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On the usage of the curvature for the comparison of planar curves

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Abstract

Given two curves, on the plane or in space, or surfaces, looking for a deformation from one into another such that the transformation is gradual and continuous is an open and interesting problem both from a theoretical stand point and for the applications that can be envisaged. In particular, if the two starting shapes have some particular features, we expect that also the intermediate ones preserve the same characteristics. This thesis focuses on planar closed curves, aiming to find deformations between two planar closed curves, preserving the closeness. Our approach is treating curves in the perspective of Differential Geometry, that is represented by parameterizations. We show how the simple idea of linearly interpolating between the two parameterizations is too much dependent on the mutual position of the curves, implicit in the parameterizations. This suggested us to consider a deformation based on intrinsic properties of curves, in particular we took in account the curvature. The fundamental theorem of planar curves states that, given the signed curvature function with respect to arc length, it is possible to reconstruct the curve from it up to rigid motions which preserve the orientation. Motivated by these observations we linearly interpolate the curvature of source and target curves, parameterized with respect to the arc length, and we reconstruct the corresponding intermediate curves. Unfortunately, the curvature interpolation not always leads to closed curves. We overwhelmed this limitations replacing each intermediate open curve with the closed curve as close as possible to the open one measuring the same length. To do this we need to define an appropriate distance between curves. A distance based on the mutual position of the curves and dependent on their particular parameterizations is not feasible for our purposes, so as measure of distance between two curves we consider the distance between their curvatures. This paradigm shift leads to find the curve with the curvature as close as possible to the curvature of the open, interpolated, one. This decision is supported by the proof that there is a link between close curvatures and close curves, or to be more precise, that the distance between curvatures gives a bound for the distance between corresponding curves with a particular mutual position, where distances are computed with appropriate metric. We also show that solving this problem in the smooth setting is very difficult since it is not a classical variational problem, so we propose a simple example where we try to solve the variational problem in the smooth setting and then we conclude giving an approximated solution for the general case.

Sommario

Date due curve, nel piano o nello spazio, o sue superfici, la ricerca di una deformazione dell'una nell'altra tale che la trasformazione sia graduale e continua è un problema ancora aperto ed è interessante sia da un punto di vista teorico che applicativo. In particolare, se le due entità geometriche hanno qualche caratteristica particolare, ci aspettiamo che anche quelle intermedie la preservino. Questa tesi si occupa di curve chiuse piane ed ha l'obiettivo di trovare deformazioni tra due curve piane e chiuse che conservino la chiusura. Noi trattiamo le curve nella prospettiva della Geometria Differenziale, cioè rappresentandole mediante parametrizzazioni. Viene messo in evidenza come l'idea della semplice interpolazione lineare tra due parametrizzazioni è troppo dipendente dalla posizione reciproca delle curve, implicita nella parametrizzazione. Questo fatto suggerisce di considerare deformazioni basate sulle proprietà intrinseche delle curve, in particolare la curvatura. Il Teorema Fondamentale delle curve piane afferma inoltre che, data la funzione di curvatura rispetto all'ascissa curvilinea, è possibile ricostruire la curva a meno di movimenti rigidi che ne preservano l'orientazione. Motivati da queste osservazioni interpoliamo linearmente la curvatura delle due curve iniziali, parametrizzate rispetto all'ascissa curvilinea e ricostruiamo le corrispondenti curve intermedie. Sfortunatamente, l'interpolazione della curvatura non porta sempre alla ricostruzione di curve chiuse. Noi proponiamo di superare il problema sostituendo ogni curva aperta intermedia con una curva chiusa che sia il più vicina possibile a quella aperta e che abbia la stessa lunghezza. Per far questo abbiamo bisogno di definire un'appropriata distanza tra le curve. Una distanza basata sulla posizione reciproca delle curve e dipendente dalla loro particolare parametrizzazione non è utile per i nostri scopi, così come misura della distanza tra due curve consideriamo la distanza tra le loro curvature. Questo paradigma sposta il problema alla ricerca di una curva che abbia curvatura il più vicino possibile alla curvatura della curva aperta ottenuta dall'interpolazione. Questa decisione è supportata dalla dimostrazione che esiste un legame tra curvature vicine e curve vicine, o per essere più precisi, che la distanza tra curvature fornisce un limite superiore per la distanza tra le corrispondenti curve poste in una particolare posizione reciproca, dove le distanze sono calcolate con metriche opportune. Mostriamo anche come risolvere il problema nel caso continuo sia molto complicato non essendo uno dei classici problemi variazionali. Proponiamo quindi un semplice esempio dove cerchiamo di risolvere il problema variazionale nel caso continuo e concludiamo dando una soluzione approssimata per il caso generale.

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Introduction

The problem of interpolating between two curves, on the plane or in space, or two surfaces, making the deformation from one into another as-natural-as possible and the intermediate shapes visually pleasing is an open issue and is interesting both from a theoretical stand point and for the applications we can think of it. A typical problem in animation, for instance, is the interpolation between two poses of the same character. It is mandatory that also the shapes in between are natural poses otherwise the transformation would result counter-intuitive and, thus, not acceptable. Roughly speaking, what we require from the transformation is to be gradual and continuous, in particular, if the two starting shapes have some particular features, we expect that also the intermediate ones preserve the same characteristics.

This thesis focuses on planar shapes, not touching the extension to surfaces and leaving it to future work. In particular, the studied deformation is performed between two planar closed curves, and the characteristic that we want to preserve is to be closed. The first step in the construction of the deformation is to choose how to represent our objects, the curves, and the answer is to take the perspective of the Differential Geometry, where curves are represented by their parameterizations. In literature, a large part of works on deformations between shapes has a linear interpolation approach. Even for our purposes the simplest idea could be to linearly interpolate between the parameterizations fixing the correspondence between the first points of the curves, but it is not difficult to show with simple examples (Figure 5.1 and top row of Figure 5.2) how this deformations are too much dependent from the mutual position of the curves, implicit in the parameterizations, and how, changing the parameterizations changes also completely the deformation, becoming either very natural (top row of Figure 5.1 and top left of Figure 5.2) or absolutely unnatural (bottom row of Figure 5.1 and top right of Figure 5.2). This fact suggested us to consider a deformation based on intrinsic properties of the curves, namely their curvatures. We know, from the fundamental theorem of planar curves, that, given the signed curvature function with respect to arc length, it is possible to reconstruct the curve from the curvature up to rigid motions which preserve the orientation. Motivated by these observations we linearly interpolate the curvature between source and target curves, parameterized with respect to the arc length, and we reconstruct the corresponding intermediate curves.

The more intuitive results are soon visible in the bottom row of Figure 5.2, where it is shown clearly that this transformation is not dependent on the mutual position and that the two deformations produce exactly the same intermediate curves (up to rigid motions). Unfortunately, the curvature interpolation not always produces intermediate closed curves (Figure 5.5), unlike the parametrization-based interpolation. The solution is to replace each intermediate open curve with the closed curve which is as close as possible to the open one and has the same length.

What does it mean, in this context, "as-close-as possible"? In other words, how do we measure the similarity between two shapes? Define a notion of similarity between shapes is complicated by the fact that similarity is often based on human perception, and it is not easily formalized [37]. We can in fact answer to the questions: "are they similar"? are they close to each other? using qualitative or quantitative criteria, which are deeply different between them. The first ones are based on perception, on our eyesight, while the quantitative ones requires the introduction of a metric which allows us to measure the similarity of the two objects. The next problem is then to decide what kind of distance we want to consider since this implies which curves are considered equal. As already said, smooth curves are represented by parameterizations, and they contain, implicitly, the positions of the curves in the plane. But this could not be the only problem. In fact, given the trace of a curve, it is the image of infinite curves, obtained one from each other by reparameterizations. Then we can require that our distance would be dependent or independent from the mutual positions of the curves and/or it would be or not invariant under reparameterizations.

Requiring that the distance does not depend on the particular parameterization of the curves is a well known problem in literature. An example is the work of Bogacki et al. [9], where the authors studied the distances in the quotient space where the equivalence relation is given by reparameterizations. Finding the best mutual position in the computation of the distance is a hard problem. To avoid to take this last issue in account we decided to use, as a measure of distance between two curves, a quantity independent from their mutual positions, the curvature, and, thus, to find the curve with curvature as close as possible to the curvature of the open one. This choice is supported from the proof that there is a link between close curvatures and close curves, or to be more precise, that the distance between curvatures gives a bound for the distance between corresponding curves with a particular mutual position, where distances are computed with appropriate metric. Unfortunately to solve this problem in the smooth setting is very difficult since it is a variational problem for which no existing theorem of functional analysis can help us. For this reason we transferred the problem in the discrete setting, discretizing the curves to have approximated arc length parametrization, curvatures and, by consequence, minimization problem.

Summarizing, this thesis deals with two problems on planar curves: the first one is to find the closest curve to an open one such that it is closed and of same length; the second one derives from the necessity to define this “closest” curve, and then studying the distance between two curves and all the linked problems.

The thesis is organized in five chapters as follows. In Chapter 1 we recall the basic notions of Differential Geometry on smooth planar curves, such as the arc length parametrization and curvature. In Chapter 2 we describe the discrete counterparts of the notions introduced in Chapter 1, focusing on discrete curvature and its convergence to the smooth one. In Chapter 3 we recall the fundamental theorem of planar curve and we prove the link existing between the distance of two curves and the distance of their respective curvature; we then analyze the problem of choosing a good distance. In Chapter 4 we show that, also in the discrete setting, as in the continuous case, it is not possible to reconstruct a curve from its curvature if it is not expressed with respect to the arc length and we look for a distance to compare curves based on intrinsic properties of the curve, edges and angles, which was not possible to try in the smooth setting. Chapter 5 is dedicated to the curvature-based interpolation and the problem of minimizing the distance between curvatures to find a closed curve with curvature as close as possible to the interpolated one. The problem is described in the smooth setting, emphasizing the difficulty to solve it as variational problem, so it is proposed a simple example where we try to solve the variational problem in the smooth setting and then an approximated solution for general situation.

Chapter 1

Smooth planar curves: basic notions

This chapter deals with planar smooth curves as considered in Differential Geometry, which implies to represent them by parameterizations. An important aspect of differential geometry is the study of intrinsic properties of its objects, that is those characteristics which do not depend on their particular embedding in the space. Through all this work the intrinsic properties of plane curves will play a fundamental role. This section contains the basic notions of Differential Geometry about planar curves used in this work; all definitions can be found in [12] and [2].

Definition 1.0.1. *A parameterized differentiable plane curve is a differentiable map* $\alpha\colon I \to \boldsymbol{R}^2$ *of an open interval* I *of the real line* $\boldsymbol{R}$ *into* $\boldsymbol{R}^2$*.*

Definition 1.0.2. *Let* $\alpha\colon I \to \boldsymbol{R}^2$ *be a parameterized differentiable curve. For each* $t \in I$ *where* $\alpha'(t) \neq 0$*, there is a well-defined straight line, which contains the point* $\alpha(t)$ *and the vector* $\alpha'(t)$*. This line is called the* tangent line *to* α *at* t*.*

Definition 1.0.3. *A parameterized differentiable curve* $\alpha\colon I \to \boldsymbol{R}^2$ *is said to be* regular *if* $\alpha'(t) \neq 0$ *for all* $t \in I$*.*

Definition 1.0.4. *Given* $t \in I$*, the* arc length *of a regular parameterized curve* $\alpha\colon I \to \boldsymbol{R}^2$*, from the point* t_0*, is by definition*

$$s(t) = \int_{t_0}^{t} \|\alpha'(u)\| du.$$

Definition 1.0.5. *Given a parameterized differentiable curve* $\alpha\colon I \to \boldsymbol{R}^2$*, the length of* α *in the interval* $[a, b] \subset I$ *is defined as*

$$l_\alpha = \int_a^b \|\alpha'(u)\| du.$$

Definition 1.0.6. *Let* $\alpha\colon I \to \boldsymbol{R}^2$ *and* $\beta\colon J \to \boldsymbol{R}^2$ *two differentiable curves.* β *is said to be a* positive reparametrization *(resp.* negative reparametrization*) of* α *if there exists a differentiable function* $h\colon J \to I$ *such that* $h'(u) > 0$ *(resp.* $h'(u) < 0$*) for all* $u \in J$*, and* $\beta = \alpha \circ h$*.*

An useful tool to study the differential geometry of plane curves is the *complex structure* of $\mathbf{R}^2$. This structure is the linear function $J\colon \mathbf{R}^2 \to \mathbf{R}^2$ given by

$$J(p_1, p_2) = (-p_2, p_1).$$

From a geometrical point of view, J can be interpreted as the $\frac{\pi}{2}$ counterclockwise rotation .

Definition 1.0.7. $J\alpha'(t)$ *is the normal vector to the curve at the point* $\alpha(t)$*.*

Definition 1.0.8. *Let* $\alpha\colon I \to \boldsymbol{R}^2$ *a regular curve. The* curvature k *of* α *is given by the following formula*

$$k_\alpha(t) = \frac{\alpha''(t) \cdot J\alpha'(t)}{\|\alpha'\|^3}.$$

More explicitly, if $\alpha(t) = (\alpha_1(t), \alpha_2(t))$*, then*

$$k_\alpha(t) = \frac{\alpha_1'(t)\alpha_2''(t) - \alpha_2'(t)\alpha_1''(t)}{({\alpha_1'}^2(t) + {\alpha_2'}^2(t))^{\frac{3}{2}}}. \tag{1.1}$$

The radius of curvature *of* α *is defined as*

$$\rho(t) = \frac{1}{|k_\alpha(t)|}. \tag{1.2}$$

Definition 1.0.9. *A differentiable function on a closed interval* $[a, b]$ *is the restriction of a differentiable function defined on an open interval containing* $[a, b]$*.*

Definition 1.0.10. *A closed plane curve is a regular parameterized curve* $\alpha\colon [a, b] \to \boldsymbol{R}^2$ *such that*

$$\alpha(a) = \alpha(b), \quad \alpha^{(k)}(a) = \alpha^{(k)}(b).$$

Definition 1.0.11. *The curve* α *is simple if it has no further self-intersections, that is, if* $s_1, s_2 \in [a, b)$ *and* $s_1 \neq s_2$*, then* $\alpha(s_1) \neq \alpha(s_2)$*.*

Definition 1.0.12. *Let* $\alpha\colon I \to \boldsymbol{R}^2$ *a regular curve and* t_0 *a fixed point in* I*. Let* θ_0 *be a number such that*

$$\frac{\alpha'(t_0)}{\|\alpha'(t_0)\|} = (\cos(\theta_0), \sin(\theta_0))$$

then there exists an unique differentiable function $\theta\colon I \to \boldsymbol{R}$ *such that* $\theta(t_0) = \theta_0$ *and*

$$\frac{\alpha'(t)}{\|\alpha'(t)\|} = (\cos(\theta(t)), \sin(\theta(t)))$$

for $t \in I$. *The angle* θ *is said* ***rotation angle*** *determined by* θ_0.

Lemma 1.0.13. *The rotation angle and the curvature of a regular plane curve* α *are linked by the formula*

$$\theta'(t) = \|\alpha'(t)\| k_\alpha(t).$$

The rotation angle and the curvature of a regular unit-speed plane curve $\alpha\colon [0, L] \to \mathbf{R}^2$ are linked by the formula

$$\theta'(s) = k_\alpha(s) \tag{1.3}$$

that is assuming $\theta(0) = \theta_0$

$$\theta(s) = \int_0^s k(s)ds + \theta_0.$$

Definition 1.0.14. *The total signed curvature of a unit-speed plane curve* $\alpha\colon [0, L] \to \boldsymbol{R}^2$ *with curvature function* $k(s)$ *is*

$$TSC(\alpha) = \int_0^L k(s)ds$$

If α is closed then

$$\int_0^L k(s)ds = \theta(L) - \theta(0) = 2\pi I. \tag{1.4}$$

The integer I is called the *rotation index* of the curve α.

Theorem 1.0.15. *The rotation index of a simple closed curve is* ± 1, *where the sign depends on the orientation of the curve.*

At the beginning of the Chapter, we talked about the intrinsic properties of geometric objects. The next two Propositions 1.0.16 and 1.0.17 proves that for a plane curve they are arc length, length and curvature.

Proposition 1.0.16. *Let* $\alpha\colon (a, b) \to \boldsymbol{R}^2$ *a curve and* $\beta\colon (c, d) \to \boldsymbol{R}^2$ *a reparametrization. Then* $l_\alpha = l_\beta$ *and* $k_\alpha = \pm k_\beta$ *(positive (resp. negative) if the reparametrization is positive (resp. negative)).*

Proof. Let $\beta = \alpha \circ h$ with $h\colon (a,b) \to (c,d)$ such that $\dot{h} > 0$, then

$$\begin{aligned} l_\beta &= \int_c^d \|\beta'(u)\| du = \int_c^d \|\alpha'(h(u))h'(u)\| du \\ &= \int_c^d \|\alpha'(h(u))\| h'(u) du = \int_a^b \|\alpha'(t)\| dt = l_\alpha. \end{aligned}$$

Analogously if $h' < 0$

$$\begin{aligned} l_\beta &= \int_c^d \|\beta'(u)\| du = -\int_c^d \|\alpha'(h(u))\| h'(u) du \\ &= \int_d^c \|\alpha'(h(u))\| h'(u) du = \int_a^b \|\alpha'(t)\| dt = l_\alpha. \end{aligned}$$

For curvatures we have:

$$k_\beta = \frac{\beta'' \cdot J\beta'}{\|\beta'\|^3} = \frac{(\alpha'' h'^2 + \alpha' h'') \cdot h' J\alpha'}{|h'|^3 \|\alpha'\|^3} = \frac{h'^3(\alpha'' \cdot J\alpha')}{|h'|^3 \|\alpha'\|^3} = \pm k_\alpha.$$

QED

Proposition 1.0.17. *Let $\alpha\colon (a,b) \to \boldsymbol{R}^2$ and $\beta\colon (c,d) \to \boldsymbol{R}^2$ two curves with unit speed and β a reparametrization of α. Then*

$$\beta(u) = \alpha(\pm u + u_0),$$

for some $u_0 \in \boldsymbol{R}$.

Proof. By hypothesis there exists a differentiable function $h\colon (c,d) \to (a,b)$ such that $\beta = \alpha \circ h$ and $h' \neq 0$ for all $u \in (c,d)$. Then

$$1 = \|\beta'(u)\| = \|\alpha'(h(u))h'(u)\| = \|\alpha'(h(u))\| |h'(u)| = |h'(u)|.$$

This implies that $h'(u) = \pm 1$ and than $h(u) = \pm u + u_0$ since the sign of $h'(u)$ is constant.

QED

Chapter 2

From smooth to discrete setting

In the previous chapter we recalled the basic notions of smooth curves. The theory of planar curves in the continuous setting is almost complete and well known, but often for applications it is inadequate. This lead to the origin, some decades ago, of what is now called *Discrete Differential Geometry*. It is in fact a new area of Mathematics where differential geometry meets and interacts with discrete geometry (which deals with polytopes, simplicial complexes, etc.) [8]. But the transition from the smooth theory to the discrete one is not devoid of problems. The objects of discrete differential geometry can obviously be interpreted as approximations of the smooth ones, but give to them only this interpretation is quite reductive. In fact, it has been observed that when the notions of smooth geometry are discretized "properly", the discrete objects are not merely approximations of the smooth ones, but have properties of their own which make them coherent entities by themselves. The link between the discrete and the smooth theory can be summarized in this way: the smooth theory can always be recovered as a limit of the discrete one, while there is no natural way to discretize notions from the smooth theory to obtain the best discretization [8]. For the same notion many different discretizations are possible, all having the same smooth limit. To chose the best discretization is not easy since we can assume two different points of view. From a theoretical point of view we would desire the preservation of all the fundamental properties of the smooth theory, while from a practical point of view we are interested in a good convergence properties.

Let us observe that the difficulty in discretization arises from the loss of the differentiability, which is the fundamental characteristic of differential geometry. The discrete differential geometry aims at the development of discrete equivalents of notions and methods of classical differential geometry. Since computers work with discrete representations of data, it is understandable why many of the applications of discrete differential geometry are found within computer science, particularly in the areas of computational geometry, graphics and geometry processing.

After this brief overview on discrete differential geometry, we concentrate our attention on the objects of our study: the discrete planar curves. Although they are discrete by nature, as already said they often represent smooth curves of which they are approximations. If we give a consistent definition of its geometric invariants (e.g., tangent vector, normal vector and curvature) we can work with discrete curves analogously to the smooth case. But we will see that the discretization has the drawback that sometimes it does not preserve all properties of the smooth analogous. For a smooth parameterized curve, its tangent and normal vectors and its curvature are well and uniquely defined. But it is not obvious how to give analogue definitions for discrete curves. In fact in differential geometry these concepts are strictly linked to the parameterization and its derivatives. But discrete curves are piecewise linear curves and we lose the differentiability in a finite number of points. This fact makes necessary an approximation of these quantities. For example, if the tangent vector to a point on a segment can be simply defined as the unit direction of the segment (the normal is orthogonal to that one), for vertices it is not so easy. As mentioned above, in literature there is not a unique definition of these concepts, conversely there is a great variety of evenly valid definitions. This fact forces us to decide which definition is better to use for each particular problem that we are considering, and this is true especially for curvature.

This chapter is dedicate to a review of all basic definitions about discrete plane curves. In particular the first section is based on definitions given in [7] and [17]. The second section summarize some of the more common used discretizations of the smooth curvature, while in the third we talk about the value of the curvature along edges, introducing an alternative definition. The last section deals with the convergence of the different discrete curvature definitions.

2.1 Discrete curves, Tangent and Normal Vectors

Definition 2.1.1. *A* ***discrete curve*** *in* $\boldsymbol{R}^n$ *is a map* $P\colon I \to \boldsymbol{R}^n$ *of an interval* $I \subseteq \boldsymbol{Z}$.

Let us denote $P(i) = p_i$.

The first case of multiple definitions of the same notion is the one of regular curves.

Definition 2.1.2. *[7] A discrete curve is called* **regular** *if any three successive points are pairwise different.*

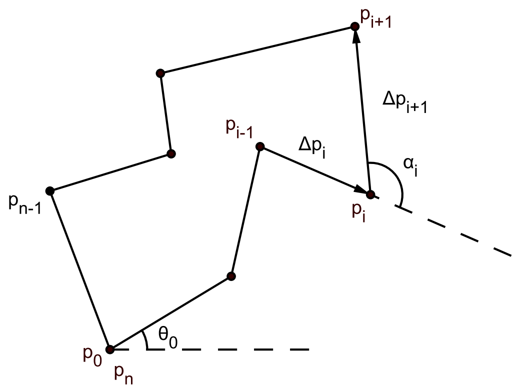


Figure 2.1: A piecewise linear curve with $n+1$ points, with $p_0 = p_n$.

Definition 2.1.3. *[17] A discrete curve is called* ***regular*** *if any three successive points are pairwise disjoint.*

Definition 2.1.4. *The* ***length*** *of a discrete curve is defined as*

$$L(P) = \sum_{k,k+1 \in I} \|p_{k+1} - p_k\|.$$

Definition 2.1.5. *[17] The discrete curve P is said to be* ***periodic*** *(or* ***closed****) if $I = \boldsymbol{Z}$ and if there is a $q \in I$ such that $P(k) = P(k+q)$ for all $k \in I$.*

Definition 2.1.6. *[7] A discrete curve is parametrized by* ***arclength*** *if*

$$\|\Delta p_k\| = 1 \quad \textit{for all} \quad k-1, k \in I$$

where $\Delta p_k = p_k - p_{k-1}$ (see Figure 2.1).

Sometimes if a curve has $\Delta p_k = c$ for all $k \in I$, with c constant different from zero, it is called *arc-length* parameterized. In [17] Hoffmann states that, unlike the smooth case, once a discrete curve is given, it is not possible to reparameterize it by arc length. This definition is quite restrictive and we prefer to consider a different definition of arc length which is based on the parameterization of the discrete curve as a piecewise linear curve. In this way it is possible to avoid the constraint of edges of same length.

Piecewise linear parameterizations Let P be a polygonal curve composed of $n+1$ vertices $p_0, \ldots, p_n$, n edges $\Delta p_i = p_i - p_{i-1}$ of lengths $l_i = \|\Delta p_i\|$, with $i = 1, \ldots, n$, and total length $L = \sum_{i=1}^{n} l_i$. Let $\tau = [t_0, \ldots, t_n]$ the partition of the interval $[0, L]$ obtained by subdividing in intervals of length equal to the edge lengths, that is $t_0 = 0$, $t_1 = \|p_1 - p_0\| = l_1$ and $t_i = \sum_{j=1}^{i} l_j$.

We can now parameterize the polygon P on $[0, L]$ as the piecewise linear curve on τ

$$\tilde{\gamma}\colon [0, L] \to \mathbf{R}^2 \tag{2.1}$$

such that $\tilde{\gamma}(t_i) = p_i$ and

$$\tilde{\gamma}|_{[t_i, t_{i+1}]}(t) = p_i + (t - t_i)\frac{p_{i+1} - p_i}{t_{i+1} - t_i}$$

This parameterization is the *arc length parameterization* of the polygon P since the distance of a point $p \in P$ from the origin p_0 is exactly equal to the corresponging parameter t.

A parameterization very similar to the arc length one and used also in the smooth setting is the *reduced arc length parameterization* [9], where the parameter domain is the interval $[0, 1]$ and the parameter $s_i \in [0, 1]$ corresponding to $t_i \in [0, L]$ is nothing else that $s_i = \frac{t_i}{L}$. For polygonal curves the parameters $s_i \in [0, 1]$ are defined as

$$s_i = \sum_{j=1}^{i} l_j \Big/ \sum_{j=1}^{n} l_j$$

and the partition of the interval $[0, 1]$ is then $\sigma = [s_0, \ldots, s_n]$ to which corresponds the piecewise linear parameterization over σ

$$\hat{\gamma}\colon [0, 1] \to \mathbf{R}^2 \tag{2.2}$$

defined by $\hat{\gamma}(s_i) = p_i$ and

$$\hat{\gamma}|_{[s_i, s_{i+1}]}(t) = p_i + (s - s_i)\frac{p_{i+1} - p_i}{s_{i+1} - s_i}$$

that is $\hat{\gamma}(s) = \tilde{\gamma}(sL)$.

Definition 2.1.7. *A discrete curve is closed if* $p_0 = p_n$*.*

This definition is not helpful in applications where instead one prefers to use the following conditions (2.3) (see [32]). Let P a piecewise linear curve with $n + 1$ vertices, we call α_i the exterior angle at the vertex p_i obtained by edges Δp_i and Δp_{i+1}. Fixed θ_0, for $i = 1, \ldots, n$ we define θ_i recursively as $\theta_i = \theta_{i-1} + \alpha_i$ (see Figure 2.1). Then P is closed if and only if

$$\begin{cases} \sum_{k=1}^{n} l_i \cos(\theta_{i-1}) = 0 \\[2ex] \sum_{k=1}^{n} l_i \sin(\theta_{i-1}) = 0 \end{cases} \tag{2.3}$$

We use these conditions in the Chapter 5.

2.1.1 Tangent and Normal Vectors

To define the tangent vector to a polygonal curve it is necessary to distinguish between tangent vector to the edge and to the vertex.

