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A New BGK Model for Relativistic Kinetic Theory of Monatomic and Polyatomic Gases

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Abstract. We propose a possible generalization of the BGK collisional term in the relativistic kinetic theory and we compare our model with previous ones. The present model has the advantage in that it satisfies the conservation of particle number, energy-momentum and the H-theorem in the Eckart frame without constraints on particular local equilibrium state. In the last part of the paper we extend the model to include also the polyatomic gas.

1. Introduction

In the kinetic theory one of most complex questions is to give an explicit expression for the collisional term. In the classical case the simplest case for the collisional term is the one proposed in the well-known paper by Bhatnagar-Gross-Krook [1] that is known as BGK model. The BGK model is simple, but has the main ingredients needed to satisfy a collisional term, i.e. the conservation of mass, momentum and energy and the H-theorem. Of course the model is oversimplified because only one relaxation time is present in the associated production terms related to the moments, and the phenomenological coefficients obtained via Chapman-Enskog procedure as heat conductivity, and viscosity are not so good in comparison with experiments. Nevertheless BGK model has been largely used not only in Boltzmann equation (see e.g. [2]), but also in the context of Rational Extended Thermodynamics (ET) to determine the production terms of the differential system [3]. In particular, this can be said if we take into account the recent progress obtained in ET to include polyatomic gas in the theory [4].

In the relativistic framework the BGK model was reconsidered first by Marle [5] and successively a different proposal was given by Anderson and Witting (AW) [6]. In this paper we discuss first the previous proposal and then we will propose, both for monatomic and for polyatomic gases, a variant of AW model proving the conservation of particles number, momentum-energy and the H-Theorem in the Eckart frame instead of the Landau-Lifshitz one.

The results presented here can be useful to evaluate the production terms in the relativistic ET of monatomic gas [7] and for polyatomic gas [8].

2. The relativistic Boltzmann Chernikov equation

Let us consider, as starting point, the Boltzmann Chernikov equation for the determination of the distribution function $f(x^\alpha, p^\beta)$:

$$p^\alpha \partial_\alpha f = Q, \quad (1)$$



where x^α is the space-time coordinate, p^α is the four-momentum ($p_\alpha p^\alpha = m^2 c^2$), m is the particle mass, c is the light velocity $\partial_\alpha = \partial/\partial x^\alpha$, and Q is the collisional term ($\alpha = 0, 1, 2, 3$). Repeated indexes, as usual, denote summation from 0 to 3. There is now the problem of determining the production term Q appearing in this equation. There are different possible models in literature for Q (see, e.g. [2, 9]). In any way a physical Q must vanish in any local equilibrium state and moreover, it must guarantee the conservation of particle number and momentum-energy and need to satisfy the H-theorem.

It is well known that from (1), we can define the moment equations. In particular the following first five ones:

$$\partial_\alpha V^\alpha = P, \quad \partial_\alpha T^{\alpha\beta} = P^\beta, \quad (2)$$

with

$$V^\alpha = mc \int_{\mathbb{R}^3} f p^\alpha d\mathbf{P}, \quad T^{\alpha\beta} = c \int_{\mathbb{R}^3} f p^\alpha p^\beta d\mathbf{P}, \quad (3)$$

and

$$P = mc \int_{\mathbb{R}^3} Q d\mathbf{P}, \quad P^\alpha = c \int_{\mathbb{R}^3} Q p^\alpha d\mathbf{P}, \quad (4)$$

where $d\mathbf{P} = dp^1 dp^2 dp^3 / p^0$. As V^α is the particle-particle flux and $T^{\alpha\beta}$ is the energy-momentum tensor, we need that (2) must be conservation laws and therefore we need that the collisional term Q must satisfy the conditions $P = 0, P^\alpha = 0$, i.e.

$$\int_{\mathbb{R}^3} Q d\mathbf{P} = 0, \quad \int_{\mathbb{R}^3} Q p^\alpha d\mathbf{P} = 0, \quad \alpha = 0, 1, 2, 3. \quad (5)$$

