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**Global normal forms and global properties in function
spaces for second order Shubin type operators**

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Presentata da	Giorgia Tranquilli
Coordinatore Dottorato	Prof. Giuseppe Rodriguez
Relatore	Prof. Todor V. Gramchev

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Abstract

We investigate the reduction to global normal forms of second order Shubin (or Γ) type differential operators $P(x, D)$ in functional spaces on \mathbb{R}^n . We describe the isomorphism properties of normal form transformations, introduced by L. Hörmander for the study of affine symplectic transformations acting on pseudodifferential operators, in spaces like the Schwartz class, the weighted Shubin-Sobolev spaces and the Gelfand-Shilov spaces. We prove that the operator $P(x, D)$ and the normal form $P_{NF}(x, D)$ have the same regularity/solvability and spectral properties. We also study the stability of global properties of the normal forms under perturbations by zero order Shubin type pseudodifferential operators and, more generally, by operators acting on $\mathcal{S}(\mathbb{R}^n)$ and admitting discrete representations. Finally, we study Cauchy problems on \mathbb{R}^n globally in time for second order hyperbolic equations $\partial_t^2 + P(x, D) + R(x, D)$, where $P(x, D)$ is a second order self-adjoint globally elliptic Shubin pseudodifferential operator and $R(x, D)$ is a first order pseudodifferential operator.

Declaration

I declare that to the best of my knowledge the contents of this thesis are original and my work except where indicated otherwise.

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Introduction

The main goal of the Ph.D. thesis is to address the issue of reduction to global normal forms of operators of the type

$$-\Delta + \langle \mathcal{A}(x), D_x \rangle + \mathcal{B}(x), \quad x \in \mathbb{R}^n,$$

where $\mathcal{A}(x) = \{\mathcal{A}_{j,\ell}(x)\}_{j,\ell=1}^n$ (respectively, $\mathcal{B}(x) = \{\mathcal{B}_{j,\ell}(x)\}_{j,\ell=1}^n$) is a matrix with entries $\mathcal{A}_{j,\ell}(x)$ (respectively, $\mathcal{B}_{j,\ell}(x)$) being real or complex polynomials of degree k (respectively, $2k$). In particular, if $k = 1$, we recapture the second order Shubin type differential operators of the type

$$P(x, D) = -\Delta + \langle Ax, D_x \rangle + \langle Bx, x \rangle + \langle M, D_x \rangle + \langle N, x \rangle + r, x \in \mathbb{R}^n,$$

where A, B are real or complex matrices, $M, N \in \mathbb{C}^n$, $r \in \mathbb{C}$, $\langle \xi, \eta \rangle = \sum_{j=1}^n \xi_j \eta_j$, via conjugation with a normal form transformation (NFT) E

$$E^{-1} \circ P(x, D) \circ E = P_{NF}(x, D),$$

In the case of symmetric Shubin type operators we classify the NFT given by unitary maps $E : L^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$, defined as composition of multiplication with $e^{i(\langle Qx, x \rangle + \langle \alpha, x \rangle)}$ (where $Q \in M_n(\mathbb{R})$, $Q^t = Q$, $\alpha \in \mathbb{R}^n$), translations $T_\beta u(x) = u(x + \beta)$, the action of the orthogonal group $SO(\mathbb{R}^n)$ and global Fourier integral operators with quadratic phase function $\phi(x, \eta)$ generating linear symplectic transformation in \mathbb{R}^{2n}

$$Jv(x) = \int_{\mathbb{R}^{2n}} e^{i\phi(x, \xi) - iy\xi} v(y) dy \bar{d}\xi.$$

Such types of global maps have been introduced by Hörmander in the context of the theory of the pseudodifferential operators on \mathbb{R}^n (cf. [37], Chapter 18, pp. 157-159), see also [23]. In order to demonstrate that the original operator $P(x, D)$ and the normal form operator $P_{NF}(x, D)$ have the same regularity/solvability and spectral properties, we have to investigate isomorphism properties of such type of NFT between function spaces on \mathbb{R}^n , like the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, the weighted Shubin-Sobolev spaces $\mathcal{Q}^s(\mathbb{R}^n)$, $s \in \mathbb{R}^n$, the Gelfand-Shilov spaces $S_v^\mu(\mathbb{R}^n)$, $\mu, \nu > 0$, $\mu + \nu \geq 1$. We are also interested in the study of the stability of the properties of the normal forms under perturbations

$$P(x, D) + b(x, D)$$

with zero order Shubin(Γ -) pseudodifferential operators and we investigate the corresponding reduction

$$E^{-1} \circ b(x, D) \circ E = b_{NF}(x, D).$$

We mention as another source of motivation an approach, based on using different type of global normal forms of evolution PDEs on \mathbb{R}^n for deriving global estimates in weighted spaces on \mathbb{R}^n , proposed by M. Ruzhansky and M. Sugimoto [54]. We introduce discrete representations for the action of Shubin type pseudo.differential operators based on eigenfunction expansions associated to self-adjoint Shubin differential operators. We mention that in a recent book Ruzhansky and Turunen [55] have proposed a series of fundamental new results on pseudodifferential operators on compact Lie groups and homogeneous spaces based on discrete representations, see also [17], [56] and the references therein.

We also investigate and classify completely self-adjoint anisotropic Γ el-

liptic operators, which are given for $n = 1$ by

$$D_x^2 + \left(\sum_{j=0}^k a_j x^{k-j} \right) D_x + \sum_{\ell=0}^{2k} b_\ell x^{2k-j}, \quad k \geq 2, x \in \mathbb{R},$$

with NFT containing term of the type $e^{iax^{k+1}}$, $a \in \mathbb{R}$ if $a_0 \in \mathbb{R}$. However, in contrast to the case of the Shubin type operators, we are not able to consider perturbation with p.d.o. since one has to study FIO with phase functions having at least cubic growth in x ($e^{ix^{k+1}}$, $k \geq 2$).

Clearly we are also inspired, at least indirectly, by ideas coming from the celebrated Poincaré normal form theory, the Hamiltonian dynamics, the integrability problems, semi-classical analysis. In fact, if we write the symbol of the perturbation of the modified multidimensional harmonic oscillator

$$\sum_{j=1}^n \omega_j (\xi_j^2 + x_j^2) + p(x, \xi)$$

and consider the function above locally near the origin $(0,0)$, we find in the literature an impressive list of fundamental results, we cite [11], [48], [2], [47] and the references therein. However, we are interested more in global properties in the context of the theory of Shubin type p.d.o. on \mathbb{R}^n which lead to different notion of normal form.

Next, we consider perturbations of some non self-adjoint Γ -elliptic operators modeled on 1D complex harmonic oscillators $D_x^2 + \omega x^2$, $\omega \in \mathbb{C}$, $\Re \omega > 0$, $\Im \omega \neq 0$. Classes of such operators have been studied in the framework of the theory of the spectral properties of non self-adjoint operator, for more details (see [14],[15], [43]). Here we need to use NFT containing non unitary maps of the type $e^{i(\langle Qx, x \rangle + i \langle \alpha, x \rangle)}$ where $\Im Q \neq 0$ or $\Im \alpha \neq 0$. Such maps are not defined on $L^2(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ and if $\Im Q = 0$ and $\Im \alpha \neq 0$, the maps preserve the Gelfand-Shilov spaces $\mathcal{S}_\mu^\mu(\mathbb{R}^n)$ for $1/2 \leq \mu < 1$ while for $\Im Q \neq 0$ they do not act even on $\mathcal{S}_{1/2}^{1/2}(\mathbb{R}^n)$ but on some scales of Banach spaces in $\mathcal{S}_{1/2}^{1/2}(\mathbb{R}^n)$.

We mention that such maps belong to Bargmann-Fock type spaces of entire functions, which play an important role in the realm of the time-frequency analysis and Toeplitz operators, cf. see the recent work of Gröchenig and Toft [31] and Toft [63], and the references therein. The main novelty of Chapter 3 is the detailed study of reduction to normal forms of multidimensional complex anisotropic Γ differential operators under suitable symmetry and separation of variables type conditions. Finally, we investigate perturbations with Shubin operators having symbols in $S_{1/2}^{1/2}(\mathbb{R}^n)$. We mention that pseudodifferential operators with $S_{1/2}^{1/2}(\mathbb{R}^n)$ type symbols have been studied in different context cf. [63] and [64]. As a consequence of our investigations we are able to present new results on spectral properties of $P(x, D)$.

We also investigate operators with real non negative principal symbol and non empty (Shubin type) characteristic set

$$\Sigma_P = \{(x, \xi) \in \mathbb{R}^{2n} \setminus \mathbf{0}; p_2(x, \xi) := \|\xi\|^2 + \langle Ax, \xi \rangle + \langle Bx, x \rangle = 0\} \neq \emptyset, \quad \|\xi\|^2 = \langle \xi, \xi \rangle.$$

We have two cases: first, the matrix A is symmetric. In that case we are able to reduce P via unitary NFT to

$$P_{NF} = -\Delta u + i \langle M, D_x \rangle + (\rho + i\sigma)x_1 + i\tau x_2, \quad M \in \mathbb{R}^n, \rho, \sigma, \tau \in \mathbb{R}.$$

In the case $\tau = 0$, we obtain Airy type normal form and we are able to describe completely the hypoellipticity in $\mathcal{S}(\mathbb{R}^n)$. We observe that the symbol of the Airy normal form is not hypoelliptic symbol if $n \geq 2$. It turns out that for $n \geq 2$ the usual functional frame of Shubin spaces $\mathcal{Q}^s(\mathbb{R}^n)$ is not good for the study of P_{NF} . We outline the notion of new type of spaces suitable for the Airy type normal form operators and we derive new anisotropic type subelliptic estimates in such spaces. In the case $\tau\rho \neq 0$ we are not able to study the global regularity or solvability on \mathbb{R}^n as we encounter serious difficulties for deriving hypoellipticity-solvability results on \mathbb{R}^n , since the Fourier transform

of the normal form becomes a polynomial perturbation of $\partial_{\bar{z}_1}$ in $\mathbb{C} \times \mathbb{R}^{n-2}$, $z_1 = x_1 + ix_2$.

The second case is when A admits nonzero skew-symmetric part. In that case, we are able, under suitable sharp conditions, to reduce to normal forms given by multidimensional twisted Laplacian type operators

$$-\Delta + \sum_{j=1}^n \left(\tau_j (x_{2j} D_{x_{2j-1}} - x_{2j-1} D_{x_{2j}}) + \frac{\tau_j^2}{4} (x_{2j-1}^2 + x_{2j}^2) \right) + r, \quad \tau_j \in \mathbb{R}, r \in \mathbb{R}.$$

and to show new results on regularity and solvability, generalizing previous results of Dasgupta, Wong [13], Gramchev, Pilipovic and Rodino [23].

Finally, we investigate the Cauchy problem for the second order hyperbolic operator $D_t^2 + P(x, D_x) + R(x, D)$ globally on \mathbb{R}^n , where P is a self-adjoint globally elliptic or twisted Laplacian type Shubin operator and R is a first order pseudodifferential operator. We derive new results on global in time well posedness of the Cauchy problem in functional frame containing the Schwartz class, the weighted Shubin spaces and the Gelfand–Shilov spaces.

The thesis is organized as follows: Chapter 1 is dedicated to some preliminaries on Γ p.d.o. It follows the exposition in the book of Nicola and Rodino [50].

Chapter 2 is concerned with second order self-adjoint differential operators. We recall that Sjöstrand [59] studied in details second order differential operators of Shubin type (see also [38] and [4] and the references therein). We start by proposing some refinements, namely, we derive a necessary and sufficient condition for the reduction of $P(x, D)$ to harmonic oscillators by conjugations defined by multiplication with quadratic oscillation type maps and orthogonal transformations. More precisely, we show that there are two different patterns of behaviour, namely, the classification depends on whether the matrix $A = \{a_{jk}\}_{j,k=1}^n$ is symmetric, or stated in an equivalent

way, whether the linear differential form

$$\langle Ax, dx \rangle = \sum_{j,k} a_{jk} x_j dx_k \quad \text{is closed.}$$

Set $A = A_{\text{symm}} + A_{\text{skew}}$, $A_{\text{symm}} = A_{\text{symm}}^T$, $A_{\text{skew}} = -A_{\text{skew}}^T$. If $A_{\text{skew}} = 0$ we show that all globally elliptic operator are reduced to

$$H_{\omega} = -\Delta + \sum_{j=1}^n \omega_j^2 x_j^2 + r, \quad \text{for some } \omega_j > 0, j = 1, \dots, n, r \in \mathbb{R},$$

using the NFT of the type

$$U_{\nu}(x) = e^{-i\frac{1}{4}(\langle Qx, x \rangle + \langle \alpha, x \rangle)} \nu(S_0 x),$$

and U is an automorphism of each of the following spaces $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{Q}^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, $S_{\nu}^{\mu}(\mathbb{R}^n)$, $\mu \geq \nu$. Moreover, U preserves the Shubin type operators, namely $b(x, D) \in OPI^m$ iff $U^* \circ b(x, D) \circ U \in OPI^m$.

We also propose, motivated by the paper of Greendfield and Wallach on global hypoellipticity of commuting differential operators on compact Riemannian manifold cf [30], new discrete representation of the action of $b(x, D)$ provided $b(x, D)$ commutes with P and the eigenvalues of P are simple, which is equivalent to the non resonance condition

$$\omega_1, \dots, \omega_n \text{ are linearly independent over } \mathbb{Q}.$$

First, we derive easily the complete description of the centralizer $P(x, D)$ in the algebra of the Γ pseudodifferential operators. However, replacing the commutativity with less restrictive and more natural condition in the the pseudodifferential operator theory

$$[P(x, D), b(x, D)] = R(x, D) \in OPI^{-\infty}(\mathbb{R}^n),$$

$OPI^{-\infty}(\mathbb{R}^n)$ being the space of regularizing Γ operators, small divisors type problems arise and we have to impose Diophantine conditions on $\omega = (\omega_1, \dots, \omega_n)$,

namely, there exist $C > 0$, $\tau \geq 0$, such that

$$|\langle \omega, \alpha \rangle| \geq \frac{C}{|\alpha|^\tau}, \quad \alpha \in \mathbb{Z}^n \setminus \mathbf{0},$$

in order to demonstrate the decomposition

$$b(x, D) = b_{comm, P}(x, D) + b_\infty(x, D), \quad \text{where } [P, b_{comm, P}] = 0, \quad b_\infty(x, D) \in OPG^{-\infty}.$$

In the third chapter, we address the reduction to normal form for complex second order Shubin type operators. We prove, using complex WKB methods and the theory of complex ordinary differential equations, that one can find a NFT U with quadratic exponential growth reducing P to the complex harmonic oscillator

$$U^{-1} \circ P \circ U \circ e = D_x^2 + \omega x^2 + r, \quad \omega \in \mathbb{C}, \Im \omega \neq 0.$$

We describe completely the spectral properties of P . In particular, we reduce our operator to three cases

$$L_+ = D_x^2 + i\varepsilon x D_x + (1 + i\delta)x^2 + ipD_x + qx + \tau, \quad (1)$$

$$L_- = D_x^2 + i\varepsilon x D_x + (-1 + i\delta)x^2 + ipD_x + qx + \tau, \quad (2)$$

$$L_0 = D_x^2 + i\varepsilon x D_x + i\delta x^2 + ipD_x + qx + \tau, \quad (3)$$

where $\varepsilon, \delta \in \mathbb{R}$, $q, \tau \in \mathbb{C}$. Next, we investigate in details normal forms of multidimensional anisotropic operators. Finally, perturbations with $S_{1/2}^{1/2}(\mathbb{R}^n)$ Shubin type operators are discussed. As it concerns the one-dimensional anisotropic Shubin operators, we mention a paper of Nicola and Rodino [51] where the authors study global hypoellipticity in $\mathcal{S}(\mathbb{R})$. We stress the fact that our proofs are based on the reduced to the normal form operators

$$L_{2k, \rho} = D_x^2 + x^{2k} + \sum_{j=0}^{2k-3} \rho_j x^{2k-3-j}. \quad (4)$$

where $\rho \in \mathbb{R}^{2k-2}$.

In Chapter 4, we study the normal form of classes of Shubin type degenerate operators with non negative principal symbol. We stress that our classes of degenerate operators is different from those studied in [34], [49]. Two novelties appear: first, for the Airy type normal form operators we derive global hypoellipticity in $\mathcal{S}(\mathbb{R}^n)$ and anisotropic subelliptic type estimates in new spaces. Our proofs borrow ideas from the approach of F. Trèves [65] for showing local subelliptic estimates for first order p.d.o. of principal type.

Second novelty occurs in the global hypoellipticity result for twisted Laplacian type operators: discrete phenomena appear for the zero order term, in contrast with the hypoellipticity results in [49], where the estimates depend only on the sub-principal symbol.

We have again two cases: when $A_{skew} = 0$ and $\dim(\Sigma_P) \geq 1$ we find the Airy type operator and the $\bar{\partial}$ operator

$$P_{NF} = -\Delta + i\langle M, D_x \rangle + \rho x_1 + i\sigma x_1 + i\tau x_2.$$

In $\tau = 0$ we obtain the multidimensional Airy type operator, while if $\tau\rho \neq 0$ then we have that the Fourier transform of P_{NF} is written as polynomial perturbation of $\bar{\partial}$ type operator. We prove complete classification of the Airy type normal form case.

In the second case $A_{skew} \neq 0$ we reduce to normal forms generalizations of twisted Laplacian type operators and derive a complete description of the spectral properties and the global hypoellipticity and solvability in $\mathcal{S}(\mathbb{R}^n)$ and the Gelfand–Shilov spaces $S_\mu^\mu(\mathbb{R}^n)$, $\mu \geq 1/2$.

In the last chapter we study the well-posedness for a second order of

Cauchy problem

$$\begin{cases} \partial_t^2 u + P(x, D)u + R(x, D)u = 0, & t \geq 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0 \in S'(\mathbb{R}^n), & u_t(0, x) = u_1 \in S'(\mathbb{R}^n). \end{cases}$$

We prove the well posedness in $C^\infty([0, \infty[; \mathcal{S}'(\mathbb{R}^n))$, $C^\infty([0, \infty[; \mathcal{Q}'(\mathbb{R}^n))$, $C^\infty([0, \infty[; \mathcal{S}_V^\mu(\mathbb{R}^n))$. One of the fundamental ingredients of the proofs is the systematic use of discrete Fourier analysis defined by the eigenfunction expansions for self-adjoint globally elliptic differential operators in functional spaces, following the approach used by Gramchev, Pilipovic and Rodino [23].

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Chapter 1

Basic notions

In the first section of this chapter we recall some definitions and properties of function and distribution spaces on \mathbb{R}^n , which we shall use in this thesis (see the book of Rodino [52] for details). In the second one we give an introduction to the theory of pseudodifferential operators and its symbolic calculus (see the book of Rodino e Nicola 2010 [50]). The last section is devoted to some notions of the spectral theory.

1.1 Distribution and Function spaces

1.1.1 The spaces \mathcal{S} and \mathcal{S}'

In this section, we introduce the *Schwartz space* $\mathcal{S}(\mathbb{R}^n)$ and its dual spaces the *class of temperate (tempered) distribution* $\mathcal{S}'(\mathbb{R}^n)$.

Definition 1. *The Schwartz class $\mathcal{S}(\mathbb{R}^d)$ is defined as the space of all smooth functions in \mathbb{R}^d such that*

$$\sup_{x \in \mathbb{R}^d} |x^\beta \partial^\alpha f(x)| < \infty, \quad \text{for all } \alpha, \beta \in \mathbb{N}^d.$$

Before recalling some properties of the Schwartz space, we need to define the space \mathcal{P} , the space of C^∞ functions with polynomial growths at the infinity.

Definition 2. We define \mathcal{P}^0 , as the set of all continuous functions φ such that

$$|\varphi(x)| \leq C(1 + |x|^2)^N,$$

for some constants C and N . While we write $\varphi \in \mathcal{P}$ if φ is a C^∞ function such that $\partial^\alpha \varphi \in \mathcal{P}^0$, for all $\alpha \in \mathbb{Z}_+^d$.

As explained by the following Lemma, (we refer the reader to Lemma 1.4 in [57] for more details), \mathcal{S} is closed under the operations of differentiation and multiplication by C^∞ functions with polynomial growths at the infinity.

Lemma 1. One has

i) For any $\alpha \in \mathbb{Z}_+^d$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then $\partial^\alpha \varphi \in \mathcal{S}(\mathbb{R}^d)$ with

$$|\partial^\alpha \varphi|_k \leq |\varphi|_{k+|\alpha|},$$

for all $k \in \mathbb{Z}_+$.

ii) For any $\psi \in \mathcal{P}$ there exist two sequence C_k and N_k such that If $\varphi \in \mathcal{S}(\mathbb{R}^d)$ then $\varphi\psi \in \mathcal{S}(\mathbb{R}^d)$ with

$$|\varphi\psi|_k \leq C_k |\varphi|_{k+2N_k},$$

for all $k \in \mathbb{Z}_+$. In particular, if $\psi(x) = x^\alpha$ one has $|x^\alpha \varphi|_k \leq 2^k(\alpha!) |\varphi|_{k+|\alpha|}$.

With $|u|_k$ we denote the inductive norm of the Schwartz class.

We point out that one motivation for the introduction of $\mathcal{S}(\mathbb{R}^d)$ lies in the fact that when dealing integrals of such function, all the difficult operations

will be valid thanks to the decay of Schwartz functions at infinity. For this reason we give the properties of this function class, (for more details see Theorem 1.6 in [57]).

Proposition 1. *Let $1 \leq p < \infty$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. One has $\mathcal{S}(\mathbb{R}^d) \subset \bigcap_p L^p(\mathbb{R}^d)$ with $Norm_{L^p}(\varphi) \leq (2\pi)^n |\varphi|_{2n}$. Moreover*

i) For any $1 \leq p < \infty$, $u \in L^p(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then $u\bar{\varphi} \in L^1(\mathbb{R}^d)$ with

$$|(u, \varphi)| \leq (2\pi)^n Norm_{L^p}(u) |\varphi|_{2d}.$$

ii) For any measurable u such that $u\bar{\varphi} \in L^1(\mathbb{R}^d)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$

$$(u, \varphi) = 0 \text{ for all } \varphi \in \mathcal{S}(\mathbb{R}^d) \Rightarrow u = 0 \text{ a.e.}$$

iii) If $\varphi \rightarrow U(\varphi)$ is a semilinear form on $\mathcal{S}(\mathbb{R}^d)$ satisfying $|U(\varphi)| \leq C \|\varphi\|_{L^2(\mathbb{R}^d)}$ then there exists a unique $u \in L^2(\mathbb{R}^d)$ such that $U(\varphi) = (u, \varphi)$ for $\varphi \in \mathcal{S}(\mathbb{R}^d)$, and $\|u\|_{L^2(\mathbb{R}^d)} \leq C$.

It is well known that the Fourier transform

$$\mathcal{F}\varphi(\xi) = \hat{\varphi}(\xi) = \int e^{-i\langle x, \xi \rangle} f(x) \bar{d}x, \quad \bar{d}x = (2\pi)^{(-d/2)} dx$$

defines an isomorphism of $\mathcal{S}(\mathbb{R}^d)$ and an isometry of $L^2(\mathbb{R}^d)$. The inverse Fourier transform is

$$\mathcal{F}^{-1}\varphi(x) = \int e^{i\langle x, \xi \rangle} \hat{f}(\xi) d\xi.$$

The following Theorem, (see Theorem 1.8 in [57] for more details), establishes some properties of the Fourier transform in the Schwartz class.

Theorem 1. *For any $\varphi \in \mathcal{S}(\mathbb{R}^d)$, one has $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^d)$. Moreover, the Fourier transform $\hat{\varphi}$ of $\varphi \in \mathcal{S}(\mathbb{R}^d)$, satisfies*

i) $\widehat{D^\alpha \varphi}(\xi) = \xi^\alpha \widehat{\varphi}(\xi)$ and $\widehat{x^\alpha \varphi}(\xi) = (-1)^{|\alpha|} D^\alpha \widehat{\varphi}(\xi)$, for any $\alpha \in \mathbb{Z}_+^d$.

ii) (Parseval formula) $(\widehat{\varphi}, \widehat{\psi}) = (2\pi)^n (\varphi, \psi)$, for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Now we are able to define the space of *temperate distributions*.

Definition 3. We say that u is a temperate distribution, and we write $u \in \mathcal{S}'(\mathbb{R}^d)$ if u is a semilinear form $\varphi \rightarrow (u, \varphi)$ on $\mathcal{S}(\mathbb{R}^d)$, with two constants $C \in \mathbb{R}$ and $N \in \mathbb{Z}_+$ such that

$$|(u, \varphi)| \leq C |\varphi|_N \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R}^d).$$

It is well known that every Lebesgue space $L^p(\mathbb{R}^d)$ is a subspace of $\mathcal{S}'(\mathbb{R}^d)$. Thus, we may construct the extension of the Fourier transform to $\mathcal{S}'(\mathbb{R}^d)$.

Theorem 2. Let $u \in \mathcal{S}'(\mathbb{R}^d)$. Then the formulas

$$(\widehat{u}, \varphi) = (u, \widehat{\varphi}), \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R}^d),$$

defines distributions $\widehat{u} \in \mathcal{S}'(\mathbb{R}^d)$. Moreover if $u \in L^2(\mathbb{R}^d)$ it implies $\widehat{u} \in L^2(\mathbb{R}^d)$ via Parseval formula:

$$(\widehat{u}, \widehat{v}) = (2\pi)^n (u, v), \quad \text{for } u, v \in L^2(\mathbb{R}^d).$$

We may also extend to the temperate distributions space the operator of differentiation. Thus, if $u \in \mathcal{S}'(\mathbb{R}^d)$, the following formula

$$(D^\alpha u, \varphi) = (u, D^\alpha \varphi), \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R}^d)$$

defines a distribution $D^\alpha u \in \mathcal{S}'(\mathbb{R}^d)$, for any $\alpha \in \mathbb{Z}_+^d$.

Remark 1. We note that this operation extends the usual differentiation of functions. We remark the important fact that differentiation is always possible in the space of distributions. We can now always differentiate a function, even when it is not classically differentiable.

Unfortunately, it has been proved that it is impossible to define in general the product of two distributions with the usual properties of products of functions. For example the operation of multiplication will be restricted to the following two situations:

- i) when u and ψ are functions;
- ii) when $\psi \in \mathcal{D}$ and $u \in \mathcal{S}'(\mathbb{R}^d)$, the formula

$$(\psi u, \varphi) = (u, \bar{\psi} \varphi), \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R}^d)$$

defines a distribution $\psi u \in \mathcal{S}'(\mathbb{R}^d)$.

1.1.2 The spaces \mathcal{D} and \mathcal{D}'

As explained by the previous section to have a good theory of Fourier transformation, we need the control of growth at infinity of temperate distributions. However, if one gives up the Fourier transformations to keep only the operations of differentiation and multiplication by smooth functions, one can consider much wider classes of distributions. Before introducing the definition of this distribution class, we need to define the functional space of $\mathcal{D}(\Omega)$.

Definition 4. Let $\Omega \subseteq \mathbb{R}^d$ be an open set of \mathbb{R}^d . $C_0^\infty(\Omega)$ denotes the linear subset consisting of those functions in $C_0^\infty(\mathbb{R}^d)$ which have compact support in Ω .

Now we are able to define the space $\mathcal{D}'(\Omega)$.

Definition 5. Let $\Omega \subseteq \mathbb{R}^d$ be an open set of \mathbb{R}^d . One can define a distribution in Ω as follows: $u \in \mathcal{D}'(\Omega)$ if u is a linear form on $C_0^\infty(\Omega)$ continuous in the

sense that for each compact set $K \subset \Omega$, there exist two constants C_k, N_k such that

$$|(u, \varphi)| \leq C_k |\varphi|_{N_k} \quad \text{for } \varphi \in C_0^\infty(\Omega) \text{ and } \text{supp } \varphi \subset K,$$

where the notation $\text{supp } \varphi$ means the support of φ , and it is defined as the intersection of all closed subsets in whose complement φ vanishes.

The product $\psi \varphi \in \mathcal{D}'(\Omega)$ is defined by the formula $(\psi u, \varphi) = (u, \overline{\psi} \varphi)$ for any $u \in \mathcal{D}'(\Omega)$ and $\psi \in C^\infty(\Omega)$.

Remark 2. The obvious inclusion $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$, is strict. For example the Gaussian $e^{-|x|^2} \in \mathcal{S}(\mathbb{R}^d)$; but it is not in $\mathcal{D}(\mathbb{R}^d)$.

Finally, we recall the *Paley-Wiener-Schwartz* theorem, which shows that distributions with compact support can be recognized by their Fourier transform, (see for more details Theorem 1.13 in [57] or see [52])

1.1.3 Sobolev spaces H^s

One property of the Fourier transform states that for a temperate distribution u , $u \in L^2(\mathbb{R}^d)$ is equivalent to $\hat{u} \in L^2(\mathbb{R}^d)$. Moreover, there is a correspondence between differentiation of u and multiplication of \hat{u} by a polynomial, there is also a correspondence between the smoothness of u and the growth of \hat{u} at infinity. This fact is used to define the *Sobolev spaces* which are more convenient than the classical $C^k(\mathbb{R}^d)$.

Definition 6. Let $s \in \mathbb{R}$, and $u \in \mathcal{S}'(\mathbb{R}^d)$. We will write $u \in H^s(\mathbb{R}^d)$, if $\langle \xi \rangle^s \hat{u}(\xi) \in L^2(\mathbb{R}^d)$. In other words, $u \in H^s(\mathbb{R}^d)$ if

$$\|u\|_{H^s(\mathbb{R}^d)}^2 = \int \langle \xi \rangle^s |\hat{u}(\xi)|^2 d\xi < \infty,$$

where $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$.

We note that we have the following inclusions: $H^s(\mathbb{R}^d) \subset H^t(\mathbb{R}^d)$ if $s \geq t$ and $\mathcal{S}(\mathbb{R}^d) \subset H^\infty(\mathbb{R}^d) \subset H^{-\infty}(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$, where $H^\infty = \bigcap_s H^s$ and $H^{-\infty} = \bigcup_s H^s$. These inclusions are strict, for example it is not true that $\mathcal{S} = H^\infty$. In fact if one take $u(x) = (1 + |x|^2)^{-d}$, $u \in H^\infty$ but $u \notin \mathcal{S}$.

As explained by the following Proposition (we refer the reader to Proposition 1.14 in [57]), the Sobolev spaces measure the same smoothness as the $C^k(\mathbb{R}^d)$ up to a fixed shift of exponents.

Proposition 2. *For all $s \in \mathbb{R}$ one has*

$$u \in H^{s+1}(\mathbb{R}^d) \Leftrightarrow u, D_{x_1}u, \dots, D_{x_d}u \in H^s(\mathbb{R}^d),$$

with the equality $\|u\|_{H^{s+1}(\mathbb{R}^d)}^2 = \|u\|_{H^s(\mathbb{R}^d)}^2 + \sum_j \|D_{x_j}u\|_{H^s(\mathbb{R}^d)}^2$. Moreover, for any $k \in \mathbb{Z}_+ \cup \{\infty\}$,

$$i) \quad u \in H^k(\mathbb{R}^d) \Leftrightarrow D^\alpha u \in L^2(\mathbb{R}^d), \text{ for all } |\alpha| \leq k.$$

$$ii) \quad \text{If } s > \frac{d}{2} + k \text{ and } u \in H^k(\mathbb{R}^d) \Rightarrow D^\alpha u \text{ are bounded continuous functions for } |\alpha| \leq k.$$

Using the Riesz's representation theorem, the H^s distributions can also be characterized as the continuous semi-linear forms of H^{-s} . More precisely we have the following Proposition, (for more details see Proposition 1.15 [57]).

Proposition 3. *If $u \in H^s(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, then*

$$|(u, \varphi)| \leq \|u\|_{H^s(\mathbb{R}^d)} \|\varphi\|_{H^{-s}(\mathbb{R}^d)}.$$

Conversely, if $u \in \mathcal{S}'(\mathbb{R}^d)$ satisfies $|(u, \varphi)| \leq C \|\varphi\|_{H^{-s}(\mathbb{R}^d)}$ for some constant C and all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, then $u \in H^s(\mathbb{R}^d)$ with $\|u\|_{H^s(\mathbb{R}^d)} \leq C$.

When we study the linear partial differential equation with variable coefficients, using $H^s(\mathbb{R}^d)$ distributions, we will have to consider products of

distributions with the coefficients of the equation. To measure the smoothness of these products, we first compute the Fourier transforms, as explained by the following Lemma, (we refer the reader to Lemma 1.17 in [57] for more details).

Lemma 2. *One has*

$$\hat{u}v(\xi) = \int \hat{u}(\xi - \eta)\hat{v}(\eta) \, d\eta = \int \hat{u}(\zeta)\hat{v}(\xi - \zeta) \, d\zeta,$$

for any $u, v \in L^2(\mathbb{R}^d)$, as well as for any $u \in H^\infty(\mathbb{R}^d)$ and $v \in H^{-\infty}(\mathbb{R}^d)$. Moreover the Leibniz formula

$$D^\alpha(uv) = \sum_{\beta} \binom{\alpha}{\beta} (D^\beta u)(D^{\alpha-\beta} v),$$

holds for any $u, v \in H^{|\alpha|+r}(\mathbb{R}^d)$, $r > n/2$

From the Lemma 2 and the Peetre's inequality we get the following continuity properties, as stated by the following Corollary, (for more detail see Corollary 1.19 in [57])

Corollary 1. *Let $s \in \mathbb{R}$ and $\varphi \in H^\infty(\mathbb{R}^d)$. Then*

$$u \in H^s(\mathbb{R}^d) \Rightarrow \varphi u \in H^s(\mathbb{R}^d) \text{ with } \|\varphi u\|_{H^s(\mathbb{R}^d)} \leq 2^{|s|-(d/2)} \|\varphi\|_{H^{|s|+d}(\mathbb{R}^d)} \|u\|_{H^s(\mathbb{R}^d)}.$$

Moreover, if

$$a(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

is a linear partial differential operator of order m with coefficients $a_\alpha(x) \in H^\infty(\mathbb{R}^d)$ then $a(x, D)$ maps continuously $H^s(\mathbb{R}^d)$ into $H^{s-m}(\mathbb{R}^d)$, for any $s \in \mathbb{R}$.

1.1.4 Gelfand-Shilov spaces S_V^μ

If one would like to know more precisely how fast the decay of $f \in \mathcal{S}(\mathbb{R}^d)$ is at the infinity, then it is convenient to use the spaces $S_V^\mu(\mathbb{R}^d)$, subspaces of

$\mathcal{S}(\mathbb{R}^d)$. We will define them in terms of simultaneous estimates of exponential type for $f(x)$ and $\hat{f}(\xi)$. We note the symmetrical role of the variables x and ξ .

Definition 7. Let $\mu > 0$ and $\nu > 0$. The function $f(x)$ is in $S_{\nu}^{\mu}(\mathbb{R}^d)$ if $f(x) \in \mathcal{S}(\mathbb{R}^d)$ and there exists a constant $\varepsilon > 0$ such that

$$|f(x)| \lesssim e^{-\varepsilon|x|^{1/\nu}}, \quad (1.1)$$

$$|\hat{f}(\xi)| \lesssim e^{-\varepsilon|\xi|^{1/\mu}}. \quad (1.2)$$

We note that we have the inclusions $S_{\nu}^{\mu}(\mathbb{R}^d) \subset S_{\nu'}^{\mu'}(\mathbb{R}^d)$ for $\mu \leq \mu', \nu \leq \nu'$. It is also well known that application of the Fourier transform interchanges the indices μ and ν in the above definition. In fact we have the following Proposition (see Theorem 6.1.2 in [50])

Proposition 4. For $f \in \mathcal{S}(\mathbb{R}^d)$, we have $f \in S_{\nu}^{\mu}(\mathbb{R}^d)$ if and only if $\hat{f} \in S_{\mu}^{\nu}(\mathbb{R}^d)$.

Remark 3. In particular the symmetric spaces $S_{\mu}^{\mu}(\mathbb{R}^d)$, $\mu > 0$, are invariant under the action of the Fourier transform. It will be clear that Definition 7 does not change meaning, if referred to $f \in L^2(\mathbb{R}^d)$, or $f \in \mathcal{S}'(\mathbb{R}^d)$, provided (1.1), (1.2) make sense.

Now we are interested in passing from the estimates (1.1), (1.2) to estimates involving only $f(x)$. The first step is to convert exponential bounds in the factorial bounds. For this reason we recall the following Proposition (see for more details Proposition 6.1.5 in [50])

Proposition 5. The following conditions are equivalent:

i) the condition (1.1) holds, i.e., there exists a constant $\varepsilon > 0$ such that

$$|f(x)| \lesssim e^{-\varepsilon|x|^{1/\nu}}. \quad (1.3)$$

ii) there exists a constant $C > 0$ such that

$$|x^\alpha f(x)| \lesssim C^{|\alpha|} (\alpha!)^\nu, \quad \alpha \in \mathbb{N}^d. \quad (1.4)$$

The following Theorem, (we refer the reader to Theorem 6.1.6 in [50] for more details), give us an equivalent definition of $S_v^\mu(\mathbb{R}^d)$.

Theorem 3. Assume $\mu > 0, \nu > 0, \mu + \nu \geq 1$. For $f \in \mathcal{S}(\mathbb{R}^d)$ the following conditions are equivalent:

i) $f \in S_v^\mu(\mathbb{R}^d)$.

ii) There exists a constant $C > 0$ such that

$$|x^\alpha f(x)| \lesssim C^{|\alpha|} (\alpha!)^\nu, \quad \alpha \in \mathbb{N}^d; \quad (1.5)$$

$$|\xi^\beta \hat{f}(\xi)| \lesssim C^{|\beta|} (\beta!)^\mu, \quad \beta \in \mathbb{N}^d. \quad (1.6)$$

iii) There exists a constant $C > 0$ such that

$$\|x^\alpha f(x)\|_{L^2(\mathbb{R}^d)} \lesssim C^{|\alpha|} (\alpha!)^\nu, \quad \alpha \in \mathbb{N}^d; \quad (1.7)$$

$$\|\xi^\beta \hat{f}(\xi)\|_{L^2(\mathbb{R}^d)} \lesssim C^{|\beta|} (\beta!)^\mu, \quad \beta \in \mathbb{N}^d. \quad (1.8)$$

iv) There exists a constant $C > 0$ such that

$$\|x^\alpha f(x)\|_{L^2(\mathbb{R}^d)} \lesssim C^{|\alpha|} (\alpha!)^\nu, \quad \alpha \in \mathbb{N}^d; \quad (1.9)$$

$$\|\partial^\beta f(x)\|_{L^2(\mathbb{R}^d)} \lesssim C^{|\beta|} (\beta!)^\mu, \quad \beta \in \mathbb{N}^d. \quad (1.10)$$

v) There exists a constant $C > 0$ such that

$$\|x^\alpha \partial^\beta f(x)\|_{L^2(\mathbb{R}^d)} \lesssim C^{|\alpha|+|\beta|} (\alpha!)^\nu (\beta!)^\mu, \quad \alpha, \beta \in \mathbb{N}^d. \quad (1.11)$$

vi) There exists a constant $C > 0$ such that

$$|x^\alpha \partial^\beta f(x)| \lesssim C^{|\alpha|+|\beta|} (\alpha!)^\nu (\beta!)^\mu, \quad \alpha, \beta \in \mathbb{N}^d. \quad (1.12)$$

We note that the assumption $\mu + \nu \geq 1$ in Theorem 3 is not restrictive. In fact in the Theorem 4 we will prove that for $\mu + \nu < 1$ the spaces $S_{\nu}^{\mu}(\mathbb{R}^d)$ contain only the zero function. For this aim we give the following Propositions (see Proposition 6.1.1, Proposition 6.1.8 and Proposition 6.1.9 in [50])

Proposition 6. *Let $\mu > 0, \nu > 0$ and $f \in \mathcal{S}(\mathbb{R}^d)$. Then the estimates (vi) in Theorem 3 are valid if and only if there exist positive constants C and ε such that*

$$|\partial^{\beta} f(x)| \lesssim C^{|\beta|} (\beta!)^{\mu} e^{-\varepsilon|x|^{(1/\nu)}}, \quad \beta \in \mathbb{N}^d. \quad (1.13)$$

Hence, when $\mu + \nu \geq 1$ the estimates give an equivalent definition of $S_{\nu}^{\mu}(\mathbb{R}^d)$.

Proposition 7. *Assume $0 < \mu < 1, \nu > 0$ and $f \in \mathcal{S}(\mathbb{R}^d)$. Let (1.13) be satisfied for suitable constants $C > 0$ and $\varepsilon > 0$. Then f extends to an entire function $f(x+iy)$ in \mathbb{C}^d , with*

$$|f(x+iy)| \lesssim e^{-\varepsilon|x|^{(1/\nu)} + \delta|y|^{1/(1-\mu)}}, \quad (1.14)$$

where δ is a suitable positive constant. In the case $\mu = 1, \nu > 0$, f extends to an analytic function $f(x+iy)$ in the strip $\{x+iy \in \mathbb{C}^d : |y| < T\}$ with

$$|f(x+iy)| \lesssim e^{-\varepsilon|y|^{(1/\nu)}}, \quad |y| < T, \quad (1.15)$$

for suitable $T > 0$.

Therefore, if $\mu + \nu \geq 1$ and $\mu < 1$ then every $f \in S_{\nu}^{\mu}(\mathbb{R}^d)$ extends to the complex domain as an entire function satisfying (1.14), while if $\mu + \nu \geq 1$ and $\mu = 1$ then every $f \in S_{\nu}^{\mu}(\mathbb{R}^d)$ extends to the complex domain as an holomorphic function in a strip satisfying (1.15).

The following Theorem, (see Theorem 6.1.10 in [50] for more details), answers the question of the triviality of the classes $S_{\nu}^{\mu}(\mathbb{R}^d)$, when $\mu + \nu < 1$. This result can be expressed by the statement that a function $f(x)$ and its Fourier transform $\hat{f}(\xi)$ cannot both be small at infinity.

Theorem 4. *Let $f \in \mathcal{S}(\mathbb{R}^d)$ satisfy*

$$|f(x)| \lesssim e^{-\varepsilon|x|^{1/\nu}}, \quad |\hat{f}(\xi)| \lesssim e^{-\varepsilon|\xi|^{1/\mu}},$$

for some $\varepsilon > 0, \mu > 0, \nu > 0$ and $\mu + \nu < 1$. Then $f \equiv 0$. In other words, according to the Definition, the classes $\mathcal{S}_\nu^\mu(\mathbb{R}^d)$ are trivial if $\mu + \nu < 1$.

Moreover, if $\mu + \nu < 1$ each conditions (ii)-(vi) in the Theorem 3 and (1.13), implies $f \equiv 0$.

We recall that the symmetric Gelfand-Shilov spaces $\mathcal{S}_\mu^\mu(\mathbb{R}^d)$, $\mu \geq 1/2$, are invariant under the Fourier transform (see Proposition 4). These spaces play an important role in the applications to the study of Shubin operator. It is convenient for $\mathcal{S}_\mu^\mu(\mathbb{R}^d)$ to reformulate (v), Theorem 3, in the following form, (see Proposition 6.1.12 in [50]).

Proposition 8. *Let $\mu \geq \frac{1}{2}$. A function $f \in \mathcal{S}(\mathbb{R}^d)$ belongs to $\mathcal{S}_\mu^\mu(\mathbb{R}^d)$ if and only if there exists a constant $C > 0$ such that*

$$\|x^\alpha \partial^\beta f(x)\|_{L^2(\mathbb{R}^d)} \lesssim C^N N^{N\mu}, \quad \text{for } |\alpha| + |\beta| \leq N, N = 0, 1, 2, \dots$$

1.2 Symbols and pseudodifferential operators

1.2.1 Symbol classes

Before introducing the definition of pseudodifferential operator we need to define the general notion of symbol classes. We begin with the following concepts of *sub-linear weight* and *temperate weight*

Definition 8. *A positive continuous function $\phi(x, \xi)$, $(x, \xi) \in \mathbb{R}^{2d}$, is called a sub-linear weight if*

$$1 \leq \phi(x, \xi) \lesssim 1 + |x| + |\xi|, \quad (1.16)$$

A positive continuous function $\phi(x, \xi)$ is called temperate weight, if there exists $s > 0$, such that

$$\phi(x+y, \xi + \eta) \lesssim \phi(x, \xi)(1 + |y| + |\eta|)^s, \quad (1.17)$$

Now we define the symbol classes $S(M; \phi, \psi)$

Definition 9. Let $\phi(x, \xi)$ and $\psi(x, \xi)$ be sub-linear and temperate weights. Let $M(x, \xi)$ be a temperate weight. With $S(M; \phi, \psi)$ we denote the space of all smooth functions $a(x, \xi)$, $(x, \xi) \in \mathbb{R}^{2d}$ such that for every $\alpha, \beta \in \mathbb{N}^d$,

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \lesssim M(x, \xi) \psi(x, \xi)^{-|\alpha|} \phi(x, \xi)^{-|\beta|}. \quad (1.18)$$

In this thesis we will use a particular kind of symbols, the so called Γ -symbols. They are defined by taking $\phi(z) = \psi(z) = \langle z \rangle$, where $z = (x, \xi) \in \mathbb{R}^{2d}$, in the Definition 9.