Definition 2.1.8. *[7] The edge tangent vector of a regular curve is defined as*

$$S_k = \frac{\Delta p_k}{\|\Delta p_k\|}.$$

Remark 2.1.9. *Working in $\boldsymbol{R}^2$, it is convenient to identify $\boldsymbol{R}^2 \cong \boldsymbol{C}$.*

Definition 2.1.10. *If the discrete curve P is arc-length parameterized as in Definition 2.1.6, the tangent vector to the vertex p_k is the average of the edge tangent vectors:*

$$T_k := \frac{1}{2}(S_k + S_{k+1}).$$

For arbitrary curves Hoffmann [17] suggests a better choice:

Definition 2.1.11. *The vertex tangent vector of a discrete curve $P\colon I \to \boldsymbol{R}^2 \simeq \boldsymbol{C}$ is given by*

$$T_k := 2\frac{\Delta p_k \Delta p_{k+1}}{\Delta p_k + \Delta p_{k+1}}$$

T is the harmonic mean of the edge tangent vectors not normalized.

Remark 2.1.12. *If P is arc-length parameterized as in Definition 2.1.6 then*

$$T_k = 2\frac{\Delta p_k \Delta p_{k+1}}{\Delta p_k + \Delta p_{k+1}} = 2\frac{\Delta p_k \Delta p_{k+1}(\Delta \bar{p}_k + \Delta \bar{p}_{k+1})}{\|\Delta p_k + \Delta p_{k+1}\|^2} =$$

$$= 2\frac{\Delta p_k \Delta p_{+1}}{\|\Delta p_k + \Delta p_{k+1}\|^2} = \frac{S_k + S_{k+1}}{1 + \langle S_k, S_{k+1}\rangle}.$$

which shows that for arc-length parameterized curve the vertex tangent vector has the same direction of the averaged edge tangent vectors.

But these definitions of vertex tangent vector are not the only ones. In [6] and [20] it is estimated by considering the polygonal curve as a discrete approximation of a smooth curve $\gamma(s)$. In that definition, the discrete tangent vector is a second order approximation of the tangent vector of the original smooth curve (for a proof see [20]).

Definition 2.1.13. *[20] Let γ be a smooth regular plane curve with points $p_{k-1} = \gamma(t_{k-1}), p_k = \gamma(t_k)$, $p_{k+1} = \gamma(t_{k+1})$ on the curve such that p_{k-1} and p_{k+1} are located on the opposite side of p_k. Then the following vector is a second order approximation of the unit tangent vector of γ at $p_k = \gamma(t_k)$*

$$T_k = \frac{\Delta p_k \|\Delta p_{k+1}\|^2 + \Delta p_{k+1}\|\Delta p_k\|^2}{\|\Delta p_k\|\|\Delta p_{k+1}\|(\|\Delta p_k\| + \|\Delta p_{k+1}\|)}$$

This definition coincides with the definition given in [6]:

Definition 2.1.14. *Given a smooth regular plane curve γ, its first derivative at t_k can be approximated, by using the same notations introduced in the previous Definition 2.1.13, by*

$$\gamma'(t_k) \approx T_k = \frac{p_{k+1} - p_k}{\|\Delta p_{k+1}\|} + \frac{p_k - p_{k-1}}{\|\Delta p_k\|} - \frac{p_{k+1} - p_{k-1}}{\|\Delta p_k\| + \|\Delta p_{k+1}\|}.$$

Let us observe that in this definition the vertex tangent vector is defined by a linear combination of forward and backward differences.

Definition 2.1.15. *[17] By identifing the plane $\boldsymbol{R}^2$ with the complex line $\boldsymbol{C}$, the normal vector is defined for vertices and edges as $i \in \boldsymbol{C}$ times the corresponding tangent vectors.*

2.2 Curvature

Curvature is one of the most important geometric properties of a curve (if not the most important one) and differential geometry furnishes different but equivalent ways to define and characterize it. The drawback of this fact is that in the transition to the discrete setting, choosing to discretize by following one definition or another one, one obtains different definitions of discrete curvature not equivalent between them. For this reason the literature proposes several methods to estimate the curvature of a discrete plane curve. All these definitions can be classified in three groups, depending on the definition of smooth curvature that they discretize. It is not our purpose to review all of them, but interested people can find more details in [22]. We focused on the more common used definitions. Let us observe that the definitions of curvature of the three groups are given on vertices.

2.2.1 First group: methods based on the tangent direction

These methods estimate the derivative of the tangent direction with respect to the arc length, that is $k(s) = \theta'(s)$ (as defined in (1.3)).

The most used definition of this group [6], [10] is

$$k_i = \frac{2\alpha_i}{\|\Delta p_i\| + \|\Delta p_{i+1}\|} \tag{2.4}$$

(see Figure 2.1). In [10] we find the construction from which it derives. Since also the Definition 2.1.13 of vertex tangent vector is based on this construction, we give a brief synthesis of it. In [10] Borrelli et al. consider a smooth regular curve γ, fix a point p_0 in γ and represent locally γ by the graph $(x, f(x))$ of a smooth function f, with $p_0 = (0,0)$ and $f'(0) = 0$. By expressing $(x, f(x))$ by polar coordinates they prove the following lemma:

Lemma 2.2.1. *Let $f(x)$ a smooth regular function with $k = f''(0)$ and $\nu = f'''(0)$. For a point $p = (\eta\cos\theta, \eta\sin\theta)$ on the graph of f near the origin, one has:*

$$\theta = \frac{k\eta}{2} + \frac{\nu\eta^2}{6} + o(\eta^2) \quad \text{if} \quad x \geq 0 \tag{2.5}$$

$$\theta = \frac{k\eta}{2} - \frac{\nu\eta^2}{6} + o(\eta^2) \quad \text{if} \quad x \leq 0. \tag{2.6}$$

Theorem 2.2.2. *Let p_{i-1}, p_i and p_{i+1} be three points, with η_{i-1} (η_{i+1}) the distance from p_i to p_{i-1} (p_{i+1}). Also let $\bar{\eta}_i = \frac{\eta_{i+1}+\eta_{i-1}}{2}$. The angular defect α_i (see Figure 2.2) at p_i and the curvature k satisfy:*

- *if $\eta_{i-1} = \eta_{i+1} = \eta$:*
$$\frac{\pi - \varphi_i}{\eta} = k + o(\eta) \tag{2.7}$$

- *if $\eta_{i-1} \neq \eta_{i+1}$*
$$\frac{\pi - \varphi_i}{\bar{\eta}_i} = k + o(1) \tag{2.8}$$

But expressing the Lemma 2.2.1 as in [20]:

$$\theta = \frac{k\eta}{2} + \frac{\nu\eta^2}{6} + \mathcal{O}(\eta^3) \quad \text{if} \quad x \geq 0$$
$$\theta = \frac{k\eta}{2} - \frac{\nu\eta^2}{6} + \mathcal{O}(\eta^3) \quad \text{if} \quad x \leq 0.$$

the equation (2.7) for $\eta_{i-1} = \eta_{i+1}$ becomes

$$\frac{\pi - \varphi_i}{\eta} = k + \mathcal{O}(\eta^2). \tag{2.9}$$

Proof: If $\eta_{i-1} = \eta_{i+1}$, applying equations (2.5) and (2.6) to p_{i-1} and p_{i+1} we obtain

$$\theta_{i+1} + \theta_{i-1} = \frac{k}{2}\eta + \mathcal{O}(\eta^3),$$

but it holds also

$$\theta_{i+1} + \theta_{i-1} = \pi - \varphi_i$$

and calling $\alpha_i = \pi - \varphi_i$ we have

$$2\frac{\alpha_i}{\eta} = k + \frac{\mathcal{O}(\eta^3)}{\eta} = k + \mathcal{O}(\eta^2).$$

QED

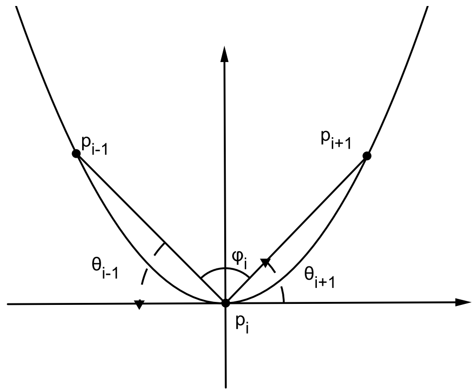


Figure 2.2: Graph of the curve γ and angular defect.

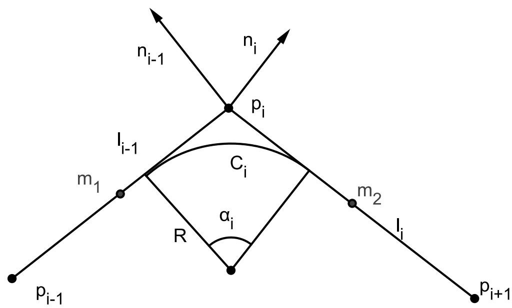


Figure 2.3: Integral interpretation of curvature

Integral interpretation In literature, quantities as Gaussian and Mean curvature of meshes can be found computed as integral mean around a vertex [13]. Following this idea also for curvature of curves we obtain the same definition (2.4).

Let p_{i-1}, p_i and p_{i+1} three sampled points on a smooth curve and let m_1 and m_2 the mid points of segments $\overline{p_{i-1}p_i}$ and $\overline{p_i p_{i+1}}$ respectively (see Figure 2.3) of length respectively l_{i-1} and l_i. Let $E = \overline{m_1 p_i} \cup \overline{p_i m_2}$ be the union of the segments $\overline{m_1 p_i}$ and $\overline{p_i m_2}$, then

$$k(p_i) = \frac{\int_E k(p)dl}{l(E)}. \tag{2.10}$$

But k is the smooth curvature function and is not defined on non smooth curves,

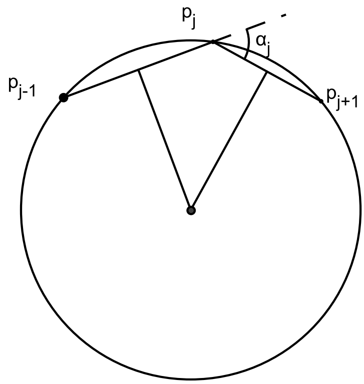


Figure 2.4: The vertex osculating circle.

so we replace the vertex by a small circular arc C_i of radius R that tangentially joins the incident edges to p_i (see Figure 2.3).

Computing the integral (2.10) and considering that the curvature is zero everywhere except in C_i where it measure $\frac{1}{R}$, we have

$$k(p_i) = \frac{\int_E k(p)dl}{l(E)} = \frac{\int_{C_i} k(p)dl}{\frac{l_{i-1}}{2} + \frac{l_i}{2}} = \frac{\alpha_i R \frac{1}{R}}{\frac{l_{i-1}}{2} + \frac{l_i}{2}} = \frac{2\alpha_i}{l_{i-1} + l_i}.$$

QED

2.2.2 Second group: methods based on the radius of curvature

Methods of this group compute the discrete curvature by an estimation of the osculating circle, that is discretizing $k(s) = \frac{1}{\rho(s)}$. Also for this group there is not a unique definition because there exist different ways to discretize the osculating circle. In this section we point out the most common used definitions [17], [7].

- The *vertex osculating circle* of a discrete curve at a point p_j is given by the unique circle through the point and its two nearest neighbours p_{j-1} and p_{j+1}. Thus, the curvature at p_j is defined as

$$k_j = \frac{1}{R_j} = \frac{2\sin(\alpha_j)}{\|\Delta p_j + \Delta p_{j-1}\|} \tag{2.11}$$

  This curvature definition matches the definition 2.1.11 of vertex tangent vector as shown by the following lemma (a proof can be found in [17]):

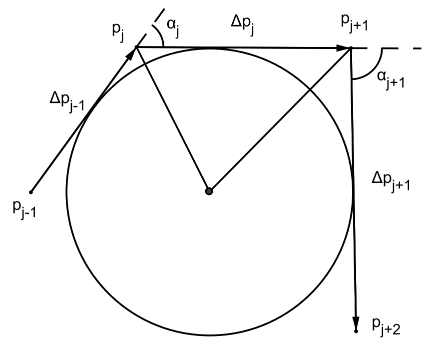


Figure 2.5: The edge osculating circle

Lemma 2.2.3. *The vertex tangent vector at a point p_j is always tangential to the vertex osculating circle at that point.*

The problem of this definition is that if the curve is parameterized by arc length (as in Definition 2.1.6) the maximum value of the curvature is 2 (since the minimum value of the radius is $\frac{1}{2}$ obtained at the limit case of p_{j-1} overlapped to p_{j+1}). This problem can be solved by using the following definition of osculating circle.

- The *edge osculating circle* is the circle touching three successive edges $\Delta p_{j-1}, \Delta p_j, \Delta p_{j+1}$ (or their extensions) with matching orientations. It has its center at the intersection of the angular bisectors of $\angle(-\Delta p_{j-1}, \Delta p_j)$ and $\angle(-\Delta p_j, \Delta p_{j+1})$ and touches the straight line through Δp_j (see Figure 2.5). Its osculating radius is

$$R = \frac{\|\Delta p_j\|}{\tan\frac{\alpha_j}{2} + \tan\frac{\alpha_{j+1}}{2}}. \tag{2.12}$$

  This definition involves four successive points and hence it can not be used for space curves.

- The *edge osculating circles for arc-length parameterized discrete curves (or for curves of equal edge length)* can be defined as a mix of both previous definitions.

  The osculating circle is the circumference which is tangent to both edges at their mid points m_1 and m_2 and has radius

$$R = 2\tan\frac{\alpha_j}{2}.$$

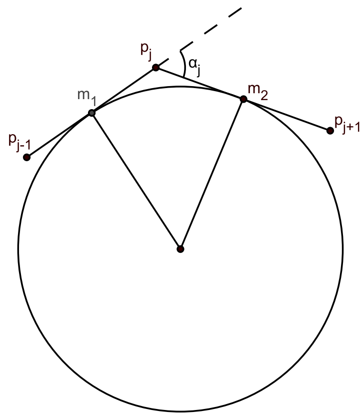


Figure 2.6: Edge osculating circle for arc length curves.

This definition is good only for curves with edges of same length since it is a sufficient condition to the existence of the circle tangent to both edges in their mid points.

2.2.3 Third group: methods based on coordinate functions derivation

In [22] are described three methods belonging to this group, but we prefer to mention only the new method introduced in that work: each coordinate is fitted as a quadratic function of the arc–length, and the curvature is estimated by derivation of that function. Let $\gamma\colon I \to \mathbf{R}^2$ a C^3 curve parameterized by arc length and consider a piecewise-linear approximation $P = \{p_0, \ldots, p_m\}$. If p_j is the origin of the curve (i.e., $\gamma(0) = p_j$) then

$$\begin{cases} x(s) = x(0) + x'(0)s + \frac{1}{2}x''(0)s^2 + g_1(s)s^3 \\ \\ y(s) = y(0) + y'(0)s + \frac{1}{2}y''(0)s^2 + g_2(s)s^3 \end{cases} .$$

with $g_i \to 0$ when $s \to 0$. Since $p_i = (x_i, y_i)$ are samples of the curve associated to the value of arc-length s_i, it is possible to write

$$\begin{cases} x_i = x_j + x'_j s_i + \frac{1}{2}x''_j s_i^2 + g_1(s_i)s_i^3 + \eta_{x,1} \\ \\ y_i = y_j + y'_j s_i + \frac{1}{2}y''_j s_i^2 + g_2(s_i)s_i^3 + \eta_{y,i} \end{cases} .$$

where $\eta_i = (\eta_{x,i}, \eta_{y,i})$ is the noise corresponding to the point p_i. The values x'_j, y'_j, x''_j, y''_j are estimated from the samples by using a weighted least squares approach.

Define Δl_k the length of the vector $p_k p_{k+1}$, the estimate for the arc length s_i is defined as $\Delta l_i^j = \sum_{k=j}^{i-1} \Delta l_k$ when $i > j$, and $\Delta l_i^j = -\sum_{k=i}^{j-1} \Delta l_k$ when $i < j$. Considering $2q+1$ points centered around p_j, and associating to $\Delta_{j-q}^j, \ldots, \Delta_{j+q}^j$ the $x_{j-q} - x_j, \ldots, x_{j+q} - x_j, y_{j-q} - y_j \ldots, y_{j+q} - y_j$, we look for the quadratic functions

$$\begin{cases} x(s) = x_j + x_j' s + \frac{1}{2} x_j'' s^2 \\ y(s) = y_j + y_j' s + \frac{1}{2} y_j'' s^2 \end{cases}$$

that better fits these data in the weighted least squares sense, that is that minimize

$$E_x(x_j', x_j'') = \sum_{i=j-q}^{j+q} w_i \left(x_i - x_j - x_j' \Delta l_i^j - \frac{1}{2} x_j'' (\Delta l_i^j)^2 \right)^2$$

and similarly for y_j' and y_j''. The w_i are positive, large for small $|\Delta l_i^j|$ and small for larges $|\Delta l_i^j|$. When $q > 1$ this problem has a well-known solution which is

$$\begin{cases} x_j' = \frac{ce-bf}{ac-b^2}, \quad x_j'' = \frac{af-be}{ac-b^2} \\ \\ y_j' = \frac{cg-bh}{ac-b^2}, \quad y_j'' = \frac{ah-bg}{ac-b^2} \end{cases}$$

where

$$\begin{cases} a = \sum_{i=j-q}^{j+q} w_i^2 (\Delta l_i^j)^2 \\ b = \frac{1}{2} \sum_{i=j-q}^{j+q} w_i^2 (\Delta l_i^j)^3 \\ c = \frac{1}{4} \sum_{i=j-q}^{j+q} w_i^2 (\Delta l_i^j)^4 \\ e = \sum_{i=j-q}^{j+q} w_i^2 \Delta l_i^j (x_i - x_j) \\ f = \frac{1}{2} \sum_{i=j-q}^{j+q} w_i^2 (\Delta l_i^j)^2 (x_i - x_j) \\ g = \sum_{i=j-q}^{j+q} w_i^2 \Delta l_i^j (y_i - y_j) \\ h = \frac{1}{2} \sum_{i=j-q}^{j+q} w_i^2 (\Delta l_i^j)^2 (y_i - y_j) \end{cases}$$

The curvature estimator is given by

$$k(p_i) = \frac{eh - fg}{ac - b^2} \tag{2.13}$$

This definition in mentioned for completeness, but in this work we do not use it since its expression is much more complicate then the other two because it depends on more than two points (the two adjacent ones).

2.3 Curvature along edges

In the previous section we defined the curvature on vertices, in this one we deal with the value of the curvature along edges. From the point of view of differential geometry the value along edges is obviously zero since they are segments. This lead to a definition of curvature $k(p)$ along a discrete curve P as an impulse function, zero everywhere except at vertices p_i, $i \in [1, \dots, n]$:

$$k(p) = \begin{cases} k_i & \text{if} \quad p = p_i \\ 0 & \text{elsewhere} \end{cases}$$

As mentioned in the Introduction, when one discretizes a notion of differential geometry, this is done with the idea to preserve the properties of smooth setting. Focusing our attention on discrete closed curves, we would like to preserve the property (1.4) of the total signed curvature.

Let us see if that property is satisfied for curvatures defined on vertices as in (2.4) or (2.11) and zero along edges.

Being these curvatures defined on a finite number of points, the property (1.4) is discretized as a sum and assumes the form

$$\sum_{i=1}^{n} k_i = 2\pi I \tag{2.14}$$

where I is the turning number.

If on vertices we take the curvature (2.11)

$$k(p_i) = \frac{2 \sin \alpha_i}{\|\Delta p_i + \Delta p_{j-1}\|}$$

the property (2.14) is not satisfied. Instead, with definition (2.4)

$$k(p_i) = \frac{2\alpha_i}{\|\Delta p_{i-1}\| + \|\Delta p_i\|}$$

the discrete total signed curvature $TCS(P)$ satisfies (2.14) if we consider the arc length parameterization as in Definition 2.1.6, that is curves with unit edge lengths; in fact in this case

$$TCS(P) = \sum_{i=1}^{n} k_i = \sum_{i=1}^{n} \alpha_i = 2\pi I.$$

But if edge lengths are equal but different from one or if they are different from each other, the property (2.14) again does not hold.

To overcome the constraint of unit edge lengths we introduce a new definition of curvature, which along edges assumes values different from zero. Let us consider a parameterized piecewise linear closed curve $\tilde{\gamma}(t)$ over the partition $\tau = [t_0, \ldots, t_n]$ of the interval $[0, L]$ as defined in (2.1). We define its curvature function as the piecewise linear function over τ

$$\tilde{k}\colon [0, L] \to \mathbf{R} \tag{2.15}$$

such that at the vertex $p_i = \tilde{\gamma}(t_i)$ it assumes the value

$$\tilde{k}(t_i) := k_i = \frac{2\alpha_i}{l_i + l_{i+1}}$$

and along the edge Δp_i, that is for $t \in [t_{i-1}, t_i]$ it is the linear interpolation of the values at the vertices p_{i-1} and p_i

$$\tilde{k}(t) = \frac{k_{i-1}(t_i - t) + k_i(t - t_{i-1})}{t_i - t_{i-1}}.$$

Since this curvature is a C^0 function on $[0, L]$ we verify that it satisfies the property (1.4). Being

$$\int_0^L k(t)dt = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} k(t)dt$$

we study the integral on the generic interval $[t_i, t_{i+1}]$.

$$\begin{aligned}
\int_{t_i}^{t_{i+1}} k(t)dt &= \int_{t_i}^{t_{i+1}} \left(\frac{k_i t_{i+1} - t_i k_{i+1}}{t_{i+1} - t_i} + \frac{k_{i+1} - k_i}{t_{i+1} - t_i} t \right) dt \\
&= \frac{k_i t_{i+1} - t_i k_{i+1}}{t_{i+1} - t_i}(t_{i+1} - t_i) + \frac{k_{i+1} - k_i}{t_{i+1} - t_i} \frac{t_{i+1}^2 - t_i^2}{2} \\
&= \frac{k_i(t_{i+1} - t_i)}{2} + \frac{k_{i+1}(t_{i+1} - t_i)}{2} \\
&= \frac{2\alpha_i}{l_i + l_{i+1}} \frac{t_{i+1} - t_i}{2} + \frac{2\alpha_{i+1}}{l_{i+1} + l_{i+2}} \frac{t_{i+1} - t_i}{2} \\
&= \frac{\alpha_i}{l_i + l_{i+1}} l_{i+1} + \frac{\alpha_{i+1}}{l_{i+1} + l_{i+2}} l_{i+1}
\end{aligned}$$

Then, by adding all terms of the sum we obtain

$$\sum_{i=0}^{n-1}\left(\frac{\alpha_i l_{i+1}}{l_i+l_{i+1}}+\frac{\alpha_{i+1} l_{i+1}}{l_{i+1}+l_{i+2}}\right)=$$

$$\sum_{i=1}^{n}\left(\frac{\alpha_i l_i}{l_i+l_{i+1}}+\frac{\alpha_i l_{i+1}}{l_i+l_{i+1}}\right)=$$

$$\sum_{i=1}^{n}\alpha_i = 2\pi I.$$

The same definition of curvature function can be used for reduced arc length parameterizations (2.2) on $[0,1]$

$$\hat{k}\colon [0,1]\to \mathbf{R} \tag{2.16}$$

with

$$\hat{k}(s_i) := k_i = \frac{2\alpha_i}{l_i+l_{i+1}}$$

$$\hat{k}(s) = \frac{k_{i-1}(s_i-s)+k_i(s-s_{i-1})}{s_i-s_{i-1}} \quad \text{for } s\in[s_{i-1},s_i].$$

A similar definition of piecewise linear curvature can be obtained by computing curvatures on vertices by (2.11) but it does not satisfy the property (1.4). Thus, when in the following we refer to the piecewise linear curvature function, we are considering (2.15) or (2.16).

This definition could be considered not correct if compared with the value of the smooth curvature of a segment, of course, but it has the big advantage to be a piecewise linear approximation of the smooth signed curvature with respect to the arc length (see next section 2.4.3) which preserves the property (1.4). Moreover, if we consider that our original purpose is to interpolate the smooth curvatures of the source and target smooth curves, this discrete curvature is a very good approximation of the smooth one also along edges, implying that the interpolation of these discrete curvatures is very close to the interpolation of the original smooth ones. To conclude, let us observe that to work with a continuous function is easier then with impulse ones.

Remark 2.3.1. *It is necessary to make an observation about the value of the curvature at the end points of a curve. A discrete curve is represented as* $P=[p_0,\dots,p_n]$ *and it can be open or closed. If it is closed we compute* $k_0=k_n$ *by the same formula, using* $p_{-1}=p_{n-1}$ *and* $p_{n+1}=p_1$*. If it is open we have*

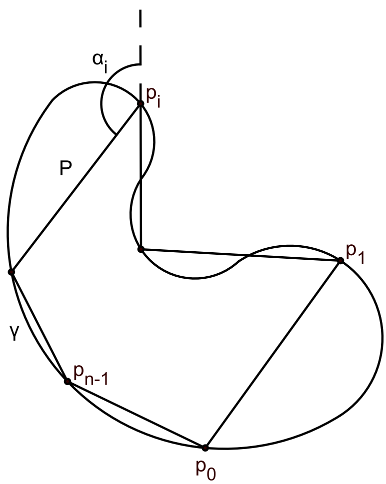


Figure 2.7: Approximation of a smooth curve γ by a polygon P.

two possibilities. The first one is do not compute the curvature values at the extremes points p_0 and p_n, while the second one arises when we need to compute the distance between the curvature of a closed curve and the curvature of an open one. To this last one then impose $k_0 = k_n$; in our particular case (see Chapter 5) the values of the curvature at the extremes of the open curve are obtained by interpolation, but in a general situation it would be necessary to assign in some way a value, maybe zero or maybe by considering as exterior angle the one obtained by extending the first and the last edges.

2.4 Convergence of the definitions of curvature

In this section we study the convergence order of the three discrete curvatures defined in (2.4), (2.11) and (2.15), but to do that, it is necessary to consider the discrete curve P not as an independent discrete object but as an approximation of a smooth curve γ.

More precisely, given a smooth curve $\gamma\colon [a,b] \to \mathbf{R}^2$ with curvature function $k\colon [a,b] \to \mathbf{R}$, we sample γ at $n+1$ points which correspond to the $n+1$ parameters points t_i uniformly distributed on I (i.e., $t_i = (b-a)i/n$). The points $p_i = \gamma(t_i)$ form the polygon $P = [p_0, \dots, p_n]$ as shown in Figure 2.7.

Remark 2.4.1. *In this thesis, whenever we consider a piecewise linear approximation of a smooth curve, it is obtained by an equally spaced sampling on the parameter domain.*

Our objective is to analyze as the error in computing the discrete curvature at the sampled points decreases while reducing the length of the segments used to approximate the smooth curve, that is increasing the sampling. In general, a sequence $\{x_i\}_{i=1}^{\infty}$, obtained by an iterative method, converges with order β to the value x if, calling $\mathrm{err}_i = |x_i - x|$, we have

$$\lim_{i\to\infty} \frac{|\mathrm{err}_{i+1}|}{|\mathrm{err}_i|^{\beta}} = c$$

where c and β are constants.

Instead if we want to compute the numerical approximation of an exact value u, we have to consider the link between the approximation value u_h and a parameter h, which can be the grid size or the time step. The numerical method is of order β if there exists a number c independent on h such that

$$|u_h - u| \leq ch^{\beta}$$

at least for sufficiently small h.

