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Boundary control and observation of coupled parabolic PDEs

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. . . alla mia famiglia . . .

Abstract

Reaction-diffusion equations are parabolic Partial Differential Equations (PDEs) which often occur in practice, e.g., to model the concentration of one or more substances, distributed in space, under the influence of different phenomena such as local chemical reactions, in which the substances are transformed into each other, and diffusion, which causes the substances to spread out over a surface in space. Certainly, reaction-diffusion PDEs are not confined to chemical applications but they also describe dynamical processes of non-chemical nature, with examples being found in thermodynamics, biology, geology, physics, ecology, etc.

Problems such as parabolic Partial Differential Equations (PDEs) and many others require the user to have a considerable background in PDEs and functional analysis before one can study the control design methods for these systems, particularly boundary control design.

Control and observation of coupled parabolic PDEs comes in roughly two settings-depending on where the actuators and sensors are located “in domain” control, where the actuation penetrates inside the domain of the PDE system or is evenly distributed everywhere in the domain and “boundary” control, where the actuation and sensing are applied only through the boundary conditions.

Boundary control is generally considered to be physically more realistic because actuation and sensing are nonintrusive but is also generally considered to be the harder problem, because the “input operator” and the ”output operator” are unbounded operators.

The method that this thesis develops for control of PDEs is the so-called backstepping control method. Backstepping is a particular approach to stabilization of dynamic systems and is particularly successful in the area of nonlinear control. The backstepping method achieves Lyapunov stabilization, which is often achieved by collectively shifting all the eigenvalues in a favorable direction in the complex plane, rather than by assigning individual eigenvalues. As the reader will soon learn, this task can be achieved in a rather elegant way, where the control gains are easy to compute symbolically, numerically, and in some cases even explicitly.

In addition to presenting the methods for boundary control design, we present the dual methods for observer design using boundary sensing. Virtually every one of our control designs for full state stabilization has an observer counterpart. The observer gains are easy to compute symbolically or even explicitly in some cases. They are designed in such a way that the observer error system is exponentially stabilized. As in the case of finite-dimensional observer-based control, a separation principle holds in the sense that a closed-loop system remains stable after a full state stabilizing feedback is replaced by a feedback that employs the observer state instead of the plant state.

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Introduction

This introductory Chapter is intended to present the motivations behind the development of this Thesis along with a brief description of the Thesis' structure. Finally, a list of the Author's publications derived from the present work are listed.

Motivations

Reaction-diffusion equations are parabolic Partial Differential Equations (PDEs) which often occur in practice, e.g., to model the concentration of one or more substances, distributed in space, under the influence of different phenomena such as local chemical reactions, in which the substances are transformed into each other, and diffusion, which causes the substances to spread out over a surface in space. Certainly, reaction-diffusion PDEs are not confined to chemical applications (see e.g. [20]), but they also describe dynamical processes of non-chemical nature, with examples being found in thermodynamics, biology, geology, physics, ecology, etc. (see e.g. [60, 61]).

In the present work, the problems of stabilization and observation are considered for several classes of linear reaction-diffusion PDEs, including the challenging scenario of coupled PDEs provided that only boundary information is available for measurements. A preliminary result involving the boundary stabilization of coupled reaction-diffusion-advection equations is also provided.

The adopted treatment does not rely on any discretization or finite-dimensional approximation of the underlying PDEs and it preserves the infinite-dimensional structure of the system during the entire design process. The proposed synthesis is mainly based on the so-called "backstepping" approach [13]. Basically, the backstepping approach deals with an invertible Volterra integral transformation, mapping the system dynamics onto a predefined exponentially stable target dynamics. Backstepping is a versatile and powerful approach to boundary control and observer design, applicable to a broad spectrum of linear PDEs, and under certain circumstances controllers and observers are derived in explicit forms [13].

The backstepping-based boundary control of scalar reaction-diffusion processes was studied, e.g., in [17], [67] whereas scalar wave processes were studied, e.g., in [14], [58]. Complex-valued PDEs such as the Schrodinger equation were dealt with by means of such an approach [16]. Synergies between the backstepping methodology and the flatness approach were exploited in [18], [19] to control parabolic PDEs with spatially and time-varying coefficients in spatial domains of dimension 2 and higher. In addition, an interesting feature of backstepping is that it admits a synergic integration with robust control paradigms such as the sliding mode control methodology (see, e.g., [10]).

The implementation of backstepping controllers usually requires the full state information. From the practical standpoint, the available measurements of Distributed Parameter Systems (DPSs) are typically located at the boundary of the spatial domain, that motivates the need of the state observer design [46, 70]. For linear infinite-dimensional systems, the Luenberger observer theory was established by replacing matrices with linear operators [69, 72, 70], and the observer design was confined to determining a gain operator that stabilizes the associated observation error dynamics. In contrast to finite-dimensional systems, finding such a gain operator was not trivial even numerically because operators were not generally represented with a finite number of parameters.

Observer design methods that would be capable of yielding the observer gains in the analytical form have only recently been investigated. In this context, the backstepping method appears to be a particularly effective systematic observer design approach [13, 43]. For scalar systems governed by parabolic PDEs defined on a 1-dimensional (1D) spatial domain, a systematic observer design approach, using boundary sensing, is introduced in [43]. Recently, the backstepping-based observer design was presented in [44] for reaction-diffusion processes with spatially-varying reaction coefficients while measuring a certain integral average value of the state of the plant. In [71, 11], backstepping-based observer design was addressed for reaction-diffusion processes evolving in multi-dimensional spatial domains.

More recently, high-dimensional systems of *coupled* PDEs were considered in the backstepping boundary control and observer design settings. The most intensive efforts of current literature were oriented towards coupled hyperbolic processes of the transport-type [5, 8, 37, 29, 30].

In [5], a 2×2 linear hyperbolic system was stabilized by a scalar observer-based output-feedback boundary control input, with an additional feature that an unmatched disturbance, generated by an *a-priori* known exosystem, was rejected. In [29], a 2×2 system of coupled linear heterodirectional hyperbolic equations was stabilized by observer-based output feedback. The underlying design was extended in [8] to a particular type of 3×3 linear systems, arising in modeling of multiphase flow, and to the quasilinear case in [30]. In [37], backstepping observer-based output-feedback design was presented for a system of $n + 1$ coupled first-order linear heterodirectional hyperbolic PDEs (n of which featured rightward convecting transport, and one leftward) with a single boundary input.

Some specific results on the backstepping based boundary stabilization of parabolic coupled PDEs have additionally been presented in the literature [6, 28, 75, 32, 33]. In [28], two parabolic reaction-diffusion processes, coupled through the corresponding boundary conditions, were dealt with. The stabilization of the coupled equations was reformulated in terms of the stabilization problem for a unique process, which possessed piecewise-continuous diffusivity and (space-dependent) reaction coefficient and which was viewed as the "cascade" between the two original systems. The problem was then solved by using a scalar boundary control input and by employing a non conventional backstepping approach with a discontinuous kernel function. In [6], the Ginzburg-Landau equation with the imaginary and real parts expanded, thus being specified to a 2×2 parabolic system with equal diffusion coefficients, was dealt with. In [75], the linearized 2×2 model of thermal-fluid convection was treated by using a singular perturbations approach combined with backstepping and Fourier series expansion. In [33], the boundary stabilization of the linearized model of an incompressible magnetohydrodynamic flow in an infinite rectangular 3D channel, also recognized as Hartmann flow, was achieved by reducing the original system to a set of coupled diffusion equations with the same diffusivity parameter and by applying backstepping. In [32], an observer that estimated the velocity, pressure, electric potential and current fields in a Hartmann flow was presented where the observer gains were designed using multi-dimensional backstepping. In [41], a backstepping observer was designed for a system of two diffusion-convection-reaction processes coupled through the corresponding boundary conditions.

Thus motivated, the primary concern of this work is to extend the backstepping synthesis developed in [13] [43] for scalar unstable reaction-diffusion processes.

A constructive observer-based output-feedback synthesis procedure, with all controllers and observers given in explicit form, presents the main contribution of this thesis. This generalization is far from being trivial because the underlying backstepping-based treatment gives rise to more complex development of finding out an analytical solution in the form of Bessel-like matrix series.

Thesis' overview

The Thesis is organized into two distinct parts. The first one, namely **State of the Art** which provide a brief summary of the necessary theoretical notions useful for understanding the remainder the Thesis, and the second one, namely **Author's contribution** in which all the Author's works, developed during this research, are deeply discussed. A brief overview of each Chapter of the Thesis is reported below.

- Part **I. State of the Art:**
  - **Chapter 1. Sate Feedback Control**.
    In this Chapter a problem of boundary stabilization of a class of linear parabolic partial integro-differential equations (P(I)DEs) in one dimension is considered using the method of backstepping. The problem is formulated as a design of an integral operator whose kernel is required to satisfy a hyperbolic P(I)DE. The kernel P(I)DE is then converted into an equivalent integral equation and by applying the method of successive approximations, the equation's well posedness and the kernel's smoothness are established. For one particular case feedback laws are constructed explicitly and the closed-loop solutions are found in closed form. Also a brief excursus of the related literarture of the problem of boundary stabilization for coupled reaction-diffusion Partial Differential Equations (PDEs) is illustrated.
  - **Chapter 2. State Observation**.
    In this Chapter we design exponentially convergent observers for a class of parabolic partial integro-differential equations (P(I)DEs) with only boundary sensing available. The problem is posed as a problem of designing an invertible coordinate transformation of the observer error system into an exponentially stable target system. Observer gain (output injection function) is shown to satisfy a well-posed hyperbolic PDE that is closely related to the hyperbolic PDE governing backstepping control gain for the state-feedback problem. For one problemm the observer gains are obtained in closed form. Also a brief excursus of the related literarture of the problem of observation for coupled reaction-diffusion Partial Differential Equations (PDEs) is illustrated.
  - **Chapter 3. Output Feedback**.
    The observer gains calculated in the previous section are then used for an output-feedback design in anti-collocated setting of sensor and actuator. Explicit solutions of a closed loop system is found in a particular case. Also a brief excursus of the related literarture of the problem of output feedback for reaction-diffusion Partial Differential Equations (PDEs) is illustrated.
- Part **II. Author's contribution:**
  - **Chapter 1. Scalar Reaction-Diffusion Equation**.

In this chapter we study the stabilization problem in the space $L_2(0, 1)$ for a class of parabolic PDEs of the reaction-diffusion type equipped with destabilizing Robin-type boundary conditions. The considered class of PDEs is also affected by a matching boundary disturbance with an a-priori known constant upperbound to its magnitude. The problem is solved by means of a suitable synergic combination between the infinite-dimensional backstepping methodology and the sliding mode control approach. A constructive Lyapunov analysis supports the presented synthesis, and simulation results validate the developed technique.

Also we give an overview of the available results and methods in the field of second-order sliding mode based boundary control synthesis for uncertain and perturbed distributed parameter systems. We particularly aim at showing how the same basic algorithm (the combined Twisting/PD algorithm) can be applied to solve different problems involving parabolic and hyperbolic-type equations. Then, we deal with a reaction-diffusion process by also providing some novelty in that a destabilizing mixed-type boundary condition, which was not considered in the previous work [25], is taken into account. The effectiveness of the developed controller is supported by simulation results.

– **Chapter 2. Coupled Reaction-Diffusion Equation**.

In this chapter the problem of boundary stabilization is considered for some classes of coupled parabolic linear PDEs of the reaction-diffusion type. With reference to n coupled equations, each one equipped with a scalar boundary control input, a state feedback law is designed with actuation at only one end of the domain, and exponential stability of the closed-loop system is proven. The treatment is addressed separately for the case in which all processes have the same diffusivity and for the more challenging scenario where each process has its own diffusivity and a different solution approach has to be taken. The backstepping method is used for controller design, and, particularly, the kernel matrix of the transformation is derived in explicit form of series of Bessel-like matrix functions by using the method of successive approximations to solve the corresponding PDE. Thus, the proposed control laws become available in explicit form. Additionally, the stabilization of an underactuated system of two coupled reaction-diffusion processes is tackled under the restriction that only a scalar boundary input is available. Capabilities of the proposed synthesis and its effectiveness are supported by numerical studies made for three coupled systems with distinct diffusivity parameters and for underactuated linearized dimensionless temperature-concentration dynamics of a tubular chemical reactor, controlled through a boundary at low fluid superficial velocities when convection terms become negligible.

Also the state observation problem is tackled for a system of n coupled reaction-diffusion PDEs, possessing the same diffusivity parameter and equipped with boundary sensing devices. Particularly, a backstepping-based observer is designed and the exponential stability of the error system is proved with an arbitrarily fast convergence rate. The transformation kernel matrix is derived in the explicit form by using the method of successive approximations, thereby yielding the observer gains in the explicit form, too.

Finally we consider the problem of output feedback boundary stabilization for n coupled plants, distributed over the one-dimensional spatial domain $[0, 1]$ where they are governed by linear reaction-diffusion Partial Differential Equations (PDEs). All plants are equipped with its own scalar boundary control input, acting at one end of the domain. First, a state-feedback law is designed to exponentially stabilize the closed-loop system with an arbitrarily fast convergence rate. Then, collocated and anti-collocated observers are designed, using a single boundary measurement for each plant. The exponential convergence of the observed state towards the actual one is demonstrated for both observers, with a convergence rate that can be made as fast as desired. Finally, the state-feedback controller and the selected, either collocated or anti-collocated, observer are coupled together to yield an output-feedback stabilizing controller. The distinct treatments are proposed separately for the case in which all processes have the same diffusivity and for the more challenging scenario where each process has its own diffusivity. The backstepping method is used for both controller and observer designs, and, particularly, the kernel matrices of the underlying transformations are derived in analytical form by using the method of successive approximations to solve the corresponding kernel PDEs. Thus, the resulting control laws and observers become available in explicit form.

– **Chapter 3.Coupled Reaction-Diffusion-Advection Equation**. In this chapter we consider the problem of boundary stabilization for a system of n coupled parabolic linear PDEs with advection term added. Particularly, we design a state feedback law with actuation on only one end of the domain and prove exponential stability of the closed-loop system with an arbitrarily fast convergence rate. The backstepping method is used for controller design, and the transformation kernel matrix is derived in explicit form by using the method of successive approximations to solve the corresponding PDE. Thus, the suggested control law is also made available in explicit form.