Moreover defining the entropy four vector h^α and the entropy production Σ :

$$h^\alpha = -k_B c \int_{\mathbb{R}^3} f \ln f p^\alpha d\mathbf{P}, \quad \Sigma = -k_B c \int_{\mathbb{R}^3} Q \ln f d\mathbf{P}, \quad (6)$$

(k_B is the Boltzmann constant), then from (1) we obtain the entropy law:

$$\partial_\alpha h^\alpha = \Sigma. \quad (7)$$

The H-theorem requires that the entropy production must be non-negative ($\Sigma \geq 0$) and therefore we need to require that Q satisfies also the inequality

$$\int_{\mathbb{R}^3} Q \ln f d\mathbf{P} \leq 0. \quad (8)$$

To sum up, we have that any physical collisional term Q must satisfy the five equations (5) and the inequality (8).

3. Eckart and Landau-Lifshitz frames

We recall that in the relativistic hydrodynamics there are two possible choices of the 4-velocity. The most popular one is the Eckart 4-velocity and the other one is the Landau-Lifshitz 4-velocity (see e.g. [9, 10]).

– The *Eckart frame* or *particle frame* corresponds to the frame in which there is no dissipative contribution to the rest-mass density current and to the energy density. In this case the four velocity U^β is collinear with V^β and we have:

$$V^\alpha = V_E^\alpha, \quad \left(T^{\alpha\beta} - T_E^{\alpha\beta}\right) U_\beta = 0, \quad (9)$$

where the index E indicates a generic local equilibrium state.

In the Eckart frame it is possible to rewrite V^α and $T^{\alpha\beta}$ in terms of the usual physical quantities:

$$\begin{aligned} V^\alpha &= \rho U^\alpha \\ T^{\alpha\beta} &= \sigma^{\langle\alpha\beta\rangle} + (p + \Pi)h^{\alpha\beta} + \frac{1}{c^2}(q^\alpha U^\beta + q^\beta U^\alpha) + \frac{e}{c^2}U^\alpha U^\beta, \end{aligned} \quad (10)$$

where U^α is the Eckart four-velocity ($U^\alpha U_\alpha = c^2$), $\rho = nm = c^{-1} \sqrt{V^\alpha V_\alpha}$ is the number density, p is the pressure, Π is the dynamical pressure, $h^{\alpha\beta}$ is the projector tensor $h^{\alpha\beta} = -g^{\alpha\beta} + \frac{1}{c^2}U^\alpha U^\beta$ with $g^{\alpha\beta} = \text{diag}(1, -1, -1, -1)$ being the metric tensor, the symbol $\langle \dots \rangle$ denotes the 3-dimensional traceless part of a tensor, $\sigma^{\langle\alpha\beta\rangle} = T^{\mu\nu} \left(h_\mu^\alpha h_\nu^\beta - \frac{1}{3}h^{\alpha\beta} h_{\mu\nu} \right)$ is the viscous deviatoric stress, e is the energy, and $q^\alpha = -h_\mu^\alpha U_\nu T^{\mu\nu}$ is the heat flux. From the previous definitions we have $h^{\alpha\beta} U_\beta = 0$, $q^\alpha U_\alpha = 0$, $\sigma^{\langle\alpha\beta\rangle} U_\beta = 0$ and therefore only 14 fields are independent.

– The *Landau-Lifshitz frame* or *energy frame* represents the frame in which there is no net energy flux, i.e., it is the frame defined by the conditions

$$(V^\alpha - V_E^\alpha) U_{L\alpha} = 0, \quad \left(T^{\alpha\beta} - T_E^{\alpha\beta}\right) U_{L\alpha} U_{L\beta} = 0, \quad (11)$$

where U_L^α indicates the Landau-Lifshitz four-velocity.