In our specific case each successive step is characterized by the doubling (minus one) of the sampling, always at equally spaced parameters. We call err^i the approximation error at the step i and h_i the maximum edge length at the same step i. Then, solving the system

$$\begin{cases} \mathrm{err}^i = cl_i^{\alpha} \\ \mathrm{err}^{i+1} = cl_{i+1}^{\alpha} \end{cases} \tag{2.17}$$

where c is a constant, we find

$$c = \exp\left\{\frac{\log \mathrm{err}^{i+1} \log l_i - \log l_{i+1} \log \mathrm{err}^i}{\log l_i - \log l_{i+1}}\right\}$$

and the order of convergence is α

$$\alpha = \frac{\log \frac{\mathrm{err}^i}{c}}{\log l_i}.$$

The three curvatures of which we want to study the convergence are of two types. The first two, (2.4) and (2.11) are defined only on vertices, thus we study for them the piecewise convergence on vertices (see subsection 2.4.1 and 2.4.2) , while for the third we consider the functions's convergence (subsection 2.4.3).

2.4.1 Convergence of the curvature based on the rotation angle

Let us consider the curvature defined in (2.4). Theorem 2.2.2 gives the formal proof of its convergence to the real value (which is quadratic at least for constant edge lengths).

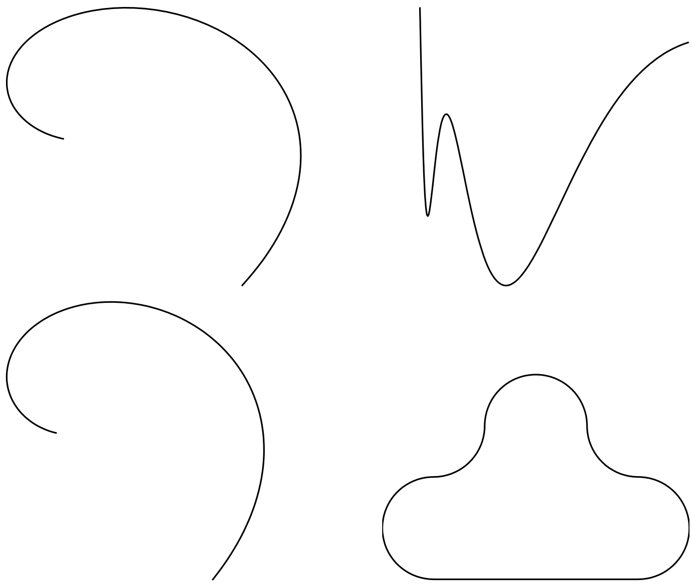


Figure 2.8: The four test functions, α_1 and α_2 on the top row and α_3 and α_4 on the bottom row.

To test the numerical pointwise convergence, for both curvature definitions: (2.4) and (2.11) in the next subsection 2.4.2, we use four different parameterized curves such that two of them are parameterized by arc length and in particular one of these is an arc splines curve. At each step we sample uniformly the parameter domain, starting from six points at the first step and doubling the sampling at each next one. Two of the first six points are the two extremes of the curve and since three of our test curves are open curves, they are not considered in the evaluation of the convergence and the real curvature is computed only in the four interior points. Then, with a loop of six iterations, such that in everyone we double the sampling, we consider the discrete curvature only at the same four original interior points. We observed that the discrete values converge in few steps to the real ones, but the exact convergence order is given by solving the system (2.17).

For each test function we report a log log plot of the error with respect to the maximum edge, this is done for each of the four interior points. The dotted line represents the trend of the quadratic convergence and it is used to compare with the convergence order of tests.

- The first test curve (top left of Figure 2.8) is parameterized by

$$\alpha_1(t) = \left(\int_0^t \frac{\cos(u) - 1}{u} du + \log(s) + 0.577216, \int_0^t \frac{\sin(u)}{u} du \right)$$

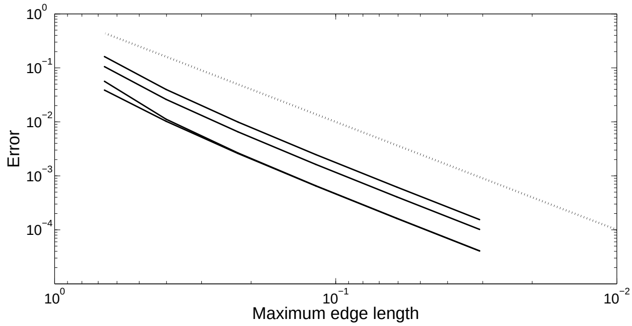


Figure 2.9: Convergence order of curvature rotation angle based of α_1

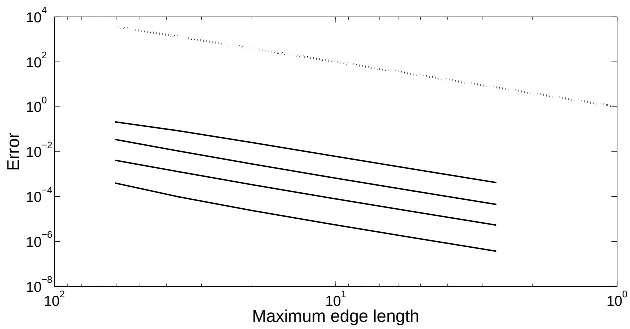


Figure 2.10: Convergence order of curvature rotation angle based of α_2

with $t \in [1, 6]$.

Its error converges very fast to zero and after few steps the order of convergence stabilizes to 2, as shown in Figure 2.9.

- Also this second curve (top right of Figure 2.8) is not parameterized by arc length, it has equations

$$\alpha_2(t) = (e^t + 2, \cos t + \sin(3t));$$

with $t \in [\frac{\pi}{4}; \frac{3\pi}{2}]$. Its convergence order is shown in Figure 2.10.

- The third curve (bottom left of Figure 2.8) is very close to the first one but it is parameterized but arc length

$$\alpha_3(s) = \left(\int_{\log 1}^{s} \cos(e^u) du, \int_{\log 1}^{s} \sin(e^u) du \right)$$

with $s \in [0, \log 6 - \log 1]$ and curvature e^s.

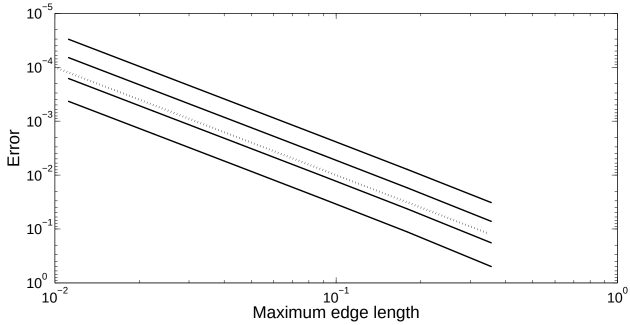


Figure 2.11: Convergence order of curvature rotation angle based of α_3

- This last case is different from the previous ones because the curve is an arc spline.

  Definition 2.4.2. *Circular arc spline, in short circular spline or arc spline, is a curve composed of some number of tangentially joined circular arcs and straight line segments [19].*

  Its curvature is not continuous, implying that it is not G^2 but only G^1, that is it has continuous unit tangent vector [19]. The test arc spline curve is shown on the bottom right of Figure 2.8 and has curvature

$$k = \begin{cases} 1 & (0, \pi) \\ -1 & (\pi, \frac{3\pi}{2}) \\ 1 & (\frac{3\pi}{2}, \frac{5\pi}{2}) \\ -1 & (\frac{5\pi}{2}, 3\pi) \\ 1 & (3\pi, 4\pi) \\ 0 & (4\pi, 4\pi + 4) \end{cases}$$

  While the behavior of the first three curves is very similar to each other and the convergence order is clearly quadratic, the curve α_4 shows a strange trend. It is due to the sharp variation of curvature between adjacent pieces of curve, particularly for low samplings of the curve.

  Nonetheless, the top graph in Figure 2.12 shows that after few steps the convergence order follows the trend of the quadratic convergence, except for one point, whose graph stops very early. This is due to the fact that the

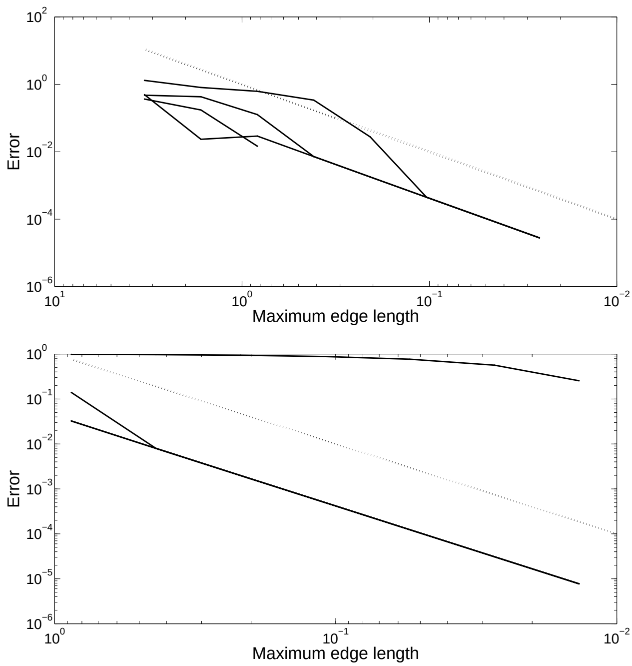


Figure 2.12: Convergence order of convergence rotation angle based for two different sampling of the arc spline curve α_4.

point lies on a segment and after three steps also the two adjacent points used for discrete curvature computation stay on the segment making an angle equal to zero. Thus the approximated curvature is equal to the real one and it loses meaning to talk about order of convergence.

But if we change the sampling, for example starting from 20 points, we observe a different behavior of one point (Figure 2.12, bottom graph). This is a drawback of the non continuity of the curvature, which involves such kind of non intuitive result when the sampled point is very close to one of the points where curvature is not continues.

Remark 2.4.3. *The problem of not continuity of the curvature has already been studied in literature because it is linked to practical problems such as the engineering one of the construction of railways and roads. In these situations the continuity of the curvature is necessary and the problem becomes to find a good way to connect two curves such that the curvature is continuous. One solution proposed in literature is to take as connection curve a clothoid, whose curvature is a linear function of the arc length. But clothoids are transcendental curves that cannot be represented as NURBS, making difficult their use in applications. In [23] it is presented a method that does not produce a clothoid, but it is much easier to apply* *than methods which do, since it produces a low-degree NURBS curve rather than a transcendental curve. Thanks to Meek at al. who in [26] give an arc spline approximation of a clothoid, that arc spline is smoothed to give a G^2 NURBS curve which approximates the clothoid as much as desired. This observation is to point out that if one need to consider curves with continuous curvature, there exist the theoretical and the practical solution to substitute the arc spline. But it is possible also to ignore this problem since very hardly in numerical computation we find exactly one of the point where curvature is not defined, usually we are on one of its sides having the behavior seen in the previous Figure 2.12.*

2.4.2 Convergence of the curvature based on the osculating circle

Following the same method of previous subsection, we compute the convergence order of curvature defined at (2.11) for the same test curves. For the first three curves we represent only graphs of their convergence order because it is clearly quadratic.

We can observe as the convergence order of the arc spline curve is worse than the convergence order of the curvature based on the rotation angle.

2.4.3 Convergence of piecewise linear curvature

The last curvature definition of which we want to study the convergence is the piecewise linear curvature function defined in (2.15) as the function $\tilde{k}\colon [0, L] \to \mathbf{R}$ on $\tau = [t_0, \dots, t_n]$ with

$$\tilde{k}(t_i) := \kappa_i = \frac{2\alpha_i}{\|p_i - p_{i-1}\| + \|p_{i+1} - p_i\|} \tag{2.18}$$

for $i = 0, \dots, n$, or likewise $\hat{k}$ defined on $[0, 1]$.

Remark 2.4.4. *To have a better understanding of the importance of this definition, let us observe that this function is the curvature of a piecewise linear curve* $\tilde{\gamma}\colon [0, L] \to \boldsymbol{R}^2$ *parameterized by arc length as defined in* (2.1)*. By considering* $\tilde{\gamma}$ *as the approximation of a smooth curve* γ *as described at the beginning of this Section (see Remark 2.4.1), then* $\tilde{\gamma}$ *is its arc length parameterized approximation. This means that the function* $\tilde{k}$ *is an approximation of the smooth signed curvature function of the curve* γ *with respect to the arc length. Moreover in [16] it is proved as the Hausdorff distance between* γ *and* $\tilde{\gamma}$ *is of order* $1/n^2$*. By considering the reduced arc length parameterization* (2.2) $\hat{\gamma}$ *on the partition* σ *of* $[0, 1]$*, the previous convergence result holds also for this one.*

Being $\tilde{k}$ (or $\hat{k}$) continuous functions, we do not compute the piecewise convergence but a metric convergence, that is we compute the error using a distance defined on continuous function spaces, in particular the L^2 norm. This means that at the increasing of the sampling, the approximation error is computed as the L^2 norm between the real value and the approximating one.

But to compute a distance between the real curvature of a curve and the curvature of its piecewise linear approximation in a meaningful way, we need to have the correct correspondence between points, and to do that we need the arc length parameterizations or equivalently the reduced arc length. While this condition is perfect for theory, it is not always available in practice when we deal with smooth curves. To tackle this problem we approximate the smooth curve γ with a piecewise linear curve $\hat{\gamma}$ obtained by a sampling of $2 \cdot 10^5$ points, parameterize this curve by reduced arc length as in (2.2) and compute its curvature function $\hat{k}$ as in equation (2.16). $\hat{k}$ is considered as the arc length approximation of the smooth curvature and then in the following of the paragraph we refer to $\hat{k}$ as the real curvature.

To see the convergence order we sample the smooth curve γ at an increasing number n of points, ranging from 25 to 50000, construct their arc length parameterizations $\hat{\gamma}_n$ and use these points to compute their $\hat{k}_n$, which are the approximated curvature functions. We compute the discrete L^2 norm between them by sampling these curvature functions at $2 \cdot 10^5$ points.

Our numerical results suggest that $d_2(k, \hat{k}) = \mathcal{O}(1/n^2)$, but proving this approximation order remains future work. Figure 2.17 shows this approximation order for a test curve (upper left) represented in polar coordinates with a very oscillating curvature.

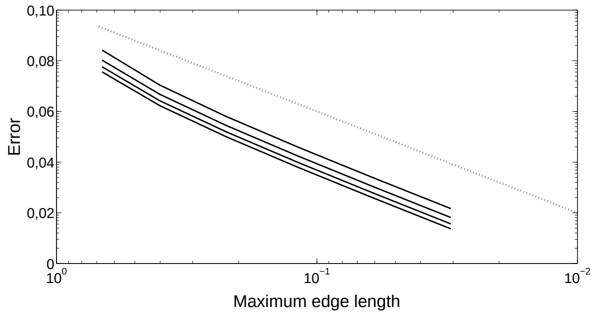


Figure 2.13: Convergence order of the curvature based on the osculating circle for the curve α_1.

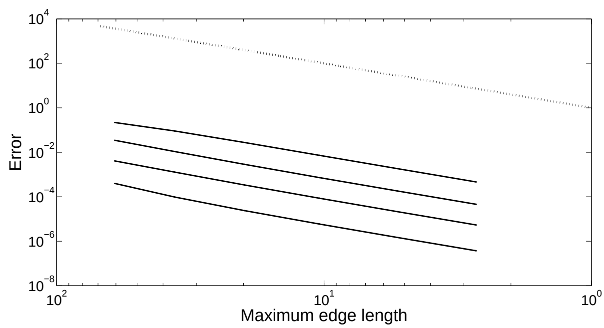


Figure 2.14: Convergence order of the curvature based on the osculating circle for the curve α_2.

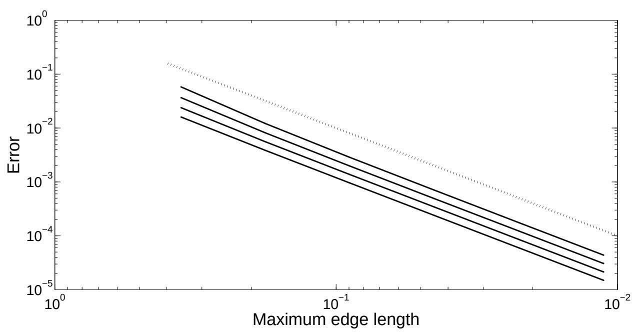


Figure 2.15: Convergence order of the curvature based on the osculating circle for the curve α_3.

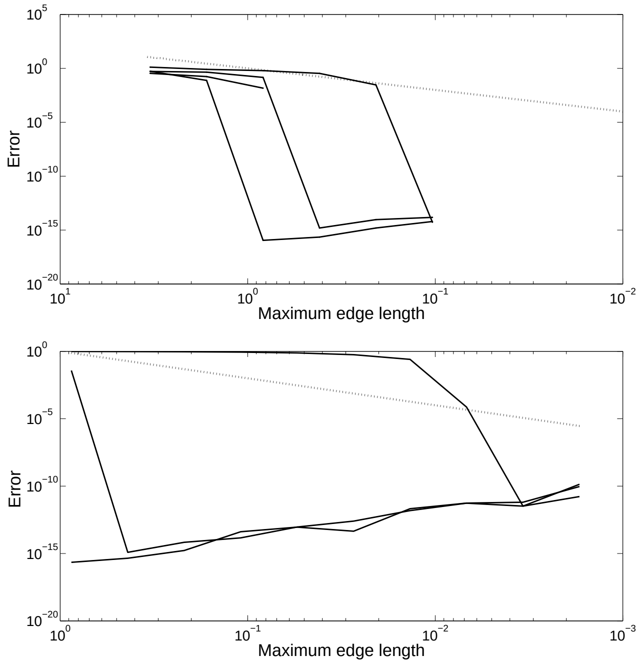


Figure 2.16: Convergence order of the curvature based on the osculating circle for the curve α_4.

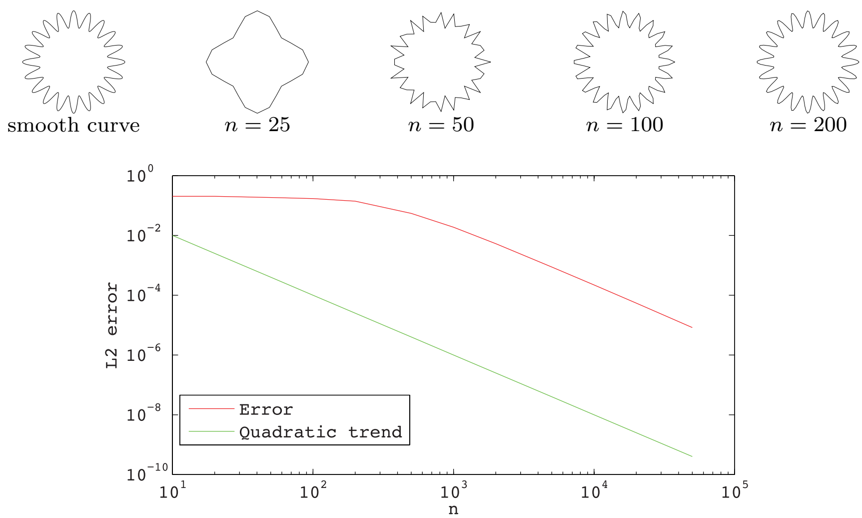


Figure 2.17: Convergence of the piecewise linear signed curvature function in the L_2 norm.

Chapter 3

Smooth curves: reconstruction and similarity

This chapter is subdivided in four sections. The first one recalls the well known Fundamental Theorem of planar curves 3.1.4, which is at the basis of our work. The other three sections study in the smooth setting which could be the best distance to use in the minimization process to replace the open curve with a closed curves as-close-as possible to it. In particular, in the second section we recall the more used distances in literature and we discuss the reasons for which we do not use them for our purposes; the third contains two new theorems which show a link between the distance of two curves and the the distance of their corresponding curvatures, under the condition to take arc length parameterized curves, and the attempts to generalize the theorems of the previous section to generic parameterizations. The last section try to find some result on similarity of curves by considering the first (and second) derivative of the parameterizations, beeing the curvature is expressed through them.

3.1 Reconstruction of the curve from curvature

The curvature of a planar curve is probably its most important characteristic since it is invariant for reparameterizations and rigid motions. Moreover, if a curvature function is expressed with respect to the arc length, it is possible to reconstruct the curve (which has that function as curvature), up to rigid motions in the plane, by solving a systems of ordinary differential equations.

Let us recall some of most important results for ODE. For more details and missing proofs see [29].

Definition 3.1.1. *Let* (X, d) *be a metric space. A function* $f\colon X \to X$ *is called* Lipschitz function *if there exists a number* $\rho > 0$ *such that, for each couple* x, y *of points of* X*, we have*

$$d(f(x), f(y)) \leq \rho d(x, y).$$

Definition 3.1.2. *Given a function* f *of two variables* s *and* t*, we say that* $\lim_{s\to s_0} f(s,t) = l(t)$ *uniformly with respect to* $t \in T \subset \boldsymbol{R}$ *if, fixed* $\epsilon > 0$*, there exists* $\delta = \delta(\epsilon)$ *such that:*

$$|f(s,t) - l(t)| < \epsilon \quad \forall t \in T \quad \textit{if} \quad 0 < |s - s_0| < \delta$$

or equivalently if

$$\sup_{t\in T} |f(s,t) - l(t)| \to 0 \quad \textit{for} \quad s \to s_0.$$

Theorem 3.1.3. *(local existence and uniqueness) Let* $\boldsymbol{f}\colon \boldsymbol{R}^{n+1} \supseteq D \to \boldsymbol{R}^n$*, with* D *open subset, and let us consider the problem*

$$\begin{cases} \boldsymbol{y}'(t) = \boldsymbol{f}(t, \boldsymbol{y}) \\ \boldsymbol{y}(\tau) = \boldsymbol{\xi}. \end{cases} \tag{3.1}$$

If:

- $\boldsymbol{f}$ *is continuous in* D
- $\boldsymbol{f}$ *is locally Lipschitz in* D*, with respect to* $\boldsymbol{y}$ *and uniformly in* t

then, for each point $(\tau, \boldsymbol{\xi}) \in D$ *exists a neighborhood* I_δ *of* τ*, in which it is defined a solution* $\boldsymbol{\varphi}$ *of the Cauchy problem* (3.1)*. This solution is unique in the sense that every other solution coincides with* $\boldsymbol{\varphi}$ *in the common interval of definition.*

Coming back to curves let us see in detail how to reconstruct a curve given its curvature. Let α be an arc length parametrized curve. Its *Frenet's frame* is given by $\{T, N\}$, where $T = \alpha'$ and $N = JT$, and the corresponding *Frenèt equations* are

$$\begin{cases} T' = kN \\ N' = -kT \end{cases}$$

which express the first derivatives of the basis $\{T, N\}$ as linear combination of its vector field. They are simply a system of ordinary differential equations and for this reason applying Theorem 3.1.3 of local existence and uniqueness and fixing some initial conditions, it is possible to reconstruct the curve with respect to the arc length. But we recall also a more interesting result. In fact the following theorem gives an explicit way to reconstruct a planar curve from its curvature.

Theorem 3.1.4. *(Fundamental Theorem of planar curves) [2]*

- Uniqueness*: Let α and $\tilde{\alpha}$ be two unit-speed regular curves in $\boldsymbol{R}^2$ defined on the same interval $I = (0, l)$. Let us assume that α and $\tilde{\alpha}$ have the same curvature. Then there is an orientation-preserving Euclidean motion F of $\boldsymbol{R}^2$ mapping α into $\tilde{\alpha}$.*

- Existence*: A unit-speed curve $\alpha\colon I \to \boldsymbol{R}^2$ whose curvature is a given piecewise-continuous function $k\colon I \to \boldsymbol{R}$ is parameterized by*

$$\begin{cases} \alpha(s) = (\int \cos\theta(t)dt + c, \int \sin\theta(t)dt + d), \\ \theta(s) = \int k(t)dt + \theta_0, \end{cases} \tag{3.2}$$

where c, d and θ_0 are the initial conditions.

Remark 3.1.5. *Let us observe that the reconstruction of the curve is unique up to rigid motions. To remove these degrees of freedom it is necessary to fix initial conditions on the start point (c and d) and on the start tangent vector (θ_0).*

Remark 3.1.6. *Let us recall that the* Frenèt equations *exist also for curves in $\boldsymbol{R}^3$. In that case the* Frenèt frame *involves a third vector field (the binormal vector) and besides the curvature it is necessary to know the torsion. It is then possible to construct the analogue* Frenèt equations *and to solve them by Theorem 3.1.3. But for curves in $\boldsymbol{R}^3$ there not exists a theorem analogue to Theorem 3.1.4 which gives an explicit reconstruction. It is a property unique of curves in $\boldsymbol{R}^2$. A proof of Theorem 3.1.4 can be found in [12].*

3.2 Similarity of curves

In the Introduction we mentioned the problem deriving from the interpolation of curvatures of closed curves, that is intermediate curvature functions whose corresponding curves are open. Each open curve must be replaced with a closed curve, of same length, such that it is as-close-as possible to the open one. To satisfy this last condition, we need a metric which computes the “closeness” between them.

A possible measure can be the distance between the two curves. In literature can be found different definitions of distances between two curves parameterized on the same interval I. In this section we mention the most commonly used and why they are not suitable for our purposes.

Remark 3.2.1. *If we are interested in similarity of two curves we would like to consider equivalent two curves which vary for a scaling. The simplest thing to do is then to scale one of them so that they have same length.*

The Hausdorff distance The Hausdorff distance between two parameterized curves $\alpha(t)$ and $\beta(t)$ is a distance between the points set $\{\alpha(t): t \in I\}$ and $\{\beta(t): t \in I\}$ defined as

$$d_H(\alpha, \beta) := \max \left\{ \sup_{p \in \alpha(t)} \inf_{q \in \beta(t)} \|p - q\|, \sup_{q \in \beta(t)} \inf_{p \in \alpha(t)} \|p - q\| \right\}. \tag{3.3}$$

Remark 3.2.2. *Observe that we defined the curves on the same domain only for uniformity with the rest of the section since to compute this distance it is not necessary. Moreover, it does not require that the curves are represented by parameterizations, since it computes the euclidean distance from each point of a curve to all points of the other curve, and to do that the parameterizations are not necessary (for example it is defined exactly in the same way for discrete curves (see Section 4.2).*

The big problem of this distance is that it does not consider the orientation of the curves [9] and the distance can be small also if the curves are evidently different, as shown in the next example.