Definition 10. Let $m \in \mathbb{R}$. We define $\Gamma^m(\mathbb{R}^d)$, as the set of all function $a(z) \in C^\infty(\mathbb{R}^{2d})$ satisfying,

$$|\partial_z^\gamma a(z)| \lesssim \langle z \rangle^{m-|\gamma|}, \quad (1.19)$$

for all $\gamma \in \mathbb{N}^{2d}$.

For completeness, we also define the G -classes (introduced by Parenti and Cordes). Note that, differently from the Γ -classes, the symbols in the G -classes have independent asymptotic behaviour in x and ξ . However, they can be seen as a special case of Definition 9 with $\psi = \langle \xi \rangle$ and $\phi = \langle x \rangle$.

Definition 11. Let $m, n \in \mathbb{R}$. We define $G^{m,n}(\mathbb{R}^d)$, as the set of all functions $a(x, \xi) \in C^\infty(\mathbb{R}^{2d})$ satisfying the estimates

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \lesssim \langle \xi \rangle^{m-|\alpha|} \langle x \rangle^{n-|\beta|}, \quad (1.20)$$

for all $\alpha, \beta \in \mathbb{N}^d$,

For the symbolic calculus in $S(\mathcal{M}; \phi, \psi)$, it is very important that the sub-linear and temperate weights satisfy the following *strong uncertainty principle*:

$$\psi(x, \xi)\phi(x, \xi) \gtrsim (1 + |x| + |\xi|)^\delta \quad \text{for some } \delta > 0. \quad (1.21)$$

It is well known (see [50]), that if (1.21) holds, then the symbols in $S(Mh^n; \phi, \psi)$, where $h(x, \xi)$ is the *Plank function*, decay at infinity, together with their derivatives provided that n is large enough. Then, one can introduce a notion of *asymptotic expansion* in $S(\mathcal{M}; \phi, \psi)$

Definition 12. Let $n \in \mathbb{N}$ and $a \in S(\mathcal{M}; \phi, \psi)$, for any given sequence of symbols $a_n \in S(Mh^n; \phi, \psi)$, we write

$$a(x, \xi) \sim \sum_n a_n(x, \xi) \quad (1.22)$$

if, for every $N \geq 1$,

$$a(x, \xi) - \sum_{j=0}^{N-1} a_j(x, \xi) \in S(\mathcal{M}; \phi, \psi). \quad (1.23)$$

The right-hand side of (1.22) is called *asymptotic expansion* of a .

As explained by the following Proposition (we refer the reader to Proposition 1.1.16 in [50] for more details), if the strong uncertainty principle is satisfied we have an asymptotic expansion modulo a Schwartz function.

Proposition 9. Assume the strong uncertainty principle (1.21). Let $a_n \in S(Mh^n; \phi, \psi)$, $n \in \mathbb{N}$. Then there exists a symbol $a(x, \xi) \in S(\mathcal{M}; \phi, \psi)$ such that

$$a(x, \xi) \sim \sum_n a_n(x, \xi)$$

Moreover a is uniquely determined modulo Schwartz functions.

1.2.2 Pseudodifferential operators

Oscillatory integrals

In this section we want to sketch a theory of oscillatory integrals which will be used to define pseudodifferential operators. For more details we refer the reader to [57]. Note that these integrals are in general not absolutely convergent.

Definition 13. (*Spaces of amplitudes*) Let $m \geq 0$. With A^m we denote the space of all functions $a \in C^\infty(\mathbb{R}^n)$ such that

$$\sup_{x \in \mathbb{R}^n} \langle x \rangle^{-m} |\partial_x^\alpha a(x)| < \infty \quad \text{for all } \alpha \in \mathbb{Z}_+^n. \quad (1.24)$$

In this space we will use the following seminorms

$$\|a\|_{A^m, k} = \max_{|\alpha| \leq k} \|\langle x \rangle^{-m} \partial^\alpha a\|_{L^\infty(\mathbb{R}^n)}. \quad (1.25)$$

The following Theorem (given by [57] Theorem 2.3) defines the notion of *oscillatory integrals*

Theorem 5. Let q be a non degenerate real quadratic form on \mathbb{R}^n (i.e. $\nabla q(x) \neq 0$, for $x \neq 0$) $a \in A^m$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, such that $\varphi(0) = 1$. Then the limit

$$\lim_{\varepsilon \rightarrow 0} \int e^{iq(x)} a(x) \varphi(\varepsilon x) dx \quad (1.26)$$

exists, is independent of φ (as long as $\varphi(0) = 1$), and is equal to $\int e^{iq(x)} a(x) dx$, when $a \in L^1(\mathbb{R}^n)$. When $a \notin L^1(\mathbb{R}^n)$ the limit in (1.26) is still denoted by $\int e^{iq(x)} a(x) dx$, and fulfils

$$\left| \int e^{iq(x)} a(x) dx \right| \leq C_{q,m} \|a\|_{A^m, m+n+1}, \quad (1.27)$$

where the constant $C_{q,m}$ depends only on the quadratic form q and on the order m .

Some properties of oscillatory integrals are listed below (see Theorem 2.5 [57])

Proposition 10. -

i) *Change of the variable: if A is an invertible real matrix, then*

$$\int e^{iq(Ay)} a(Ay) |\det A| dy = \int e^{iq(x)} a(x) dx.$$

ii) *Integration by parts: if $a \in A^m$, $b \in A^\ell$ and $\alpha \in \mathbb{Z}_+^n$, then*

$$\int e^{iq(x)} a(x) \partial^\alpha b(x) dx = \int b(x) (-\partial)^\alpha (e^{iq(x)} a(x)) dx.$$

iii) *Differentiation under \int : if $a \in A^m(\mathbb{R}^n \times \mathbb{R}^p)$ then $\int e^{iq(x)} a(x) dx \in A^m(\mathbb{R}^p)$ and*

$$\partial_y^\alpha \int e^{iq(x)} a(x, y) dx = \int e^{iq(x)} \partial_y^\alpha a(x, y) dx \quad \text{for all } \alpha \in \mathbb{Z}_+^p.$$

iv) *Interchange of the \int : if $a \in A^m(\mathbb{R}^n \times \mathbb{R}^p)$ as in iii) and if r is a non degenerate real quadratic form on \mathbb{R}^p ,*

$$\int e^{ir(y)} \left(\int e^{iq(x)} a(x, y) dx \right) dy = \int e^{i(q(x)+r(y))} a(x, y) dx dy.$$

Pseudodifferential operators

Definition 14. Let $m \in \mathbb{R}$. We say that $a(x, \xi)$ belongs to the set of symbols $S^m(\mathbb{R}^{2n})$ if and only if $a(x, \xi) \in C^\infty(\mathbb{R}^{2n})$ and the following is satisfied

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \lesssim \langle \xi \rangle^{m-|\alpha|}, \quad (1.28)$$

$\forall \alpha, \beta \in \mathbb{N}^n$. We also denote $\bigcup_m S^m$ by S^∞ and $\bigcap_m S^m$ by $S^{-\infty}$.

We define, also, for these symbol classes the asymptotic expansion.

Definition 15. Let $m \in \mathbb{R}$ and $a_j \in S^{m_j}(\mathbb{R}^{2n})$ for $j \in \mathbb{N}$, where

$$m = m_0 \geq m_1 \geq \dots \geq m_n \rightarrow -\infty.$$

We say that $\sum_j a_j$ is an asymptotic expansion of a and we write $a \sim \sum_{j=0}^{\infty} a_j$, if

$$\forall N \in \mathbb{Z}^+, \quad a - \sum_{j=0}^{N-1} a_j \in S^{m_N}(\mathbb{R}^{2n}).$$

The following Proposition, (we refer the reader to Theorem 10.9 in [18] for more details), give us the conditions to have the asymptotic expansion modulo a symbol in $S^{-\infty}$.

Proposition 11. Let $m \in \mathbb{R}$ and $a_j \in S^{m_j}(\mathbb{R}^{2n})$, $j \in \mathbb{N}$, with $m_0 = m$ and $m_j \searrow -\infty$. Then, there exists $a \in S^m(\mathbb{R}^{2n})$ such that $a \sim \sum_j a_j$. The symbol a is unique modulo $S^{-\infty}$.

Now we can introduce the adjoint symbol a^* and the compound symbol $a\#b$ (see Theorem 2.7 [57])

Theorem 6. Let $a \in S^m(\mathbb{R}^{2n})$ and $b \in S^\ell(\mathbb{R}^{2n})$. Then the oscillatory integrals

$$\begin{aligned} a^*(x, \xi) &= \int_{\mathbb{R}^{2n}} e^{-iy\eta} \bar{a}(x-y, \xi-\eta) dy \bar{d}\eta, \\ a^t(x, \xi) &= \int_{\mathbb{R}^{2n}} e^{-iy\eta} a(x-y, -\xi+\eta) dy \bar{d}\eta, \\ a\#b(x, \xi) &= \int_{\mathbb{R}^{2n}} e^{-iy\eta} a(x, \xi-\eta) b(x-y, \xi) dy \bar{d}\eta, \end{aligned}$$

define symbols in $S^m(\mathbb{R}^{2n})$, in $S^m(\mathbb{R}^{2n})$ and in $S^{m+\ell}(\mathbb{R}^{2n})$, respectively. Furthermore, we have the following asymptotic expansion:

$$\begin{aligned} a^* &\sim \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} D_x^{\alpha} a(x, \xi), \\ a^t &\sim \sum_{\alpha} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_{\xi}^{\alpha} D_x^{\alpha} \bar{a} \\ a\#b &\sim \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} a D_x^{\alpha} b. \end{aligned}$$

Now we can define pseudodifferential operators.

Definition 16. Let $a \in S^m(\mathbb{R}^{2d})$. We call pseudodifferential operator of symbol a the operator defined on $\mathcal{S}(\mathbb{R}^n)$ as follows

$$a(x, D)f(x) = \int_{\mathbb{R}^n} e^{ix\xi} a(x, \xi) \hat{f}(\xi) \bar{d}\xi. \quad (1.29)$$

It is known that the pseudodifferential operator $a(x, D)$ in (1.29) is a continuous linear operator on the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. In fact the following Theorem (Theorem 3.1 in [57]) holds

Theorem 7. If $a \in S^\infty(\mathbb{R}^{2n})$ and $f \in S(\mathbb{R}^n)$ the formula

$$a(x, D)f(x) = \int_{\mathbb{R}^n} e^{ix\xi} a(x, \xi) \hat{f}(\xi) \bar{d}\xi,$$

defines a function $a(x, D)f \in S(\mathbb{R}^n)$ and there exist constants $N \in \mathbb{Z}_+^n$ and $C_k > 0$, for $k \in \mathbb{Z}_+^n$, depending on a such that $\|a(x, D)f\|_{p_k} \leq C_k \|f\|_{p_{k+N}}$.

We note that the pseudodifferential operator $a(x, D)$ (1.29) can be written in the oscillatory integral form, in fact we recall the following (Proposition 10.7 [18])

Proposition 12. Let $a \in S^m(\mathbb{R}^{2n})$ and $u \in S(\mathbb{R}^n)$. We have that

i) $q(y, \xi) := -y\xi$ is a non degenerate real quadratic form,

ii) $b(x, y, \xi) := e^{ix\xi} a(x, \xi) u(y) \in A^m(\mathbb{R}_y^n \times \mathbb{R}_\xi^n)$ for all $x \in \mathbb{R}^n$.

iii)

$$\begin{aligned} a(x, D)u(x) &= \int_{\mathbb{R}^n} e^{ix\xi} a(x, \xi) \hat{u}(\xi) \bar{d}\xi \\ &= \int_{\mathbb{R}^{2n}} e^{-iy\xi} \left(e^{ix\xi} a(x, \xi) u(y) \right) dy \bar{d}\xi. \end{aligned}$$

Now we want to extend the definition of pseudodifferential operator to the space of Schwartz distribution $\mathcal{S}'(\mathbb{R}^n)$. For this reason we recall the following Theorem (see Theorem 3.2 in [57])

Theorem 8. *For any $a, b \in \mathcal{S}^\infty(\mathbb{R}^{2n})$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$ one has*

$$i) (a^*(x, D)f, g)_{L^2(\mathbb{R}^{2n})} = (f, a(x, D)g)_{L^2(\mathbb{R}^{2n})}.$$

$$ii) (a\sharp b(x, D)f, g) = (a(x, D)b(x, D)f, g).$$

We are now able to extend the operator $a(x, D) : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ as an operator from $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$.

Definition 17. *Given $a \in \mathcal{S}^\infty$, we call pseudodifferential operator of symbol a the operator $a(x, D) : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ defined as follows*

$$(a(x, D)u, v) = u(a^t(x, D)v).$$

From the definition and elementary properties of Sobolev spaces $H^s(\mathbb{R}^n)$ it is well known that a pseudodifferential operator of order m maps continuously $H^s(\mathbb{R}^n)$ into $H^{s-m}(\mathbb{R}^n)$ as stated by the following Theorem (see Theorem 3.6 [57])

Theorem 9. *Let $a \in S^m$. Then for every $s \in \mathbb{R}$*

$$a(x, D) : H^s(\mathbb{R}^n) \rightarrow H^{s-m}(\mathbb{R}^n),$$

and there exists a constant C_s such that $\|a(x, D)u\|_{H^s} \leq C_s \|u\|_{H^{s-m}}$.

Elliptic and hypoelliptic symbols

One of the main achievements of the symbol calculus is a construction of an approximated inverse for elliptic operators. This has been done for a general class of hypoelliptic symbols. For this reasons we need the other particular symbol classes.

Definition 18. Let $m \in \mathbb{R}$ and $\ell \leq m$. The symbol $a \in S^m$ is hypoelliptic, if there exists a radius $R > 0$, such that

$$\exists c_0 > 0, \forall x \in \mathbb{R}^n, \forall \xi, |\xi| \geq R, \quad |a(x, \xi)| \geq c_0 \langle \xi \rangle^\ell, \quad (1.30)$$

and

$$\begin{aligned} \forall \alpha, \beta \in \mathbb{N}^n \exists c_{\alpha, \beta} > 0 \forall x \in \mathbb{R}^n, \forall \xi, |\xi| \geq R, \\ |\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq c_{\alpha, \beta} |a(x, \xi)| \langle \xi \rangle^{-|\alpha|}. \end{aligned} \quad (1.31)$$

If $\ell = m$ then a is an elliptic symbol of order m .

Remark 4. We note that, if the symbol a is elliptic then the bound from below condition (1.30) implies the condition on the derivatives (1.31).

Now we define the notion of *parametrix*:

Definition 19. Let $a(x, D)$ a pseudodifferential operator. $a(x, D)$ has a left parametrix (right parametrix, respectively) if there exists a pseudodifferential operator $p(x, D)$ such that

$$p(x, D)a(x, D) = I + r(x, D) \quad \left(\text{resp. } a(x, D)p(x, D) = I + r(x, D) \right),$$

where $r \in S^{-\infty}$ and I is the identity operator on $\mathcal{S}'(\mathbb{R}^n)$. If there exist left and right parametrix then we call $p(x, D)$ the parametrix of the operator $a(x, D)$.

We recall the following Theorem on the parametrix construction for operator with elliptic symbol, (see for details Theorem 2.10 [57])

Theorem 10. Let $a \in S^m$. Then the following statements are equivalent

- i) There exists a $b \in S^{-m}$ such that $a \# b - 1 \in S^{-\infty}$;

- ii) There exists a $b \in S^{-m}$ such that $b\sharp a - 1 \in S^{-\infty}$;
- iii) There exists a $b_0 \in S^{-m}$ such that $ab_0 - 1 \in S^{-1}$;
- iv) There exists an $\varepsilon > 0$ such that $|a(x, \xi)| \geq \varepsilon \langle \xi \rangle^m$ for $|\xi| \geq 1/\varepsilon$.

Now we may construct a parametrix for a hypoelliptic operator. This is done by constructing formal series of symbols to whom to apply Proposition 11. For this reason, we recall the following Proposition (see [18])

Proposition 13. *Let a be a hypoelliptic symbol of type (m, ℓ) . We set*

$$p_0(x, \xi) = a^{-1}(x, \xi)\varphi(\xi),$$

with $\varphi \in C^\infty(\mathbb{R}^n)$, $\varphi(\xi) = 0$ for $|\xi| \geq 2R$.

- i) p_0 is an element of $S^{-\ell}(\mathbb{R}^{2n})$;
- ii) for all $\alpha, \beta \in \mathbb{N}^n$, $p_0 \partial_\xi^\alpha \partial_x^\beta a \in S^{-|\alpha|}$.
- iii) We define for $h \geq 1$,

$$p_h(x, \xi) = - \left(\sum_{|\gamma|+j=h, j < h} \frac{(-i)^{|\gamma|}}{\gamma!} \partial_x^\gamma a(x, \xi) \partial_\xi^\gamma p_j(x, \xi) \right) p_0(x, \xi) \quad (1.32)$$

Then $p_h(x, \xi) \in S^{-\ell-h}(\mathbb{R}^{2n})$ for all h .

Now we are able to recall the following Theorem (Theorem 10.20 [18])

Theorem 11. *Let a be a hypoelliptic symbol of type (m, ℓ) . Then there exists $p \in S^{-\ell}(\mathbb{R}^{2n})$ such that*

$$p(x, D)a(x, D) = I + r(x, D)$$

$$a(x, D)p(x, D) = I + s(x, D),$$

where $r(x, D)$ and $s(x, D)$ are regularizing operators (i.e. r and s are symbols in $S^{-\infty}$).

1.3 Γ - differential operators

1.3.1 Γ - pseudodifferential operators

The main subject of this section, and in general of this thesis, is the study of some properties for relevant classes of operators, including as basic examples partial differential operators with polynomial coefficients in \mathbb{R}^d

$$\sum_{|\alpha|+|\beta|\leq m} c_{\alpha,\beta} x^\beta D^\alpha.$$

For this reason in this section, attention is confined to Γ -pseudodifferential operators corresponding to the Γ -symbols defined in Definition 10. More generally we can introduce the following classes

Definition 20. Let $m \in \mathbb{R}$ and $0 < \rho \leq 1$. We define $\Gamma_\rho^m(\mathbb{R}^d)$, as the set of all functions $a(z) \in C^\infty(\mathbb{R}^{2d})$ satisfying,

$$|\partial_z^\gamma a(z)| \lesssim \langle z \rangle^{m-\rho|\gamma|}, \quad (1.33)$$

for all $\gamma \in \mathbb{N}^{2d}$.

In this definition we assume $\rho > 0$, so the strong uncertainty principle (1.21) is satisfied. Note that $\Gamma_1^m(\mathbb{R}^d) = \Gamma^m(\mathbb{R}^d)$, if $a \in \Gamma_\rho^{m_1}(\mathbb{R}^d)$ and $b \in \Gamma_\rho^{m_2}(\mathbb{R}^d)$, then $ab \in \Gamma_\rho^{m_1+m_2}(\mathbb{R}^d)$, $\partial_z^\alpha a \in \Gamma_\rho^{m_1-\rho|\alpha|}(\mathbb{R}^d)$ and for all $\alpha \in \mathbb{N}^{2d}$

$$\bigcap_m \Gamma_\rho^m(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d).$$

One can also introduce a notion of *asymptotic expansion* in Γ_ρ^m .

Definition 21. Let $a_j \in \Gamma_\rho^{m_j}(\mathbb{R}^d)$, $j = 1, 2, \dots, m_j \rightarrow -\infty$ and $a \in C^\infty(\mathbb{R}^n)$. We will write

$$a \sim \sum_{j=1}^{\infty} a_j, \quad (1.34)$$

if, for any integer $r \geq 2$,

$$a - \sum_{j=1}^{r-1} a_j \in \Gamma_{\rho}^{\tilde{m}}(\mathbb{R}^d), \quad (1.35)$$

where $\tilde{m} = \max_{j \geq r} m_j$. The right-hand side of (1.34) is called asymptotic expansion of a .

As explained by the following Proposition (we refer the reader to Proposition 23.1 in [58] for more details), we have existence and uniqueness (modulo a Schwartz function) of a symbol having a given asymptotic expansion.

Proposition 14. *Let $a_j \in \Gamma_{\rho}^{m_j}(\mathbb{R}^d)$, $j = 1, 2, \dots, m_j \rightarrow -\infty$ as $j \rightarrow +\infty$. Then there exists a function a such that $a \sim \sum_{j=1}^{\infty} a_j$. If another function b has the same property, then $a - b \in \mathcal{S}(\mathbb{R}^n)$.*

Now we can define another symbol classes $\Gamma_{cl}^m(\mathbb{R}^d)$

Definition 22. *Let $m \in \mathbb{R}$. We define $\Gamma_{cl}^m(\mathbb{R}^d)$, as the subset of $\Gamma^m(\mathbb{R}^{2d})$ of all symbols $a(z)$ which admit asymptotic expansion*

$$a(z) \sim \sum_{k=0}^{\infty} a_{m-k}(z), \quad (1.36)$$

for a sequence of functions $a_{m-k} \in C^{\infty}(\mathbb{R}^{2d} \setminus \{0\})$ which are positively homogeneous of degree $m - k$.

We will denote by $OPI^m(\mathbb{R}^d)$, $OPI_{\rho}^m(\mathbb{R}^d)$ and $OPI_{cl}^m(\mathbb{R}^d)$ the classes of pseudodifferential operators with symbols in $\Gamma^m(\mathbb{R}^d)$, $\Gamma_{\rho}^m(\mathbb{R}^d)$ and $\Gamma_{cl}^m(\mathbb{R}^d)$ respectively. Note that the pseudodifferential operator

$$a(x, D)u(x) = \int e^{ix\xi} a(x, \xi) \hat{u}(\xi) \bar{d}\xi,$$

maps $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$ continuously and $\mathcal{S}'(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$.

We now recall the composition formula: If $a \in \Gamma_{\rho}^{m_1}(\mathbb{R}^d)$, $b \in \Gamma_{\rho}^{m_2}(\mathbb{R}^d)$ with

$0 < \rho \leq 1$ then the operator $c(x, D) = a(x, D)b(x, D)$ belongs to $OPI_{\rho}^{m_1+m_2}(\mathbb{R}^d)$ with symbol

$$c(x, \xi) \sim \sum_{\alpha} (\alpha!)^{-1} \partial_{\xi}^{\alpha} a(x, \xi) D_x^{\alpha} b(x, \xi),$$

For these operators we want to construct an approximate inverse for the elliptic operators. For this reason we define the notion of Γ -elliptic symbols:

Definition 23. *Let $m \in \mathbb{R}$ and $a \in \Gamma^m(\mathbb{R}^d)$. We say that a is Γ -elliptic if there exists $R > 0$ such that*

$$|z|^m \lesssim |a(z)| \quad \text{for } |z| \geq R. \quad (1.37)$$

When applying this definition in $\Gamma_{cl}^m(\mathbb{R}^d) \subset \Gamma^m(\mathbb{R}^d)$, we shall rather rely on the following equivalent notion of Γ -ellipticity (see for more details Proposition 2.1.5 [50]).

Proposition 15. *The symbol $a \in \Gamma_{cl}^m(\mathbb{R}^d)$ is Γ -elliptic if and only if the principal part a_m satisfies*

$$a(z) \neq 0 \quad \text{for every } z \neq 0. \quad (1.38)$$

Now we may construct a parametrix for a Γ -elliptic operator. For this reason we recall the following Theorem (Theorem 2.1.6 [50])

Theorem 12. *Let $a \in \Gamma^m(\mathbb{R}^d)$ be Γ -elliptic. Then there exists $b \in \Gamma^{-m}(\mathbb{R}^d)$ such that $b(x, D)$ is a parametric of $a(x, D)$, i.e.*

$$a(x, D)b(x, D) = I + S_1(x, D) \quad b(x, D)a(x, D) = I + S_2(x, D),$$

where $S_1(x, D)$ and $S_2(x, D)$ are regularizing operators. Hence $a(x, D)$ is globally regular i.e. $u \in \mathcal{S}'(\mathbb{R}^d)$ and $a(x, D)u \in \mathcal{S}(\mathbb{R}^d)$ imply $u \in \mathcal{S}(\mathbb{R}^d)$.

Γ -operator in $\mathcal{Q}^s(\mathbb{R}^n)$

The natural functional frameworks of Γ -operators are *weighted Sobolev spaces*, the Shubin spaces $\mathcal{Q}^s(\mathbb{R}^n)$. We can define $\mathcal{Q}^s(\mathbb{R}^n)$ using arbitrary Γ -symbols, as stated the following Proposition (see for more details Proposition 2.1.9 [50]):

Proposition 16. *Let $T \in \text{OP}\Gamma^s(\mathbb{R}^d)$ have a Γ -elliptic symbol. Then*

$$\mathcal{Q}^s(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) : Tu \in L^2(\mathbb{R}^d)\} \quad (1.39)$$

We can provide $\mathcal{Q}^s(\mathbb{R}^d)$ of the structure of Hilbert space by this scalar product

$$(u, v)_{\mathcal{Q}^s(\mathbb{R}^d)} = (Tu, Tv)_{L^2(\mathbb{R}^d)} + (Ru, Rv)_{L^2(\mathbb{R}^d)}, \quad (1.40)$$

where R is a regularizing operator associated to a parametrix $\tilde{T} \in \text{OP}\Gamma^{-s}(\mathbb{R}^d)$ of T , namely $\tilde{T}T = I + R$.

From the definition and elementary properties of Sobolev spaces, we have the following Theorem (see Theorem 2.1.10 [50]).

Theorem 13. *Every $A \in \text{OP}\Gamma^m(\mathbb{R}^d)$ defines, for all $s \in \mathbb{R}$, a continuous operator*

$$a(x, D) : \mathcal{Q}^s(\mathbb{R}^d) \rightarrow \mathcal{Q}^{s-m}(\mathbb{R}^d).$$

One of the properties of the $\mathcal{Q}^s(\mathbb{R}^d)$ spaces is the compactness of the map $A : \text{OP}\Gamma^s(\mathbb{R}^d) \rightarrow \text{OP}\Gamma^t(\mathbb{R}^d)$, for $A \in \text{OP}\Gamma^m(\mathbb{R}^d)$ whenever $s - t > m$. In particular if A is regularizing, ($A \in \text{OP}\Gamma^{-\infty}(\mathbb{R}^d) := \bigcap_m \text{OP}\Gamma^m(\mathbb{R}^d)$), then it is continuous and compact from $\mathcal{Q}^s(\mathbb{R}^d)$ to $\mathcal{Q}^t(\mathbb{R}^d)$, for any $s, t \in \mathbb{R}$. Moreover, for every $s \in \mathbb{R}$ we have the continuous immersions

$$j : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{Q}^s(\mathbb{R}^d), \quad j : \mathcal{Q}^s(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d).$$

The following equivalent definition, when $s \in \mathbb{N}$, is peculiar for the spaces $\mathcal{Q}^s(\mathbb{R}^d)$, as stated by the following Theorem (for more details see Theorem 2.1.12 [50])

Theorem 14. *Let $s \in \mathbb{R}$. An equivalent definition of the space $\mathcal{Q}^s(\mathbb{R}^d)$ is given by*

$$\mathcal{Q}^s(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) : x^\beta D^\alpha u \in L^2(\mathbb{R}^d), \text{ for } |\alpha| + |\beta| \leq s\} \quad (1.41)$$

with the equivalent norm

$$\|u\|_{\mathcal{Q}^s(\mathbb{R}^d)} = \sum_{|\alpha|+|\beta| \leq m} \|x^\beta D^\alpha u\|_{L^2(\mathbb{R}^d)}. \quad (1.42)$$

We can give a more precise statement of Theorem 12, in the context of weighted spaces $\mathcal{Q}^s(\mathbb{R}^d)$. This is the following Theorem 15 (we refer the reader to Theorem 2.1.13 in [50] for more details).

Theorem 15. *Let $A \in OPG^m(\mathbb{R}^d)$ with Γ -elliptic symbol and assume $u \in \mathcal{S}'(\mathbb{R}^d)$, $Au \in \mathcal{Q}^s(\mathbb{R}^d)$. Then $u \in \mathcal{Q}^{s+m}(\mathbb{R}^d)$ and for every $t < s + m$,*

$$\|u\|_{\mathcal{Q}^{s+m}(\mathbb{R}^d)} \leq C \left(\|Au\|_{\mathcal{Q}^s(\mathbb{R}^d)} + \|u\|_{\mathcal{Q}^t(\mathbb{R}^d)} \right) \quad (1.43)$$

for a positive constant $C_{s,t}$.

In particular if m is a positive integer, we may refer to equivalent norm (1.42) and rewrite (1.43) for $s = 0$, $t = 0$:

$$\sum_{|\alpha|+|\beta| \leq m} \|x^\beta D^\alpha u\|_{L^2(\mathbb{R}^d)} \leq C \left(\|Au\|_{L^2(\mathbb{R}^d)} + \|u\|_{L^2(\mathbb{R}^d)} \right) \quad (1.44)$$

If we denote by A_s the restriction of $A : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{Q}^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, or equivalently the extension of $A : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ to $\mathcal{Q}^s(\mathbb{R}^d)$, then the operator A_s is a Fredholm operator and for this reason we recall the following Theorem (for more details see Theorem 2.1.14 [50])

Theorem 16. Consider $A \in O\Gamma^m(\mathbb{R}^d)$ with Γ -elliptic symbol. Then:

i) $A_s \in \text{Fred}\left(Q^s(\mathbb{R}^d), Q^{s-m}(\mathbb{R}^d)\right)$.

ii) $\text{ind}A_s = \dim\text{Ker}(A) - \dim\text{Ker}(A^*)$, $\text{ind}A_s = \dim\text{Ker}(A) - \dim\text{Ker}(A^t)$, where A^* is the formal adjoint and A^t is the transposed. Observe that the index is then independent of s .

iii) If $T \in O\Gamma^n(\mathbb{R}^d)$, with $n < m$, Then $A_s + T_s \in \text{Fred}\left(Q^s(\mathbb{R}^d), Q^{s-m}(\mathbb{R}^d)\right)$ and $\text{ind}(A_s + T_s) = \text{ind}(A_s)$.

iv) If $A : Q^s(\mathbb{R}^d) \rightarrow Q^{s-m}(\mathbb{R}^d)$, is invertible for some $s \in \mathbb{R}$; then it is invertible for all $s \in \mathbb{R}$, and the inverse is an operator in $O\Gamma^{-m}(\mathbb{R}^d)$.

We have denoted by $\text{ind}A_s$ the index of the Fredholm operator A_s , (namely $\text{ind}A_s = \dim\text{Ker}A_s - \dim\text{CoKer}A_s$).

In conclusion, we consider the notion of hypoelliptic symbol in the frame $\Gamma_\rho^m(\mathbb{R}^d)$, $0 < \rho \leq 1$.

Definition 24. Let $m \in \mathbb{R}$. We say $a(z) \in \Gamma_\rho^m(\mathbb{R}^d)$, $0 < \rho \leq 1$, is Γ_ρ -hypoelliptic, if there exist $m_1 \in \mathbb{R}$, $m_1 \leq m$ and $R > 0$ such that

$$|z|^{m_1} \lesssim |a(z)|, \quad \text{for } |z| \geq R, \quad (1.45)$$

and for every $\gamma \in \mathbb{N}^{2d}$,

$$|\partial_z^\gamma a(z)| \lesssim |a(z)| \langle z \rangle^{-\rho|\gamma|}, \quad \text{for } |z| \geq R. \quad (1.46)$$

If $m_1 = m$ in (1.45) then (1.46) holds. It follows that if a symbol is Γ -elliptic then it is also Γ_ρ -elliptic

Now we may construct a parametrix for a hypoelliptic operator. For this reason, we recall the following Theorem (see Theorem 2.1.16 [50])

Theorem 17. *Let $a \in \Gamma_\rho^m$, $0 < \rho \leq 1$ be a Γ_ρ -hypoelliptic for some $m_1 < m$ in (1.45). Then there exists $b \in \Gamma^{-m_1}(\mathbb{R}^d)$ such that*

$$a(x, D)b(x, D) = I + s_1(x, D) \quad b(x, D)a(x, D) = I + s_2(x, D),$$

where $s_1(x, D)$ and $s_2(x, D)$ are regularizing operators. Hence $a(x, D)$ is global regular. Moreover, if we assume $u \in \mathcal{S}'(\mathbb{R}^d)$, $a(x, D)u \in \mathcal{Q}^s(\mathbb{R}^d)$, then we have $u \in \mathcal{Q}^{s+m_1}(\mathbb{R}^d)$ and for every $t < s + m_1$

$$\|u\|_{\mathcal{Q}^{s+m_1}(\mathbb{R}^d)} \leq C \left(\|Au\|_{\mathcal{Q}^s(\mathbb{R}^d)} + \|u\|_{\mathcal{Q}^t(\mathbb{R}^d)} \right), \quad (1.47)$$

for a positive constant C depending on s and t .

Γ -elliptic differential operator

It is well known that all differential operators in $OPI_\rho^m(\mathbb{R}^d)$, $0 < \rho \leq 1$ have polynomial coefficients. In fact we have the following Proposition (see Proposition 2.2.1 [50])

Proposition 17. *Assume $p(x, \xi) \in \Gamma_\rho^m(\mathbb{R}^d)$ is of the form*

$$p(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha,$$

for some $a_\alpha(x) \in C^\infty(\mathbb{R}^d)$. Then $a_\alpha(x)$ is a polynomial.

Let us then consider

$$P = \sum_{|\alpha|+|\beta| \leq m} c_{\alpha, \beta} x^\beta D^\alpha, \quad c_{\alpha, \beta} \in \mathbb{C}. \quad (1.48)$$

with symbols

$$p(z) = \sum_{|\gamma| \leq m} c_\gamma z^\gamma, \quad \text{where } \gamma = (\beta, \alpha), z = (x, \xi). \quad (1.49)$$

The principal part is given by

$$p_m(z) = \sum_{|\gamma|=m} c_\gamma z^\gamma, \quad (1.50)$$

and the equivalent Γ -ellipticity condition (1.38) is

$$p_m(z) = \sum_{|\gamma|=m} c_\gamma z^\gamma \neq 0, \quad \text{for } z \neq 0. \quad (1.51)$$

We may now consider the case of a generic ordinary differential operator with polynomial coefficients in \mathbb{R} . The Γ -ellipticity condition (1.51) reads

$$p_m(x, \xi) = \sum_{\alpha+\beta=m} c_{\alpha,\beta} x^\beta \xi^\alpha \neq 0, \quad \text{for } (x, \xi) \in \mathbb{R}^2 \setminus \mathbf{0}. \quad (1.52)$$

Factorizing we obtain

$$p_m(x, \xi) = c(\xi - r_1 x)(\xi - r_2 x) \dots (\xi - r_m x), \quad (1.53)$$

with $\Im r_j \neq 0$, $j = 1, \dots, m$, $c \neq 0$. Hence we may write our operator P (after a multiplication by c^{-1})

$$P = (D_x - r_1 x)(D_x - r_2 x) \dots (D_x - r_m x) + \sum_{\alpha+\beta < m} a_{\alpha,\beta} x^\beta D^\alpha, \quad (1.54)$$

for some constants $a_{\alpha,\beta} \in \mathbb{C}$. We may regard P as a Fredholm operator

$$P : Q^m(\mathbb{R}) \rightarrow L^2(\mathbb{R}).$$

Thus, we obtain the following result (see for more details Theorem 2.2.2 [50], and the reference therein).

Theorem 18. *Consider P in (1.53), (1.54) and assume $\Im r_j > 0$ for $j = 1, \dots, m^+$, $\Im r_j < 0$ for $j = m^+ + 1, \dots, m$, $m = m^+ + m^-$. Then P is a Fredholm operator with*

$$\text{ind} P = m^+ - m^-.$$

From this Theorem, concerning the existence of a non-trivial solution $u \in \mathcal{S}'(\mathbb{R})$, hence $u \in \mathcal{S}(\mathbb{R})$, of $Pu = 0$, we may obtain the following conclusions (see [50]):

- If $\text{ind}P > 0$ then $\dim \text{Ker}(P) > 0$ and a non trivial solution exists.
- If $m^+ = 0, m^- = m$ then $\text{ind}P = 0$ and a non trivial solution does not exist.

A motivating example is the *Harmonic oscillator* of Quantum Mechanics:

$$H = -\Delta + |x|^2. \quad (1.55)$$

We begin here to compute eigenvalues and eigenfunctions of H . We have the following result (see Theorem 2.2.3 [50])

Theorem 19. *The equation*

$$Pu = Hu - \lambda u = -\Delta u + |x|^2 u - \lambda u = 0, \quad u \in \mathcal{S}'(\mathbb{R}^d),$$

admits for

$$\lambda = \lambda_k = \sum_{j=1}^d (2k_j + 1), \quad k = (k_1, \dots, k_n) \in \mathbb{Z}_+^d,$$

the solution in $\mathcal{S}(\mathbb{R}^d)$,

$$u_k(x) = \prod_{j=1}^d P_{k_j}(x_j) \exp(-|x|^2/2), \quad (1.56)$$

where $P_r(t)$ is the r -th Hermite polynomial. Note that the family $u_k, k \in \mathbb{Z}_+^d$, forms an orthogonal system in $L^2(\mathbb{R}^d)$.

Remark 5. *Because of the completeness of the Hermite functions u_k we know from the spectral theory (for more details see the Theorem 24) that for $\lambda \neq \lambda_k$ the map*

$$P = H - \lambda : Q^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$$

is an isomorphism, with inverse

$$P^{-1} = (H - \lambda)^{-1} : L^2(\mathbb{R}^d) \rightarrow Q^2(\mathbb{R}^d),$$

belonging to $OP\Gamma^{-2}(\mathbb{R}^d)$ (see Theorem 16 (iv)).

Γ -operator in Gelfand-Shilov $S_{\nu}^{\mu}(\mathbb{R}^d)$

Let us consider the linear partial differential operators with polynomial coefficients in \mathbb{R}^d (1.48) and assume the fulfilment of the Γ -ellipticity condition

$$p_m(x, \xi) \neq 0 \quad \text{for } (x, \xi) \neq (0, 0). \quad (1.57)$$

The following result holds in Gelfand-Shilov spaces (see [20] and Theorem 6.2.1 in [50])

Theorem 20. *Let P in (1.48) satisfy the Γ -ellipticity condition (1.57). If $u \in \mathcal{S}'(\mathbb{R}^d)$ is a solution of $Pu = f$ with $f \in S_{\mu}^{\mu}(\mathbb{R}^d)$, $\mu \geq \frac{1}{2}$ then $u \in S_{\mu}^{\mu}(\mathbb{R}^d)$. In particular, the equation $Pu = 0$, with $u \in \mathcal{S}'(\mathbb{R}^d)$ implies $u \in S_{\frac{1}{2}}^{\frac{1}{2}}(\mathbb{R}^d)$.*

Recently, Gramchev, Pilipovic and Rodino [24] have characterized the Gelfand-Shilov spaces $S_{\mu}^{\mu}(\mathbb{R}^d)$ by the decay of the Fourier coefficients associated to the eigenvalues. For this reason we recall the following theorem (for more details see Theorem 1.2 [24])

Theorem 21. *Suppose that P in (1.48) is Γ -elliptic and normal operator, $PP^* = P^*P$. Then $\text{spec}(PP^*) = \{\lambda_1^2 \leq \dots \leq \lambda_j^2 \leq \dots\}$, $\lambda_j \geq 0$ with an orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$. Let $\mu \geq 1/2$. Then for any $u \in \mathcal{S}'(\mathbb{R}^d)$ we have*

$$u \in S_{\mu}^{\mu}(\mathbb{R}^d) \iff \sum_{j=1}^{\infty} |a_j|^2 e^{\varepsilon |\lambda_j|^{1/(m\mu)}} < \infty, \quad (1.58)$$

for some $\varepsilon > 0 \iff \sum_{j=1}^{\infty} |a_j|^2 e^{\varepsilon j^{1/(m\mu)}} < \infty$, for some $\varepsilon > 0 \iff$ there exist $C > 0, \varepsilon > 0$ such that

$$|a_j| \leq C e^{-\varepsilon j^{2/d\mu}}, \quad j \in \mathbb{N}, \quad (1.59)$$

where $a_j = u(\varphi_j)$ are the Fourier coefficients of u .

1.4 Spectral Theory

1.4.1 Unbounded Operators in Hilbert spaces

In this section we recall some basic facts about unbounded operators in Hilbert spaces.

Let H_1 and H_2 be Hilbert spaces and suppose we are given an unbounded operator

$$A : H_1 \rightarrow H_2$$

The *adjoint operator*

$$A^* : H_2 \rightarrow H_1$$

is defined if the domain of A (denote by D_A) is dense in H_1 and, in this case, D_{A^*} is the set of all $v \in H_2$, for which there exists $g \in H_1$ such that

$$(Au, v) = (u, g), \quad u \in D_A.$$

It is clear that g is uniquely defined and by definition $A^*v = g$. In particular, we have the identity

$$(Au, v) = (u, A^*v), \quad u \in D_A, v \in D_{A^*}.$$

Let H be a Hilbert space. An operator $A : H \rightarrow H$ is called *symmetric* if

$$(Au, v) = (u, Av), \quad u, v \in D_A,$$

while an operator $A : H \rightarrow H$ is called *self-adjoint* if $A = A^*$. Note that a self-adjoint operator is symmetric. The converse is in general not true.

Let $A : H_1 \rightarrow H_2$. We define the graph of A as the linear subspace $\{(u, Au) \in$

$H_1 \times H_2 : u \in D_A\}$ of $H_1 \times H_2$. A is called *closed* if its graph is a closed subspace in $H_1 \oplus H_2$.

A is called *closable* if the closure of its graph is still a graph of a linear operator, which is then denoted by \bar{A} .

Note that, any symmetric operator $A : H \rightarrow H$ has a closure if D_A is dense, and its closure \bar{A} is, also, symmetric, (see Proposition 4.1.1 in [50]).

A densely defined symmetric operator A is called *essentially self-adjoint* if \bar{A} is self-adjoint. Note that, it is equivalent to saying that $A^* = \bar{A}$, because for any closable operator A we have $\bar{A}^* = A^*$. A criterion for essential self-adjointness is given by the following Proposition (for more details see Theorem 26.1 [58])

Proposition 18. *A symmetric operator $A : H \rightarrow H$ with dense domain is essentially self-adjoint if and only if the following inclusions*

$$\begin{aligned} \text{Ker}(A^* - iI) &\subset D_{\bar{A}}, \\ \text{Ker}(A^* + iI) &\subset D_{\bar{A}}. \end{aligned}$$

hold.

Now we can introduce the *resolvent set* and the *spectrum* of an operator A .

Definition 25. *Let A be a closed densely defined operator on a complex Hilbert space H . We define the resolvent set of A as the set $\rho(A)$ of complex numbers λ such that $A - \lambda I$ is a bijection $D_A \rightarrow H$, with a bounded inverse $R_A(\lambda) = (A - \lambda I)^{-1}$. $R_A(\lambda)$ is called resolvent operator of A . The spectrum of A is the complementary set of $\rho(A)$ in \mathbb{C}*

$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

It is well known that self-adjoint operators have a real spectrum. For this reason we recall the following Proposition (see Proposition 4.1.2 [50])

Proposition 19. *If A is self-adjoint, then $\sigma(A) \subset \mathbb{R}$.*

As explained by the following proposition (see for more details Proposition 4.13 in [50]), if there exists $\lambda \in \rho(A)$ such that $R_A(\lambda)$ is compact then A has a compact resolvent.

Proposition 20. *If A has a compact resolvent, then $R_A(\lambda)$ is compact for all $\lambda \in \rho(A)$.*

We note that self-adjoint operators with compact resolvent have very simple spectrum. In order to show this, we recall the following theorem, about the spectrum of compact operators, combined with the subsequent lemma (we refer the reader to Theorem 4.1.4 and Lemma 4.1.5 in [50] for more details).

Theorem 22. *Let A be a compact operator on a complex Hilbert space H . Then $\sigma(A)$ is at most countable set with no accumulation point from 0 . Each non zero $\lambda \in \sigma(A)$ is an eigenvalue with finite multiplicity. If A is also self-adjoint, then all eigenvalues are real and H has an orthonormal basis made of eigenvectors of A .*

Lemma 3. *Let A be a closed densely defined operator on a complex Hilbert space H such that $\rho(A) \neq \emptyset$. Then for any $\lambda_0 \in \rho(A)$ we have*

$$\rho(A) = \{\lambda_0\} \cup \{\lambda \in \mathbb{C} : \lambda \neq \lambda_0 \text{ and } (\lambda - \lambda_0)^{-1} \in \rho(R_A(\lambda_0))\}.$$

Now we are able to describe the spectrum of a self-adjoint operator with compact resolvent (see Theorem 4.1.6 [50]).

Theorem 23. *Let A be a densely defined self-adjoint operator on a complex Hilbert space H . If A has compact resolvent, then $\sigma(A)$ is a sequence of real isolated eigenvalues, diverging to ∞ . Each eigenvalue has finite multiplicity and H has an orthonormal basis made of eigenfunctions of A .*

1.4.2 Spectrum of hypoelliptic symmetric operator

In this section we want to describe the spectrum of operators with hypoelliptic symbols. For this reason we recall the following definition.