In the Landau-Lifshitz frame V^α and $T^{\alpha\beta}$ have the following decompositions:

$$\begin{aligned} V^\alpha &= \rho \left(U_L^\alpha - \frac{1}{p+e} q^\alpha \right) \\ T^{\alpha\beta} &= \sigma^{\langle\alpha\beta\rangle} + (p + \Pi)h_L^{\alpha\beta} + \frac{e}{c^2} U_L^\alpha U_L^\beta, \end{aligned} \quad (12)$$

with projector $h_L^{\alpha\beta} = -g^{\alpha\beta} + \frac{1}{c^2}U_L^\alpha U_L^\beta$. The expressions (10) and (12) are equivalent if the following relation between Eckart and Landau-Lifshitz 4-velocities holds:

$$U_L^\alpha = U^\alpha + \frac{1}{e+p} q^\alpha,$$

and if we neglect second order terms in the non-equilibrium variables $\Pi, q^\alpha, \sigma^{\langle\alpha\beta\rangle}$.

4. The relativistic BGK approximation

In the relativistic framework the most important generalization of BGK approximation was made by Marle [5] and successively by Anderson and Witting [6].

4.1. The Marle BGK model

The Marle model [5] is an extension of the non-relativistic BGK model in the Eckart frame:

$$Q = -\frac{m}{\tau} (f - f_E), \quad (13)$$

where τ is the relaxation time in the rest frame where the momentum of particles is zero, and $f_E = f_J$ is the Jüttner equilibrium distribution (see [2, 8]):

$$f_J = \exp\left(\xi - \frac{1}{k_B T} U_\beta p^\beta\right), \quad \xi = -1 + \frac{m}{k_B} \frac{g_r}{T}. \quad (14)$$

In (14) g_r denotes the relativistic chemical potential

$$g_r = \frac{e + p}{\rho} - TS.$$

T and S are respectively the temperature and the entropy density ($\rho S = h^\alpha u_\alpha$). We notice that g_r is in reality the chemical potential except for c^2 term. In fact taking into account that the energy is composed of two parts: Part due to the internal energy ε and part to the rest-energy:

$$e = \rho(\varepsilon + c^2),$$

we have

$$g_r = g + c^2, \quad g = \varepsilon + \frac{p}{\rho} - TS,$$

where g denotes the usual chemical potential.

In the Marle BGK approximation, by inserting (13), the conditions (5) and the inequality (8) become, respectively:

$$\int_{\mathfrak{R}^3} (f - f_E) d\mathbf{P} = 0, \quad \int_{\mathfrak{R}^3} (f - f_E) p^\alpha d\mathbf{P} = 0, \quad (15)$$

$$\int_{\mathfrak{R}^3} (f - f_E) \ln f d\mathbf{P} \geq 0. \quad (16)$$

The second of (15) is equivalent to the first condition of the Eckart frame definition (9) and the inequality (16) holds for analogous arguments of classical BGK (see e.g. [2]). The most weak condition is (15)₁ that becomes a constraint on the distribution function f . In fact the five conditions (15) can be interpreted as the equations for the five unknown particular local equilibrium variables T, g_r and U_α appearing in the Jüttner equilibrium distribution function (14) and these coefficients in reality depend themselves on f . The existence of the solution of this problem was proved in [11]. Another weak point of the Marle model is that the relaxation time becomes unbounded in the case of particles with zero rest mass [2].

4.2. The Anderson and Witting BGK model

The Anderson-Witting model provides another expression of the Q , described in the Landau-Lifshitz frame and has been widely used:

$$Q = - \frac{U_{L\mu} p^\mu}{c^2 \tau} (f - f_E), \quad (17)$$

where $U_{L\mu}$ indicates the four-velocity according with the Landau-Lifshitz definition. In the AW model the conditions that guarantee the conservation laws (5) and the H-theorem (8) become respectively:

$$U_{L\mu} \int_{\mathfrak{R}^3} (f - f_E) p^\mu d\mathbf{P} = 0, \quad U_{L\mu} \int_{\mathfrak{R}^3} (f - f_E) p^\mu p^\alpha d\mathbf{P} = 0, \quad (18)$$

$$U_{L\mu} \int_{\mathbb{R}^3} (f - f_E) \ln f p^\mu d\mathbf{P} \geq 0. \quad (19)$$