Example 3.2.3. *Let $\alpha(t) = (t, 0)$ and $\beta(t) = (1 - t, 0)$ be two curves defined on $[0, 1]$. Then $d_H(\alpha, \beta) = 0$ also if they are different curves (having same trace but different orientations).*

Distances based on parameterizations To overcome this problem we can consider distances strongly based on parameterizations of the curves, that is by considering curves as functions from $I \subset \mathbf{R}$ to $\mathbf{R}^2$. In this way we can take typical distances of functional spaces as L^p, with $1 \leq p < \infty$, or L^∞, with respective distances d_p and d_∞. Moreover, if the curves are at least $C^2(I)$ continuous, we can also consider the distance $\tilde{d}$ which makes the set of functions $C^2(I)$ a Banach space. Thus, let $\alpha(t)$ and $\beta(t)$ be two parametric curves defined on the same interval I, define

$$d_p(\alpha, \beta) := \left(\int_I \|\alpha(t) - \beta(t)\|^p dt \right)^{1/p} \quad \text{for} \quad 1 \leq p < \infty \tag{3.4}$$

$$d_\infty(\alpha, \beta) := \sup_{t \in I} \|\alpha(t) - \beta(t)\| \tag{3.5}$$

$$\tilde{d}(\alpha, \beta) := d_\infty(\alpha, \beta) + d_\infty(\alpha', \beta') + d_\infty(\alpha'', \beta'') \tag{3.6}$$

Remark 3.2.4. *These distances have an important drawback, that is they are strictly dependent on the particular parameterization used to define the curve. While each curve identify a shape (its trace), it has associated infinite parameterizations. When we use these three distances, we compare points which* *correspond to the same parameter value, and if at the change of the parameter the two corresponding points are not going through the curves with the same speed, we can have unexpected results.*

Example 3.2.5. *Let us consider for example two curves*

$$\begin{array}{rccc} \alpha\colon & [0,1] & \to & \boldsymbol{R}^2 \\ & t & \mapsto & (t,t^2) \end{array} \qquad \begin{array}{rccc} \beta\colon & [0,1] & \to & \boldsymbol{R}^2 \\ & u & \mapsto & (\sqrt{u},u) \end{array}$$

with the same trace (an arc of parabola), parameterized on the same domain $[0,1]$ *but not with respect to the arc length. Then their distance computed by* (3.4)*,* (3.5) *or* (3.6) *is not equal to zero.*

This fact shows that the previous distances can not be defined on the quotient space whose classes are composed by curves which change for a reparameterization. That is, calling X the set of planar curves, the quotient space $\tilde{X} = X/\sim$ is defined by the following equivalence relation: if $\alpha, \beta \in X$, with $\alpha\colon I \to \mathbf{R}^2$ and $\beta\colon J \to \mathbf{R}^2$, then $\alpha \sim \beta$ if and only if exists $\varphi\colon I \to J$ such that $\alpha = \beta \circ \varphi$. These distances are acceptable only for the canonical representation of curves, that is the arc length parameterization, which exists for every curve. The strong dependency of the arc length on the length of the curve could be a problem in defining the distances between curves on a variable domain [9]. It can be overcome by using the **reduced arc length parameterization**

$$\begin{array}{rccc} \tilde{\alpha}\colon & [0,1] & \to & \mathbf{R}^2 \\ & t & \mapsto & \alpha(tL(\alpha)) \end{array} \tag{3.7}$$

and by defining the distances (3.4), (3.5) and (3.6) on the fixed interval $[0,1]$.

Frechèt distance In the previous paragraph we introduced some good definition of distance between two planar curves under the constraint that they are represented with respect to their arc length parameterizations. But if we desire a distance independent on the parameterization, that is a distance defined on the quotient space $\tilde{X}$, we have to look elsewhere. Fortunately it can be easily found in literature [9]), [15] and it is the *Frechèt metric.*

Definition 3.2.6. *Let* $\alpha\colon I\boldsymbol{R}^2$ *and* $\beta\colon J \to \boldsymbol{R}^2$ *be two curves, then the* ***Frechèt metric*** *is defined as*

$$d_F(\alpha,\beta) = \inf_{\varphi\colon I\to J} \max_{t\in I} d(\alpha(t),\beta(\varphi(t))) \tag{3.8}$$

where d *is the euclidean norm.*

All distances mentioned in the previous paragraphs are computed by making explicit use of parameterizations, and parameterizations contain implicitly the position of the curves in the plane. This implies distances strongly dependent on the mutual position of the curves. We can easily understand how important is this problem if we compute the distance between two curves which have the same shape but vary for a rigid motion which preserves the orientation. We would like to can consider equal these curves, or formally, we require that the metric is independent on mutual position of the curves. In literature can be found a perfect distance for this purpose and it is the Gromov-Hausdorff distance, which extends the idea of the Hausdorff distance.

Gromov Hausdorff distance

Definition 3.2.7. *[27] Given two compact metric spaces X and Y, we define the* ***Gromov-Hausdorff distance***

$$d_{GH}(X,Y) = \inf_{f,g} d_H(f(X), g(Y))$$

where d_H is the Hausdorff distance, $f(X)$ (resp. $g(Y)$) denotes an isometric embedding of X into some metric space Z (resp. an isometric embedding of Y into Z) and the infimum is taken over all possible such embeddings.

This definition makes intractable this distance, but it can be reformulated in terms of distances in X and Y, that is:

$$d_{GH}(Y,X) = \frac{1}{2} \inf_{\substack{\varphi\colon X \to Y \\ \psi\colon Y \to X}} \max\{\mathrm{dis}\varphi, \mathrm{dis}\psi, \mathrm{dis}(\varphi,\psi)\}$$

where

$$\mathrm{dis}\varphi = \sup\nolimits_{x,x' \in X} |d_X(x,x') - d_Y(\varphi(x),\varphi(x'))|$$

$$\mathrm{dis}\psi = \sup\nolimits_{y,y' \in Y} |d_Y(y,y') - d_X(\psi(y),\psi(y'))|$$

are the distortion of the embeddings φ and ψ, respectively, and

$$\mathrm{dis}(\varphi,\psi) = \sup_{x\in X, y \in Y} |d_X(x,\psi(y)) - d_Y(y,\varphi(x))|$$

This problem involves two embeddings and three distortion terms, and it is usually solved by discretizing X and Y.

In theory it solves our problem, but in practice it is still too difficult to treat since we have to solve a shape-matching problem without to know the second shape which is the result of the minimization.

3.3 Link between curves and curvature

Seen the big difficulty to solve the problem with the Gromov-Hausdorff distance, an alternative idea could be to consider a distance which depends only on the intrinsic properties of the curves.

We have already seen in Proposition 1.0.16 that the curvature is an intrinsic property of a curve and in Theorem 3.1.4 that it determines the curve if expressed w.r.t. the arc length. This motivates us to look at the curvature as the possible solution. The natural question is now if there exists some kind of link between "closeness" of curvatures and "closeness" of corresponding curves.

A positive answer arises thanks to the proof of the following Theorems 3.3.1 and 3.3.2, under the constraint to consider curves parameterized by arc length and with a particular mutual position.

3.3.1 New theorems on link between closeness of curves and curvatures

Let us consider the space of curves in $\mathbf{R}^2$

$$\mathcal{F} = \{\alpha\colon I \subset \mathbf{R} \to \mathbf{R}^2 \colon \alpha \in C^2(I), \|\alpha'\| = 1\}$$

with the metric $\tilde{d}$ defined in (3.6)

$$\tilde{d}(\alpha,\tilde{\alpha}) = d_\infty(\alpha,\tilde{\alpha}) + d_\infty(\alpha',\tilde{\alpha}') + d_\infty(\alpha'',\tilde{\alpha}'') \tag{3.9}$$

and the space of functions $\mathcal{K} = \{k\colon I \to \mathbf{R} \ |k \in C^0(I), \text{bounded}\}$ with the distance d_∞

$$d_\infty(k,\tilde{k}) = \sup_{s\in I} |k(s) - \tilde{k}(s)|. \tag{3.10}$$

or

$$d_2(k,\tilde{k}) = \left(\int_I |k(s) - \tilde{k}(s)| ds\right)^{1/2} \tag{3.11}$$

Theorem 3.3.1. *If two curves $\alpha,\tilde{\alpha} \in \mathcal{F}$ are $\tilde{d}$-close and their curvatures k_α and $k_{\tilde{\alpha}}$ are bounded, then k_α and $k_{\tilde{\alpha}}$ are d_∞-close. Viceversa, if two functions $k,\tilde{k} \in \mathcal{K}$ are d_∞-close, then there exist two curves α and $\tilde{\alpha}$, with $k_\alpha = k$ and $k_{\tilde{\alpha}} = \tilde{k}$, which are $\tilde{d}$-close. The same result holds true also by taking distance d_2* (3.11) *for curvatures.*

Proof. Let $\alpha,\tilde{\alpha} \in \mathcal{F}$ such that $\tilde{d}(\alpha,\tilde{\alpha}) = \epsilon$. We want to prove that for all $\epsilon \geq 0$ as small as we want, there exists $\delta_\epsilon = \epsilon(1 + \sup_{s\in I} |k_{\tilde{\alpha}}(s)|)$ such that

$$\tilde{d}(\alpha,\tilde{\alpha}) = \epsilon \quad \Rightarrow \quad d_\infty(k_\alpha, k_{\tilde{\alpha}}) \leq \delta_\epsilon.$$

Since $\tilde{d}(\alpha,\tilde{\alpha}) = \epsilon$, then

$$\begin{aligned} &\|\alpha(s) - \tilde{\alpha}(s)\| \leq \epsilon \\ &\|\alpha'(s) - \tilde{\alpha}'(s)\| \leq \epsilon \\ &\|\alpha''(s) - \tilde{\alpha}''(s)\| \leq \epsilon. \end{aligned}$$

for all $s \in I$. By setting $v = \alpha' - \tilde{\alpha}'$, $Jv = J\alpha' - J\tilde{\alpha}'$ and $w = \alpha'' - \tilde{\alpha}''$, we obtain

$$k_\alpha = \alpha'' \cdot J\alpha' = (w + \tilde{\alpha}'') \cdot (Jv + J\tilde{\alpha}') = w \cdot Jv + k_{\tilde{\alpha}} + w \cdot J\tilde{\alpha}' + \tilde{\alpha}'' \cdot Jv$$

which implies that

$$\sup_{s\in I}|k_\alpha(s)-k_{\tilde\alpha}(s)| = \sup_{s\in I}|w(s)\cdot Jv(s)+w(s)\cdot J\tilde\alpha'(s)+\tilde\alpha''(s)\cdot Jv(s)| \le$$
$$\sup_{s\in I}|w(s)\cdot Jv(s)+w(s)\cdot J\tilde\alpha'(s)|+\sup_{s\in I}|\tilde\alpha''(s)\cdot Jv(s)| \le$$
$$\sup_{s\in I}\|w(s)\|\|J\alpha'(s)\|+\sup_{s\in I}\|\tilde\alpha''(s)\|\|Jv(s)\| \le$$
$$\epsilon+\epsilon\sup_{s\in I}\|\tilde\alpha''(s)\| = \epsilon+\epsilon\sup_{s\in I}|k_{\tilde\alpha}(s)| =$$
$$\epsilon(1+\sup_{s\in I}|k_{\tilde\alpha}(s)|)=\delta_\epsilon.$$

Let us show now the other implication:
Let us suppose that $\sup_{s\in I}|k(s)-\tilde k(s)|=\delta$ such that $\epsilon=\delta|I|$ and $0\le\epsilon\le\pi$, where $|I|$ is the length of the interval I. Let us take two curves α and $\tilde\alpha$ such that $k_\alpha=k$, $k_{\tilde\alpha}=\tilde k$, which have the same starting point ($\alpha(0)=\tilde\alpha(0)$) and whose tangent vectors start with the same direction ($\theta(0)=\tilde\theta(0)$). Then

$$\sup_{s\in I}|\theta(s)-\tilde\theta(s)| = \sup_{s\in I}\left|\int_0^s k(u)du-\int_0^s \tilde k(u)du\right| \le$$
$$\sup_{s\in I}\int_0^s|k(u)-\tilde k(u)|du \le \sup_{s\in I}\int_0^s\delta du =$$
$$\sup_{s\in I}s\delta=\delta|I|=\epsilon.$$

This implies that

$$\sup_{s\in I}\|\alpha'(s)-\tilde\alpha'(s)\| = \sup_{s\in I}\|(\cos\theta(s)-\cos\tilde\theta(s),\sin\theta(s)-\sin\tilde\theta(s))\| =$$

using Prostaferesi's formulas

$$2\sup_{s\in I}\left\|(-\sin(\frac{\theta(s)+\tilde\theta(s)}{2})\sin(\frac{\theta(s)-\tilde\theta(s)}{2}),\sin(\frac{\theta(s)-\tilde\theta(s)}{2})\cos(\frac{\theta(s)+\tilde\theta(s)}{2}))\right\| \le$$
$$2\sup_{s\in I}\sqrt{\sin^2(\frac{\theta(s)-\tilde\theta(s)}{2})\left(\sin^2(\frac{\theta(s)+\tilde\theta(s)}{2})+\cos^2(\frac{\theta(s)+\tilde\theta(s)}{2})\right)} =$$
$$2\sup_{s\in I}\left|\sin\frac{\theta(s)-\tilde\theta(s)}{2}\right| \le \sup_{s\in I}\left|2\frac{\theta(s)-\tilde\theta(s)}{2}\right| \le \epsilon.$$

It follows that

$$\sup_{s\in I}\|\alpha(s)-\tilde\alpha(s)\| = \sup_{s\in I}\|\alpha(s)-\alpha(0)+\tilde\alpha(0)-\tilde\alpha(s)\| =$$
$$\sup_{s\in I}\left\|\int_0^s\alpha'(s)ds-\int_0^s\tilde\alpha'(s)ds\right\| \le \sup_{s\in I}\int_0^s\|\alpha'(s)-\tilde\alpha'(s)\|ds \le \epsilon|I|.$$

To finish the proof it remains to show that $\sup_{s\in I}\|\alpha''(s)-\tilde{\alpha}''(s)\|$ is small.

$$\sup_{s\in I}\|\alpha''(s)-\tilde{\alpha}''(s)\|=\sup_{s\in I}\|k_\alpha J\alpha'-k_{\tilde{\alpha}}J\tilde{\alpha}'\|=$$
$$\sup_{s\in I}\|k_\alpha(Jv+J\tilde{\alpha}')-k_{\tilde{\alpha}}J\tilde{\alpha}'\|=\sup_{s\in I}\|J\tilde{\alpha}'(k_\alpha-k_{\tilde{\alpha}})+k_\alpha Jv\|\leq$$
$$\sup_{s\in I}(\|J\tilde{\alpha}'(k_\alpha-k_{\tilde{\alpha}})\|+\|k_\alpha Jv\|)\leq$$
$$\sup_{s\in I}|k_\alpha-k_{\tilde{\alpha}}|\|J\tilde{\alpha}'\|+\sup_{s\in I}|k_\alpha|\|Jv\|\leq\delta+\epsilon\sup_{s\in I}|k_\alpha|.$$

To prove the last part of the Theorem, let $\epsilon\geq 0$ such that $\tilde{d}(\alpha,\tilde{\alpha})\leq\epsilon$, then

$$(d_2(k_\alpha,k_{\tilde{\alpha}}))^2=\int_I|k_\alpha-k_{\tilde{\alpha}}|^2ds=\int_I|w\cdot Jv+w\cdot J\tilde{\alpha}+\tilde{\alpha}''\cdot Jv|^2ds\leq$$
$$\int_I|(w\cdot Jv+w\cdot J\tilde{\alpha}')+\tilde{\alpha}''\cdot Jv|^2ds\leq\int_I|w\cdot J\alpha'+\tilde{\alpha}''\cdot Jv|^2ds\leq$$
$$\int_I(|w\cdot J\alpha'|^2+|\tilde{\alpha}''\cdot Jv|^2+2|\tilde{\alpha}''\cdot Jv||w\cdot J\alpha'|)ds\leq$$
$$\int_I(\epsilon^2+\epsilon^2\|\tilde{\alpha}''\|^2+2\epsilon^2\|\tilde{\alpha}''\|)ds=\epsilon^2\int_I(1+3\|\tilde{\alpha}''\|^2)ds\leq$$
$$\epsilon^2(1+3\sup_{s\in I}|k_{\tilde{\alpha}}(s)|^2)I.$$

Viceversa, to prove that curvatures d_2-close implies curves $\tilde{d}$-close, it is sufficient to prove that

$$\left(\int_I|k_\alpha(s)-k_{\tilde{\alpha}}(s)|^2ds\right)^{1/2}=\epsilon$$

implies

$$\sup_{s\in I}|\theta(s)-\tilde{\theta}(s)|\leq\delta_\epsilon=\epsilon\sqrt{|I|}.$$

since the last part of the proof follows as before. From Holder's inequality [1]

[1] Given two functions $f\in L^p$ and $g\in L^q$ the Holder's inequality states that

$$\|fg\|_1\leq\|f\|_p\|g\|_q$$

or equivalently

$$\int_S\|fg\|d\mu\leq\left(\int_S\|f(x)\|^pd\mu\right)^{1/p}\left(\int_S\|f(x)\|^qd\mu\right)^{1/q}.$$

with $p = q = 2$, $f(s) = k_\alpha(s) - k_{\tilde{\alpha}}(s)$ and $g(s) = 1$, we have

$$\sup_{s\in I} |\theta(s) - \tilde{\theta}(s)| = \sup_{s\in I} \left| \int_0^s (k(t) - \tilde{k}(t))dt \right| \leq$$
$$\sup_{s\in I} \int_0^s |k(t) - \tilde{k}(t)|dt \leq \sup_{s\in I} \left(\sqrt{\int_0^s |k(t) - \tilde{k}(t)|^2 dt} \sqrt{\int_0^s dt} \right)$$
$$\sup_{s\in I} \epsilon\sqrt{s} = \epsilon\sqrt{|I|}.$$

QED

Another result is stated in the following theorem.

Theorem 3.3.2. *If two curvature functions $k, \tilde{k} \in \mathcal{K}$ are d_∞-close (resp. d_2-close), then there exist two curves α and $\tilde{\alpha}$ with $\alpha(0) = \tilde{\alpha}(0)$, $\alpha'(0) = \tilde{\alpha}'(0)$ and $k_\alpha = k$ and $k_{\tilde{\alpha}} = \tilde{k}$, which are d_∞-close (resp. d_2-close).*

Proof. From the proof of Theorem 3.3.1 we deduce the statement for d_∞.

For d_2, let us suppose that

$$\left(\int_I |k_\alpha(s) - k_{\tilde{\alpha}}(s)|^2 ds \right)^{1/2} = \epsilon$$

then from Holder's inequality

$$\left(\int_I |\theta(s) - \tilde{\theta}(s)|^2 ds \right)^{1/2} \leq \left(\int_I (\epsilon\sqrt{s})^2 ds \right)^{1/2} \leq \frac{\epsilon|I|}{\sqrt{2}}.$$

This implies that

$$\int_I \|\alpha'(s) - \tilde{\alpha}'(s)\|^2 ds = \int_I \|(\cos\theta(s) - \cos\tilde{\theta}(s), \sin\theta(s) - \sin\tilde{\theta}(s))\|^2 ds$$
$$= \int_I 4\sin^2\left(\frac{\theta(s) - \tilde{\theta}(s)}{2} \right) ds$$
$$\leq \int_I 4 \left| \frac{\theta(s) - \tilde{\theta}(s)}{2} \right|^2 ds \leq \frac{(\epsilon|I|)^2}{2}.$$

Let us prove now that

$$\int_I \|\alpha(s) - \tilde{\alpha}(s)\|^2 ds \leq \frac{\epsilon|I|^2}{\sqrt{2}}$$

In fact

$$\int_I \|\alpha(s) - \tilde{\alpha}(s)\|^2 ds =$$
$$\int_I \left\| \left(\int_I (\cos\theta(s) - \cos\tilde{\theta}(s)) ds, \int_I (\sin\theta(s) - \sin\tilde{\theta}(s)) ds \right) \right\|^2 ds$$

and from Prostaferesis formulas and calling

$$\frac{\theta(s) - \tilde{\theta}(s)}{2} = \rho_1 \quad \text{and} \quad \frac{\theta(s) + \tilde{\theta}(s)}{2} = \rho_2$$

it is equal to

$$= \int_I \left\| \left(\int_I 2\sin(\rho_1)\sin(\rho_1) ds, \int_I 2\sin(\rho_1)\cos(\rho_2) ds \right) \right\|^2 ds$$

$$= \int_I \left[\left(\int_I 2\sin(\rho_1)\sin(\rho_2) ds \right)^2 + \left(\int_I 2\cos(\rho_2)\sin(\rho_1) ds \right)^2 \right] ds$$

$$= 4\int_I \left[\left| \int_I \sin(\rho_2)\sin(\rho_1) ds \right|^2 + \left| \int_I \sin(\rho_1)\cos(\rho_2) ds \right|^2 \right] ds$$

$$\leq 4\int_I \left[\left(\int_I |\sin(\rho_2)\sin(\rho_1)|\, ds \right)^2 + \left(\int_I |\sin(\rho_1)\cos(\rho_2)|\, ds \right)^2 \right] ds$$

by Holder's inequality

$$\leq 4\int_I \left[\int_I |\sin(\rho_2)|^2\, ds \int_I |\sin(\rho_1)|^2\, ds + \int_I |\cos(\rho_2)|^2\, ds \int_I |\sin(\rho_1)|^2\, ds \right] ds$$
$$= 4|I| \int_I \left[\int_I |\sin(\rho_1)|^2\, ds \right] ds \leq 4|I| \int_I \left[\int_I |\rho_1|^2\, ds \right] ds$$
$$= 4|I| \int_I \left[\int_I \left| \left(\frac{\theta(s) - \tilde{\theta}(s)}{2} \right) \right|^2 ds \right] ds \leq \frac{(\epsilon|I|)^2}{2}|I|^2.$$

QED

Remark 3.3.3. *Starting from close curvatures, by integrating and fixing appropriate initial conditions, we can show that also the corresponding curves are close. Instead, to prove that two close curves have close curvatures requires a distance which considers not only the parameterizations, but also the first two derivatives, from which derives the curvature.*

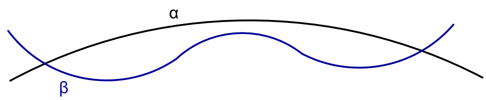


Figure 3.1: Two close curves but with completely different curvatures.

In Figure 3.1 are represented two curves α and β of same length which have small L^∞ and L^2 distances between them but curvatures very different from each other, even with opposite signs. The origin of this fact is the impossibility to prove that $\sup_{s\in I}\|\alpha(s)-\beta(s)\|<\epsilon$ implies $\sup_{s\in I}\|\alpha'(s)-\beta'(s)\|<\epsilon$, as shown by next example.

Example 3.3.4. *For any $\epsilon > 0$ let $f\colon[-1,1]\to\boldsymbol{R}$ be a monotonic function with*

$$f(-1)=-\epsilon,\quad f(1)=\epsilon,\quad f(0)=0,\quad f'(0)=1$$

and define the two curves

$$\begin{aligned} p(t) =&(t,f(t))\\ q(t) =&(t,-f(t)). \end{aligned}$$

Their distance computed by metric d_∞ (3.5) is 2ϵ, and that distance is the same after reparameterizing both curves with respect to the arc length, obtaining curves $\tilde{p}(s)$ and $\tilde{q}(s)$. After this reparametrization, there exists some $s*$, corresponding to $t=0$, such that $\tilde{p}(s*)=\tilde{q}(s*)=(0,0)$ and $\tilde{p}'(s*)=(1,1)/\sqrt(2)$ and $\tilde{q}'(s*)=(1,-1)/\sqrt(2)$. This implies

$$d_\infty(\tilde{p}',\tilde{q}')=\sup_{s\in I}\|\tilde{p}'-\tilde{q}'\|\geq\|\tilde{p}'(s*)-\tilde{q}'(s*)\|=\sqrt{2},$$

so that the distance of the derivatives of these curves is constant for arbitrarily small $\epsilon>0$.

3.3.2 Problems with reparameterizations

In the previous section (Remark 3.2.4 and Example 3.2.5) we mentioned the risk to work with general parameterizations rather then with arc length ones because of the possibility of unexpected correspondence between points. In this subsection we try to generalize 3.3.1 and 3.3.2 to general parameterizations concluding that it is not an obvious step.

Let us consider the following basic question: given two functions defined on the same domain which represent the curvatures of two curves, is it possible to understand if they are representing the same curve with respect to different parameterizations? A positive answer to this question motivate us to ask also if close curvatures imply close corresponding curves and possibly viceversa, that is if theorems 3.3.2 and 3.3.1 can be generalized to not necessary arc length parameterizations.

The existence of the Frechèt metric (3.8) makes us hopeful that a positive answer exists for the first question, which formally can be formulated as the problem to find a metric such that the distance between curvatures is zero if and only if they are the curvatures of two curves which vary for a reparameterization. If the two curvature functions are given with respect to the arc length, we can immediately see if they are the same curve or not because the arc length parametrization is intrinsic for every curve (Theorem 1.0.17). But if they are given with respect to two generic parameterizations, it is not trivial to compare them and we need an appropriate definition of distance. Let us show that the Frechèt metric (3.8) does not satisfies our expectations.

Proposition 3.3.5. *Let $k_1(t)$ and $k_2(u)$ two curvatures defined on the intervals I and J respectively. The fact that their Frechèt distance is equal to zero*

$$\inf_{\varphi\colon J\to I}\max_{u\in J}|k_1(\varphi(u)) - k_2(u)| = 0$$

does not imply that the curves $\alpha(t)$ with curvature $k_\alpha(t) = k_1(t)$ and $\beta(u)$ with curvature $k_\beta(u) = k_2(u)$ satisfy

$$\alpha(\varphi(u)) = \beta(u).$$

Proof. We are looking for a function $\varphi\colon J \to I$ such that $k_1(\varphi(u)) = k_2(u)$ and by restricting to a neighborhood of t such that the function $k_1(t)$ is invertible, we always can find

$$\varphi(u) = k_1^{-1}k_2(u).$$

By applying this reparameterization to the curve $\alpha(t)$ and computing the curvature, by Theorem 1.0.16 it is exactly $k_2(u)$, also if the original curve from which we computed the curvature $k_2(u)$ can be different.

QED

The previous proposition tell us that the Frechèt distance applied to generic curvatures is not useful to understand if they are curvatures of the same curve with respect to different parameterizations because we can have different curves with the same curvature function. Moreover, given curvatures, it is not possible to apply the Frechèt distance to the original corresponding curves because it is not possible to reconstruct the curve from curvature if it is not expressed with respect to the arc length. In fact, if we use the explicit reconstruction of Theorem 3.1.4 we reconstruct a different curve from the original one.

Let $k_\alpha(t)$ be the curvature of a generic parameterized curve $\alpha(t)$ and $\beta(s) = \alpha(\varphi(s))$ is its arc length reparametrization with $t = \varphi(s)$, then $k_\alpha(t) =$

$k_\alpha(\varphi(s)) = k_\beta(s)$. But using Theorem 3.1.4 to reconstruct explicitly the curve from $k_\alpha(t)$, we obtain

$$\theta_\alpha(t) = \int k_\alpha(t)dt + \theta_0 = \int k_\alpha(\varphi(s))\varphi'(s)ds + \theta_0$$

$$= \int k_\beta(s)\varphi'(s)ds + \theta_0 \neq \theta_\beta(s)$$

and this implies a different reconstructed curve.

Until we are not able to answer to the first question, we can not take into account the second one. Both stay open problems.

Remark 3.3.6. *Let us observe that being necessary to have arc length parameterizations, to construct examples we used arc spline curves (Definition 2.4.2). In this case, when we compute distances between curves or curvatures, it is necessary to remove from the domain those parameters points in which the function is not defined. This is not a problem because they are a finite number of points. The drawback when working with curves only piecewise C^2 is that the distance $\tilde{d}$ does not make it a Banach space.*

3.4 Curves represented by their first and second derivative

In the previous section we stumble in the research of a metric to deduce if two curvature functions are the curvatures of the same curve w.r.t. different parameterizations. But curvatures are expressed by first and second derivatives, so in this section we try to understand if they menage where curvatures fail.

Let us consider then two couples of functions $\{f'(t), f''(t)\}$ and $\{g'(u), g''(u)\}$, or simply $\{f'(t)\}$ and $\{g'(u)\}$ (being the second derivative trivially obtainable from them) defined on the interval $[0, 1]$, can we deduce if $f(t)$ and $g(u)$ represent the same curve up to a reparametrization?