Author's Publications

Bibliography

[1] [Baccoli et al., 2015] Antonello Baccoli, Yury Orlov, Alessandro Pisano, and Elio Usai (2016) *Backstepping based boundary control of coupled reaction-advection-diffusion equations having the same diffusivity parameters* Under revision for submission on: 2nd IFAC Workshop on Control of Systems Governed by Partial Differential Equations

[2] [Baccoli et al., 2015] Antonello Baccoli, Yury Orlov, Alessandro Pisano, and Elio Usai (2016) *Output feedback stabilization of coupled reaction-diffusion processes* Under revision on: SIAM Journal on Control and Optimization (SICON)

[3] [Baccoli et al., 2015] Daniela Dejaco, Antonello Baccoli, Alessandro Pisano, Elio Usai, Martin Horn, Giorgio Cau, Pierpaolo Puddu, Fabio Serra (2015) *Numerical Investigation of Packed-Bed Thermal Energy Storage Systems with Prediction-based Adjustment of the Heat Transfer Fluid Flow* Proc. of the Third International Conference On Advances in Computing, Control and Networking - ACCN 2015, 28-29 December, 2015, Bangkok, Thailand ISBN no.978-1-63248-082-8

[4] [Baccoli et al., 2015] Antonello Baccoli, Yury Orlov, Alessandro Pisano, and Elio Usai (2015) *Anti-collocated backstepping observer design for a class of coupled reaction-diffusion PDEs.* Journal of Control Science and Engineering 2015 pages 1-10. DOI:10.1155/2015/164274

[5] [Baccoli et al., 2015] Antonello Baccoli, Yury Orlov, Alessandro Pisano, and Elio Usai (2015) *Sliding mode boundary control of a class of perturbed reaction-diffusion processes* Proc. International Workshop on Recent Advances in Sliding Modes RASM Page(s): 1 - 6 Conference Location : Istanbul DOI:10.1109/RASM.2015.7154644 Publisher: IEEE

[6] [Baccoli et al., 2014] Antonello Baccoli, Yury Orlov, Alessandro Pisano, and Elio Usai (2014) *On the boundary control of distributed parameter systems by second-order sliding-mode technique. Recent advances and new results.* Published in:Variable Structure Systems (VSS) 13th International Workshop. Date of Conference:June 29 2014-July 2 2014 Page(s):1 - 6 Conference Location :Nantes DOI:10.1109/VSS.2014.6881133 Publisher: IEEE

[7] [Baccoli et al., 2014] Antonello Baccoli, Yury Orlov, and Alessandro Pisano (2014) *Boundary control of coupled reaction-diffusion processes with constant parameters* Volume 54, April 2015, Pages 80–90 Received 12 March 2014, doi:10.1016/j.automatica.2015.01.032 Published in: AUTOMATICA

[8] [Baccoli et al., 2014] Antonello Baccoli, Yury Orlov, and Alessandro Pisano (2014) *On the boundary control of coupled reaction-diffusion equations having the same diffusivity parameters* Published in: Decision and Control (CDC), 2014 IEEE 53rd Annual

Conference on Date of Conference: 15-17 Dec. 2014 Page(s): 5222 - 5228 ISBN:978-1-4799-7746-8 Conference Location : Los Angeles, CA DOI: 10.1109/CDC.2014.7040205 Publisher: IEEE

Part I

State of the Art

Chapter 1

Sate Feedback Control

1.1 Generalized Reaction-Diffusion Equation with spatially varying coefficients

1.1.1 Problem Formulation

We consider the following class of linear parabolic partial integro-differential equations (P(I)DEs):

$$\begin{aligned} u_t(x,t) &= \epsilon u_{xx}(x,t) + b(x)u_x(x,t)\lambda(x)u(x,t) \\ &+ g(x)u(0,t) + \int_0^x f(x,y)u(y,t)dy \end{aligned} \tag{1.1}$$

$$\tag{1.2}$$

for $x \in (0,1), t > 0$ with boundary conditions

$$u_x(0,t) = qu(0,t), \tag{1.3}$$

$$u(1,t) = U(t) \;\; or \;\; u_x(1,t) = U(t) \tag{1.4}$$

and under the assumption

$$\begin{aligned} \epsilon &> 0 \quad q \in \mathbb{R} \\ b, \lambda, g &\in C^1([0,1]) \\ f &\in C^1([0,1]X[0,1]) \end{aligned}$$

where the $U(t)$ is the control input. The control objective is to stabilize the equilibrium $u(x,t) = 0$. The 1.1 is in fact a P(I)DE, but for convenience we abuse the terminology and call it a PDE. The problem is formulated as a design of an integral operator state transformation whose kernel is shown to satisfy a well posed hyperbolic PDE. The kernel well posedness for $b = g = f = 1/q = 0$ was shown by [4]. Integral operator transformations for linear parabolic PDEs can be traced as far back as the papers of [3] and [2] who were studying solvability and open-loop controllability of the problem with $b = g = f = 1/q = 0$.

Simplification

Before we start, without loss of generality we set

$$b(x) = 0 \tag{1.5}$$

since it can be eliminated from the equation with the transformation

$$u(x,t) \to u(x,t)e^{-\frac{1}{2\epsilon}\int_0^x b(\tau)d\tau} \tag{1.6}$$

and the appropriate changes of the parameters

$$\begin{aligned} \lambda(x) &\to \lambda(x) + \frac{b'(x)}{2} + \frac{b^2(x)}{4\epsilon} \\ g(x) &\to g(x)e^{-\frac{1}{2\epsilon}\int_0^x b(\tau)d\tau} \\ q &\to q - \frac{b(0)}{2\epsilon} \\ f(x,y) &\to f(x,y)e^{-\frac{1}{2\epsilon}\int_y^x b(\tau)d\tau} \end{aligned} \tag{1.7}$$

1.1.2 PDE for the Kernel

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)dy \tag{1.8}$$

that transforms system (1.1)-(1.4) into the system

$$\begin{aligned} w_t(x,t) &= \epsilon w_{xx}(x,t) - cw(x,t), \quad x \in (0,1) &(1.9)\\ w_x(0,t) &= qw(0,t), &(1.10)\\ w(1,t) &= 0 \;\; or \;\; w_x(1,t) = 0 &(1.11) \end{aligned}$$

which is exponentially stable for $c \geq \epsilon \bar{q}^2$ (respectively, $c \leq \epsilon \bar{q}^2 + \frac{\epsilon}{2}$ where $\bar{q} = max(0, -q)$. The free parameter can be used to set the desired rate of stability. Once we find the transformation (1.8) (namely $k(x,y)$), the boundary condition (1.11) gives the controller in the form

$$u(1,t) = U(t) = \int_0^1 k_1(y)u(y,t)dy \tag{1.12}$$

for the Dirichlet actuation and

$$u_x(1,t) = U(t) = k_1(1)u(1,t) + \int_0^1 k_2(y)u(y,t)dy \tag{1.13}$$

for the Neumann actuation. Here, we denoted

$$\begin{aligned} k_1(y) &= k(1,y) &(1.14)\\ k_2(y) &= k_x(1,y) &(1.15)\\ & &(1.16) \end{aligned}$$

Differentiating (1.8) and using the Leibnitz differentiation rule we get:

$$\frac{d}{dx}\int_0^x f(x,y)\,dy = f(x,x) + \int_0^x f_x(x,y)\,dy. \tag{1.17}$$

Introducing the following notation:

$$\begin{aligned} k_x(x,x) &= \frac{\partial}{\partial x}\,k(x,y)\big|_{y=x}, &(1.18)\\ k_y(x,x) &= \frac{\partial}{\partial y}\,k(x,y)\big|_{y=x}, &(1.19)\\ \frac{d}{dx}k(x,x) &= k_x(x,x) + k_y(x,x), &(1.20) \end{aligned}$$

$$
\begin{aligned}
w_t(x,t) &= u_t(x,t) - \int_0^x k(x,y)\{\epsilon u_{yy}(y,t) + \lambda(y)u(y,t) + g(y)u(0,t) \\
&+ \int_0^y f(y,\xi)u(\xi,t)d\xi\}dy \qquad (1.21) \\
&= u_t(x,t) - \epsilon k(x,x)u_x(x,t) + \epsilon k(x,0)u_x(0,t) \\
&+ \epsilon k_y(x,x)u(x,t) - \epsilon k_y(x,0)u(0,t) \\
&- \int_0^x (\epsilon k_{yy}(x,y) + \lambda(y))u_y(y,t)\, dy - u(0,t)\lambda \int_0^x k(x,y)g(y)\, dy \\
&- \int_0^x u(y,t)\left(\int_y^x k(x,\xi)f(\xi,y)d\xi\right) dy \qquad (1.22)
\end{aligned}
$$

Spatial derivatives of 1.8 $w_x,\ w_{xx}$ are :

$$
\begin{aligned}
w_x(x,t) &= u_x(x,t) - k(x,x)u(x,t) - \int_0^x k_x(x,y)u(y,t)\, dy, \qquad (1.23) \\
w_{xx}(x,t) &= u_{xx}(x,t) - u(x,t)\frac{d}{dx}k(x,x) - k(x,x)u_x(x,t) - k_x(x,x)u(x,t) \\
&\quad - \int_0^x k_{xx}(x,y)u(y,t)\, dy. \qquad (1.24)
\end{aligned}
$$

Substituting (1.22)-(1.24) into (1.9) - (1.10) and using (1.1) - (1.3) with $b(x) = 0$ we obtain the following equation:

$$
\begin{aligned}
0 &= \int_0^x \{\epsilon k_{xx}(x,y) - \epsilon k_{yy}(x,y) - (\lambda(y) + c)k(x,y) + f(x,y)\}u(y,t)dy \\
&- \int_0^x u(y,t)\int_y^x k(x,\xi)f(\xi,y)d\xi dy \\
&+ \{\lambda(x) + c + 2\epsilon\frac{d}{dx}k(x,x)\}u(x,t) + \epsilon q k(x,0)u(0,t) \\
&+ \{g(x) - \int_0^x k(x,y)g(y)dy - \epsilon k_y(x,0)\}u(0,t)dy \qquad (1.25)
\end{aligned}
$$

For this equation to be verified for all $u(x,t)$ the following PDE for $k(x,y)$ must be satisfied:

$$
\epsilon k_{xx}(x,y) - \epsilon k_{yy}(x,y) = (\lambda(y) + c)k(x,y) - f(x,y) + \int_y^x k(x,\xi)f(\xi,y) \qquad (1.26)
$$

for $(x,y) \in T$ with boundary conditions

$$
\begin{aligned}
\epsilon k_y(x,0) &= \epsilon q k(x,0) + g(x) - \int_0^x k(x,y)g(y)dy \qquad (1.27) \\
k(x,x) &= -\frac{1}{2\epsilon}\int_0^x (\lambda(y) + c)dy \qquad (1.28)
\end{aligned}
$$

Here, we denote $T = x, y : 0 < y < x < 1$

We will prove well posedness of (1.26) - (1.28) in the next two sections

1.1.3 Converting the Kernel PDE into an Integral Equation

We derive now an integral equation equivalent to the system (1.26)-(1.28). We introduce the standard change of variables [1]

$$\xi = x + y \quad \eta = x - y \tag{1.29}$$

we have

$$\begin{aligned}
k(x,y) &= G(\xi,\eta) && (1.30)\\
k_x &= G_\xi + G_\eta && (1.31)\\
k_{xx} &= G_{\xi\xi} + 2G_{\xi\eta} + G_{\eta\eta} && (1.32)\\
k_y &= G_\xi - G_\eta && (1.33)\\
k_{yy} &= G_{\xi\xi} - 2G_{\xi\eta} + G_{\eta\eta} && (1.34)\\
& && (1.35)
\end{aligned}$$

transforming problem (1.26)-(1.28) to the following PDE:

$$\begin{aligned}
4\epsilon G_{\xi\eta}(\xi,\eta) &= a\left(\frac{\xi-\eta}{2}\right)G(\xi,\eta) - f\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)\\
&+ \int_{\frac{\xi-\eta}{2}}^{\frac{\xi+\eta}{2}} G\left(\frac{\xi+\eta}{2}+\tau,\frac{\xi-\eta}{2}-\tau\right) f\left(\tau,\frac{\xi-\eta}{2}\right)d\tau && (1.36)\\
\epsilon G_\xi(\xi,\xi) &= \epsilon G_\eta(\xi,\xi) + \epsilon q G(\xi,\xi) + g(\xi) - \int_0^\xi G(\xi+\tau,\xi-\tau)g(\tau)d\tau && (1.37)\\
G(\xi,0) &= -\frac{1}{4\epsilon}\int_0^\xi a\left(\frac{\tau}{2}\right)d\tau && (1.38)\\
& && (1.39)
\end{aligned}$$

Here, we introduced $T_1 = \xi,\eta : 0 < \xi < 2, 0 < \eta < min(\xi, 2-\xi)$ and $a(\tau) = \lambda(\tau) + c$ Integrating (1.36) with respect to η from 0 to η and using (1.38), we obtain

$$\begin{aligned}
G_\xi(\xi,\eta) &= -\frac{1}{4\epsilon}a\left(\frac{\xi}{2}\right) + \frac{1}{4\epsilon}\int_0^\eta a\left(\frac{\xi-s}{2}\right)G(\xi,s)ds\\
&+ \frac{1}{4\epsilon}\int_0^\eta\int_\xi^{\xi+\eta-s} G(\tau,s) f\left(\frac{\tau-s}{2},\xi-\frac{\tau+s}{2}\right)d\tau ds\\
&- \frac{1}{4\epsilon}\int_0^\eta f\left(\frac{\xi+\tau}{2},\frac{\xi-\tau}{2}\right)d\tau && (1.40)
\end{aligned}$$

Integrating (1.40) with respect to ξ from η to ξ gives

$$\begin{aligned}
G(\xi,\eta) &= G(\eta,\eta) - \frac{1}{4\epsilon}\int_\eta^\xi a\left(\frac{\tau}{2}\right)d\tau - \frac{1}{4\epsilon}\int_\eta^\xi\int_0^\eta f\left(\frac{s+\tau}{2},\frac{s-\tau}{2}\right)d\tau ds\\
&+ \frac{1}{4\epsilon}\int_\eta^\xi\int_0^\eta\int_\mu^{\mu+\eta-s} G(\tau,s) f\left(\frac{\tau-s}{2},\mu-\frac{\tau+s}{2}\right)d\tau ds d\mu\\
&+ \frac{1}{4\epsilon}\int_\eta^\xi\int_0^\eta a\left(\frac{\tau-s}{2}\right)G(\tau,s)dsd\tau && (1.41)
\end{aligned}$$

To find $G(\eta,\eta)$, we use (1.37) to write

$$\begin{aligned}
\frac{d}{d\xi}G(\xi,\xi) &= G_\xi(\xi,\xi)+G_\eta(\xi,\xi) \\
&= 2G_\xi(\xi,\xi)-qG(\xi,\xi)-\frac{1}{\epsilon}g(\xi) \\
&+ \frac{1}{\epsilon}\int_0^\xi G(\xi+s,\xi-s)g(s)ds \qquad (1.42)
\end{aligned}$$

Using (1.41) with $\eta=\xi$ we can write (1.42) in the form of differential equation for $G(\xi,\xi)$

$$\begin{aligned}
\frac{d}{d\xi}G(\xi,\xi) &= -qG(\xi,\xi)-\frac{1}{2\epsilon}\int_0^\xi f\left(\frac{\xi+\tau}{2},\frac{\xi-\tau}{2}\right)d\tau \\
&- \frac{1}{2\epsilon}a\frac{\xi}{2}+\frac{1}{2\epsilon}\int_0^\xi a\left(\frac{\xi-s}{2}\right)G(\xi,s)ds \\
&+ \frac{1}{2\epsilon}\int_0^\xi\int_\xi^{2\xi-s}G(\tau,s)f\left(\frac{\tau-s}{2},\xi-\frac{\tau+s}{2}\right)d\tau ds \\
&- \frac{1}{\epsilon}g(\xi)+\frac{1}{\epsilon}\int_0^\xi G(\xi+s,\xi-s)g(s)ds \qquad (1.43)
\end{aligned}$$

Integrating (1.43) using the variation of constants formula and substituting the result into (1.41), we obtain an integral for G

$$G(\xi,\eta) = G_0(\xi,\eta)+F[G](\xi,\eta) \qquad (1.44)$$

where G_0 and $F[G]$ are given by

$$\begin{aligned}
G_0(\xi,\eta) &= -\frac{1}{4\epsilon}\int_\eta^\xi a\left(\frac{\tau}{2}\right)d\tau-\frac{1}{2\epsilon}\int_0^\eta e^{q(\tau-\eta)}\left[a\left(\frac{\tau}{2}\right)+2g(\tau)\right]d\tau \\
&- \frac{1}{4\epsilon}\int_\eta^\xi\int_0^\eta f\left(\frac{s+\tau}{2},\frac{s-\tau}{2}\right)d\tau ds \\
&- \frac{1}{2\epsilon}\int_0^\eta e^{q(\tau-\eta)}\int_0^\tau f\left(\frac{\tau+s}{2},\frac{\tau-s}{2}\right)ds d\tau \qquad (1.45)
\end{aligned}$$

$$\begin{aligned}
F[G](\xi,\eta) &= \frac{1}{2\epsilon}\int_0^\eta e^{q(\tau-\eta)}\int_0^\tau a\left(\frac{\tau-s}{2}\right)G(\tau,s)ds d\tau \\
&+ \frac{1}{4\epsilon}\int_\eta^\xi\int_0^\eta a\left(\frac{\tau-s}{2}\right)G(\tau,s)ds d\tau \\
&+ \frac{1}{2\epsilon}\int_0^\eta\int_s^{2\eta-s}e^{q\left(\frac{\tau+s}{2}-\eta\right)}g\left(\frac{\tau-s}{2}\right)G(\tau,s)d\tau ds \\
&+ \frac{1}{4\epsilon}\int_\eta^\xi\int_0^\eta\int_\mu^{\mu+\eta-s}f\left(\frac{\tau-}{2},\mu-\frac{\tau+s}{2}\right)G(\tau,s)d\tau ds d\mu \\
&+ \frac{1}{2\epsilon}\int_0^\eta e^{q(\mu-\eta)}\int_0^\mu\int_\mu^{2\mu-s}f\left(\frac{\tau-s}{2},\mu-\frac{\tau+s}{2}\right)G(\tau,s)d\tau ds d\mu \qquad (1.46)
\end{aligned}$$