The conditions (18) are coincident with (11) that define the Landau-Lifshitz frame and the inequality (19) is satisfied (see [2] for the proof). Therefore the AW model satisfies all the requirements necessary for the collisional term and moreover in the classical limit converges to the classical BGK model. The weak point is that the frame is the Landau-Lifshitz frame that is less used in literature also for the complexity in the conservation of number of particle [10]. A comparison between the two BGK models and also on the Marle-Grad 14 moments was studied in [12]. In particular in [12], the Cauchy problem has been studied for the linearized kinetic equations with the Marle and Anderson-Witting models, and compared the resulting dispersion relations with the 14-moment theory.

5. A new relativistic BGK approximation for monatomic gases

The question arises if it is possible to construct a relativistic BGK in the Eckart frame without the problematic present in the Marle model. We notice that, if we substitute in the AW model the 4-velocity of Eckart the condition (18)₂ is not anymore satisfied because the left side is not zero but proportional to the heat flux 4-vector q^α . We want to prove in this paper that it is possible to construct a sort of generalization of the AW model that satisfies automatically the definition of the Eckart frame (9), satisfies the H-theorem, and reduces to the BGK in the classical limit.

Before introducing it, let us consider the following equilibrium moment of third order:

$$A_E^{\alpha\beta\mu} = \frac{c}{m} \int_{\mathbb{R}^3} f_E p^\alpha p^\beta p^\mu d\mathbf{P} = a U^\alpha U^\beta U^\mu + b \left(h^{\alpha\beta} U^\mu + h^{\alpha\mu} U^\beta + h^{\beta\mu} U^\alpha \right). \quad (20)$$

Inserting the Jüttner distribution into (20), it is simple to verify that the coefficients appearing in (20) are

$$a = \rho \left(3 \frac{K_3(\gamma)}{\gamma K_2(\gamma)} + 1 \right), \quad b = k_B n T \frac{K_3(\gamma)}{K_2(\gamma)} = p \frac{K_3(\gamma)}{K_2(\gamma)},$$

with

$$\gamma = \frac{mc^2}{k_B T}, \quad (21)$$

and $K_n(\gamma)$ denotes the modified Bessel functions of second kind.

We propose as variant of Relativistic BGK this form of the collisional term:

$$Q = \frac{U_\alpha p^\alpha}{c^2 \tau} \left(f_E - f - f_E p^\mu q_\mu \frac{1}{bmc^2} \right). \quad (22)$$

Comparing these expression with (17), we see that we have now the extra term $-f_E p^\mu q_\mu \frac{1}{bmc^2}$ inside the parenthesis and the use of Eckart 4-velocity instead of the Landau-Lifshitz one. We can prove

Theorem 1 (Conservation Laws) *The collisional term (22) conserves the number of particle and the energy momentum in the Eckart frame i.e. $P = P^\alpha = 0$.*

The proof is immediate. Inserting (22) in (4) and taking into account the Eckart frame definition (9) we have:

$$\begin{aligned}
P &= \frac{m}{c\tau} U_\alpha \int_{\mathbb{R}^3} \left(f_E - f - f_E p^\mu q_\mu \frac{1}{bmc^2} \right) p^\alpha d\mathbf{P} = \\
&= -\frac{1}{b\tau c^4} U_\alpha T_E^{\alpha\mu} q_\mu = -\frac{e}{b\tau c^4} U^\mu q_\mu = 0, \\
P^\beta &= \frac{1}{c\tau} U_\alpha \int_{\mathbb{R}^3} \left(f_E - f - f_E p^\mu q_\mu \frac{1}{bmc^2} \right) p^\alpha p^\beta d\mathbf{P} = \\
&= \frac{1}{c^2\tau} U_\alpha \left(T_E^{\alpha\beta} - T^{\alpha\beta} - \frac{1}{bc^2} A_E^{\alpha\beta\mu} q_\mu \right) = -\frac{1}{c^2\tau} (q^\beta - q^\beta) = 0.
\end{aligned} \tag{23}$$