Remark 3.4.1. *It is important to observe that working with derivatives of parameterizations we are forced to consider the mutual position of the curves. So, if we want to be in the same situation considered in Theorems 3.3.1 and 3.3.2, we can rotate one of the first derivatives to have $f'(0)/\|f'(0)\| = g'(0)/\|g'(0)\|$ and when we integrate to find f and g we fix the same starting point.*

The easiest case is when $\|f'\| = c_1$ and $\|g'\| = c_2$, with c_1 and c_2 positive constants, since it means that the curves are reduced arc length parameterized. If $c_1 = c_2$ they represent the same curve up to a translation and eventually up to a change of orientation which can be determined by f' and g'. Instead if $c_1 \neq c_2$ then they represent different curves. But if we are interested in the shape of the curves up to scalings, we can scale one of them to have the same length and to repeat the previous reasoning.

Instead, if $\|f'\|$ and $\|g'\|$ are not constant, we can compute the arc length parametrization for both curves and repeat the previous considerations. If the obtained curves are different, using one of the distances based on parameterization defined in Section 3.2 we can see if they are close between them or not. The problem of these solutions is that also if in theory for every parameterized curve there exists its arc length parametrization, in practice often it is difficult to find it analytically and it is necessary to use approximations.

To overcome this problem, we can look for a distance between functions f' and g' such that it is zero if they are reparametrizations of the same curve, without using arc length parameterization. A good candidate could be the Frechèt metric (3.8) as it considers explicitly all possible reparametrizations.

Suppose then $d_F(f,g) = 0$, that is there exists a change of parameter φ such that $f(t) = g(\varphi(t))$. Is it possible to show that $d_F(f,g) = 0$ implies $d_F(f',g') = 0$, obviously with a different change of parameter function? The answer is negative as shown by this example.

Let $f\colon [0,1] \to \mathbf{R}^2$ and $g\colon [0,1] \to \mathbf{R}^2$ given by $f(t) = (t,t^2)$ and $g(u) = (\sqrt{u}, u)$ two parameterizations of the same parabola.

Since there exists $\varphi(t)\colon [0,1] \to [0,1]$ such that

$$d_F(f,g) = \inf_{a\colon [0,1]\to[0,1]} \max_{t\in[0,1]} \sqrt{\left(t - \sqrt{\varphi(t)}\right)^2 + (t^2 - \varphi(t))^2} = 0$$

with $u = \varphi(t) = t^2$. But the Frechèt distance between $f'(t) = (1, 2t)$ and $g'(u) = (\frac{1}{2\sqrt{u}}, 1)$ is

$$d_F(f',g') = \inf_{a\colon [0,1]\to[0,1]} \max_{t\in[0,1]} \sqrt{\left(1 - \frac{1}{2\sqrt{\varphi(t)}}\right)^2 + (2t-1)^2}$$

and $\left(1 - \frac{1}{2\sqrt{\varphi(t)}}\right)^2 + (2t-1)^2 = 0$ for

$$\varphi(t) = \frac{1 \pm \sqrt{-(2t-1)^2}}{4(2t^2 - 2t + 1)}$$

which is a complex function except for $t = 1/2$. So using this distance it does not exists a reparametrization in $[0,1] \subset \mathbf{R}$ for first derivatives.

Using the same functions we can show also that $d_F(f',g') = 0$ does not implies $d_F(f,g) = 0$. The functions

$$\begin{array}{rccc} f'\colon & [0,1] & \to & [0,1] \\ & t & \mapsto & (t,t^2) \end{array}$$

and

$$\begin{array}{cccc} g'\colon & [0,1] & \to & [0,1] \\ & u & \mapsto & (\sqrt{u}, u) \end{array}$$

have Frechèt distance equal to zero with reparametrization $u = \varphi(t) = t^2$ but their integrals

$$f = (\frac{t^2}{2}, \frac{t^3}{3}) \quad \text{and} \quad g = (\frac{2}{3}t^{\frac{2}{3}}, \frac{t^2}{2})$$

have Frechèt distance greater then zero because it does not exist $u = \varphi(t)$ such that

$$\begin{cases} \frac{t^2}{2} = \frac{2}{3}\varphi(t)^{\frac{3}{2}} \\ \\ \frac{t^3}{3} = \frac{1}{2}\varphi(t)^2 \end{cases}$$

Clearly, the problem is the absence of the first derivative of the change of parameters φ', so we try to define a distance which takes into consideration also this one.

Given $f\colon I \to \mathbf{R}^2$ and $g\colon J \to \mathbf{R}^2$, the function $d_{\bar{F}}$ defined by

$$d_{\bar{F}}(f, g) = \inf_{\varphi\colon I \to J} \max_{t \in I} \|f(t) - g(\varphi(t))\varphi'(t)\|$$

is a distance? Unfortunately no, since we prove that $d_{\bar{F}}$ does not satisfy the triangular inequality.

Proof: Given $f\colon I \to \mathbf{R}^2$, $g\colon J \to \mathbf{R}^2$ and $h\colon \bar{J} \to \mathbf{R}^2$, we want to prove that

$$d_{\bar{F}}(f, g) + d_{\bar{F}}(g, h) \ngeq d_{\bar{F}}(f, h).$$

$$\begin{aligned} d_{\bar{F}}(f, g) &= \inf_{\varphi\colon I \to J} \max_{t \in I} \|f(t) - g(\varphi(t))\varphi'(t)\| \\ d_{\bar{F}}(g, h) &= \inf_{\psi\colon J \to \bar{J}} \max_{u \in J} \|g(u) - h(\psi(u))\psi'(u)\| \\ d_{\bar{F}}(f, h) &= \inf_{\tau\colon I \to \bar{J}} \max_{t \in I} \|f(t) - h(\tau(t))\tau'(t)\| \end{aligned}$$

then, given $\epsilon > 0$, there exist $\varphi\colon I \to J$ and $\psi\colon J \to \bar{J}$ such that

$$\max_{t \in I} \|f'(t) - g'(\varphi(t))\varphi'(t)\| < d_{\bar{F}}(f', g') + \frac{\epsilon}{3}$$

and

$$\max_{u \in J} \|g'(u) - h'(\psi(u))\psi'(u)\| < d_{\bar{F}}(g', h') + \frac{\epsilon}{3}$$

There exists $t_\epsilon \in I$ such that

$$\|f'(t_\epsilon) - h'(\psi\varphi(t_\epsilon))(\psi\varphi)'(t_\epsilon)\| > \max_{t \in I} \|f'(t) - h'(\psi\varphi(t))(\psi\varphi)'(t)\| - \frac{\epsilon}{3}.$$

Calling $\varphi(t_\epsilon) = u_\epsilon$ and $\psi(u_\epsilon) = v_\epsilon$ we have

$$\|f'(t_\epsilon) - g'(u_\epsilon)\varphi'(t_\epsilon)\| \le \max_{t\in I} \|f'(t) - g'(\varphi(t))\varphi'(t)\| \le d_{\bar{F}}(f', g') + \frac{\epsilon}{3}.$$

and

$$\|g'(u_\epsilon) - h'(\psi(u_\epsilon))\psi'(u_\epsilon)\| \le \max_{u\in J} \|g'(u) - h'(\psi(u))\psi'(u)\| \le d_{\bar{F}}(g', h') + \frac{\epsilon}{3}.$$

Then

$$\begin{aligned}
&d_{\bar{F}}(f', g') + d_{\bar{F}}(g', h') > \\
&\|f'(t_\epsilon) - g'(u_\epsilon)\varphi'(t_\epsilon)\| + \|g'(u_\epsilon) - h'(\psi(u_\epsilon))\psi'(u_\epsilon)\| - \frac{2\epsilon}{3} \ge \\
&\|f'(t_\epsilon) - g'(u_\epsilon)\varphi'(t_\epsilon) + g'(u_\epsilon) - h'(\psi(u_\epsilon))\psi'(u_\epsilon)\| - \frac{2\epsilon}{3}
\end{aligned}$$

from which it is not possible to obtain

$$\|f'(t_\epsilon) - h'(\psi(u_\epsilon))\psi'(u_\epsilon)\|.$$

QED

Chapter 4

Reconstruction and distances in the discrete setting

In the previous chapter we recalled the fundamental theorem of planar curves, that is the reconstruction from curvature, we studied distances between curves and proved the link existing for distances between curves and distances between corresponding curvatures. In this chapter we deal with the problem of reconstruction of a curve from its discrete curvature and the last section is dedicated to the discretization of the distances between curves already studied in the smooth setting in Chapter 3 and we try to find a new definition of distance based on the intrinsic properties of the curves.

4.1 Reconstruction from curvature

Given a smooth signed curvature function with respect to the arch length, after fixing a starting point and an initial direction, we reconstruct uniquely the orthonormal frames $\{T, N\}$ and then the curve by integration (Theorem 3.1.4). In the discrete setting the arc length is nothing else that the edge lengths, while the curvature is linked not only to edge lengths but also to the exterior angles. Thus, given edge lengths $l_i = \|\Delta p_i\|$ and exterior angles α_i of a polygon P, for $i = 1, \ldots, n$, fixed a starting point p_0 and the direction of Δp_1, which corresponds to fix the starting angle θ_0, we reconstruct uniquely the polygon P by reconstructing its points p_i by

$$p_i = p_{i-1} + l_i \begin{pmatrix} \cos(\theta_i) \\ \sin(\theta_i) \end{pmatrix}, \quad \text{for}\, i = 1, \ldots, n$$

where $\theta_i = \theta_{i-1} + \alpha_i$ (see Figure 2.1).

But what can we deduce of the original curve given only its curvature? That is, if of a polygon P it is known only the curvature, is it possible to reconstruct P (up to rigid motion)? If we can find the polygon, is it unique? Or there exists at least another curve different from it (which does not change for a rigid motion) but with the same curvature?

We study these problems separately.

Firstly we consider the following situation. Let be given a set of ordered curvatures $k_1, \dots, k_n$ defined as in (2.4) or (2.11). If they belong to an open curve it is not possible to reconstruct the original curve since we have $n+1$ degrees of freedom. For closed curve the result is not better then for open ones as next Theorem 4.1.1 shows.

Theorem 4.1.1. *Let P be a closed curve with $n+1$ vertices $\{p_0, \dots, p_n\}$, edge lengths l_i and exterior angles α_i, for $i = 1, \dots, n$. Let us call*

$$k_i = \frac{2\alpha_i}{l_i + l_{i+1}}$$

$$\tilde{k}_i = \frac{2\sin\alpha_i}{\|p_{i+1} - p_{i-1}\|}$$

and $k_P = \{k_1, \dots, k_n\}$, $\tilde{k}_P = \{\tilde{k}_1, \dots, \tilde{k}_n\}$. Let Q be another closed curve with $n+1$ vertices $\{q_0, \dots, q_n\}$, edge lengths m_i and exterior angles β_i, and k_Q and $\tilde{k}_Q$ the corresponding curvatures. Then $k_P = k_Q$ or $\tilde{k}_P = \tilde{k}_Q$ do not imply $P = Q$.

Proof: The curves are equal if they have equal edge length and exterior angles. An example proves the theorem. Given the quadrilaterals of Figure 4.1, they are different since they do not have equal edge lengths

$$\begin{array}{ll} \{l_1, \dots, l_4\} = \{2, 4, 2, 4\} & \{m_1, \dots, m_4\} = \{3, 3, 3, 3\} \\ \{\alpha_1, \dots, , \alpha_4\} = \{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}\} & \{\beta_1, \dots, \beta_4\} = \{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, 2\frac{\pi}{3}\}. \end{array}$$

however the polygons P and Q have equal curvatures:

$$k_P = \left\{ \frac{2\pi}{3(4+2)}, \frac{4\pi}{3(4+2)}, \frac{2\pi}{3(4+2)}, \frac{4\pi}{3(4+2)} \right\}$$
$$k_Q = \left\{ \frac{2\pi}{3(3+3)}, \frac{4\pi}{3(3+3)}, \frac{2\pi}{3(3+3)}, \frac{4\pi}{3(3+3)} \right\}$$

To show that also $\tilde{K}$ does not determine the curve uniquely we consider two quadrilaterals inscribed in the same circle (Figure 4.2). Being $\tilde{k}$ the inverse of the radius of the vertex osculating circle, they have the same curvature at every vertex.

QED

We conclude that these two definitions of curvature do not admit a unique reconstruction of the curve.

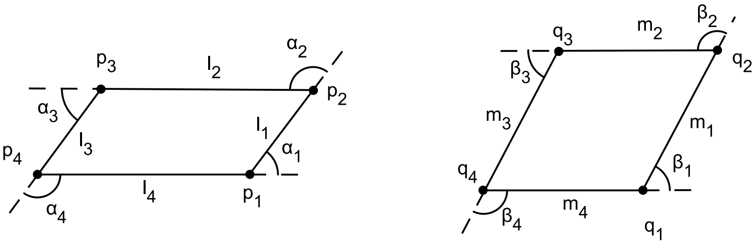


Figure 4.1: Polygons P and Q different between them but with same curvatures (2.4).

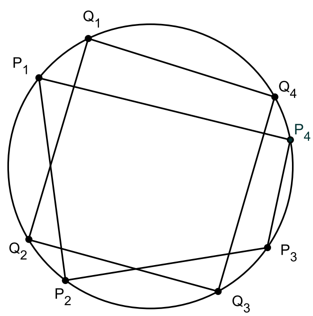


Figure 4.2: Polygons P and Q different between them but with same curvatures (2.11).

Existence of a closed curve with a prescribed curvature Given a set of n curvatures as defined in (2.4), are we able to understand if they derive from a closed polygon? That is, can we understand if there exist at least one closed curve whose curvature assumes exactly those values? Let us study the problem for the simplest polygon, the triangle.

Given a set of three real values k_1, k_2, k_3, can we establish if they are the curvature, computed by (2.4), of a triangle? If they have been obtained by a triangle, we would be able to compute its edge length l_i and angles α_i, such that they satisfy the system of nonlinear inequalities

$$\begin{cases} k_1(l_1+l_2)=2\alpha_1 \\ k_2(l_2+l_3)=2\alpha_2 \\ k_3(l_3+l_1)=2\alpha_3 \\ \sum_{i=1}^{3}\alpha_i=2\pi \\ l_1+l_2\cos(\alpha_1)+l_3\cos(\alpha_1+\alpha_2)=0 \\ l_2\sin(\alpha_1)+l_3\sin(\alpha_1+\alpha_2)=0 \\ l1,l2,l3\geq 0 \end{cases} \tag{4.1}$$

Let us observe that the last two equations in the system are the closing conditions of the curve seen in (2.3). By expressing the angles as functions of curvatures and edges, we can study the following system (4.2) instead of the system (4.1) and compute the angles α_i from k_i and l_i

$$\begin{cases} l_1(k_1+k_3)+l_2(k_1+k_2)+l_3(k_2+k_3)=4\pi \\ l_1+l_2\cos(\frac{k_1(l_1+l_2)}{2})+l_3\cos(\frac{k_1(l_1+l_2)}{2}+\frac{k_2(l_2+l_3)}{2})=0 \\ l_2\sin(\frac{k_1(l_1+l_2)}{2})+l_3\sin(\frac{k_1(l_1+l_2)}{2}+\frac{k_2(l_2+l_3)}{2})=0 \\ l1,l2,l3\geq 0 \end{cases} \tag{4.2}$$

We looked for a formal solution of systems (4.1) and (4.2) but without good results. Then we tried with the help of the symbolic tool of MatLab. To find the solution of the system (4.2) we define the symbolic variables l_i and the parameters k_i, fix the positivity constraints for l_i and by the function *solve* we look for the solution with respect to the l_i:

$$\begin{array}{l} \text{syms}l_1 \quad l_2 \quad l_3 \quad k_1 \quad k_2 \quad k_3 \\ \text{assume}(l_1,l_2,l_3\geq 0) \\ \text{solve}(eq_1,eq_2,eq_3,l_1,l_2,l_3) \end{array}$$

but again without success since we obtain the following warning message:

Warning: explicit solution could not be found.

Thus we try a numerical approach, that is not solving the system for generic parameters k_1,k_2,k_3, but assigning to them real curvature values. We test the curvatures of 12 different triangles, all stored with 15 decimal places to avoid

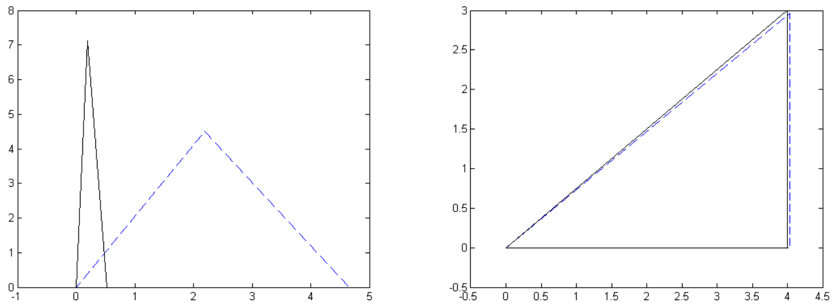


Figure 4.3: The original (smooth lines) and the reconstructed triangles (dashed lines), different to each other.

as much as possible numerical errors. These tests confirm that there is not uniqueness of the solution because for two of them, the reconstructed triangles are different from the original ones, as shown in Figure 4.3.

But the biggest problem is that for two of them the solver does not find the solution, and it is not clear the reason also because they are not almost degenerate triangles. This seems to confirm that is not possible to understand if three given values can be the curvatures of a triangle, that is if at least the existence can be guaranteed. Moreover, we considered an open polygon with three edges and I computed the third curvature's value by considering as angle the one obtained by extending the first and the last edges and as edges the correct ones. Given these values, the system finds a triangle.

A similar approach applied to system (4.1) does not give analogous good results because very often it does not reconstruct triangles but open polygons. Since for a triangle the closing constraints are equivalent to the Carnot formulas, we use these last ones in the system but it appears an equation more which create more problems.

Not obtaining good results for triangles, we try to solve the problem for polygons at least quadrangular with the same criterion as above, but when we apply the previous method it appears a warning message:

Warning: 3 equations in 5 variables. New variables might be introduced.

It seems that it does not exist a closed-form solution.

Reconstruction from piecewise linear curvature Let us see in this paragraph if we have the same problems with the piecewise linear curvature $\tilde{k}$ defined in (2.15) or with $\hat{k}$ defined in (2.16), which corresponds to the reduced arc length parameterization of a polygon.

Obviously if we reconstruct the curve by integration of $\hat{k}$ or $\tilde{k}$ as in Theorem 3.1.4 we do not find the polygon but a smooth curve. But from $\tilde{k}$ we can reconstruct the polygon up to rigid motions if we consider that knowing $\tilde{k}$ means to know the values of the curvature at the vertices and the partition of the parameter domain, that is the edge lengths. Knowing in this way the edge length, the exterior angle α_i at the vertex p_i can be recovered from the value

$$\tilde{k}(t_i) = \frac{2\alpha_i}{\|p_i - p_{i-1}\| + \|p_{i+1} - p_i\|}.$$

Specified the position of the first vertex p_0 and the direction of the first edge Δp_1 by the rotation angle θ_0 (which measures the angle between the positive x-axis and the the edge Δp_i), we fix the position in the plane. Defined $\theta_i = \theta_{i-1} + \alpha_i$ the angle of the global rotation from the origin to the edge t_i, t_{i+1}, the vertices p_i of the polygon P are reconstructed by recursion as

$$p_{i+1} = p_i + \|t_{i+1} - t_i\| \begin{pmatrix} \cos\theta_i \\ \sin\theta_i \end{pmatrix}$$

If the curvature is $\hat{k}$ on the partition σ, we can reconstruct (up to uniform scalings and rigid motions) the polygon P, because the edge lengths of P are proportional to the distances $s_i - s_{i-1}$ and the exterior angles α_i at the vertices of P can be recovered from the values $\tilde{k}(s_i)$ using

$$\tilde{k}(s_i) = \frac{2\alpha_i}{\|p_i - p_{i-1}\| + \|p_{i+1} - p_i\|}.$$

and the polygon can be reconstructed as

$$p_i = p_{i-1} + |s_i - s_{i-1}| \begin{pmatrix} \cos\theta_i \\ \sin\theta_i \end{pmatrix}$$

4.2 Distances for discrete curves

As said in the Introduction, after interpolation of the curvature of two closed curves we often obtain an open curve which we want to subtitute. The first attempt to find a closed curve to substitute the open one, was to find a closed curve which minimizes an appropriate distance from the open one. Appropriate distance means that it must be independent on their mutual position, since our study of distance is linked to the analysis of the similarity of the shapes that the curves represent. In the previous Chapter 3 we observed that no smooth distance between curves was good for our purposes (neither Gromov-Hausdorff because too complicate). So we decided to take the curvature as measure of distance between two curves and to minimize with respect to it. But before to think to the curvature as solution to our problem, we brought the distance minimization problem in the discrete setting to see if it was possible to obtain some result. In this section we study the distances commonly used for discrete curves and we see if one of them can be the solution to our problem.

Let us observe that since we are interested in shape similarity, we scale the curves to have the same length which anyhow preserves the proportion between edges.

In literature the most used distances between discrete curves are Hausdorff and Frechèt distances.

Hausdorff distance The Hausdorff distance is defined exactly as in the continuous setting (Definition 3.3) and keeps the same problem of loss of orientation of the curves.

Discrete Frechèt metric In Section 3.2 we studied the Frechèt distance for smooth curves, but in practice very often this distance is computed for their polygonal approximating curves [14]. In 1992 Alt and Godau published the work [3] where they studied the computational properties of the Frechèt distance d_F (Definition 3.2.6). Their algorithm is based on a parametric search technique and has computational time $\mathcal{O}(pq \log 2pq)$, where p and q are the number of segments which compose respectively the two polygonal curves.

In [14] it is described a discrete variation of the Frechèt distance, called δ_F, whose idea is to look at all possible couplings between the ordered vertices of the two polygonal curves. They show that δ_F is a good approximation of d_F, in particular that $\delta_F \geq d_F$ and that the difference between these measures is bounded by the length of the longest edge of the curves. The computational time to measure δ_F is $\mathcal{O}(pq)$.

Let P and Q be two polygonal curves and $\sigma(P) = (p_1, \ldots, p_n)$ and $\sigma(Q) = (q_1, \ldots, q_m)$ the corresponding sequences of points of the curves.

Definition 4.2.1. *A coupling L between P and Q is a sequence $(p_{a_1}, q_{b_1}), (p_{a_2}, q_{b_2}), \ldots, (p_{a_r}, q_{b_r})$ of distinct pairs from $\sigma(P) \times \sigma(Q)$ such that $a_1 = b_1 = 1$, $a_r = n$, $b_r = m$, and for all $i = 1, \ldots, q$ we have $a_{i+1} = a_i$ or $a_{i+1} = a_i + 1$, and $b_{i+1} = b_i$ or $b_{i+1} = b_i + 1$.*

This definition solves the drawback of the Hausdorff distance because each coupling respects the order of the points in P and Q. The length $|L|$ of the coupling L is the length of the longest link in L, that is,

$$|L| = \max d(p_{a_i}, q_{b_i})$$

$i = 1, \ldots, r$.

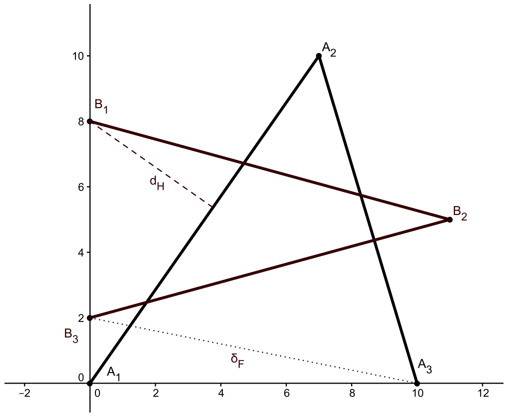


Figure 4.4: The smooth lines are the curves P and Q, whose orientations are indicated by the increasing order of vertices indices. The dashed line represents the Hausdorff distance d_H while the dotted line the Frechèt distance δ_F.

Definition 4.2.2. *Given two polygonal curves P and Q, their discrete Frechèt distance δ_F is defined to be*

$$\delta_F(P,Q) = \min\{|L| : L \textit{ is a coupling between } P \textit{ and } Q\}$$

The explicit algorithm to compute δ_F can be found in [14].

Example 4.2.3. *This example shows a comparison between the Hausdorff and the Frechèt distances. Let P and Q be two curves with $\sigma(P) = \{(0,0),(7,10),(10,0)\}$ and $\sigma(Q) = \{(0,8),(11,5),(0,2)\}$ (Figure 4.4). Denoting by d_H the Hausdorff distance and by δ_F the Frechèt distance as in definition 4.2.2, it is easy to show (see figure 4.4) that they assume values very far to each other, in fact $d_H = 4.59$ while $\delta_F = 10.1$ since the best coupling of the curves is $L = \{(A_1,B_1),(A_2,B_2),(A_3,B_3)\}$.*

Distances parameterization based If we consider the arc length parameterization (2.1) or (2.2) of the curve we can use the maximum norm or the L^p norm as seen in Chapter 3 for smooth arc length parameterized curves, without to solve the problem of the mutual position in the plane.

4.2.1 Metric independent on the mutual position of the curves

All distances previously mentioned are strictly dependent on the mutual positions of the curves exactly as in the smooth setting. Instead we require that if T is an Euclidean motion of $\mathbf{R}^2$ that does not change the orientation of the curve γ, then $d(\gamma, T\gamma) = 0$. To overcome this problem we have two possibilities. One is to use the Gromov-Hausdorff distance, which, as in the smooth setting is too much complicate. In fact, our objective is not to measure the distance between two given curves, but given an open one, we want to find a closed curve as minimizer of their distance. The second possibility is to find a distance which depends only on intrinsic properties of the curves, which obviously can not have a smooth analogue.

Looking for a metric dependent on intrinsic properties of curves Let be given two curves P and Q with $n+1$ vertices, edges lengths l_i and m_i resp. and exterior angles α_i and β_i resp, with $i = 1, \dots, n$ or $i = 1, \dots, n-1$ depending if the curves are closed or open. If we compute a distance based on edges and angles and the two curves are one open and one closed, we must to assign some value to the angle at the first and last vertices of the open curve (obviously the same value). This is a similar problem to the one mentioned in the Remark 2.3.1 of previous Chapter for curvatures. In this definition we define the n-th exterior angle of the open curve as the angle obtained by extension of the first and the last edges. After several attempts we found a possible distances

$$
\begin{aligned}
d_{p_3}\ (P,Q) = \\
&\tfrac{1}{2}\sup_{i\in I}[\sqrt{(l_{i+1}\cos\alpha_i - m_{i+1}\cos\beta_i)^2 + (l_{i+1}\sin\alpha_i - m_{i+1}\sin\beta_i)^2} + \\
&\sqrt{(l_i\cos\alpha_i - m_i\cos\beta_i)^2 + (l_i\sin\alpha_i - m_i\sin\beta_i)^2}]
\end{aligned}
$$

The meaning of this distances are explained for a triangle in Figure 4.5: For each i we overlap the points p_i e q_i and the edges l_{i-1} and m_{i-1} and measure the distance between p_{i+1} and q_{i+1} in this configuration. But computing this distance in clockwise or counterclockwise gives different results, so we consider the averaged value obtained by considering also the overlap of p_i and q_i and edges l_{i+1} and m_{i+1} and computing the distance p_{i-1} and q_{i-1} in this configuration.