Lemma 1. *Any* $G(\xi,\eta)$ *satisfying* (1.36)-(1.38) *also satisfies integral* (1.44)

1.1.4 Analysis of the integral equation by a Successive Approximation Series

Using the result of the previous section we can now compute a uniform bound on the solutions by the method of successive approximations. With G_0 defined in (1.45), let

$$G_{n+1}(\xi,\eta) = F[G_n], \quad n = 0, 1, 2, \ldots \tag{1.47}$$

and denote

$$\begin{aligned} \bar{\lambda} &= sup_{x\in[0,1]}|\lambda(x)| \\ \bar{g} &= sup_{x\in[0,1]}|g(x)| \\ \bar{f} &= sup_{(x,y)\in[0,1]\times[0,1]}|f(x,y)| \end{aligned} \tag{1.48}$$

We estimate now $G_n(\xi,\eta)$

$$\begin{aligned} |G_0(\xi,\eta)| &\leq \frac{1}{4\epsilon}(\bar{\lambda}+c)(\xi-\eta) + \frac{1}{2\epsilon}(\bar{\lambda}+c+2\bar{g})\eta + \frac{1}{4\epsilon}\bar{f}\eta^2 + \frac{1}{4\epsilon}\bar{f}(\xi-\eta)\eta \\ &\leq \frac{1}{\epsilon}(\bar{\lambda}+c+\bar{f}+\bar{g})(1+e^{-q}) = M \end{aligned} \tag{1.49}$$

Suppose that

$$|G_n(\xi,\eta)| \leq M^{n+1}\frac{(\xi+\eta)^n}{n!} \tag{1.50}$$

Then, we have the following estimate.

$$\begin{aligned} |G_{n+1}(\xi,\eta)| &\leq \frac{M^{n+1}}{4\epsilon}\frac{1}{n!}\{(\bar{\lambda}+c)\int_\eta^\xi\int_0^\eta(\tau+s)^n ds d\tau \\ &+ 2(\bar{\lambda}+c)\int_0^\eta e^{q(\tau-\eta)}\int_0^\tau(\tau+s)^n ds d\tau \\ &+ \bar{f}\int_\eta^\xi\int_0^\eta\int_\mu^{\mu+\eta-s}(\tau+s)^n d\tau ds d\mu \\ &+ 2\bar{g}\int_0^\eta\int_s^{2\eta-s} e^{q(\frac{\tau+s}{2}-\eta)}(\tau+s)^n d\tau ds + 2\bar{f} \\ &\times \int_0^\eta e^{q(\mu-\eta)}\int_0^\mu\int_\mu^{2\mu-s}(\tau+s)^n d\tau ds d\mu\} \\ &\leq \frac{M^{n+1}}{4\epsilon}\frac{1}{n!}\{(\bar{\lambda}+c)(2+2(1+e^{-q})) + 2\bar{f} + 4(1+e^{-q})\bar{g} \\ &+ 2(1+e^{-q})\bar{f}\}\frac{(\xi+\eta)^{n+1}}{n+1} \\ &\leq M^{n+2}\frac{(\xi+\eta)^{n+1}}{(n+1)!} \end{aligned} \tag{1.51}$$

So, by induction, (1.50) is proved. Note also that $G_n(\xi,\eta)$ is $C^2(T_1)$ which follows from (1.45) - (1.46). Therefore, the series

$$G(\xi,\eta) = \sum_{n=0}^{\infty} G_n(\xi,\eta) \tag{1.52}$$

converges absolutely and uniformly in T_1 and its sum G is a twice continuously differentiable solution of (1.44) with a bound

$$|G(\xi,\eta)| \le Me^{M(\xi+\eta)} \tag{1.53}$$

The uniqueness of this solution can be proved by the following argument. Suppose $G^{'}(\xi,\eta)$ and $G^{''}(\xi,\eta)$ are two different solutions of (1.44). Then $\Delta G(\xi,\eta) = G^{'}(\xi,\eta) - G^{''}(\xi,\eta)$ satisfies the homogeneous integral (1.46) in which G_n and G_{n+1} are changed to ΔG. Using the above result of boundedness we have $|\Delta G(\xi,\eta)| \le 2Me^{2M}$. Using this inequality in the homogeneous integral equation and following the same estimates as in (1.51) we get that $\Delta G(\xi,\eta)$ satisfies for all n

$$|\Delta G(\xi,\eta)| \le 2M^{n+1}e^{2M}\frac{(\xi+\eta)^n}{n!} \to 0 \quad as \quad n \to \infty \tag{1.54}$$

Thus, $\Delta G = 0$ which means that (1.51) is a unique solution to (1.44). By direct substitution we can check that it is also a unique (by Lemma 1) solution to PDE (1.36)- (1.38). Thus, we proved the following result, which generalizes [4].

Theorem 1. *The* (1.26) *with boundary conditions* (1.27)-(1.28) *has a unique* $C^2(T)$ *solution. The bound on the solution is*

$$|k(x,y)| \le Me^{2Mx} \tag{1.55}$$

where M *is given by* (1.49) *To prove stability we need to prove that the transformation* (1.8) *is invertible. The proof that for* (1.8) *an inverse transformation with bounded kernel exists can be found in [4]. The other way to prove it is to directly find and analyze the PDE for the kernel of the inverse transformation. We take this route because we need the inverse kernel for further quantitative analysis. Let us denote the kernel of the inverse transformation by* $l(x,y)$*. The transformation itself has the form*

$$u(x,t) = w(x,t) + \int_0^x l(x,y)w(y,t)dy \tag{1.56}$$

Substituting (1.56) *into* (1.9)-(1.11) *and using* (1.1)-(1.4)*, we obtain the following PDE governing* $l(x,y)$

$$\epsilon l_{xx}(x,y) - \epsilon l_{yy}(x,y) \;=\; -(\lambda(x)+c)l(x,y) - f(x,y) - \int_y^x l(\tau,y)f(x,\tau) \tag{1.57}$$

for $(x,y) \in T$ *with boundary conditions*

$$\epsilon l_y(x,0) \;=\; \epsilon q l(x,0) + g(x) \tag{1.58}$$

$$l(x,x) \;=\; -\frac{1}{2\epsilon}\int_0^x (\lambda(y)+c)dy \tag{1.59}$$

This hyperbolic PDE is a little bit simpler than the one for k (the boundary condition does not contain an integral term), but has a very similar structure. So, we can apply the same approach of converting the PDE to an integral equation and using a method of successive approximations to show that the inverse kernel exists and has the same properties as we proved for the direct kernel.

Theorem 2. *The* (1.57) *with boundary conditions* (1.58)-(1.59) *has a unique* $C^2(T)$ *solution. The bound on the solution is*

$$|l(x,y)| \le Me^{2Mx} \tag{1.60}$$

where M *is given by* (1.49)

1.1.5 Properties of the Closed-Loop System

Theorems 1 and 2 establish the equivalence of norms of u and w in both L_2 and H_1. From the properties of the damped heat (1.9) - (1.11) exponential stability in both L_2 and H_1 follows. Furthermore, it can be proved that if the kernels (1.14)- (1.15) are bounded than the system (1.1) - (1.4) with a boundary condition (1.12) or (1.13) is well posed. Thus, we get the following main result.

Theorem 3. *For any initial data $u_0(x) \in L_2(0,1)$ (respectively, $H_1(0,1)$) that satisfy the compatibility conditions*

$$u_{0x}(0) = qu_0(0) \quad u_0(1) = \int_0^1 k_1(y)u_0(y)dy \tag{1.61}$$

system (1.1) - (1.4) *with Dirichlet boundary control* (1.12) *has a unique classical solution $u(x,t) \in C^{2,1}((0,1) \times (0,\infty))$ and is exponentially stable at the origin $u(x,t) = 0$*

$$||u(t)||_{\mathcal{L}} \leq Ce^{-(c-\epsilon\bar{q}^2)t}||u_0||_{\mathcal{L}} \tag{1.62}$$

where C is a positive constant independent of u_0 and $\mathcal{L}$ is either L_2 H_1. For any initial data $u_0(x) \in L_2(0,1)$ (respectively, $H_1(0,1)$) that satisfy the compatibility conditions $u_{0x}(0) = qu_0(0)$

$$u_{0x}(1) = k_1(1)u_0(1) + \int_0^1 k_2(y)u_0(y)dy \tag{1.63}$$

system (1.1) - (1.4) *with Dirichlet boundary control* (1.13) *has a unique classical solution $u(x,t) \in C^{2,1}((0,1) \times (0,\infty))$ and is exponentially stable at the origin $u(x,t) = 0$*

$$||u(t)||_{\mathcal{L}} \leq Ce^{-(c-\epsilon\bar{q}^2-1/2)t}||u_0||_{\mathcal{L}} \tag{1.64}$$

See [67].

1.2 Reaction-Diffusion Equation with constant coefficients

1.2.1 Unstable Heat Equation

Let $\lambda(x) = \lambda_0 = const$, $g(x) = 0$, $f(x, y) = 0$, $q = +\infty$ In this, system (1.1)-(1.4) takes the form of the unstable heat equation case

$$
\begin{aligned}
u_t(x,t) &= \epsilon u_{xx}(x,t) + \lambda_0 u(x,t), && (1.65)\\
u(0,t) &= 0, && (1.66)\\
u(1,t) &= U(t) \quad or \quad u_x(1,t) = U(t) && (1.67)
\end{aligned}
$$

The open-loop system (1.1)-(1.4) (with $u(1,t) = 0$ or $u_x(1,t) = 0$ is unstable with arbitrarily many unstable eigenvalues (for large λ_0/ϵ). Although this constant coefficient problem may appear easy, the explicit (closed-form) boundary stabilization result in the case of arbitrary ϵ, λ_0 is not available in the literature.

The kernel PDE (1.26)-(1.28), in this case, takes the following form:

$$
\begin{aligned}
k_{xx}(x,y) - k_{yy}(x,y) - \lambda k(x,y) &= 0 \quad (x,y) \in T && (1.68)\\
k(x,0) &= 0 && (1.69)\\
k(x,x) &= -\frac{\lambda x}{2} && (1.70)
\end{aligned}
$$

where we denote $\lambda = (\lambda_0 + c)/\epsilon$. Let us solve this equation directly by the method of successive approximations. Integral (1.44) for G becomes

$$
G(\xi,\eta) = -\frac{\lambda}{4}(\xi - \eta) + \frac{\lambda}{4}\int_\eta^\xi \int_0^\eta G(\tau, s)\, ds d\tau \qquad (1.71)
$$

Now, set

$$
G_0(\xi,\eta) = -\frac{\lambda}{4}(\xi - \eta) G_{n+1} = \frac{\lambda}{4}\int_\eta^\xi \int_0^\eta G_n(\tau, s)\, ds d\tau \qquad (1.72)
$$

Fortunately, we can find the general term G_n in closed form

$$
G_n(\xi,\eta) = -\frac{(\xi-\eta)\xi^n\eta^n}{(n!)^2(n+1)}\left(\frac{\lambda}{4}\right)^{n+1} \qquad (1.73)
$$

Now, we can calculate the series (1.51):

$$
G(\xi,\eta) = \sum_{n=0}^{\infty} G_n(\xi,\eta) = -\frac{\lambda}{2}(\xi-\eta)\frac{I_1(\sqrt{\lambda\xi\eta})}{\sqrt{\lambda\xi\eta}} \qquad (1.74)
$$

where I_1 is a modified Bessel function of order one. Writing (1.74) in terms of x, y gives the following solution for $k(x,y)$:

$$
k(x,y) = -\lambda y \frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}} \qquad (1.75)
$$

Plant:

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t) \tag{1.79}$$
$$u(0,t) = 0 \tag{1.80}$$

Controller:

$$U(t) = -\lambda \int_0^1 y \frac{I_1\left(\sqrt{\lambda(1-y^2)}\right)}{\sqrt{\lambda(1-y^2)}} u(y,t)\, dy. \tag{1.81}$$

Transformation:

$$w(x,t) = u(x,t) + \lambda \int_0^x y \frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}} u(y,t)\, dy, \tag{1.82}$$
$$u(x,t) = w(x,t) - \lambda \int_0^x y \frac{J_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}} w(y,t)\, dy. \tag{1.83}$$

Target System:

$$w_t(x,t) = w_{xx}(x,t) \tag{1.84}$$
$$w(0,t) = 0, \tag{1.85}$$
$$w(1,t) = 0 \tag{1.86}$$

Table 1.1: Summary of control design for the reaction-diffusion equation.

which gives the gain kernels

$$k_1(y) = k(1,y) = -\lambda y \frac{I_1\left(\sqrt{\lambda(1-y^2)}\right)}{\sqrt{\lambda(1-y^2)}} \tag{1.76}$$

$$k_2(y) = k_x(1,y) = -\lambda y \frac{I_2\left(\sqrt{\lambda(1-y^2)}\right)}{1-y^2} \tag{1.77}$$

Now, let us construct an inverse-optimal controller. First, we need to find the kernel of the inverse transformation. Noticing that in our case $l(x,y) = -k(x,y)$ when λ is replaced by $-\lambda$ we immediately obtain

$$l(x,y) = -\lambda y \frac{I_1\left(\sqrt{-\lambda(x^2-y^2)}\right)}{\sqrt{-\lambda(x^2-y^2)}} = -\lambda y \frac{J_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}} \tag{1.78}$$

where J_1 is the usual (nonmodified) Bessel function of the first order.

A summary of the control design for the plant (1.65)- (1.67) is presented in Table 1.1.

1.3 Control for coupled reaction-diffusion PDE

Reaction-diffusion equations are parabolic Partial Differential Equations (PDEs) which often occur in practice, e.g., to model the concentration of one or more substances, distributed in space, under the influence of different phenomena such as local chemical reactions, in which the substances are transformed into each other, and diffusion, which causes the substances to spread out over a surface in space. Certainly, reaction-diffusion PDEs are not confined to chemical applications (see e.g. [20]), but they also describe dynamical processes of non-chemical nature, with examples being found in thermodynamics, biology, geology, physics, ecology, etc. (see e.g. [60, 61]).

Recently, high-dimensional systems of *coupled* PDEs were considered in the backstepping boundary control design settings. The most intensive efforts of current literature were oriented towards coupled hyperbolic processes of the transport-type [5, 8, 37, 29, 30]. Some specific results on the backstepping based boundary stabilization of parabolic coupled PDEs have additionally been presented in the literature [6, 28, 75, 32, 33].

In [28], two parabolic reaction-diffusion processes, coupled through the corresponding boundary conditions, were dealt with. The stabilization of the coupled equations was reformulated in terms of the stabilization problem for a unique process, which possessed piecewise-continuous diffusivity and (space-dependent) reaction coefficient and which was viewed as the "cascade" between the two original systems. The problem was then solved by using a scalar boundary control input and by employing a non conventional backstepping approach with a discontinuous kernel function. In [6], the Ginzburg-Landau equation with the imaginary and real parts expanded, thus being specified to a 2×2 parabolic system with equal diffusion coefficients, was dealt with. In [75], the linearized 2×2 model of thermal-fluid convection was treated by using a singular perturbations approach combined with backstepping and Fourier series expansion. In [33], the boundary stabilization of the linearized model of an incompressible magnetohydrodynamic flow in an infinite rectangular 3D channel, also recognized as Hartmann flow, was achieved by reducing the original system to a set of coupled diffusion equations with the same diffusivity parameter and by applying backstepping.