Theorem 2 (H-Theorem) *The collisional term (22) satisfies the H-theorem up to second order terms with respect to non-equilibrium variables.*

Proof- For our collisional term (22) we have from (6)₂:

$$\Sigma = -\frac{k_B}{c\tau} U_\alpha \int_{\mathbb{R}^3} \left(f_E - f - f_E p^\mu q_\mu \frac{1}{bmc^2} \right) \ln f p^\alpha d\mathbf{P}. \tag{24}$$

Let us consider the right hand side of this relation, but with $\ln f_E$ instead of $\ln f$, that is,

$$-\frac{k_B}{c\tau} U_\alpha \int_{\mathbb{R}^3} \left(f_E - f - f_E p^\mu q_\mu \frac{1}{bmc^2} \right) \ln f_E p^\alpha d\mathbf{P}. \tag{25}$$

Thanks to eq. (14) it becomes

$$-\frac{k_B}{c\tau} U_\alpha \int_{\mathbb{R}^3} \left(f_E - f - f_E p^\mu q_\mu \frac{1}{bmc^2} \right) \left[-1 + \frac{mg_r}{k_B T} - \frac{1}{k_B T} U_\beta p^\beta \right] p^\alpha d\mathbf{P}$$

which is zero, thanks to (23). Consequently, we can subtract (25) from (24) so that it becomes

$$\Sigma = -\frac{k_B}{c\tau} U_\alpha \int_{\mathbb{R}^3} \left(f_E - f - f_E p^\mu q_\mu \frac{1}{bmc^2} \right) \ln \frac{f}{f_E} p^\alpha d\mathbf{P}.$$

Now we sum to this expression, (23)₂ contracted by $-\frac{k_B}{bmc^3} q_\beta$; so it becomes

$$\Sigma = \frac{k_B}{c\tau} U_\alpha \int_{\mathbb{R}^3} \left(-f_E + f + f_E p^\mu q_\mu \frac{1}{bmc^2} \right) \left(\ln \frac{f}{f_E} + p^\beta q_\beta \frac{1}{bmc^2} \right) p^\alpha d\mathbf{P}.$$

Now, Taylor's expansion of $\ln \frac{f}{f_E}$ around $f = f_E$, dropped at the first order expansion, is

$$\ln \frac{f}{f_E} = \ln \frac{f_E}{f_E} + \frac{1}{f_E} (f - f_E) + O(2) \tag{26}$$

so that we have

$$\Sigma = \frac{k_B}{c\tau} U_\alpha \int_{\mathbb{R}^3} \frac{1}{f_E} \left(-f_E + f + f_E p^\mu q_\mu \frac{1}{bmc^2} \right)^2 p^\alpha d\mathbf{P} + O(3)$$

from which, taking into account that $U_\alpha p^\alpha > 0$, it follows $\Sigma \geq 0$.

5.1. Classical Limit of (22)

Taking into account that the time and space components of U^α and of p^α are

$$U^\alpha \equiv \bar{\Gamma}(c, v^i), \quad p^\alpha \equiv m\Gamma(c, \xi^i), \quad \bar{\Gamma} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad \Gamma = \frac{1}{\sqrt{1 - \frac{\xi^2}{c^2}}}$$

(v^i and ξ^i are respectively the macroscopic and the microscopic velocity), and evaluating (22) in the rest frame $v^i = 0$, we have:

$$Q = \frac{m\Gamma}{\tau} \left(f_E - f + f_E \frac{1}{bc^2} \Gamma q_i \xi^i \right).$$

Taking into account that [8]

$$\lim_{c \rightarrow \infty} \Gamma = 1, \quad \lim_{c \rightarrow \infty} b = p, \quad \lim_{c \rightarrow \infty} f_E = \frac{1}{m^3} f_M, \quad \lim_{c \rightarrow \infty} f = \frac{1}{m^3} f^C,$$

where f_M is the Maxwellian and f^C denote the classical distribution function solution of

$$\partial_t f^C + \xi^i \partial_i f^C = Q^C, \quad x^0 = ct$$

we obtain:

$$Q^C = \lim_{c \rightarrow \infty} \frac{m^2 Q}{\Gamma} = \frac{1}{\tau} (f_M - f^C).$$

Therefore our BGK relativistic variant converges to the classical BGK!