Unfortunately these distances did not show any interesting property so we decide to forget this way and to consider as measure of similarity between curves another intrinsic property, the curvature.

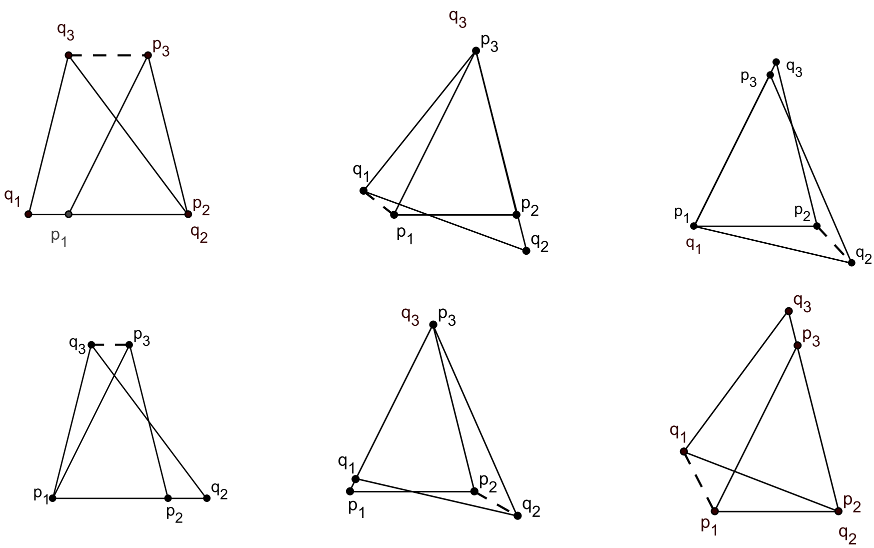


Figure 4.5: This figure shows the meaning of the distance d_{p3} for a triangle.

Chapter 5

Interpolation of curvature

5.1 Related works

Shape blending, or shape morphing, is a very active research field in Computer Graphics dealing with the mathematical theory and the algorithms able to construct a gradual and continuous transformation between two planar or solid shapes.

Solutions to the metamorphosis problem can be found under many domains, such as two-dimensional images [38], planar polygons and polylines (i.e., piecewise linear curves) [32], [33], [35], polyhedra [1] and free form curves [31], [34].

Problems related to 2–D shape blending arise in different areas of computer graphics, such as shape recognition [11], [5], where the primary concern is determining how similar two complete objects are. Shape blending can be associated to the computer vision problem of contour identification [40], [21], solved in this last work by energy minimization. Another important area where shape blending is used is animation, that is the two shapes to be blended are key frames in a character animation and it is fundamental that the in between shapes are similar [4].

Typically, the problem of shape blending is divided in two parts: the first addresses the problem of the correspondence between the two shapes [24], [33], [41], while the second one wants to find the interpolation between them, solved in many cases via a linear interpolation approach.

When dealing with polygonal curves [1], [28], [33], we call the two parts of the problems as *vertex correspondence* and *vertex path problems.*

While dealing with the blending of planar shapes, two main classes of solutions to this problems have been proposed: the ones taking into account only the boundary of the shapes (i.e., by considering just the curves describing it) and the ones that uses information also on the interior of the shape (e.g., its topological skeleton or the area).

Unfortunately the simple approach of using a linear interpolation scheme of the vertices positions produces very bad results because of the independence of each vertex's path from the paths of the other ones, and we can have unexpected intermediate results (Figures 5.1 and 5.2).

One first attempt of tackling this problem taking into account the mutual relations of the vertices was presented by Sederberg and colleagues [33] who optimized the obtained interpolation by minimizing the bending and stretching energy. Instead in [32] they propose an interpolation of intrinsic properties of the polygons, edge lengths and angles between edges, while minimizing the variation of edge lengths under polygon closing constraints, leading to a closed form solution. The main limitation of both solutions is that they cannot guarantee interpolated shapes not self-intersecting.

In [34] the algorithm first introduced in [33] is generalized to shapes bounded by B-spline curves. The correspondence problem is solved by inserting new knots such that the work required to bend and stretch one shape into the other is minimized and such that the two curves have the same number of knots, while the intermediate B-splines are obtained by linear interpolation of control points and knot.

Shapira and collaborators in [35] introduced the idea of decomposing the two polygons into star-shaped pieces with a skeleton connecting them, taking for the first time in account information on the whole of the shape and not only its boundary. In fact, the star-skeleton explicitly models the interdependence between all the vertices of the polygons. The deformation is obtained by blending the skeletons and then reconstructing the corresponding polygon.

A different approach is proposed in [1], where the blending is applied to the interior of the shapes to achieve a sequence of in-between shapes which is locally least-distorting. The morph proposed in this work is rigid in the sense that local volumes are least-distorting as they vary from their source to target configurations. It can be synthesized in few steps. Given a boundary vertex correspondence, the two shapes are decomposed into isomorphic simplicial complexes and for each one they find a closed-form expression of the paths of interior and boundary vertices. For each pair of simplices the transformation is an affine one and the vertices path is obtained as the minimization of a functional.

It is only in [36] that one can find the idea of linearly interpolating the curvatures, after making sure to have the same number of vertices for both curves, introducing new points if needed. The curve closing step of their algorithm is just done applying the method of [32].

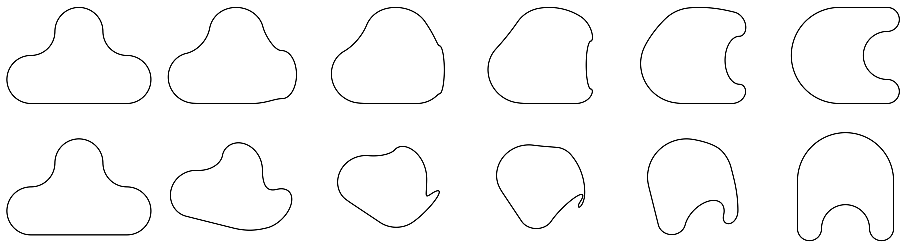


Figure 5.1: Two deformations of γ_0 (left figure) into γ_1 (right figure), where γ_1 changes in the bottom row for a rotation of ninety degrees. Both deformations are parameterization-based interpolations.

5.2 Curvature interpolation

Let be given two planar parametric curves $\gamma_0\colon I_0 \to \mathbf{R}^2$ and $\gamma_1\colon I_1 \to \mathbf{R}^2$. A blending between them is obtained by finding for any $t \in [0,1]$ a curve $\gamma_t\colon I_t \to \mathbf{R}^2$ such that the mapping $t \mapsto \gamma_t$ is continuous in t and reproduces γ_0 for $t=0$ and γ_1 for $t=1$. The blending is then obtained by interpolating between γ_0 and γ_1 by varying the parameter t.

By assuming that γ_0 and γ_1 are parameterized over a common interval I, then an easy way to blend the curves is given by linear interpolation of the parameterizations, that is $\gamma_t(s) = (1-t)\gamma_0(s) + t\gamma_1(s)$. But this intuitive deformation has two drawbacks. First, it depends on the particular parameterizations of γ_0 and γ_1 and second, it is strictly dependent on the mutual position of the curves since it is implicit in the parameterizations. The problem of mutual position can lead either to naturally (first row of Figure 5.1 and top right of Figure 5.2) or to unnaturally looking intermediate curves (second row of Figure 5.1 and top left of Figure 5.2).

These drawbacks suggest us to look in a different direction, basing the deformation on some intrinsic property of curves, as can be the curvature. A similar approach is suggested in [36], where the intermediate curves are defined by linearly interpolating the signed curvature functions of γ_0 and γ_1 and by reconstructing the intermediate curve γ_t from the interpolated curvature values. The more intuitive and pleasant interpolation of their shapes can be seen with the simple example of the deformation of a segment into a semicircumference of same length as in Figure 5.2, which shows the difference between the parameterization-based (top row) and the curvature-based deformations (bottom row). Can we obtain similar positive results for closed simple curves?

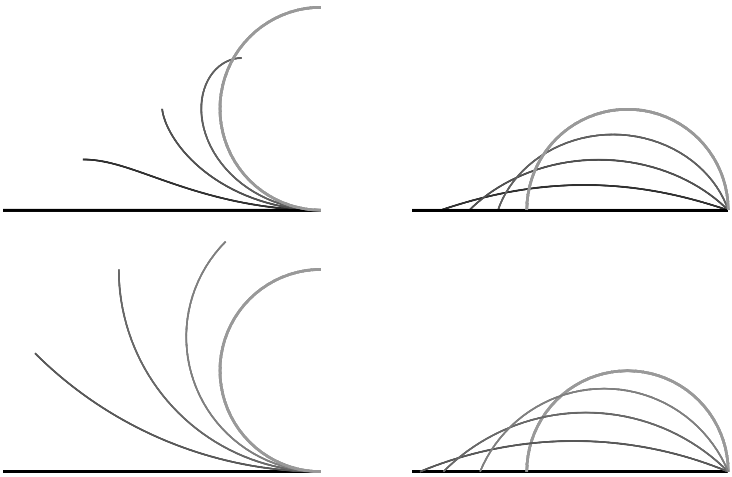


Figure 5.2: Two deformations of a segment into a semicircumference. The first row shows a parameterization based interpolation where we can notice different result depending on the mutual position. The second row shows an interpolation curvature-based which gives intermediate curve exactly equal.

5.3 Curvature interpolation of simple closed curves

Let be given two closed simple curves γ_0 and γ_1 at least piecewise C^2 with same length, parameterized by arc length in the interval $[0, l]$. Let k_0 and k_1 their signed curvature functions (bounded) and interpolate them linearly, that is at time $t \in [0, 1]$ the interpolated curvature function $k_t \colon I \to \mathbf{R}$ is

$$k_t(s) = (1-t)k_0(s) + tk_1(s).$$

This curvature k_t is the curvature of some curve $\gamma_t(s) \colon [0, l] \to \mathbf{R}^2$ with respect to arc length, and by integration, from fundamental theorem of planar curves (Theorem 3.1.4 of Chapter 3) we have:

$$\theta_t(s) = \int k_t(u)du + \theta_t^0 = (1-t)\theta_0(s) + t\theta_1(s)$$
$$\gamma_t(s) = \left(\int \cos\theta_t(s)ds + c_1, \int \sin\theta_t(s)ds + c_2 \right).$$

Let us observe that to can apply Theorem 3.1.4 we need to have arc length parametrizations and then the equal lengths of the curves to can interpolate their curvatures. The solution to this problem is to scale both curves to be unit curves, reconstruct the intermediate unit curves and in a second step, to account for this simplification, re-scale uniformly these curves by the interpolated length

$L_t = (1-t)L_0 + tL_1$ so as to get a smooth blend from the original curves γ_0 and γ_1.

What kind of curves we obtain by this reconstruction? Which properties has γ_t? Which is its length? Does it preserve the properties of the original curves, that is, is it closed and simple?

The length of $\gamma_t(s)$ is

$$L(\gamma_t(s)) = \int_0^l \|\dot{\gamma}_t(s)\| ds = \int_0^l ds = l$$

exactly equal to the length of the original curves.

If a curve of length l parameterized by arc length is closed and simple, then from Theorem 1.0.15 it holds

$$\int_0^l k(s)ds = 2\pi$$

but the converse is not true as shown by next examples.
The the arc length parameterized curve $\alpha\colon [0, 2\pi + 1] \to \mathbf{R}^2$

$$\alpha(s) = \begin{cases} (\cos s, \sin s) & s \in [0, \pi] \\ (0, \pi - s) & s \in [\pi, \pi + 1] \\ (1 + \cos(s-1), -1 + \sin(s-1)) & s \in [\pi + 1, 2\pi + 1] \end{cases}$$

has total curvature 2π but it is open (Figure 5.3, left), while the arc length parameterized curve $\beta\colon [0, 2 + 2\pi] \to \mathbf{R}^2$

$$\beta(s) = \begin{cases} (1 - s, 1) & s \in [0, 1] \\ (\cos(s - 1 + \frac{\pi}{2}), \sin(s - 1 + \frac{\pi}{2})) & s \in [1, , 1 + 2\pi] \\ (1 + 2\pi - s, 1) & s \in [1 + 2\pi, 2 + 2\pi]. \end{cases}$$

has total curvature equal to 2π but it is self-intersecting (Figure 5.3, right).

This implies that also if the interpolated curvature k_t satisfy

$$\int_0^l k_t(s)ds = 2\pi$$

it can be open and/or self-intersecting.

Can we then understand if the curve $\gamma_t(s)$ reconstructed from $k_t(s)$, with $s \in [0, l]$ preserves the property to be closed? That is, by Definition 1.0.10 that $\gamma(0) = \gamma(l)$ and $\gamma^{(k)}(0) = \gamma^{(k)}(l)$ for each k?

If a curve $\gamma(s)$ at least $C^2([0, l])$ is closed, then $k(0) = k(l)$ and $\theta(0) = \theta(l) + 2h\pi$ where $h \in \mathbf{Z}$, otherwise if it is only piecewise C^2 and closed then $\theta(0) = \theta(l) + 2h\pi$. But starting from the interpolated curvature $k_t(s) =$

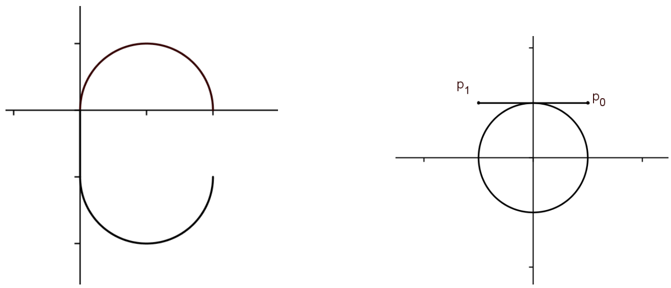


Figure 5.3: Two curves which show that holding true that $\int_0^l k(s)ds = 2\pi$ does not imply for corresponding curve to be close (left) nor simple (right).

$(1-t)k_0(s) + tk_1(s)$, with k_0 and k_1 corresponding to two closed curves γ_0 and γ_1, can we deduce that the reconstructed γ_t is closed? By hipothesis, $k_0(0) = k_0(l)$ and $k_1(0) = k_1(l)$, which implies

$$\begin{aligned}
&k_t(0) = (1-t)k_0(0) + tk_1(0) = (1-t)k_0(l) + tk_1(l) = k_t(l)\\
&\theta_t(0) = (1-t)\theta_0(0) + t\theta_1(0) = (1-t)(\theta_0(l) + 2h_0\pi) + t(\theta_1(l) + 2h_1\pi) =\\
&\quad \theta_t(l) + 2h_0\pi + t2\pi(h_1 - h_0)
\end{aligned}$$

But if $h_0 \neq h_1$ then $t(h_1 - h_0) \notin \mathbf{Z}$ which implies

$$\gamma_t'(0) = (\cos\theta_t(0), \sin\theta_t(0)) \neq (\cos\theta_t(l), \sin\theta_t(l)) = \gamma_t'(l).$$

But supposing that the curves γ_0 and γ_1 are simple, then $h_0 = h_1 = 1$ which implies $\gamma_t'(0) = \gamma_t'(l)$. Unfortunately it is not possible to prove that $\gamma_t(0) = \gamma_t(l)$ because there can be different points where the rotation angles formed by tangents are the same.

These considerations lead us to the intuition that the reconstructed curve can be open, but without a theoretical proof we need at least an example which confirms this fact. To create this example we need to work with closed arc length parameterized curves, but in practice it is not easy to find two such curves. To overcome this problem we use closed arc splines (Definition 2.4.2).

Let $\gamma_0(s)$ and $\gamma_1(s)$ two arc splines defined on the interval $[0, 4\pi + 4]$ (see

Figure 5.4)

$$\gamma_0(s)=\begin{cases}(2+\cos(s-\frac{\pi}{2}),-1+\sin(s-\frac{\pi}{2})) & s\in[0,\pi]\\ (2+\cos(\frac{\pi}{2}-s),1+\sin(\frac{\pi}{2}-s)) & s\in[\pi,\pi+\frac{\pi}{2}]\\ (\cos(s-\frac{3\pi}{2}),1+\sin(s-\frac{3\pi}{2})) & s\in[\pi+\frac{\pi}{2},2\pi+\frac{\pi}{2}]\\ (-2+\cos(\frac{5\pi}{2}-s),1+\sin(\frac{5\pi}{2}-s)) & s\in[2\pi+\frac{\pi}{2},3\pi]\\ (-2+\cos(s-\frac{5\pi}{2}),-1+\sin(s-\frac{5\pi}{2})) & s\in[3\pi,4\pi]\\ (s-4\pi-2,-2) & s\in[4\pi+4]\end{cases}$$

$$\gamma_1(s)=\begin{cases}(1+\cos(-s-\frac{\pi}{2}),\sin(-s-\frac{\pi}{2})) & s\in[0,\pi]\\ (1+\frac{1}{2}\cos(2s-\frac{5\pi}{2}),\frac{3}{2}+\frac{1}{2}\cos(2s-\frac{5\pi}{2})) & s\in[\pi,\pi+\frac{\pi}{2}]\\ (\frac{3\pi}{2}-s+1,2) & s\in[\frac{3\pi}{2}+2]\\ (-1+2\cos(\frac{s}{2}-1-\frac{\pi}{4}),2\sin(\frac{s}{2}-1-\frac{\pi}{4})) & s\in[\frac{3\pi}{2}+2,\frac{7\pi}{2}+2]\\ (s-\frac{7\pi}{2}-3,-2) & s\in[\frac{7\pi}{2}+2,\frac{7\pi}{2}+4]\\ (1+\frac{1}{2}\cos(2s-8-\frac{15\pi}{2}),-\frac{3}{2}+\frac{1}{2}\sin(2s-8-\frac{15\pi}{2})) & s\in[\frac{7\pi}{2}+4,4\pi+4]\end{cases}$$

Their curvatures are

$$k_0(s)=\begin{cases}1 & s\in(0,\pi)\\ -1 & s\in(\pi,\frac{3\pi}{2})\\ 1 & s\in(\frac{3\pi}{2},\frac{5\pi}{2})\\ -1 & s\in(\frac{5\pi}{2},3\pi)\\ 1 & s\in(3\pi,4\pi)\\ 0 & s\in(4\pi+4)\end{cases}\qquad k_1(s)=\begin{cases}-1 & s\in(0,\pi)\\ 2 & s\in(\pi,\frac{3\pi}{2})\\ 0 & s\in(\frac{3\pi}{2},\frac{3\pi}{2}+2)\\ \frac{1}{2} & s\in(\frac{3\pi}{2}+2,\frac{7\pi}{2}+2)\\ 0 & s\in(\frac{7\pi}{2}+2,\frac{7\pi}{2}+4)\\ 2 & s\in(\frac{7\pi}{2}+4,4\pi+4)\end{cases}$$

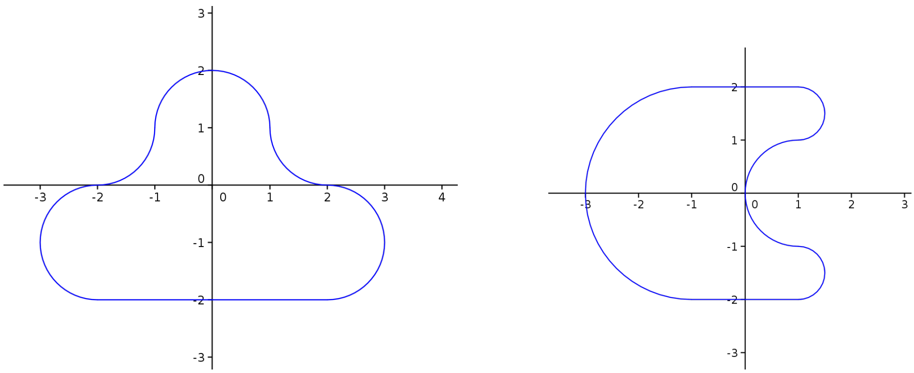


Figure 5.4: γ_0 on the left and γ_1 on the right side.

The interpolated curvature is

$$k_t(s) = \begin{cases} 1-2t & s \in (0,\pi) \\ -1+3t & s \in (\pi, \frac{3\pi}{2}) \\ 1-t & s \in (\frac{3\pi}{2}, \frac{3\pi}{2}+2) \\ 1-\frac{1}{2}t & s \in (\frac{3\pi}{2}+2, \frac{5\pi}{2}) \\ -1+\frac{3}{2}t & s \in (\frac{5\pi}{2}, 3\pi) \\ 1-\frac{1}{2}t & s \in (3\pi, 4\pi) \\ \frac{1}{2}t & s \in (4\pi, \frac{7\pi}{2}+2) \\ 0 & s \in (\frac{7\pi}{2}+2, \frac{7\pi}{2}+4) \\ 2t & s \in (\frac{7\pi}{2}+4, 4\pi+4) \end{cases} \tag{5.1}$$

We now reconstruct the curve starting from interpolated curvature function γ_t (5.1). Let us observe that the curve must be an arc spline and then it must have continuous first derivative. This condition is expressed by imposing that the angle $\theta_t(s)$, formed by the tangent and the x-axis, is the same at the interior contact points. By integration of interpolated curvature k_t we find $\theta_t(s)$, which is a vector with components $\theta_t^i = k_t^i s + a_i$, with i ranging from 1 to the number of pieces which compose the parameter domain. The vector of a_i is composed by those constants of integration whose values are computed to guarantee the same rotation angle at the contact points, condition necessary, as already said, to have C^1 continuity. Moreover, for a more pleasant visualization of the deformation process, we fix a_1 to be the linear interpolation of the first rotation angles of the original curves, permitting in this way to avoid unexpected rotations of the shape during the process.

$$\theta_t(s) = \begin{cases} (1-2t)s + t\pi & s \in (0,\pi) \\ (-1+3t)s + (2\pi - 4\pi t) & s \in (\pi, \frac{3\pi}{2}) \\ (1-t)s + (2\pi t - \pi) & s \in (\frac{3\pi}{2}, \frac{3\pi}{2}+2) \\ (1-\frac{1}{2}t)s + (-\pi - t + \frac{5t\pi}{4}) & s \in (\frac{3\pi}{2}+2, \frac{5\pi}{2}) \\ (-1+\frac{3}{2}t)s + (4\pi - t - \frac{15t\pi}{4}) & s \in (\frac{5\pi}{2}, 3\pi) \\ (1-\frac{1}{2}t)s + (-2\pi - t + \frac{9t\pi}{4}) & s \in (3\pi, 4\pi) \\ \frac{t}{2}s + (2\pi - t - \frac{7t\pi}{4}) & s \in (4\pi, \frac{7\pi}{2}+2) \\ 2\pi & s \in (\frac{7\pi}{2}+2, \frac{7\pi}{2}+4) \\ 2ts + (2\pi - 8t - 7t\pi) & s \in (\frac{7\pi}{2}+4, 4\pi+4) \end{cases}$$

During the reconstruction process we can make the error to integrate with respect to s by considering t as a parameter and give to it its numerical value only after reconstruction. Instead, we must consider that for some value of t the curvature can be zero and this can leads to have zero at the denominator when we integrate $\cos(\theta_t)$ and $\sin(\theta_t)$. Thus, for every value of t we insert a control to see if some component of k_t is equal to zero or not, since this implies a different reconstruction process. The curve is reconstructed piecewise and to attach these pieces at the contact points we impose that the last and the first points of two adjacent arcs coincide. These constraints can be obtained by computing the correct constants of integration b_i and c_i for every contact parameter s_i, except at the first and last point. Fixed b_1 and c_1 (to fix the starting point), for $i = 2, \ldots, n$ we compute b_i and c_i in a recursive way

$$\begin{pmatrix} b_i \\ c_i \end{pmatrix} = \gamma_t^{i-1}(s_{i-1}) - \gamma_t^i(s_{i-1}) + \begin{pmatrix} b_{i-1} \\ c_{i-1} \end{pmatrix}$$

where the top index i stays for the i-th piece of the reconstructed curve.

Figure 5.5 shows the intermediate reconstructed curves from the interpolated curvature of curves γ_0 and γ_1, and in particular that during the interpolation the curve can be open and self-intersecting.

Problem of self-intersection A successive problem that arises when we work with deformations of curves is the risk to obtain self-intersections. In fact,

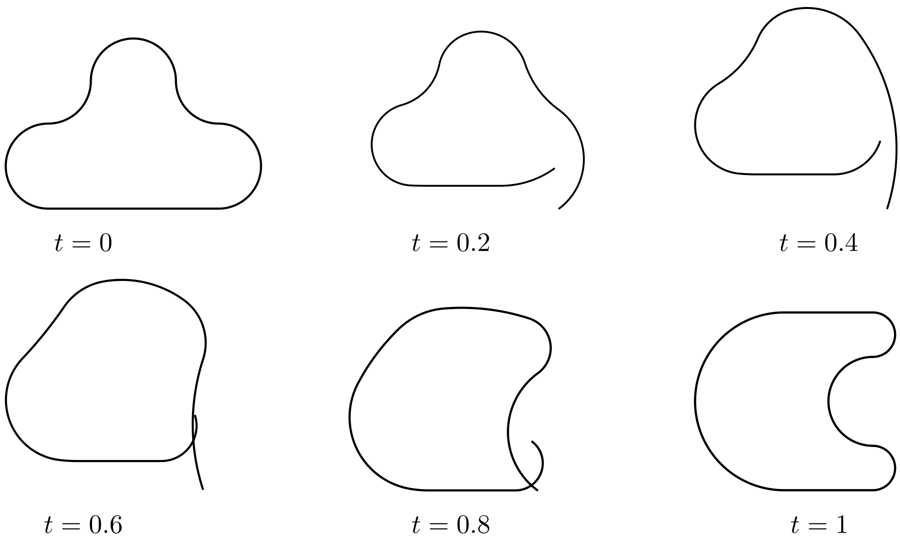


Figure 5.5: The reconstructed curves corresponding to the intermediate parameter values $t = 0.2, 0.4, 0.6, 0.8$ from the interpolation of the curvatures of source (left) and target (right) curves.

the constraints for the solution of our problem concern only the closing of the curve and there is not a mathematical formula to describe algebraically this property.

5.4 Mathematical formulation of the problem

Given two closed curves of same length and parameterized by arc length, we would like to construct a blending between them based on the linear interpolation of the curvature, such that the intermediate curves are closed and of same length of the original two. But we have just seen that to obtain closed curves is not always available by a simple reconstruction from the linear interpolated curvature (Figure 5.5).

Surazhsky and Elber [36] fix this problem by adapting the strategy of Sederberg et al. [32] to close γ_t in a post-processing step.

We propose a different solution. Since γ_t is not necessarily closed, we would like to take as intermediate curve the closed curve $\tilde{\gamma}_t$ which is closest to γ_t with respect to some metric. But as already said, this involves the problem of the mutual position whose variation changes the value of the distance, unless we do not find a way to chose the best one, or the difficult problem with the Gromov-Hausdorff distance.

A more promising solution is to consider a distance based on intrinsic properties, as curvature. It is strengthened by the fact that curvature has two properties: it determines completely the curve up to rigid motions (Theorem 3.1.4) and Theorems 3.3.1 and 3.3.2 show the link between close curvatures and close corresponding curves. So we look for a closed curve which do not minimize the distance from the open curve but whose curvature is the closest to the curvature of the open one.

