $u \in W_0^{s,p}(\Omega)$

$$\psi(u) \geq \theta_0 \|u\|^p.$$

Proof. By Lemma 2.8 we have for all $u \in W_0^{s,p}(\Omega)$

$$\psi(u) \geq \|u\|^p - \lambda_1 \|u\|_p^p \geq 0.$$

To complete the proof, we argue by contradiction: we assume that there exists a sequence (u_n) in $W_0^{s,p}(\Omega)$ such that $\|u_n\| = 1$ for all $n \in \mathbb{N}$, and $\psi(u_n) \to 0$. By reflexivity of $W_0^{s,p}(\Omega)$ and the compact embedding $W_0^{s,p}(\Omega) \hookrightarrow L^p(\Omega)$, passing if necessary to a subsequence we have $u_n \rightharpoonup u$ in $W_0^{s,p}(\Omega)$ and $u_n \to u$ in $L^p(\Omega)$, as well as $u_n(x) \to u(x)$ for a.e. $x \in \Omega$. By convexity

$$\liminf_n \|u_n\|^p \geq \|u\|^p,$$

while by the dominated convergence theorem

$$\lim_n \int_\Omega \theta(x)|u_n|^p \, dx = \int_\Omega \theta(x)|u|^p \, dx.$$

Thus we obtain $\psi(u) = 0$. Two cases may occur:

(a) if $u = 0$, then we obtain

$$\|u_n\|^p = \psi(u_n) + \int_\Omega \theta(x)|u_n|^p \, dx \to 0,$$

against $\|u_n\| = 1$;

(b) if $u \neq 0$, then u is a minimizer of the Rayleigh quotient in Lemma 2.8, hence by simplicity of the principal eigenvalue we can find $\mu \in \mathbb{R}$ such that $u = \mu u_1$, in particular $|u(x)| > 0$ for all $x \in \Omega$, which in turn implies

$$\|u\|^p = \int_\Omega \theta(x)|u|^p \, dx < \lambda_1 \|u\|_p^p,$$

against the variational characterization of λ_1.

So we have

$$\inf_{\|u\|=1} \psi(u) = \theta_0 > 0,$$

and noting that ψ is p-homogeneous we complete the proof. □

3. Proof of the main result

In this section we prove Theorem 1.2. First we establish a variational framework for problem (1.1) introducing two truncated, nonsmooth energy functionals. For all $(x, t) \in \Omega \times \mathbb{R}$ set

$$j_\pm(x, t) = j(x, \pm t^\pm),$$

and for all $u \in W_0^{s,p}(\Omega)$ set

$$\varphi_\pm(u) = \frac{\|u\|^p}{p} - \int_\Omega j_\pm(x, u) \, dx.$$

The functionals $\varphi_\pm$ select constant sign solutions of (1.1), as explained by the following lemmas:

Lemma 3.1. *The functional $\varphi_+ : W_0^{s,p}(\Omega) \to \mathbb{R}$ is locally Lipschitz continuous. Moreover, if $u \in W_0^{s,p}(\Omega) \setminus \{0\}$ is a critical point of φ_+, then $u \in C^\alpha(\overline{\Omega})$ is a solution of* (1.1) *such that*

(i) $u(x) > 0$ *for all* $x \in \Omega$*;*
(ii) *for all* $y \in \partial\Omega$

$$\liminf_{\Omega \ni x \to y} \frac{u(x)}{\operatorname{dist}(x, \Omega^c)^s} > 0.$$

Proof. First we note that $j_+(\cdot, t)$ is measurable in Ω for all $t \in \mathbb{R}$ and $j_+(x, \cdot)$ is locally Lipschitz continuous in $\mathbb{R}$ for a.e. $x \in \Omega$, with generalized subdifferential $\partial j_+(x, \cdot)$ such that for all $t \in \mathbb{R}$,

$$\partial j_+(x,t) \begin{cases} = \{0\} & \text{if } t < 0 \\ \subseteq \{\mu\xi : \mu \in [0,1],\, \xi \in \partial j(x,0)\} & \text{if } t = 0 \\ = \partial j(x,t) & \text{if } t > 0. \end{cases} \tag{3.1}$$

Moreover, there exists $c_4 > 0$ such that for a.e. $x \in \Omega$, all $t \in \mathbb{R}$, and all $\xi \in \partial j_+(x,t)$

$$|\xi| \le c_4 |t|^{p-1}. \tag{3.2}$$

Indeed, by (3.1), the inequality above holds if $t < 0$. Now fix $\varepsilon > 0$. By (H3) (H5) we can find $\rho > 0$ such that for a.e. $x \in \Omega$, all $t > \rho$, and all $\xi \in \partial j(x,t)$ we have

$$0 \le \xi \le (\|\theta\|_\infty + \varepsilon) t^{p-1},$$

while by (H4) (H5) we can find $\delta \in (0, \rho)$ such that for a.e. $x \in \Omega$, all $0 < t < \delta$, and all $\xi \in \partial j(x,t)$ we have

$$0 \le \xi \le (\|\eta_2\|_\infty + \varepsilon) t^{p-1},$$

and by (H2) (H5) for a.e. $x \in \Omega$, all $\delta \le t \le \rho$, and all $\xi \in \partial j(x,t)$ we have

$$0 \le \xi \le \|a_\rho\|_\infty \le \frac{\|a_\rho\|_\infty}{\delta^{p-1}} t^{p-1}.$$