6. A Relativistic BGK approximation for polyatomic gases

In [8] starting from the classical ideas for polyatomic gases introduced in [4, 13–17] we proposed a generalized Boltzmann-Chernikov equation that has the same form of (1) but the extended distribution function $f \equiv f(x^\alpha, p^\beta, \mathcal{I})$ depends on an extra variable \mathcal{I} that takes into account the energy due to the internal degrees of freedom of a molecule. We consider instead of (3), the following moments:

$$\begin{aligned} V^\alpha &= mc \int_{\mathbb{R}^3} \int_0^{+\infty} f p^\alpha \phi(\mathcal{I}) d\mathbf{P} d\mathcal{I}, \\ T^{\alpha\beta} &= \frac{1}{mc} \int_{\mathbb{R}^3} \int_0^{+\infty} f (mc^2 + \mathcal{I}) p^\alpha p^\beta \phi(\mathcal{I}) d\mathbf{P} d\mathcal{I}. \end{aligned} \quad (27)$$

The meaning of (27)₂ is that the energy and the momentum in relativity are components of the same tensor and we expect that, besides the energy at rest, there is a contribution due to the internal structure, as in the case of a classical polyatomic gas. $\phi(\mathcal{I})$ is the state density of the internal mode, that is, $\phi(\mathcal{I}) d\mathcal{I}$ represents the number of the internal states of a molecule having the internal energy between \mathcal{I} and $\mathcal{I} + d\mathcal{I}$.

The macroscopic internal energy

$$\varepsilon = \frac{e}{mn} - c^2,$$

in the classical limit, when γ given by (21) tend to infinity, converges to the one of a classical polyatomic gas [8]:

$$\lim_{\gamma \rightarrow \infty} \varepsilon = \frac{D}{2} \frac{k_B}{m} T,$$

provided that the measure

$$\phi(\mathcal{I}) = \mathcal{I}^a,$$

where the constant is given by

$$a = \frac{D-5}{2},$$

and $D = 3 + f^i$ is related to the degrees of freedom of a molecule given by the sum of the space dimension 3 for the translational motion and the contribution from the internal degrees of freedom $f^i \geq 0$ related to the rotation and vibration. For monatomic gases $D = 3$ and $a = -1$.

As in the monatomic case we introduce the following equilibrium triple tensor

$$\begin{aligned} A_E^{\alpha\beta\mu} &= \frac{c}{m} \int_{\mathbb{R}^3} \int_0^{+\infty} f_E p^\alpha p^\beta p^\mu \left(1 + \frac{\mathcal{I}}{mc^2}\right)^2 \phi(\mathcal{I}) d\mathbf{P} d\mathcal{I} = \\ &= a U^\alpha U^\beta U^\mu + b \left(h^{\alpha\beta} U^\mu + h^{\alpha\mu} U^\beta + h^{\beta\mu} U^\alpha \right), \end{aligned} \quad (28)$$

where f_E is the generalized Jüttner equilibrium distribution function obtained in [8] via *Maximum Entropy Principle*:

$$f_E = \exp \left[\xi - \left(1 + \frac{\mathcal{I}}{mc^2}\right) \frac{1}{k_B T} U_\beta p^\beta \right], \quad \xi = -1 + \frac{m g_r}{k_B T}, \quad (29)$$