Definition 26. *A symbol $a \in S(M; \phi, \psi)$ is called (global) hypoelliptic, if there exist a temperate weight $M_0(x, \xi)$ and a radius $R > 0$, such that*

$$|a(x, \xi)| \gtrsim M_0(x, \xi) \quad \text{for } |x| + |\xi| \geq R, \quad (1.60)$$

and, for every $\alpha, \beta \in \mathbb{N}^d$

$$|\partial_{\xi}^{\alpha} \partial_x^{\beta} a(x, \xi)| \lesssim |a(x, \xi)| \psi(x, \xi)^{-|\alpha|} \phi(x, \xi)^{-|\beta|}, \quad |x| + |\xi| \geq R. \quad (1.61)$$

We denote the class of such symbols by $\text{Hypo}(M, M_0)$.

Now we study the spectrum of operators with hypoelliptic symbols. For this reason we recall the following proposition (see for more details Proposition 4.2.5 [50])

Proposition 21. *Assume the strong uncertainty principle (1.21). Consider a pseudodifferential operator A with symbol in $H(M, M_0)$, with $M_0(x, \xi) \rightarrow +\infty$ at infinity. Then its closure \bar{A} in L^2 has either spectrum $\sigma(\bar{A}) = \mathbb{C}$ or has a compact resolvent.*

Now we can recall the main result: the spectral theorem for operators with hypoelliptic symbols (see Theorem 4.2.9 [50])

Theorem 24. *Assume the strong uncertainty principle (1.21). Consider a pseudodifferential operator A with real-valued symbol in $H(M, M_0)$, with $M_0(x, \xi) \rightarrow +\infty$ at infinity. Its closure \bar{A} in L^2 has spectrum given by a sequence of real eigenvalues either diverging to $+\infty$ or $-\infty$. The eigenvalues have all finite multiplicity and the eigenfunctions belong to $\mathcal{S}(\mathbb{R}^d)$. Moreover $L^2(\mathbb{R}^d)$ has an orthonormal basis made of eigenfunctions of \bar{A} .*

1.4.3 Weyl Asymptotics

In this section, we consider the classes $OP\Gamma_\rho^m(\mathbb{R}^d)$, $m > 0$, $0 < \rho \leq 1$, (see Definition 20), and we recall the asymptotic distribution of the eigenvalues of some special classes of elliptic self-adjoint operators.

In particular, we denote by λ_j the eigenvalues of an operator A and we define the so-called *counting function*

$$N(\lambda) = \#\{j : \lambda_j \leq \lambda\}.$$

We can describe the asymptotic behaviour of the $N(\lambda)$. For this reason we recall the following theorem (for more details see Theorem 4.6.3 [50])

Theorem 25. *Let $a \in \Gamma_\rho^m(\mathbb{R}^d)$, $m > 0$, $0 < \rho \leq 1$ be real valued,*

$$a(x, \xi) = a_m(x, \xi) + a_{m-\rho}(x, \xi) \quad \text{for } |x| + |\xi| \text{ large,}$$

where $a_m(x, \xi)$ is real value and satisfies $0 < a_m(tx, t\xi) + t^m a_m(x, \xi)$, for $t > 0$, $(x, \xi) \in \mathbb{R}^d$, and $a_{m-\rho}(x, \xi) \in \Gamma_\rho^{m-\rho}(\mathbb{R}^d)$. Then the counting function $N(\lambda)$ of the operator $a(x, D)$ has the asymptotic behaviour

$$N(\lambda) \sim C \lambda^{\frac{2d}{m}} \quad \text{as } \lambda \rightarrow +\infty, \quad (1.62)$$

where C is given by

$$C = \frac{(2\pi)^{-d}}{2d} \int_{\mathbb{S}^{2d-1}} a_m(\Theta)^{-\frac{2d}{m}} d\Theta. \quad (1.63)$$

We note that, by using the homogeneity of a_m , we can rewrite the formula (1.62) as

$$N(\lambda) \sim \int_{a_m(x, \xi) \leq \lambda} \bar{d}x \bar{d}\xi \quad \text{as } \lambda \rightarrow +\infty, \quad (1.64)$$

which is, up to the factor $(2d)^{-1}$, the volume of the set

$$\{(x, \xi) \in \mathbb{R}^{2d} : a_m(x, \xi) \leq \lambda\}.$$

We can also deduce the asymptotic behaviour of eigenvalues and we recall the well known Theorem (for more details see Theorem 4.6.4 [50])

Theorem 26. *Under the hypotheses of Theorem 25 we have*

$$\lambda_j \sim C^{-\frac{m}{2d}} j^{\frac{m}{2d}} \quad \text{as } j \rightarrow +\infty, \quad (1.65)$$

where C is given in (1.63).

Chapter 2

Normal forms and conjugations for second order self-adjoint globally elliptic operators

We consider second order self-adjoint differential operators generalizing the harmonic oscillator

$$P(x, D) = -\Delta + \langle Ax, D_x \rangle + \langle Bx, x \rangle + \langle M, D_x \rangle + \langle N, x \rangle + r, \quad (2.1)$$

where $A, B \in M_n(\mathbb{R})$, B is symmetric, $M, N \in \mathbb{R}^n$, $r \in \mathbb{C}$. We decompose the matrix A to symmetric and skew-symmetric components

$$A = A_{symm} + A_{skew}, \quad A_{symm} = \frac{1}{2}(A + A^T), \quad A_{skew} = \frac{1}{2}(A - A^T), \quad (2.2)$$

with $A_{symm}^T = A_{symm}$ and

$$A_{skew}^T = -A_{skew}. \quad (2.3)$$

There exist non zero skew-symmetric matrices only for $n \geq 2$. We note that (2.3) implies that the quadratic form

$$\langle A_{skew}x, x \rangle = 0, \quad x \in \mathbb{R}^n, \quad (2.4)$$

and if A_{skew} is non singular it is called symplectic.

The hypothesis $A = A_{symm}$ is equivalent to the closedness of the 1-form $Ax dx$, namely, there exists a quadratic form $U(x)$ such that $\nabla U = Ax$. One notes that $A_{skew} \neq 0$ is equivalent to $Ax dx$ is not closed.

Example 1. Consider the following symmetric Shubin operator on \mathbb{R}^2

$$L(x, D) = -\Delta + \sigma(x_1 D_{x_1} + x_2 D_{x_2}) + \tau(x_2 D_{x_1} - x_1 D_{x_2}) + b_1 x_1^2 + b_2 x_2^2, \quad \sigma, \tau, b_1, b_2 \in \mathbb{R}. \tag{2.5}$$

One checks easily that L is globally elliptic iff $\min\{b_1, b_2\} > \sigma^2/4 + \tau^2/4$, $A_{skew} \neq 0$ if $\tau \neq 0$ while for $\sigma = 0, b_1 = b_2 = |\tau|^2/4 > 0$ we recapture self adjoint generalizations of the twisted Laplacian L ($\tau = -1$) and its transposed L^t ($\tau = 1$), cf. [13], [25] and the references therein.

We also investigate perturbations $P + b(x, D)$ of (2.1) with zero order pseudodifferential operators $b(x, D)$ on \mathbb{R}^n of Shubin type

$$b(x, D)u = \int_{\mathbb{R}^n} e^{ix\xi} b(x, \xi) \hat{u}(\xi) \overline{d\xi}. \tag{2.6}$$

Our goals could be summarized as follows

- To derive reductions of P (and $P + b$) to simpler normal forms by means of transformations associated to affine symplectic maps introduced by Hörmander [37] for general classes of p.d.o. on \mathbb{R}^n , namely, via unitary maps of $L^2(\mathbb{R}^n)$ generated by $e^{i\langle Qx, x \rangle}$, Q being real symmetric $n \times n$ matrix, $e^{i\langle \alpha, x \rangle}$, $\alpha \in \mathbb{R}^n$, translations, the action of $SO(n)$ and global FIO with quadratic phase functions. .
- To prove that the corresponding normal form transformations (NFT) preserve the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ and and the Gelfand–Shilov spaces $S_\mu^\mu(\mathbb{R}^n)$, $\mu \geq 1/2$, and the classes of Shubin type pseudodifferential operators.

- To apply the normal forms for getting novel discrete representation of the action of Shubin type p.d.o. and for the study of the spectral properties as well as the hypoellipticity and the solvability of P in $\mathcal{S}(\mathbb{R}^n)$ and $S_V^\mu(\mathbb{R}^n)$.

The first set of new result of the thesis could be summarized as follows: we show that the only obstruction to the reduction of the second order self-adjoint globally elliptic operator P to a multidimensional harmonic oscillator H_ω is the presence of non zero skew-symmetric mixed term in $\langle A_{skew}x, D_x \rangle$. More precisely, we solve completely the problems related to the reduction to harmonic oscillator normal form of P and to aforementioned issues provided there are no rotations in the mixed term perturbation $\langle Ax, D \rangle$, namely

$$A \text{ is symmetric.} \tag{2.7}$$

Clearly the symmetry condition above is not superfluous if $n \geq 2$. We point out that a globally elliptic self-adjoint operator in \mathbb{R}^2 containing, as the twisted Laplacian, a skew-symmetric part in A , is an example when no separation of variables is possible as outlined above.

The second main goal is to study normal forms for anisotropic versions of elliptic self-adjoint Shubin type operators which might be viewed as perturbations of anisotropic harmonic oscillators of the type $-\Delta + \sum_{j=1}^n \omega_j^2 x_j^{2k}$, $\omega_j > 0$, $k \geq 2$. It turns out that it is not enough to ask for the symmetry of the mixed term. One needs additional separation of variables type hypothesis in order to derive and classify the possible normal forms and the spectral and hypoellipticity-solvability properties. However we do not consider perturbations with zero order p.d.o. since the conjugation encounters nontrivial issues like the appearance of global Fourier integral operators with phase functions admitting superquadratic (at least cubic) growth for $|x| \rightarrow \infty$.

2.1 Preliminaries on symmetric Shubin type second order operators

Using standard arguments on quadratic forms and integration by parts we characterize completely the second order linear symmetric Shubin operators.

Proposition 22.

$$P = -\Delta + \langle Ax, D_x \rangle + \langle Bx, x \rangle + \langle L, D_x \rangle + \langle M, x \rangle + p, \quad (2.8)$$

where $A, B \in M_n(\mathbb{R})$ are symmetric matrix, and $L, M \in \mathbb{C}^n$, $p \in \mathbb{C}$. Then following assertions are equivalent

- i) P is symmetric,
- ii) $L, M \in \mathbb{R}^n$, $p + i \frac{\text{tr}(A_{\text{symm}})}{2} \in \mathbb{R}$.

Finally, P is globally elliptic iff the symmetric matrix

$$B - \frac{A_{\text{symm}}^2}{4} + \frac{A_{\text{skew}}^2}{4} + \frac{1}{4}(A_{\text{symm}}A_{\text{skew}} - A_{\text{skew}}A_{\text{symm}}) > 0. \quad (2.9)$$

Proof. One has, taking into account that $\text{tr}(A) = \text{tr}(A_{\text{symm}})$,

$$P^*(x, D) = -\Delta + \langle Ax, D_x \rangle + \langle Bx, x \rangle + \langle \bar{L}, D_x \rangle + \langle \bar{M}, x \rangle + \bar{p} - i \frac{\text{Tr}(A_{\text{symm}})}{2}, \quad (2.10)$$

which yields the equivalence i) \Leftrightarrow ii). As it concerns the global ellipticity, we write explicitly the principal symbol and we obtain the hypothesis. \square

Remark 6. We observe that if P is defined by (2.8) its transposed P^t is defined as follows

$$P^t = -\Delta - \langle Ax, D_x \rangle + \langle Bx, x \rangle - \langle L, D_x \rangle + \langle M, x \rangle + p. \quad (2.11)$$

It is easy to check that if $P = P^*$ we have $P^t = P$ iff $A = 0$.

2.2 Reduction to a global normal form

Let P be a second order globally symmetric linear differential operator of Shubin type in \mathbb{R}^n

$$P = -\Delta + \langle Ax, D_x \rangle + \langle Bx, x \rangle + \langle L, D_x \rangle + \langle M, x \rangle + p - i \frac{\text{Tr} A_{\text{symm}}}{2}, \quad (2.12)$$

where $A, B \in M_n(\mathbb{R})$, B is symmetric matrix, and $L, M \in \mathbb{R}^n$, $p \in \mathbb{R}$.

We propose refinements of results of Sjöstrand [59] on the classification of second order Shubin differential operators.

Theorem 27. *Suppose that*

$$A = A_{\text{symm}}. \quad (2.13)$$

Then the following assertions are equivalent for P defined by (2.12):

- i) P is globally elliptic.*
- ii) There exists an unitary transformation $U : L^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$, where*

$$U_V(x) = e^{-i\frac{1}{4}(\langle A_{\text{symm}}x, x \rangle + \langle \alpha, x \rangle)}_V(S_0x), \quad \alpha \in \mathbb{R}^n, S_0 \in SO(\mathbb{R}^n), \quad (2.14)$$

such that

$$U^* \circ P \circ U = H_\omega = -\Delta + \sum_{j=1}^n \omega_j^2 x_j^2 + r = -\Delta + \langle D_\omega^2 x, x \rangle + r \quad (2.15)$$

for some $\omega = (\omega_1, \dots, \omega_n)$, $\omega_j > 0$, $j = 1, \dots, n$, $r \in \mathbb{R}$.

The spectrum of P which coincides with the spectrum of H_ω is given by

$$\text{spec}(H_\omega) = \{\lambda_\omega(k) + r := \sum_{j=1}^n \omega_j(2k_j + 1) + r = 2 \langle \omega, k \rangle + |\omega| + r, \quad k \in \mathbb{Z}_+^n\} \quad (2.16)$$

with an orthonormal basis of eigenfunctions

$$H_k^\omega(x) := \left(\prod_{j=1}^n \omega_j \right)^{1/2} H_k(D_\omega^{1/2} x) = \left(\prod_{j=1}^n \omega_j \right)^{1/2} \prod_{j=1}^n H_{k_j}(\omega_j^{1/2} x_j). \quad (2.17)$$

48 and setting

$$\{\omega_1, \dots, \omega_n\} = \{\theta_j : j = 1, \dots, d, \theta_j \neq \theta_\ell, 1 \leq j < \ell \leq d\}$$

with $\text{mult}(\theta_j) = n_j$, $n_1 + n_2 + \dots + n_d = n$, we rewrite (2.16)

$$\text{spec}(H_\omega) := \{s_\theta(q) + r : s(q) = \sum_{s=1}^d n_s \theta_s (2q_s + 1) = 2 \langle \theta, q \rangle + n_1 \theta_1 + \dots + n_d \theta_d, q \in \mathbb{Z}_+^d\} \quad (2.18)$$

with the multiplicity of $s_\theta(q) + r$ expressed in the following way

$$\text{mult}(s_\theta(q)) = \left(\sum_{k^1 \in \mathbb{Z}_+^{n_1}; |k^1|=q_1} 1 \right) \dots \left(\sum_{k^d \in \mathbb{Z}_+^{n_d}; |k^d|=q_d} 1 \right) \quad (2.19)$$

provided that $\theta_1, \dots, \theta_d$ are non resonant.

Finally, $H_\omega + r$ is invariant under the linear action of the subgroup $SO_\omega(\mathbb{R}^n)$ defined as follows:

$$SO_\omega(\mathbb{R}^n) := \bigoplus_{j=1}^k SO(\mathbb{R}^{n_j}). \quad (2.20)$$

In particular, if $\omega_j \neq \omega_\ell$, $j \neq \ell$, $SO_\omega(\mathbb{R}^n)$ consist of 2^n symmetries

$$x = (x_1, \dots, x_n) \mapsto (\varepsilon_1 x_1, \dots, \varepsilon_n x_n), \quad \varepsilon_j \in \{1, -1\}, \quad j = 1, \dots, n.$$

Next, we show that the NFT U is an automorphism in the scale of function spaces in the theory of the Shubin type operators.

Theorem 28. *Let U be a unitary transformation defined by (2.14). Then U and the convolution map $U*$ defined by*

$$\begin{aligned} U * v(x) &= e^{-i\frac{1}{4}(\langle A_{\text{symm}} x, x \rangle + \langle \alpha, x \rangle)} * v(S_0 x) \\ &= \int_{\mathbb{R}^n} e^{-i\frac{1}{4}(\langle A_{\text{symm}}(x-y), (x-y) \rangle + \langle \alpha, x-y \rangle)} v(Sy) dy \end{aligned} \quad (2.21)$$

are automorphisms of $B(\mathbb{R}^n)$, where

$$\begin{aligned} B(\mathbb{R}^n) &= \mathcal{S}(\mathbb{R}^n), \quad B(\mathbb{R}^n) = Q^s(\mathbb{R}^n), s \in \mathbb{R}, \\ B(\mathbb{R}^n) &= S_v^\mu(\mathbb{R}^n), \mu \geq \nu \geq 1/2. \end{aligned} \quad (2.22)$$

Finally, we show that the conjugation with U is an automorphism in the space of Shubin class symbols and p.d.o. of fixed order m .

Theorem 29. *Let $b(x, D)$ be a Shubin p.d.o. of order $m \in \mathbb{R}$. Then*

$$U^* \circ b(x, D) \circ U = b_{NF}(x, D) \quad (2.23)$$

where b_{NF} is a Shubin p.d.o. of order m whose symbol $b_{NF}(x, \xi) = \tilde{b}_{NF}(S_0^t x, S_0 \xi)$ satisfies

$$\tilde{b}_{NF}(x, \xi) \sim \sum_{\alpha \in \mathbb{Z}_+^n} \frac{1}{\alpha!} (A \partial_\xi)^\alpha D_\xi b(x, \xi + 2Ax + a) \quad (2.24)$$

If we want summarize the results above, we can say that the spectrum of the original perturbed operator $P(x, D) + b(x, D)$, the global solvability and global hypoellipticity in function spaces $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{Q}^s(\mathbb{R}^n)$, $S_\mu^\mu(\mathbb{R}^n)$, is characterized by the family of equivalent NF, using the action of $SO_\omega(\mathbb{R}^n)$

$$S^T \circ U^* \circ (P(x, D) + b(x, D)) \circ U \circ S = H_\omega + r + b_{NF}(Sx, S^T D). \quad (2.25)$$

In view of the fact that $H_\omega + r$ is globally hypoelliptic in $\mathcal{S}(\mathbb{R}^n)$ and $S_\mu^\mu(\mathbb{R}^n)$ (e.g., see [58], [6]) and globally solvable iff

$$\lambda_\omega(k) - r \neq 0, \quad k \in \mathbb{Z}_+^n, \quad (2.26)$$

we can derive the following perturbation result on the solvability of $P + b$.

Proposition 23. *Suppose that (2.26) holds. Then there exists a small positive constant C such that*

$$\max_{\alpha, \beta \in \mathbb{Z}_+^n, |\alpha| \leq n+1, |\beta| \leq n+1} \sup_{(x, \xi) \in \mathbb{R}^{2n}} |\partial_x^\beta \partial_\xi^\alpha b_{NF}(x, \xi)| < C \min_{k \in \mathbb{Z}_+^n} |\lambda_\omega(k) + r| \quad (2.27)$$

then $P + b$ is invertible in $L^2(\mathbb{R}^n)$ and solvable in $\mathcal{S}(\mathbb{R}^n)$.

Next, we derive the one the main novel results, motivated by and borrowing ideas from the fundamental paper Greendfield and Wallach [30]. In this paper, they study the global hypoellipticity of commuting differential operators on compact Riemannian manifold using discrete representations for commuting normal differential operators. In fact, we derive perturbations the discrete representation in the case when of commuting operators and simple eigenvalues and application . We characterize completely the global properties of $P + b$, if the commutator $[P, b] = 0$, provided the eigenvalues of P are simple.

Theorem 30. *Suppose that the eigenvalues of P are simple, i.e.,*

$$\omega_1, \dots, \omega_n \quad \text{are non resonant.} \tag{2.28}$$

Then there is a unique map $\mathbb{N} \ni j \mapsto k(j) \in \mathbb{Z}_+^n$ such that

$$\lambda_j = \lambda_{k(j)} = 2 \langle \omega, k(j) \rangle + |\omega| + r, \quad j \in \mathbb{N}, \quad \lambda_1 < \lambda_2 < \dots < \lambda_j < \dots \tag{2.29}$$

($k(j) = j - 1$ when $n = 1$) and if

$$[P(x, D), b(x, D)] = 0, \tag{2.30}$$

then $b(x, D)$ has the following discrete representation

$$b(x, D)u = \sum_{k \in \mathbb{Z}_+^n} b(k) \Phi_k(x) = \sum_{j=1}^{\infty} \tilde{b}_j \varphi_j(x), \tag{2.31}$$

where $\Phi_k = U(H_k \omega)$ stands for the orthonormal basis of P , $\tilde{b}_j = b(k(j))$, $\varphi_j(x) = \Phi_{k(j)}(x)$, $j \in \mathbb{N}$ with b_k, \tilde{b}_j satisfying

$$\sup_{k \in \mathbb{Z}_+^n} |b_k| = \sup_{j \in \mathbb{N}} |\tilde{b}_j| < +\infty. \tag{2.32}$$

Finally, we claim that

$$P(x, D) + b(x, D) \text{ is global hypoelliptic in } S_{\mu}^{\mu}(\mathbb{R}^n), \quad \mu \geq 1/2. \tag{2.33}$$

We stress the important point concerning (2.33): we do not require that $b(x, D)$ is Gelfand–Shilov Shubin type p.d.o. as in [5], [6], [7], [8]. For example, if $g(t)$ is smooth bounded function which is not analytic, satisfying $|g^{(j)}(t)| = O(|t|^{-j})$, $t \rightarrow \infty$, $j \in \mathbb{Z}_+$, $g(P)$ is well defined Shubin p.d.o. with principal symbol $g(P(x, \xi))$ which commutes with P .

One is led to conjecture, taking into account the fact that the p.d.o. calculus rules are valid modulo smoothing operators, that if P and q commute modulo regularizing operator we can assume that b equals commuting operator plus a regularizing one.

2.2.1 Proof of the NF result

In the proof of Theorem 27 we will make use of the following well known lemmas:

Lemma 4. *Let P be the operator*

$$P = -\Delta + \sum_{j=1}^n a_j x_j D_{x_j} + \langle Bx, x \rangle + \langle L, D_x \rangle + \langle M, x \rangle + p, \quad (2.34)$$

where $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, $B \in M_n(\mathbb{R})$ is symmetric, $L, M \in \mathbb{R}^n$ and $p \in \mathbb{C}$. Then $u = e^{i\psi(x)}v(x)$, where $\psi(x) = \frac{1}{4} \sum_{j=1}^n a_j x_j^2$ satisfies

$$P(e^{i\psi}v(x)) = e^{i\psi(x)} (-\Delta + \langle Cx, x \rangle + \langle L, D_x \rangle + \langle N, x \rangle + q)v(x),$$

where $C = B - \frac{1}{4} \sum_{j=1}^n a_j x_j^2$, $N = M + \text{diag}\{a_1, \dots, a_n\}L \in \mathbb{R}^n$, $q = p - \frac{i}{2} \sum_{j=1}^n a_j$.

Lemma 5. *Let P be the operator*

$$P = D_x^2 + \sum_{j=1}^n c_j x_j^2 + \langle L, D_x \rangle + \langle M, x \rangle + p, \quad (2.35)$$

where $c = (c_1, \dots, c_n) \in \mathbb{R}^n$, $L, M \in \mathbb{R}^n$ and $p \in \mathbb{R}$. Then $u = e^{i\psi(x)}v(x)$, where $\psi(x) = -\frac{\langle L, x \rangle}{2}$ satisfies

$$P(e^{i\psi(x)}v(x)) = e^{i\psi(x)} \left(D_x^2 + \sum_{j=1}^n c_j x_j^2 + \langle M, x \rangle + p - \frac{\|L\|^2}{4} \right) v(x).$$

Now we can prove Theorem 27.

Proof. First we reduce A to diagonal form $\text{diag} \{a_1, \dots, a_n\}$ by orthogonal transformation $x \rightarrow S_1 x$, $S_1 \in SO(n)$, so the operator P becomes

$$D_x^2 + \sum_{j=1}^n a_j x_j D_{x_j} + \langle B' x, x \rangle + \langle L', D_x \rangle + \langle M', x \rangle + p', \quad (2.36)$$

where $B' = S_1^t B S_1$, $L' = S_1^t L$ and $M' = S_1^t M$.

Now we apply Lemma 4 and reduce (2.36) to

$$D_x^2 + \langle B'' x, x \rangle + \langle L'', D_x \rangle + \langle M'', x \rangle + p''. \quad (2.37)$$

with B'' symmetric. Next, we reduce B'' to diagonal form $\text{diag} \{b_1, \dots, b_n\}$, by means of orthogonal transformation $x \rightarrow S_2 x$, $S_2 \in SO(n)$, so the operator (2.37) becomes

$$P''' = D_x^2 + \sum_{j=1}^n b_j x_j^2 + \langle L''', D_x \rangle + \langle M''', x \rangle + p'', \quad (2.38)$$

where $L''' = S_2^t L''$ and $M''' = S_2^t M''$. We note that the global ellipticity is invariant under the previous transformations which means that the global ellipticity is equivalent to $b_j > 0$ for $j = 1, \dots, n$. Next, we apply Lemma 5 and transform the operator (2.38) to

$$\tilde{P} = D_x^2 + \sum_{j=1}^n b_j x_j^2 + \langle \tilde{L}, x \rangle + \ell. \quad (2.39)$$

Now we apply the translation $T_\gamma(x_j) = x_j + \gamma_j$, where $\gamma_j = -\frac{\tilde{L}_j}{2}$, $j = 1, \dots, n$ and we obtain the normal form operator

$$P_{NF} = -\Delta + \sum_{j=1}^n \omega_j^2 x_j^2 + r, \quad \omega_j = \sqrt{b_j}, \quad j = 1, \dots, n, \quad r = \ell - \sum_{j=1}^n \frac{\tilde{L}_j}{2}. \quad (2.40)$$

As it concerns the explicit form of the spectrum and the eigenfunctions, the dilations $x_j = \omega_j^{-1/2} y_j$, $j = 1, \dots, n$ transform P_{NF} to $\sum_{j=1}^n \omega_j (D_{y_j}^2 + y_j^2) + r$. The invariance under the action of $SO^\omega(\mathbb{R}^n)$ follows from the definition of the centralizer of matrices and the restriction to the corresponding orthogonal groups. The proof is complete. \square

2.3 Normal form transformations in Gelfand-Shilov spaces

We introduce decreasing scales of Banach spaces defining $S_V^\mu(\mathbb{R}^n)$:

$$HS_V^\mu(L^p(\mathbb{R}^n); \rho, \sigma) = \left\{ f \in S_V^\mu(\mathbb{R}^n) : \|f\|_{L^p; \mu, \nu; \rho, \sigma} := \sup_{\alpha, \beta \in \mathbb{Z}_+^n} \left(\frac{\rho^{|\alpha|} \sigma^{|\beta|}}{\alpha!^\mu \beta!^\nu} |x^\beta \partial^\alpha f|_{L^p} \right) \right\}, \quad (2.41)$$

If $\mu < 1$ we will use the fact that the functions of $S_V^\mu(\mathbb{R}^n)$ are restrictions of entire functions $f \in \mathcal{O}(\mathbb{C}^n)$ belonging for some $a, b > 0$ to the following Banach space of entire functions $\mathcal{O}_V^\mu(\mathbb{R}^n; a, b) =$

$$\left\{ f \in \mathcal{O}(\mathbb{C}^n) : \|f\|_{\mathcal{O}; \mu, \nu; a, b} := \sup_{z=x+iy \in \mathbb{C}^n} \left(|f(x+iy)| e^{a|x|^{1/\nu} - b|y|^{1/(1-\mu)}} \right) < +\infty \right\} \quad (2.42)$$

Another useful $S_V^\mu(\mathbb{R}^n)$ norms containing exponential decay are defined by

$$\|f\|_{\mu, \nu; a, b} := \sup_{\alpha \in \mathbb{Z}_+^n} \left(\frac{a^{-|\alpha|}}{\alpha!^\mu} |e^{b|x|^{1/\nu}} \partial_x^\alpha f(x)|_{L^\infty} \right), \quad a, b > 0. \quad (2.43)$$

Next we derive some estimates typical for decreasing scales of Banach spaces.

Proposition 24. *Let $\rho > \tilde{\rho}, \sigma > \tilde{\sigma}$. Then*

$$HS_V^\mu(L^p(\mathbb{R}^n); \rho, \sigma) \hookrightarrow HS_V^\mu(L^p(\mathbb{R}^n); \tilde{\rho}, \tilde{\sigma}).$$

Moreover, the Gelfand–Shilov spaces are defined as inductive limits

$$S_v^\mu(\mathbb{R}^n) = \bigcup_{\rho \searrow 0, \sigma \searrow 0} HS_v^\mu(L^p(\mathbb{R}^n); \rho, \sigma), \quad 1 \leq p \leq +\infty.$$

for all $\mu, \nu > 0$, $\mu + \nu \geq 1$ and

$$S_v^\mu(\mathbb{R}^n) = \bigcup_{a \searrow 0, b \searrow 0} \mathcal{O}S_v^\mu(\mathbb{R}^n; a, b),$$

provided $0 < \mu < 1$, $\mu + \nu \geq 1$.

Proof. The proof for the $L^\infty(\mathbb{R}^n)$ based norms is well known, while for the $L^p(\mathbb{R}^n)$ based norm we use the Sobolev embedding type theorems. \square

We recall that by the Sobolev embedding theorem we get that if $f \in H^{[n/2]+1}(\mathbb{R}^n)$ we have

$$|f(x)| \leq \varkappa_n \sqrt{\|f\|^2 + \sum_{k=1}^n \|\partial_{x_k}^{[n/2]+1} f\|^2}, \quad x \in \mathbb{R}^n, \varkappa_n := \left(\int_{\mathbb{R}^n} \frac{1}{1 + \sum_{k=1}^n \xi_k^{2[n/2]+2}} d\xi \right)^{1/2} \quad (2.44)$$

Next, we derive L^2 Sobolev type embedding theorem in the Gelfand–Shilov spaces. We write for brevity $\|f\|_{\rho, \sigma} := \|f\|_{L^2; \mu, \nu, \rho, \delta}$ for fixed $\mu, \nu > 0$.

Theorem 31. *Let $\mu, \nu > 0$, $\mu + \nu \geq 1$. Then $HS_v^\mu(L^\infty(\mathbb{R}^n); \rho, \sigma) \hookrightarrow HS_v^\mu(L^2(\mathbb{R}^n); \rho, \sigma)$ and the following estimates hold:*

$$\|f\|_{L^\infty; \mu, \nu; \varepsilon \rho, \delta \sigma} \leq \varkappa_n K_{\mu, \nu}(\rho, \sigma, \varepsilon, \delta) \|f\|_{\rho, \sigma}, \quad f \in HS_v^\mu(L^2(\mathbb{R}^n); \rho, \sigma), \quad (2.45)$$

where

$$K_{\mu, \nu}(\rho, \sigma, \varepsilon, \delta) = \sqrt{1 + (\rho^{-1} + \sigma)^{2[n/2]+2} C_{\mu, \nu}^2(\varepsilon, \delta)}, \quad C_{\mu, \nu}(\varepsilon, \delta) > 0, \quad (2.46)$$

for $\rho, \sigma > 0$, $0 < \varepsilon < 1$, $\delta = 1$ if $\nu \geq 1$, $0 < \delta < 1$ if $\nu < 1$.

Proof. We will make use of the following lemma:

Lemma 6. *Let $\mu, \nu > 0$, $\mu + \nu \geq 1$, $\rho, \sigma > 0$, $\gamma \in \mathbb{Z}_+^n \setminus 0$, $0 < \varepsilon < 1$, $0 < \delta \leq 1$, with $\delta < 1$ if $\nu < 1$. Then the following estimates hold:*

$$\|\partial_x^\gamma u\|_{\varepsilon\rho, \delta\sigma} \leq (\rho^{-1} + \sigma)^{|\gamma|} K_\mu(\varepsilon, |\gamma|) K_{\max\{0, 1-\nu\}}(\delta, |\gamma|) \|u\|_{\rho, \sigma}, \quad (2.47)$$

where

$$K_t(\omega, s) = \begin{cases} \sup_{k \geq 0} (\omega^k ((k+1) \dots (k+s))^t) < +\infty & \text{if } 0 < \omega < 1, \\ 1 & \text{if } t = 0, 0 < \omega \leq 1 \end{cases}, \quad s \in \mathbb{N}. \quad (2.48)$$

Proof. We have

$$\begin{aligned} [u]_{\mu, \nu; \varepsilon\rho, \delta\sigma}^{\alpha, \beta; \gamma}(x) &= \frac{|\partial_x^\gamma (x^\beta \partial_x^\alpha u(x))| (\varepsilon\rho)^{|\alpha|} (\delta\sigma)^{|\beta|}}{\alpha!^\mu \beta!^\nu} \\ &\leq \sum_{j \leq \gamma} \binom{\gamma}{j} \frac{\beta!}{(\beta - \gamma + j)!} \frac{|x^{\beta - \gamma + j} \partial_x^{\alpha + j} u(x)| (\varepsilon\rho)^{|\alpha|} (\delta\sigma)^{|\beta|}}{\alpha!^\mu \beta!^\nu} \\ &= \varepsilon^{|\alpha|} \delta^{|\beta|} \sum_{j \leq \gamma} \binom{\gamma}{j} \rho^{-|j|} \sigma^{|\gamma - j|} \left(\frac{\beta!}{(\beta - \gamma + j)!} \right)^{1-\nu} \left(\frac{(\alpha + j)!}{\alpha!} \right)^\mu \\ &\times \left(\left| x^{\beta - \gamma + j} \partial_x^{\alpha + j} u(x) \right| \frac{\rho^{|\alpha + j|} \sigma^{|\beta - \gamma + j|}}{(\alpha + j)!^\mu (\beta + \gamma - j)!^\nu} \right) \\ &\leq \varepsilon^{|\alpha|} \left(\prod_{s=1}^{|\gamma|} (|\alpha| + s) \right)^\mu \delta^{|\beta|} \left(\prod_{s=1}^{|\gamma|} (|\beta| + s) \right)^{\max\{0, 1-\nu\}} \sum_{j \leq \gamma} \binom{\gamma}{j} \rho^{-|j|} \sigma^{|\gamma - j|} \\ &\times \left(\left| x^{\beta - \gamma + j} \partial_x^{\alpha + j} u(x) \right| \frac{\rho^{|\alpha + j|} \sigma^{|\beta - \gamma + j|}}{(\alpha + j)!^\mu (\beta + \gamma - j)!^\nu} \right) \end{aligned}$$

which, in view of the identity $\sum_{j \leq \gamma} \binom{\gamma}{j} \rho^{-|j|} \sigma^{|\gamma - j|} = (\rho^{-1} + \sigma)^{|\gamma|}$ yields the desired estimates. \square

Now we get immediately (2.45), (2.46). Indeed, taking into account the identity

$$\partial_{x_k}^\ell (x^\beta \partial_x^\alpha u) = \sum_{j=0}^{\ell} \binom{\ell}{j} \beta_k \dots (\beta_k - j + 1) x^{\beta - j e_k} \partial_x^{\alpha + (\ell - j) e_k} f, \quad k = 1, \dots, n,$$

(2.44) for $x^\beta \partial^\alpha f(x)$

$$\sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha f(x)| \leq \varkappa_n \sqrt{\|x^\beta \partial_x^\alpha f\|^2 + \sum_{k=1}^n \|\partial_{x_k}^{[n/2]+1}(x^\beta \partial_x^\alpha f)\|^2}$$

and (2.47), (2.48), we obtain

$$\|u\|_{L^\infty; \mu, \nu; \varepsilon \rho, \delta \sigma} \leq \varkappa_n \sqrt{1 + n(\rho^{-1} + \sigma)^{2[n/2]+2} K_\mu^2(\varepsilon, [\frac{n}{2}] + 1) K_{\max\{0, 1-\nu\}}^2(\delta, [\frac{n}{2}] + 1)} \|u\|_{\rho, \sigma}$$

which yields (2.45) with $C_{\mu, \nu}(\varepsilon, \delta) = \sqrt{n} K_\mu(\varepsilon, [\frac{n}{2}] + 1) K_{\max\{0, 1-\nu\}}(\delta, [\frac{n}{2}] + 1)$. □

Next, we classify the action of quadratic oscillations on Banach spaces in the full scale of Gelfand–Shilov spaces $S_V^\mu(\mathbb{R}^n)$, $\mu, \nu > 0$, $\mu + \nu \geq 1$, recapturing as a particular case the automorphism of the NFT in the symmetric Gelfand–Shilov spaces $S_\mu^\mu(\mathbb{R}^n)$, $\mu \geq 1/2$. Since the Fourier transform \mathcal{F} is isomorphism $\mathcal{F} : S_V^\mu(\mathbb{R}^n) \mapsto S_\mu^\nu(\mathbb{R}^n)$ one readily obtains that a map $U : S_V^\mu(\mathbb{R}^n) \mapsto S_\mu^\nu(\mathbb{R}^n)$ is isomorphism iff the convolution map $U* : S_\mu^\nu(\mathbb{R}^n) \mapsto S_V^\mu(\mathbb{R}^n)$ is isomorphism. Hence it is enough to study the map U .

Theorem 32. *Let $E : L^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$ be the unitary map defined by $Eu(x) = e^{i\langle Ax, x \rangle} u(x)$, $x \in \mathbb{R}^n$, A being real symmetric matrix. We have.*

1. Let $0 < \nu \leq \mu$, $\mu + \nu \geq 1$. Then

$$E \text{ is an automorphism of } S_V^\mu(\mathbb{R}^n). \tag{2.49}$$

2. Let now $0 < \mu < \nu$, $\mu + \nu \geq 1$. Then

$$E : S_V^\mu(\mathbb{R}^n) \mapsto S_V^\nu(\mathbb{R}^n). \tag{2.50}$$

and (2.50) is sharp. In particular, E does not preserve $S_V^\mu(\mathbb{R}^n)$.

Proof. In view of the invariance of $S_V^\mu(\mathbb{R}^n)$ under linear maps $x \mapsto Sx$ and the reduction to the case A diagonal, it is enough to consider the one-dimensional case and $E = e^{ix^2}$. Let $f \in S_V^\mu(\mathbb{R})$. Then one can find positive constants C_0 , a and b such that

$$|f^{(k)}(x)| \leq C_0 a^k k!^\mu e^{-b|x|^{1/\nu}}, \quad k \in \mathbb{Z}_+, x \in \mathbb{R}. \quad (2.51)$$

We get by the Faa' di Bruno type formula

$$\begin{aligned} (e^{ix^2} f(x))^{(k)} &= \sum_{j=0}^k \binom{k}{j} \left(\frac{d}{dx}\right)^j (e^{ix^2}) f^{(k-j)}(x) \\ &= e^{ix^2} f^{(k)}(x) + e^{ix^2} f^{[k-1]}(x) \end{aligned} \quad (2.52)$$

where

$$f^{[k-1]}(x) = e^{ix^2} \sum_{j=1}^k \sum_{j/2 \leq \ell \leq j} \Theta_{j,\ell}^k i^\ell x^{2\ell-j} f^{(k-j)}(x) \quad (2.53)$$

$$\Theta_{j,\ell}^k = \binom{k}{j} \binom{\ell}{2\ell-j} (j-\ell)! 2^{2\ell-j}. \quad (2.54)$$

Clearly

$$|\Theta_{j,\ell}^k| \leq 2^{k+3\ell-j} (j-\ell)!, \quad k, j, \ell \in \mathbb{N}, j/2 \leq \ell \leq j \leq k. \quad (2.55)$$

In view of (2.53) and (2.54) we get that for every $\varepsilon \in]0, b[$ the following estimates hold

$$e^{(b-\varepsilon)|x|^{1/\nu}} |(e^{ix^2} f(x))^{(k)}| \leq C_0 \left(a^k k!^\mu + \sum_{j=1}^k \sum_{j/2 \leq \ell \leq j} \Theta_{j,\ell}^k a^{k-j} (k-j)!^\mu |x|^{2\ell-j} e^{-\varepsilon|x|^{1/\nu}} \right). \quad (2.56)$$

Since $\sup_{t \geq 0} (t^s e^{-\varepsilon t}) = s^s \varepsilon^{-s} e^{-s}$, $s > 0$, $\varepsilon > 0$, we obtain that

$$\sup_{t \geq 0} (t^{2\ell-j} e^{-\varepsilon t^{1/\nu}}) = \varepsilon^{-\nu(2\ell-j)} (\nu(2\ell-j))^{\nu(2\ell-j)} e^{-\nu(2\ell-j)} = \left(\frac{\nu}{\varepsilon}\right)^{\nu(2\ell-j)} (2\ell-j)^{\nu(2\ell-j)}, \quad (2.57)$$

for $j, \ell \in \mathbb{N}, j/2 \leq \ell \leq j$. By the Stirling formula we can find $C = C_\nu > 0$ such that

$$\sup_{t \geq 0} \left(t^{2\ell-j} e^{-\varepsilon t^{1/\nu}} \right) \leq C^{2\ell-j} \varepsilon^{-\nu(2\ell-j)} (\ell - j/2)!^{2\nu}, \quad j, \ell \in \mathbb{N}, j/2 \leq \ell \leq j. \quad (2.58)$$

Plugging (2.58) into (2.56) we show, taking into account (2.51) and (2.54), that

$$e^{(b-\varepsilon)|x|^{1/\nu}} \frac{|D^k(e^{ix^2} f(x))|}{a^k k!^\mu} \leq C_0 \left(1 + 4^k \sum_{j=1}^k \sum_{j/2 \leq \ell \leq j} a^{-j} (\varepsilon^{-\nu} C)^{2\ell-j} \frac{(j-\ell)! (\ell-j/2)!^{2\nu}}{j!^\mu} \right) \quad (2.59)$$

Now we see that the hypotheses $0 < \nu \leq \mu$ is necessary and sufficient condition for estimating the RHS in (2.59) by C^k , $k \in \mathbb{N}$. Indeed, $\mu \geq \nu$ and $\mu + \nu \geq 1$ implies $2\mu \geq 1$ and therefore

$$\frac{(j-\ell)! (\ell-j/2)!^{2\nu}}{j!^\mu} \leq \frac{(j-\ell)!^{1-2\mu} ((j-\ell)! (\ell-j/2)!)^{2\mu}}{j!^\mu} \leq \frac{((j/2)!)^{2\mu}}{j!^\mu} \leq 1$$

which implies that

$$|D^k(e^{i\alpha x^2} f(x))| \leq C_0 \left(a + \frac{C}{\varepsilon^\nu} \right)^k k!^\mu e^{-(b-\varepsilon)|x|^{1/\nu}}, \quad k \in \mathbb{Z}_+, 0 < \varepsilon \ll 1. \quad (2.60)$$

The proof in the case $\nu > \mu$ follows from the fact $2\nu \geq 1$ (otherwise $\mu < \nu < 1/2$ contradicts $\mu + \nu \geq 1$) by the estimates

$$\frac{(j-\ell)! (\ell-j/2)!^{2\nu}}{j!^\mu} \leq \frac{(j-\ell)! (\ell-j/2)!^{2\nu}}{j!^\nu} \leq \frac{((j/2)!)^{2\nu}}{j!^\nu} \leq 1.$$

We propose another proof if $\mu < 1$. We use the norms of the spaces $\mathcal{O}S_\nu^\mu(\mathbb{R}^n; a, b)$, namely $f \in S_\nu^\mu(\mathbb{R}^n)$, $\mu < 1$ means that f is an entire function in \mathbb{C}^n and there exist positive constants C_0, a, b such that

$$|f(x+iy)| \leq C_0 e^{-a|x|^{1/\nu} + b|y|^{1/(1-\mu)}}, \quad x, y \in \mathbb{R}^n. \quad (2.61)$$

Without loss of generality we consider the onedimensional case $n = 1$.