Chapter 2

State Observation

2.1 Generalized Reaction-Diffusion Equation with spatially varying coefficients

2.1.1 Problem statement

We consider the following class of linear parabolic partial integro-differential equations (P(I)DEs):

$$\begin{aligned} u_t(x,t) &= \epsilon u_{xx}(x,t) + b(x)u_x(x,t)\lambda(x)u(x,t) \\ &+ g(x)u(0,t) + \int_0^x f(x,y)u(y,t)dy \end{aligned} \tag{2.1}$$

for $x \in (0,1), t > 0$ with boundary conditions

$$u_x(0,t) = qu(0,t), \tag{2.2}$$

$$u(1,t) = U(t) \;\; or \;\; u_x(1,t) = U(t) \tag{2.3}$$

and under the assumption

$$\begin{aligned} \epsilon &> 0 \quad q \in \mathbb{R} \\ \lambda, g &\in C^1[0,1] \\ f &\in C^1([0,1]X[0,1]) \end{aligned} \tag{2.4}$$

Without loss of generality we can set $b(x) = 0$ since it can be eliminated from the equation with the transformation

$$u(x,t) \to u(x,t)e^{-(1/2\epsilon)\int_0^x b(\tau)d\tau} \tag{2.5}$$

and the appropriate changes of parameters q, $\lambda(x)$, $g(x)$ and $f(x,y)$. The PDE (2.1)-(2.3) is actuated at $x = 1$ (using either Dirichlet or Neumann actuation) by a boundary input $U(t)$ that can be any function of time or a feedback law. The problem is to design an exponentially convergent observer for the plant with only boundary measurements available. The observer design depends on the type (Dirichlet/Neumann) and the location of measurement and actuation. We consider two setups: the anti-collocated setup, when sensor and actuator are placed at the opposite ends, and the collocated case, when sensor and actuator are placed at the same end. There is not much technical difference between the cases of Dirichlet and Neumann actuation. We use the backstepping state-feedback results of 1.2

2.1.2 Observer design for anti-collocated setup

Suppose the only available measurement of our system is at $x = 0$, the opposite end to actuation. We propose the following observer for system (2.1) -(2.3) with Dirichlet actuation:

$$\begin{aligned} \hat{u}_t(x,t) &= \epsilon\hat{u}_{xx}(x,t) + \lambda(x)\hat{u}(x,t) + g(x)u(0,t) + \int_0^x f(x,y)\hat{u}(y,t)dy \\ &+ p_1(x)[u(0) - \hat{u}(+n0)], \end{aligned} \tag{2.6}$$

$$\hat{u}_x(0,t) = qu(0,t) + p_{10}[u(0) - \hat{u}(0)], \tag{2.7}$$

$$\hat{u}(1,t) = U(t). \tag{2.8}$$

Here $p_1(x)$ and p_{10} are output injection functions (p_{10} is a constant) to be designed. Note that we introduce output injection not only in Eq. (2.6) but also at the boundary where measurement is available. We also implicitly use the additional output injection here in a form $q(u(0,t) - \hat{u}(0,t))$ that cancels the dependency on q in the error dynamics.

The observer error

$$\tilde{u} = u - \hat{u} \tag{2.9}$$

satisfies the following PDE:

$$\begin{aligned}
\tilde{u}_t(x,t) &= \epsilon \tilde{u}_{xx}(x,t) + \lambda(x)\tilde{u}(x,t) + \int_0^x f(x,y)\tilde{u}(y,t)dy - p_1(x)\tilde{u}(0,t), && (2.10)\\
\tilde{u}_x(0,t) &= -p_{10}\tilde{u}(0,t), && (2.11)\\
\tilde{u}(1,t) &= 0. && (2.12)
\end{aligned}$$

Observer gains $p_1(x)$ and p_{10} should be now chosen to stabilize system (2.10) - (2.12) . We solve the problem of stabilization of (2.10) - (2.12) by the same integral transformation approach as the (state feedback) boundary control problem reviewed in 1.1. We look for a backstepping-like coordinate transformation

$$\tilde{u}(x) = \tilde{w}(x) - \int_0^x p(x,y)\tilde{w}(y)dy \tag{2.13}$$

that transforms system (2.10) - (2.12) into the exponentially stable (for $\tilde{c} \geq 0$) system

$$\begin{aligned}
\tilde{w}_t &= \epsilon \tilde{w}_{xx} - \tilde{c}\tilde{w}(x,t) && (2.14)\\
\tilde{w}_x(0) &= 0, && (2.15)\\
\tilde{w}(1) &= 0 && (2.16)
\end{aligned}$$

The free parameter $\tilde{c}$ can be used to set the desired observer convergence speed. It is in general different from the analogous coefficient c in control design since one usually wants the estimator to be faster than the state feedback closed-loop dynamics. By substituting (2.13) into (2.10)-(2.12) we obtain a set of conditions on the kernel $p(x,y)$ in the form of the hyperbolic PDE

$$\epsilon p_{yy}(x,y) - \epsilon p_{xx}(x,y) = (\lambda(x)+c)p(x,y) - f(x,y) + \int_y^x p(\xi,y)f(x,\xi)d\xi \tag{2.17}$$

for $(x,y) \in T = x, y : 0 < y < x < 1$, with the boundary conditions

$$\begin{aligned}
\frac{d}{dx}p(x,x) &= \frac{1}{2\epsilon}(\lambda(x)+c), && (2.18)\\
p(1,y) &= 0. && (2.19)
\end{aligned}$$

that yield

$$\begin{aligned}
\tilde{w}_t &= \epsilon \tilde{w}_{xx} - \tilde{c}\tilde{w}(x,t) - \epsilon p(x,0)\tilde{w}_x(0,t) + (\epsilon p_y(x,0) - p_1(x))\tilde{w}(0,t) && (2.20)\\
\tilde{w}_x(0,t) &= (p(0,0) - p_{10})\tilde{w}(0,t), && (2.21)\\
\tilde{w}(1,t) &= 0 && (2.22)
\end{aligned}$$

Comparing this with (2.14)- (2.16), it follows that the observer gains should be chosen as

$$p_{10}(x,y) = p(0,0), \tag{2.23}$$
$$p_1(x) = \epsilon p_y(x,0). \tag{2.24}$$

The problem is first to prove that PDE (2.17)-(2.19) is well-posed. Once the solution $p(x,y)$ to the problem (2.17)-(2.19) is found, the observer gains can be obtained from (2.30)-(2.30).

The condition 2.30 is obtained by differentiating 2.13 with respect to x, setting $x = 0$, and substituting 2.10 and 2.15 in the resulting equation. The condition 2.31 is obtained by setting $x = 1$ in 2.13 and substituting 2.12 and 2.16 in the resulting equation.

Let us make a change of variables

$$\bar{x} = 1 - y, \quad \bar{y} = 1 - x, \quad \bar{p}(\bar{x},\bar{y}) = p(x,y) \tag{2.25}$$

In these new variables problem 2.12 and 2.16 becomes

$$\epsilon\bar{p}_{\bar{x}\bar{x}}(\bar{x},\bar{y}) - \epsilon\bar{p}_{\bar{y}\bar{y}}(\bar{x},\bar{y}) = (\bar{\lambda}(\bar{y}) + \tilde{c})\bar{p}(\bar{x},\bar{y}) - \bar{f}(\bar{x},\bar{y}) + \int_{\bar{y}}^{\bar{x}} \bar{p}(\bar{x},\xi)\bar{f}(\xi,\bar{y})d\xi \tag{2.26}$$
$$\bar{p}(\bar{x},\bar{x}) = -\frac{1}{2\epsilon}\int_0^{\bar{x}} (\bar{\lambda}(\xi) + \tilde{c})d\xi \tag{2.27}$$
$$\bar{p}(\bar{x},0) = 0 \tag{2.28}$$

Theorem 4. *Eq.* (2.17) *with boundary conditions* (2.19)- (2.18) *has a unique* $C^2(T)$ *solution. The kernel* $r(x,y)$ *of the inverse transformation*

$$\tilde{w}(x,t) = \tilde{u}(x,t) + \int_0^x r(x,y)\tilde{u}(y,t)dy \tag{2.29}$$

is also a unique $C^2(T)$ *function.*

The fact that the observer gain in transposed and switched variables satisfies the same class of PDEs as control gain is reminiscent of the duality property of state-feedback and observer design problems for linear finite-dimensional systems. The difference be- tween the equations for observer and control gains is due to the fact that the observer error system does not contain terms with $g(x)$ and q because $u(0,t)$ is mesured. The observer gains in the new coordinates are given by

$$p_{10}(x,y) = \bar{p}(1,1), \tag{2.30}$$
$$p_1(x) = -\epsilon\bar{p}_{\bar{x}}(1,1-x). \tag{2.31}$$
$$\tag{2.32}$$

The exponential stability of the target system (2.14)-(2.16) and invertibility of transformation (2.13) (established in Theorem 4 imply the exponential stability of (2.10)-(2.12) both in L_2 and H_1. The result is formulated in the following theorem.

Theorem 5. *Let* $p(x,y)$ *be the solution of system* (2.17)-(2.19). *Then for any* $\tilde{u}_0(x) \in L_2(0,1)$ *system* (2.10)-(2.12) *with* $p_1(x)$ *and* p_{10} *given by Eq.* (2.30)-(2.31) *has a unique classical solution* $u(x,t) \in C^{2,1}((0,1)\times(0,\infty))$. *Additionally, the origin* $\tilde{u}(x,t) = 0$ *is exponentially stable in the* $L_2(0,1)$ *and* $H_1(0,1)$ *norms.*

2.2 Reaction-Diffusion Equation with constant coefficients

2.2.1 Observer design for Reaction-Diffusion PDEs

Starting from the equation

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) + \lambda u(x,t), && (2.33)\\ u_x(0,t) &= 0, && (2.34)\\ u(1,t) &= U(t). && (2.35) \end{aligned}$$

The open-loop system (2.33)-(2.35) with ($U = 0$) is unstable with arbitrarily many unstable eigenvalues. Let us consider the anti-collocated setup. Eqs. (2.26) - (2.28) for the observer gain takes the form

$$\begin{aligned} \bar{p}_{\bar{x}\bar{x}}(\bar{x},\bar{y}) - \bar{p}_{\bar{y}\bar{y}}(\bar{x},\bar{y}) &= \lambda \bar{p}(\bar{x},\bar{y}) && (2.36)\\ \bar{p}(\bar{x},\bar{x}) &= -\frac{\lambda}{2}\bar{x} && (2.37)\\ \bar{p}(\bar{x},0) &= 0 && (2.38) \end{aligned}$$

where $\lambda = (\lambda_0 + c)/\epsilon$ The solution to (2.36)-(2.38) is

$$\bar{p}(\bar{x},\bar{y}) = -\lambda \bar{y} \frac{I_1\left(\sqrt{\lambda(\bar{x}^2 - \bar{y}^2)}\right)}{\sqrt{\lambda(\bar{x}^2 - \bar{y}^2)}} \tag{2.39}$$

I_1 is the modified Bessel function of the first order. Using (2.30)-(2.31) we obtain the observer gains

$$\begin{aligned} p_1(x) &= \epsilon \frac{\lambda(1-x)}{x(2-x)} I_2\left(\sqrt{\lambda x(2-x)}\right) && (2.40)\\ p_{10} &= -\frac{\lambda}{2} && (2.41) \end{aligned}$$

2.3 Observer design for coupled reaction-diffusion PDE

The implementation of backstepping controllers usually requires the full state information. From the practical standpoint, the available measurements of Distributed Parameter Systems (DPSs) are typically located at the boundary of the spatial domain, that motivates the need of the state observer design [46, 70]. For linear infinite-dimensional systems, the Luenberger observer theory was established by replacing matrices with linear operators [69, 72, 70], and the observer design was confined to determining a gain operator that stabilizes the associated observation error dynamics. In contrast to finite-dimensional systems, finding such a gain operator was not trivial even numerically because operators were not generally represented with a finite number of parameters.

Observer design methods that would be capable of yielding the observer gains in the analytical form have only recently been investigated. In this context, the backstepping method appears to be a particularly effective systematic observer design approach [13]. Recently, the backstepping-based observer design was presented in [44] for reaction-diffusion processes with spatially-varying reaction coefficients while measuring a certain integral average value of the state of the plant. In [71, 11], backstepping-based observer design was addressed for reaction-diffusion processes evolving in multi-dimensional spatial domains.

More recently, high-dimensional systems of *coupled* PDEs were considered in the backstepping boundary control and observer design settings. The most intensive efforts of current literature were oriented towards coupled hyperbolic processes of the transport-type [5, 8, 37, 29, 30]. In [32], an observer that estimated the velocity, pressure, electric potential and current fields in a Hartmann flow was presented where the observer gains were designed using multi-dimensional backstepping. In [41], a backstepping observer was designed for a system of two diffusion-convection-reaction processes coupled through the corresponding boundary conditions.

Chapter 3

Output Feedback

3.1 Output Feedback for Reaction-Diffusion PDEs

The exponentially convergent observers developed in previous sections are independent of the control input and can be used with any controller. In this section we combine these observers with their natural dual controllers-backstepping controllers-to solve the output-feedback problem fully by backstepping.

Theorem 6. *Let $k_1(x)$ be the solution of* (1.26) - (1.28) *and $p_1(x)$, p_{10} be the solutions of* (2.17)-(2.19), *and let the assumptions 2.4, $\tilde{c} \geq 0$, and $c \geq max(0, -\epsilon q|q|)$ hold. Then for any $u_0, \hat{u}_0 \in L_2(0,1)$ the system consisting of plant* (2.1)-(2.3), *the controller*

$$u(1,t) = \int_0^1 k_1(y)\hat{u}(y,t)dy \tag{3.1}$$

and the observer (2.6)-(2.8) *has a unique classical solution $u(x,t)$, $\hat{u}(x,t) \in C^{((0,1)\times(0,\infty))}$ and is exponentially stable at the origin, $u(x,t) = 0$, $\hat{u}(x,t) = 0$ in the $L_2(0,1)$ and $H_1(0,1)$ norms.*

3.1.1 Output Feedback for single Reaction-Diffusion PDEs

We can now write the explicit solution to the output-feedback problem. The gain kernel for the state-feedback problem has been found previously analytically:

$$k(x,y) = -\lambda y \frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}} \tag{3.2}$$

Using (3.2), (2.40)-(2.41) and Theorem 6 we get the following result.

Theorem 7. *The controller*

$$U(t) = -\int_0^1 k(1,y)\hat{u}(y,t)dy = -\int_0^1 \lambda \frac{I_1\left(\sqrt{\lambda(1-y^2)}\right)}{\sqrt{\lambda(1-y^2)}}\hat{u}(y,t)dy \tag{3.3}$$

with the observer

$$\hat{u}_t(x,t) = \epsilon\hat{u}_{xx}(x,t) + \lambda_0\hat{u}(x,t) + \epsilon\frac{\lambda(1-x)}{x(2-x)}I_2\left(\sqrt{\lambda x(2-x)}\right)[u(0,t)-\hat{u}(0,t)] \tag{3.4}$$

$$\hat{u}_x(0,t) = -\frac{\lambda}{2}[u(0,t)-\hat{u}(0,t)] \tag{3.5}$$

$$\hat{u}(1,t) = -\int_0^1 \lambda \frac{I_1\left(\sqrt{\lambda(1-y^2)}\right)}{\sqrt{\lambda(1-y^2)}}\hat{u}(y,t)dy \tag{3.6}$$

stabilizes the zero solution of system (2.33) - (2.34)

3.2 Output feedback boundary stabilization for coupled reaction-diffusion PDE

In [5], a 2×2 linear hyperbolic system was stabilized by a scalar observer-based output-feedback boundary control input, with an additional feature that an unmatched disturbance, generated by an *a-priori* known exosystem, was rejected. In [29], a 2×2 system of coupled linear heterodirectional hyperbolic equations was stabilized by observer-based output feedback. The underlying design was extended in [8] to a particular type of 3×3 linear systems, arising in modeling of multiphase flow, and to the quasilinear case in [30]. In [37], backstepping observer-based output-feedback design was presented for a system of $n + 1$ coupled first-order linear heterodirectional hyperbolic PDEs (n of which featured rightward convecting transport, and one leftward) with a single boundary input.