The expression of the coefficients appearing in (28)₃ is

$$\begin{aligned} a &= \frac{\rho}{\gamma A(\gamma) K_2(\gamma)} \int_0^{+\infty} [3K_3(\gamma^*) + \gamma^* K_2(\gamma^*)] \phi(\mathcal{I}) d\mathcal{I}, \\ b &= \frac{nk_B T}{A(\gamma) K_2(\gamma)} \int_0^{+\infty} K_3(\gamma^*) \phi(\mathcal{I}) d\mathcal{I}, \\ A(\gamma) &= \frac{\gamma}{K_2(\gamma)} \int_0^{+\infty} \frac{K_2(\gamma^*)}{\gamma^*} \phi(\mathcal{I}) d\mathcal{I}, \end{aligned}$$

where we put

$$\gamma^* = \gamma \left(1 + \frac{\mathcal{I}}{mc^2}\right).$$

We consider as a variant of Relativistic Polyatomic BGK this form of the collisional term:

$$Q = \frac{U_\alpha p^\alpha}{c^2 \tau} \left(f_E - f - f_E p^\mu q_\mu \frac{1 + \frac{\mathcal{I}}{mc^2}}{bmc^2} \right). \quad (30)$$

We want to prove the following:

Theorem 3 (Conservation Laws in Polyatomic Gas) *The collisional term (30) conserves the number of particle and the energy momentum, i.e.*

$$\begin{aligned} P &= \partial_\alpha V^\alpha = mc \int_{\mathbb{R}^3} \int_0^{+\infty} Q \phi(\mathcal{I}) d\mathbf{P} d\mathcal{I} = 0, \\ P^\beta &= \partial_\alpha T^{\alpha\beta} = c \int_{\mathbb{R}^3} \int_0^{+\infty} Q p^\beta \left(1 + \frac{\mathcal{I}}{mc^2}\right) \phi(\mathcal{I}) d\mathbf{P} d\mathcal{I} = 0. \end{aligned} \quad (31)$$

The proof is immediate. Inserting (30) in (31) and taking into account the Eckart frame (9) we have:

$$\begin{aligned}
P &= \frac{m}{c\tau} U_\alpha \int_{\mathbb{R}^3} \int_0^{+\infty} \left(f_E - f - f_E p^\mu q_\mu \frac{1 + \frac{\mathcal{I}}{mc^2}}{bmc^2} \right) p^\alpha \phi(\mathcal{I}) d\mathbf{P} d\mathcal{I} = \\
&= -\frac{1}{b\tau c^4} U_\alpha T_E^{\alpha\mu} q_\mu = -\frac{e}{b\tau c^4} U^\mu q_\mu = 0, \\
P^\beta &= \frac{1}{c\tau} U_\alpha \int_{\mathbb{R}^3} \int_0^{+\infty} \left(f_E - f - f_E p^\mu q_\mu \frac{1 + \frac{\mathcal{I}}{mc^2}}{bmc^2} \right) p^\alpha p^\beta \cdot \\
&\quad \cdot \left(1 + \frac{\mathcal{I}}{mc^2} \right) \phi(\mathcal{I}) d\mathbf{P} d\mathcal{I} = \\
&= \frac{1}{c^2\tau} U_\alpha \left(T_E^{\alpha\beta} - T^{\alpha\beta} - \frac{1}{bc^2} A_E^{\alpha\beta\mu} q_\mu \right) = -\frac{1}{c^2\tau} (q^\beta - q^\beta) = 0.
\end{aligned} \tag{32}$$

Theorem 4 (H-Theorem in Polyatomic Gas) *Let*

$$h^\alpha = -k_B c \int_{\mathbb{R}^3} \int_0^{+\infty} f \ln f p^\alpha \phi(\mathcal{I}) d\mathbf{P} d\mathcal{I}$$

be the entropy four-vector. In correspondence we have the entropy law (7) with the entropy production

$$\Sigma = -k_B c \int_{\mathbb{R}^3} \int_0^{+\infty} Q \ln f \phi(\mathcal{I}) d\mathbf{P} d\mathcal{I}.$$