One notes that

$$\begin{aligned} |e^{i(x+iy)^2} f(x+yi)| &= |e^{-2xy} f(x+yi)| \\ &\leq C_0 e^{-a|x|^{1/\nu} + b|y|^{1/(1-\mu)} - 2xy}, \quad x, y \in \mathbb{R}. \end{aligned} \quad (2.62)$$

If $\nu \leq \mu$, by the Hölder inequality we get

$$2|xy| = 2|\varepsilon^\nu x| |\varepsilon^{-\nu} y| \leq 2 \left(\nu \varepsilon |x|^{1/\nu} + (1-\nu) \varepsilon^{-\nu/(1-\nu)} |y|^{1/(1-\nu)} \right),$$

by which, combined with (2.62), we derive the following exponential decay in x estimate for all $x, y \in \mathbb{R}$, $0 < \varepsilon \ll 1$. One observes that $1/(1-\nu) \leq 1/(1-\mu)$ if $\nu \leq \mu$ which yields $|y|^{1/(1-\nu)} \leq |y|^{1/(1-\mu)} + 1$, $y \in \mathbb{R}$, and therefore

$$|e^{i(x+iy)^2} f(x+yi)| \leq C_0 e^{2(1-\nu)\varepsilon^{-\nu/(1-\nu)}} e^{-(a-2\nu\varepsilon)|x|^{1/\nu} + (b+2(1-\nu)\varepsilon^{-\nu/(1-\nu)})|y|^{1/(1-\mu)}}, \quad (2.63)$$

for $x, y \in \mathbb{R}$. \square

We stress the fact that the transformation $e^{i\langle Ax, x \rangle}$ does not preserves the $H^{s_1, s_2}(\mathbb{R}^n)$, with $s_1 \in \mathbb{N}$, $s_2 \geq 0$

2.4 NFT representation of pseudodifferential operators

Without loss of generality we assume $S_0 = I_n$ and $\alpha = 0$ in the NFT U which yields $b(x, \xi) = \tilde{b}(x, \xi)$. Hence, setting $Q = 4^{-1}A_{\text{symm}}$, we have

$$\begin{aligned} b_{NF}(x, D)v &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\xi + \langle Qx, x \rangle - \langle Qy, y \rangle} b(x, \xi) v(y) dy \, \bar{d}\xi \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\xi + \langle Qx, x \rangle - \langle Qy, y \rangle} b(x, \xi) \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(y-z)\eta} \eta_\nu(z) d\eta dz \right) dy \, \bar{d}\xi. \end{aligned} \quad (2.64)$$

60 Taking into account the standard oscillatory integral methods and the identity

$$e^{i(x-y)\xi+i(y-z)\eta} = e^{i(x-z)\eta} e^{i(x-y)(\xi-\eta)},$$

we exchange the integration and get

$$b_{NF}(x, D)v = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-z)\eta} b_{NF}(x, \eta)v(z) dz \bar{d}\eta, \quad (2.65)$$

where

$$b_{NF}(x, \eta) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)(\xi-\eta)+\langle Qx,x \rangle - \langle Qy,y \rangle} b(x, \xi) dy \bar{d}\xi. \quad (2.66)$$

Since Q is symmetric we can write

$$\langle Qx,x \rangle - \langle Qy,y \rangle = \langle (x-y), Q(x+y) \rangle. \quad (2.67)$$

Indeed, if $S \in SO(\mathbb{R}^n)$ diagonalizes Q , i.e., $S^T QS = \text{diag}\{q_1, \dots, q_n\}$, setting $x = S\tilde{x}$, $y = S\tilde{y}$, we obtain that $\langle Qx,x \rangle - \langle Qy,y \rangle$ is equal to

$$\begin{aligned} \langle S^T QS\tilde{x}, \tilde{x} \rangle - \langle S^T QS\tilde{y}, \tilde{y} \rangle &= \sum_{j=1}^n q_j (\tilde{x}_j^2 - \tilde{y}_j^2) \\ &= \langle (\tilde{x} - \tilde{y}), \text{diag}\{q_1, \dots, q_n\}(\tilde{x} + \tilde{y}) \rangle \\ &= \langle (\tilde{x} - \tilde{y}), S^T QS(\tilde{x} + \tilde{y}) \rangle = \langle (x-y), Q(x+y) \rangle. \end{aligned} \quad (2.68)$$

Hence we can rewrite (2.66) as follows:

$$\begin{aligned} b_{NF}(x, \eta) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)(\xi-\eta+Q(x+y))} b(x, \xi) dy \bar{d}\xi \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{iy\xi} b(x, \xi + \eta - 2Qx + Qy) dy \bar{d}\xi \end{aligned} \quad (2.69)$$

after the change of the variables $\xi \mapsto \xi - \eta + Q(x+y) = \xi - \eta + 2Qx - Q(x-y)$, $y \mapsto x - y$. Next, following the approach of Shubin used in [58], we apply the

Taylor formula to

$$b(x, \xi + \eta - 2Qx + Qy) = \sum_{|\alpha| \leq N} \frac{\xi^\alpha}{\alpha!} \partial_\eta^\alpha b(x, \eta - 2Qx + Qy) + R_N(x, \xi, \eta, y) \quad (2.70)$$

$$R_N(x, \xi, \eta, y) = \sum_{|\beta|=N+1} \frac{1}{\beta!} \int_0^1 (1-t)^N \partial_\eta^\beta b(x, t\xi + \eta - 2Qx + Qy) dt. \quad (2.71)$$

We have

$$\begin{aligned} B_\alpha(x, \eta) &:= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{iy\xi} \frac{\xi^\alpha}{\alpha!} \partial_\eta^\alpha b(x, \eta - 2Qx + Qy) dy \bar{d}\xi \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} D_y^\alpha \left(e^{iy\xi} \right) \frac{1}{\alpha!} \partial_\eta^\alpha b(x, \eta - 2Qx + Qy) dy \bar{d}\xi \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{iy\xi} \frac{(-1)^{|\alpha|}}{\alpha!} D_y^\alpha \left(\partial_\eta^\alpha b(x, \eta - 2Qx + Qy) \right) dy \bar{d}\xi \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{iy\xi} \frac{(-1)^{|\alpha|}}{\alpha!} (QD_y \eta)^\alpha \partial_\eta^\alpha b(x, \eta - 2Qx + Qy) dy \bar{d}\xi \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{iy\xi} \bar{d}\xi \right) \frac{(-1)^{|\alpha|}}{\alpha!} (QD_\eta)^\alpha \partial_\eta^\alpha b(x, \eta - 2Qx + Qy) dy. \end{aligned} \quad (2.72)$$

We recall that

$$(2\pi)^{-n} \int_{\mathbb{R}^n} e^{iy\xi} d\xi = \delta(y),$$

$\delta(y)$ being the Dirac delta function centered in 0 . Therefore

$$B_\alpha(x, \eta) = \frac{(-1)^{|\alpha|}}{\alpha!} (QD_\eta)^\alpha \partial_\eta^\alpha b(x, \eta - 2Qx). \quad (2.73)$$

On the other hand, for the same arguments in [58], we derive that

$$\tilde{B}_{N+1}(x, \eta) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{iy\xi} R_{N+1}(x, \eta, y, \xi) dy \bar{d}\xi \quad (2.74)$$

belongs to $\Gamma^{m-N-1}(\mathbb{R}^{2n})$. We have shown that

$$b_{NF}(x, \eta) - \sum_{|\gamma| \leq N} \frac{(-1)^{|\alpha|}}{\alpha!} (QD_\eta)^\gamma \partial_\eta^\gamma b(x, \eta - 2Qx) \in \Gamma^{m-N-1}(\mathbb{R}^{2n})$$

for every $N \in \mathbb{N}$.

We observe that

$$|\partial_x^\beta \partial_\eta^\alpha B_\gamma(x, \eta)| = \frac{1}{\gamma!} \left| \partial_x^\beta \left((QD_\eta)^\gamma \partial_\eta^{\alpha+\gamma} b(x, \eta - 2Qx) \right) \right|. \quad (2.75)$$

We plug the identity

$$\begin{aligned} \partial_x^\beta (a(x, \eta - 2Qx)) &= (\partial_x - 2Q\partial_\xi)^\beta a(x, \xi)|_{\xi=\eta-2Qx} \\ &= \sum_{\delta \leq \beta} \binom{\beta}{\delta} (-2)^{|\delta|} (Q\partial_\xi)^\delta \partial_x^{\beta-\delta} a(x, \xi)|_{\xi=\eta-2Qx} \end{aligned} \quad (2.76)$$

in (2.75) for $a(x, \xi) = (QD_\xi)^\gamma \partial_\xi^{\alpha+\gamma} b(x, \xi)$, and readily obtain the estimate

$$\begin{aligned} |\partial_x^\beta \partial_\eta^\alpha B_\gamma(x, \eta)| &\leq \frac{1}{\gamma!} \sum_{\delta \leq \beta} \binom{\beta}{\delta} \\ &\quad \times 2^{|\delta|} \left| (Q\partial_\xi)^{\delta+\gamma} \partial_x^{\beta-\delta+\alpha+\gamma} b(x, \xi) \Big|_{\xi=\eta-2Qx} \right| \\ &\leq \frac{1}{\gamma!} \sum_{\delta \leq \beta} \binom{\beta}{\delta} 2^{|\delta|} \|Q\|_\infty^{|\delta|+|\gamma|} \\ &\quad \times C_{\delta+\gamma, \beta-\delta+\alpha+\gamma} \langle x, \eta - 2Qx \rangle^{m-2|\gamma|-|\alpha|-|\beta|} \end{aligned} \quad (2.77)$$

for all $(x, \xi) \in \mathbb{R}^{2n}$, $\alpha, \beta, \gamma \in \mathbb{Z}_+^n$, with $C_{p,q}$ defined by

$$C_{\alpha, \beta} := \sup_{(x, \xi) \in \mathbb{R}^{2n}} \left(\langle x, \xi \rangle^{-m+|\alpha|+|\beta|} |\partial_x^\beta \partial_\xi^\alpha b(x, \xi)| \right) \quad (2.78)$$

and $\|Q\|_\infty := \max_{j,k=1,\dots,n} |q_{jk}|$.

Combining (2.78) with the estimates

$$0 < \inf_{(x, \eta) \in \mathbb{R}^{2n}} \frac{\langle x, \eta - 2Qx \rangle}{\langle x, \eta \rangle} \leq \sup_{(x, \eta) \in \mathbb{R}^{2n}} \frac{\langle x, \eta - 2Qx \rangle}{\langle x, \eta \rangle} < +\infty$$

we obtain that $b_{NF}(x, \eta) \in \Gamma^m(\mathbb{R}^{2n})$. The proof is complete.

2.5 Discrete representation of pseudodifferential operators

In this section P will stand for a globally elliptic self-adjoint Shubin pseudodifferential operator of order $m > 0$, semibounded from below. We recall that (for more details see [58], [50]) the operator P has a discrete spectrum diverging to $+\infty$.

$$\text{spec}(P) = \{\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k < \dots \rightarrow +\infty\}, \quad (2.79)$$

and one can choose an orthonormal basis of $L^2(\mathbb{R}^n)$ the eigenfunctions $\{\varphi_j\}_{j=1}^\infty$.

We recall several notions on discrete representations of onedimensional Shubin operators developed by Chodosh [10].

Definition 27. *If we define the discrete difference operator Δ on a function $K : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{R}$ by*

$$(\Delta K)(j, k) = K(j+1, k+1) - K(j, k)$$

(writing Δ^α to signify applying the difference operator α times), then we will say that a function K is a symbol matrix of order r if for all $\alpha, N \in \mathbb{N}_0$ there is $C_{\alpha, N} > 0$ such that

$$|(\Delta^\alpha K)(m, n)| \leq C_{\alpha, N} (1+m+n)^{r-\alpha} (1+|m-n|)^{-N}, \quad m, n \in \mathbb{N}_0.$$

We denote the set of symbol matrices of order r , $\mathbf{SM}^r(\mathbb{N}_0)$.

We introduce discrete representation depending on the basis $\{\varphi_j\}_{j=1}^\infty$ of P . One of our motivations comes from the methods on Fourier analysis relative to elliptic differential operators on compact manifolds developed by Greenfield and Wallach in 1973 cf. [30].

Definition 28. For every Shubin type pseudodifferential operator Q of order m on \mathbb{R}^n , we define the following infinite dimensional matrix:

$$K_P^Q : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{R}$$

$$(\alpha, \beta) \rightarrow K_P^Q(\alpha, \beta) = \langle Q\varphi_\alpha, \varphi_\beta \rangle.$$

We note that if P is the onedimensional harmonic oscillator and $\{\varphi_\alpha\}$, $\alpha \in \mathbb{N}_0$, are the Hermite functions, we recapture the definition of matrices of order m in [10].

Moreover, one is led to ask the following question: can we characterize the action of the operator Q via the infinite matrix K_P^Q for larger classes of operators?

Our normal form result yields an easy generalization for second order self-adjoint Shubin differential operators with symmetric mixed term.

Proposition 25. Let $n = 1$ and P be as in Theorem 27. We denote by $\{\varphi_\alpha\}_{\alpha=1}^\infty$ the basis of eigenfunctions of P . Then $Q(x, D) \in \Gamma^m$ iff K_P^Q belongs to $SM^{m/2}(\mathbb{N}_0)$.

Proof. It follows from the application of the NFT U in Theorem 27, and the fact that $\varphi_\alpha = UH_\alpha$, namely

$$K_P^Q(\alpha, \beta) = \langle Q\varphi_\alpha, \varphi_\beta \rangle = \langle QUH_\alpha, UH_\beta \rangle = \langle U^*QUH_\alpha, H_\beta \rangle,$$

where $\{H_\gamma\}_{\gamma=1}^\infty$ are the Hermite functions. By Theorem 28 we know that $Q \in \Gamma^m$ if and only if $U^* \circ Q \circ U \in \Gamma^m$. Hence we obtain

$$K_{\alpha, \beta; P}^Q = K_{\alpha, \beta; \mathcal{H}_1}^{U^* \circ Q \circ U} = K(\alpha, \beta),$$

with $K(\alpha, \beta)$ being the matrix of $Q_{NF}(x, D) = U^* \circ Q(D) \circ U$ according to Definition 27 and we can apply the result in [10]. □

Next, motivated and inspired again by the paper [30], where global hypoellipticity results for commuting differential operators with a normal elliptic operator on a compact manifold have been derived, we propose a new result in the context of the theory of the Shubin type p.d.o. under a Diophantine type condition on the eigenvalues of P .

Theorem 33. *Assume that the spectrum of P consists of simple eigenvalues and there exist $C > 0$ $\tau \geq 0$ such that*

$$|\lambda_j - \lambda_\ell| \geq \frac{C}{(|j| + |\ell|)^\tau}, \quad j \neq \ell. \quad (2.80)$$

Let Q be a Shubin pseudodifferential operator of order m such that

$$[P, Q] \in O\Gamma^{-\infty}(\mathbb{R}^n). \quad (2.81)$$

Then the matrix $K_P^Q := \{\kappa_{j,\ell}\}_{j,\ell=1}^\infty$ admits the following uniquely determined splitting

$$K_P^Q = \text{diag}\{k_{j,j}\}_{j=1}^\infty + \tilde{\kappa}, \quad \tilde{\kappa}_{j,\ell} = (1 - \delta_{j\ell})\kappa_{j,\ell}, \quad j, \ell \in \mathbb{N}. \quad (2.82)$$

with $\tilde{\kappa}_{j,\ell} = O((|j| + |\ell|)^{-N})$, $j + \ell \rightarrow +\infty$, $\forall N \in \mathbb{N}$.

Proof. From the Theorem 25 we have $K_P^P = \text{diag}\{\lambda_j\}_{j=1}^\infty$. Moreover using standard linear algebra matrix arguments we obtain that the matrix representation of the commutator $[P, Q]$

$$K_P^{[P,Q]} = K_P^Q K_P^P - K_P^P K_P^Q = R \in SM^{-\infty}(\mathbb{N}). \quad (2.83)$$

Clearly $R_{j,j} = 0$ while $R_{j,\ell} = (\lambda_j - \lambda_\ell)K_{j,\ell}$, when $j \neq \ell$. Hence

$$K_{j,\ell} = \frac{1}{\lambda_j - \lambda_\ell} R_{j,\ell}, \quad j \neq \ell.$$

Here occurs the necessity to use the condition (2.80), because it is well known that $R_{j,\ell} = O((|j| + |\ell|)^{-\infty})$ and so we need the conditions (2.80) to avoid that

$(\lambda_j - \lambda_\ell)$ becomes $O((j + \ell)^{-\infty})$ for some subsequence of (k, ℓ) , $k, \ell \in \mathbb{N}$, $k \neq \ell$. One notes that if $[P, Q] = 0$, one gets that $(\lambda_j - \lambda_\ell)K_{j,\ell} = 0$ for $j \neq \ell$ and so we have $K_{j,\ell} = 0$ without any additional condition. \square

Remark 7. Let $P_{NF} = \mathcal{H}_\omega + r$, with $\omega = (\omega_1, \dots, \omega_n)$ being Diophantine. Then $\lambda_k \neq \lambda_j$, $j \neq k$ and the Diophantine type condition, We can consider a zero order Shubin operator $b(x, D)$ (not necessarily self-adjoint) such that $[b, P] \in \Gamma^{-\infty}$. The the spectrum of $P + b$ is discrete and we can write $b(x, D) = b_{diag}(x, D) + b_{-\infty}(x, D)$, with $[P, b_{diag}] = 0$ and $b_{-\infty} \in \Gamma^{-\infty}$.

2.6 Discrete representation of centralizers of operators via eigenfunction expansions

Let H be a separable Hilbert spaces. Let P be a self-adjoint unbounded operator, semibounded from below. It is well known that P has a discrete spectrum and we can write the eigenvalues of P like a sequence $\{\mu_1 < \mu_2 < \dots \mu_k < \dots \infty\}$. We denote by m_j the multiplicity of μ_j and set $B^{m_j}(\mu_j)$ the eigenspace corresponding to μ_j , $j \in \mathbb{N}$. We choose and fix an orthonormal basis of double index $\theta_{\ell,j}$, $j = 1, \dots, m_\ell$, $\ell = 1, 2, \dots$, and define the eigenfunctions expansion of $u \in D(P)$

$$u = \sum_{\ell=1}^{\infty} \sum_{j=1}^{m_\ell} u_{\ell,j} \theta_{\ell,j} = \sum_{\ell=1}^{\infty} \langle \vec{u}^\ell, \vec{\theta}^\ell \rangle,$$

where $\vec{u}^\ell = (u_{\ell,1}, \dots, u_{\ell,m_\ell}) \in \mathbb{C}^{m_\ell}$, $\vec{\theta}^\ell = (\theta_{\ell,1}, \dots, \theta_{\ell,m_\ell}) \in (B^{m_\ell}(\mu_\ell))^{m_\ell}$, $u_{\ell,j} = \langle u, \theta_{\ell,j} \rangle$, Moreover, we have

$$H = \bigoplus_{\ell=1}^{+\infty} B^{m_\ell}(\mu_\ell)$$

and

$$Pu = \sum_{\ell=1}^{\infty} \sum_{j=1}^{m_\ell} u_{\ell,j} \mu_\ell \theta_{\ell,j} = \sum_{\ell=1}^{\infty} \langle \mu_\ell I_{m_\ell} \vec{u}^\ell, \vec{\theta}^\ell \rangle_{\mathbb{R}^{m_\ell}},$$

where $\vec{\theta}^\ell = (\theta_{\ell,1}, \dots, \theta_{\ell,m_\ell})$, are the eigenfunctions associated to the eigenvalue μ_ℓ with multiplicity m_ℓ and $\vec{u}^\ell = (u_{\ell,1}, \dots, u_{\ell,m_\ell})$ and $\ell = 1, \dots, \infty$. If $A_\ell \in SO(\mathbb{R}^{m_\ell})$ (or $A_\ell \in SU(\mathbb{R}_\ell^m)$) we set

$$\vec{\theta}^\ell := A_\ell \underline{\theta}^\ell, \quad \vec{u}^\ell = A_\ell \underline{u}^\ell.$$

So we obtain the invariance of P under the action of the infinite matrix

$$A = \oplus A_\ell \tag{2.84}$$

namely

$$\begin{aligned} P \circ Au &= \sum_{\ell=1}^{\infty} \langle \mu_\ell \vec{u}^\ell, \vec{\theta}^\ell \rangle = \sum_{\ell=1}^{\infty} \langle \mu_\ell I_{m_\ell} A \underline{u}^\ell, A \underline{\theta}^\ell \rangle \\ &= \sum_{\ell=1}^{\infty} \mu_\ell \langle A^t A \underline{u}^\ell, \underline{\theta}^\ell \rangle = \sum_{\ell=1}^{\infty} \mu_\ell \langle I_{m_\ell} \underline{u}^\ell, \underline{\theta}^\ell \rangle = Pu. \end{aligned}$$

Remark 8. Note that if the eigenvalues are simple ($m_\ell = 1$), the choice of the eigenfunctions $\theta_\ell = \theta_{\ell,1}$ is unique modulo multiplied by ± 1 ($e^{i\beta}$, $\beta \in \mathbb{R}$, if H is a complex space)

We consider the centralizer of P

$$Z(P) = \{Q : PQ = QP\} \tag{2.85}$$

the set of all linear operators $Q : D(Q) \rightarrow \mathbb{C}$, where $D(Q)$ contains all eigenfunction, commuting with P on $D(Q) \cap D(P) \subset H$. Then we have recall a well known assertion

Proposition 26. Let $Q \in Z(P)$. Then

$$Qu = \sum_{\ell=1}^{\infty} \langle M^\ell \vec{u}^\ell, \vec{\theta}^\ell \rangle_{\mathbb{R}^{m_\ell}},$$

where $M^\ell \in M_{m_\ell}(\mathbb{C})$ and $M^\ell = (M^\ell)^*$, if $Q^* = Q$.

Proof. We know that since Q is invariant on $B^{m_\ell}(\mu_\ell)$, $\ell \in \mathbb{N}$, we have

$$Qu = \sum_{\ell=1}^{\infty} \langle M^\ell \vec{u}^\ell, \vec{\theta}^\ell \rangle_{\mathbb{R}^{m_\ell}}, \tag{2.86}$$

where $M^\ell = \{\mu_{j,k}^\ell\}_{j,k=1,\dots,m_\ell} \in M_{m_\ell}(\mathbb{C})$, $\vec{\theta}^\ell = (\theta_{\ell,1}, \dots, \theta_{\ell,m_\ell})$, $\vec{u}^\ell = (u_{\ell,1}, \dots, u_{\ell,m_\ell})$.
 Next, we consider the action of A More precisely, if $\vec{\theta}^\ell = A^\ell \vec{\theta}^\ell$ and $\vec{u}^\ell = A^\ell \vec{u}^\ell$, we have

$$\begin{aligned} Q \circ Au &= \sum_{\ell=1}^{\infty} \langle M^\ell A^\ell \vec{u}^\ell, A^\ell \vec{\theta}^\ell \rangle \\ &= \sum_{\ell=1}^{\infty} \langle {}^t(A^\ell) M^\ell A^\ell \vec{u}^\ell, \vec{\theta}^\ell \rangle. \end{aligned} \tag{2.87}$$

We complete the proof by standard linear algebra arguments. □

Remark 9. In general we can not reduce a matrix M in a diagonal form by conjugation with $S \in SU(m_\ell)$, if M is not normal (i.e. $M^*M \neq M^*M$). Moreover, we can reduce M^ℓ to a lower triangular form (upper-triangular form) by a conjugation with a matrix $S \in SU(m_\ell)$. In general, case one encounters families matrices with non trivial Jordan blocks.

2.7 Representation of Shubin Operators

We describe completely the discrete representation of the centralizer of P in the space of pseudodifferential operators defined by the continuous action on $\mathcal{S}(\mathbb{R}^n)$.

Theorem 34. Suppose that $Q : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ is a linear continuous operator, that commutes with a globally elliptic self-adjoint operator P . Then we have

i) the following decomposition

$$L^2(\mathbb{R}^n) = B^{m_1}(\mu_1) \oplus B^{m_2}(\mu_2) \oplus \dots \oplus B^{m_k}(\mu_k) \oplus \dots$$

where $\mu_1 < \mu_2 < \dots < \mu_k < \dots$ are the eigenvalues, m_j is the multiplicity of the eigenvalue μ_j , $j = 1, 2, \dots$, further $B^{m_j}(\mu_j)$ is the eigenspace of the eigenvalue μ_j , $j = 1, 2, \dots$ with a orthonormal basis of eigenfunctions

$$\theta_{\ell,j}, \text{ such that } P\theta_{\ell,j} = \mu_\ell \theta_{\ell,j} \quad \ell \in \mathbb{N}, j = 1, \dots, m_\ell;$$

ii) $Q_{m_\ell} := Q|_{B^{m_\ell}(\mu)}$ is invariant on $B^{m_\ell}(\mu)$;

iii) There exist matrices $\Gamma^\ell = \{\gamma_{rs}^\ell\}_{r,s=1}^{m_\ell} \in M_{m_\ell}(\mathbb{K})$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, such that

$$\begin{pmatrix} Q\theta_{\ell,1} \\ \vdots \\ Q\theta_{\ell,m_\ell} \end{pmatrix} = \begin{pmatrix} \gamma_{1,1}^\ell & \cdots & \gamma_{1,m_\ell}^\ell \\ \vdots & & \vdots \\ \gamma_{m_\ell,1}^\ell & \cdots & \gamma_{m_\ell,m_\ell}^\ell \end{pmatrix} \begin{pmatrix} \theta_{\ell,1} \\ \vdots \\ \theta_{\ell,m_\ell} \end{pmatrix} = \Gamma^\ell \begin{pmatrix} \theta_{\ell,1} \\ \vdots \\ \theta_{\ell,m_\ell} \end{pmatrix}. \quad (2.88)$$

and the family of matrices $\{\Gamma^\ell\}_{\ell=1}^\infty$ satisfies

$$\sup_{\ell \geq 1} \max_{\xi \in \mathbb{R}^{m_\ell}, |\xi|=1} \left(\ell^{-N} \|\Gamma^\ell \xi\| \right) < \infty, \quad \text{for some } N > 0. \quad (2.89)$$

In particular, if Q is a symmetric Shubin operator the matrices Γ^ℓ , $\ell \in \mathbb{N}$, are symmetric (respectively, Hermitian) if H is real (respectively, complex) Hilbert space.

Proof. Set $\psi_{\ell,j} = Q\theta_{\ell,j}$. We get

$$P\psi_{\ell,j} = PQ\theta_{\ell,j} = QP\theta_{\ell,j} = Q\mu_\ell \theta_{\ell,j} = \mu_\ell Q\theta_{\ell,j} = \mu_\ell \psi_{\ell,j}.$$

Hence $\psi_{j,\ell} = Q\theta_{j,\ell} \in B^{m_\ell}(\mu_\ell)$, $j = 1, \dots, m_\ell$ and since Q is invariant on $B^{m_\ell}(\mu_\ell)$ we deduce i), ii) and iii) (2.88).

We prove (2.89) using the topology on $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$. Finally, the last sentence is proved by standard linear algebra. \square

2.8 Anisotropic form of the 1D harmonic oscillator

The main goal of this section is to investigate the reduction to normal forms and to classify the normal forms of second order self-adjoint anisotropic 1D differential operators. In this section we consider the following operator:

$$Pu = D_x^2 u + a(x)D_x u + b(x)u = -u'' + ia(x)u' + b(x)u, \quad x \in \mathbb{R}, \quad (2.90)$$

where $a(x)$ (resp. $b(x)$) is polynomial of degree k (resp. $2k$). Since $P = P^*$ necessarily

$$a(x) = \sum_{j=0}^k a_j x^{k-j}, \quad a_0 \in \mathbb{R}, a_j \in \mathbb{R}, j = 1, \dots, k, \quad (2.91)$$

$$b(x) = \sum_{j=0}^{2k} b_j x^{2k-j}, \quad b_0 \in \mathbb{R}, b_j \in \mathbb{C}, j = 1, \dots, 2k. \quad (2.92)$$

We recall the global anisotropic ellipticity condition (cf. [3], [6])

$$p_2(x, \xi) = \xi^2 + a_0 x^k \xi + b_0 x^{2k} \neq 0, \quad \text{for } (x, \xi) \neq (0, 0). \quad (2.93)$$

Straightforward calculations imply that the operator P defined in (2.90) is symmetric iff $b_j \in \mathbb{R}, j = 0, 1, \dots, k$ and

$$\sum_{j=k+1}^{2k} b_j x^{k-j} - \frac{i}{2} a'(x) \text{ is real valued.} \quad (2.94)$$

This polynomial have real coefficients, while the global anisotropic ellipticity condition is equivalent to

$$q_0 := b_0 - \frac{1}{4} a_0^2 > 0. \quad (2.95)$$

We define a “canonical” normal form operator depending on $2k - 2$ real parameters, which will generate all globally elliptic self-adjoint operators defined above. Given $k \geq 1$ and $\rho = (\rho_0, \dots, \rho_{2k-3}) \in \mathbb{R}^{2k-2}$ if $k \geq 2$ we set

$$L_{2k, \rho} = D_x^2 + x^{2k} + \sum_{j=0}^{2k-3} \rho_j x^{2k-3-j}. \quad (2.96)$$

Recall that by well known result on the spectral theory of the Schödinger operators (e.g., cf. [58], [3] and the references therein) we know that the operator $L_{2k,\rho}$ is essentially self-adjoint with discrete spectrum

$$\text{spec}(L_{2k,\rho}) = \{\lambda_1(\rho) < \dots < \lambda_j(\rho) \dots \rightarrow \infty\}, \quad (2.97)$$

with the corresponding orthonormal basis of eigenfunctions $\{\varphi_j^\rho(x)\}_{j=1}^\infty \in L^2(\mathbb{R})$. The eigenfunctions belong to the limiting Gelfand–Shilov space $\mathcal{S}_{1/(k+1)}^{k/(k+1)}(\mathbb{R})$. Moreover the entire function extensions $\varphi_j(z)$, $z = x + yi \in \mathbb{C}$ satisfy the following property:

there exists absolute constants $c_0 = c_0(k)$, $\varepsilon = \varepsilon(k) > 0$ and positive numbers M_j , $j \in \mathbb{N}$ such that

$$|\varphi_j(z)| \leq M_j e^{-\varepsilon|x|^{k+1} + c_0|y|^{(k+1)}} \quad (2.98)$$

We have

Proposition 27. *The following properties are equivalent:*

- i) P is globally elliptic self-adjoint operator.
- ii) there exists a unitary map

$$U : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R}), \quad U := T_c \circ E_A v(x) = e^{-iA(x+c)/2} v(x+c),$$

where $A(x) = \int_0^x a(t)dt$, $c \in \mathbb{R}$, dilation $D_\tau v(x) = v(\tau^{-1}x)$, $\tau > 0$, and $r \in \mathbb{R}$, such that

$$U^t \circ D_\tau^{-1} \circ P \circ U \circ D_\tau v(x) = \tau^2 L_{2k,\rho} + r. \quad (2.99)$$

Clearly (2.99) implies that

$$\text{spec}(P) = \{\lambda_j = \tau^2 \lambda_j(\rho) + r\}$$

and the eigenfunctions are

$$\{\psi_j(x) = \kappa_j e^{-iA(x+c)/2} \varphi_j^p(x+c),\}_{j=1}^\infty$$

where κ_j are constants such that $\|\psi_j\| = 1$.

Proof. We have

$$\tilde{P} = E_A^* \circ P \circ E_A u = D_x^2 + \sum_{j=0}^{2k} q_j x^{2k-j}, \quad q_j \in \mathbb{R}, q_0 > 0, q_j \in \mathbb{R}, j = 1, \dots, 2k.$$

Next, we set $y = x + \frac{q_1}{2kq_0}$ and reduce \tilde{P} to

$$\tilde{P}' = D_y^2 + q_0 y^{2k} + \sum_{j=0}^{2k-3} \tilde{q}_j y^{2k-2-j} + r, \quad \tilde{q}_j \in \mathbb{R}, j = 0, \dots, 2k-3, r \in \mathbb{R}.$$

We conclude the proof by applying the dilation $y = \tau^{-1}x$, $\tau = q_0^{2k+2}$. \square

Next we investigate the multidimensional anisotropic case

$$P = -\Delta + \langle A(x), D_x \rangle + B(x), \quad x \in \mathbb{R}^n, \quad (2.100)$$

where $A(x) = (A_1(x), \dots, A_n(x))$,

$$A_j(x) = \sum_{\ell=0}^k A_j^{k-\ell}(x), \quad A_j^{k-\ell}(x) = \sum_{|\alpha|=k-\ell} A_{j,\alpha} x^\alpha, \quad (2.101)$$

$$B(x) = \sum_{\ell=0}^{2k} B_{2k-\ell}(x), \quad B_{2k-\ell}(x) = \sum_{|\alpha|=2k-\ell} B_\alpha x^\alpha, \quad (2.102)$$

with

$$A_k(x) = (A_1^k(x), \dots, A_n^k(x)), B_{2k}(x) \text{ real valued.} \quad (2.103)$$

Here we impose a symmetry type condition (an analogue to $A_{skew} = 0$ or its geometric version). We suppose that $\langle A(x), dx \rangle$ is closed form i.e. there exist

$$V(x) \in C^\infty(\mathbb{R}^n : \mathbb{R}) \text{ such that } \nabla V(x) = A(x) \quad (2.104)$$

with $V(x) = \sum_{j=0}^{2k+1} V_{2k+1-j}(x)$, $V_{2k+1-j}(x)$ real homogeneous polynomial of degree $2k+1-j$, $j = 0, 1, \dots, 2k+1$.

Under the hypothesis (2.104) we have

Theorem 35. *The operator P is symmetric iff*

$$Q(x) := B(x) - \frac{1}{4} \|\nabla V_{2k+1}(x)\|^2 + \frac{i\Delta V(x)}{2} \text{ is real for } x \in \mathbb{R}^n \quad (2.105)$$

Moreover, P is self-adjoint globally elliptic operator iff

$$B_{2k}(x) - \frac{1}{4} \|\nabla V_{2k+1}(x)\|^2 = B_{2k}(x) - \frac{1}{4} \|A^k(x)\|^2 > 0, \quad x \in \mathbb{R}^n \setminus \mathbf{0}. \quad (2.106)$$

Suppose now that (2.105) and (2.106) hold. We assume in addition that there exist $S \in SO(n)$ and n polynomials of degree $2k$

$$q_j(t) = \sum_{\ell=0}^{2k} q_{j;\ell} t^{2k-j}, \quad q_{j;\ell} \in \mathbb{R}, \ell = 0, 1, \dots, 2k, q_{j;\ell} > 0, j = 1, \dots, n, \quad (2.107)$$

such that

$$Q(Sy) = \sum_{j=1}^n q_j(y_j). \quad (2.108)$$

Then we can find n positive numbers ω_j , $r \in \mathbb{R}$, $\gamma \in \mathbb{R}^n$ such that there exists a unitary map $U : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$, $U := T_\gamma \circ E_V u(x) = e^{-iV(x+\gamma)/2} v(Sx + \gamma)$, $\rho \in \mathbb{R}^n$, dilation $D_\omega v(x) = v(\omega_1^{-1}x_1, \dots, \omega_n^{-1}x_n)$, $\tau > 0$, and $r \in \mathbb{R}$, such that

$$D_\omega^{-1} \circ S^t \circ U^t \circ P \circ U \circ S \circ D_\omega v(x) = \sum_{j=1}^n \omega_j^2 L_{2k, c_j} + r. \quad (2.109)$$

Proof. After the conjugation with $e^{-\frac{i}{2}V(x)}$ and the change $x \mapsto Sx$ we are reduced to the sum of n 1D model anisotropic operators and we conclude as in the previous assertion.

We conclude the proof by applying the following well known assertions.

Lemma 7. *Let P_j be unbounded self-adjoint operator in a Hilbert spaces H_j $j = 1, \dots, n$, and semi-bounded from below. Let $H = H_1 \otimes \dots \otimes H_n$. The operator P*

$$\text{spec}(P_j) = \{\lambda_{j,\ell} : \ell = 1, 2, \dots\},$$

with bases $\{\varphi_{j,\ell}\}_{\ell=1}^{\infty}$, $j = 1, \dots, n$. Then

$$P = P_1 \oplus \dots \oplus P_n,$$

has a discrete spectrum defined by

$$\text{spec}(P) = \{\lambda_{1,\ell} + \lambda_{2,\ell} + \dots + \lambda_{n,\ell} \in \mathbb{N}\},$$

with

$$\{\varphi_{\ell}\}_{\ell=1}^{\infty} = \{\varphi_{1,\ell} \otimes \dots \otimes \varphi_{n,\ell}\}.$$

□

Chapter 3

Normal forms for classes of second order non self-adjoint Shubin operators

The main goal of this chapter is to investigate the reduction and the classification of the normal forms of classes of second order non self-adjoint Shubin type differential operators.

We start with the one dimensional case. As a model (and a candidate for a complex NF) we have in mind the non self-adjoint (complex) harmonic oscillator

$$Pu = \mathcal{H}_\omega := D_x^2 u + \omega x^2 u = -u''(x) + \omega x^2 u(x), \quad \omega = z^2, z \in \mathbb{C} \setminus 0, \arg(z) < \frac{\pi}{2}. \quad (3.1)$$

If $\arg(z) \leq \pi/4$ we have an example of the classes of globally elliptic quadratic Shubin operators with non negative real part, which are subject of recent intensive investigations in the context of semiclassical analysis, subelliptic estimates, partial regularity, existence of symmetries etc. (cf. [4], [59] and the references therein). A part from the interest "per se" in the theory of

the pseudodifferential operators on \mathbb{R}^n , we mention another motivation from recent papers on the spectral properties of non self-adjoint differential operators (cf. [14], [15], [44] and the references therein). It is well known that \mathcal{H}_ω has a discrete spectrum, the set of eigenfunctions is complete in the sense that its linear span is dense in $L^2(\mathbb{R})$, but it does not form a Riesz basis, namely, one can not find a bounded invertible linear operator $K : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ such that $K^{-1} \circ \mathcal{H}_\omega \circ K$ becomes a normal operator (for more details see [15] and the references therein). In fact, one may choose complex eigenfunctions $\Phi_k(x)$ (complex Hermite functions) $\Phi_k(x) = \omega^{1/4} H_k(\omega^{1/2}x) \sigma(x)$, $H_k(t)$ being the Hermite polynomials of degree k , $k \in \mathbb{Z}_+$, and $\sigma(x) = e^{-\frac{\omega^{1/2}}{2}x^2}$ is a complex weight function, which form a (pseudo) orthogonal basis, or bi-orthogonal in $L^2(\mathbb{R})$, namely

$$\int_{\mathbb{R}} \Phi_j(x) \Phi_k(x) dx = \langle \Phi_j, \Phi_k^* \rangle_{L^2} = \delta_{jk}, \quad j, k \in \mathbb{N}. \quad (3.2)$$

where δ_{jk} stands for the Kronecker symbol, while $\Phi^*(x) = \overline{\Phi_k(x)}$. Note that the integral in the LHS of (3.2) is without any conjugation. One defines the (non-orthogonal) projector as $P_k u = \langle u, \Phi_k^* \rangle_{L^2} \Phi_k$. The authors prove that

$$\lim_{k \rightarrow \infty} \|P_k\|^{1/k} = c > 1 \text{ or in equivalent form } \lim_{k \rightarrow \infty} \frac{\ln(\|P_k\|)}{k} = d > 0 \quad (3.3)$$

where the norm of the projector P_k defined by

$$\|P_k\|^2 = \int_{\mathbb{R}} |\Phi_k(x)|^2 dx, \quad k \in \mathbb{N}. \quad (3.4)$$

and apply this estimates to show that there exists $t_\omega > 0$, proportional to d , such that

$$e^{-tP} = \sum_{j=0}^{\infty} e^{-t\lambda_j} P_j \text{ is norm convergent if } t > t_\omega \text{ and divergent if } 0 < t < t_\omega. \quad (3.5)$$

Recently, Mityagin, Siegel and Viola (see [44]) have studied operators T , in a separable Hilbert space H , which are unbounded, densely defined, and

have compact resolvent. It is well known that the spectrum of T consists of at most countable number of isolated eigenvalues accumulating at infinity. They define the spectral projection P_λ via the Cauchy integral formula

$$P_\lambda = \frac{1}{2\pi i} \int_{|\zeta - \lambda| = \varepsilon} (\zeta - T)^{-1} d\zeta,$$

where $\varepsilon > 0$ is smaller than the distance from λ to any other eigenvalue.

It is well known, in contrast with the case when T is normal, the spectral projection of a non normal operator T are, in general, not orthogonal. As a consequence in the study of simple differential operators on the real line which admit minimal complete system of eigenvectors $\{u_k\}_{k=0}^\infty$ which is not a Riesz basis. The authors have proved the existence of a natural class of non self-adjoint operators for which

$$\lim_{k \rightarrow \infty} \frac{1}{k^\sigma} \log |P_k| = c > 0, \quad (3.6)$$

where σ can take any value in $(0, 1)$, (see for more details Theorems 2.6 and 3.7 in [44]). These operators arise as non self-adjoint perturbations B of a self-adjoint Schrödinger operator T . They study the non self-adjoint operator

$$M_\varepsilon u = D_x^2 u + x^2 u + i2\varepsilon x D_x u = -u''(x) + x^2 u(x) + 2\varepsilon x u'(x), \quad \varepsilon \in \mathbb{R} \setminus \{0\}. \quad (3.7)$$

a skew-adjoint perturbation $B = 2iax$, $a \in \mathbb{R} \setminus \{0\}$ of the harmonic oscillator $D_x^2 + x^2$. We point out that both operators \mathcal{H}_ω and M_ε are Γ elliptic.

Finally, we mention that in [15] the authors show growth more rapid than any power of k for the norms of spectral projections $\{P_k\}_{k=0}^\infty$ for the anisotropic operators

$$\mathcal{A}_{m,\omega} = -\frac{d^2}{dx^2} + \omega x^m,$$

acting on $L^2(\mathbb{R})$, $\omega \in \mathbb{C} \setminus \mathbb{R}$ and $\arg(\omega) < C(m)$, for more details see Theorem 3 in [15].

Coming back to the issue of NF for non self-adjoint operators, taking into account that in the one dimensional case all solutions of $Pu - \lambda u = 0$ are entire functions (restricted on the real line), it is easy to show that, if P is Γ -elliptic, we can reduce P to \mathcal{H}_ω modulo composition under a transformation of the type $u(x) = e^{(ax^2+bx)}v(x+c)$, for some $a, b, c \in \mathbb{C}$, $\Im a \neq 0$ and/or $\Im b \neq 0$, which acts in the space of the entire functions on \mathbb{C} . The main challenge is to describe completely the spectral properties. Here the Gelfand-Shilov spaces $S_\mu^\mu(\mathbb{R})$, $\mu < 1$, shall play a fundamental role, since all functions from these spaces are extended as entire functions and all possible eigenfunctions of complex harmonic oscillators on the real line

$$P = D_x^2 + \alpha x D_x + \beta x^2 + \gamma D_x + \delta x + r, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}, \quad (3.8)$$

belong to $S_{1/2}^{1/2}(\mathbb{R})$ (see [6]).

We show that if $\omega_1 := \Re\left(\beta - \frac{\alpha^2}{4}\right) \geq 0$ then all perturbations which keep the Γ ellipticity have discrete spectrum and compact resolvent while in the case $\omega_1 < 0$ there exists a threshold ε_0 depending on $\omega_2 = \Im\left(\beta - \frac{\alpha^2}{4}\right)$ such that if $|\Im \alpha| < \varepsilon_0$ the spectrum is discrete and we find explicitly the eigenfunctions whereas $|\Im \alpha| > \varepsilon_0$ implies that the spectrum of P coincides with \mathbb{C} . For the critical value $|\Im \alpha| = \varepsilon_0$ the operator is not globally elliptic. The crucial ingredient of our proofs is the use of the complex JWKB method.

One easy consequence of our NF is the extension of some of the results in the aforementioned works for a slightly larger classes of operators.

We also consider non self-adjoint globally elliptic operators in the multi-dimensional case

$$P = -\Delta + \langle Ax, D_x \rangle + \langle Bx, x \rangle + \langle g, D_x \rangle + \langle h, x \rangle + r. \quad (3.9)$$

where $B, A \in M_n(\mathbb{C})$, $g, h \in \mathbb{C}^n$, $r \in \mathbb{C}$. Here $\langle \xi, \eta \rangle = \xi_1 \eta_1 + \dots + \xi_n \eta_n$, $\xi, \eta \in$

\mathbb{C}^n . Clearly $\Re B$ and $\Im B$ are symmetric, i.e., $B^T = B$ with respect to the complex pseudo inner product above.

We derive reductions via conjugations with NFT of operators given by (3.9) to multidimensional complex harmonic oscillator modulo complex translations

$$H_{\omega}u := -\Delta u + \sum_{j=1}^n \omega_j x_j^2 u, \quad \omega_j \in \mathbb{C} \setminus 0, \Re \omega_j > 0, \quad (3.10)$$

provided $\Re A$, $\Im A$, $\Re B$, $\Im B$ satisfy suitable commutator and separation of variables type conditions. Thus we are able to classify the spectral properties of (3.9).

Next, we investigate the reduction to NF of anisotropic versions of (3.7)

$$P = D_x^2 + \sum_{j=0}^k a_j x^{k-j} D_x + \sum_{\ell=0}^{2k} b_{\ell} x^{2k-\ell} \quad (3.11)$$

where $a_j \in \mathbb{C}$, $a_0 \neq 0$, $b_{\ell} \in \mathbb{C}$, $b_0 \neq 0$. We are able to classify completely the spectral properties of (3.11). Here again we rely heavily on the complex JWKB methods but unlike the quadratic harmonic oscillator we are not able to write down explicitly the eigenvalues and the eigenfunctions. However, under additional restrictions on the complex coefficients, we show that the spectrum and the bi-orthogonal eigenfunctions are generated by the canonical self-adjoint NF anisotropic operator $L_{2k,c}$, $c \in \mathbb{R}^{2k-2}$ (defined in the previous chapter) by means of complex dilations and translation and multiplication by exponential terms of the type $e^{\eta x^{k+1} + O(|x|^k)}$, $\Re \eta \neq 0$. We take advantage of the hypoellipticity in $S_{1/(k+1)}^{k/(k+1)}(\mathbb{R})$ of anisotropic elliptic operators of the type $-\Delta + (\|x\|^2)^k$, see [5].

The major novelty of our investigations this section concerns the case of multidimensional anisotropic non self-adjoint operators. We derive normal forms via transformations acting in Gelfand-Shilov spaces of entire functions

of operators defined by

$$P = D_x^2 + \langle A(x), D_x \rangle + B(x), \quad x \in \mathbb{R}^n, \quad (3.12)$$

where $A(x) = (A_1(x), \dots, A_n(x))$, A_j are polynomials with complex coefficients of degree k , B is polynomial with complex coefficients of degree $2k$, $\langle A(x), dx \rangle$ is closed 1 form, with $A(x)$ and $B(x)$ rather restrictive satisfying symmetry and separation of variables type conditions.