Finally, for $t = 0$, by Lemma 2.2 (ii), (3.1) and the computations above we have for a.e. $x \in \Omega$, $\xi \in \partial j_+(x, 0)$,

$$|\xi| \le (\|\eta_2\|_\infty + \varepsilon) t^{p-1}.$$

So (3.2) is achieved. Now we see that j_+ satisfies hypothesis (H6). So, by Lemma 2.3, the functional $J_+ : L^p(\Omega) \to \mathbb{R}$ defined by

$$J_+(u) = \int_\Omega j_+(x,t)\, dx$$

is locally Lipschitz continuous and for all $u \in L^p(\Omega)$, $w \in \partial J_+(u)$ we have $w \in N_+(u)$, where

$$N_+(u) = \{w \in L^{p'}(\Omega) : w(x) \in \partial j_+(x, u(x)) \text{ for a.e. } x \in \Omega\}.$$

The continuous embedding $W_0^{s,p}(\Omega) \hookrightarrow L^p(\Omega)$, with reverse embedding $L^{p'}(\Omega) \hookrightarrow W_0^{s,p}(\Omega)^*$, implies that J_+ is locally Lipschitz continuous in $W_0^{s,p}(\Omega)$ and the inclusion $\partial J_+(u) \subseteq N_+(u)$ still holds for all $u \in W_0^{s,p}(\Omega)$. By Lemma 2.2 (iii)–(v), then, φ_+ is locally Lipschitz continuous in $W_0^{s,p}(\Omega)$ and for all $u \in W_0^{s,p}(\Omega)$

$$\partial\varphi_+(u) \subseteq A(u) - N_+(u). \tag{3.3}$$

Now let $u \in W_0^{s,p}(\Omega)\setminus\{0\}$ be a critical point of φ_+. By (3.3) we can find $w \in N_+(u)$ such that $A(u) = w$ in $W_0^{s,p}(\Omega)^*$. By (3.2) we have for a.e. $x \in \Omega$

$$|w(x)| \le c_4|u(x)|^{p-1}.$$

By Lemma 2.7 we have $u \in C^\alpha(\overline{\Omega})$. Moreover, by the previous estimate and the strong maximum principle for the fractional p-Laplacian (see [13, Theorem 1.4]) we have $u(x) > 0$ for all $x \in \Omega$, which proves (i).

By (3.1), the latter estimate implies $w(x) \in \partial j(x, u(x))$ for a.e. $x \in \Omega$, hence by Definition 2.4 u solves (1.1).

Finally, by the Hopf lemma for the fractional p-Laplacian [13, Theorem 1.5], we have for all $y \in \partial\Omega$

$$\liminf_{\Omega\ni x\to y} \frac{u(x)}{\operatorname{dist}(x,\Omega^c)^s} > 0,$$

which yields (ii) and completes the proof. □

An analogous argument leads to the following result.

Lemma 3.2. *The functional $\varphi_- : W_0^{s,p}(\Omega) \to \mathbb{R}$ is locally Lipschitz continuous. Moreover, if $u \in W_0^{s,p}(\Omega) \setminus \{0\}$ is a critical point of φ_-, then $u \in C^\alpha(\overline{\Omega})$ is a solution of* (1.1) *such that*

(i) *$u(x) < 0$ for all $x \in \Omega$;*

(ii) *for all $y \in \partial\Omega$*

$$\limsup_{\Omega\ni x\to y} \frac{u(x)}{\operatorname{dist}(x,\Omega^c)^s} < 0.$$

We can now prove our main result.

Proof of Theorem 1.2. We deal first with the positive solution, which is detected as a global minimizer of the truncated functional φ_+. By (H1) (H3), for any $\varepsilon > 0$ we can find $\rho > 0$ such that for a.e. $x \in \Omega$, all $t > \rho$ and all $\xi \in \partial j_+(x,t)$

$$|\xi| \le (\theta(x) + \varepsilon)t^{p-1}$$

(recall that $\partial j_+(x,t) = \partial j(x,t)$ by (3.1)). By (H2) and (3.1) again, there exists $a_\rho \in L^\infty(\Omega)_+$ such that for a.e. $x \in \Omega$, all $t \le \rho$ and all $\xi \in \partial j_+(x,t)$

$$|\xi| \le a_\rho(x).$$

So, for a.e. $x \in \Omega$, all $t \in \mathbb{R}$ and all $\xi \in \partial j_+(x,t)$ we have

$$|\xi| \le a_\rho(x) + (\theta(x) + \varepsilon)|t|^{p-1}. \tag{3.4}$$

By the Rademacher theorem and [10, Proposition 2.2.2], for a.e. $x \in \Omega$ the mapping $j_+(x,\cdot)$ is differentiable for a.e. $t \in \mathbb{R}$ with

$$\frac{d}{dt}\, j_+(x,t) \in \partial j_+(x,t).$$

So, integrating and using (3.4) we obtain for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$

$$j_+(x,t) \le a_\rho(x)|t| + (\theta(x) + \varepsilon)\frac{|t|^p}{p}. \tag{3.5}$$