Let us describe formally the problem. Let be given a curvature function $k\colon [0,L] \to \mathbf{R}$ from which we reconstruct the open curve $\gamma\colon [0,L] \to \mathbf{R}^2$ with respect to the arc length. We would like to find a curvature function $\tilde{k}$ such that it minimizes the distance from k and the corresponding curve $\tilde{\gamma}$ is closed. We are then interested in a constrained variational type problem where we are looking for a curve $\tilde{\gamma}$ arc length parameterized on $[0,L]$, at least piecewise $C^2([0,L])$, with $\tilde{\gamma} = (\tilde{x}, \tilde{y})$, such that its curvature function

$$\tilde{k}(s) = \tilde{x}'(s)\tilde{y}''(s) - \tilde{y}'(s)\tilde{x}''(s)$$

minimizes the functional L^2-distance from $k(s)$

$$\begin{aligned} J(\tilde{\gamma}) &= \|\tilde{k}(s) - k(s)\|_2 = \left(\int_0^L |\tilde{k}(s) - k(s)|^2 ds\right)^{1/2} \\ &= \left(\int_0^L |\tilde{x}'(s)\tilde{y}''(s) - \tilde{y}'(s)\tilde{x}''(s) - k(s)|^2 ds\right)^{1/2} \end{aligned} \tag{5.2}$$

or the L^∞ norm

$$H(\tilde{\gamma}) = \sup_{s\in[0,L]} |\tilde{k}(s) - k(s)| = \sup_{s\in[0,L]} |\tilde{x}'(s)\tilde{y}''(s) - \tilde{y}'(s)\tilde{x}''(s) - k(s)| \tag{5.3}$$

under the constraints of arc length parameterizations and closure (boundary conditions)

$$\begin{cases} \|\tilde{\gamma}'(s)\| = 1 \\ \tilde{\gamma}^{(k)}(0) = \tilde{\gamma}^{(k)}(L). \end{cases} \tag{5.4}$$

We work on problem (5.2) rather then on problem (5.3) because the space L^∞ does not preserve all properties of L^p spaces. The theoretical solving of this problem is very difficult because it is a vectorial variational problem and moreover the function depends also on the second derivative.

Generalities on Variational Problems Given a functional, that is an application which associate a real number to a function, we call variational problem the research of the function which minimizes that functional. An important class of variational problems is expressed as the research of a function $y\colon [a,b] \to \mathbf{R}$ in $C^1([a,b])$ which minimizes the functional

$$J(y) = \int_a^b F(x,y,y')dx$$

where $F \in C^2([a,b] \times \mathbf{R} \times \mathbf{R})$ and satisfies the boundary conditions $y(a) = y_1$ and $y(b) = y_2$. The necessary condition for the function $\hat{u}$ to be a minimum of the functional $J(u)$ is that the function $\hat{u}$ satisfies the Euler-Lagrange equation:

$$\frac{\partial F}{\partial u}(x, \hat{u}(x), \hat{u}'(x)) - \frac{d}{ds}\frac{\partial F}{\partial u'}(x, \hat{u}(x), \hat{u}'(x)) = 0$$

A sufficient condition on the critical point $\hat{u}$ to be a minimizer of the functional $J(u)$ is that $F \in C^2([a,b] \times \mathbf{R} \times \mathbf{R})$ is convex with respect to u and u'. If F is strictly convex then $\hat{u}$ is the unique minimum.

If the problem is constrained by integral type or implicit constraints, that is

$$\int_a^b G(x,y,y')dx = c, \quad c \in \mathbf{R}$$

or

$$G(x,y) = 0,$$

the problem is solved by introducing the Lagrange multipliers.

But a solution of the Euler-Lagrange equation can lead to false conclusions if the existence of a minimizer is not established beforehand.

Typically, the method for constructing a proof of the existence of a minimizer for J is called *direct method* and a condition often used is that the functional is coercive, that is

$$\lim_{\|x\| \to +\infty} \frac{J(x)}{\|x\|} = +\infty.$$

All we said can be generalized to functions $F \in C^1(\Omega, \mathbf{R}^m, \mathbf{R}^m)$, with $\Omega \subset \mathbf{R}^n$. The natural spaces where variational problems are defined are the Sobolev spaces $W^{1,p}, p > 1$ or $H^s(\Omega, \mathbf{R}^m)$, in fact they describe in a more appropriate way the particular studied problem and have some useful properties, such as the completeness for H^s.

Sobolev spaces Let Ω an open subset of $\mathbf{R}^n$.

Definition 5.4.1. *Let $C_0^\infty(\Omega)$ the space of functions infinitely derivable on Ω and identically zero outside a closed and limited set strictly contained in Ω.*

Definition 5.4.2. *Let us call $L^1_{loc}(\Omega)$ the space of functions locally integrable in Ω, that is the space of functions $f \in L^1_{loc}(K)$ for every closed and limited set K strictly contained in Ω.*

Definition 5.4.3. *Let $f \in L^1_{loc}(\Omega)$, fixed a multi-index*

$$\alpha = (\alpha_1, \ldots, \alpha_n)$$

*with $\alpha_i \in \boldsymbol{Z}^+$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$, we say that f has **weak derivative** $D^\alpha f$ in Ω if there exists a function $g \in L^1_{loc}(\Omega)$ such that*

$$\int_\Omega f D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega g \varphi dx \quad \forall \varphi \in C_0^\infty(\Omega)$$

where

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}}.$$

In this case we write $g = D^\alpha f$.

Definition 5.4.4. *The **Sobolev space** is defined as*

$$W^{k,p}(\Omega) = \left\{ f \in L^1_{loc}(\Omega) : D^\alpha f \in L^p, 0 \leq \alpha \leq k \right\}.$$

We write $H^k = W^{k,2}$.

The Sobolev space is a Banach space when equipped with the norm

$$\|f\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \max_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty(\Omega)}, & p = \infty \end{cases}$$

With this norm, $H^k(\Omega)$ is an Hilbert space [1] and this further motivates the choice of the minimization of the distance L^2.

Minimization of the distance between curvatures The constrained variational problem (5.2) is not included in the typical variation problems and for this reason it is difficult to find a theoretical solution for it. In fact as already said it is a vectorial problem with second derivatives. However the functional $J(\tilde{\gamma})$ can be studied as $J(\tilde{\gamma}')$, that is by considering $\tilde{\gamma}'$ as a function and $\tilde{\gamma}''$ as its first derivative:

$$J(\tilde{\gamma}') = \int_0^L F(s, \tilde{\gamma}', \tilde{\gamma}'')ds.$$

Unfortunately the functional $J(\tilde{\gamma}')$ is not coercive and this makes complicate to study the existence of the minimizer with the direct method and moreover it is not convex. These are two important restrictions for known solution of variational problems.

[1] If a vectorial space X has an inner product and it is complete with respect to the norm generated by the inner product is called a Hilbert space.

Our problem can be studied at least in three different ways. Let

$$J(x',y') = \int_0^L |-y'x''+x'y''-k|^2 ds$$

with boundary conditions $x'(0) = x'(L)$ and $y'(0) = y'(L)$, and constraints

$$\begin{cases} \int_0^L x'(s)ds = 0 \\ \int_0^L y'(s)ds = 0 \\ x'(s)^2 + y'(s)^2 = 1 \end{cases}$$

This is an over-determined system and there is no known theoretical solutions. To simplify the notation we call

$$a = -y'x'' + x'y'' - k$$

and we solve the minimization problem in the explicit way by using the Euler-Lagrange equation:

$$\frac{\partial L}{\partial \tilde{\gamma}'} - \frac{d}{ds}\frac{\partial L}{\partial \tilde{\gamma}''} = 0$$

$$\begin{cases} 2|a|y''sgn(a) + \lambda_1 + \lambda_3 2x' + \frac{d}{ds}(2|a|sgn(a)y') = 0 \\ \\ -2|a|x''sgn(a) + \lambda_2 + \lambda_3 2y' - \frac{d}{ds}(2|a|sgn(a)x') = 0 \end{cases}$$

$$\begin{cases} 4|a|y''sgn(a) + \lambda_1 + 2\lambda_3 x' + 2a'y'(sgn(a))^2 + 2|a|\delta(a)y' = 0 \\ \\ -2|a|x''sgn(a) + \lambda_2 + 2\lambda_3 y' - 2a'x'(sgn(a))^2 + 2|a|\delta(a)x' = 0 \end{cases}$$

where δ is the Dirac delta function.

We can simplify the problem by changing the vectorial problem into a scalar one, by considering $\tilde{\gamma}'(s) = (x'(s), y'(s)) \in S^1$ since we are assuming that $\tilde{\gamma}$ is arc length parameterized. Then, switching to the complex line, $\tilde{\gamma}'(s) = \tilde{x}'(s) + i\tilde{y}'(s) = e^{i\theta(s)}$. Since we want a curve closed and C^1, we want $\theta(L) = \theta(0) + 2\pi h$, $h \in \mathbf{Z}$. Moreover

$$\begin{cases} \tilde{x}'(s) = \cos(\theta(s)) \\ \\ \tilde{y}'(s) = \sin(\theta(s)) \end{cases}$$

and

$$\begin{cases} \tilde{x}''(s) = -\sin(\theta(s))\theta'(s) \\ \\ \tilde{y}''(s) = \cos(\theta(s))\theta'(s) \end{cases}$$

which implies $\tilde{k}(s) = \theta'(s)$ and then, forgetting the square root we want to minimize the scalar function

$$J(\theta) = \int_0^L |\theta'(s) - k(s)|^2 ds$$

under the constraints

$$\begin{cases} \int_0^L \cos(\theta(s))ds = 0 \\ \int_0^L \sin(\theta(s))ds = 0 \\ \theta(L) = \theta(0) + 2\pi h \end{cases}$$

But again the functional is not coercive and the problem is over-determined so that there is no known solution.

A third way derives from $\|\tilde{\gamma}'\|^2 = 1$. Let us call $\tilde{x}' = \xi(s)$ and $\tilde{y}'(s) = \eta(s)$. If we suppose symmetry with respect to both axes, then

$$\eta = \pm\sqrt{1-\xi^2}$$
$$\eta(s) = -\eta(s+\frac{L}{2}), \quad s \in [0, \frac{L}{2}].$$

By deriving ξ and η we have

- in $[0, \frac{L}{2}]$: $\eta = \sqrt{1-\xi^2}$, $\eta' = -\xi'(1-\xi^2)^{-\frac{1}{2}}$
- in $[\frac{L}{2}, L]$: $\eta = -\sqrt{1-\xi^2}$, $\eta' = \xi'(1-\xi^2)^{-\frac{1}{2}}$

and we can divide the integral in the sum of two equal integrals

$$2\int_0^{L/2} |-\sqrt{1-\xi^2}\xi' - \xi\xi'(1-\xi^2)^{-\frac{1}{2}} - k|^2 ds$$

which has a singularity which further complicates the solution.

Being not easy to find a theoretical result for the general problem, we approach it in two different ways. The first approach is to study the problem only for a particular case (Section 5.5), while the second approach gives an approximated solution for the general problem (Section 5.6).

5.5 Particular example

Having seen in the previous section that the general minimization problem can not be solved easily using common theorems of variational calculus, in this section we try to understand if at least for a particular case we find it, maybe restricting the class of closed curves in which we look for the closest one. We consider then the following problem.

Let be given as open curve an arc of circumference, whose curvature k_c is constant for every parameterization, and look for a closed curve whose curvature is as close as possible to k_c with respect to the metric L^2 (3.11). To further simplify the problem, we do not search in the set of all smooth closed curves but in the restricted class of cubic B-splines, which ensure a good approximation of every smooth curve. The choice of this example is not casual. To define a meaningful distance between curvatures it is necessary to compare them at correct corresponding points. In fact, if we have two generic parameterizations of the same curve defined on the same domain and compute their curvature functions, then the distance between curves, exactly as the distance between curvatures, is not zero (see Example 3.2.5 of the arc of parabola). A way to solve the problem is to use arc length parameterizations which often are difficult to find explicitly. Although B-splines are not parameterized by arc length, we do not have the problem of the correct correspondence between curvature values of the B-spline and of the open curve because this last one is an arc of circumference which has the same curvature values at each point.

Let $\gamma\colon [0, 3\pi] \to \mathbf{R}^2$ an arc of circumference of length 3π and radius 2, and thus curvature $\frac{1}{2}$ at each point. We are looking for a closed cubic B-spline of length 3π whose curvature is as close as possible to $1/2$ in the norm L^2. The intuition suggests that the circumference β, 3π long, of radius $\frac{3}{2}$ and curvature $\frac{2}{3}$, is the expected curve. The best way to prove this assertion would be to give an analytical solution, but we start with a numerical approach to confirm our intuition.

B-splines Let us spend some words to recall the notion of B-spline. A B-spline of order $k+1$ is a piecewise polynomial function S(t) of degree k in a variable t, whose construction requires $n+1$ control points $p_0, \ldots, p_n$ and a knot vector $t_0, \ldots, t_m$, with $m = k + n + 1$. Its equation is

$$S(t) = \sum_{i=0}^{n} p_i B_i^{k+1}(t), \quad t \in [t_k, t_{n+1}] \tag{5.5}$$

whose functions basis B_i^k can be constructed by recursion as

$$B_i^k(t) = B_i^{k-1}(t)\frac{t - t_i}{t_{i+k-1} - t_i} + B_{i+1}^{k-1}(t)\frac{t_{i+k} - t}{t_{i+k} - t_{i+1}}, \quad \text{if} \quad k > 1$$
$$B_i^1(t) = \begin{cases} 1 & \text{if} \qquad t \in [t_i, t_{i+1}) \\ 0 & \text{otherwise.} \end{cases}$$

There is not a unique way to define the knot vector, the only constraint is that its knots must be in ascending order. If the knots are equally spaced (i.e.,

$t_{i+1} - t_i$ is a constant for $i = 0, \ldots, m-2$), the knot vector is said uniform but other possibilities are to use a chord length or centripetal subdivisions. Let us observe that often $t_0 = 0$ and $t_{m-1} = 1$. When all knots are distinct the spline is C^{k-1} on the domain, instead if two knots are coincident, the spline is C^{k-2} and the order decreases by 1 for each additional repeated knot. In the particular case of a closed B-spline the first k control points must be added at the end of the ordered set of control points, and also the knots vector must be increased with k knots more. An useful property of B-splines is that one computes their derivatives by the formula

$$S'(t) = \sum_{i=0}^{n} k \frac{p_i - p_{i-1}}{t_{i+k} - t_i} B_i^k. \tag{5.6}$$

Coming back to our particular case, we construct a closed cubic B-spline by determining four points p_0, p_1, p_2, p_3 (with the idea to increase the number of control points if the experiments do not show good results) such that the ordered set $p_0, \ldots, p_6$ are the control points under the conditions:

$$p_0 = p_4, \quad p_1 = p_5, \quad p_2 = p_6.$$

Moreover, we considered uniform B-spline with knot vector $v = (t_0, \ldots, t_m) = (-k, \ldots, 0, \ldots, n+1) = (-3, \ldots, 7)$. The subset I of $\mathbf{R}$ in which we evaluate the spline is $I = [v(k+1), v(n+2)] = [0, 4]$.

Formally the problem is the following: let $k_c = 0.5$ be the curvature of the open curve γ of length 3π, we look for a spline

$$Sp(t) = (\sum_{i=0}^{6} x_i B_i^4(t), \sum_{i=0}^{6} y_i B_i^4(t)), \qquad t \in I$$

such that its curvature function k_S minimizes the L^2 distance from the curvature k

$$\min_{(x_i, y_i) \in \mathbf{R}^2} \|k_S(t) - k\|_2 \tag{5.7}$$

under the constraints of fixed spline's length (5.8) and closing constraints expressed by the repetition of the first three control points (5.9)

$$\int_I \|Sp'(t)\| \, dt = 3\pi \tag{5.8}$$

$$\begin{array}{ll} x_i = x_{i+4}, & \text{for} \quad i = 0, 1, 2 \\ y_i = y_{i+4}, & \text{for} \quad i = 0, 1, 2 \end{array} \tag{5.9}$$

Remark 5.5.1. *We do not use the constraint of arc length bacause B−spline are not parameterized by arc length but we impose that the length of the curve is equal to the open one.*

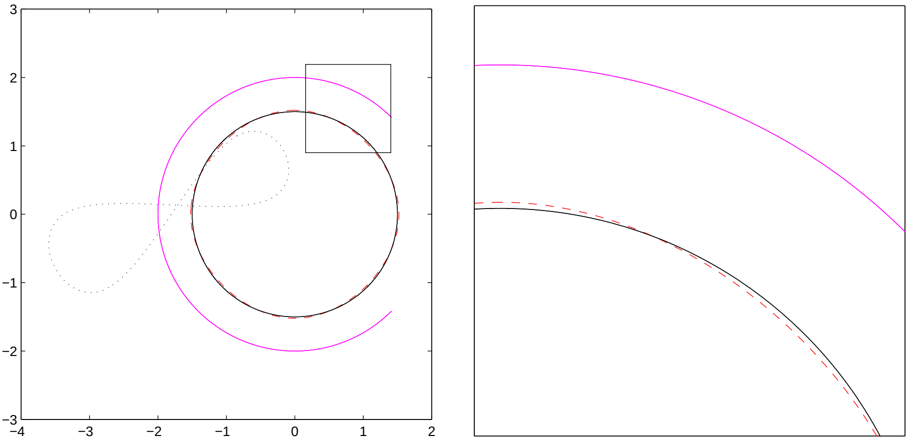


Figure 5.6: Spline with 7 control points. The expected circumference of perimeter 3π is the black line, cases $1, 2, 3$ give the same spline represented by the dashed line and the last case gives the eight function.

Implementation The problem has been implemented in MatLab by considering separately the x-components $\{x_i\}$ and the y-components $\{y_i\}$ of the control points and only at the minimization step they are composed in a unique vector $w = (x_0, \ldots, x_6, y_0, \ldots, y_6)$.

Being a non linear minimization problem, we used the MatLab minimization solver *fmincon*, which finds a local minimum starting from an initial point fixed by the user. We construct three different functions, one is the objective function (5.7), one represents the linear equality constraints (5.9) and one for the non linear constraint (5.8). We approximate the integrals of the objective function and of the nonlinear constraint as sum of rectangles, whose basis are equal to the width of an equally spaced sampling of I.

The function *fmincon* is a local solver and then it requires a start point, that is an initial set of control points. In particular we try six different combinations to see the dependence on this choice. The first three cases consider control points on the x and y axes and fixed to be symmetric with respect to the origin; the points in the first two cases are at the same distance from the origin and they are of the type $\{(x_1, 0), (0, x_1), (-x_1, 0), (0, -x_1)\}$ while in the third case they are of this form $\{(x_1, 0), (0, x_2), (-x_1, 0), (0, -x_2)\}$. The cases four and five consider the vertices of a square centered in the origin $\{(x_1, x_1), (-x_1, x_1), (-x_1, -x_1), (x_1, -x_1)\}$ and the last case the vertices of a rectangle $\{(x_1, x_2), (-x_1, x_2), (-x_1, -x_2), (x_1, -x_2)\}$.

In Figure 5.6 and 5.7 we can notice as, except for the last set of initial control points, the splines are very close to the expected circumference. The distance between the curvature of the open curve and the closed expected circle

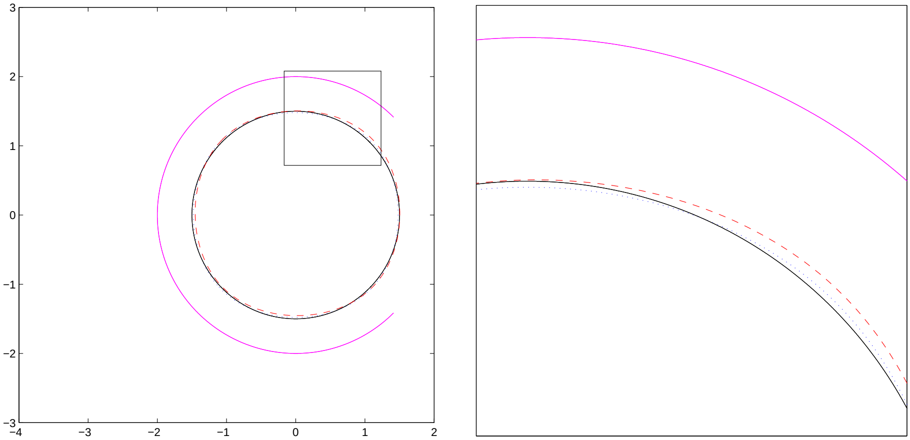


Figure 5.7: Spline with 7 control points. The expected circumference of perimeter 3π is the black line, case 4 is represented by the dotted line while case 5 by the dashed line.

$\mathcal{C}$ of radius 3/2 is

$$\int_0^{3\pi} |k - k_{\mathcal{C}}| ds = \int_0^{3\pi} \left| \frac{1}{2} - \frac{2}{3} \right|^2 ds = \frac{\pi}{12} \simeq 0.2618. \tag{5.10}$$

But if we look at the measure of the distance between the curvature of the splines obtained with the previous method and the curvature of the open curve, it is 0.1554, which seams counterintuitive since it is less then the distance computed with the expected circle. But there is no error because while in (5.10) we are integrating on the domain expressed with respect to the arc length, in the spline's case not. To obtain a more coherent result we would to consider the arc length parameterization of the spline but it is not possible to compute it analytically, so we make an approximation of its arc length by piecewise linear approximation. In this way we can compute correctly the measure of the distance between curvatures, obtaining a value close to 0.3587, bigger then the expected value, also if it approximates very well the circumference.

The situation changes considering 9 control points, that is six different points with the first three repeated. At these six points we give as initial values $\{(0,3),(-2,2),(-2,-2),(0,-3),(2,-2),(2,2)\}$ and, considering at the end of the process the arc length approximation, it happens exactly what we expect: the value of the distance between curvatures is 0.2729 and the figure 5.8 shows that the spline is a perfect approximation of the circumference.

Remark 5.5.2. *It is important to make an observation about the approximation by arc length considered in computing the distance between curvatures. Let N be the number of control points and n the number of uniformly sampled points*

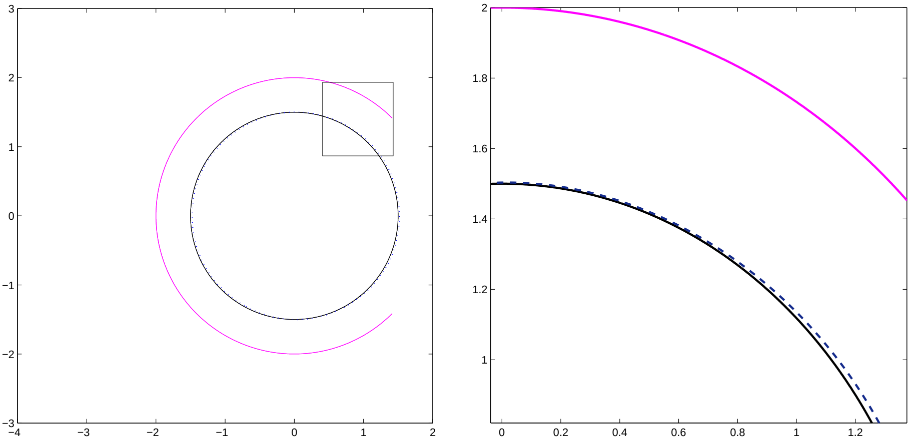


Figure 5.8: Spline with 9 control points. The expected circumference and the spline are almost overlapped.

in the interior of the interval I as defined above, let us say $t_0, \dots, t_{n+1}$. *To each parameter value* t_i *corresponds a parameter value* s_i *in the arc length parameterization and the curvatures computed at corresponding points are equal. But to obtain a distance coherent with the distance between the open curve and the expected circumference* (5.10) *we need to use the arc length parameterization. The numerical curvature values do not change, what is changing is the parameterization of the spline (also if approximated) and then the value of the integral because of the change of extreme of integration. The approximated arc length is done by computing the distance between each two consecutive sampled points on the spline, and it will be more exact as the sampling increase (see Remark 2.4.4).*

Now that we have a clear idea of the shape of the spline which minimizes the curvature, that is a circumference, we try to solve the problem in an analytic way. To construct a cubic B-spline by taking seven control points (4+3) means to have eight unknowns making very difficult the analytical solution. So we simplify the problem by reducing the number of unknowns to only two. This simplification follows from results of numerical experiments. In fact we have seen that the closed splines approximate the circumference, and also varying the control points (obtained from different initial points), all splines are close to each other. Since between them there is also the spline obtained by control points symmetric with respect to the origin, we obtain the simplification by restricting the control points to be of the type $p_0 = (x_1, 0), p_1 = (0, x_2), p_2 = (-x_1, 0), p_3 = (0, -x_2)$.

The analytical solution of the problem by cubic B-spline assures C^2 regularity but it is very hard. To simplify the problem, we look for the solution for quadric B-splines, which assure only piecewise C^2 continuity. All we said until now

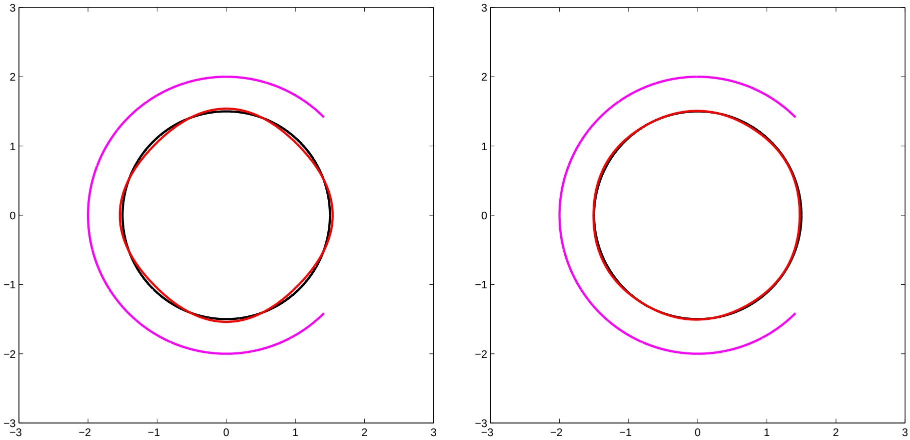


Figure 5.9: Quadric spline with 6 control points (left) and with 8 control points (right).

holds also for quadric spline, but obviously taking only six control points the shape of the spline is more *stretched*, but with 8 the result is very nice (see Figure 5.9).