Finally, we study perturbations of complex quadratic harmonic oscillators with some regularizing pseudodifferential operators $b(x, D)$, whose symbols belong to the limit Gelfand–Shilov space $S_{1/2}^{1/2}(\mathbb{R}^n)$ and show that despite the quadratic exponential growth of the NFT we can still conjugate $b(x, D)$ with $e^{\varepsilon x^2}$, provided $|\varepsilon| \ll 1$.

The chapter is organized as follows: Section 1 is concerned with the classification of the NF and the study of the spectral properties of the perturbations of the complex harmonic oscillator. In the second section we study NF and spectral properties of perturbations of multidimensional complex harmonic oscillators. The classification of the NF and the study of the spectral properties of non–self–adjoint 1D anisotropic operators is done in Section 3 whereas in Section 4 we address the same issues for some classes of multidimensional anisotropic operators. In the final section we demonstrate a result on the conjugation of classes of $S_{1/2}^{1/2}(\mathbb{R}^n)$ operators via quadratic exponential NFT.

3.1 NF of perturbations of 1D complex harmonic oscillators

We consider the second order Shubin differential operator on the real line

$$P = D_x^2 + \alpha x D_x + \beta x^2 + \gamma D_x + \delta x + r. \quad (3.13)$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.

We assume, without loss of generality, that $\Re\alpha = 0$, $\Re\gamma = 0$ - otherwise, as in Chapter 2, we apply the conjugation with a unitary NFT

$$e^{i(\frac{\Re(\alpha)}{4}x^2 + \frac{\gamma}{2}x)} \circ P \circ e^{-i(\frac{\Re(\alpha)}{4}x^2 + \frac{\gamma}{2}x)}. \quad (3.14)$$

Therefore, we consider the operator P in this form as a perturbation of a non self-adjoint operator

$$L = D_x^2 + i\varepsilon x D_x + \omega x^2 + ip D_x + qx + r, \quad (3.15)$$

where $\omega = \omega_1 + \omega_2 i \in \mathbb{C} \setminus 0$, $\omega_1 > 0$, if $\omega_2 = 0$, $\varepsilon, p \in \mathbb{R}$, $q = q_1 + q_2 i \in \mathbb{C}$, $q_1 = 0$ if $\omega_1 \neq 0$, $q_2 = 0$ if $\omega_1 = 0$, $r \in \mathbb{C}$.

We derive reductions to normal forms taking advantage of the limiting Gelfand–Shilov $S_{1/2}^{1/2}(\mathbb{R})$ regularity of the eigenfunctions of second order globally elliptic differential operators.

We introduce, following the arguments used in [14], [15], complex inner product with complex weight function $\sigma(x)$, $\sigma(x) \neq 0$, $x \in \mathbb{R}$,

$$\langle f, g \rangle_{\mathbb{C}, \sigma} = \int_{\mathbb{R}} f(x) g(x) \sigma^2(x) dx. \quad (3.16)$$

One checks easily that if for some $s \in \mathbb{R}$, $t \geq 0$, $C > 0$

$$|\sigma(x)| \leq C e^{sx^2 + t|x|}, \quad x \in \mathbb{R} \quad (3.17)$$

then (3.16) is well defined for all $f, g \in S_{1/2}^{1/2}(\mathbb{R})$ (respectively, $f, g \in S_{\mu}^{\mu}(\mathbb{R})$, $1/2 \leq \mu < 1$), decaying exponentially

$$\begin{aligned} \max\{|f(x)|, |g(x)|\} &= O(e^{-\eta x^2}, \quad x \rightarrow \infty) \\ &(\text{respectively, } O(e^{-\eta|x|^{1/\mu}}, \quad x \rightarrow \infty), \quad k \in \mathbb{N}, \quad (3.18) \end{aligned}$$

for all $0 < s < \eta$ (respectively, $\eta > 0$), provided $s > 0$ (respectively, $s \leq 0$).

We consider first the simpler case of $\varepsilon = 0$.

Proposition 28. *Suppose that $\varepsilon = 0$. Then there exists a transformation*

$$T = T_{p/2, q/(2\omega)} w(x) = e^{\frac{p}{2}x} w(x - \frac{q}{2\omega}i), \quad p, q \in \mathbb{C}, \quad (3.19)$$

$T_{a,b}^{-1} = T_{-a,-b}$ such that

$$T_{-p/2, -q/(2\omega)} \circ L \circ T_{p/2, q/(2\omega)} = D_x^2 + \omega x^2 + \tilde{r}, \quad (3.20)$$

where $\tilde{r} = r - \frac{p^2}{4} + \frac{q^2}{4\omega}$. L has a discrete spectrum and the eigenvalues are given by

$$\lambda_k = \omega^{1/2}(2k+1) + r, \quad k \in \mathbb{Z}_+, \quad (3.21)$$

with eigenfunctions defined by

$$\varphi_k(x) = \kappa e^{-\frac{p}{2}x} H_k(\omega^{1/4}(x - \frac{q}{2\omega}i)), \quad k \in \mathbb{N}, \quad (3.22)$$

with $\kappa^{-1} = \int_{\mathbb{R}} H_k^2(\omega^{1/4}(x - \frac{q}{2\omega}i)) dx = \omega^{-1/4}$, forming bi-orthogonal basis with respect to the complex weighted inner product (3.16)

$$\sigma(x) = e^{-\frac{p}{2}x + \omega^{1/4}(x - \frac{q}{2\omega}i)^2}, \quad k \in \mathbb{N}, \quad (3.23)$$

Finally, we claim that the spaces $S_\mu^\mu(\mathbb{R})$, $1/2 \leq \mu < 1$, are invariant under the action of $T_{p/2, q/(2\omega)}$ and there exist $C > 0$ such that

$$\|T_{p/2, q/(2\omega)} u\|_{\mu; \rho - \delta, \sigma + \delta} \leq e^{C(1-\mu)(\frac{C}{\delta})^{\mu/(1-\mu)}} \|u\|_{\mu; \rho, \sigma}, \quad (3.24)$$

for $u \in BS_\mu^\mu(\mathbb{R}; \rho, \sigma)$, $0 < \delta \ll 1$, $1/2 \leq \mu < 1$.

Proof. We have

$$\begin{aligned} L(e^{\frac{p}{2}x} v(x)) &= D_x^2(e^{\frac{p}{2}x} v(x)) + \omega x^2(e^{\frac{p}{2}x} v(x)) + ipD_x(e^{\frac{p}{2}x} v(x)) \\ &\quad + qx(e^{-i\frac{p}{2}x} v(x)) \\ &= e^{\frac{p}{2}x} \left(D_x^2 + \omega x^2 + qx + r - \frac{p^2}{4} \right) v(x) = e^{\frac{p}{2}x} \tilde{L}v(x), \end{aligned}$$

and the translation $x \mapsto x + \frac{q}{2\omega}$ transforms \tilde{L} into (3.20). We conclude by observing that for every $u \in BS_\mu^\mu(\mathbb{R}; \rho, \sigma)$, $\rho, \sigma > 0$, $1/2 \leq \mu \leq 1$ we get

$$\begin{aligned} |e^{\frac{p}{2}x} u(x + a_1 + (y + a_2)i)| &\leq C_0 e^{-\rho|x+a_1|^{1/\mu} + \sigma|y+a_2|^{1/\mu} + \frac{p}{2}x} \\ &\leq C_1 e^{-\rho|x|^{1/\mu} + \sigma|y|^{1/\mu} + C_1(|x|+|y|)} \\ &\leq C_1 e^{-(\rho-\delta)|x|^{1/\mu} + (\sigma+\delta)|y|^{1/\mu} - \delta(|x|^{1/\mu} + |y|^{1/\mu}) + C_1(|x|+|y|)} \\ &\leq C_1 \varkappa_\mu^2 e^{-(\rho-\delta)|x|^{1/\mu} + (\sigma+\delta)|y|^{1/\mu}}, \end{aligned} \quad (3.25)$$

where

$$\varkappa_\mu = \sup_{t \geq 0} \left(e^{-\delta t^{1/\mu} + C_1 t} \right) = e^{(1-\mu)C_1 \frac{C_1^{\mu/(1-\mu)}}{\delta^{\mu/(1-\mu)}}}, \quad (3.26)$$

which implies (3.25). \square

Remark 10. We note that if $\Re p \neq 0$ or $\Re q \neq 0$ the transformation T is not defined on $L^2(\mathbb{R})$ or $S_\mu^\mu(\mathbb{R})$ for $\mu \geq 1$. The crucial point for the requirement $\mu < 1$ is the fact that the all $f \in S_\mu^\mu(\mathbb{R})$ are restrictions on \mathbb{R} of entire functions and T acts continuously in $S_\mu^\mu(\mathbb{R})$ using equivalent topology for entire functions. For more details see [20],[5].

Let us consider the second order operator

$$L = D_x^2 + i\varepsilon x D_x + \omega x^2 + ip D_x + qx + \tau, \quad (3.27)$$

where $\omega = \omega_1 + \omega_2 i \in \mathbb{C}$, $\omega \neq 0$, $\varepsilon, p \in \mathbb{R}$, $q, \tau \in \mathbb{C}$.

Actually, using dilation $x = |\omega_1|^{1/4} y$, if ω_1 is different from zero, we can consider 3 classes of operators (modulo multiplication with $|\omega_1|^{1/2}$, when $\omega_1 \neq 0$), reducing to 3 cases: $\omega_1 = 1$, $\omega_1 = -1$, $\omega_1 = 0$, namely

$$L_+ = D_x^2 + i\varepsilon x D_x + (1 + i\delta)x^2 + ip D_x + qx + \tau, \quad (3.28)$$

$$L_- = D_x^2 + i\varepsilon x D_x + (-1 + i\delta)x^2 + ip D_x + qx + \tau, \quad (3.29)$$

$$L_0 = D_x^2 + i\varepsilon x D_x + i\delta x^2 + ip D_x + qx + \tau, \quad (3.30)$$

for $\varepsilon, \delta, p \in \mathbb{R}$, $q, \tau \in \mathbb{C}$, $\delta \neq 0$ in (3.30).

We classify completely the class of operators (3.27) when $\varepsilon \neq 0$.

Theorem 36. *The operator L (3.27) is globally elliptic iff*

$$\text{either } \omega_1 \geq 0 \text{ or } \omega_1 < 0 \text{ and } \omega_2^2 + \omega_1 \varepsilon^2 \neq 0, \quad (3.31)$$

or stated equivalently

$$L_+ \text{ is globally elliptic for all } \delta, \varepsilon \in \mathbb{R}, \quad (3.32)$$

$$L_- \text{ is globally elliptic iff } \delta^2 \neq \varepsilon^2, \quad (3.33)$$

$$L_0 \text{ is globally elliptic iff } \delta \neq 0. \quad (3.34)$$

Furthermore, there exist $a, b \in \mathbb{C}$ such that the transformation $T_{a,b}$

$$u(x) = e^{-\frac{\varepsilon}{4}(x+b)^2} T_{a,b} v(x) = e^{-\frac{\varepsilon}{4}(x+b)^2} e^{ax} v(x+b) \quad (3.35)$$

transforms L to

$$H_\theta v(y) + rv(y) = -v''(y) + \theta x^2 v(y) + r, \quad \theta \in \mathbb{C} \setminus 0, 2\Re\sqrt{\theta} \neq |\varepsilon|, r \in \mathbb{C}. \quad (3.36)$$

Under the assumption (3.31) the following properties are equivalent:

i) P has a discrete spectrum with simple eigenvalues in $\Re z > 0$ with eigenfunctions forming a Schauder basis (but not Riesz basis) and compact resolvent.

ii) Either $\omega_1 \geq 0$ or $\omega_1 < 0$, $\omega_2 \neq 0$ and $|\varepsilon| < \varepsilon_0 = -\omega_2^2/\omega_1$.

iii) The normal form operator H_θ in (3.36) satisfies

$$2\Re\sqrt{\theta} > |\varepsilon|. \quad (3.37)$$

In particular, if *i*) or *ii*) holds, the eigenvalues are given by

$$\lambda_k = \theta^{1/2}(2k-1) + r, \quad \theta = \omega_1 + \frac{\varepsilon^2}{4} + i\omega_2, k \in \mathbb{N} \quad (3.38)$$

with eigenfunctions forming bi-orthogonal basis with respect to the complex weight inner product (3.23)

$$\sigma(x) = e^{\frac{\varepsilon}{4}x^2 + ax}. \quad (3.39)$$

defined by

$$\varphi_k(x) = \kappa e^{-\frac{\varepsilon}{4}x^2} e^{-ax} H_k(\theta^{1/4}(x-b)), \quad k \in \mathbb{N}, \quad (3.40)$$

with

$$\kappa^{-1} = \left(\int_{\mathbb{R}} H_k^2(\theta^{1/4}(x-b)) dx \right)^{1/2} = \theta^{-1/4}.$$

Finally,

$$\text{if } \omega_1 < 0, |\varepsilon| > \varepsilon_0, \text{ then } \text{spec}(L) = \mathbb{C}. \quad (3.41)$$

Remark 11. We can rewrite the result in Theorem 36 for L_+ , L_- and L_0 under the hypothesis of global hypoellipticity:

L_+ and L_0 have discrete spectrum and eigenfunctions defined in (3.21), (3.40) (3.42)

L_- has discrete spectrum as (3.42) iff $\delta^2 > \varepsilon^2$ and $\text{spec}(L_-) = \mathbb{C}$ for $\delta^2 < \varepsilon^2$,

3.1.1 Proof of the complex harmonic oscillator NF

First we study the Γ ellipticity. Since the principal symbol of P is

$$p_2(x, \xi) = \xi^2 + \omega_1 x^2 + ix(\varepsilon \xi + \omega_2). \quad (3.43)$$

one checks easily that that $p_2(x, \xi) \neq 0$ for $(x, \xi) \neq (0, 0)$ iff $\omega_2^2 + \omega_1 \varepsilon^2 \neq 0$.

The idea of the proof is quite natural. We know by the theory of complex ODE that all solutions of

$$Pu - \lambda u = 0, \quad \lambda \in \mathbb{C}, \quad (3.44)$$

are extended to entire functions. Therefore, straightforward calculations show that $u(x)$ solves (3.44) iff $v(x)$, defined by

$$u(x) = e^{-\varepsilon \frac{(x+b)^2}{4} - \frac{p(x+b)}{2}} v(x+b) \quad (3.45)$$

solves

$$\mathcal{H}_\omega v + (r - \lambda)v = 0, \quad \lambda \in \mathbb{C}, \quad (3.46)$$

where

$$\omega = \beta - \varepsilon^2/4, \quad a = p/2, \quad b = \frac{q}{2\omega}. \quad (3.47)$$

The spectrum of \mathcal{H}_ω is discrete, with eigenvalues $\lambda_k = \omega^{1/2}(2k-1) + r$, and basis the complex Hermite functions $H_k(\omega^{1/4}y)$. One checks immediately (returning to u via (3.35))

$$\varphi_k(x) := e^{-\varepsilon \frac{(x+b)^2}{4} - \frac{px}{2}} H_k(\omega^{1/4}(x+b)) \in L^2(\mathbb{R}) \text{ if (3.37) holds.} \quad (3.48)$$

In order to complete the proof we have to apply complex WKB methods (cf. [66]). .

Lemma 8. *Let $\lambda \in \mathbb{C}$. Then all solutions of*

$$Lu - \lambda u = 0 \quad (3.49)$$

are restrictions on the real line of entire functions. Moreover, we can find a basis $\varphi_{exp}^\pm(x; \lambda)$, $\varphi_{dec}^\pm(x; \lambda)$, $x \in \mathbb{C}$ satisfying

$$\begin{aligned} \varphi_{exp}^\pm(x) &= a_{exp}^\pm(x) \exp\left(\varepsilon \frac{x^2}{4} \pm \int_{x_0}^x \sqrt{(\omega_1 + i\omega_2)t^2} dt + O(|x|)\right), \quad x \in \Gamma_\pm, |x| \rightarrow \infty, \\ \varphi_{dec}^\pm(x) &= a_{dec}^\pm(x) \exp\left(\varepsilon \frac{x^2}{4} \mp \int_{x_0}^x \sqrt{(\omega_1 + i\omega_2)t^2} dt + O(|x|)\right), \quad x \in \Gamma_\pm, |x| \rightarrow \infty. \end{aligned}$$

with Γ_{\pm} being a complex conic neighbourhood of \mathbb{R}^{\pm} , and the amplitudes $a_{dec}^{\pm}(x), a_{exp}^{\pm}(x)$ are holomorphic in Γ^{\pm} and are $O(|x|^{-1/2})$, $x \in \Gamma^{\pm}$, $|x| \rightarrow \infty$.

Proof. We consider the following transformation

$$Eu(x) := e^{-\frac{p}{2}x}u(x),$$

So the operator L in (3.15) becomes

$$E^* \circ L \circ E = \hat{L} = D_x^2 + i\varepsilon x D_x + (\omega_1 + i\omega_2)x^2 + hx + \ell,$$

where $h = q - i\varepsilon \frac{p}{2}$ and $\ell = r - \frac{p^2}{4}$.

We can apply the transformation

$$E_{\varepsilon}v(x) = e^{\varepsilon x^2/4}v(x),$$

so we have

$$E_{\varepsilon}^{-1} \circ \hat{L} \circ E_{\varepsilon} = \tilde{L} = D_x^2 + (\tilde{\omega}_1 + i\tilde{\omega}_2)x^2 + \tilde{h}x + \tilde{\ell}, \quad (3.50)$$

where $\tilde{\omega}_1 = \omega_1 + \frac{\varepsilon^2}{2}$, and $\tilde{\omega}_2 = \omega_2$, $\tilde{h} = h$, $\tilde{\ell} = \ell - \frac{\varepsilon}{2}$. The equation (3.49) is equivalent to

$$\tilde{L}v - \lambda v = -v'' + (\tilde{\omega}x^2 + \tilde{h}x + \tilde{\ell} - \lambda)v = 0 \quad (3.51)$$

Now we can apply the complex *WKB* method for the equation above. More precisely, we can find two linearly independent solutions for $x \rightarrow +\infty$ (respectively, $-\infty$), namely

$$\begin{aligned} \Psi_{exp}^+(x; \lambda) &= b_{exp}^+(x) e^{\int_0^x \sqrt{(\tilde{\omega}_1 + i\tilde{\omega}_2)t^2 + \tilde{h}t + \tilde{\ell} - \lambda} dt}, \\ \Psi_{dec}^+(x; \lambda) &= b_{dec}^+(x) e^{-\int_0^x \sqrt{(\tilde{\omega}_1 + i\tilde{\omega}_2)t^2 + \tilde{h}t + \tilde{\ell} - \lambda} dt}, \end{aligned} \quad (3.52)$$

for $x \in \Gamma_+$ (respectively,

$$\begin{aligned}\Psi_{exp}^-(x; \lambda) &= b_{exp}^-(x) e^{-\int_0^x \sqrt{(\tilde{\omega}_1 + i\tilde{\omega}_2)t^2 + \tilde{h}t + \tilde{\ell} - \lambda} dt}, \\ \Psi_{dec}^-(x; \lambda) &= b_{dec}^-(x) e^{\int_0^x \sqrt{(\tilde{\omega}_1 + i\tilde{\omega}_2)t^2 + \tilde{h}t + \tilde{\ell} - \lambda} dt},\end{aligned}\tag{3.53}$$

for $x \in \Gamma_-$) and the amplitudes $b_{dec}^\pm(x), b_{exp}^\pm(x)$ are holomorphic in Γ^\pm and are $O(|x|^{-1/2})$, $x \in \Gamma^\pm$, $|x| \rightarrow \infty$. We point out that (3.52) (respectively, (3.53)) implies

$$\begin{aligned}|\Psi_{exp}^+(x)| &\sim \varkappa_{exp}^+ |x|^{-1/2} e^{\int_0^x \operatorname{Re} \sqrt{(\tilde{\omega}_1 + i\tilde{\omega}_2)t^2 + \tilde{h}t + \tilde{\ell} - \lambda} dt} \\ &= \varkappa_{exp}^+ |x|^{-1/2} e^{\operatorname{Re} \left(\sqrt{(\tilde{\omega}_1 + i\tilde{\omega}_2)x^2/2} \right) + O(|x|)} \\ |\Psi_{dec}^+(x)| &\sim \varkappa_{dec}^+ |x|^{-1/2} e^{-\int_0^x \operatorname{Re} \sqrt{(\tilde{\omega}_1 + i\tilde{\omega}_2)t^2 + \tilde{h}t + \tilde{\ell} - \lambda} dt} \\ &= \varkappa_{dec}^+ |x|^{-1/2} e^{-\operatorname{Re} \left(\sqrt{(\tilde{\omega}_1 + i\tilde{\omega}_2)x^2/2} \right) + O(|x|)}\end{aligned}\tag{3.54}$$

for $x \in \Gamma_+$, $|x| \rightarrow \infty$, where $\varkappa_{exp/dec}^+ \in \mathbb{R}^+$,

(respectively,

$$\begin{aligned}|\Psi_{exp}^-(x)| &\sim \varkappa_{exp}^- |x|^{-1/2} e^{-\int_0^x \operatorname{Re} \left(\sqrt{(\tilde{\omega}_1 + i\tilde{\omega}_2)t^2 + \tilde{h}t + \tilde{\ell} - \lambda} \right) dt} \\ &= \varkappa_{exp}^- |x|^{-1/2} e^{-\operatorname{Re} \left(\sqrt{(\tilde{\omega}_1 + i\tilde{\omega}_2)x^2/2} \right) + O(|x|)} \\ |\Psi_{dec}^-(x)| &\sim \varkappa_{dec}^- |x|^{-1/2} e^{\int_0^x \operatorname{Re} \sqrt{(\tilde{\omega}_1 + i\tilde{\omega}_2)t^2 + \tilde{h}t + \tilde{\ell} - \lambda} dt} \\ &= \varkappa_{dec}^- |x|^{-1/2} e^{+\operatorname{Re} \left(\sqrt{(\tilde{\omega}_1 + i\tilde{\omega}_2)x^2/2} \right) + O(|x|)}\end{aligned}\tag{3.55}$$

for $x \in \Gamma_-$, $|x| \rightarrow \infty$ and $\varkappa_{exp/dec}^- \in \mathbb{R}^+$).

Let C_{\mp}^\pm be the transition matrix from $\mp\infty$ to $\pm\infty$. Thus

$$\Psi^+ = \begin{pmatrix} \Psi_{exp}^+ \\ \Psi_{dec}^+ \end{pmatrix} = C_-^+ \Psi^- = C_-^+ \begin{pmatrix} \Psi_{exp}^- \\ \Psi_{dec}^- \end{pmatrix}, \quad C_-^+ = (C_+^-)^{-1}.$$

One observes that

$$\int_0^x \sqrt{(\tilde{\omega}_1 + i\tilde{\omega}_2)t^2 + \tilde{h}t + \tilde{\ell} - \lambda} dt \simeq \sqrt{(\tilde{\omega}_1 + i\tilde{\omega}_2)} \frac{x^2}{2} + O(|x|).$$

Coming back to the operator P , and using the fact that the solutions of $Pu - \lambda u = 0$ are entire functions we obtain

$$\varphi_{exp}^+(x; \lambda) \sim e^{\varepsilon \frac{x^2}{4}} \psi_{exp}^+(x),$$

$$\varphi_{exp}^-(x; \lambda) \sim e^{\varepsilon \frac{x^2}{4}} \psi_{exp}^-(x),$$

$$\varphi_{dec}^+(x; \lambda) \sim e^{\varepsilon \frac{x^2}{4}} \psi_{dec}^+(x),$$

$$\varphi_{dec}^-(x; \lambda) \sim e^{\varepsilon \frac{x^2}{4}} \psi_{dec}^-(x).$$

and so we can find a solution of our equation (3.49). □

Now we are able to prove Theorem 36. We study the spectral properties using the previous Lemma. We have three cases:

- If $\varepsilon < -Re\sqrt{(\omega_1 + \frac{\varepsilon^2}{4} + i\omega_2)}$ then $Ker(P - \lambda I) \subset S_{1/2}^{1/2}(\mathbb{R})$ for any $\lambda \in \mathbb{C}$, so $spec(P) = \mathbb{C}$.
- If $\varepsilon > Re\sqrt{(\omega_1 + \frac{\varepsilon^2}{4} + i\omega_2)}$ then $Ker(P - \lambda I) = \{0\}$, so the $indP \neq 0$, so the $spec(P) = \mathbb{C}$.
- If $|\varepsilon| < Re\sqrt{(\omega_1 + \frac{\varepsilon^2}{4} + i\omega_2)}$ then we have a compact resolvent so the spectrum of P is discrete.

If $\omega_1 > 0$, the operator P is a globally elliptic, so we can reduce P to normal form in the operator

$$\tilde{P} = \sqrt{\omega} \left(D_z^2 + z^2 + \frac{\tilde{r}}{\sqrt{\omega}} \right).$$

That operator has compact resolvent and its eigenvalues are

$\lambda_k = \sqrt{\omega}(2k+1) + r$, $k \in \mathbb{Z}_+$. We write the eigenfunctions

$$\tilde{u}_k(z) = \tilde{c}_k H_k(z) e^{-z^2/2},$$

where H_k are the Hermite functions. So for the operator P (3.13) a basis of eigenfunctions is defined as follows:

$$u_k(x) = c_k e^{-i(\alpha \frac{x^2}{4} + \frac{\gamma x}{2})} H_k(\sqrt[4]{\omega}(x + \delta/(2\omega))) e^{-\frac{(x+\delta)/(2\omega)^2}{2}}.$$

The proof is complete.

3.2 Normal forms of 1D anisotropic non self-adjoint operators

In view of the results in Chapter 2 for self-adjoint anisotropic operators without loss of generality, applying first the unitary NFT, used in Chapter 2, we consider operators of the following form

$$Pu = D_x^2 u + ia(x)D_x u + b(x)u = -u'' + a(x)u' + b(x)u, \quad x \in \mathbb{R}, \quad (3.56)$$

where

$$a(x) = \sum_{j=0}^k a_j x^{k-j}, \quad a_j \in \mathbb{R}, j = 0, 1, \dots, k, \quad (3.57)$$

$$b(x) = \sum_{j=0}^{2k} b_j x^{2k-j}, \quad b_j \in \mathbb{C}, j = 0, 1, \dots, 2k, b_0 \neq 0 \quad (3.58)$$

We recall the global anisotropic ellipticity condition (cf. [3], [6])

$$p_2(x, \xi) = \xi^2 + ia_0 x^k \xi + b_0 x^{2k} \neq 0, \quad \text{for } (x, \xi) \neq (0, 0). \quad (3.59)$$

It is well known that if $u \in \mathcal{S}'(\mathbb{R}^n)$, $Pu = f \in S_v^\mu(\mathbb{R})$, with

$$\mu \geq \frac{k}{k+1} \quad \nu \geq \frac{1}{k+1},$$

then also $u \in S_V^\mu(\mathbb{R})$ cf. [6].

We recall that all solutions of

$$Pu - \lambda u = 0, \quad x \in \mathbb{R}, \lambda \in \mathbb{C} \quad (3.60)$$

are extended to entire functions in \mathbb{C} because the coefficients are polynomials. Therefore, applying the transformation $E_{-A} = e^{-A(x)/2}$, where $A(x) = \int_0^x a(t) dt$, to the operator (3.56) we get

$$\tilde{P} = E_A^{-1} \circ P \circ E_A = D_x^2 + Q(x), \quad (3.61)$$

where

$$Q(x) = b(x) + \frac{a(x)^2}{4} - \frac{\partial_x(a(x))}{2}i = \sum_{j=0}^{2k} q_j x^{2k-j}, \quad q_j \in \mathbb{C}, j = 0, 1, \dots, 2k. \quad (3.62)$$

We require that

$$\Re q_0 > 0, \text{ if } \Im q_0 = 0. \quad (3.63)$$

Then one checks easily that P is globally (anisotropically) elliptic iff

$$\Im b_0^2 + \frac{1}{4} a_0^2 \Re b_0 \neq 0 \text{ if } \Re b_0 \leq 0. \quad (3.64)$$

We note that $Q(x) = q_0(x + \frac{q_1}{2kq_0})^{2k}$ plus polynomial of degree $\leq 2k - 2$.

Hence

$$\tilde{Q}(y) := Q(y - \frac{q_1}{2kq_0}) = q_0 y^{2k} + \sum_{j=0}^{2k-3} \tilde{q}_j x^{2k-j} + r, \quad \tilde{q}_j \in \mathbb{C}, j = 0, 1, \dots, 2k-3, r \in \mathbb{C}. \quad (3.65)$$

which leads to

$$T_{\frac{-q_1}{2kq_0}}^{-1} \circ \tilde{P} \circ T_{\frac{-q_1}{2kq_0}} = -\Delta + \tilde{Q}(x) \quad (3.66)$$

Proposition 29. *Set $J = E_A \circ T_{\frac{-q_1}{2kq_0}} \circ D_{q_0}^{1/(2k+2)}$. Then*

$$J^{-1} \circ P \circ J = \tau L_{2k,w}(x, D_x) + r, w_j := \frac{\tilde{q}_j}{\tau^{2k-j}}, \quad j = 0, 1, \dots, 2k-3. \quad (3.67)$$

Assume now that

$$w_j := \rho_j \in \mathbb{R}, \quad j = 0, \dots, 2k-3. \quad (3.68)$$

Then every solution of $Pu - \lambda u = 0$ is given by

$$u(x) = e^{A \frac{\left((q_0)^{1/(2k+2)} \left(x - \frac{q_0}{2kq_1} \right) \right)}{2}} v \left((q_0)^{1/(2k+2)} \left(x - \frac{q_0}{2kq_1} \right) \right) \quad (3.69)$$

with v solving

$$L_{2k,\rho} v + \frac{r - \lambda}{\tau^2} v = 0. \quad (3.70)$$

Proof. One observes that

$$\begin{aligned} J^{-1} \circ P \circ J &= T_{-\frac{q_1}{2kq_0}}^{-1} \circ \tilde{P} \circ T_{\frac{-q_1}{2kq_0}} \\ &= D_{q_0}^{-1/(2k+2)} \circ T_{\frac{q_1}{2kq_0}} \circ \tilde{P} \circ T_{\frac{-q_1}{2kq_0}} \circ D_{q_0}^{1/(2k+2)} \\ &= D_{q_0}^{-1/(2k+2)} \circ (-\Delta + \tilde{Q}) \circ D_{q_0}^{1/(2k+2)} = \tau^2 L_{2k,\rho} + r. \end{aligned} \quad (3.71)$$

□

Next, we study the spectral properties.

Theorem 37. *Suppose that*

$$|a_0| > 2\Re(q_0^{1/2}). \quad (3.72)$$

Then

$$\text{spec}(P) = \mathbb{C}.$$

Finally, in the case k odd we have

$$\begin{aligned} \text{Ker}(P - \lambda) \cap \mathcal{S}'(\mathbb{R}) &= \{u \in \mathcal{O}(\mathbb{C}) : Pu - \lambda u = 0\}, \\ \text{Ker}(P^* - \lambda) \cap \mathcal{S}'(\mathbb{R}) &= \{0\} \end{aligned}$$

(respectively

$$\begin{aligned} \text{Ker}(P^* - \lambda) \cap \mathcal{S}'(\mathbb{R}) &= \{u \in \mathcal{O}(\mathbb{C}) : Pu - \lambda u = 0\}, \\ \text{Ker}(P - \lambda) \cap \mathcal{S}'(\mathbb{R}) &= \{0\} \end{aligned}$$

provided $a_0 < -2\Re(q_0^{1/2})$ (respectively $a_0 > 2\Re(q_0^{1/2})$) while for k even

$$\text{Ker}(P - \lambda) \cap \mathcal{S}'(\mathbb{R}) = \text{Ker}(P^* - \lambda) \cap \mathcal{S}'(\mathbb{R}) = \{0\}.$$

Remark 12. We note that under the hypothesis (3.72) we do not require the zero imaginary part condition.

Next, we investigate the case of discrete spectrum.

Theorem 38. Suppose that that

$$|a_0| < 2\Re(q_0^{1/2}). \quad (3.73)$$

Then P has compact resolvent and discrete spectrum. Moreover, if the zero condition holds, we have:

- The spectrum of P is discrete and given by

$$\text{spec}(P) = \{\lambda_j : \lambda_j = q_0^{1/(k+1)} \lambda_j(\vec{\rho}) + r, j \in \mathbb{N}\}. \quad (3.74)$$

with a basis of eigenfunctions ψ_j , $j \in \mathbb{N}$ defined by

$$\psi_j(x) = c_j e^{-iA(x)/2} \varphi_j(q_0^{1/(2k+2)}(x + \frac{q_1}{2kq_0})), \quad j \in \mathbb{N} \quad (3.75)$$

with $c_j > 0$ defined by $\|\psi_j\|_{L^2} = 1$

3.2.1 Proof of the assertions on the spectral properties

The first ingredient of our proof consists of the use of the complex WKB (or JWKB) method. We derive an important auxiliary assertion

Lemma 9. *Let $\lambda \in \mathbb{C}$. Then all solutions of*

$$Pu - \lambda u = 0 \quad (3.76)$$

are restrictions on the real line of entire functions. Moreover, we can find two basis $\varphi_{exp}^{\pm}(x; \lambda)$, $\varphi_{dec}^{\pm}(x; \lambda)$, $x \in \mathbb{C}$ of $\text{Ker}(P - \lambda)$ satisfying

$$\begin{aligned} \varphi_{exp}^{\pm}(x) &= a_{exp}^{\pm}(x) \exp\left(\frac{a_0 x^{k+1}}{2(k+1)} \pm \int_{x_0}^x \sqrt{q_0 t^{2k}} dt + O(|x|^k)\right), \quad x \in \Gamma_{\pm}, |x| \rightarrow \infty, \\ \varphi_{dec}^{\pm}(x) &= a_{dec}^{\pm}(x) \exp\left(\frac{a_0 x^{k+1}}{2(k+1)} \mp \int_{x_0}^x \sqrt{q_0 t^{2k}} dt + O(|x|^k)\right), \quad x \in \Gamma_{\mp}, |x| \rightarrow \infty. \end{aligned}$$

with Γ_{\pm} being a complex conic neighbourhood of \mathbb{R}^{\pm} , and the amplitudes $a_{dec}^{\pm}(x)$, $a_{exp}^{\pm}(x)$ are holomorphic in Γ^{\pm} and are $O(|x|^{-k/2})$, $x \in \Gamma^{\pm}$, $|x| \rightarrow \infty$.

Proof. We can apply the transformation $E_{A(x)} = e^{-A(x)/2}$, where $A(x) = \int_0^x a(t) dt$, to the operator (3.56) and we are reduced to the study of \tilde{P} (3.61). Thus the equation (3.76) is equivalent to

$$\tilde{P}v - \lambda v = -v'' + \tilde{Q}(x)v = 0, \quad (3.77)$$

where $\tilde{Q}(x) = Q(x) - \lambda$.

Now we can apply the complex *WKB* method for the equation above. More precisely, we can find two linearly independent solutions for $x \rightarrow +\infty$ (respectively, $-\infty$), namely

$$\begin{aligned} \psi_{exp}^+(x; \lambda) &= b_{exp}^+(x) e^{\int_0^x \sqrt{q_0 t^{2k} + q_1 t^{2k-1} + \dots + q_{2k} - \lambda} dt}, \\ \psi_{dec}^+(x; \lambda) &= b_{dec}^+(x) e^{-\int_0^x \sqrt{q_0 t^{2k} + q_1 t^{2k-1} + \dots + q_{2k} - \lambda} dt}, \end{aligned} \quad (3.78)$$

for $x \in \Gamma_+$ (respectively,

$$\begin{aligned}\Psi_{exp}^-(x; \lambda) &= b_{exp}^-(x) e^{-(-1)^{k-1} \int_0^x \sqrt{q_0 t^{2k} + q_1 t^{2k-1} + \dots + q_{2k} - \lambda} dt}, \\ \Psi_{dec}^-(x; \lambda) &= b_{dec}^-(x) e^{(-1)^{k-1} \int_0^x \sqrt{q_0 t^{2k} + q_1 t^{2k-1} + \dots + q_{2k} - \lambda} dt},\end{aligned}\tag{3.79}$$

for $x \in \Gamma_-$) and the amplitudes $b_{dec}^\pm(x), b_{exp}^\pm(x)$ are holomorphic in Γ^\pm and are $\mathcal{O}(|x|^{-k/2})$, $x \in \Gamma^\pm$, $|x| \rightarrow \infty$.

We point out that the WKB method and (3.78) (respectively, (3.79)) imply

$$\begin{aligned}|\Psi_{exp}^+(x)| &\sim \varkappa_{exp}^+ |x|^{-k/2} e^{\int_0^x \operatorname{Re} \sqrt{q_0 t^{2k} + q_1 t^{2k-1} + \dots + q_{2k} - \lambda} dt} \\ &= \varkappa_{exp}^+ |x|^{-k/2} e^{\operatorname{Re}(\sqrt{q_0} x^{k+1}/(k+1)) + \mathcal{O}(|x|^k)} \\ |\Psi_{dec}^+(x)| &\sim \varkappa_{dec}^+ |x|^{-k/2} e^{-\int_0^x \operatorname{Re} \sqrt{q_0 t^{2k} + q_1 t^{2k-1} + \dots + q_{2k} - \lambda} dt} \\ &= \varkappa_{dec}^+ |x|^{-k/2} e^{-\operatorname{Re}(\sqrt{q_0} x^{k+1}/(k+1)) + \mathcal{O}(|x|^k)}\end{aligned}\tag{3.80}$$

for $x \in \Gamma_+$, $|x| \rightarrow \infty$, where $\varkappa_{exp/dec}^+ \in \mathbb{R}^+$,

(respectively,

$$\begin{aligned}|\Psi_{exp}^-(x)| &\sim \varkappa_{exp}^- |x|^{-k/2} |e^{-(-1)^{k-1} \int_0^x \sqrt{q_0 t^{2k} + q_1 t^{2k-1} + \dots + q_{2k} - \lambda} dt}| \\ &= \varkappa_{exp}^- |x|^{-k/2} e^{(-1)^{k-1} \operatorname{Re}(\sqrt{q_0} x^{k+1}/(k+1)) + \mathcal{O}(|x|^k)} \\ |\Psi_{dec}^-(x)| &\sim \varkappa_{dec}^- |x|^{-k/2} |e^{\int_0^x \sqrt{q_0 t^{2k} + q_1 t^{2k-1} + \dots + q_{2k} - \lambda} dt}| \\ &= \varkappa_{dec}^- |x|^{-k/2} e^{-(-1)^{k-1} \operatorname{Re}(\sqrt{q_0} x^{k+1}/(k+1)) + \mathcal{O}(|x|^k)}\end{aligned}\tag{3.81}$$

for $x \in \Gamma_-$, $|x| \rightarrow \infty$ and $\varkappa_{exp/dec}^- \in \mathbb{R}^+$), taking into account that

$$\int_0^x \sqrt{q_0 t^{2k} + q_1 t^{2k-1} + \dots + q_{2k} - \lambda} dt \simeq \pm (-1)^{k-1} \sqrt{q_0} \frac{x^{k+1}}{k+1} + \mathcal{O}(|x|^k), \quad x \in \Gamma^\pm.$$

Set $C_{\mp}^\pm * \{C_{\mp}^\pm(jk)\}_{j,k=1,2}$ to be the transition matrix from \mp to \pm , namely

$$\Psi^\pm = \begin{pmatrix} \Psi_{exp}^\pm \\ \Psi_{dec}^\pm \end{pmatrix} = C_{\mp}^\pm \Psi^\mp = C_{\mp}^\pm \begin{pmatrix} \Psi_{exp}^\mp \\ \Psi_{dec}^\mp \end{pmatrix},$$

with $C_-^+ = (C_+^-)^{-1}$.

Coming back to the operator P , we obtain that

$$\varphi_{exp}^\pm(x) = e^{-A(x)/2} \psi_{exp}^\pm(x),$$

$$\varphi_{dec}^\pm(x) = e^{-A(x)/2} \psi_{dec}^\pm(x),$$

$$\varphi_{exp}^\pm(x) \sim \chi_{exp}^\pm |x|^{-k/2} e^{-a_0 x^{k+1}(2(k+1)) + Re(\sqrt{q_0}|x|^{k+1}/(k+1)) + O(|x|^k)}, \quad x \in \mathbb{R}, |x| \rightarrow \infty,$$

$$\varphi_{dec}^\pm(x) \sim e \chi_{dec}^\pm |x|^{-k/2} e^{-a_0 x^{k+1}(2(k+1)) - Re(\sqrt{q_0}|x|^{k+1}/(k+1)) + O(|x|^k)}, \quad x \in \mathbb{R}, |x| \rightarrow \infty$$

and $\varphi^\pm = C_\mp^\pm \varphi^\mp$, $\varphi^\pm = (\varphi_{exp}^\pm, \varphi_{dec}^\pm)$. Evidently the estimates above, combined with standard arguments in the spectral theory and the explicit formulas for the eigenvalues of the complex harmonic oscillator complete the proof of both theorems. \square

3.3 Normal forms and spectral properties of multidimensional Shubin operators

We consider the operator in (3.12). Set $A = A_1 + iA_2$, $B = B_1 + iB_2$, $A_j, B_j \in M_n(\mathbb{R})$, B_j , $j = 1, 2$ are symmetric. We assume that

$$B_1 > 0, \quad (3.82)$$

$$A^T = A, \quad \text{namely, } A_1 \text{ and } A_2 \text{ are symmetric,} \quad (3.83)$$

$$B_1 - \frac{A_1^2}{4} > 0 \quad (3.84)$$

$$\Theta_1 := B_1 - \frac{A_1^2}{4} + \frac{A_2^2}{4} \quad \text{and} \quad \Theta_2 := B_1 + \frac{A_1 A_2}{4} + \frac{A_2 A_1}{4} \quad \text{are commuting.} \quad (3.85)$$

We note that (3.83) implies that Θ_1 and Θ_2 are symmetric. By well known classical theorems on the centralizers of finite matrices we obtain that (3.85)

is equivalent to the existence of $S \in SO(n)$ which diagonalizes simultaneously Θ_1 and Θ_2

$$S^T \Theta_j S = \text{diag} \{ \omega_{j;1}, \dots, \omega_{j;n} \}, \quad \omega_{j;\ell} \in \mathbb{R}, \quad j = 1, 2, \ell = 1, \dots, n. \quad (3.86)$$

One checks easily that (3.84) yields $\omega_{1,\ell} > 0$, $\ell = 1, \dots, n$. Under the hypothesis above we have

Theorem 39. *The operator P defined by (3.12) is Γ elliptic and one can find $\alpha \in \mathbb{C}^n$, $r \in \mathbb{C}$ such that the transformation*

$$J = e^{-\frac{i}{4} \langle S^T A S x, x \rangle + \langle \alpha, x \rangle} T_\beta \quad (3.87)$$

reduces P to a multidimensional complex harmonic oscillator

$$J^{-1} \circ P \circ J = -\Delta + \sum_{\ell=1}^n (\omega_{1,\ell} + \omega_{2;\ell} i) x_j^2 + r. \quad (3.88)$$

Moreover, the spectrum is discrete and we can express explicitly the eigenvalues and the eigenfunctions of P via J and the Hermite functions.

The proof follows from the fact that

$$e^{-\frac{i}{4} \langle S^T A S x, x \rangle} \circ P \circ e^{-\frac{i}{4} \langle S^T A S x, x \rangle} = -\Delta + \sum_{\ell=1}^n (\omega_{1,\ell} + \omega_{2;\ell} i) x_j^2 + \text{lower order terms.} \quad (3.89)$$

So we have separation of the variables and we apply the results for the one-dimensional harmonic oscillator n times.

Finally, we outline a strategy for the study of normal forms in the multi-dimensional anisotropic case

$$P = -\Delta + \langle a(x), D_x \rangle + b(x), \quad x \in \mathbb{R}^n, \quad (3.90)$$

where

$$a(x) = (a_1(x), \dots, a_n(x)), \quad (3.91)$$

with

$$a_j = \sum_{|\beta| \leq k} a_{j,\beta} x^\beta, \quad a_{j,\beta} \in \mathbb{C},$$

$$b(x) = \sum_{\beta \leq 2k} b_\beta x^\beta \quad b_\beta \in \mathbb{C}.$$

We have to require in the multidimensional case, in order to be able to cancel the mixed term by transformation of the type $e^{-\frac{i}{2}U(x)}$ to (3.90), the existence of global polynomial potential $U(x)$ of the linear form $\langle a(x), dx \rangle$, namely $\nabla U(x) = a(x)$, or equivalently the form

$$a_1(x)dx_1 + \dots + a_n(x)dx_n \text{ is closed in } \mathbb{C}^n. \quad (3.92)$$

Under the assumption above we get

$$\tilde{P} = e^{\frac{i}{2}U(x)} \circ P \circ e^{-\frac{i}{2}U(x)} = -\Delta + \tilde{B}(x), \quad (3.93)$$

where

$$\begin{aligned} \tilde{B}(x) &= b(x) - \frac{1}{4} \langle a(x), a(x) \rangle + \frac{i}{2} \Delta U(x) \\ &= \sum_{\alpha \leq 2k} \tilde{b}_\alpha x^\alpha. \end{aligned} \quad (3.94)$$

Now we need a symmetry type condition yielding to separation of the variables: there exists $S \in SO(n)$ and n polynomials of degree $2k$

$$\tilde{b}_\ell(t) = \sum_{j=0}^{2k} \tilde{b}_{\ell,j} t^{2k-j}, \quad \tilde{b}_{\ell,j} \in \mathbb{C}, j = 0, 1, \dots, 2k, \Re \tilde{b}_{\ell,0} > 0, \ell = 1, \dots, n, \quad (3.95)$$

such that

$$\tilde{B}(Sy) = \sum_{\ell=1}^n \tilde{b}_\ell(y_\ell). \quad (3.96)$$

So (3.93) becomes

$$\tilde{P} = S^T \circ \tilde{P} \circ S = -\Delta_y + \sum_{\ell=1}^n \tilde{b}_\ell(y_\ell). \quad (3.97)$$

Finally, we assume that for each ℓ the zero imaginary part condition holds and thus translations $y_\ell \rightarrow y_\ell + \tilde{b}_{\ell;1}/(2k\tilde{b}_{\ell;0})$ and dilations $y_\ell \rightarrow \tilde{b}_{\ell;0}^{-1/(2(k+1))}y_\ell$ reduce to the normal form

$$P_{NF} = \sum_{\ell=1}^n \omega_\ell L_{2k,\rho^\ell}(y_\ell, D_{y_\ell}) + r, \quad \rho^\ell \in \mathbb{R}^{2k-2}, \ell = 1, \dots, n, r \in \mathbb{C}. \quad (3.98)$$

where $\omega_\ell = \tilde{b}_{\ell;0}^{-1/(2(k+1))}$, and $L_{2k,\rho}$ stands for the canonical one dimensional anisotropic NF of order $2k$, $\rho = (\rho_0, \dots, \rho_{2k-3}) \in \mathbb{R}^{2k-2}$. We recall (see in Chapter 2) the $\lambda_1(\rho) < \dots < \lambda_j(\rho) < \dots < +\infty$ are the eigenvalues of L_{2k,ρ^ℓ} with a basis of eigenfunctions $\{\varphi_j^\rho(x)\}_{j=1}^\infty \in L^2(\mathbb{R})$.