The recent authors' work [35] dealt with the state-feedback controller design for coupled reaction-diffusion processes equipped with Neumann (rather than Dirichlet) boundary conditions. The same publication also addressed a state-feedback stabilization problem for two coupled reaction-diffusion processes, which were underactuated by a scalar boundary input applied just to one of the processes.

Part II

Author's Contributions

Chapter 1

Scalar Reaction-Diffusion Equation

1.1 Sliding-mode boundary control of a class of perturbed parabolic PDEs

We study the stabilization problem in the space $L_2(0,1)$ for a class of parabolic PDEs of the reaction-diffusion type equipped with destabilizing Robin-type boundary conditions. The considered class of PDEs is also affected by a matching boundary disturbance with an a-priori known constant upperbound to its magnitude. The problem is solved by means of a suitable synergic combination between the infinite-dimensional backstepping methodology and the sliding mode control approach. A constructive Lyapunov analysis supports the presented synthesis, and simulation results validate the developed technique.

1.1.1 Introduction

Many important engineering systems and industrial processes are governed by partial differential equations (PDEs) and are often subject to a significant degree of uncertainty. Therefore, a growing interest is arising towards extending sliding mode control to infinite-dimensional systems. Presently, the discontinuous control synthesis in the infinite-dimensional setting is well documented [52, 54, 57] and it is generally shown to retain the main robustness features as those possessed by its finite-dimensional counterpart.

Reaction-diffusion systems, in particular, are mathematical models which explain, e.g., how the concentration of one or more substances distributed in space changes under the influence of two processes: local chemical reactions in which the substances are transformed into each other, and diffusion which causes the substances to spread out over a surface in space. Reaction-diffusion systems are thus naturally arising in many chemical applications but they can also describe dynamical processes of non-chemical nature, with examples being found in thermodynamics, biology, geology, physics, ecology, etc. (see e.g. [60], [61]).

The boundary control problem for several classes of parabolic reaction-diffusion processes was studied, e.g., in [47, 63, 50]. The main contribution of the present work is that of enhancing the robustness features by admitting the presence of a boundary disturbance, unknown in shape, which will be completely rejected by a suitably designed discontinuous boundary feedback input. A similar disturbance rejection task was pursued in [25] with reference to a standard diffusion equation equipped by Neumann-type boundary conditions.

More precisely, In the present paper we consider the space- and time-varying scalar field $z(\xi,t)$ evolving in a Hilbert space $H = L_2(0,1)$, with the monodimensional (1D) spatial variable $\xi \in [0,1]$ and time variable $t \geq 0$. Let it be governed by the following reaction-diffusion boundary-value problem equipped with controlled and perturbed Boundary Conditions (BCs)

$$\begin{aligned} z_t(\xi,t) &= \theta z_{\xi\xi}(\xi,t) + \lambda z(\xi,t) && (1.1)\\ z_\xi(0,t) &= -qz(0,t), && (1.2)\\ z_\xi(1,t) &= v(t) + \psi(t) && (1.3) \end{aligned}$$

where $v(t)$ is the manipulable boundary control input, $\theta \in \Re^+$, $\lambda \in \Re$ and $q \in \Re$ are the system's parameters, and $\psi(t) : [0,\infty) \to \Re$ is a nonvanishing disturbance. A critical feature of the above class of systems is the presence of the Robin-type BC (1.2), which is destabilizing when $q > 0$. For sufficiently large positive λ and q, system (1.1)-(1.3) possesses arbitrarily many unstable eigenvalues yielding an open-loop unstable behavior. In [67], the problem was solved in the case $\psi(t) = 0$ by means of the infinite-dimensional backstepping approach. Here, we combine the infinite-dimensional backstepping (see [13] for an overview) and the sliding mode control technique for guaranteeing, additionally, the rejection of the boundary disturbance $\psi(t)$ under the assumption that an a priori known constant $M > 0$ exists such that $|\psi(t)| \leq M$. Lyapunov analysis will be presented to support the treatment, and simulation results will be presented to corroborate the theoretical analysis.

The rest of the paper is organized as follows. Section II collects the problem formulation and presents an outline of the proposed solution. Section III illustrates the constructive Lyapunov-based synthesis of the boundary stabilizing controller. Section IV deals with the simulation results and, finally, Section V collects some concluding remarks.

Notation and instrumental Lemmas

The notation used throughout is fairly standard. $H^l(0,1)$, with $l = 0, 1, 2, \ldots$, denotes the Sobolev space of absolutely continuous scalar functions $z(\zeta)$ on $(0,1)$ with square integrable derivatives $z^{(i)}(\zeta)$ up to the order l and the H^l-norm

$$\|z(\cdot)\|_l = \sqrt{\int_0^1 \Sigma_{i=0}^l [z^{(i)}(\zeta)]^2 d\zeta}. \tag{1.4}$$

Throughout the paper we shall also utilize the standard notation $H^0(0,1) = L_2(0,1)$.

Lemma 2. *[25] Let $y(\xi) \in H^1(0,1)$. Then, the following inequality holds:*

$$y^2(i) \leq y^2(1-i) + 2\|y(\cdot)\|_0 \|y_\xi(\cdot)\|_0, \quad i = 0, 1 \tag{1.5}$$

(1.6)

Lemma 3. *Extended triangle inequality. Let $a, b, \gamma \in \Re$ with $\gamma > 0$. Then, the following inequalities hold:*

$$-\left(\frac{\gamma}{2}a^2 + \frac{1}{2\gamma}b^2\right) \leq ab \leq \frac{\gamma}{2}a^2 + \frac{1}{2\gamma}b^2 \tag{1.7}$$

1.1.2 Problem formulation and solution outline

We consider the space- and time-varying scalar field $z(\xi, t)$ evolving in a Hilbert space $H = L_2(0,1)$, with the monodimensional (1D) spatial variable $\xi \in [0,1]$ and time variable $t \geq 0$. Let it be governed by the boundary value problem (1.1)-(1.3), where $v(t)$ is the manipulable boundary control input, θ, λ and q are the system's parameters, and $\psi(t)$ is a nonvanishing disturbance. For a large positive λ and q the system can have arbitrarily many unstable eigenvalues. The initial condition (IC) is

$$z(\xi, 0) = z^0(\xi) \in H^4(0,1). \tag{1.8}$$

The class of initial functions and admissible disturbances is specified by the following assumption.

Assumption 1. *The initial function $z^0(\xi)$ is compatible to the following perturbed BC's*

$$z^0_\xi(0) = -qz^0(0), \;\; z^0_\xi(1) = \psi(0), \tag{1.9}$$

whereas the disturbance $\psi(t)$ is such that an a priori known constant $M > 0$ exists such that

$$|\psi(t)| \leq M, \qquad \forall t \geq 0. \tag{1.10}$$

The proposed control strategy is developed through the two following steps:
Step 1. Following [67], the backstepping transformation

$$x(\xi,t) = z(\xi,t) - \int_0^{\xi} k(\xi,y)z(y,t)dy \tag{1.11}$$

is employed to map system (1.1)-(1.3) into the target dynamics

$$\begin{aligned} x_t(\xi,t) &= \theta x_{\xi\xi}(\xi,t) - cx(\xi,t) && (1.12)\\ x_\xi(0,t) &= -qx(0,t), && (1.13)\\ x_\xi(1,t) &= u(t) + \psi(t) && (1.14) \end{aligned}$$

where $u(t)$ is a new manipulable input and c is an arbitrarily chosen positive constant.
Step 2. A first-order sliding mode algorithm is employed to design a discontinuous boundary input $u(t)$ providing the exponential stabilization of the target dynamics (and, therefore, of the original dynamics as well) in the Hilbert space $L_2(0,1)$.

1.1.3 Main result

The proposed boundary control algorithm takes the form

$$v(t) = -\frac{\lambda^*}{2}z(1,t) + \int_0^1 k_\xi(1,y)z(y,t)dy + u(t) \tag{1.15}$$

where

$$\lambda^* = \frac{\lambda + c}{\theta}, \tag{1.16}$$

$$\begin{aligned} k_\xi(1,y) &= -\frac{\lambda^*}{2}z(1,t) - \lambda^* \int_0^1 \frac{I_1(\sqrt{\lambda^*(1-y^2)})}{\sqrt{\lambda^*(1-y^2)}} z(y,t)\,dy \\ &- \quad \lambda^* \int_0^1 \frac{I_2(\sqrt{\lambda^*(1-y^2)})}{1-y^2} z(y,t)dy \\ &+ \quad \frac{q\lambda^{*2}}{2\sqrt{\lambda^*+q^2}} \int_0^1 \left[\int_0^{1-y} e^{-\frac{q\tau}{2}} \sinh\left(\frac{\sqrt{\lambda^*+q^2}}{2}\tau\right) \times \right. \\ &\times \quad \left. \frac{(2-\tau)I_0\left(\sqrt{\lambda^*(1+y)(1-y-\tau)}\right)}{\sqrt{\lambda^*(1+y)(1-y-\tau)}} d\tau \right] z(y,t)\,dy, \end{aligned} \tag{1.17}$$

The control signal $u(t)$ entering (1.15) is given by

$$u(t) = -U_1 \text{sign}\, x(1,t) - U_2 x(1,t) \tag{1.18}$$

where U_1, U_2 are constants parameter subject to appropriate inequalities that shall be derived throughout the paper, and sign $\cdot$ stands for the multi-valued function sign z : $\mathbb{R} \to [-1,1]$ such that

$$\text{sign}\, z \in \begin{cases} 1 & z > 0 \\ [-1,1] & z = 0 \\ -1 & z < 0 \end{cases} \tag{1.19}$$

Relation (1.18) depends on $x(1,t)$. The explicit form of $x(1,t)$ in terms of the original systems coordinates, can be derived by (1.11) as follows

$$x(1,t) = z(1,t) - \int_0^1 k(1,y)z(y,t)dy \tag{1.20}$$

where

$$
\begin{aligned}
k(1,y) &= -\lambda\frac{I_1(\sqrt{\lambda^*(1-y^2)})}{\sqrt{\lambda^*(1-y^2)}} \\
&+ \frac{q\lambda^*}{\sqrt{\lambda^*+q^2}}\int_0^{1-y} e^{-q\tau/2} sinh\left(\frac{\sqrt{\lambda^*+q^2}}{2}\tau\right)\times \\
&\times\ I_0\left(\sqrt{\lambda^*(1+y)(1-y-\tau)}\right)d\tau \qquad (1.21)
\end{aligned}
$$

Remark 1. *Since the applied control input is discontinuous, the precise meaning of the solutions of the distributed parameter system (1.1)-(1.3), driven by the discontinuous controller (1.15)-(1.21), is then specified in the sense of Filippov [62]. Extension of the Filippov concept towards the infinite-dimensional setting may be found in [52, 64]. As in the finite-dimensional case, a motion along the discontinuity manifold, if any, is referred to as a sliding mode.*

The next Theorem investigates the convergence features of the proposed boundary control design and summarizes the main result of this paper.

Theorem 8. *Consider the boundary value problem (1.1)-(1.3), satisfying the Assumption 1, along with the boundary control strategy (1.15)-(1.21). Let the controller parameters be chosen according to the following inequalities*

$$c > \theta q^2 \qquad (1.22)$$

$$U_1 > M \qquad (1.23)$$

$$U_2 > q \qquad (1.24)$$

Then, the global exponential stability of the closed-loop system (1.1)-(1.3), (1.15)-(1.21), in the space $L_2(0,1)$ is in force in accordance with

$$\|z(\cdot,t)\|_0 \leq A\|z(\cdot,0)\|_0 e^{-(c-\theta q^2)t}, \qquad (1.25)$$

where A is a positive constant independent of $z(x,0)$.

Proof of Theorem 8

The proof is composed of three steps

1. Backstepping transformation

To map system (1.1)-(1.3) into the target dynamics (1.12)-(1.14) the backstepping transformation (1.11) is employed. In order to transfer the original PDE (1.1) into the target PDE (1.12), the associated Kernel PDE was derived in [67, Sect. VIII] as follows:

$$
\begin{aligned}
k_{\xi\xi}(\xi,y) - k_{yy}(\xi,y) - \lambda^* k(\xi,y) &= 0, &(1.26)\\
k_y(\xi,0) &= -qk(\xi,0) &(1.27)\\
k(\xi,\xi) &= -\frac{\lambda^*}{2}\xi &(1.28)
\end{aligned}
$$

Conditions (1.26)-(1.28) are compatible and form a well-posed PDE with the continuously differentiable solution [67, Sect. VIII]

$$\begin{aligned}
k(x,y) &= -\lambda x \frac{I_1(\sqrt{\lambda^*(x^2-y^2)})}{\sqrt{\lambda^*(x^2-y^2)}} \\
&+ \frac{q\lambda^*}{\sqrt{\lambda^*+q^2}} \int_0^{x-y} e^{q\tau/2} sinh\left(\frac{\sqrt{\lambda^*+q^2}}{2}\tau\right) \times \\
&\times I_0\left(\sqrt{\lambda^*(x+y)(x-y-\tau)}\right) d\tau \qquad (1.29)
\end{aligned}$$

By differentiating (1.11) with respect to the spatial variable ξ it yields

$$x_\xi(\xi,t) = z_\xi(\xi,t) - k(\xi,\xi)z(\xi,t) - \int_0^\xi k_\xi(\xi,y)z(y,t)dy \qquad (1.30)$$

Evaluating (1.30) at $\xi = 0$ and $\xi = 1$ one obtains

$$x_\xi(0,t) = z_\xi(0,t) - k(0,0)z(x,t) \qquad (1.31)$$

$$x_\xi(1,t) = z_\xi(1,t) - k(1,1)z(1,t) - \int_0^1 k_\xi(1,y)z(y,t)dy \qquad (1.32)$$

Substituting into (1.31) the boundary condition (1.2), noticing that by (1.11) the equality $z(0,t) = x(0,t)$ is in force, and observing that, by (1.28), $k(0,0) = 0$, one derives that the boundary condition (1.13) is satisfied.

Substituting into (1.32) the boundary condition (1.3), and observing that, by (1.28), $k(1,1) = -\frac{\lambda^*}{2}$, one obtains

$$x_\xi(1,t) = v(t) + \psi(t) + \frac{\lambda^*}{2}z(1,t) - \int_0^1 k_\xi(1,y)z(y,t)dy \qquad (1.33)$$

Therefore, choosing $v(t)$ as in (1.15)-(1.17), where (1.17) is obtained by spatial differentiation of (1.29) at $\xi = 1$, one derives that the boundary condition (1.14) is also satisfied. Thus, by means of the backstepping transformation (1.11) complemented by the boundary feedback (1.15)-(1.17) system (1.1)-(1.3) is transferred into the target dynamics (1.12)-(1.14).