For all $u \in W_0^{s,p}(\Omega)$, we have

$$\varphi_+(u) \ge \frac{\|u\|^p}{p} - \int_\Omega \Big(a_\rho(x)|u| + (\theta(x) + \varepsilon)\frac{|u|^p}{p}\Big)\, dx \quad \text{(by (3.5))}$$

$$
\begin{aligned}
&\geq \frac{1}{p}\Big(\|u\|^p - \int_\Omega \theta(x)|u|^p\,dx\Big) - \|a_\rho\|_\infty \|u\|_1 - \frac{\varepsilon}{p}\|u\|_p^p \\
&\geq \frac{1}{p}\Big(\theta_0 - \frac{\varepsilon}{\lambda_1}\Big)\|u\|^p - c_5\|u\| \quad (\theta_0, c_5 > 0),
\end{aligned}
$$

where in the final passage we have used Lemmas 2.8, 2.9, and the continuous embedding $W_0^{s,p}(\Omega) \hookrightarrow L^1(\Omega)$. If we choose $\varepsilon \in (0, \theta_0\lambda_1)$, the latter tends to ∞ as $\|u\| \to \infty$, hence φ_+ is coercive in $W_0^{s,p}(\Omega)$.

Moreover, the functional $u \mapsto \|u\|^p/p$ is convex, hence weakly lower semi continuous in $W_0^{s,p}(\Omega)$, while J_+ is continuous in $L^p(\Omega)$, which, by the compact embedding $W_0^{s,p}(\Omega) \hookrightarrow L^p(\Omega)$ and the Eberlein-Smulyan theorem, implies that J_+ is sequentially weakly continuous in $W_0^{s,p}(\Omega)$. So, φ_+ is sequentially weakly l.s.c. in $W_0^{s,p}(\Omega)$. As a consequence, there exists $u_+ \in W_0^{s,p}(\Omega)$ such that

$$
\varphi_+(u_+) = \inf_{u \in W_0^{s,p}(\Omega)} \varphi_+(u) = m_+. \tag{3.6}
$$

By Lemma 2.2 (vi), u_+ is a critical point of φ_+. We claim now that

$$
m_+ < 0. \tag{3.7}
$$

Indeed, by (H4), for any $\varepsilon > 0$ we can find $\delta > 0$ such that for a.e. $x \in \Omega$, all $t \in [0, \delta)$, and all $\xi \in \partial j_+(x,t)$

$$
\xi \geq (\eta_1(x) - \varepsilon)t^{p-1}.
$$

As above, integrating we have

$$
j_+(x,t) \geq \frac{\eta_1(x) - \varepsilon}{p} t^p. \tag{3.8}
$$

Let $u_1 \in W_0^{s,p}(\Omega) \cap C^\alpha(\overline{\Omega})$ be defined as in Lemma 2.8. We can find $\mu > 0$ such that $0 < \mu u_1(x) \leq \delta$ for all $x \in \Omega$. Then we have

$$
\begin{aligned}
\varphi_+(\mu u_1) &\leq \frac{\mu^p}{p}\|u_1\|^p - \frac{\mu^p}{p}\int_\Omega (\eta_1(x) - \varepsilon)u_1^p\,dx \quad \text{(by (3.8))} \\
&= \frac{\mu^p}{p}\Big(\int_\Omega (\lambda_1 - \eta_1(x))u_1^p\,dx + \varepsilon\Big) \quad \text{(by Lemma 2.8)}.
\end{aligned}
$$

Recalling that $\eta_1 \geq \lambda_1$ with $\eta_1(x) > \lambda_1$ for all x in a subset of Ω with positive measure, and that $u_1(x) > 0$ for all $x \in \Omega$, we see that

$$
\int_\Omega (\lambda_1 - \eta_1(x))u_1^p\,dx < 0.
$$

So, for $\varepsilon > 0$ small enough, the estimates above imply $\varphi_+(\mu u_1) < 0$. Thus, we have (3.7).

In particular, from (3.6) we have $u_+ \neq 0$. Now Lemma 3.1 implies that $u_+ \in C^\alpha(\overline{\Omega})$, $u_+(x) > 0$ for all $x \in \Omega$, for all $y \in \partial\Omega$

$$
\liminf_{\Omega \ni x \to y} \frac{u_+(x)}{\operatorname{dist}(x, \Omega^c)} > 0,
$$

and finally that u_+ is a solution of (1.1).

An analogous argument, applied to φ_- with the support of Lemma 3.2, proves the existence of another solution $u_- \in C^\alpha(\overline{\Omega})$ such that $u_-(x) < 0$ for all $x \in \Omega$,

$$
\limsup_{\Omega \ni x \to y} \frac{u_-(x)}{\operatorname{dist}(x, \Omega^c)} < 0
$$

for all $y \in \partial\Omega$. So the proof is concluded. □

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