Set T to be the composition of the maps $e^{-\frac{i}{2}U(x)}$, S , the translations and the dilations defined for the last reduction, Under the hypotheses (3.95), (3.98) we have

Theorem 40. *P admits compact resolvent and discrete spectrum which coincides with the spectrum of*

$$P_{NF} = T^{-1} \circ P \circ T,$$

and

$$\text{spec}(P_{NF}) = \{\lambda_\alpha = \sum_{\ell=1}^n \lambda_{\ell,\alpha_\ell} + r, \alpha \in \mathbb{N}^n\},$$

with $\lambda_{\ell,\alpha_\ell} = \omega_\ell \lambda_{\alpha_\ell}(\rho^\ell)$, $\ell = 1, \dots, n$, $\alpha \in \mathbb{N}^n$, and a complete system of eigenfunctions $\{\Psi_\alpha(x) = c_\alpha T \Phi_\alpha^{\bar{p}}(x)\}_{\alpha \in \mathbb{N}}$ (which is not a Riesz basis), where

$$\Phi_\alpha^{\bar{p}}(y) = \varphi_{\alpha_1}(\rho^1)(y_1) \otimes \dots \otimes \varphi_{\alpha_n}(\rho^1)(y_n),$$

$c_\alpha^{-1} = \|\|T \Phi_\alpha\|\|_{L^2}$, $\alpha \in \mathbb{N}^n$. The eigenfunctions belong to the limiting Gelfand–Shilov space $S_{1/(k+1)}^{k/(k+1)}(\mathbb{R}^n)$

The proof is straightforward since the compact resolvent property and the global ellipticity follow from (3.95). Furthermore, (3.95) and (3.98) imply

that $\Psi_\alpha \in L^2(\mathbb{R}^n)$, while the global hypoellipticity yields $\Psi_\alpha \in S_{1/(k+1)}^{k/(k+1)}(\mathbb{R}^n)$, $\alpha\mathbb{N}^n$. The completeness of the system of eigenfunctions is obtained by the general results on the spectrum of unbounded operators

$$P = P_1 \oplus \dots \oplus P_n$$

in the separable Hilbert space $H = H_1 \otimes \dots \otimes H_n$, with P_j being unbounded operator on a separable Hilbert space H_j having discrete spectrum, $j = 1, \dots, n$.

Remark 13. *We are not able to construct in the multidimensional case explicit examples of globally elliptic Shubin operators whose spectrum coincides with \mathbb{C} since there are no multidimensional analogues to the complex WBK methods for linear ODE.*

3.4 Perturbation with $S_{1/2}^{1/2}(\mathbb{R}^n)$ smoothing operator

The main goal of this section is to show that despite of the presence of quadratic exponential growth in the NFT for perturbations of complex harmonic oscillators we can conjugate classes of Shubin p.d.o. with symbols $p(x, \xi) \in S_{1/2}^{1/2}(\mathbb{R}^n)$ provided the quadratic growth of the NFT is smaller than the quadratic exponential decay of the Schwartz kernel canceled by the quadratic decay of p.d.o. (see [63] where such symbols appear in different context).

We consider a p.d.o. $P(x, D)$ whose symbol is defined by

$$p(x, \xi) = Q(x, \xi)e^{-ax^2 - b\xi^2}, \quad (x, \xi) \in \mathbb{R}^{2n}, \quad a > 0, b > 0. \quad (3.99)$$

where

$$Q(x, \xi) = \sum_{|\alpha|+|\beta|\leq m} q_{\alpha\beta} x^\alpha \xi^\beta$$

is polynomial. Let A be a non zero $n \times n$ symmetric matrix and set

$$\varepsilon_\pm := \max_{x \in \mathbb{R}^n, \|x\|=1} (\pm \langle Ax, x \rangle) > 0 \quad (3.100)$$

Then we have

Proposition 30. *The conjugation*

$$\tilde{P}(x, D) = e^{\langle Ax, x \rangle} \circ P(x, D) \circ e^{-\langle Ax, x \rangle} \quad (3.101)$$

is well defined in the Schwartz class and $\tilde{P}(x, D)$ is smoothing operator with Schwartz kernel $K_{\tilde{P}}(x, y) \in S_{1/2}^1(\mathbb{R}^{2n})$ given by

$$K_{\tilde{P}}(x, y) = \tilde{Q}(x, x-y) e^{-\langle Ax, x \rangle - ax^2 - \frac{1}{4b}(x-y)^2 + \langle Ay, y \rangle} \quad (3.102)$$

where $\tilde{Q}(x, z)$ is polynomial of degree m , provided

$$\begin{aligned} 1 - (4ba - 4b\varepsilon_- + 1)(1 - 4b\varepsilon_+) = \\ -4ab + 4b\varepsilon_- + 16b^2a\varepsilon_+ - 16b^2\varepsilon_+\varepsilon_- + O(\max\{|\varepsilon_+| + |\varepsilon_-|\}) < 0, \end{aligned} \quad (3.103)$$

for $|\varepsilon_-|, |\varepsilon_+|$ small enough. In particular, if $n = 1$, $A = -\varepsilon$, we have $\varepsilon_\pm = \pm\varepsilon$ and we can find sharp estimates on the range of ε :

$$\varepsilon \in]\frac{2ab - \sqrt{5}ab}{4b^2}, \frac{2ab + \sqrt{5}ab}{4b^2}[=] -(\sqrt{5} - 2)\frac{a}{4b}, (2 + \sqrt{5})\frac{a}{4b}[. \quad (3.104)$$

Proof. Recall formula for the inverse Fourier transform of $e^{-b\xi^2}$

$$\int_{\mathbb{R}^n} e^{iz\xi} e^{-b\xi^2} d\xi = \pi^{n/2} b^{-n/2} e^{-\frac{1}{4b}z^2}. \quad (3.105)$$

Next, we observe that we can calculate explicitly the Schwartz kernel $K(x, y) = K_P(x, x-y) \in S_{1/2}^1(\mathbb{R}^{2n})$. Indeed,

$$\begin{aligned} K_P(x, z) &= \int_{\mathbb{R}^n} e^{ixz} p(x, \xi) \bar{d}\xi = e^{-ax^2} Q(x, D_z) \int_{\mathbb{R}^n} e^{ixz} e^{-b\xi^2} \bar{d}\xi \\ &= \pi^{n/2} b^{-n/2} e^{-ax^2} Q(x, D_z) (e^{-\frac{1}{4b}z^2}) = \tilde{Q}(x, z) e^{-ax^2 - \frac{1}{4b}z^2}, \end{aligned} \quad (3.106)$$

where

$$\tilde{Q}(x, z) = \pi^{n/2} b^{-n/2} \sum_{|\alpha|+|\beta|\leq m} q_{\alpha\beta} x^\alpha (2b)^{n/2} h_\beta\left(\frac{1}{\sqrt{2b}}z\right) \quad (3.107)$$

with $h_\beta(t) = h_{\beta_1}(t_1) \dots h_{\beta_n}(t_n)$, $h_k(t) = e^{t^2/2}(e^{-t^2/2})^{(k)}$ standing for the Hermite polynomial of degree k . Next, since $e^{\langle Ax, x \rangle} \circ P(x, D) \circ e^{-\langle Ax, x \rangle} u$

$$\begin{aligned} &= \int_{\mathbb{R}^n} e^{\langle Ax, x \rangle} K_P(x, x-y) \circ e^{-\langle Ay, y \rangle} u(y) dy \\ &= \pi^{n/2} b^{-n/2} \int_{\mathbb{R}^n} \tilde{Q}(x, x-y) e^{\langle Ax, x \rangle - ax^2 - \frac{1}{4b}(x-y)^2 - \langle Ay, y \rangle} u(y) dy. \end{aligned} \quad (3.108)$$

We estimate $\Phi(x, y) := \langle Ax, x \rangle - ax^2 - \frac{1}{4b}(x-y)^2 - \langle Ay, y \rangle$ using (3.100).

$$\begin{aligned} \Phi(x, y) &\leq (\varepsilon_- - a - \frac{1}{4b})x^2 + \frac{1}{2b}\|x\|\|y\| + (\varepsilon_+ - \frac{1}{4b})y^2 \\ &= \frac{1}{4b}((4b\varepsilon_- - 4ba - 1)x^2 + 2\|x\|\|y\| + (4b\varepsilon_+ - 1)y^2) \end{aligned} \quad (3.109)$$

Clearly the RHS of (3.109) is negative quadratic form iff $\varepsilon_- < a + \frac{1}{4b}$, $\varepsilon_+ < \frac{1}{4b}$ and

$$1 - (4ab + 1 - 4b\varepsilon_-)(1 - 4b\varepsilon_+) = -4ab + \mathcal{O}(|\varepsilon_+| + |\varepsilon_-|) < 0, \quad (3.110)$$

which yields (3.103). In the case $\varepsilon_\pm = \pm\varepsilon$ we have optimal bound for $|\varepsilon|$ since we are reduced to

$$1 - (4ab + 1 - 4b\varepsilon)(1 + 4b\varepsilon) = -4ab - 16ab^2\varepsilon + 4b^2\varepsilon^2 < 0, \quad (3.111)$$

which is true if and only if

$$\varepsilon \in]\frac{2ab - \sqrt{5ab}}{4b^2}, \frac{2ab + \sqrt{5ab}}{4b^2}[=] - (\sqrt{5} - 2)\frac{a}{4b}, (2 + \sqrt{5})\frac{a}{4b}[. \quad (3.112)$$

□

Remark 14. If $a = b$ and $\langle Ax, x \rangle = \varepsilon\|x\|^2$ we get that the necessary and sufficient condition in order the conjugation (3.101) to be well defined is given by

$$\varepsilon \in] - \frac{\sqrt{5} - 2}{4}, \frac{\sqrt{5} + 2}{4}[\quad \text{and} \quad |\varepsilon| < a. \quad (3.113)$$

Chapter 4

Normal forms and global hypoellipticity for degenerate Shubin operator

The main goal of the present chapter is to study reduction to normal forms, the global hypoellipticity and the global solvability of classes of degenerate (second order differential operators of Shubin type $P = -\Delta + \langle Bx, D_x \rangle + \langle Cx, x \rangle +$ lower order terms, with non-negative principal symbol. Such operators do not fall in the classes of second order operators satisfying subelliptic estimates in Shubin spaces cf. [34], [49]. Few results on the global regularity and solvability in $\mathcal{S}(\mathbb{R}^n)$ for general classes of Shubin operators with non hypoelliptic symbols are available in the literature.

First we study the operator P when B is symmetric. The main body of new results covers the case when the zero set of the principal symbol $p_2(x, \xi)$ is n -dimensional, which turns out to be equivalent to the property that the zero set is a Lagrangian linear subspace of \mathbb{R}^{2n} with respect to the

canonical symplectic form $\omega = \sum_{j=1}^n d\xi_j \wedge dx_j$. The first main result could be summarized as follows: broadly speaking, such operators are reduced to a generalized multidimensional Airy operator

$$A = -\Delta + (\rho + i\sigma)x_1, \quad \rho, \sigma \in \mathbb{R},$$

or to Fourier $\bar{\partial}$ type normal form

$$\hat{D} = -\Delta + (\rho + i\sigma)x_1 + \tau x_2, \quad \tau \in \mathbb{R}, \rho\tau \neq 0.$$

We call the operator A a generalized Airy one, because for $n = 1$ we recapture a modified Airy equation $Au(x_1) = -u''(x_1) + (\rho + \sigma i)x_1u(x_1)$, while the use of the term Fourier $\bar{\partial}$ normal form is motivated from the fact that $\hat{D}u$ is Fourier transform (modulo some constants) of

$$\rho \partial_{\bar{z}_1} \hat{u}(\xi) + (|z_1|^2 + \|\xi'\|^2) \hat{u}(\xi)$$

for $\rho = -\tau \neq 0$, $\sigma = 0$, $z_1 = \xi_1 + i\xi_2$, $\xi' = (\xi_3, \dots, \xi_n)$, if $n \geq 3$. We recall that the Airy functions were one of the fundamental tools for the mathematical study of diffraction problems and initial boundary value problems for strictly hyperbolic equations, when the smooth boundary of the domain is convex with respect to the characteristics of the hyperbolic operator cf. [62], [26] and the references therein.

We propose complete classification of the hypoellipticity and solvability in $\mathcal{S}(\mathbb{R}^n)$ for the generalized Airy NF. Moreover, outline a natural functional scale of seemingly new function spaces which are essentially a direct sum of anisotropic Shubin spaces $\mathcal{Q}_\Lambda^s(\mathbb{R}_{x_1})$, $\Lambda(x_1, \xi_1) = (1 + x_1^2 + \xi_1^4)^{1/4}$, with norm $|\Lambda(x_1, D_{x_1})u|_{L^2}$, and SG or G weighted Sobolev spaces $H^{r_1, r_2}(\mathbb{R}_{x'}^{n-1})$ in $x' = (x_2, \dots, x_n)$. We do not dwell upon generalizations in Gelfand-Shilov spaces since highly non has to use complicated functional–analytic arguments

on anisotropic Gelfand–Shilov spaces. Concerning the Fourier $\bar{\partial}$ NF, the problems become quite involved and challenging since we enter in the realm of the complex analysis and global properties of perturbations of $\partial_{\bar{z}_1}$. One is led to conjecture, in view of the fact that $\bar{\partial}$ is not globally hypoelliptic in $\mathcal{S}(\mathbb{R}^n)$, the same is true for the Fourier $\bar{\partial}$ NF. We can describe the kernel of \hat{D} in $C^\infty(\mathbb{R}^n)$ and a possible result on non hypoellipticity depends on whether or not one can find an entire function on the complex plane with suitable cubic decay properties in a half plane.

We point out to another problem here: since the spectrum of the NF is not discrete it is not possible to use the discrete approach.

The second case is when $B_{skew} \neq 0$. We mention as a motivating example the twisted Laplacian L on \mathbb{R}^2 :

$$L = -\Delta + x_1 D_{x_2} - x_2 D_{x_1} + \frac{1}{4}(x_1^2 + x_2^2). \quad (4.1)$$

The twisted Laplacian appears in harmonic analysis naturally in the context of Wigner transforms and Weyl transforms [67], [13], [25] and also in physics. The transpose L' of the twisted Laplacian L is given by

$$L = -\Delta + \frac{1}{4}(x_1^2 + x_2^2) + x_2 D_{x_1} - x_1 D_{x_2}. \quad (4.2)$$

In the paper [16], it is shown that L is globally hypoelliptic in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$, while global hypoellipticity and global solvability in Gelfand–Shilov spaces $S_\mu^\mu(\mathbb{R}^2)$ has been shown in [23], [25], [22] for more general operators of the type

$$L_\tau(x, D_x) = -\Delta + \tau(x_2 D_{x_1} - x_1 D_{x_2}) + \frac{\tau^2}{4}(x_1^2 + x_2^2) = (D_{x_1} + \frac{\tau}{2}x_2)^2 + (D_{x_2} - \frac{\tau}{2}x_1)^2, \quad (4.3)$$

where $\tau \in \mathbb{R}$, $\tau \neq 0$. In our notation (4.3) corresponds to a homogeneous operator in \mathbb{R}^2 with $B = B_{skew} = \begin{pmatrix} 0 & -\tau \\ \tau & 0 \end{pmatrix}$ and $C = \begin{pmatrix} \frac{\tau^2}{4} & 0 \\ 0 & \frac{\tau^2}{4} \end{pmatrix}$.

First, we focus our attention the the case when the characteristic set is a Lagrangian set, with respect to the standard canonical form in $T^*\mathbb{R}^n$, namely, when the matrix B in the mixed term $\langle Bx, D_x \rangle$, is symmetric, namely $B_{skew} = 1/2(B - B^T) = 0$. This case corresponds to (global) involutive characteristics.

We note that the characteristic set of L and L^t is two dimensional and has double characteristics which are symplectic.

Our main result in the skew-symmetric case could be summarized as follows: if the space dimension is even $2n$ and the $rank(B_{skew})$ is maximal we reduce our operator to multidimensional twisted Laplacian perturbed by first order terms, and classify completely the spectral properties and derive sharp necessary and sufficient conditions for the global hypoellipticity and global solvability of the operator. Stability under perturbations by zero order p.d.o. is studied as well.

4.1 Generalized Airy and Fourier $\bar{\partial}$ type normal form

We consider the operator $P = -\Delta + \langle Bx, D_x \rangle + \langle Cx, x \rangle + l.o.t.$, when $B_{skew} = 0$. We require that the characteristic set is not empty

$$\Sigma_P = \{(x, \xi) \in \mathbb{R}^{2n} \setminus 0; p_2(x, \xi) := \|\xi\|^2 + \langle Bx, \xi \rangle + \langle Cx, x \rangle = 0\} \neq \emptyset, \quad (4.4)$$

and

$$p_2(x, \xi) \geq 0, (x, \xi) \in \mathbb{R}^{2n} \setminus 0 \quad \text{and} \quad B = B_{symm} \quad \text{i.e.,} \quad B = B^t. \quad (4.5)$$

Theorem 41. *Suppose that (4.5) holds. Then the following properties are equivalent*

- i) $\dim \Sigma_P = n$;

$$ii) \ C = \frac{1}{4}B^2;$$

iii) Σ_P is Lagrangian with respect to the canonical symplectic form

$$\omega = \sum_{j=1}^n d\xi_j \wedge dx_j;$$

iv) There exists a unitary operator $T : L^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$, which is a linear automorphism in $\mathcal{S}(\mathbb{R}^n), \mathcal{Q}^s(\mathbb{R}^n), S_\mu^\mu(\mathbb{R}^n)$, $s \in \mathbb{R}$, $\mu > 0$, defined, for some $S \in SO(n)$, $\alpha, \beta \in \mathbb{R}^n$, by

$$Tv(x) = e^{i\frac{1}{2}\langle S^t B S x + 2\alpha, x \rangle} v(S^t x + \beta) \quad (4.6)$$

such that

$$T^{-1} \circ P \circ T = \tilde{P} = -\Delta + i\langle M, D_x \rangle + \rho x_1 + i\sigma x_1 + i\tau x_2 + r(\rho, \sigma, \tau) \quad (4.7)$$

for some $M \in \mathbb{R}^n$, $\rho, \sigma, \tau \in \mathbb{R}$, and $r(\rho, \sigma, \tau) \in \mathbb{C}$. In particular, we have $\tau = 0$ if $\rho = 0$ or $n = 1$ and $r(\rho, \sigma, \tau) = 0$ if $\rho\tau \neq 0$, $\Re r(\rho, \sigma, \tau) = 0$ if $\tau = 0$, $\rho \neq 0$, $\Im r(\rho, \sigma, \tau) = 0$ if $\rho = 0$. Moreover,

$$\tau = 0 \quad \text{iff } 2S^t \Re q + A \Re p \text{ and } 2S^t \Im q + A \Im p \text{ are linearly dependent.} \quad (4.8)$$

Proof. First we rewrite the principal symbol, taking into account that B is symmetric:

$$\begin{aligned} p_2(x, \xi) &= \langle \xi + \frac{1}{2}Bx, \xi + \frac{1}{2}Bx \rangle + \langle Cx, x \rangle - \frac{1}{4}\langle Bx, Bx \rangle \\ &= \|\xi + \frac{1}{2}Bx\|^2 + \langle C'x, x \rangle, \quad C' = C - \frac{1}{4}B^2. \end{aligned} \quad (4.9)$$

Since C' is symmetric, we can diagonalized C' , $x = S_0 y$, with $S_0 \in SO(n)$, such that $S_0^t C' S_0 = \text{diag}\{c_1, \dots, c_n\}$, with signature (n_+, n_-) , $0 \leq n_\pm \leq n$, $n_+ + n_- \leq n$, $c_j = \mu_j > 0$, $j = 1, \dots, n_+$, $c_j = -\nu_j < 0$, $j = n_+ + 1, \dots, n_-$, $c_j = 0$, $j = n_+ + n_- + 1, \dots, n$. Set $\eta = \xi - \frac{1}{2}Bx$. Hence we get

$$p_2(x, \xi) = p_2(S_0 y, \eta - B S_0 y) = \sum_{j=1}^n \eta_j^2 + \sum_{k=1}^{n_+} \mu_k y_k^2 - \sum_{k=n_++1}^{n_++n_-} \nu_j y_j^2, \quad (4.10)$$

with the convention $\Sigma_s^{s-1} \cdot = 0$.

We need a well known fact from the theory of the quadratic forms (see [41] for more details).

Lemma 10. *Let p_2 be the quadratic form defined by (4.9). Then*

$$\dim \Sigma_P = \begin{cases} 2n-1 & \text{if } n_- > 0 \\ n-n_+ & \text{if } n_- = 0 \end{cases} \quad (4.11)$$

In particular,

$$\dim \Sigma_P = n \quad \text{iff } n_+ = n_- = 0, \text{ i.e., } C' = C - \frac{1}{4}B^2 = 0. \quad (4.12)$$

Proof. Let $n_- > 0$. The zero set of the quadratic form (4.9) is a union of graphs of two functions of $2n-1$ variables,

$$y_{n_++1} = \pm \frac{1}{v_{n_++1}} \sqrt{\sum_{j=1}^n \eta_j^2 + \sum_{k=1}^{n_+} \mu_k y_k^2 - \sum_{k=n_++2}^{n_++n_-} v_j y_j^2}.$$

In the case $n_- = 0$ the zero set is the $n-n_+$ dimensional linear subspace defined by $\eta_1 = \dots \eta_n = y_1 = \dots y_{n_+} = 0$. □

Since the dimension of a Lagrangian manifold in \mathbb{R}^{2n} is n , and B is symmetric, the lemma above yields the equivalence relations i) \Leftrightarrow ii) \Leftrightarrow iii), where the Lagrangian manifold is defined by $\xi = Bx$, B symmetric (e.g., see Hörmander, [37]).

Suppose now that $C' = 0$. Then we have that

$$\tilde{P} = e^{-i\frac{1}{2}\langle Bx, x \rangle} \circ P \circ e^{i\frac{1}{2}\langle Bx, x \rangle}$$

becomes

$$\tilde{P}u(x) = \left(-\Delta + \langle \alpha, D_x \rangle + i\langle \beta, D_x \rangle + \langle M, x \rangle + i\langle N, x \rangle + r \right) u(x), \quad (4.13)$$

where $G = \alpha + i\beta$ and $F = M + iN$ with $\alpha, \beta, M, N \in \mathbb{R}^n$.

Now we consider the transformation $u(x) = e^{\varphi(x)}v(x)$, where $\varphi(x) = -i(\langle \alpha, x \rangle)/2$ so we have:

$$\begin{aligned} \tilde{P}(e^{\varphi(x)}v(x)) &= e^{\varphi(x)} \left((\nabla\varphi(x))^2 - 2\langle \nabla\varphi(x), D_x \rangle + i\delta\varphi(x) - \Delta + \langle \alpha, D_x \rangle \right. \\ &\quad + \langle \alpha, \nabla\varphi(x) \rangle + i\langle \beta, D_x \rangle + i\langle \beta, \nabla\varphi(x) \rangle + \langle M, x \rangle \\ &\quad \left. + i\langle N, x \rangle + r \right) v(x). \end{aligned}$$

So our operator becomes

$$e^{-\varphi(x)} \circ \tilde{P} \circ e^{\varphi(x)} = Lv(x) = \left(-\Delta + i\langle \beta, D_x \rangle + \langle M, x \rangle + i\langle N, x \rangle + \ell \right) v(x). \quad (4.14)$$

We consider a matrix $S_1 \in SO(n)$, we apply the orthogonal transformation $x \rightarrow S_1x$ such that

$$M = (m_1, \dots, m_n) \rightarrow \tilde{M} = (\rho, 0, \dots, 0), \quad \text{with } \rho \in \mathbb{R},$$

which transforms L to:

$$\tilde{L}\tilde{v}(x) = \left(-\Delta + i\langle \tilde{\beta}, D_x \rangle + \rho x_1 + i\langle \tilde{N}, x \rangle + \ell \right) \tilde{v}(x), \quad (4.15)$$

where $\tilde{v}(x) = v(S_1x)$, $\tilde{N} = S_1^{-1}N$, $\tilde{\beta} = S_1^{-1}\beta$. We consider a matrix $S_2 \in SO(n)$, preserving $(1, 0, \dots, 0)$, i.e., $S_2 = \begin{pmatrix} 0 & 0_{1 \times n-1} \\ 0_{n-1 \times 1} & \tilde{S}_2 \end{pmatrix}$, with $\tilde{S}_2 \in SO(n-1)$, we apply the orthogonal transformation in \mathbb{R}^{n-1} $x' \rightarrow S_2x'$ such that

$$\tilde{N} = (\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_n) \rightarrow \bar{N} = (\sigma, \tau, 0, \dots, 0), \quad \text{where } \sigma = \tilde{n}_1, \text{ with } \sigma, \tau \in \mathbb{R},$$

so our operator L becomes:

$$L'v'(x) = \left(-\Delta + i\langle \beta', D_x \rangle + \rho x_1 + i\sigma x_1 + i\tau x_2 + \ell \right) v'(x), \quad (4.16)$$

where $v'(x) = \tilde{v}(S_2x)$, $\beta' = S_2^{-1}\tilde{\beta}$. Setting $S = S_1S_2$, we have shown that each one of i), ii), iii) implies iv). Reversing the arguments of the conjugation to the normal form, we prove that iv) leads to i), ii), iii). \square

Remark 15. We call the NF generalized Airy NF (respectively, generalized Fourier $\bar{\partial}$) if (4.7) holds, namely $\tau = 0$ (respectively, $\rho\tau \neq 0$).

4.2 Weighted spaces

Let $s_1, r_1, r_2 \in \mathbb{Z}_+$ with s_1 even. We define $HSC^{s_1; r_1, r_2}(\mathbb{R} \times \mathbb{R}^{n-1})$ as the spaces such that

$$HSC^{s_1; r_1, r_2}(\mathbb{R} \times \mathbb{R}^{n-1}) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{s_1; r_1, r_2} < \infty \right\}, \quad (4.17)$$

where $x = (x_1, x')$, $x_1 \in \mathbb{R}$, $x' = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$,

$$\begin{aligned} \|f\|_{s_1; r_1, r_2}^2 := & \sum_{2\beta_1 + \alpha_1 \leq s_1} \|x_1^{\beta_1} D_{x_1}^{\alpha_1} f\|_{L^2(\mathbb{R}^n)}^2 \\ & + \sum_{|\beta'| \leq r_2, |\alpha'| \leq r_1} \|x'^{\alpha'} D_{x'}^{\beta'} f\|_{L^2(\mathbb{R}^n)}^2. \end{aligned} \quad (4.18)$$

We can extend this definition for $s_1, r_1, r_2 \in [0, +\infty)$ by interpolation and by duality arguments for arbitrary s_1, s_2, r_1 .

Proposition 31. Let $s_1, r_1, r_2 \in \mathbb{Z}_+$, with s_1 even. Then the following norms are equivalent to the norm (4.17)

i)

$$\sum_{\tilde{s}_1/s_1 + \tilde{r}_1/r_1 + \tilde{r}_2/r_2 \leq 1} \sum_{2\beta_1 + \alpha_1 \leq \tilde{s}_1; |\beta'| \leq \tilde{r}_2; |\alpha'| \leq \tilde{r}_1} \|x_1^{\beta_1} D_{x_1}^{\alpha_1} x'^{\alpha'} D_{x'}^{\beta'} f\|_{L^2(\mathbb{R}^n)} \quad (4.19)$$

ii)

$$\|\langle x_1, D_{x_1}^2 \rangle^{s_1/2} f\|_{L^2(\mathbb{R}^n)} + \|\langle x', D_{x'} \rangle^{\tilde{r}_1} \langle D_{x'} \rangle^{\tilde{r}_2} f\|_{L^2(\mathbb{R}^n)}, \quad (4.20)$$

where $\tilde{s}_1 \leq s_1, \tilde{r}_1 \leq r_1, \tilde{r}_2 \leq r_2$.

Lemma 11. Let $s_1, r_1, r_2 \in \mathbb{Z}_+$, and let $u \in HSC^{s_1; (r_1, r_2)}$. Then

$$\langle x_1, D_{x_1}^2 \rangle^{\tilde{s}_1/2} \langle x' \rangle^{\tilde{r}_1} \langle D_{x'} \rangle^{\tilde{r}_2} u \in L^2(\mathbb{R}^n). \quad (4.21)$$

We recall that $\langle x_1, \xi_1^2 \rangle = \sqrt{1 + x_1^2 + \xi_1^4}$.

The proofs are based on standard arguments for L^2 estimates of p.d.o.

4.3 Multidimensional Airy operator

We derive the main result on hypoellipticity and solvability in $\mathcal{S}(\mathbb{R}^n)$ for degenerate second order Shubin operators P , with symmetric matrix B in the mixed term, and Lagrangian characteristic set. It is reduced to generalized Airy type normal form. We consider the operator P in the Airy form. Without loss of generality we suppose that $M = 0$ and so we want to study the equation

$$Pu = (-\Delta + \alpha x_1)u = f, \quad (4.22)$$

where $f \in \mathcal{S}(\mathbb{R}^n)$.

Theorem 42. *Suppose that $\alpha := \rho + \sigma i \neq 0$. Then we have*

- i) *The symbol $p(x, \xi) = \xi^2 + \alpha x_1$ is not hypoelliptic symbol in the sense of Shubin iff $n \geq 2$ or $n = 1$ and $\sigma = 0$.*
- ii) *The operator P is globally hypoelliptic in $\mathcal{S}(\mathbb{R}^n)$ provided $\sigma \neq 0$*

We propose more precise estimates in the weighted spaces $HSC^{s;r_1,r_2}(\mathbb{R}^n)$.

Theorem 43. *We consider the operator P , then*

- i) *Let $\Im(\alpha) \neq 0$, then $\text{spec}(P) = \text{spec}(P^*) = \emptyset$ and*

$$(P - \lambda)^{-1} : HSC^{s_1;r_1,r_2}(\mathbb{R}^n) \rightarrow HSC^{s_1+2;r_1,r_2+2}(\mathbb{R}^n) \quad \forall \lambda \in \mathbb{C}. \quad (4.23)$$

- ii) *Suppose now that $\Im(\alpha) = 0$. Then $\text{spec}(P) = \text{spec}(P^*) = \mathbb{R}$ and*

$$\text{Ker}(P - \lambda) \cap \mathcal{S}'(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \mid u = J\mathcal{A}(\xi'), \mathcal{A} \in \mathcal{S}'(\mathbb{R}^{n-1}) \right\}, \quad (4.24)$$

where

$$J = \int e^{ix\xi} e^{\frac{1}{i\alpha}(\xi_1^2 + \xi_1 \xi'^2) - \lambda \xi_1} \bar{d}\xi = \int e^{ix'\xi'} Ai\left(\frac{x_1}{\sqrt[3]{\alpha}} + \frac{\xi'^2}{\sqrt[3]{\alpha^4}} - \frac{\lambda}{\sqrt[3]{\alpha^4}}, \xi'\right) \bar{d}\xi, \tag{4.25}$$

where

$$Ai(z) \simeq \frac{e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}z^{1/4}}$$

Proof. We apply the Fourier transform and the problem (4.22) becomes:

$$\hat{P}\hat{u} = (\xi^2 + i\alpha\partial_{\xi_1})\hat{u} = \hat{f}. \tag{4.26}$$

The problem whether (4.26) is hypoelliptic and solvable in $\mathcal{S}(\mathbb{R}^n)$ is, as far as we know, open if $n \geq 2$. Even the one dimensional case of Airy equation seems not be covered explicitly in the works on anisotropic pseudodifferential equations.

We observe that the symbol of the operator (4.26) is not hypoelliptic in the sense of Shubin, in fact it does not satisfy the hypoellipticity condition

$$|\partial_{(x,\xi)}^\gamma p(x, \xi)| < C |p(x, \xi)| \langle (x, \xi) \rangle^{-|\gamma|}, \quad \text{for } |x| + |\xi| \geq R. \tag{4.27}$$

Moreover it doesn't fall in the class of the SG- operators.

Dividing the equation (4.26) by $i\alpha$, we get

$$L\hat{u} = (\partial_{\xi_1} + a(\xi_1^2 + \xi'^2))\hat{u} = g = \frac{\hat{f}}{i\alpha}, \tag{4.28}$$

where $a = \frac{1}{i\alpha}$. We note that we have

$$\Im(\alpha) \neq 0 \iff \Re(a) \neq 0.$$

Let $\Re(a) \neq 0$, and without loss of generality we can consider $\Re(a) > 0$. Formally we can write the solution of the problem (4.28)

$$v = L^{-1}g = e^{-a(\frac{\xi_1^3}{3} + \xi_1 \xi'^2)} \int_{-\infty}^{\xi_1} e^{a(\frac{t^3}{3} + t \xi'^2)} g(t, \xi') dt. \tag{4.29}$$

We put $s_1 = r_1 = r_2 = 0$ and we want to prove that

$$\sum_{2\beta_1 + \alpha_1 \leq 2} \|\xi_1^{\alpha_1} D_{\xi_1}^{\beta_1} v\| + \sum_{|\beta'| \leq 0, |\alpha'| \leq 2} \|x'^{\alpha'} v\| < C \|g\|_{L^2(\mathbb{R}^n)}, \quad (4.30)$$

where v is the solution of $(L - \lambda)v = g$, so we have

$$\begin{aligned} |\xi'^2 v| &\leq \int_{-\infty}^{\xi_1} |e^{-a[\frac{\xi_1^3 - t^3}{3} + (\xi_1 - t)(i\frac{\lambda}{\alpha} + \xi'^2)]} \xi'^2 g(t, \xi')| dt \\ &\leq \int_{-\infty}^{\xi_1} |e^{-a[c(\xi_1 - t)^3 + (\xi_1 - t)(i\frac{\lambda}{\alpha} + \xi'^2)]} \xi'^2 g(t, \xi')| dt. \end{aligned}$$

So we can write the last integral like

$$\|\xi'^2 v\|_{L^2(\mathbb{R}^n)} \leq \int_{\mathbb{R}} |K_1(\xi_1, \xi', t) g(t, \xi')| dt,$$

where

$$K_1(\xi_1, \xi', t) = \int_{\mathbb{R}} H(\xi_1 - t) e^{-a[\frac{\xi_1^3 - t^3}{3} + (\xi_1 - t)(i\frac{\lambda}{\alpha} + \xi'^2)]} \xi'^2 dt.$$

We bound the last integral, putting $s = (\xi_1 - t)\xi'^2$, by

$$c \int_0^{\infty} e^{-\Re a(\frac{s^3}{|\xi'|^6} + s)} ds, \quad c > 0$$

Now we can apply the Schur lemma and we have

$$\|\xi'^2 v\|_{L^2(\mathbb{R}^n)} \leq C \|g\|_{L^2(\mathbb{R}^n)}.$$

In the same way we have

$$\begin{aligned} |\xi_1^2 v| &\leq |\xi_1^2 e^{-a[\frac{\xi_1^3}{3} + \xi_1(\xi'^2 + i\frac{\lambda}{\alpha})]} \int_{-\infty}^{\xi_1} e^{a[\frac{t^3}{3} - t\xi'^2 - i\frac{\lambda}{\alpha}]} g(t, \xi')| dt \\ &\leq \int_{\mathbb{R}} |K_2(\xi_1, \xi', t) g(t, \xi')| dt, \end{aligned}$$

where

$$K_2(\xi_1, t) := \int_{\mathbb{R}} H(\xi_1 - t) e^{-a[\frac{\xi_1^3 - t^3}{3} + (\xi_1 - t)(i\frac{\lambda}{\alpha} + \xi'^2)]} \xi_1^2 dt.$$

We have two cases, the first when $-N \leq \xi_1 \leq N$ and so we can apply the Schur's lemma and we found

$$\|\xi'^2 v\|_{L^2(\mathbb{R}^n)} \leq C_2 \|g\|_{L^2(\mathbb{R}^n)}.$$

If $|\xi_1| > N$ then we use de l'Hôpital theorem and the Schur lemma and we have

$$\|\xi_1^2 v\|_{L^2(\mathbb{R}^n)} \leq C_3 \|g\|_{L^2(\mathbb{R}^n)}.$$

The proof of i) is complete.

Let now $\mathfrak{S}(\alpha) = 0$. We consider the homogeneous equation $(P - \lambda)u = 0$. We can apply the Fourier transform, only for the variable $x' \in \mathbb{R}^{n-1}$, to the problem and we obtain

$$\begin{aligned} (-\partial_{x_1}^2 + \xi'^2)\tilde{u}(x_1, \xi') + (\alpha x_1 - \lambda)\tilde{u}(x_1, \xi') &= 0 \\ -\left(\tilde{u}''(x_1, \xi') - \alpha\left(x_1 + \frac{\xi'^2}{\alpha} - \frac{\lambda}{\alpha}\right)\tilde{u}(x_1, \xi')\right) &= 0 \end{aligned} \quad (4.31)$$

Now we put $t = x_1 + \frac{\xi'^2}{\alpha} - \frac{\lambda}{\alpha}$ and we have

$$-\left(v''(t, \xi') - \alpha t v(t, \xi')\right) = 0,$$

where $v(t, \xi') = \tilde{u}\left(t - \frac{\xi'^2}{\alpha} + \frac{\lambda}{\alpha}, \xi'\right)$.

We put $z = \sqrt[3]{\alpha}t$, and our problem becomes

$$-\sqrt[3]{\alpha^2}\left(\tilde{v}''(z, \xi') - z\tilde{v}(z, \xi')\right) = 0, \quad (4.32)$$

where $\tilde{v}(z, \xi') = v\left(\frac{z}{\sqrt[3]{\alpha}}, \xi'\right)$.

From the theory of the Airy function the solutions are given by

$$\tilde{v}(z, \xi') = C_1 Ai(z) + C_2 Bi(z). \quad (4.33)$$

More details shall be given in the next section. □

4.3.1 The one dimensional case: Airy operator

If we are in the one dimensional case we recapture the Airy type operator

$$Au = -u'' + \alpha x u, \quad x \in \mathbb{R}. \quad (4.34)$$

We recall that the symbol of the operator A (4.34) is *hypoelliptic* if

$$|\partial_{(x,\xi)}^\gamma p(x,\xi)| < C |p(x,\xi)| \langle (x,\xi) \rangle^{-|\gamma|}, \quad (4.35)$$

(see [58], [3] and the references therein).

We have

Theorem 44. *The operator A (4.34) is hypoelliptic iff the $\Im(\alpha) \neq 0$*

Proof. The hypoellipticity condition is

$$|\partial_{(x,\xi)}^\gamma a(x,\xi)| < C |a(x,\xi)| \langle (x,\xi) \rangle^{-|\gamma|}, \quad \text{for } |x| + |\xi| \geq R, \quad (4.36)$$

where $a(x,\xi) = \xi^2 + \alpha x$. If $(x,\xi) \neq (0,0)$ then $a(x,\xi) = 0$ iff $\Im(\alpha) = 0$, this concludes the proof. \square

We consider now the following problem:

$$Au = -u''(x) + \alpha xu(x) = -(u(x)'' - \alpha xu(x)) = 0. \quad (4.37)$$

We put $x = \alpha^{-\frac{1}{3}}z$, and our problem becomes

$$-\alpha^{\frac{2}{3}}(v''(z) - zv(z)) = 0, \quad (4.38)$$

where $v(z) = u(\alpha^{-\frac{1}{3}}z)$. From the theory of the Airy function the solution of (4.38) is

$$v(z) = C_1 Ai(z) + C_2 Bi(z), \quad (4.39)$$

where

$$Ai(z) \simeq \frac{e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}z^{\frac{1}{4}}} \quad \text{and} \quad Bi(z) \simeq \frac{e^{\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}z^{\frac{1}{4}}} \quad \text{for } z \rightarrow \infty \quad (4.40)$$

$$Ai(-z) \simeq \frac{\sin(\frac{2}{3}z^{3/2} + \frac{\pi}{4})}{2\sqrt{\pi}z^{\frac{1}{4}}} \quad Bi(-z) \simeq \frac{\cos(\frac{2}{3}z^{3/2} + \frac{\pi}{4})}{2\sqrt{\pi}z^{\frac{1}{4}}} \quad \text{for } z \rightarrow -\infty. \quad (4.41)$$

For the problem (4.37) the solutions are:

$$u(x) = -\alpha^{\frac{2}{3}} \left(C_1 Ai\left(\frac{x}{\sqrt[3]{\alpha}}\right) + C_2 Bi\left(\frac{x}{\sqrt[3]{\alpha}}\right) \right). \quad (4.42)$$

We prove the following theorem:

Theorem 45. *If $\Im(\alpha) \neq 0$ then the $\text{spec}(A) = \emptyset$*

Proof. We apply the Fourier transform to our operator A and we obtain:

$$\hat{A} = \xi^2 + i\alpha\partial_\xi.$$

Dividing the last equation for $i\alpha$ we get

$$\hat{L} = \frac{1}{i\alpha}\hat{A} = \partial_\xi + a\xi^2, \tag{4.43}$$

where $a = \frac{1}{i\alpha}$. We note that we have

$$\Im(\alpha) \neq 0 \iff \Re(a) \neq 0.$$

So we have, if we suppose that $\Re(a) > 0$

$$\begin{aligned} (\hat{L} + i\lambda)^{-1}\hat{u} &= e^{-(a\frac{\xi^3}{3} + i\frac{\lambda}{\alpha}\xi)} \int_{-\infty}^{\xi} e^{-a(\frac{t^3}{3} - i\frac{\lambda}{\alpha})} g(t) dt \\ &= \int_{\mathbb{R}} K(\xi, t) g(t) dt, \end{aligned} \tag{4.44}$$

where $K(\xi, t) = H(\xi - t)e^{-\frac{a}{3}(\xi^3 - t^3) - i\frac{\lambda}{\alpha}(\xi - t)} \leq e^{-\frac{c\Re(a)}{3}(\xi - t)^3 + \Im(\frac{\lambda}{\alpha})(\xi - t)}$. And so if we put $s = \xi - t$, our problem becomes

$$K(\xi, t) \leq \int_0^\infty e^{-\frac{c\Re(a)}{3}s^3 + \Im(\frac{\lambda}{\alpha})s} ds,$$

and if we put $\sqrt[3]{\frac{c\Re(a)}{3}}s \rightarrow s$ the last integral becomes

$$\frac{c\Re(a)}{3} \int_0^\infty e^{-s^3 + \mu s} ds,$$

where $\mu = \Im(\frac{\lambda}{\alpha})\sqrt[3]{\frac{3}{c\Re(a)}}$. We have two cases:

if $\mu \leq 1$, we put $s = |\mu|^{-1}r$ and the integral becomes

$$\frac{1}{|\mu|} \int_0^\infty e^{-\frac{r^3}{|\mu|^3} + r} dr,$$

If $\mu > 1$ we consider

$$e^{2\sqrt{\frac{\mu^3}{3^3}}} \int_0^\infty e^{-s^3 + \mu s - \sqrt{\frac{\mu^3}{3^3}}} ds,$$

now we put $\tau = s - \sqrt{\frac{\mu}{3}}$. So we have

$$e^{2\sqrt{\frac{\mu^3}{3^3}}} \int_{-\sqrt{\frac{\mu}{3}}}^\infty e^{-(\tau + \sqrt{\frac{\mu}{3}})^3 + \mu(\tau + \sqrt{\frac{\mu}{3}}) - 2\sqrt{\frac{\mu^3}{3^3}}} d\tau.$$

□

4.4 On the global $\bar{\partial}$ type normal form

In this section we dwell upon the global regularity and the solvability properties of the Fourier $\bar{\partial}$ normal form operator

$$D_b = -\Delta + (\alpha + i\beta)x_1 + i\gamma x_2, \quad (4.45)$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha, \gamma \neq 0$. We want to study the following equation

$$D_b u = \left(-\Delta + (\alpha + i\beta)x_1 + i\gamma x_2 + r \right) u = f, \quad (4.46)$$

in the framework of weighted spaces like $H^{s_1, s_2}(\mathbb{R}^n)$ (Cordes type), $\mathcal{Q}^s(\mathbb{R}^n)$ (Shubin type) or $\mathcal{S}(\mathbb{R}^n)$.