2. Stability of the target dynamics

To assess the stability of the target dynamics (1.12)-(1.14) along with the discontinuous feedback (1.18), (1.23)-(1.24) we consider the next Lyapunov function.

$$V(t) = \frac{1}{2}\|x(\cdot,t)\|_0^2. \qquad (1.34)$$

In light of (1.12), the corresponding time derivative takes the form:

$$\dot{V}(t) = \int_0^1 x(\nu,t)x_t(\nu,t)d\nu = \theta\int_0^1 x(\nu,t)x_{\xi\xi}(\xi,t)d\nu - c\int_0^1 x^2(\nu,t)d\nu \qquad (1.35)$$

Integration by parts of the first term in the last row of (1.35) yields

$$\theta\int_0^1 x(\nu,t)x_{\xi\xi}(\nu,t)d\nu = \theta[x(1,t)x_\xi(1,t) - x(0,t)x_\xi(0,t)] - \theta\int_0^1 x_\xi^2(\nu,t)d\nu \qquad (1.36)$$

Considering the BCs (1.13) and (1.14) into (1.36) one obtains

$$\begin{aligned}
&\theta \int_0^1 x(\nu,t)x_{\xi\xi}(\nu,t)d\nu \\
= \quad & \theta x(1,t)x_\xi(1,t) - \theta x(0,t)x_\xi(0,t) - \theta\|x_\xi(\cdot,t)\|_0^2 \\
= \quad & -\theta U_1|x(1,t)| - \theta U_2 x^2(1,t) + \theta x(1,t)\psi(t) + \theta q x^2(0,t) - \theta\|x_\xi(\cdot,t)\|_0^2
\end{aligned} \tag{1.37}$$

Now substituting (1.37) into (1.35), and rearranging, one straightforwardly derives the next

$$\begin{aligned}
\dot V(t) \quad = \quad & -\theta U_1|x(1,t)| - \theta U_2 x^2(1,t) + \theta\psi(t)x(1,t) + \theta q x^2(0,t) \\
- \quad & \theta\|x_\xi(\cdot,t)\|_0^2 - c\|x(\cdot,t)\|_0^2
\end{aligned} \tag{1.38}$$

By Agmon's inequality (1.5), specified with $y(\cdot) = x(\cdot)$ and $i = 0$, one has that

$$x^2(0,t) \quad \leq \quad x^2(1,t) + 2\|x(\cdot,t)\|_0\|x_\xi(\cdot,t)\|_0 \tag{1.39}$$

Inequality (1.7) allows us to further manipulate (1.39) as follows

$$x^2(0,t) \leq x^2(1,t) + \gamma_1\|x(\cdot,t)\|_0^2 + \frac{1}{\gamma_1}\|x_\xi(\cdot,t)\|_0^2 \tag{1.40}$$

where γ_1 is an arbitrary positive constant. Thus by (1.40) we conclude that

$$|q\theta x^2(0,t)| \leq \theta q x^2(1,t) + \theta q\gamma_1\|x(\cdot,t)\|_0^2 + \frac{\theta q}{\gamma_1}\|x_\xi(\cdot,t)\|_0^2 \tag{1.41}$$

Substituting (1.41) and (1.10) into (1.38), and making simple manipulations, one derives the next estimation

$$\begin{aligned}
\dot V(t) \quad \leq \quad & -\theta(U_1 - M)|x(1,t)| - \theta(U_2 - q)x^2(1,t) \\
- \quad & (c - \theta q\gamma_1)\|x(\cdot,t)\|_0^2 - \theta\left(1 - \frac{q}{\gamma_1}\right)\|x_\xi(\cdot,t)\|_0^2
\end{aligned} \tag{1.42}$$

In order to have a negative definite right hand side of $\dot V(t)$, we select the arbitrary constant γ_1 as follows

$$\gamma_1 = q \tag{1.43}$$

and require that the next system of inequalities should be satisfied

$$\left\{ \begin{array}{c} U_1 - M > 0 \\ U_2 - q > 0 \\ c - \theta q\gamma_1 = c - \theta q^2 > 0 \end{array} \right. \tag{1.44}$$

which straightforwardly give rise to (1.22)-(1.24).

Considering together (1.42), (1.34) and (1.44), it is then trivial to derive the next differential inequality for the Lyapunov function $V(t)$

$$\dot V(t) \leq -2(c - \theta q^2)V(t) \tag{1.45}$$

which implies that $V(t)$ exponentially escapes to zero. Therefore, the target system dynamics (1.12)-(1.14) along with the discontinuous feedback (1.18), (1.23)-(1.24) is exponentially stable in the space $L_2(0,1)$ in the sense that it can be found a constant A, independent on $x(\cdot,0)$, such that

$$\|x(\cdot,t)\|_0 \leq A\|x(\cdot,0)\|_0 e^{-(c-\theta q^2)t}, \tag{1.46}$$

3. Stability of the original system

To prove stability of the original we need to prove that the transformation (1.11) is invertible

By performing analogous developments as those made for the derivation of the kernel PDE (1.26)-(1.28) of the direct transformation (1.11), the kernel PDE of the inverse transformation

$$z(\xi,t)=x(\xi,t)+\int_0^{\xi} l(\xi,y)x(y,t)dy \tag{1.47}$$

can be derived as

$$\begin{aligned} l_{xx}(x,y)-l_{yy}(x,y)+\lambda^* l(x,y) &= 0 && (1.48)\\ l_y(x,0) &= q l(x,0) && (1.49)\\ l(x,x) &= -\frac{\lambda^*}{2}x && (1.50) \end{aligned}$$

By comparison between (1.26)-(1.28) and (1.48)-(1.50) one immediately notice that in this case $l(x,y)=-k(x,y)$ when λ^* and q are replaced by $-\lambda^*$ and $-q$. Now it suffices to explicitly denote the dependence of the solutions $l(x,y)=l(x,y;\lambda^*,q)$ and $k(x,y)=k(x,y;\lambda^*,q)$ on λ^* and q and verify that the substitution $l(x,y;\lambda^*,q)=-k(x,y;-\lambda^*,-q)$ transfers (1.48)-(1.50) into (1.26)-(1.28).

By taking into account (1.29), it thus yields that the boundary value problem (1.48)-(1.50) admits the continuously differentiable solution

$$\begin{aligned} l(x,y) &= -\lambda x\frac{J_1(\sqrt{\lambda^*(x^2-y^2)})}{\sqrt{\lambda^*(x^2-y^2)}}+\\ &\quad \frac{q\lambda^*}{\sqrt{q^2-\lambda^*}}\int_0^{x-y} e^{q\tau/2} sinh\left(\frac{\sqrt{q^2-\lambda^*}}{2}\tau\right)\times\\ &\times\ J_0\left(\sqrt{\lambda^*(x+y)(x-y-\tau)}\right)d\tau && (1.51) \end{aligned}$$

Recall that the exponential stability of (1.12)-(1.14), (1.18), (1.23)-(1.24) was previously demonstrated, showing the exponential decay of $\|x(\cdot,t)\|_0$ as time goes to infinity. With this in mind, it is followed [67] to derive analogous convergence properties for the original system (1.1)-(1.3), (1.15)-(1.21) as well. The estimates $|k(x,y)|\le Me^{2Mx}$ and $|l(x,y)|\le Me^{2Mx}$ are established for some positive constant M [67]. A straightforward generalization of [67, Th 4] yields that the above two upper estimates, coupled together, establish the equivalence of norms of $z(x,t)$ and $x(x,t)$ in $L_2(0,1)$ thereby ensuring that there exists a positive constant A independent of $z(\xi,0)$ such that (1.25) straightforwardly follows from (1.46). This completes the proof of Theorem 8. ■

Remark 2. *It can be proven that the proposed controller also guarantees the annihilation of $z_\xi(1,t)$, which means that the average component of the discontinuous control, which can be extracted through low-pass filtering (see, e.g., [66]), will converge to the sign-reversed disturbance $-\psi(t)$. This disturbance estimation capability can be useful for FDI purposes to reveal the occurrence of additive actuators faults. This property can be ensured by an additional more involved Lyapunov analysis proving convergence in the Sobolev space $H_2(0,1)$ rather than just in the Hilbert space $L_2(0,1)$, which is skipped for the sake of brevity. In the simulation Section, however, this capability will be verified.*

1.1.4 Simulations

Consider the perturbed reaction diffusion equation (1.1)-(1.3), with parameters $\theta = 1$, $\lambda = 5$, $q = 1$ and the boundary disturbance $\psi(t)$ set to

$$\psi(t) = 3 + 5\mathtt{sin}(5t). \tag{1.52}$$

The magnitude of the disturbance $\psi(t)$ can be upper-estimated as $M = 8$, according to (1.10). The initial conditions have been selected as $x(\xi, 0) = 12sin(\pi\xi) + 6sin(3\pi\xi)$.

For solving the closed-loop PDE, a standard finite-difference approximation method is used by discretizing the spatial solution domain $\xi \in [0, 1]$ into a finite number of N uniformly spaced solution nodes $\xi_i = ih$, $h = 1/(N+1)$, $i = 1, 2, ..., N$. The value $N = 40$ has been used. The resulting 40-th order discretized system is solved by fixed-step Euler method with step $T_s = 10^{-4}$.

Figure 1.1 depicts the spatiotemporal profile of the solution $z(\xi, t)$ in the open loop case (i.e., with $v(t) = 0$, TEST 1) which confirms the open-loop instability of the considered system.

In a preliminary test (TEST 2), the control (1.15)-(1.17) is applied by choosing $c = 2$ and setting $u(t) = 0$. The effect of the unrejected disturbance $\psi(t)$ is clearly visible in Figure 1.2, which shows that persistent residual oscillations affect the spatiotemporal profile of the solution $z(\xi, t)$. TEST 2 shows that the discontinuous control $u(t)$ is responsible for the rejection of the disturbance $\psi(t)$, which cannot be provided by a purely backstepping based control. Figure 1.3 shows the corresponding norm $\|z(\cdot, t)\|_0$, which exhibits steady state oscillations.

The complete controller (1.15)-(1.18), combining the backstepping-based and sliding-mode control terms, has been then implemented in the final TEST 3, with the parameters $c = 2$, $U_1 = 16$, $U_2 = 2$, which are selected in accordance with (1.22)-(1.24).

Figure 1.4 depicts the spatiotemporal profile of the solution $z(\xi, t)$. It can be seen that the closed loop trajectory is quickly regulated at the zero value, as expected. Figure 1.5 shows the corresponding time profile of the norm $\|z(\cdot, t)\|_0$. Finally, Figure 1.6 shows that the average component $v_{av}(t)$ of the discontinuous control $v(t)$, extracted through the simple low-pass filtering

$$\tau\dot{v}_{av}(t) + v_{av}(t) = v(t), \quad \tau = 0.01, \tag{1.53}$$

estimates rather accurately the reversed disturbance $-\psi(t)$ (see Remark 2).

1.1.5 Conclusions

A class of unstable and perturbed reaction-diffusion processes has been stabilized in the space $L_2(0, 1)$ by means of a synergic combination between the infinite-dimensional backstepping and the sliding mode control methodologies. An interesting topic for future generalization of the present result is that of widening the controlled class of systems by including the convection term and by covering the case of spatially and/or temporally varying parameters. Other research directions will be explored as well, namely the use of second-order sliding mode control to alleviate chattering, and the design of robust observers for reconstructing the state $z(\xi, t)$ using boundary sensing.

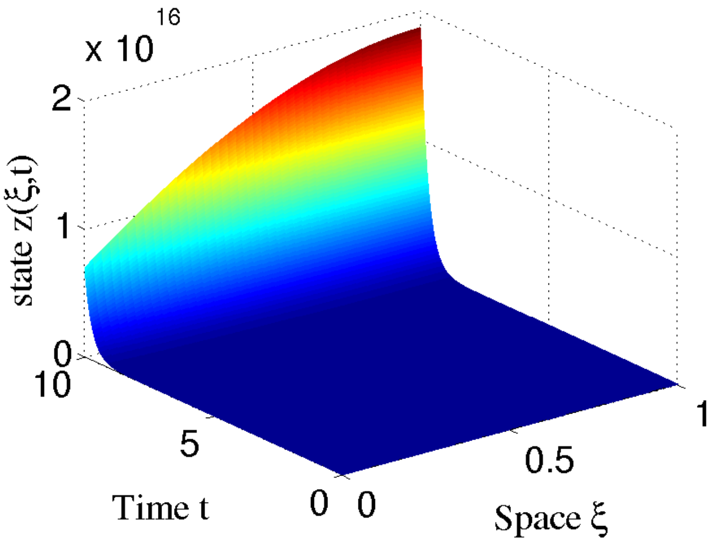

Figure 1.1: Solution $z(\xi, t)$ in the open-loop test with v(t)=0 (TEST 1)

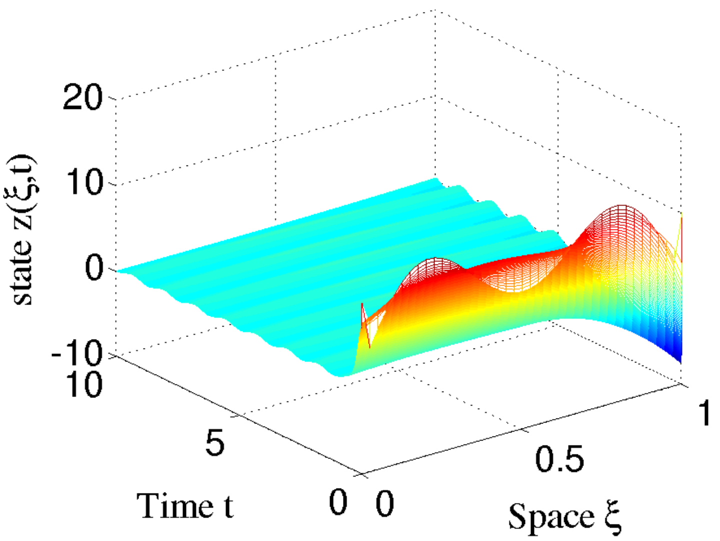

Figure 1.2: Solution $z(\xi, t)$ in the closed-loop test with $u(t) = 0$ (TEST 2).

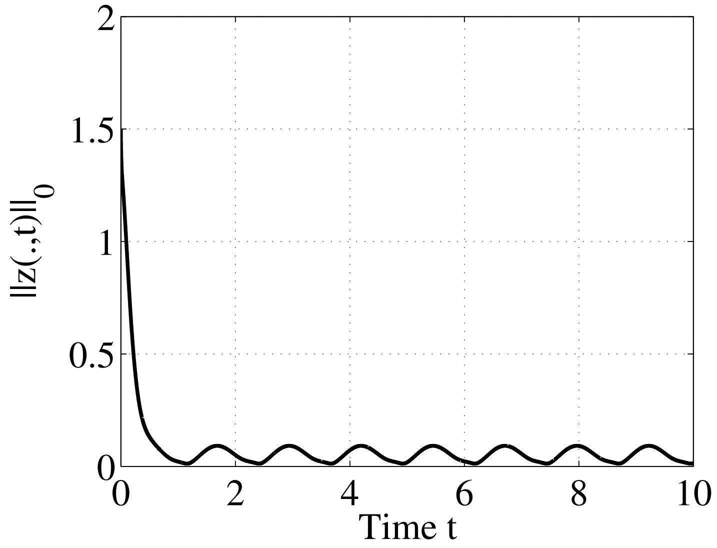

Figure 1.3: L_2 norm $\|z(\cdot,t)\|_0$ in the closed loop test with with $u(t) = 0$ (TEST 2).

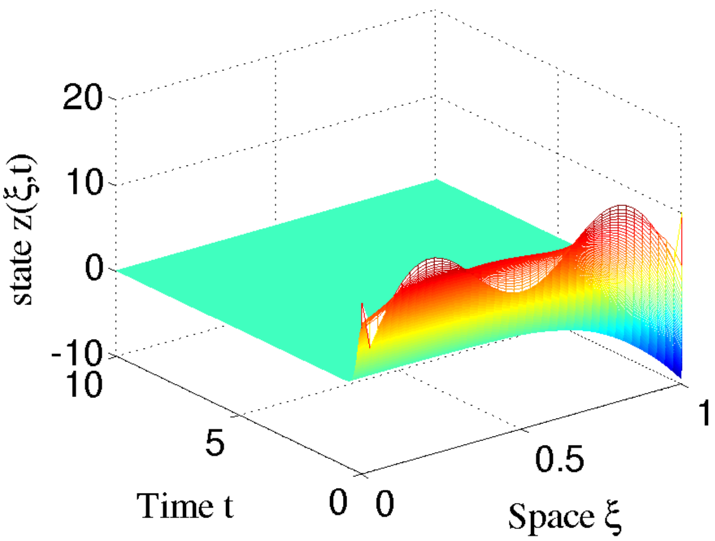

Figure 1.4: Solution $z(\xi, t)$ in the closed loop test with the complete controller (TEST 3).