Theoretical solution for quadric B-spline Let us consider a quadric B-spline defined by six control points

$$S(t) = \sum_{i=0}^{5} p_i B_i^3 \quad [t_2, t_5] = [0, 3]$$

under the constraints

$$\begin{cases} p_0 = p_4 \\ p_1 = p_5 \\ \int_{t_2}^{t_5} \sqrt{S'(t) \cdot S'(t)} dt = 3\pi. \end{cases}$$

and the simplification (justified above) of $p_0 = (x, 0)$, $p_1 = (0, y)$, $p_2 = (-x, 0)$, $p_3 = (0, -y)$. By using definition (5.6) until the order 1, we have

$$S'(t) = \begin{pmatrix} l - xB_2^1 + (2t-3)xB_3^1 + xB_4^1 + (4-t)xB_5^1 \\ (1-2t)yB_2^1 - yB_3^1 + (2t-5)yB_4^1 + (4-t)yB_5^1 \end{pmatrix}$$

and the integral of the constraint can be divided in the sum

$$\int_{t_2}^{t_5} \|S'(t)\| dt = \sum_{i=2}^{4} \int_{t_i}^{t_{i+1}} \|S'(t)\| dt = 3\pi$$

whose result is

$$\frac{3}{2}\sqrt{x^2+y^2}+\frac{x^2}{4y}\log\left(\frac{y+\sqrt{x^2+y^2}}{x}\right)+\frac{y^2}{8x}\log\left(\frac{x+\sqrt{x^2+y^2}}{y}\right)=3\pi.$$

Theoretically it can be solved by the implicit function theorem [30] but not explicitly unfortunately. The explicit expression would permit us to express one variable as function of the other one and to substitute it in the objective function

$$\int_{t_2}^{t_5} |k(t)-3/2|^2 dt$$

where

$$k(t)=\frac{2xy(B_2^1+B_3^1+B_4^1)+2xyB_5^1(t-4)}{[D]^{3/2}}$$

and

$$\begin{aligned} D = \ & \left(x^2+(1-2t)^2y^2\right)B_2^1+\left(y^2+(2t-3)^2x^2\right)B_3^1 \\ & +\left(x^2+(2t-5)^2\right)B_4^1+(4-t)^2(x^2+y^2)B_5^1 \end{aligned}$$

5.6 Approximated solution

In the previous Section 5.4 we pointed out that the minimization problems (5.2) or (5.3) are difficult to solve in the continuous setting and there is no relevance of such a proof in literature. Moreover, the minimization problem is expressed with respect to the arc length parameterizations, but in practice it is difficult to have their analytical expressions. To overcome this last problem we have two possibilities. One is to approximate curves by arc splines ([19, 25]) but we stay in the smooth setting where we are not able to find an explicit solution. The second one, which is the more promising and convenient, is to leave the smooth setting to work in the discrete one, by approximating curves by piecewise linear curves (see Remark 2.4.4). Hence, we propose to solve the problem in practice by an approximate solution.

In particular, in this section we study as can be given an approximated solution to the problem (5.2), that is the minimization of the L^2 distance between curvatures. We prefer to solve with respect to the L^2 norm rather then with respect to the L^∞ norm (also if both distances can be used to prove that close curvatures imply close curves, Theorems 3.3.1 and 3.3.2) because the discrete L^2 norm gives rise to convex problems, easier to solve, which is not true for the maximum norm.

Let $\gamma\colon [a,b]\to\mathbf{R}^2$ be a parametric curve, it is sampled at $n+1$ uniformly distributed parameter values $u_i=a+(b-a)i/n$, obtaining the polygon $P=[p_0,\dots,p_n]$ with points $p_i=\gamma(u_i)$ for $i=0,\dots,n$ (Figure 2.7). If the curve γ is closed, then so is P with $p_0 = p_n$. This polygonal curve P can be parameterized by reduced arc length as in (2.2) obtaining the piecewise linear curve $\hat{\gamma}$ (see remark 2.4.4).

Let us consider now two smooth curves γ_0 and γ_1 with generic parameterizations, approximated by polygonal curves P_0 and P_1 and scaled to have unit length, and construct their arc length parameterizations $\hat{\gamma}_0$ and $\hat{\gamma}_1$. Let us denote by $\sigma_0 = \{s_0, \dots, s_n\}$ and $\sigma_1 = \{t_0, \dots, t_n\}$ the partitions of the interval $[0,1]$ corresponding to $\hat{\gamma}_0$ and $\hat{\gamma}_1$ respectively. To have a perfect correspondence between points of the two curves, we consider the joint partition $\sigma = \sigma_0 \cup \sigma_1$ and refine both curves by adding points on edges which correspond to the parameters of the other curve. For example, if $t_i \in \sigma_1$ but $t_i \notin \sigma_0$, then there exists an index $j \in \{1, \dots, n\}$ such that $t_i \in [s_{j-1}, s_j]$ and we add on the edge e_i of $\hat{\gamma}_0$ the point $\hat{\gamma}_0(t_i)$. The refined polygons $\hat{P}_0 = [p_0^0, \dots, p_m^0]$ and $\hat{P}_1 = [p_0^1, \dots, p_m^1]$ have the same number of points and the same edge lengths $e_i = \|p_i^0 - p_{i-1}^0\| = \|p_i^1 - p_{i-1}^1\|$, for $i = 1, \dots, m$. Let us observe that the joint partition can be applied also to curves with different number of vertices.

To solve the minimization problems (5.2), we discretize the curves γ_0 and γ_1 as above and compute their curvature functions $\hat{k}$ as defined in (2.16), obtaining the interpolated signed curvature function

$$\tilde{k}_t = (1-t)\tilde{k}_0 + t\tilde{k}_1$$

It is important to specify that $\hat{k}_0$ and $\hat{k}_1$ are not computed on vertices after refinement since in this case we could find unexpected results dues to the zero values on vertices along edges. Instead we compute the values on original vertices while at the points added after refinement we assign the value of the interpolated curvature along the edge.

The corresponding piecewise linear curve which we reconstruct from $\hat{k}_t$ as explained in the last paragraph of Section 4.1 in general is not closed. So we want to change $\hat{k}_t$ with another curvature $\tilde{k}_t$ which is as close as possible to $\hat{k}_t$ and such that the corresponding reconstructed curve is closed. This means that we are looking for values $\epsilon_0, \dots, \epsilon_m$ such that $\tilde{k}_t(s_i) = \hat{k}_t(s_i) + \epsilon_i$ such that the curve is closed and minimize the distance

$$\|\hat{k}_t - \tilde{k}_t\|_2 = \left(\sum_{i=1}^{m} \epsilon_i^2\right)^{1/2}$$

The curvature $\hat{k}_t$ is a function of edge lengths and exterior angles, and we need to decide on what we want to focus our attention during the minimization process. In fact we can work on both of them but, if we want to preserve the length of the polygon and then the common arc length parameterization on σ_t, an obvious solution is to fix edge lengths and to change only angles.

We are then looking for a closed intermediate polygon $\hat{P}_t = [p_0^t, \ldots, p_m^t]$ with edge lengths $\|p_i^t - p_{i-1}^t\| = l_i$ for $i = 1, \ldots, m$ and exterior angles which are as-close-as-possible to the target values $\hat{\alpha}_i^t$ for $i = 1, \ldots, m$, of the open curve.

Let us observe that if we use $\hat{\alpha}_i^t = (1-t)\hat{\alpha}_i^0 + t\hat{\alpha}_i^1$, we can introduce unexpected artifacts in the result since about half of the exterior angles $\hat{\alpha}_i^0$ and $\hat{\alpha}_i^1$ at the vertices of $\hat{P}_0$ and $\hat{P}_1$ are zero, exactly as we specified for the curvatures $\hat{k}_0$ and $\hat{k}_1$. This is the reason for which we reconstruct the angles α_i^t from $\hat{k}_t$. Instead, we use the linearly interpolated signed curvature function $\hat{k}_t$ and by knowing the edge lengths, we find the angles values by the curvature formula 2.16.

In order to close the polygon we have to ensure that $p_0^t = p_m^t$. A condition to assure the closing of the curve is (2.3). Unfortunately, our decision to fix edges and change angles leads to the non-linear optimization problem

$$\min_{\tilde{\alpha}^t} \sum_{i=1}^{m} \left\| \frac{2}{l_i + l_{i+1}} (\tilde{\alpha}_i^t - \hat{\alpha}_i^t) \right\|^2,$$

subject to closure conditions (2.3)

$$\begin{cases} \sum_{k=1}^{m} l_i \cos(\theta_{i-1}) = 0 \\ \\ \sum_{k=1}^{m} l_i \sin(\theta_{i-1}) = 0 \end{cases}$$

where $\theta_i = \theta_{i-1} + \tilde{\alpha}_i$, which requires the help of a Matlab solver.

Concluded the minimization process, to obtain a smooth closed intermediate curve, we fit a closed B-spline curve in the least squares sense to the vertices of the polygon $\hat{P}_t$. For this fitting procedure, we use the nodes of σ_t as initial parameter values, but perform several iterations of parameter optimization [18] to improve the result.

5.6.1 Algorithm for minimization

Let us see now in more detail the most important steps of the algorithm.

1. For the first step we have two possibilities. Or we give the parameterizations of two curves, or two polygons P_0 and P_1 with respectively l and m vertices and construct two closed cubic B-splines S_1 and S_2 which have these polygons as control points and uniform knot vector such that the splines will be evaluated in the interval $[0, 1]$.

2. In both cases the next step is the discretization of the curves, that is for each curve we sample uniformly the parameterization domain and create the polygon by connecting the corresponding points of the curve. Then

we compute the edge lengths l_i of the polygon and we create the piecewise linear parameterization over the partition σ of $[0,1]$ as

$$s_j = \sum_{i=1}^{j} \frac{l_i}{L}$$

where L is the length of the polygon (the sum of edges).

3. The two curvature functions are computed as the discrete curvatures at the sampled points and linearly interpolated along edges to construct the piecewise linear curvature functions $\hat{k}_0$ and $\hat{k}_1$.

4. We compute the joint partition $\sigma_t = (s_1, \ldots, s_r)$, add the new points on edges of the curves and compute the value of the curvature in that points as the value of the $\hat{k}_i(s_j)$, $i = 0, 1$. The intermediate curves are obtained by the linear interpolation $\hat{k}_t$, since the knowledge of edge lengths implies uniquely determined angles α_i^t as explained above.

5. Minimization step. We are looking for a closed curve with edges length equal to the open one and with curvature as close as possible to the curvature of the open curve. But since the edge lengths are fixed, the minimization involves only angles and then we look for exterior angles as close as possible to the angles of the open curve. The minimization is made by the help of the MatLab solver *fmincon*, which is a local solver for non linear problem (in our case due to the closure conditions.)

6. To see the smooth intermediate curve we fit a cubic B-spline in the least square sense to the vertices of the closed polygon.

5.6.2 Results

Figure 5.10 shows the result of the optimization process for two cosine functions in polar coordinates with different periods for different t. For this example we need 500 samples in order to avoid approximation errors on the input curves and 100 control points on the B-spline to capture the oscillation during the fitting.

Two more extreme examples are given by the interpolation between two arc splines, the H-shape and the cross, and the two curves taken as example at the beginning of the Chapter 5.4. Even if this problem seems simpler because of the fewer oscillations of the boundary with respect to the previous example, the curvatures of the two curves are piecewise constant. Nevertheless our method successfully interpolates between the two curves, as shown in Figure 5.11 and Figure 5.13.

We use this extreme example to show how the curvature of the obtained closed curve approaches the target curvature depending on the choice of the

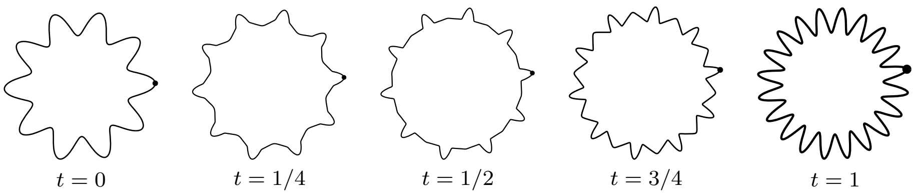


Figure 5.10: Spline of degree 5 with 100 control points and using 500 samples obtained by the curvature optimization process.

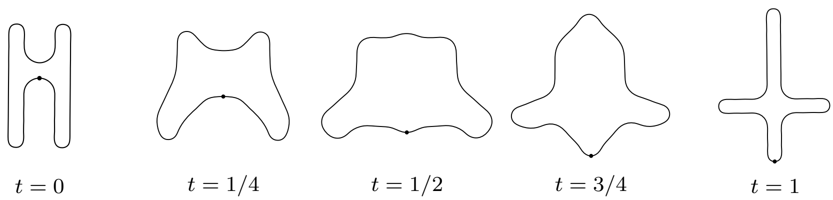


Figure 5.11: Spline of degree 5 with 50 control points and using 500 samples obtained by the curvature optimization process.

parameters (*i.e*, the degree of the spline, the number of control points and the number of samples). In Figure 5.12 we show in blue the piecewise linear curvature obtained at the halfway between the H-shaped curve and the cross ($t = 0.5$), and in red the curvature obtained by the closing process. We can notice that, by increasing the number of samples used to construct the initial polygons P_0 and P_1 and the number of control points, the curvature convergence improves even if the target curvature is discontinuous. The plot in the bottom right angle is a demonstration of this assumption.

5.6.3 Comparison with other methods

Let us do now some comparison with other methods and to do that let us consider $B-splines$ as input curves, instead of generic parametric curves, with the same number of control points and the same degree. In [36] the authors propose, as alternative method to [32], to do not consider the polygon's vertices but the vertices of the control points, imposing on them some conditions to ensure G^2 continuity of the curvature. The use of the control polygon can lead to different approaches to generate the intermediate curves, for example by linear interpolating the control polygons or by employing Sederberg [32] idea.

Figure 5.14 and Figure 5.15 show a comparison of the different methods. The topmost row shows the splines generated via a linear interpolation of the control points; the second the splines obtained applying a Sederberg interpolation to the control points; the third row the splines obtained by linear interpolation of the curvature and by applying the *closing* procedure as proposed in [36]; the

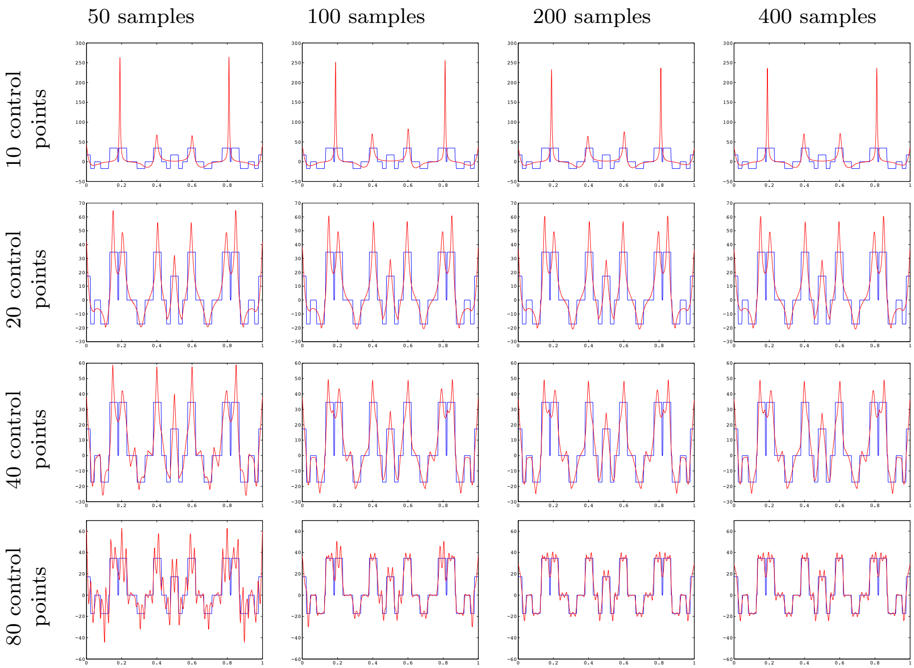


Figure 5.12: Effect of changing the number of sample points used to create the initial polygons and the number of control points of a spline of degree 3.

last one the splines which least square fit the closed polygon (our approach).

The main problem with all these approaches is that they do not guarantee to avoid self intersection of the polygon, which introduces undesired loops in the curves. These examples show that it is not clear which deformation method gives the best visual result. In Figure5.16 we show the difference (the error) between the piecewise linear interpolated curvature (at $t = 1/2$) and the corresponding signed curvature of the closed curve obtained by the four methods. As expected our method is the one with the minimum error.

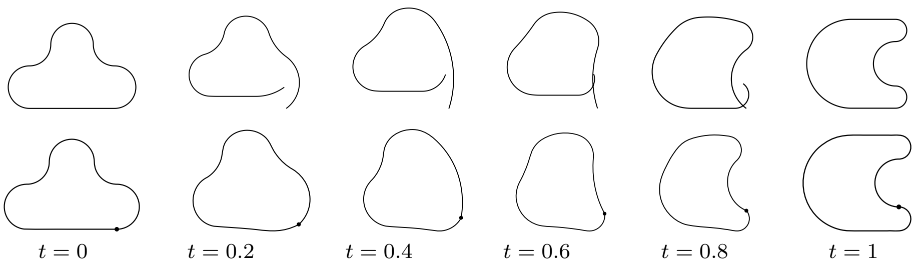


Figure 5.13: The top row shows the reconstructed curves corresponding to the intermediate parameter values $t = 0.2, 0.4, 0.6, 0.8$ from the interpolation of the curvatures of source (left) and target (right) curves, while the bottom row shows the reconstructed curves after the minimization process.

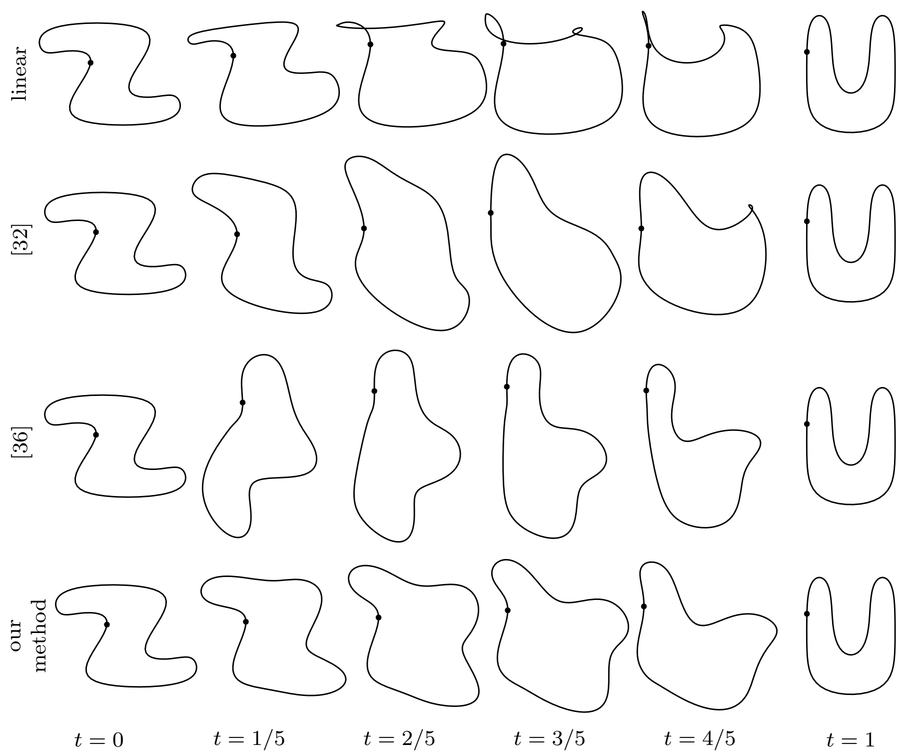


Figure 5.14: Splines of degree 5 with 10 control points and using 100 samples obtained applying the four methods to the same source and target curves.

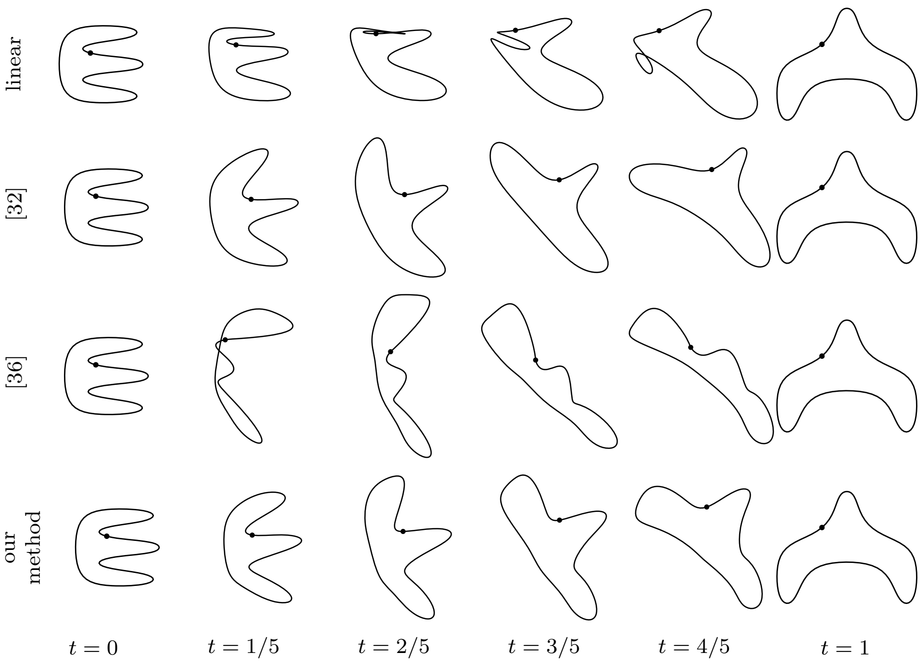
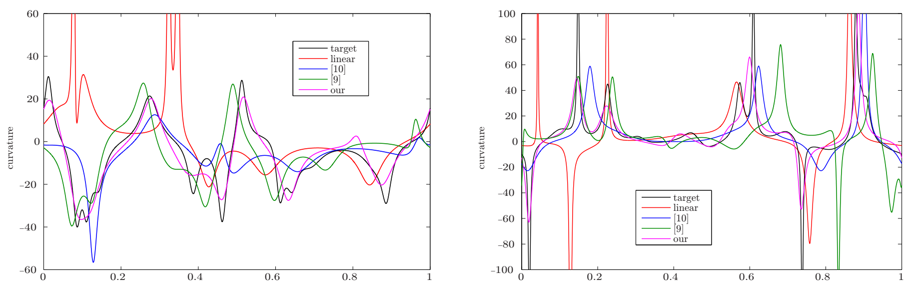


Figure 5.15: Splines of degree 5 with 16 control points and using 200 samples obtained applying the four methods to the same source and target curves.



Curvature of Figure 5.14 Curvature of Figure 5.15

Figure 5.16: A comparison of the curvature before and after the closing process for the curves of Figure 5.15 and Figure 5.14, with the four methods. In the plots the color coding is: black for target, red for linear, blue for [32], green for [36], and magenta for our method.

Chapter 6

Conclusions

We presented a new method to interpolate smooth closed curves in curvature space, using an appropriate discretization of this last one. Since the interpolated curves can be open, we introduce a new distance to assure that any interpolated curve can be approximated with respect to this distance by the closest closed piecewise linear curve. We then fit a spline in a least square sense to the sampled points of the piecewise linear curve, to obtain the final smooth result. Our method lets the user choose the desired degree of approximation via three parameters: the number of samples on the curve, the degree of the fitting spline, and the number of its control points.

An application of this method can be envisaged in writing filters for vectorial drawing packages which let the user construct an as-smooth-as-possible interpolation of two or more generic selected curves.

6.1 Limitations

When the two source and target curves are simple and closed, it is desirable to have all the interpolants both closed and simple, too. Our method always generates closed intermediate curves, but it is not able to assure the absence of self-intersections. For this it is necessary to consider the curve as the boundary of a shape; see Section 6.2.

It is hard to tell what should be the correct behaviour of a method when blending two curves with different winding numbers. This is challenging for testing our approach of wanting to be as more compliant as possible with the "natural" deformation of the curves, since it involves one or more foldings or unfoldings. We experimented by blending a circle into an "8-shaped" curve. The results are reported in Figure 6.1. One can see how all the methods, included ours, have a rather unpredictable behaviour.

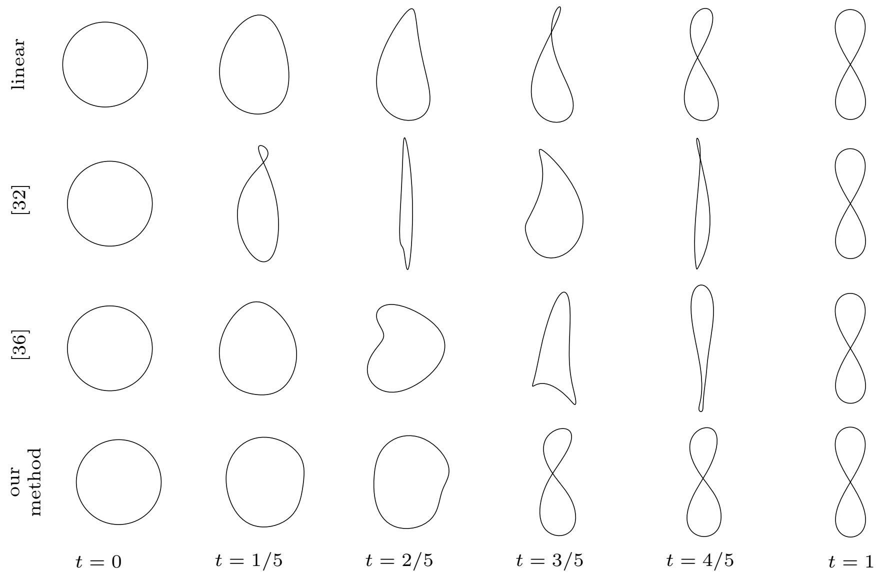


Figure 6.1: Comparison of the methods when interpolating two curves with different winding numbers. In this case, γ_0 (left) has winding number 1 and γ_1 (right) has winding number 0.

We believe that *shape matching* will further improve our results, but it is beyond the scope of this work to properly address this issue, which is a major problem in computer vision and pattern recognition [5, 39].

6.2 Future work

One promising evolution of our work is to apply our method to topologically similar shapes, using their topological skeleton, thus not being limited to examining the curve, but taking also into account its orientation (i.e., what is inside or outside the curve) to define which shape it bounds. This will extend our method from curves to shapes. If we want to blend two shapes having the same topology (i.e., that have the same graphs of the skeleton), it is possible to use information contained in the skeleton [35] to avoid self-intersections, and, more generally speaking, to keep track of the shape while blending them.

An interesting suggestion can be found in [36] where, they observed that the curvature is a local property and then the interpolation can leads to not completely pleasant intermediate shapes. To solve this drawback they subdivide the source and target shapes into the same number of pieces and interpolate linearly each corresponding pair, probably reconnecting them with the condition on control points.

It could be a good idea to see if the definition of curvature based on coordinate functions derivation (2.13) can be a valid alternative to the curvature based on rotation angle (2.4) and in that case we can try to implement the algorithm by substituting the curvature definition on vertices.

We defined in Chapter 2 a new curvature definition and we show that numerically it converges quadratically to the smooth arc length curvature. It would be interesting to show also the theoretical quadratic convergence.

It would be interesting to discretize Theorems 3.3.1 or 3.3.2 for polygonal curves.

6.3 Open Problems

In Chapter 3 we prove the theorem which states that if the distance L^∞ or L^2 between two curvatures expressed with respect to the arc length is small, then also the distance between the corresponding curves (w.r.t. the same metric) is small, under the condition that the curves have the same starting point and the same starting tangent vector. If we change metric to compute the distance between curves and we use the metric which makes the space of $C^2(I)$ functions a Banach space, then we can prove that close curves imply close curvatures. Unfortunately we have the strong constraint of arc length parameterization, which is perfect in theory but seldom explicitly available in practice. To overcome this drawback we tried to prove the same theorem for general parameterization with the help of Frechèt metric. Unfortunately the Frechet metric applied to the curvatures does not work and it is not possible to apply it to corresponding curves since if the curvatures are not expressed with respect to the arc length it is not possible to reconstruct the curves.

In Chapter 4 we saw that if are given the curvature values on vertices but we do not have any information about edges, it seems that it is not possible to deduce if they are eligible values to be the curvatures of a closed curve. We tried to find a formal solution for triangles but at the moment we do not have a proof which shows that the curve exists nor that the curve does not exist. It seems that it does not exist a closed-form solution for the system which would reconstruct a closed curve.

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