First of all, in view of the well known properties of the action of the Fourier transform operator in the aforementioned spaces, we are reduced to study

$$\hat{D}_b \hat{u} = \left(\xi^2 - (\beta - i\alpha)\partial_{\xi_1} - \gamma\partial_{\xi_2} + r \right) \hat{u} = \left(\xi'^2 + \xi''^2 - (\beta - i\alpha)\partial_{\xi_1} - \gamma\partial_{\xi_2} + r \right) \hat{u} = \hat{f}, \quad (4.47)$$

where $\xi' = (\xi_1, \xi_2)$, $\xi'' = (\xi_3, \dots, \xi_n)$.

If $\alpha \neq 0$, we apply a linear transformation $M^{-1} : \mathbb{R}_{\xi_1, \xi_2}^2 \rightarrow \mathbb{R}_{\eta_1, \eta_2}^2$ such that $(\xi_1, \xi_2) \rightarrow (\frac{1}{\alpha}\eta_1, -\frac{\beta}{\alpha\gamma}\eta_1 - \frac{1}{\gamma}\eta_2)$, so we have

$$\tilde{P}w = \left(\xi''^2 + \langle M^t M \eta, \eta \rangle + i(\partial_{\eta_1} - i\partial_{\eta_2}) \right) w = g, \quad (4.48)$$

where $w(\boldsymbol{\eta}, \boldsymbol{\xi}) = \hat{u}(M^{-1}\boldsymbol{\xi}', \boldsymbol{\xi}'')$ and $g(\boldsymbol{\eta}, \boldsymbol{\xi}'') = \hat{f}(M^{-1}\boldsymbol{\xi}', \boldsymbol{\xi}'')$.

Hence setting $z = \boldsymbol{\eta}_1 + i\boldsymbol{\eta}_2 \in \mathbb{C}$, we need to study a perturbation of the $\bar{\partial}_z$ -operator depending polynomially on $\boldsymbol{\xi}''^2 = \xi_3^2 + \dots + \xi_n^2$. If $n \geq 3$, we demonstrate the following lemma.

Lemma 12. *We have*

$$\langle M^t M \boldsymbol{\eta}, \boldsymbol{\eta} \rangle = Az^2 + 2B|z|^2 + C\bar{z}^2 = \Re(Az^2) + \Im(Az^2) + 2B|z|^2, \quad (4.49)$$

where $B \in \mathbb{R}$, $A, C \in \mathbb{C}$, such that $\bar{A} = C$ and $z \neq 0$.

Proof. In view of the well known fact that

$$\boldsymbol{\eta}_1 = \frac{z + \bar{z}}{2}, \quad \boldsymbol{\eta}_2 = \frac{z - \bar{z}}{2i}.$$

we get

$$\begin{aligned} \langle M^t M \boldsymbol{\eta}, \boldsymbol{\eta} \rangle &= a\boldsymbol{\eta}_1^2 + 2b\boldsymbol{\eta}_1\boldsymbol{\eta}_2 + c\boldsymbol{\eta}_2^2 \\ &= a\left(\frac{z + \bar{z}}{2}\right)^2 + 2b\left(\frac{z + \bar{z}}{2}\right)\left(\frac{z - \bar{z}}{2i}\right) + c\left(\frac{z - \bar{z}}{2i}\right)^2 \\ &= Az^2 + 2B|z|^2 + C\bar{z}^2 \\ &= \Re(A)(\boldsymbol{\eta}_1^2 - \boldsymbol{\eta}_2^2 + 2i\boldsymbol{\eta}_1\boldsymbol{\eta}_2) - i\Im(A)(\boldsymbol{\eta}_1^2 - \boldsymbol{\eta}_2^2 + 2i\boldsymbol{\eta}_1\boldsymbol{\eta}_2) \\ &\quad + \Re(A)(\boldsymbol{\eta}_1^2 - \boldsymbol{\eta}_2^2 - 2i\boldsymbol{\eta}_1\boldsymbol{\eta}_2) + i\Im(A)(\boldsymbol{\eta}_1^2 - \boldsymbol{\eta}_2^2 - 2i\boldsymbol{\eta}_1\boldsymbol{\eta}_2) + 2B|z|^2 \\ &= 2\Re(Az^2) + \Im(Az^2) + 2B|z|^2. \end{aligned}$$

where $A = \frac{a}{4} - \frac{c}{4} + i\frac{b}{2}$, $C = \frac{a}{4} - \frac{c}{4} - i\frac{b}{2}$ and $B = \frac{a}{4} + \frac{c}{4}$. □

So we can write the operator (4.48) as

$$\tilde{P}w = \left(i\bar{\partial}_z + \boldsymbol{\xi}''^2 + \Re(Az^2) + 2\Im(Az^2) + 2B|z|^2 \right) w = g. \quad (4.50)$$

Unlike the generalized Airy normal form, apparently there is no hope for showing global hypoellipticity in $S(\mathbb{R}^n)$ of the Fourier $\bar{\partial}$ -operator. Indeed, it is well known that

$$\bar{\partial}u = f$$

is locally hypoelliptic and solvable for every $f \in C_0^\infty(\mathbb{R}^2)$, namely any solution is in $C^\infty(\mathbb{R}^2)$. One checks easily that $\bar{\partial}$ is not globally hypoelliptic in $\mathcal{S}(\mathbb{R}^2)$.

Let us consider the perturbed $\bar{\partial}_{\bar{z}}$

$$\left(\bar{\partial}_{\bar{z}} + \frac{i}{2} \xi''^2 + i(\Re(Az^2) + \Im(Az^2) + B|z|^2) \right) w = g. \quad (4.51)$$

We can write (formally) the solutions

$$\begin{aligned} w(z) &= e^{-\left(\frac{i}{4}z^2\bar{z} + \frac{i}{2}\xi''\bar{z} + i(\Re(Az^2) + \Im(Az^2)\bar{z})\right)} \varphi(z) + \tilde{w}(z) \\ \tilde{w}(z) &= e^{-\left(\frac{i}{4}z^2 + \frac{i}{2}\xi''z + i(\Re(Az^2) + \Im(Az^2)z)\right)} \\ &\int_{\mathbb{R}^2} e^{\left(\frac{i}{4}\zeta^2\bar{\zeta} + \frac{i}{2}\xi''\bar{\zeta} + i(\Re(A\zeta^2) + \Im(A\zeta^2)\bar{\zeta})\right)} g(\zeta, \xi'') \frac{d\zeta \wedge d\bar{\zeta}}{z - \zeta}. \end{aligned} \quad (4.52)$$

where φ is entire function in z and (we can choose) compactly supported in ξ'' if $n \geq 3$ and $g \in \mathcal{S}(\mathbb{R}^n)$, compactly supported in (ξ_1, ξ_2) . The main problem is whether one can find a nonzero entire function in z_1 such that the first u belongs to $\mathcal{S}'(\mathbb{R}^2)$ in (ξ_1, ξ_2) but is not in $\mathcal{S}(\mathbb{R}^2)$.

One is led naturally to conjecture that the perturbed operator is neither globally hypoelliptic nor globally solvable in $\mathcal{S}(\mathbb{R}^2)$ but it seems that one has to overcome nontrivial technical difficulties.

4.5 Splitting to globally elliptic and Airy or $\bar{\partial}$ type normal forms

The main goal of the present section is to study the reduction to normal form for non globally elliptic operators with non negative principal symbol, with B symmetric, i.e. (4.5) holds, and the characteristic set Σ_P is not Lagrangian, namely its dimension is different from n .

Theorem 46. *Let $n \geq 2$ and suppose that (4.5) holds. Then the following properties are equivalent*

- i) $\dim \Sigma_P \neq n$;
- ii) $\dim \Sigma_P < n$;
- iii) $C - \frac{1}{4}B^2$ is non zero and non negative;
- iv) There exists a unitary operator $T : L^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$, which is a linear automorphism in $\mathcal{S}(\mathbb{R}^n), \mathcal{Q}^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, $S_\mu^\mu(\mathbb{R}^n)$, $\mu \geq 1/2$ defined, for some $S \in SO(n)$, $\alpha \in \mathbb{R}^n$, by

$$Tv(x) = e^{i\frac{1}{2}\langle S^t B S x + 2\alpha, x \rangle} v(S^t x + \beta) \quad (4.53)$$

and splitting $x = (x', x'')$, $x' \in \mathbb{R}^\ell$, $x'' \in \mathbb{R}^{n-\ell}$, $1 \leq \ell \leq n-1$, such that

$$T^* \circ P \circ T = \tilde{P} = \tilde{P}'(x', D_{x'}) + \tilde{P}''(x'', D_{x''}) \quad (4.54)$$

$$\tilde{P}'(x', D_{x'}) = -\Delta' + i\langle \tilde{\beta}', D_{x'} \rangle + \rho x_1 + i\sigma x_1 + i\tau x_2 \quad (4.55)$$

$$\begin{aligned} \tilde{P}''(x'', D_{x''}) &= -\Delta'' + \langle \text{diag}\{c_{\ell+1}, \dots, c_n\} x'', x'' \rangle + i\langle \beta'', D_{x''} \rangle \\ &+ i\langle \tilde{N}'', x'' \rangle + \tilde{r}, \end{aligned} \quad (4.56)$$

for some $M \in \mathbb{R}^n$, $\rho, \sigma, \tau \in \mathbb{R}$, with ρ, σ, τ being as in Theorem 41.

Proof. The lemma on quadratic forms in Section 4.1 and the hypothesis on the principal symbol we deduce that $C' \neq 0$. Then we can reduce this matrix to diagonal form $D_c = \text{diag}\{c_1, \dots, c_n\}$, where

$$c_1 = c_2 = \dots = c_\ell = 0 \quad \text{and} \quad 0 < c_{\ell+1} \leq \dots \leq c_n. \quad (4.57)$$

by means of orthogonal transformation $x \rightarrow Rx$, where $R \in SO(n)$. Hence the operator becomes

$$e^{-i\frac{1}{4}\langle R^t B R x, x \rangle} \circ P \circ e^{i\frac{1}{4}\langle R^t B R x, x \rangle} = -\Delta + \langle D_c x, x \rangle + \langle \alpha, D_x \rangle + i\langle \beta, D_x \rangle + \langle M, x \rangle + i\langle N, x \rangle + \ell. \quad (4.58)$$

Without loss of generality, we may assume that $\alpha = 0$, by conjugation with $e^{i\frac{1}{2}\langle\alpha,x\rangle}$ (as in Chapter 2), so we consider the operator

$$L = -\Delta + \langle D_c x, x \rangle + i\langle \beta, D_x \rangle + \langle M, x \rangle + i\langle N, x \rangle + \ell \quad (4.59)$$

Next, we split the variables x as follows

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x' \\ x'' \end{pmatrix}, \quad \text{where } x' = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_\ell \end{pmatrix} \text{ and } x'' = \begin{pmatrix} x_{\ell+1} \\ x_{\ell+2} \\ \vdots \\ x_n \end{pmatrix}.$$

We can write

$$\begin{aligned} L = L_1 + L_2 &= -\Delta' + i\langle \beta', D_{x'} \rangle + \langle M', x' \rangle + i\langle N', x' \rangle \\ &\quad - \Delta'' + i\langle \beta'', D_{x''} \rangle + \langle \text{diag} \{c_{\ell+1}, \dots, c_n\} x'', x'' \rangle + \langle M'', x'' \rangle + i\langle N'', x'' \rangle + \ell. \end{aligned} \quad (4.60)$$

First of all we investigate the operator L_1 and considering a matrix $\tilde{S}' \in SO(\ell)$, we apply the orthogonal transformation $x' \rightarrow \tilde{S}'x'$ such that

$$M' = (m_1, \dots, m_\ell) \rightarrow \tilde{M}' = (\rho, 0, \dots, 0), \quad \text{with } \rho = \|M'\| \geq 0,$$

so L_1 becomes:

$$\tilde{L}_1 = -\Delta + i\langle \tilde{\beta}', D_{x'} \rangle + \rho x_1 + i\langle \tilde{N}', x' \rangle + r, \quad (4.61)$$

where $\tilde{N}' = \tilde{S}'^{-1}N$, $\tilde{\beta}' = \tilde{S}'^{-1}\beta'$.

If $\ell \geq 2$, we consider a matrix $\tilde{T}' \in SO(\ell-1)$, we apply the orthogonal transformation in $\mathbb{R}^{\ell-1}$ $x \rightarrow \tilde{T}'x$ such that

$$\tilde{N}' = (\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_\ell) \rightarrow \bar{N} = (\sigma, \tau, 0, \dots, 0), \quad \text{where } \sigma = \tilde{n}_1, \text{ with } \sigma, \tau \in \mathbb{R},$$

which transforms L_1 to

$$\bar{L}_1 = -\Delta' + i\langle \bar{\beta}', D_{x'} \rangle + \rho x_1 + i\sigma x_1 + i\tau x_2 + r, \quad (4.62)$$

where $\bar{\beta} = \tilde{T}'^{-1}\beta$. Now for L_2 we consider the following translation

$$x_j'' \rightarrow x_j'' + \frac{M_j''}{2c_j}$$

So we have

$$\bar{L}_2 = -\Delta'' + i\langle \beta'', D_{x''} \rangle + \langle \text{diag} \{c_{\ell+1}, \dots, c_n\} x'', x'' \rangle + i\langle \tilde{N}'', x'' \rangle + \tilde{r}. \quad (4.63)$$

So the operator L (4.60) becomes:

$$\begin{aligned} \bar{L} = & -\Delta' + i\langle \bar{\beta}', D_{x'} \rangle + \rho x_1 + i\sigma x_1 + i\tau x_2 + r \\ & -\Delta'' + i\langle \beta'', D_{x''} \rangle + \langle \text{diag} \{c_{\ell+1}, \dots, c_n\} x'', x'' \rangle + i\langle \tilde{N}'', x'' \rangle + \tilde{r}. \end{aligned} \quad (4.64)$$

□

Remark 16. *We can summarize the result above as follows: we have reduced the study of the solvability and hypoellipticity of $Pu = f$ to the study of the following decomposed equation*

$$\tilde{P}u = \tilde{P}'(x', D_{x'})u + \tilde{P}''(x'', D_{x''})u = f(x), \quad x \in \mathbb{R}^n, \quad (4.65)$$

where \tilde{P}' is an Airy type operator and \tilde{P}'' is a globally elliptic operator. As far as we know, such equations have not been studied in the literature on pseudodifferential operators for solvability and hypoellipticity in functional spaces on \mathbb{R}^n . Clearly one needs to combine different techniques in order to deal with the challenging problem. At least the normal form equation serves as model case to test new techniques. We leave this study for the future.

4.6 Multidimensional twisted Laplacian type normal form

We recall that the twisted Laplacian L on \mathbb{R}^2 is the second-order partial differential operator given by

$$L = -\Delta + x_1 D_{x_2} - x_2 D_{x_1} + \frac{1}{4}(x_1^2 + x_2^2). \quad (4.66)$$

The twisted Laplacian appears in harmonic analysis naturally in the context of Wigner transforms and Weyl transforms [67], and also in physics. The transpose L^t of the twisted Laplacian L is given by

$$L = -\Delta + \frac{1}{4}(x_1^2 + x_2^2) + x_2 D_{x_1} - x_1 D_{x_2}. \quad (4.67)$$

In the paper [16], it is shown that L is globally hypoelliptic in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$, while global hypoellipticity and global solvability in Gelfand–Shilov spaces S_μ^H has been shown in [23], [25], [22] for more general operators of the type

$$L_\tau(x, D_x) = -\Delta + \tau(x_2 D_{x_1} - x_1 D_{x_2}) + \frac{\tau^2}{4}(x_1^2 + x_2^2) = (D_{x_1} + \frac{\tau}{2}x_2)^2 + (D_{x_2} - \frac{\tau}{2}x_1)^2, \quad (4.68)$$

where $\tau \in \mathbb{R}$, $\tau \neq 0$. In our notation (4.67) corresponds to a homogeneous operator in \mathbb{R}^2 with $A = A_{skew} = \begin{pmatrix} 0 & -\tau \\ \tau & 0 \end{pmatrix}$ and $B = \begin{pmatrix} \frac{\tau^2}{4} & 0 \\ 0 & \frac{\tau^2}{4} \end{pmatrix}$.

We note that the characteristic set of L and L^t is two dimensional and has double characteristics which are symplectic.

We require in this chapter the non degeneracy condition

$$A_{skew} \quad \text{has a maximal rank } n. \quad (4.69)$$

Note that the hypothesis (4.69) holds for the twisted Laplacian and its transpose L^t .

Clearly this implies that n is even. So we write $2n$ instead of n and consider the linear operator

$$P = -\Delta + \langle Ax, D_x \rangle + \langle Bx, x \rangle + \langle M, D_x \rangle + \langle N, x \rangle + r, \quad (4.70)$$

where $A \in M_{2n}(\mathbb{R})$, B is symmetric, and $M, N \in \mathbb{C}^n$, $r \in \mathbb{C}$. We recall the definition of the characteristic set

$$\Sigma_P = \{(x, \xi) \in \mathbb{R}^{4n} \setminus \mathbf{0}; p_2(x, \xi) := \|\xi\|^2 + \langle Ax, \xi \rangle + \langle Bx, x \rangle = 0\}. \quad (4.71)$$

In view of the identity

$$\langle Qx, x \rangle = \frac{1}{2} \langle (Q + Q^T)x, x \rangle$$

for all $Q \in M_n(\mathbb{R})$ one readily obtains that

$$\langle A_{skew}x, A_{symm}x \rangle = \langle A_{symm}A_{skew}x, x \rangle = \frac{1}{2} \langle A_{symm}A_{skew} - A_{skew}A_{symm}x, x \rangle. \quad (4.72)$$

We note that $A_{symm}A_{skew} - A_{skew}A_{symm}$ is symmetric.

We recall that real skew-symmetric matrices are normal matrices (they commute with their adjoint ones). The hypothesis (4.69) and the spectral theorem on $2n \times 2n$ real skew-symmetric matrices show that there exist n non-zero real numbers $\tau_1 \geq \tau_2 \geq \dots \geq \tau_n$, $j = 1, \dots, n$, two non negative integers n_+, n_- , $n_+ + n_- = n$, such that $\tau_{n_+} > 0 > \tau_{n_++1}$ if $n_+ n_- > 0$ with the convention $n_+ = 0$ (respectively, $n_- = 0$) if $\tau_1 < 0$ (respectively, $\tau_n > 0$) and an orthogonal matrix S_0 which transforms the skew-symmetric matrix to the following block

diagonal form

$$S_0^T A_{skew} S_0 = \begin{pmatrix} 0 & -\tau_1 & 0 & 0 & \dots & 0 & 0 \\ \tau_1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & -\tau_2 & \dots & 0 & 0 \\ 0 & 0 & \tau_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & -\tau_n \\ 0 & 0 & 0 & 0 & \dots & \tau_n & 0 \end{pmatrix}. \quad (4.73)$$

We need to recall that we can define an explicit symplectic transformation

$$\kappa : \mathbb{R}_y^2 \times \mathbb{R}_\eta^2 \rightarrow \mathbb{R}_x^2 \times \mathbb{R}_\xi^2,$$

defined by the generating function

$$\varphi_\omega(x, \eta) = |\omega|(\eta_1 \eta_2 + (x_1 x_2)/2 + x_2 \eta_1 + x_1 \eta_2)$$

via

$$\xi_j = \partial_{x_j} \varphi_\omega(x, \eta), \quad j = 1, 2$$

$$y_j = \partial_{\eta_j} \varphi_\omega(x, \eta), \quad j = 1, 2,$$

which leads to explicit formulas for κ

$$y_1 = |\omega| x_2 + |\omega| \eta_2$$

$$y_2 = |\omega| x_1 + |\omega| \eta_1$$

$$\eta_1 = \xi_2 / |\omega| - x_1 / 2$$

$$\eta_2 = \xi_1 / |\omega| - x_2 / 2.$$

Theorem 47. *Suppose that (4.69) holds. Then the following properties are equivalent*

i) $\dim \Sigma_P = 2n$;

ii) the matrices $B, A_{\text{symm}}, A_{\text{skew}}$ satisfy the identity

$$C := B - \frac{1}{4}A_{\text{symm}}^2 + \frac{1}{4}(A_{\text{symm}}A_{\text{skew}} - A_{\text{skew}}A_{\text{symm}}) + \frac{1}{4}A_{\text{skew}}^2 = 0; \quad (4.74)$$

iii) There exists a unitary operator $T : L^2(\mathbb{R}^{2n}) \mapsto L^2(\mathbb{R}^{2n})$, which is a linear automorphism in $\mathcal{S}(\mathbb{R}^{2n}), \mathcal{Q}^s(\mathbb{R}^{2n}), S_\mu^\mu(\mathbb{R}^{2n}), s \in \mathbb{R}, \mu \geq 1/2$, defined by

$$Tv(x) = e^{i\frac{1}{4}\langle S^t A_{\text{symm}} Sx + 2\alpha, x \rangle} v(S^t x + \beta) \quad (4.75)$$

for some $\alpha, \beta \in \mathbb{R}^{2n}$ and $S \in SO(\mathbb{R}^{2n})$ in (4.73), such that

$$\begin{aligned} T^* \circ P \circ T = \tilde{P} = & \sum_{j=1}^n \left((D_{x_{2j-1}} - \frac{\tau_j}{2} x_{2j})^2 + (D_{x_{2j}} + \frac{\tau_j}{2} x_{2j-1})^2 \right) \\ & + i \sum_{j=1}^n \rho_j (D_{x_{2j-1}} - \frac{\tau_j}{2} x_{2j}) + i \sum_{j=1}^n \sigma_j (D_{x_{2j}} + \frac{\tau_j}{2} x_{2j-1}) \\ & + \sum_{j=1}^n \gamma_j (D_{x_{2j-1}} + \frac{\tau_j}{2} x_{2j}) + \sum_{j=1}^n \delta_j (D_{x_{2j}} - \frac{\tau_j}{2} x_{2j-1}) \\ & + \tilde{r}, \end{aligned} \quad (4.76)$$

for $\rho_j, \sigma_j \in \mathbb{R}, \gamma_j, \delta_j \in \mathbb{C}, j = 1, \dots, n, \tilde{r} \in \mathbb{C}$.

- There exists a unitary operator $U : L^2(\mathbb{R}^{2n}) \mapsto L^2(\mathbb{R}^{2n})$, which is a linear automorphism in $\mathcal{S}(\mathbb{R}^{2n}), \mathcal{Q}^s(\mathbb{R}^{2n}), S_\mu^\mu(\mathbb{R}^{2n}), s \in \mathbb{R}, \mu \geq 1/2$, defined by $U = \prod_{j=1}^n |\tau_j| J_\tau \circ T$, where $\tau = (\tau_1, \dots, \tau_n), \tau_j > 0, j = 1, \dots, n_+, \tau_j < 0, j = n_+ + 1, \dots, n_+ + n_-, T$ is as in iii) and J_τ is the FIO given by

$$J_\tau v(x) = J_{\tau_1} \circ \dots \circ J_{\tau_n} v(x) := \int_{\mathbb{R}^{2n}} e^{i \sum_{j=1}^n |\tau_j| \varphi(x_{2j-1}, x_{2j}, \xi_{2j-1}, \xi_{2j})} \hat{v}(\xi) \bar{d}\xi \quad (4.77)$$

$$J_\alpha v(y) = \int_{\mathbb{R}^2} e^{i|\alpha| \varphi(y, \eta)} \hat{v}(\eta) \bar{d}\eta, \quad (4.78)$$

such that

$$\begin{aligned}
U^* \circ P \circ U &= \sum_{j=1}^{n_+} (\tau_j^2 D_{x_{2j}}^2 + x_{2j}^2) + \sum_{j=n_++1}^n (\tau_j^2 D_{x_{2j-1}}^2 + x_{2j-1}^2) \\
&+ i \sum_{j=1}^{n_+} (\rho_j D_{x_{2j}} + i \sigma_j \tau_j x_{2j}) + \sum_{j=n_++1}^n (\rho_j D_{x_{2j-1}} + \sigma_j \tau_j x_{2j-1}) \\
&+ \sum_{j=1}^{n_+} (\gamma_k D_{x_{2j-1}} + \omega_j \tau_j x_{2j-1}) + \sum_{j=n_++1}^n (\gamma_j D_{x_{2j}} + \omega_j \tau_j x_{2j})
\end{aligned} \tag{4.79}$$

for $\rho, \sigma \in \mathbb{R}^n$, $\gamma, \omega \in \mathbb{C}^n$, $\tilde{r} \in \mathbb{C}$.

Moreover,

$$P \text{ is symmetric iff } \rho, \sigma = 0, \gamma, \omega \in \mathbb{R}^n, \tilde{r} \in \mathbb{R} \tag{4.80}$$

and T has a discrete spectrum iff $\gamma = \omega = 0$.

Proof. First we rewrite the principal symbol as follows, taking into account that B is symmetric, the

$$\begin{aligned}
p_2(x, \xi) &= \langle \xi + \frac{1}{2}Ax, \xi + \frac{1}{2}Ax \rangle + \langle Bx, x \rangle - \frac{1}{4} \langle Ax, Ax \rangle \\
&= \|\xi + \frac{1}{2}A_{\text{symm}}x + \frac{1}{2}A_{\text{skew}}x\|^2 + \langle \bar{C}x, x \rangle,
\end{aligned} \tag{4.81}$$

$$\bar{C} = B - \frac{1}{4}A_{\text{symm}}^2 + \frac{1}{4}(A_{\text{symm}}A_{\text{skew}} - A_{\text{skew}}A_{\text{symm}}) + \frac{1}{4}A_{\text{skew}}^2. \tag{4.82}$$

Since B is symmetric, we can diagonalize with some matrix from with $SO(2n)$ and by the lemma on quadratic forms (replacing n by $2n$) we get that i) is equivalent to ii). As a by product we obtain that the characteristic set of p_2 is defined as

$$p_2(x, \xi) = \{(x, \xi) \in \mathbb{R}^{4n} \setminus 0; \xi = \frac{1}{2}Ax = \frac{1}{2}A_{\text{symm}}x + \frac{1}{2}A_{\text{skew}}x\} \tag{4.83}$$

which is not Lagrangian since $A_{\text{skew}} \neq 0$. Now we show the equivalence of ii) with iii).

Step 1. Using the diagonalization matrix S_0 , by the linear change $x \mapsto S_0x$ we may assume that A_{skew} is in the canonical block-diagonal form (4.73).

Step 2. Using the conjugation

$$\tilde{P} = e^{-i\frac{1}{4}\langle A_{symm}x, x \rangle} \circ P \circ e^{i\frac{1}{4}\langle A_{symm}x, x \rangle}$$

we are reduced to the case

$$\begin{aligned} \tilde{P} &= -\Delta + \langle A_{skew}x, D_x \rangle + \langle (B - \frac{1}{2}A_{skew}A_{symm})x, x \rangle + \tilde{P}_1 \\ &= \sum_{j=1}^n (D_{x_{2j-1}} - \frac{\tau_j}{2}x_{2j})^2 + (D_{x_{2j}} + \frac{\tau_j}{2}x_{2j-1})^2 \\ &\quad - \sum_{j=1}^n \tau_j(x_{2j-1}^2 + x_{2j}^2) + \langle (B - \frac{1}{2}A_{skew}A_{symm})x, x \rangle + \tilde{P}_1 \\ &= \sum_{j=1}^n (D_{x_{2j-1}} - \frac{\tau_j}{2}x_{2j})^2 + (D_{x_{2j}} + \frac{\tau_j}{2}x_{2j-1})^2 + \langle Cx, x \rangle + \tilde{P}_1 \end{aligned} \tag{4.84}$$

where

$$\tilde{P}_1 = \langle \alpha, D_x \rangle + i\langle \beta, D_x \rangle + \langle M + \frac{1}{2}\alpha A_{symm}, x \rangle + i\langle N, x \rangle + r, \tag{4.85}$$

and $G = \alpha + i\beta$ and $F = M + iN$ with $\alpha, \beta, M, N \in \mathbb{R}^{2n}$. Clearly, ii) implies $C = 0$.

Step 3. Using conjugation with e^{ipx} and translation we reduce to purely imaginary coefficients. □

4.7 Normal form transformation

The normal form transformation reduces the original operator P to the following normal form

$$\begin{aligned} P_{NF}(y, D_y)u &= \sum_{j=1}^n |\tau_j|(D_{y_j}^2 + y_j^2)u + i \sum_{j=1}^n (a_j D_{y_j} + b_j y_j)u \\ &+ \sum_{\ell=n+1}^{2n} (\gamma_\ell D_{y_\ell} + \delta_\ell y_\ell)u + ru = f, \end{aligned} \quad (4.86)$$

with $a_j, b_j \in \mathbb{R}$ for $j = 1, \dots, n$, $\gamma_\ell, \delta_\ell, r \in \mathbb{C}$, for $\ell = n+1, \dots, 2n$.

We note that the same holds for the perturbation $P(x, D) + b(x, D)$, with zero order Shubin p.d.o $b(x, D)$. We study the perturbed normal form equation

$$P_{NF}(y, D_y)u + b(y, D_y)u = f.$$

We have

Theorem 48. *The following properties are equivalent:*

- i) $P(x, D)$ is symmetric;*
- ii) $P_{NF}(y, D_y)$ is symmetric;*
- iii) $a_j = b_j = 0$, for $j = 1, \dots, n$ $\Im \gamma_j = \Im \delta_j = 0$, for $j = n+1, \dots, 2n$, $r \in \mathbb{R}$.*

Proof. The NFT preserves the symmetry properties, i.e. *i) \Leftrightarrow ii)*, the equivalence *ii) \Leftrightarrow iii)* straight forwards. \square

Next we derive complete description of the global hypoellipticity and the global solvability of the twisted Laplacian type operators. We prove the following theorem:

Theorem 49. *Let P (or equivalently P_{NF}) be symmetric. Then the following properties are equivalent:*

i) $P_{NF}(y, D_y)$ is globally hypoelliptic in $\mathcal{S}(\mathbb{R}^{2n})$;

ii) $\gamma_k = \omega_k = 0$, $k = n + 1, \dots, 2n$ and

$$r \neq - \sum_{j=1}^n |\tau_j|(2k_j + 1); \quad (4.87)$$

iii) $P_{NF}(y, D_y)$ is globally solvable in $\mathcal{S}(\mathbb{R}^{2n})$;

iv) $P_{NF}(y, D_y)$ is globally hypoelliptic in $S_{\mu}^H(\mathbb{R}^{2n})$;

v) $P_{NF}(y, D_y)$ is globally solvable in $S_{\mu}^H(\mathbb{R}^{2n})$.

Finally, if ii) holds the operator P_{NF} (and P) is essentially self-adjoint with discrete spectrum

$$\text{spec}(P_{NF})\{\lambda_k : \lambda_k = 2 \sum_{j=1}^n |\tau_j|k_j + \sum_{j=1}^n |\tau_j| + r, k' = (k'_1, \dots, k'_n) \in \mathbb{Z}_+^n\}, \quad (4.88)$$

and the multiplicity of each eigenvalue is infinity. More precisely

$$\text{Ker}(P - \lambda_k) \cap L^2(\mathbb{R}^{2n}) = \{H_{k'}(y')\psi(y''), \psi \in L^2(\mathbb{R}^n)\}. \quad (4.89)$$

Moreover, as for the twisted Laplacian in \mathbb{R}^2 , the operator P_{NF} (and P) have no compact resolvent.

Proof. Suppose that ii) holds. We use the multidimensional Hermite functions expansion

$$u(y) = \sum_{k \in \mathbb{Z}_+^{2n}} u_k H_k(y) = \sum_{k', k'' \in \mathbb{Z}_+^n} u_{k', k''} H_{k'}(y') H_{k''}(y'') \quad (4.90)$$

Clearly $Pu = f$ is equivalent to

$$\left(2 \sum_{j=1}^n |\tau_j|k_j + \sum_{j=1}^n |\tau_j| + r\right) u_k = f_k, k = (k', k'') \in \mathbb{Z}_+^{2n}. \quad (4.91)$$

One derives, in view of (4.87),

$$\begin{aligned} & \min_{k \in \mathbb{Z}_+^{2n}} \left(2 \sum_{j=1}^n |\tau_j| k_j + \sum_{j=1}^n |\tau_j| + r \right) \\ &= \min_{k' \in \mathbb{Z}_+^n} \left(2 \sum_{j=1}^n |\tau_j| k'_j + \sum_{j=1}^n |\tau_j| + r \right) = \delta_0 > 0 \end{aligned} \quad (4.92)$$

which leads to

$$|u_k| = \left| \frac{f_k}{2 \sum_{j=1}^n |\tau_j| k'_j + \sum_{j=1}^n |\tau_j| + r} \right| \leq \frac{1}{\delta_0} |f_k|. \quad (4.93)$$

The characterization of $\mathcal{S}(\mathbb{R}^{2n})$ and $S_\mu^\mu(\mathbb{R}^{2n})$ by the eigenfunction expansions yields that *ii*) implies *i*), *iii*), *iv*), *v*).

Suppose, now, that *ii*) is not true: Let $\gamma_k = \omega_k = 0$, $k = n+1, \dots, 2n$ but

$$2 \sum_{j=1}^n |\tau_j| k'_j + \sum_{j=1}^n |\tau_j| + r = 0.$$

Then $\ker(P_{NF})$ is infinite dimensional and as a consequence *i*), *ii*), *iv*), *v*) are not true.

Suppose, now, that $\sum_{k=n+1}^{2n} |\gamma_k| \neq 0$. Without loss of generality (using linear change of the variables x'') we assume $\gamma_{n+1} \neq 0$, $\gamma_j = 0$ for $j = n+2, \dots, 2n$.

We use the partial Hermite expansion with respect the variable y' , namely

$$u(y', y'') = \sum_{k' \in \mathbb{Z}_+^n} u_{k'}(y'') H_{k'}(y'). \quad (4.94)$$

Thus $Pu = f$ is equivalent to

$$\sum_{k' \in \mathbb{Z}_+^n} \left(\gamma_{n+1} D_{y_{n+1}} + \left(\sum_{\ell=1}^n \omega_{\ell y \ell} \right) u_{k'}(y'') + \lambda_{k'} u_{k'}(y'') \right) H_{k'}(y') = \sum_{k' \in \mathbb{Z}_+^n} f_{k'}(y'') H_{k'}(y') \quad (4.95)$$

i.e.

$$\gamma_{n+1} D_{y_{n+1}} u_{k'} + \left(\sum_{\ell=n+1}^{2n} \omega_{\ell y \ell} \right) u_{k'} + \lambda_{k'} u_{k'} = f_{k'}. \quad (4.96)$$

If $f_{k'} \equiv 0$, we find explicit solutions (4.96), using the fact that $D_{y_j} = \frac{1}{i} \partial_{y_j}$, and (4.96) becomes for $f_{k'} \equiv 0$

$$\gamma_{n+1} \partial_{y_{n+1}} + i \left(\sum_{\ell=n+1}^{2n} \omega_\ell y_\ell \right) u_{k'} + i \lambda_{k'} u_{k'} = 0 \quad (4.97)$$

and we write explicit solutions,

$$u_{k'}(y'') = \theta_{k'}(y''') e^{-i \frac{\omega_\ell}{2\gamma_{n+1}} y_{n+1}^2 - i \sum_{\ell=n+2}^{2n} \frac{\omega_\ell}{\gamma_{n+1}} y_\ell y_{n+1} - \frac{y_{n+1}}{\gamma_{n+1}} \lambda_{k'}} \quad (4.98)$$

where $y''' = (y_{n+2}, \dots, y_{2n})$, $\theta_{k'}(y''') \in \mathcal{S}'(\mathbb{R}^{n-1})$ and $k' \in \mathbb{Z}_+^n$.

Clearly (4.98) implies $\dim(\text{Ker}P) = \infty$, and we conclude as before that *i), iii), iv), v)* do not hold.

Finally, if $\gamma_\ell = 0$, $\ell = n+1, \dots, 2n$ and $\sum_{\ell=n+1}^{2n} |\omega_\ell| \neq 0$, we conclude that $\dim(\text{Ker}P) = \infty$, constructing infinite dimensional kernels. More precisely, if $\delta_{n+1} \neq 0$ we get

$$u_{k'} = \delta(y_{n+1} + \sum_{\ell=n+2}^{2n} \frac{\omega_\ell}{\omega_{n+1}} y_\ell + \frac{\lambda_{k'}}{\gamma_{n+1}}) \theta_{k'}(y''')$$

where $\theta_{k'}(y''') \in \mathcal{S}'(\mathbb{R}^{n-1})$ and $\delta(y_{n+1} - q(y'''))$ stands for the delta function on the hypersurface $y_{n+1} = q(y''')$. The proof is complete \square

We derive also perturbation result

Proposition 32. *Suppose that ii) of the Theorem 49 holds. Then there exists a constant $c_0 > 0$, depending only on n , such that if the normal form $b_{NF}(y, D_y)$ of the p.d.o $b(x, D)$ satisfies*

$$\sup_{(y, \eta) \in \mathbb{R}^{2n}, |\alpha| \leq n+1, |\beta| \leq n+1} \left| \partial_y^\alpha \partial_\eta^\beta b_{NF}(y, \eta) \right| \leq c_0 \delta_0, \quad (4.99)$$

where δ_0 is defined in (4.92). Then $P_{NF} + b_{NF}$ and $P + b$ are globally hypoelliptic and globally solvable in $\mathcal{S}(\mathbb{R}^{2n})$.

Proof. We use the fact that if $c_0 \ll 1$ then $\|b_{NF}\|_{L^2(\mathbb{R}^{2n})} < \delta_0$, and we have $(P + b)^{-1} = P^{-1}(1 + P^{-1}b)^{-1}$. This completes the proof. \square

Chapter 5

Applications to global Cauchy problems

The main goal of the chapter is to investigate the well-posedness in weighted Shubin spaces $\mathcal{Q}^s(\mathbb{R}^n)$ and in the Gelfand–Shilov classes $\mathcal{S}_\mu^M(\mathbb{R}^n)$ of the following second order hyperbolic Cauchy problem

$$\begin{cases} \partial_t^2 u + P(x, D)u + R(x, D)u = 0, & t \geq 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0 \in \mathcal{S}'(\mathbb{R}^n), & u_t(0, x) = u_1 \in \mathcal{S}'(\mathbb{R}^n), \end{cases} \quad (5.1)$$

where

$$P(x, D)u = \int_{\mathbb{R}^n} e^{ix\xi} p(x, \xi) \hat{u}(\xi) \bar{d}\xi, \quad \bar{d}\xi = (2\pi)^{-n} d\xi$$

is a second order self-adjoint globally elliptic (namely Γ -elliptic) pseudo-differential operator (p.d.o.) of Shubin type (namely Γ -pseudodifferential operator) and semi-bounded from below, (for more details cf. M. Shubin [58] and the Chapter 1 of this thesis), where

$$R(x, D)u = \int_{\mathbb{R}^n} e^{ix\xi} r(x, \xi) \hat{u}(\xi) \bar{d}\xi$$

is a first order Shubin type p.d.o. If $n = 2$, we also consider a second order symmetric Shubin degenerate operator, modeled by the twisted Laplacian.

We recall that the twisted Laplacian L_1 and its transposed $L_2 = L_1^t$ are not globally elliptic operators in \mathbb{R}^2 (see [16], [23]) and they admit rotation terms, namely

$$P = L_k = -\Delta + \frac{1}{4}(x_1^2 + x_2^2) + (-1)^k(x_2 D_{x_1} - x_1 D_{x_2}), \quad k = 1, 2. \quad (5.2)$$

The fundamental role of the study of the global Cauchy problem for L_k and its perturbations will play the conjugation of L_τ with the global Fourier integral operator

$$J_\tau v = \int_{\mathbb{R}^n} e^{i\tau\phi(x,\eta)} \hat{v}(\eta) \bar{d}\eta, \quad (5.3)$$

where $\phi(x, \eta)$ is homogeneous quadratic function cf. [24], see also the section dedicated to the twisted Laplacian.

We also mention the paper of Popivanov [47], where another class of degenerate Shubin operators has been studied in the presence of Diophantine phenomena.

The functional frame for the Cauchy problem is given by the weighted Shubin type spaces $\mathcal{Q}^s(\mathbb{R}^n)$ cf. [58], [50] and the Gelfand–Shilov spaces $\mathcal{S}_\mu^\mu(\mathbb{R}^n)$, $\mu \geq 1/2$, see [20], [43]. We recall that the spectrum of $P(x, D)$ is discrete

$$\text{spec}(P) = \{\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots, \lim_{k \rightarrow \infty} \lambda_k = \infty\}, \quad (5.4)$$

with all eigenvalues having finite multiplicity and with orthonormal basis of associated eigenfunctions $\{\varphi_j\}_{j=1}^\infty$ yielding Fourier eigenfunction expansions $u = \sum_{j=1}^\infty u_j \varphi_j$, $u \in \mathcal{S}'(\mathbb{R}^n)$, $u_j = \langle u, \varphi_j \rangle := u(\varphi_j)$. For more details see Theorem 24.

Remark 17. We note that we can write the spectrum of P as

$$\text{spec}(P) = \{\mu_j : \mu_1 < \dots < \mu_k < \dots, \lim_{k \rightarrow \infty} \mu_k = \infty\},$$

where the eigenvalues μ_j have the multiplicity m_j , with $j = 1, \dots, \infty$.

5.1 Previous results

More general results have been proved in the book of Boggiatto, Buzano and Rodino [3], (for more details see also [33]).

Broadly speaking, they consider more general higher order, strictly hyperbolic equations of anisotropic generalizations of Shubin operators but the solution is local in time.

The crucial ingredient is the reduction of the Cauchy problem to a first order problem

$$\begin{cases} D_t A_t = p(t, x, D_x) A_t, \\ A|_{t=0} = I, \end{cases} \quad (5.5)$$

where $p(t, x, D_x)$ is a C^∞ map from an open interval $] -T, T[$ of the real line to the $\mathcal{L}_{\mathcal{P}, \rho}^{2\rho}$, class of pseudodifferential operators of order 2ρ , $\rho \leq \frac{1}{2}$ and A_t is sought in the form of a family of Fourier integral operators also depending on the parameter t , like

$$Au(x) = \int e^{i\phi(t, x, \eta)} a(t, x, \eta) \bar{d}\eta,$$

where $\phi(t, x, \eta)$ is the phase function, smooth real valued function, and $a(t, x, \eta)$ stand for the amplitude. The goal is to construct a parametrix (approximate solution) of the problem (5.5) determining the phase function $\phi(t, x, \eta)$ and the amplitude $a(t, x, \eta)$. The phase function satisfies the Hamilton-Jacobi equation

$$\begin{cases} \partial_t \phi(t, x, \eta) = p_{2\rho}(t, x, \nabla_x \phi(t, x, \eta)), \\ \phi(t, x, \eta) = x\eta, \end{cases} \quad (5.6)$$

where $p_{2\rho}$ is the principal part of p . Typically, if $n = 1$, $p_{2\rho} = (1 + \xi^2 + x^{2k})^{1/(2k)}$, with $\rho = 1/(2k)$ and if $k = 1$ we recapture the case of Shubin operators. Here the importance of the condition of at most linear growth

at infinity $\rho \leq \frac{1}{2}$ (i.e. $2\rho \leq 1$) appears, namely, it guaranties the existence of global solution of the bi-characteristic equations and the boundless of the Hessian of $\phi_{x,\xi}$. After that, in order to compute the symbol $a(t,x,\eta)$ by giving an asymptotic expansion. Their results are being with a review of Hamilton–Jacobi theory in $S_{\mathcal{D},\rho}^m$ -frame. For this reason, we recall that

$$f \in C^\infty(I, S_{\mathcal{D},\rho}^m), \quad I =]-T, T[, \quad T > 0,$$

meaning that for any integer ν and any multi-index γ , there exists a positive constant $k_{\nu,\gamma}$ such that:

$$|\partial_t^\nu \partial_\xi^\gamma f(t, \zeta)| \leq k_{\nu,\gamma} \Lambda(\zeta)^{m-\rho|\gamma|}, \quad (5.7)$$

for $\zeta = (x, \xi) \in \mathbb{R}^{2n}$, $t \in I$, and $\Lambda(\zeta)$ is anisotropic symbol of the type $(x^{2k} + \xi^2)^{1/(2k)}$. Let $p_{2\rho}(t, x, \xi) \in C^\infty(I, S_{\mathcal{D},\rho}^m)$ be real valued and consider the Hamilton–Jacobi system

$$\begin{cases} \partial_t \xi_j = \partial_{x_j} p_{2\rho}(t, x, \xi), \\ \partial_t x_j = -\partial_{\xi_j} p_{2\rho}(t, x, \xi), \end{cases} \quad (5.8)$$

with $j = 1, \dots, n$ and the following initial condition

$$\begin{cases} x(0) = y, \\ \xi(0) = \eta. \end{cases} \quad (5.9)$$

They prove that the unique solution of this system can be expressed in terms of the class $C^\infty(]-T', T'[, S_{\mathcal{D},\rho}^\rho)$ with $0 < T' \leq T$. Thanks to the symmetry of $S_{\mathcal{D},\rho}^\rho$ the system becomes:

$$\begin{cases} \partial_t \zeta_j = \tilde{\partial}_j p_{2\rho}(t, \zeta), \\ \zeta(0) = \omega, \end{cases} \quad (5.10)$$

where $\zeta = (x, \xi)$, $\omega = (y, \eta)$, $\tilde{\partial}_j = -\partial_{\xi_j}$, $j = 1, \dots, n$, $\tilde{\partial}_j = \partial_{x_j}$, $j = n+1, \dots, 2n$. Now we recall the main theorem (for more details see Theorem 12.5 [3]).