Neglecting third and higher order terms in non-equilibrium variables we have the H-theorem

$$\Sigma \geq 0.$$

Proof - We have in the present case

$$\Sigma = -\frac{k_B}{c\tau} U_\alpha \int_{\mathbb{R}^3} \int_0^{+\infty} \left(f_E - f - f_E p^\mu q_\mu \frac{1 + \frac{\mathcal{I}}{mc^2}}{bmc^2} \right) \ln f p^\alpha \phi(\mathcal{I}) d\mathbf{P} d\mathcal{I}. \tag{33}$$

Let us consider the right hand side of this relation, but with $\ln f_E$ instead of $\ln f$, that is,

$$-\frac{k_B}{c\tau} U_\alpha \int_{\mathbb{R}^3} \int_0^{+\infty} \left(f_E - f - f_E p^\mu q_\mu \frac{1 + \frac{\mathcal{I}}{mc^2}}{bmc^2} \right) \ln f_E p^\alpha \phi(\mathcal{I}) d\mathbf{P} d\mathcal{I}. \tag{34}$$

Thanks to eq. (29) it becomes

$$\begin{aligned}
&-\frac{k_B}{c\tau} U_\alpha \int_{\mathbb{R}^3} \int_0^{+\infty} \left(f_E - f - f_E p^\mu q_\mu \frac{1 + \frac{\mathcal{I}}{mc^2}}{bmc^2} \right) \cdot \\
&\quad \cdot \left[-1 + \frac{mg_r}{k_B T} - \frac{1 + \frac{\mathcal{I}}{mc^2}}{k_B T} U_\beta p^\beta \right] p^\alpha d\mathbf{P},
\end{aligned}$$

which is zero, thanks to (32). Consequently, we can subtract (34) from (33) so that it becomes

$$\Sigma = -\frac{k_B}{c\tau} U_\alpha \int_{\mathbb{R}^3} \int_0^{+\infty} \left(f_E - f - f_E p^\mu q_\mu \frac{1 + \frac{\mathcal{I}}{mc^2}}{bmc^2} \right) \ln \frac{f}{f_E} p^\alpha \phi(\mathcal{I}) d\mathbf{P} d\mathcal{I}.$$

Now we sum to this expression, (32)₂ contracted by $-\frac{k_B}{bmc^2}q_\beta$; so it becomes

$$\Sigma = -\frac{k_B}{c\tau}U_\alpha \int_{\mathbb{R}^3} \int_0^{+\infty} \left(f_E - f - f_E p^\mu q_\mu \frac{1 + \frac{\mathcal{I}}{mc^2}}{bmc^2} \right) \cdot \left(\ln \frac{f}{f_E} + p^\beta q_\beta \frac{1 + \frac{\mathcal{I}}{mc^2}}{bmc^2} \right) p^\alpha \phi(\mathcal{I}) d\mathbf{P} d\mathcal{I}.$$

Now, Taylor's expansion of $\ln \frac{f}{f_E}$ around $f = f_E$, dropped at the first order, is given by (26), so that we have

$$\Sigma = \frac{k_B}{c\tau}U_\alpha \int_{\mathbb{R}^3} \int_0^{+\infty} \frac{1}{f_E} \left(-f_E + f + f_E p^\mu q_\mu \frac{1 + \frac{\mathcal{I}}{mc^2}}{bmc^2} \right)^2 p^\alpha \phi(\mathcal{I}) d\mathbf{P} d\mathcal{I} + O(3)$$

from which it follows $\Sigma \geq 0$.

The classical limit of the present results can be obtained in the same way described for monatomic gases and, also in this case, we find that our BGK relativistic variant converges to the classical BGK.

We observe that the Marle and the Anderson and Witting formulation can be used also in the polyatomic case without any change from the monatomic gas. For this other reason our model seems more physical. In fact the collisional term proposed in this paper changes from (22) to (30) in the passage from monatomic to polyatomic gas. This is what we expected from physical reasons as the collisional terms in polyatomic gas must depend on the internal state, i.e. must depend on \mathcal{I} .

The tendency to the equilibrium of the distribution function for our model and comparison between the solutions with the ones using by Marle and Anderson and Witting BGK models will be the subject of a future paper.

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