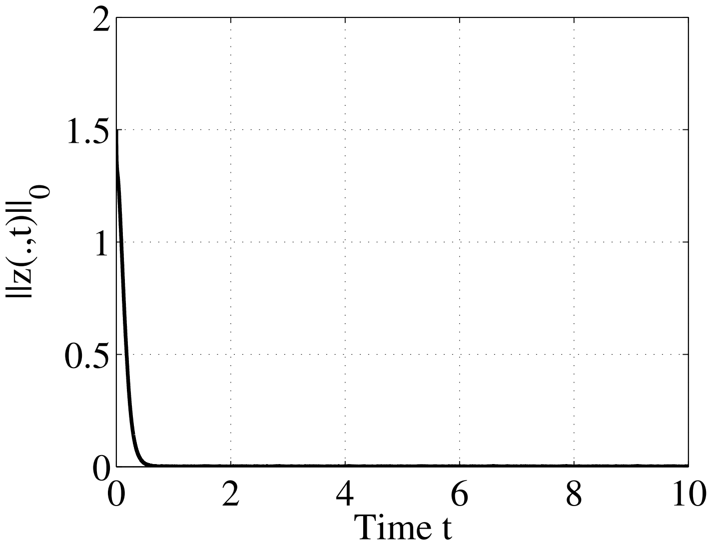

Figure 1.5: L_2 norm $\|z(\cdot,t)\|_0$ in the closed loop test with the complete controller (TEST 3).

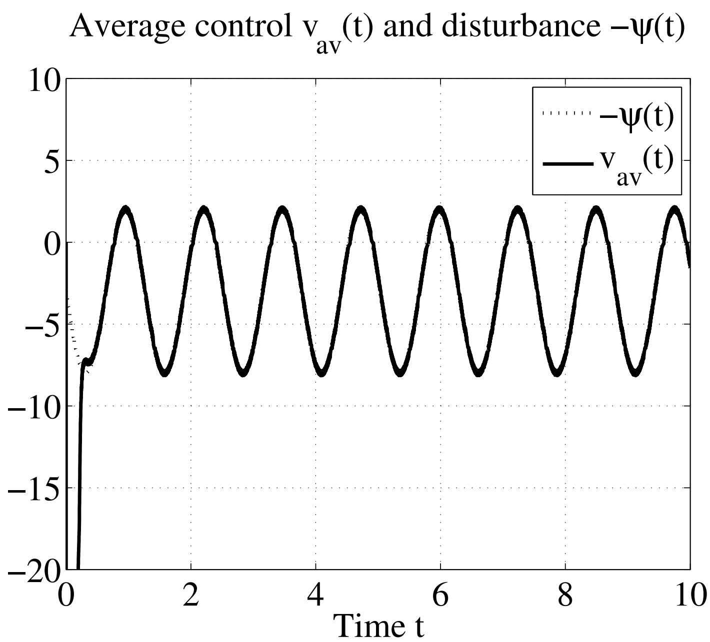

Figure 1.6: Average control $v_{av}(t)$ (continuous line) and the disturbance $-\psi(t)$ (dotted line).

1.2 Boundary control of distributed parameter systems by second-order sliding-mode technique

The primary concern of the present paper is to give an overview of the available results and methods in the field of second-order sliding mode based boundary control synthesis for uncertain and perturbed distributed parameter systems. We particularly aim at showing how the same basic algorithm (the combined Twisting/PD algorithm) can be applied to solve different problems involving parabolic and hyperbolic-type equations. Then, we deal with a reaction-diffusion process by also providing some novelty in that a destabilizing mixed-type boundary condition, which was not considered in the previous work [25], is taken into account. The effectiveness of the developed controller is supported by simulation results.

1.2.1 Introduction

Many important engineering systems and industrial processes are governed by partial differential equations (PDEs) and are often subject to a significant degree of uncertainty. Therefore, a growing interest is arising towards extending sliding mode control to infinite-dimensional systems. Presently, the discontinuous control synthesis in the infinite-dimensional setting is well documented [52, 54, 57] and it is generally shown to retain the main robustness features as those possessed by its finite-dimensional counterpart.

The boundary control problem for several classes of wave processes was studied, e.g., in [53, 58, 14, 68]. while the boundary control problem for heat processes was studied, e.g., in [47, 63, 50]. Generally, in the available literature more strict assumptions are required on the admitted uncertainties and perturbations as compared to those made in the present work. In the present paper we address the disturbance rejection problem by assuming that collocated sensing at the controlled boundary is only available, and we deal with boundary controller design based on the second-order sliding mode methodology. Other recent related works, where second order sliding mode algorithms were applied in the distributed parameter systems domain addressing the distributed (rather than boundary) control design are also worth to mention [55, 56].

The task of the present paper is to to give an overview of the available results and methods in the field of second-order sliding mode based boundary control synthesis for uncertain and perturbed distributed parameter systems. We particularly aim at showing how the same basic algorithm (the combined Twisting/PD algorithm) can be applied to solve different problems involving parabolic and hyperbolic-type equations. We shall also, at the same time, provide some advance to the field as well. We first recall a recent result [24] dealing with the discontinuous boundary regulation of a wave process with collocated boundary sensing and Neumann-type actuation. Then, we deal with a reaction-diffusion process by providing some novelty in that a destabilizing mixed-type boundary condition, which was not considered in the previous work [25], is now coped with.

The rest of the paper is organized as follows. In Section II, a family of perturbed wave processes is introduced along with the associated stabilizing discontinuous boundary controller design. Section III concerns the boundary stabilization of an uncertain and perturbed reaction diffusion process by means of a continuous, second-order sliding mode based, dynamic controller which is able to compensate for the destabilizing effect of the anticollocated boundary condition. Section IV deals with the simulation results and, finally, Section IV collects some concluding remarks.

Notation

The notation used throughout is fairly standard. $H^l(0,1)$, with $l = 0, 1, 2, \ldots$, denotes the Sobolev space of absolutely continuous scalar functions $z(\zeta)$ on $(0,1)$ with square integrable derivatives $z^{(i)}(\zeta)$ up to the order l and the H^l-norm

$$\|z(\cdot)\|_l = \sqrt{\int_0^1 \Sigma_{i=0}^l [z^{(i)}(\zeta)]^2 d\zeta}. \tag{1.54}$$

Throughout the paper we shall also utilize the standard notation $H^0(0,1) = L_2(0,1)$.

Lemma 4. *Agmon's inequality. Let $y(\xi) \in H^1(0,1)$. Then, the following inequality holds:*

$$y^2(i) \leq y^2(1-i) + 2\|y(\cdot)\|_0 \|y_\xi(\cdot)\|_0, \quad i = 0, 1 \tag{1.55}$$

Lemma 5. *Let $a, b, \gamma \in \Re$ with $\gamma > 0$. Then, the following well known inequalities hold:*

$$ab \leq \frac{\gamma}{2} a^2 + \frac{1}{2\gamma} b^2 \tag{1.56}$$

$$ab \geq -\left(\frac{\gamma}{2} a^2 + \frac{1}{2\gamma} b^2\right) \tag{1.57}$$

Also, recall the next popular inequality (a special case of the Cauchy-Schwartz inequality)

$$\int_0^1 |z(\nu)| d\nu \leq \sqrt{\int_0^1 z^2(\nu) d\nu} = \|z(\cdot)\|_0 \tag{1.58}$$

1.2.2 Boundary Controller Synthesis for perturbed wave processes

Consider the space- and time-varying scalar field $y(\xi, t)$ with the monodimensional (1D) spatial variable $\xi \in [0, 1]$ and time variable $t \geq 0$. Let it be governed by

$$y_{tt}(\xi, t) = \theta y_{\xi\xi}(\xi, t), \tag{1.59}$$

where y_{tt} and $y_{\xi\xi}$ denote second order temporal and spatial derivatives, respectively, and θ is a positive unknown coefficient. The initial conditions (ICs) are $y(\xi, 0) = y^0(\xi) \in$

$H^2(0,1)$, $y_t(\xi,0) = y_t^0(\xi) \in H^2(0,1)$. Throughout, we consider controlled and perturbed Neumann-type BC's of the form

$$\begin{aligned} y_\xi(0,t) &= c_0 y_t(0,t), \quad c_0 > 0 &(1.60)\\ y_\xi(1,t) &= u(t) + \psi(t), &(1.61)\end{aligned}$$

where c_0 is a positive constant, $u(t) \in \mathbb{R}$ is a manipulable source term (boundary control input) and $\psi(t) \in \mathbb{R}$ represents an **uncertain** sufficiently smooth disturbance. The class of initial functions and admissible disturbances is specified by the following assumption.

Assumption 2. *The initial function $y^0(\xi)$ is compatible to the next perturbed BC's*

$$y_\xi^0(0) = c_0 y_t^0(0,t), \;\; y_\xi^0(1) = \psi(0), \tag{1.62}$$

whereas the disturbance $\psi(t)$ is twice continuously differentiable, and there exists an a priori known constant $M > 0$ such that

$$|\psi(t)| \le M, \qquad \forall t \ge 0. \tag{1.63}$$

The following discontinuous controller

$$u(t) \;=\; -\lambda_1 \text{sign}\, y(1,t) - \lambda_2 \text{sign}\, y_t(1,t) - W_1 y(1,t) - W_2 y_t(1,t), \quad t > 0 \tag{1.64}$$

is currently under study. In the above relation, λ_1, λ_2, W_1 and W_2 are constant parameters subject to the inequalities

$$\lambda_2 > M, \;\; \lambda_1 > \lambda_2 + M, \;\; W_1 > \frac{1}{2} c_0, \;\; W_2 > 0. \tag{1.65}$$

The next Theorem states the convergence properties of the closed loop system.

Theorem 9. Consider the perturbed wave equation (1.59)-(1.61) with Assumption 1 above and with the boundary control strategy (1.84)-(1.65) applied. Then the solution $(y(\cdot,t), y_t(\cdot,t))$ of the resulting closed-loop boundary-value problem are globally exponentially stable in the space $H^1(0,1) \times L_2(0,1)$.

Proof. A sketch of the proof is given. A more detailed treatment can be found in [24]. Consider the next Lyapunov function

$$V_1(t) \;=\; \lambda_1 \theta |y(1,t)| + \frac{1}{2}\theta W_1 y^2(1,t) + \frac{1}{2}\|y_t(\cdot,t)\|_0^2 + \frac{1}{2}\theta \|y_\xi(\cdot,t)\|_0^2, \tag{1.66}$$

After appropriate developments, the corresponding time derivative is estimated as follows

$$\dot{V}_1(t) \;\le\; -\theta[(\lambda_2 - M)|y_t(1,t)| + W_2 y_t^2(1,t)] - \theta c_0 y_t^2(0,t) \tag{1.67}$$

Relation (1.67) establishes that, given any $R \geq V_1(0)$, the domain $V_1(t) \leq R$ will be invariant, which implies that

$$|y(1,t)| \leq \frac{R}{\lambda_1 \theta}, \ \|y_t(\cdot,t)\|_0^2 \leq 2R, \ \|y_\xi(\cdot,t)\|_0^2 \leq 2R/\theta. \tag{1.68}$$

Consider the next augmented Lyapunov function

$$V_\rho(t) \ = \ V_1(t) + \frac{1}{2}\kappa_\rho \theta W_2 y^2(1,t) + \kappa_\rho \int_0^1 y(1,t) y_t(\nu,t) d\nu \tag{1.69}$$

where κ_ρ is a positive constant small enough so as to preserve the positive definiteness of V_ρ. After suitable developments, the time derivative of $V_\rho(t)$ satisfies the inequality

$$\begin{aligned}
\dot{V}_\rho(t) \ &\leq \ -\theta(\lambda_2 - M - k_\rho\sqrt{2R})|y_t(1,t)| - \theta W_2 y_t^2(1,t) \\
&- \ \theta c_0 \left(1 - \frac{1}{2}k_\rho\right) y_t^2(0,t) \\
&- \ k_\rho \theta(\lambda_1 - \lambda_2 - M)|y(1,t)| \\
&- \ k_\rho \theta \left(W_1 - \frac{1}{2}c_0\right) y^2(1,t)
\end{aligned} \tag{1.70}$$

The inequality formed by joining (1.70) and (1.65) (along with the suitable constraint on κ_ρ being small enough) is still insufficient to establish the desired convergence properties. Therefore, consider a new augmented Lyapunov function

$$V_R(t) \ = \ V_\rho(t) + \kappa_R \int_0^1 (\nu - 1) y_\xi(\nu,t) y_t(\nu,t) d\nu \tag{1.71}$$

where $\kappa_R > 0$ is another positive parameter small enough. After appropriate developments, one derives that

$$\begin{aligned}
\dot{V}_R(t) \ &\leq \ -\theta(\lambda_2 - M - k_\rho\sqrt{2R})|y_t(1,t)| - \theta W_2 y_t^2(1,t) \\
&- \ \left[\theta c_0 \left(1 - \frac{1}{2}k_\rho\right) - \frac{1}{2}\kappa_R(1 + c_0^2\theta)\right] y_t^2(0,t) \\
&- \ k_\rho \theta(\lambda_1 - \lambda_2 - M)|y(1,t)| \\
&- \ k_\rho \theta \left(W_1 - \frac{1}{2}c_0\right) y^2(1,t) - \frac{1}{2}\kappa_R \|y_t(\cdot,t)\|_0^2 \\
&- \ \frac{1}{2}\kappa_R \theta \|y_\xi(\cdot,t)\|_0^2
\end{aligned} \tag{1.72}$$

It can be found two positive constants ρ_1 and ρ_2 (with $\rho_2 > \rho_1$) such that

$$\begin{aligned}
&\rho_1 \left(|y(1,t)| + \|y(\cdot,t)\|_1^2 + \|y_t(\cdot,t)\|_0^2\right) \leq V_R(t) \leq \\
&\qquad \rho_2 \left(|y(1,t)| + \|y(\cdot,t)\|_1^2 + \|y_t(\cdot,t)\|_0^2\right)
\end{aligned} \tag{1.73}$$

which means that the functional $V_R(t)$, being computed on the solutions (y, y_t) of the boundary-value problem (1.59), (1.60)-(1.61), (1.84), is equivalent to the $H^1(0,1) \times$

$L_2(0,1)$ norm of these solutions. It is also possible to find out a positive constant ρ_3 such that

$$\dot{V}_R(t) \leq -\rho_3 \left(|y(1,t)| + \|y(\cdot,t)\|_1^2 + \|y_t(\cdot,t)\|_0^2\right) \tag{1.74}$$

Relations (1.73) and (1.74), coupled together, result in $\dot{V}_R(t) \leq -\frac{\rho_3}{\rho_2} V_R(t)$ that establishes the exponential decay of $V_R(t)$, thereby proving, according to (1.73), the asymptotic stability of the solutions (y, y_t) of the boundary-value problem (1.59), (1.60)-(1.61), (1.84) in the space $H^1(0,1) \times L_2(0,1)$. Since the Lyapunov functional is radially unbounded, the *global* exponential stability in the $H^1(0,1) \times L_2(0,1)$-space is then concluded for the closed loop system under investigation.

□

1.2.3 Boundary Controller Synthesis for perturbed reaction-diffusion processes

Consider the space- and time-varying scalar field $x(\xi,t)$ with the monodimensional (1D) spatial variable $\xi \in [0,1]$ and time variable $t \geq 0$. Let it be governed by the boundary value problem

$$\begin{aligned} x_t(\xi,t) &= \theta x_{\xi\xi}(\xi,t) - cx(\xi,t), & (1.75)\\ x_\xi(0,t) &= qx(0,t), & (1.76)\\ x_\xi(1,t) &= u(t) + \psi(t) & (1.77) \end{aligned}$$

where x_t and $x_{\xi\xi}$ denote temporal and second order spatial derivatives, respectively, and the corresponding initial condition (ICs) are $x(\xi,0) = x^0(\xi) \in H^4(0,1)$.

θ and c are positive coefficients (called "diffusivity" and "reaction coefficient" . respectively), and q is a negative constant (a negative value for q is considered since it corresponds to the **destabilizing** case [59]). $u(t) \in \mathbb{R}$ is a manipulable source term (boundary control input) and $\psi(t) \in \mathbb{R}$ represents an uncertain sufficiently smooth disturbance. The class of initial functions and admissible disturbances is specified by the following assumption.