Theorem 50. *Let $p(t, x, \xi) = p_{2\rho}(t, x, \xi) + \tilde{p}(t, x, \xi)$, with $p_{2\rho} \in C^\infty(]-T', T'[, S_{\mathcal{D}, \rho}^{2\rho})$ be real valued, $\tilde{p} \in C^\infty(]-T', T'[, S_{\mathcal{D}, \rho}^0)$. There exists a $T' > 0$ with $T' \leq T$, and a linear map A_t , depending on the parameter $t \in]-T', T'[$ with distribution kernel $A(t, x, y) \in C^\infty(]-T', T'[, \mathcal{S}'(\mathbb{R}_x^n \times \mathbb{R}_y^n))$ such that for all $t \in]-T', T'[$:*

$$\begin{aligned} A_t &: \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n), \\ A_t &: \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n), \\ A_t &: H_{\mathcal{D}}^s(\mathbb{R}^n) \rightarrow H_{\mathcal{D}}^s(\mathbb{R}^n), \quad \text{for all } s \in \mathbb{R}. \end{aligned}$$

with continuous action, and

$$\begin{aligned} (D_t - p(t, x, D_x))A(t, x, y) &= R(t, x, y) \\ A(0, x, y) &= \delta(x - y) + \tilde{R}(x, y), \end{aligned}$$

where $R(t, x, y) \in C^\infty(]-T', T'[, \mathcal{S}'(\mathbb{R}_x^n \times \mathbb{R}_y^n))$, $\tilde{R}(x, y) \in \mathcal{S}'(\mathbb{R}_x^n \times \mathbb{R}_y^n)$. More precisely A_t can be expressed as a Fourier integral operator depending on the parameter $t \in]-T', T'[$ with phase function

$$\phi(t, x, \eta) = x\eta + \phi_0(t, x, \eta),$$

where ϕ_0 belongs to $C^\infty(]-T', T'[, S_{\mathcal{D}, \rho}^{2\rho})$, and amplitude $a(t, x, \eta) \in C^\infty(]-T', T'[, S_{\mathcal{D}, \rho}^{2\rho})$.

Remark 18. *We stress the fact that the phase function depends on the time. For this reason the results of Boggiato, Buzano and Rodino are more general but local, while with our approach we propose global results for a particular class of operators.*

5.2 Equivalent norms

One of the crucial ingredients of our proofs is the use of suitable new (semi)norms defined in $\mathcal{Q}^s(\mathbb{R}^n)$ and in $S_\mu^\mu(\mathbb{R}^n, \cdot)$ depending on the operator

$P(x, D)$. We are able to define the following norm in $Q^s(\mathbb{R}^n)$.

Definition 29. *If P is a self-adjoint globally elliptic operator of order $m > 0$, with $0 \notin \text{spec}(P)$ (i.e. P is invertible), we define*

$$\|u\|_{P,s}^2 = \|u\|_s^2 := \sum_{j=1}^{\infty} |\lambda_j|^{\frac{2s}{m}} |u_j|^2 < \infty, \quad (5.11)$$

where $\text{spec}(P) = \{\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots, \lim_{k \rightarrow \infty} \lambda_k = \infty\}$, and u_j are the Fourier coefficients.

Now we prove that the norm $\|\cdot\|_{P,s}$ is equivalent to the usual norm in $Q^s(\mathbb{R}^n)$.

Proposition 33. *Let $s \in \mathbb{R}$. Then $\|\cdot\|_{P,s}$ is equivalent to $\|\cdot\|_{Q^s(\mathbb{R}^n)}$, defined in (1.42), for every self-adjoint invertible operator P of order $m > 0$.*

Proof. It is well known (for more details we refer the reader to [58] and [50]), that $u \in Q^s(\mathbb{R}^n)$ if and only if $P^{\frac{s}{m}}u \in L^2(\mathbb{R}^n)$. Let $n \in \mathbb{N}$ then

$$\begin{aligned} \|u\|_{Q^s(\mathbb{R}^n)}^2 &= \sum_{|\alpha|+|\beta| \leq s} \|x^\beta D_x^\alpha u\|_{L^2(\mathbb{R}^n)}^2 = \sum_{|\alpha|+|\beta| \leq s} \langle x^\beta D_x^\alpha u, x^\beta D_x^\alpha u \rangle \\ &= \sum_{|\alpha|+|\beta| \leq s} \langle D_x^\alpha (x^{2\beta} D_x^\alpha u), u \rangle = \langle A_{2s} u, u \rangle. \end{aligned}$$

Hence there exists a positive operator $A_s := \sqrt{A_{2s}}$. Then

$$\begin{aligned} \|u\|_{Q^s(\mathbb{R}^n)}^2 &= \langle A_s u, A_s u \rangle = \langle A_{2s} u, u \rangle = \langle P^{\frac{2s}{m}} P^{-\frac{2s}{m}} A_{2s} u, u \rangle \\ &= \|b(x, D) P^{\frac{s}{m}} u\|_{L^2(\mathbb{R}^n)}^2 \leq C \|P^{\frac{s}{m}} u\|_{L^2(\mathbb{R}^n)}^2, \end{aligned}$$

where $b(x, D)$ is a pseudodifferential operator of order zero, with C standing for the operator norm of $b(x, D)$ in $L^2(\mathbb{R}^n)$, (see for [50], [58]). This concludes the proof. \square

We use this equivalent norm in the second section of this chapter. In fact, let $\delta > -\lambda_1$ (λ_1 is the smallest eigenvalue of the operator P), then $P + \delta$ is positive globally elliptic, invertible and $Q^s(\mathbb{R}^n)$ is defined, independently of $\delta > -\lambda_1$ cf. [58], as follows: $Q^s(\mathbb{R}^n)$ is the set of all $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|u\|_s^2 := \left\| (P + \delta)^{s/2} u \right\|_{L^2}^2 = \sum_{j=1}^{\infty} (\lambda_j + \delta)^s |u_j|^2 < +\infty. \quad (5.12)$$

Motivated by the study of the well-posedness of our Cauchy problem in the Gelfand–Shilov symmetric spaces, we introduce a new scale of Banach spaces depending on two parameters defining the Gelfand–Shilov space $S_\mu^\mu(\mathbb{R}^n)$, $\mu \geq 1/2$ using the operator $P + \delta$, $\delta > -\lambda_1$. We recall that, by Theorem 1.2 [24] on the characterization of the symmetric Gelfand–Shilov spaces $S_\mu^\mu(\mathbb{R}^n)$, we have: $u \in S_\mu^\mu(\mathbb{R}^n)$ iff for some $\varepsilon > 0$

$$\|u\|_{\mu;\varepsilon}^2 = \left\| e^{\varepsilon(P+\delta)^{1/(2\mu)}} u \right\|_{\mu;\varepsilon,P+\delta}^2 = \sum_{j=1}^{\infty} |u_j|^2 e^{2\varepsilon(\lambda_j+\delta)^{\frac{1}{2\mu}}} < +\infty. \quad (5.13)$$

We propose refinement of the semi-norms above, defining the Hilbert spaces $HS_\mu^\mu(\mathbb{R}^n : s, \varepsilon)$ as

Definition 30. We define the spaces $HS_\mu^\mu(\mathbb{R}^n : s, \varepsilon)$ as the set of all $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|u\|_{\mu;s,\varepsilon}^2 = \left\| (P + \delta)^{s/2} e^{\varepsilon(P+\delta)^{1/(2\mu)}} u \right\|_{L^2}^2 = \sum_{j=1}^{\infty} |u_j|^2 (\lambda_j + \delta)^s e^{2\varepsilon(\lambda_j+\delta)^{\frac{1}{2\mu}}} < +\infty, \quad (5.14)$$

with the inner product $\langle u, v \rangle = \sum_{j=1}^{\infty} u_j \bar{v}_j (\lambda_j + \delta)^s e^{2\varepsilon(\lambda_j+\delta)^{\frac{1}{2\mu}}}$.

One get readily the following properties:

Proposition 34. $HS_\mu^\mu(\mathbb{R}^n : s_1, \varepsilon_1) \hookrightarrow HS_\mu^\mu(\mathbb{R}^n : s_2, \varepsilon_2)$ iff $s_1 \geq s_2$, $\varepsilon_1 \geq \varepsilon_2$ and $S_\mu^\mu(\mathbb{R}^n)$ is double inductive limit of $HS_\mu^\mu(\mathbb{R}^n : s, \varepsilon)$ $\varepsilon \searrow 0$, $s \searrow -\infty$, i.e.,

$$\bigcup_{s \in \mathbb{R}, \varepsilon > 0} HS_\mu^\mu(\mathbb{R}^n : s, \varepsilon) = S_\mu^\mu(\mathbb{R}^n).$$

5.3 Well-posedness in Q^s

First we show global well-posedness for the unperturbed inhomogeneous Cauchy problem for $P(x, D)$

$$\begin{cases} \partial_t^2 u + P(x, D)u = f \in \bigcap_{j=0}^N C^j([0, +\infty[; Q^{s+1-j}(\mathbb{R}^n)), & t \geq 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0 \in Q^{s+1}(\mathbb{R}^n), & u_t(0, x) = u_1 \in Q^s(\mathbb{R}^n). \end{cases} \quad (5.15)$$

Set $J_{P\pm} := \{j \in \mathbb{N} : \pm\lambda_j > 0\}$, $J_P^0 := \{j \in \mathbb{N} : \pm\lambda_j = 0\}$. Clearly J_P^- and J_P^0 are finite or empty sets. We have

Theorem 51. *There exists a unique solution $u \in C^1([0, +\infty[; \mathcal{S}'(\mathbb{R}^n))$ of (5.15) defined by*

$$u = \partial_t S(t)[u_0] + S(t)[u_1] + \int_0^t \partial_t S(t - \tau)[f(\tau, \cdot)] d\tau, \quad (5.16)$$

where the Green function type operator $S(t) = S^-(t) + S^0(t) + S^+(t)$,

$$S^-(t)[g]_k = \sum_{j \in J_P^-} \frac{\sinh(\sqrt{-\lambda_j} t)}{\sqrt{-\lambda_j}} g_j \varphi_j(x), \quad (5.17)$$

$$S^0(t)[g]_k = t \sum_{j \in J_P^0} g_j \varphi_j(x), \quad (5.18)$$

$$S^+(t)[g]_k = \sum_{j \in J_P^+} \frac{\sin(\sqrt{\lambda_j} t)}{\sqrt{\lambda_j}} g_j \varphi_j(x), \quad (5.19)$$

with the convention $S^-(t) = 0$ (resp. $S^0(t) = 0$) if $\lambda_1 \geq 0$ (resp. $0 \notin \text{spec}(P)$).

In particular, if $u_0, u_1 \in \mathcal{S}(\mathbb{R}^n)$ and $f \in C^\infty([0, +\infty[; \mathcal{S}(\mathbb{R}^n))$,

then $u \in C^\infty([0, +\infty[; \mathcal{S}'(\mathbb{R}^n))$.

Proof. We consider the Fourier expansion $\sum_{j=1}^{\infty} u_j(t) \varphi_j$ (resp. $\sum_{j=1}^{\infty} f_j(t) \varphi_j$) in x of u (resp. f). Then our problem is reduced to the following systems of ODE:

$$\begin{cases} \ddot{u}_j(t) + \lambda_j u_j(t) = f_j(t), \\ u_j(0) = u_{0,j}, & \dot{u}_j(0) = u_{1,j}. \end{cases} \quad (5.20)$$

The solutions of (5.20) are written explicitly as follows:

$$\begin{aligned} u_j(t) &= u_{0,j} \cosh\left(\sqrt{-\lambda_j}t\right) + u_{1,j} \frac{\sinh\left(\sqrt{-\lambda_j}t\right)}{\sqrt{-\lambda_j}} \\ &+ \int_0^t \frac{\sinh\left(\sqrt{-\lambda_j}(t-\tau)\right)}{\sqrt{-\lambda_j}} f_j(\tau) d\tau, \end{aligned} \quad (5.21)$$

if $J_p^- \neq \emptyset$, $j \in J_p^-$ (i.e., $\lambda_j < 0$).

$$u_j(t) = u_{0,j} + u_{1,j}t + \int_0^t (t-\tau)f_j(\tau) d\tau \quad (5.22)$$

provided $J_p^0 \neq \emptyset$, $j \in J_p^0$ (i.e., $\lambda_j = 0$).

$$\begin{aligned} u_j(t) &= u_{0,j} \cos\left(\sqrt{\lambda_j}t\right) + u_{1,j} \frac{\sin\left(\sqrt{\lambda_j}t\right)}{\sqrt{\lambda_j}} + \\ &+ \int_0^t \frac{\sin\left(\sqrt{\lambda_j}(t-\tau)\right)}{\sqrt{\lambda_j}} f_j(\tau) d\tau, \end{aligned} \quad (5.23)$$

for $j \in J_p^+$ (i.e., $\lambda_j > 0$). Clearly (5.21), (5.22), (5.23) yield (5.17), (5.18), (5.19). \square

Set $u^{hom} = \partial_t \mathcal{S}(t)[u_0] + \mathcal{S}(t)[u_1]$, $u^{inh} = \int_0^t \partial_t \mathcal{S}(t-\tau)[f(\tau, \cdot)] d\tau$, and $U^{hom} = (u^{hom}, \partial_t u^{hom})$, $U^{inh} = (u^{inh}, \partial_t u^{inh})$.

Now we are able to enunciate the main result of this section. We note that this result recaptures a particular case of the global well-posedness results in [52] and derives new precise estimates and conservation of energy type estimates in the chosen norm for $\mathcal{Q}^s(\mathbb{R}^n)$, defined by the powers of $P + \delta$. We recall that a crucial ingredient of the following proofs is the choice of the norms (5.12).

Theorem 52. *There exist constants $c_0 \geq 0$, $C_0 > 0$, depending on λ_1 , δ and the norm $R_{\mathcal{Q}^{s+1}(\mathbb{R}^n) \rightarrow \mathcal{Q}^s(\mathbb{R}^n)}$ such that for every $u_j \in \mathcal{Q}^{s+1-j}(\mathbb{R}^n)$, $j = 0, 1$, one*

can find a unique solution $u \in \bigcap_{k=0}^1 C^k([0, +\infty[: \mathcal{Q}^{s+1-k}(\mathbb{R}^n))$ of (5.1) satisfying the energy estimate

$$\|u(t, \cdot)\|_{s+1}^2 + \|u_t(t, \cdot)\|_s^2 \leq C_0 e^{c_0 t} (\|u_0\|_{s+1}^2 + \|u_1\|_s^2). \quad (5.24)$$

Moreover,

$$u \in \bigcap_{k=0}^{\infty} C^k([0, +\infty[: \mathcal{Q}^{s+1-k}(\mathbb{R}^n)). \quad (5.25)$$

which implies the well-posedness in $\mathcal{S}(\mathbb{R}^n)$. Finally, if $R(x, D) = 0$, $\lambda_1 > 0$ and $\delta = 0$, we have the conservation of energy type phenomenon, namely

$$\|u(t, \cdot)\|_{s+1}^2 + \|u_t(t, \cdot)\|_s^2 = \|u_0\|_{s+1}^2 + \|u_1\|_s^2, \quad t \geq 0. \quad (5.26)$$

The proof of this theorem is given by the following propositions. The next assertion implies the proof of the first result in the unperturbed case, $R = 0$.

Proposition 35. Set $C_\delta := \sup_{j \in J_P^- \cup J_P^+} \frac{\lambda_j + \delta}{|\lambda_j|} < +\infty$. Then we have

$$\begin{aligned} \|u^{hom}(t)\|_{s+1}^2 + \|\partial_t u^{hom}(t)\|_s^2 &\leq \max\{2C_\delta, e^{2\sqrt{-\lambda_1}t}, (2+t)^2, C_\delta\} \\ &\times (\|u_0\|_{s+1}^2 + \|u_1\|_s^2), \quad t \geq 0. \end{aligned} \quad (5.27)$$

Proof. We note that

$$U_j^{hom}(t) = \begin{pmatrix} \cosh(\sqrt{-\lambda_j}t) & \frac{\sinh(\sqrt{-\lambda_j}t)}{\sqrt{-\lambda_j}} \\ \sqrt{-\lambda_j} \sinh(\sqrt{-\lambda_j}t) & \cosh(\sqrt{-\lambda_j}t) \end{pmatrix} \begin{pmatrix} u_{0,j} \\ u_{1,j} \end{pmatrix}, \quad (5.28)$$

for $j \in J_P^-$ when $J_P^- \neq \emptyset$;

$$U_j^{hom}(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_{0,j} \\ u_{1,j} \end{pmatrix}, \quad (5.29)$$

for $j \in J_P^0$ if $J_P^0 \neq \emptyset$; and

$$U_j^{hom}(t) = \begin{pmatrix} \cos(\sqrt{\lambda_j}t) & \frac{\sin(\sqrt{\lambda_j}t)}{\sqrt{\lambda_j}} \\ \sqrt{\lambda_j} \sin(\sqrt{\lambda_j}t) & \cos(\sqrt{\lambda_j}t) \end{pmatrix} \begin{pmatrix} u_{0,j} \\ u_{1,j} \end{pmatrix}, \quad (5.30)$$

for $j \in J_P^+$. Given $v \in \mathbb{C}^2$, $\lambda > 0$ we set

$$\|v\|_{s;\lambda}^2 = \lambda^{s+1}|v_1|^2 + \lambda^s|v_2|^2. \quad (5.31)$$

Then one observes that (5.28), (5.29), (5.30) lead to

$$\begin{aligned} \left\| U_j^{hom}(t) \right\|_{s;\lambda_j+\delta}^2 &\leq 2C_\delta e^{2\sqrt{-\lambda_j}t} \left((\lambda_j + \delta)^{s+1} |u_{0,j}|^2 \right. \\ &\quad \left. + (\lambda_j + \delta)^s |u_{1,j}|^2 \right), \end{aligned} \quad (5.32)$$

for $j \in J_P^-$ when $J_P^- \neq \emptyset$,

$$\left\| U_j^{hom}(t) \right\|_{s;\delta}^2 \leq (2+t)^2 \left(\delta^{s+1} |u_{0,j}|^2 + \delta^s |u_{1,j}|^2 \right), \quad (5.33)$$

for $j \in J_P^0$ if $J_P^0 \neq \emptyset$, and

$$\left\| U_j^{hom}(t) \right\|_{s;\lambda_j+\delta}^2 \leq C_\delta \left((\lambda_j + \delta)^{s+1} |u_{0,j}|^2 + (\lambda_j + \delta)^s |u_{1,j}|^2 \right), \quad (5.34)$$

for $j \in J_P^+$.

In particular, taking into account that the rotation matrices preserve the Euclidean norm in \mathbb{R}^2 , if $\delta = 0$ in (5.34) we obtain the equality

$$\left\| U_j^{hom}(t) \right\|_{s;\lambda_j}^2 = \lambda_j^{s+1} |u_{0,j}|^2 + \lambda_j^s |u_{1,j}|^2, \quad j \in J_P^+. \quad (5.35)$$

Summation in j completes the proof. \square

In order to show the global well-posedness for the perturbed operator $P(x, D) + R(x, D)$ we need precise estimates on U^{inh} . Without loss of generality we may assume $\lambda_1 > 0$, and we write $P(x, D) + R(x, D) = P(x, D) + \delta + (R(x, D) - \delta)$. We point out the fact that $R(x, D) - \delta$ remains first order p.d.o. perturbation.

Proposition 36. *Suppose that $f \in \bigcap_{j=0}^N([0, +\infty[: \mathcal{Q}^{s-j}(\mathbb{R}^n))$ for some $N \geq 0$. Then $u^{inh} \in \bigcap_{j=0}^{N+2}([0, +\infty[: \mathcal{Q}^{s-j}(\mathbb{R}^n))$ and one can find $c_1 > 0$ such that the following estimates hold*

$$\left\| u^{inh}(t) \right\|_{s+1}^2 + \left\| \partial_t u^{inh}(t) \right\|_s^2 \leq c_1 \int_0^t \|f(\tau, \cdot)\|_s^2 d\tau, \quad t \geq 0. \quad (5.36)$$

Proof. We use the the assumption $\lambda - 1 > 0$ and apply the same arguments as in the previous proposition combined with the Schwartz inequality. \square

Now we conclude the proof for the well-posedness in the spaces $\mathcal{Q}^s(\mathbb{R}^n)$ for the perturbed equation. We write $u = u^{hom} + v$, $g := R(x, D)u^{hom}$, and reduce the perturbed Cauchy problem to

$$\begin{cases} \partial_t^2 v + P(x, D)v = f := g - R(x, D)v, & t \geq 0, x \in \mathbb{R}^n, \\ v(0, x) = 0, & v_t(0, x) = 0. \end{cases} \quad (5.37)$$

One notes that since $R(x, D)$ is first order Shubin p.d.o. and $u^{hom} \in \bigcap_{k=0}^{\infty} C^k([0, +\infty[: \mathcal{Q}^{s+1-k}(\mathbb{R}^n))$ we obtain that the source term

$$g = R(x, D)u^{hom} \in \bigcap_{k=0}^{\infty} C^k([0, +\infty[: \mathcal{Q}^{s-k}(\mathbb{R}^n)).$$

Moreover, if the unknown solution v is required to be in $\bigcap_{k=0}^{\infty} C^k([0, +\infty[: \mathcal{Q}^{s+1-k}(\mathbb{R}^n))$, then $R(x, D)v \in \bigcap_{k=0}^{\infty} C^k([0, +\infty[: \mathcal{Q}^{s-k}(\mathbb{R}^n))$ as well. Now we apply the Green type function $S(t - \tau)$ and reduce (5.37) to

$$v(t, \cdot) = \int_0^t S(t - \tau)[g(\tau, \cdot)]d\tau + \int_0^t S(t - \tau)[R(x, D)]v(\tau, \cdot)d\tau, \quad t \geq 0, \quad (5.38)$$

The first term $h(t, x) := \int_0^t S(t - \tau)[g(\tau, \cdot)]d\tau$ belongs to $\bigcap_{k=0}^{\infty} C^k([0, +\infty[: \mathcal{Q}^{s+1-k}(\mathbb{R}^n))$ while the linear operator

$$K[v](t, x) := \int_0^t S(t - \tau)[R(x, D)]v(\tau, \cdot)d\tau, \quad t \geq 0, \quad (5.39)$$

satisfies for some $c_2 > 0$ depending only on the norm $\left\| R(x, D)_{\mathcal{Q}^{s+1}(\mathbb{R}^n) \rightarrow \mathcal{Q}^s(\mathbb{R}^n)} \right\|$ the following estimates

$$\sum_{\ell=0}^1 \left\| \partial_t^\ell K[v](t, \cdot) \right\|_{s+1-\ell} \leq c_2 \sum_{\ell=0}^1 \int_0^t \|v(\tau, \cdot)\|_{s+1-\ell} d\tau, \quad t \geq 0, \quad (5.40)$$

for all $v \in \bigcap_{\ell=0}^1 C^\ell([0, +\infty[: \mathcal{Q}^{s+1-\ell}(\mathbb{R}^n))$.

We write the following Picard scheme

$$v_{j+1}(t, \cdot) = h(t, \cdot) + K[v_j](t, \cdot), \quad t \geq 0, j = 0, 1, \dots, v_0 = 0, \quad (5.41)$$

In fact we have

$$\begin{aligned} \|v_{j+1} - v_j\|_{s+1, P} &= \|K[v_j](t, \cdot) - K[v_{j-1}](t, \cdot)\|_{s+1, P} \leq C_\delta \|K[v_j - v_{j-1}]\|_{s+1, P} \\ &= C_\delta \left\| \int_0^t S(t - \tau) [R(x, D)](v_j - v_{j-1}) d\tau \right\|_{s+1, P} \\ &\leq C_0 e^{2\sqrt{-\lambda_1}t} \left\| \int_0^t R(x, D)(v_j - v_{j-1}) d\tau \right\|_{s+1, P} \\ &\leq C_0 e^{2\sqrt{-\lambda_1}t} \kappa_R \int_0^t \|v_j(t - \tau) - v_{j-1}(t - \tau)\|_{s+1, P} d\tau \\ &\leq \dots \leq \tilde{C}_0 e^{2\sqrt{-\lambda_1}t} \tilde{\kappa}_R \int_0^t \|v_1(t - \tau)\|_{s+1, P} d\tau \leq C \frac{t^j}{j!} \end{aligned} \quad (5.42)$$

Moreover from the same reasons we have

$$\begin{aligned} \|\partial_t(v_{j+1} - v_j)\|_{s, P} &\leq C_\delta \left\| \int_0^t \partial_t S(t - \tau) [R(x, D)](v_j - v_{j-1}) d\tau \right\|_{s, P} \\ &C_0 \sqrt{-\lambda_1} e^{\sqrt{-\lambda_1}t} \kappa_R \int_0^t \|v_j - v_{j-1}\|_{s+1} d\tau \leq C \frac{t^j}{j!} \end{aligned} \quad (5.43)$$

Hence we have shown the convergence of v_j in $\bigcap_{\ell=0}^1 C^\ell([0, +\infty[: \mathcal{Q}^{s+1-\ell}(\mathbb{R}^n))$ to a solution v of the perturbed Cauchy problem, taking into account (5.40) and applying the Gronwall inequality. We obtain that $v \in \bigcap_{\ell=0}^\infty C^\ell([0, +\infty[: \mathcal{Q}^{s+1-\ell}(\mathbb{R}^n))$ via standard regularity methods for second order hyperbolic equations.

Remark 19. We point out the fact that the solution of our problem is well-posed for all $t > 0$, while in the book of Boggiatto, Buzano and Rodino [3], the Cauchy problem is studied only for $t \ll 1$.

Remark 20. We note that it is an interesting problem to study second order hyperbolic equations for Shubin type operators, depending on the time variable, and to find global analogues to local results for second order hyperbolic equations, either strictly hyperbolic but with non smooth time depending coefficients and/or weakly hyperbolic equations (e.g. see the recent paper [19] and the references therein).

5.4 Well-posedness in S_μ^μ

In this section we prove the global well-posedness in Gelfand–Shilov spaces.

Theorem 53. Let P be differential operator and $R = 0$. Then Cauchy problem is globally well posed in $S_\mu^\mu(\mathbb{R}^n)$ for $\mu \geq 1/2$.

Actually, we demonstrate an assertion that is more general, and we establish precise estimates using suitable semi norms (5.14) in $S_\mu^\mu(\mathbb{R}^n)$ defined by P . We have

Theorem 54. Consider the Cauchy problem

$$\begin{cases} \partial_t^2 u + P(x, D)u = 0, & t \in \mathbb{R}, x \in \mathbb{R}^n, \\ u(0, x) = u_0 \in HS_\mu^\mu(\mathbb{R}^n : s + 1, \varepsilon), & u_t(0, x) = u_1 \in HS_\mu^\mu(\mathbb{R}^n : s, \varepsilon), \end{cases} \quad (5.44)$$

for some $s \in \mathbb{R}$, $\varepsilon > 0$. Then there exists a unique solution u of (5.44) belonging to $\bigcap_{k=0}^1 C^k([0, +\infty[; HS_\mu^\mu(\mathbb{R}^n : s + 1 - k, \varepsilon))$ and satisfying for some $c_0 \geq 0$, $C_0 \geq 1$ the estimates

$$\|u(t, \cdot)\|_{\mu; s+1, \varepsilon}^2 + \|u_t(t, \cdot)\|_{\mu; s, \varepsilon}^2 \leq C_0 e^{c_0 t} (\|u_0\|_{\mu; s+1, \varepsilon}^2 + \|u_1\|_{\mu; s, \varepsilon}^2), \quad t \geq 0. \quad (5.45)$$

Moreover, we have the regularizing result

$$u \in \bigcap_{k=0}^{\infty} C^k([0, +\infty[: HS_\mu^\mu(\mathbb{R}^n : s+1-k, \varepsilon)) \subset C^\infty([0, +\infty[: S_\mu^\mu(\mathbb{R}^n)).$$

Finally, if $\lambda_1 > 0$ and $\delta = 0$, we have the conservation of energy type phenomenon, namely

$$\|u(t, \cdot)\|_{\mu; s+1, \varepsilon}^2 + \|u_t(t, \cdot)\|_{\mu; s, \varepsilon}^2 = \|u_0\|_{\mu; s+1, \varepsilon}^2 + \|u_1\|_{\mu; s, \varepsilon}^2, \quad t \geq 0. \quad (5.46)$$

Proof. The choice of the Banach spaces $HS_\mu^\mu(\mathbb{R}^n : s+1-k, \varepsilon)$ depending on two parameters allows a simple proof as in the $Q^s(\mathbb{R}^n)$ framework using homogeneity arguments. Indeed, multiplying in (5.32), (5.33), (5.34) by $e^{\varepsilon(\lambda_j + \delta)^{1/(2\mu)}}$ we obtain

$$\begin{aligned} \left\| U_j^{hom}(t) \right\|_{s; \lambda_j + \delta}^2 e^{2\varepsilon(\lambda_j + \delta)^{1/(2\mu)}} &\leq 2C_\delta e^{2\sqrt{-\lambda_j}t} e^{2\varepsilon(\lambda_j + \delta)^{1/(2\mu)}} \\ &\times ((\lambda_j + \delta)^{s+1} |u_{0,j}|^2 + (\lambda_j + \delta)^s |u_{1,j}|^2), \end{aligned} \quad (5.47)$$

for $j \in J_P^-$ when $J_P^- \neq \emptyset$,

$$\begin{aligned} \left\| U_j^{hom}(t) \right\|_{s; \delta}^2 e^{2\varepsilon\delta^{1/(2\mu)}} &\leq (2+t)^2 e^{2\varepsilon\delta^{1/(2\mu)}} \\ &\times ((\lambda_j + \delta)^{s+1} |u_{0,j}|^2 + \delta^s |u_{1,j}|^2), \end{aligned} \quad (5.48)$$

for $j \in J_P^0$ if $J_P^0 \neq \emptyset$,

$$\begin{aligned} \left\| U_j^{hom}(t) \right\|_{s; \lambda_j + \delta}^2 e^{2\varepsilon(\lambda_j + \delta)^{1/(2\mu)}} &\leq C_\delta e^{2\varepsilon(\lambda_j + \delta)^{1/(2\mu)}} \\ &\times ((\lambda_j + \delta)^{s+1} |u_{0,j}|^2 + (\lambda_j + \delta)^s |u_{1,j}|^2), \end{aligned} \quad (5.49)$$

for $j \in J_P^+$ and similarly with the conservation of energy if $\lambda_1 > 0$, $\delta = 0$.

$$\begin{aligned} \left\| U_j^{hom}(t) \right\|_{s; \lambda_j}^2 e^{2\varepsilon(\lambda_j)^{1/(2\mu)}} &= (\lambda_j)^{s+1} |u_{0,j}|^2 \\ &+ (\lambda_j)^s |u_{1,j}|^2 e^{2\varepsilon(\lambda_j)^{1/(2\mu)}}, \end{aligned} \quad (5.50)$$

for $j \in N$. Summation in j leads to the end of the proof. \square

5.5 The twisted Laplacian case

We recall that both L_1 and L_2 are essentially self-adjoint, their spectrum is given by a sequence of eigenvalues, which are odd natural numbers. It should be noted, however, that each eigenvalue has infinite multiplicity. Moreover, L_1 and L_2 are globally hypoelliptic in $\mathcal{S}(\mathbb{R}^2)$ and $S_\mu^\mu(\mathbb{R}^2)$, $\mu \geq 1/2$, cf. [16], [23]. On the other hand, it was shown in [23] that there exists a Fourier integral operator (FIO) J , associated to linear symplectic transformation in \mathbb{R}^4 ,

$$J(x) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{i|\omega|\Phi(x,\xi)} \hat{v}(\xi) d\xi, \quad (5.51)$$

where

$$\Phi(x, \xi) = \xi_1 \xi_2 + \frac{x_1 x_2}{2} + x_2 \xi_1 + x_1 \xi_2,$$

and J reduces L_k to simple normal form, the one dimensional harmonic oscillator in \mathbb{R}^2 , namely

$$J^{-1} \circ L_k \circ J v(y) = (\omega^2 D_{y_k}^2 + y_k^2) v(y), \quad y = (y_1, y_2) \in \mathbb{R}^2, \quad k = 1, 2, \quad (5.52)$$

with $k = 1$ if $\omega > 0$ and with $k = 2$ if $\omega < 0$, as in [23]. Moreover, KJ is an automorphism of $S(\mathbb{R}^2)$ and $S_\mu^\mu(\mathbb{R}^2)$, $\mu \geq 1/2$. Applying the same arguments, as in [23], one shows that J preserves $\mathcal{Q}^s(\mathbb{R}^2)$, $s \in \mathbb{R}$ as well. Therefore, we are reduced to the Cauchy problem for

$$(-\omega^2 \partial_{y_1}^2 + y_1^2) \otimes 1_{y_2}, \text{ if } \omega > 0 \quad \text{or} \quad 1_{y_1} \otimes (-\omega^2 \partial_{y_2}^2 + y_2^2), \text{ if } \omega < 0.$$

As in [24], we choose as orthonormal basis of eigenfunctions of $(-\partial_{y_1}^2 + y_1^2) \times 1_{y_2}$ the Hermite functions in \mathbb{R}^2 :

$$H_k(y) := H_{k_1}(y_1) H_{k_2}(y_2), \text{ with } (-\partial_{y_1}^2 + y_1^2) H_k(y) = (2k_1 + 1) H_k(y), \quad k \in \mathbb{Z}_+^2.$$

Now we are able to reduce the study of our Cauchy problem to the study of tensor type Cauchy problem, by using the reduction to global normal form.

Proposition 37. *Let $\omega > 0$ and set*

$$L = (D_{x_1} + \frac{\omega}{2}x_2)^2 + (D_{x_2} - \frac{\omega}{2}x_1)^2 + \alpha_1(D_{x_1} + \frac{\omega}{2}x_2) + \alpha_2(D_{x_2} - \frac{\omega}{2}x_1) + r,$$

with $\alpha_1, \alpha_2, r \in \mathbb{R}$. Then

i) *there exists a transformation*

$$\begin{aligned}\tilde{J} &= T_{\alpha_1/2} \circ E_{i\alpha_2/(2\omega)} \circ J, \\ T_{\alpha_1/2}v(x) &= v\left(x + \frac{\alpha_1}{2}\right) \\ E_{i\alpha_2/(2\omega)} &= e^{-i\frac{\alpha_2\omega}{2}x},\end{aligned}$$

with J is the transformation (5.51), such that

$$L = \tilde{J}^{-1} \circ \tilde{L} \circ \tilde{J},$$

where

$$\tilde{L} = \omega^2 D_{x_1}^2 + y_1^2 + \tilde{r}.$$

ii) *The transformation \tilde{J} is an isomorphism in $\mathcal{Q}^s(\mathbb{R}^n)$ and in $S_\mu^\mu(\mathbb{R}^n)$.*

Proof. i) We consider only the case with $\omega > 0$, the other case is similar.

$$L_1 = J \circ L = \omega^2 D_{y_1}^2 + y_1^2 + \alpha_1 y_1 + \alpha_2 D_{y_1} + r. \quad (5.53)$$

ii) \tilde{J} is an isomorphism because is a composition of isomorphisms in $\mathcal{Q}^s(\mathbb{R}^n)$ and in $S_\mu^\mu(\mathbb{R}^n)$.

□

We introduce scale of anisotropic Hilbert spaces defining $S_\mu^\mu(\mathbb{R}^2)$ with Shubin space index with respect to y_1 , namely $AHS_\mu^\mu(\mathbb{R}^2 : s_1, \varepsilon)$.

Definition 31. We define the spaces $AHS_\mu^\mu(\mathbb{R}^2 : s_1, \varepsilon)$ as the set of all $u \in S(\mathbb{R}^2)$, $u = \sum_{k \in \mathbb{Z}_+^2} u_k H_k(x)$ such that

$$\|u\|_{\mu; s_1, \varepsilon}^2 := \sum_{k \in \mathbb{Z}_+^2} |u_k|^2 (2k_1 + 1)^{s_1} e^{2\varepsilon|k|^{1/2\mu}} < +\infty. \quad (5.54)$$

These spaces satisfy the following properties:

Proposition 38. $AHS_\mu^\mu(\mathbb{R}^n : s_1, \varepsilon_2) \hookrightarrow AHS_\mu^\mu(\mathbb{R}^n : r_1, \varepsilon_2)$ if and only if $s_1 \geq r_1$, $\varepsilon_1 \geq \varepsilon_2$ and $S_\mu^\mu(\mathbb{R}^n)$ is double inductive limit of $AHS_\mu^\mu(\mathbb{R}^n : s_1, \varepsilon)$, $\varepsilon \searrow 0$, $s_1 \searrow -\infty$, i.e.,

$$\bigcup_{s_1 \in \mathbb{R}, \varepsilon > 0} AHS_\mu^\mu(\mathbb{R}^n : s_1, \varepsilon) = S_\mu^\mu(\mathbb{R}^n).$$

Without loss of generality, we may assume that $\omega = 1$, indeed using a dilation transformation. We are able to show the main result

Theorem 55. Consider the Cauchy problem

$$\begin{cases} \partial_t^2 v(t, y) + (-\partial_{y_1}^2 + y_1^2 + r)v(t, y) = 0 & t \in \mathbb{R}, y \in \mathbb{R}^2, \\ v(0, y) = v_0 \in AHS_\mu^\mu(\mathbb{R}^2 : s_1 + 1, \varepsilon), \quad u_t(0, y) = v_1 \in AHS_\mu^\mu(\mathbb{R}^2 : s_1, \varepsilon), \end{cases} \quad (5.55)$$

for some $s_1 \in \mathbb{R}$, $\varepsilon > 0$. Then there exists a unique solution

$$v \in \bigcap_{k=0}^1 C^k([0, +\infty[; AHS_\mu^\mu(\mathbb{R}^2 : s_1 + 1 - k, \varepsilon))$$

of (5.55) satisfying for some $c_0 \geq 0$, $C_0 \geq 1$ the energy estimate

$$\|v(t, \cdot)\|_{\mu; s_1+1, \varepsilon}^2 + \|v_t(t, \cdot)\|_{\mu; s_1, \varepsilon}^2 \leq C_0 e^{c_0 t} (\|v_0\|_{\mu; s_1+1, \varepsilon}^2 + \|v_1\|_{\mu; s_1, \varepsilon}^2). \quad (5.56)$$

Moreover $v \in \bigcap_{k=0}^\infty C^k([0, +\infty[; AHS_\mu^\mu(\mathbb{R}^2 : s_1 + 1 - k, \varepsilon)) \subset C^\infty([0, +\infty[; S_\mu^\mu(\mathbb{R}^2))$.

Next, if $\lambda > -1$ and $\delta = 0$, we have the conservation of energy type phenomenon, namely

$$\|v(t, \cdot)\|_{\mu; s_1+1, \varepsilon}^2 + \|v_t(t, \cdot)\|_{\mu; s_1, \varepsilon}^2 = \|v_0\|_{\mu; s_1+1, \varepsilon}^2 + \|v_1\|_{\mu; s_1, \varepsilon}^2, \quad t \geq 0. \quad (5.57)$$

Finally, the Cauchy problem is not well posed in $Q^s(\mathbb{R}^2)$.

Proof. We use exactly the same arguments as in the proof of Theorem 54 replacing the Fourier expansion $\sum_{j=1}^{\infty} u_j(t) \phi_j(x)$ with the double indexed Fourier expansion

$$v(t, y) = \sum_{k \in \mathbb{Z}_+^2} v_k(t) H_k(y),$$

and obtaining the following system of ODE

$$\begin{cases} \ddot{v}_k(t) + (2k_1 + 1 + r)v_k(t) = 0 \\ v_k(0) = v_{0,k}, \quad \dot{v}_k(0) = v_{1,k}. \end{cases} \quad (5.58)$$

For the sake of simplicity, we consider the case $r \geq 0$. Hence the solutions are written explicitly

$$v_k(t) = \cos(\sqrt{2k_1 + 1 + rt})v_{0,k} + \frac{\sin(\sqrt{2k_1 + 1 + rt})}{\sqrt{2k_1 + 1 + r}}v_{1,k}, \quad (5.59)$$

for $t \geq 0$, $k = (k_1, k_2) \in \mathbb{Z}_+^2$. Straightforward computations lead to the proof of the positive result.

As it concerns the non well-posedness in the scales $\mathcal{Q}^s(\mathbb{R}^n)$, in contrast to the globally elliptic case, we choose $v_0 = 0$ and $v_1 \in \mathcal{Q}^s(\mathbb{R}^2)$ in the following way: $v_{1,k} = 0$, for $k_1 \geq N + 1$, $N \in \mathbb{N}$ and

$$v_{1,k} = (2(k_1 + k_2) + 1)^{-s/2-1/2} \frac{1}{(\ln(2(k_1 + k_2) + 2))^{1/2+\delta_0}}, \quad k = (k_1, k_2) \in \mathbb{Z}_+^2, \quad (5.60)$$

$k_1 = 0, 1, \dots, n$ and $\delta_0 > 0$. It is easy to check that v_1 is in $\mathcal{Q}^s(\mathbb{R}^n)$

$$\begin{aligned} \|v_1\|_s^2 &= \sum_{k_1=0}^N \sum_{k_2=0}^{\infty} (2(k_1 + k_2) + 1)^{-1} \frac{1}{(\ln(2(k_1 + k_2) + 2))^{1+2\delta_0}} \\ &\leq (N+1) \sum_{k_2=0}^{\infty} \frac{1}{(2k_2 + 1)(\ln(2k_2 + 2))^{1+2\delta_0}} < \infty. \end{aligned}$$

Next, we show that the solution $v(t, y)$ defined by : $v_k(t) = 0$, for $k_1 \geq N + 1$

and

$$v_k(t) = (2k_1 + 1 + r)^{-1/2} (2(k_1 + k_2) + 1)^{-(s+1)/2} \frac{\sin((\sqrt{2k_1 + 1 + r})t)}{(\ln(2(k_1 + k_2) + 2))^{1/2 + \delta_0}}, \quad (5.61)$$

$t \in \mathbb{R}$, $k = (k_1, k_2) \in \mathbb{Z}_+^2$, $k_1 = 0, 1, \dots, N$, does not belong to $Q^{s+\eta}(\mathbb{R}^2)$, ($\eta > 0$)

for $0 < |t| < \frac{\pi}{2\sqrt{2N+1+r}}$. In fact

$$\begin{aligned} \|v\|_{s+\eta}^2 &= \sum_{k_1=0}^N \sum_{k_2=0}^{\infty} \|v_k\|^2 (2(k_1 + k_2) + 1)^{s+\eta} \\ &= \sum_{k_1=0}^N \frac{\sin^2((\sqrt{2k_1 + 1 + r})t)}{2k_1 + 1 + r} \sum_{k_2=0}^{\infty} \frac{(2(k_1 + k_2) + 1)^{s+\eta}}{(2(k_1 + k_2) + 1)^{s+1} (\ln(2(k_1 + k_2) + 2))^{1+2\delta_0}} \end{aligned}$$

Since for $0 < |t| < \frac{\pi}{2\sqrt{2N+1+r}}$ one has

$$\sin^2((\sqrt{2k_1 + 1 + r})t) \geq (2k_1 + 1)t^2 \cos^2((\sqrt{2k_1 + 1 + r})t) \geq \cos^2(\sqrt{2N + r + 1})t^2 > 0,$$

we have

$$\begin{aligned} \|v\|_{s+\eta}^2 &\geq \cos^2((\sqrt{2N + 1 + r})t) t^2 \sum_{k_1=0}^N \sum_{k_2}^{\infty} \frac{1}{(2(k_1 + k_2) + 1)^{1-\eta}} \frac{1}{(\ln(2(k_1 + k_2) + 2))^{1+2\delta_0}} \\ &\geq (N + 1) \cos^2((\sqrt{2N + 1 + r})t) t^2 \sum_{k_2}^{\infty} \frac{1}{(2(N + k_2) + 1)^{1-\eta}} \frac{1}{(\ln(2(N + k_2) + 2))^{1+2\delta_0}} \\ &\geq (N + 1) \cos^2((\sqrt{2N + r + 1})t) t^2 c_{\varepsilon, N} \sum_{\ell=0}^{\infty} \frac{1}{(2\ell + 2N + 1)^{1-\eta+\varepsilon}} = \infty \end{aligned}$$

if $\eta > \varepsilon > 0$ and $c_{\varepsilon, N} = \inf_{\ell \geq 0} \left(\frac{(2\ell + 2N + 1)^\varepsilon}{(\ln(2\ell + 2N + 2))^{1+2\delta_0}} \right) > 0$.

So $v(t) \notin Q^{s+\eta}(\mathbb{R}^2)$, for $0 < |t| < \frac{\pi}{2\sqrt{2N+1+r}}$, in fact the arguments of the proof imply that $v(t) \notin Q^{s+\eta}(\mathbb{R}^2)$ for all $t \neq 0$. \square

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