Assumption 3. *The initial function $x^0(\xi)$ is compatible to the next perturbed BC's*

$$x_\xi^0(0) = qx^0(0), \;\; x_\xi^0(1) = \psi(0), \tag{1.78}$$

whereas the disturbance $\psi(t)$ is twice continuously differentiable, and there exists an a priori known constant $M > 0$ such that

$$|\dot{\psi}(t)| \leq M, \qquad \forall t \geq 0. \tag{1.79}$$

Assumption 4. *There exist a priori known constants $Q > 0$ and $\Theta_m < \Theta_M$ such that*

$$|q| \leq Q, \qquad 1 < \Theta_m \leq \theta \leq \Theta_M \tag{1.80}$$

Differentiating with respect to time the original system (1.75)-(1.77) one obtains the next boundary value problem

$$
\begin{aligned}
x_{tt}(\xi,t) &= \theta x_{\xi\xi t}(\xi,t) - c x_t(\xi,t), && (1.81)\\
x_{\xi t}(0,t) &= q x_t(0,t), && (1.82)\\
x_{\xi t}(1,t) &= \dot{u}(t) + \dot{\psi}(t) && (1.83)
\end{aligned}
$$

whose augmented state vector (x, x_t) evolves in the Hilbert space $L_2(0,1) \times L_2(0,1)$.

The following dynamical controller

$$
\begin{aligned}
\dot{u}(t) &= -\lambda_1 \text{sign}\, x(1,t) - \lambda_2 \text{sign}\, x_t(1,t) - W_1 x(1,t)\\
&- W_2 x_t(1,t), \qquad u(0) = 0 && (1.84)
\end{aligned}
$$

is currently under study. In the above relation, λ_1, λ_2, W_1 and W_2 are constant parameters subject to appropriate inequalities that shall be derived throughout the paper.

Remark 3. *The proposed dynamical controller makes the explicit use of the boundary state $x(1,t)$ and of its time derivative $x_t(1,t)$. The use for feedback of the state derivative is normally impermissible to use in the static synthesis of feedback loops as it generally induces algebraic loops. However, its use becomes acceptable in the dynamic feedback when the input signal passes through a dynamic element like, e.g., an integrator and an augmented state vector (x, x_t) is defined.*

The next Theorem states the novel result of the present paper

Theorem 10. *Consider the boundary value problem (1.81)- (1.83) with the uncertain parameters and disturbance satisfying the Assumptions 3 and 4. Let the reaction coefficient c be such that*

$$
c > \max\left\{ \Theta_M Q^2, \frac{\Theta_M^2 \left(\frac{Q^2}{2} + Q\right)^2}{2(\Theta_m - 1)} \right\} \tag{1.85}
$$

Then, the dynamical boundary control strategy (1.84) with the parameters λ_1, λ_2, W_1 and W_2 selected according to the inequalities

$$
\lambda_1 - \lambda_2 > M, \quad \lambda_2 > M, \quad W_1 > \frac{1}{2}Q + \frac{cQ}{2\Theta_m}, \quad W_2 > Q \tag{1.86}
$$

guarantees the global exponential stability of the closed-loop system (1.81)-(1.83), (1.84) in the space $L_2(0,1) \times L_2(0,1)$.

Proof. Introduce the Lyapunov function

$$
V_1(t) = \lambda_1 \theta |x(1,t)| + \frac{1}{2}\theta W_1 x^2(1,t) + \frac{1}{2}\|x_t(\cdot,t)\|_0^2 \tag{1.87}
$$

By (1.81), the corresponding time derivative takes the form:

$$\begin{aligned}
\dot{V}_1(t) &= \lambda_1\theta x_t(1,t)sign(x(1,t)) + \theta W_1 x(1,t)x_t(1,t) \\
&+ \int_0^1 x_t(\nu,t)x_{tt}(\nu,t)d\nu \\
&= \lambda_1\theta x_t(1,t)sign(x(1,t)) + \theta W_1 x(1,t)x_t(1,t) \\
&+ \int_0^1 x_t(\nu,t)\left(\theta x_{\xi\xi t}(\xi,t) - c x_t(\xi,t)\right)d\nu \\
&= \lambda_1\theta x_t(1,t)sign(x(1,t)) + \theta W_1 x(1,t)x_t(1,t) \\
&+ \theta\int_0^1 x_t(\nu,t)x_{\xi\xi t}(\xi,t)d\nu - c\|x_t(\cdot,t)\|_0^2 \qquad (1.88)
\end{aligned}$$

Solving by parts the integral term in the right hand side of (1.88), and considering (1.82)-(1.84), one straightforwardly derives the next

$$\begin{aligned}
\dot{V}_1(t) &= -\theta\lambda_2|x_t(1,t)| - \theta W_2 x_t^2(1,t) + \theta x_t(1,t)\psi_t(t) - \\
&- q\theta x_t^2(0,t) - \theta\|x_{t\xi}(\cdot,t)\|_0^2 - c\|x_t(\cdot,t)\|_0^2 \qquad (1.89)
\end{aligned}$$

By Agmon's inequality (1.55), specified with $y(\cdot) = x_t(\cdot)$ and $i = 0$, followed by the application of (1.56), one derives that

$$|q\theta x_t^2(0,t)| \le \theta|q|\left(x_t^2(1,t) + \gamma_1\|x_t(\cdot,t)\|_0^2 + \frac{1}{\gamma_1}\|x_{t\xi}(\cdot,t)\|_0^2\right) \qquad (1.90)$$

where γ_1 is an arbitrary positive constant. Substituting (1.90) and (1.79) into (1.89), and making simple manipulations, one can derive the next estimation

$$\begin{aligned}
\dot{V}_1(t) &\le -\theta(\lambda_2 - M)|x_t(1,t)| - \theta(W_2 - |q|)x_t^2(1,t) \\
&-\theta\left[1 - \frac{|q|}{\gamma_1}\right]\|x_{t\xi}(\cdot,t)\|_0^2 - \left[c - |q|\theta\gamma_1\right]\|x_t(\cdot,t)\|_0^2 \qquad (1.91)
\end{aligned}$$

In order to have a semidefinite negative right hand side of $\dot{V}_1(t)$, the next system of inequalities should be satisfied

$$\begin{array}{ll}
\lambda_2 - M > 0, & W_2 - |q| > 0 \\
1 - \frac{|q|}{\gamma_1} > 0, & c - |q|\theta\gamma_1 > 0
\end{array} \qquad (1.92)$$

By taking into account relations (1.80) it is easy to derive the next sufficient conditions

$$\lambda_2 > M, \quad W_2 > Q \qquad (1.93)$$

$$\gamma_1 > Q \quad c > Q^2\Theta_M \qquad (1.94)$$

The negative semidefiniteness of $\dot{V}_1(t)$, established by (1.91) and (1.92), allows to conclude that there is some $R > V_1(0) \ge 0$ such that the domain $V_1(t) \ge R$ will be invariant for the temporal evolution of the Lyapunov function $V_1(t)$. As a trivial consequence, the next inequalities hold

$$|x(1,t)| \le \frac{R}{\lambda_1\theta}, \quad \|x_t(\cdot,t)\|_0^2 \le 2R, \qquad (1.95)$$

Now define

$$V_2(t) = V_1(t) + \frac{1}{2}\kappa_1\theta W_2 x^2(1,t) + \kappa_1 \int_0^1 x(1,t)x_t(\nu,t)d\nu \tag{1.96}$$

It must be guaranteed that $V_2(t)$ is a positive definite function. Applying inequality (1.57) yields

$$\begin{aligned}\int_0^1 x(1,t)x_t(\nu,t)\,d\nu \geq -\frac{1}{2}\left[x^2(1,t) + \|x_t(\cdot,t)\|_0^2\right] = \\ -\frac{1}{2}\left[|x(1,t)||x(1,t)| + \|x_t(\cdot,t)\|_0^2\right].\end{aligned} \tag{1.97}$$

Being coupled together, (1.95) and (1.97) immediately result in the following

$$x(1,t)\int_0^1 x_t(\eta,t)\,d\eta \geq -\frac{1}{2}\left[\frac{R}{\lambda_1\theta}|x(1,t)| + \|x_t(\cdot,t)\|_0^2\right]. \tag{1.98}$$

By (1.98), function $V_2(t)$ can be lower estimated as

$$V_2(t) \geq \left(\lambda_1\theta - \frac{\kappa_R R}{2\lambda_1\theta}\right)|x(1,t)| + \frac{1}{2}\theta(W_1 + \kappa_1 W_2)x^2(1,t) + \frac{1}{2}(1-\kappa_1)\|x_t(\cdot,t)\|_0^2. \tag{1.99}$$

Let us specify $\kappa_1 > 0$ such that

$$\kappa_1 < \mathtt{min}\ \left\{\frac{2\lambda_1^2\theta^2}{R}, 1\right\}. \tag{1.100}$$

Then, it follows from (1.99), (1.100) that the augmented functional (1.96) is lower estimated by functional (1.87) as

$$V_2(t) \geq \mu V_1(t) \tag{1.101}$$

$$\mu \ = \ \mathtt{min}\ \left\{1 - \frac{\kappa_1 R}{2\lambda_1^2\theta^2}, \frac{W_1 + \kappa_1 W_2}{W_1}, (1-\kappa_R)\right\}. \tag{1.102}$$

It means that, along with (1.87), the functional $V_2(t)$ is positive definite too. Let us now evaluate the time derivative of $V_2(t)$. After simple manipulations:

$$\begin{aligned}\dot{V}_2(t) \leq -\theta(\lambda_2 - M)|x_t(1,t)| - \theta(W_2 - |q|)x_t^2(1,t) \\ -\theta\left[1 - \frac{|q|}{\gamma_1}\right]\|x_{t\xi}(\cdot,t)\|_0^2 - \left[c - |q|\theta\gamma_1\right]\|x_t(\cdot,t)\|_0^2 \\ +\kappa_1 x_t(1,t)\int_0^1 x_t(\nu,t)d\nu - c\kappa_1 x(1,t)\int_0^1 x_t(\nu,t)d\nu \\ -\kappa_1\theta\left(\lambda_1 - \lambda_2 - M\right)|x(1,t)| - \kappa_1\theta W_1 x^2(1,t) \\ -\kappa_1\theta q x(1,t)x_t(0,t)\end{aligned} \tag{1.103}$$

By using relation (1.58), and taking into account the bound (1.95) as well, we get

$$\left|\kappa_1 x_t(1,t)\int_0^1 x_t(\nu,t)d\nu\right| \leq \kappa_1|x_t(1,t)|\int_0^1 |x_t(\nu,t)|d\nu$$
$$\leq \kappa_1|x_t(1,t)|\|x_t(\cdot,t)\|_0 \leq \kappa_1\sqrt{2R}|x_t(1,t)|$$
$$\left|c\kappa_1 x(1,t)\int_0^1 x_t(\nu,t)d\nu\right| \leq \frac{1}{2}c\kappa_1\gamma_2 x^2(1,t)$$
$$+\frac{1}{2}c\frac{\kappa_1}{\gamma_2}\|x_t(\cdot,t)\|_0^2$$
$$|\kappa_1\theta q x(1,t)x_t(0,t)| \leq \frac{1}{2}\kappa_1\theta|q|x^2(1,t) + \frac{1}{2}\kappa_1\theta|q|x_t^2(1,t)$$
$$+\frac{\gamma_3}{2}\kappa_1\theta|q|\|x_t(\cdot,t)\|_0^2 + \frac{1}{2\gamma_3}\kappa_1\theta|q|\|x_{t\xi}(\cdot,t)\|_0^2$$

Substituting the above estimations into (1.103) yields

$$\begin{aligned}
\dot{V}_2(t) \;&\leq\; -\theta\left(\lambda_2 - M - \kappa_1\frac{\sqrt{2R}}{\theta}\right)|x_t(1,t)| \\
&-\; \theta\left[W_2 - |q|\left(1+\frac{1}{2}\kappa_1\right)\right]x_t^2(1,t) \\
&-\; \theta\left[1 - \frac{|q|}{\gamma_1} - \kappa_1\frac{|q|}{2\gamma_3}\right]\|x_{t\xi}(\cdot,t)\|_0^2 \\
&-\; \left[c\left(1-\frac{\kappa_1}{2\gamma_2}\right) - |q|\theta\left(\gamma_1 + \kappa_1\frac{\gamma_3}{2}\right)\right]\|x_t(\cdot,t)\|_0^2 \\
&-\; \kappa_1\theta\left(\lambda_1 - \lambda_2 - M\right)|x(1,t)| \\
&-\; \kappa_1\theta\left(W_1 - \frac{1}{2}|q| - \frac{c}{2\theta}\gamma_2\right)x^2(1,t)
\end{aligned} \tag{1.104}$$

Now define a new augmented Lyapunon function V_3, and compute the associated time derivative

$$V_3(t) = V_2(t) + \frac{1}{2}\kappa_2\int_0^1 x^2(\nu,t)d\nu \tag{1.105}$$

$$\begin{aligned}
\dot{V}_3(t) &= \dot{V}_2(t) + \kappa_2\int_0^1 x(\nu,t)x_t(\nu,t)d\nu \\
&= \dot{V}_2(t) + \kappa_2\int_0^1 x(\nu,t)\left(\theta x_{\xi\xi}(\nu,t) - cx(\nu,t)\right)d\nu \\
&= \dot{V}_2(t) + \kappa_2\theta x(1,t)x_\xi(1,t) - \kappa_2\theta q x^2(0,t) \\
&- \kappa_2\theta\|x_\xi(\cdot,t)\|_0^2 - \kappa_2 c\|x(\cdot,t)\|_0^2
\end{aligned} \tag{1.106}$$

Let us upper estimate the magnitude of the sign-undefined terms $\kappa_2\theta x(1,t)x_\xi(1,t)$ and $\kappa_2\theta q x^2(0,t)$. Using (1.56) we get

$$|\kappa_2\theta x(1,t)x_\xi(1,t)| \leq \frac{1}{2}\theta\kappa_2 x^2(1,t) + \frac{1}{2}\kappa_2\theta x_\xi^2(1,t) \tag{1.107}$$

From Agmon's inequality

$$\theta x_\xi^2(1,t) \le 2\theta \|x_\xi(\cdot,t)\|_0 \|x_{\xi\xi}(\cdot,t)\|_0 + \theta x_\xi^2(0,t) \tag{1.108}$$

From the plant equation we have that $\theta x_{\xi\xi}(\xi,t) = x_t(\xi,t) + cx(\xi,t)$ and $x_\xi(0,t) = qx(0,t)$, hence:

$$\theta x_\xi^2(1,t) \le 2\|x_\xi(\cdot,t)\|_0 \|x_t(\xi,t) + cx(\xi,t)\|_0 + \theta q^2 x^2(0,t) \tag{1.109}$$

We can further manipulate (1.108) by applying (1.56), yielding, after substitution into (1.107), the next estimation

$$\begin{aligned} |\kappa_2 \theta x(1,t) x_\xi(1,t)| &\le \frac{1}{2}\theta\kappa_2 x^2(1,t) + \frac{1}{2}\kappa_2\gamma_4 \|x_\xi(\cdot,t)\|_0^2 \\ +\frac{1}{\gamma_4}\kappa_2 \|x_t(\xi,t)\|_0^2 &+ \frac{1}{\gamma_4}\kappa_2 c \|x(\xi,t)\|_0^2 + \frac{1}{2}\kappa_2\theta q^2 x^2(0,t) \end{aligned} \tag{1.110}$$

where γ_4 is an arbitrary positive constant. The last term of (1.110) can be further upper estimated by using (1.55) and (1.56), yielding

$$\begin{aligned} |\kappa_2 \theta x(1,t) x_\xi(1,t)| &\le \frac{1}{2}\theta\kappa_2(1+q^2) x^2(1,t) \\ &+\kappa_2 \left(\frac{\gamma_4}{2} + \frac{1}{4\gamma_5}\theta q^2\right) \|x_\xi(\cdot,t)\|_0^2 \\ +\frac{1}{\gamma_4}\kappa_2 \|x_t(\cdot,t)\|_0^2 &+ \kappa_2 \left(\frac{c}{\gamma_4} + \frac{\gamma_5}{4}\theta q^2\right) \|x(\cdot,t)\|_0^2 \end{aligned} \tag{1.111}$$

where γ_5 is an arbitrary positive constant We finally get

$$\begin{aligned} |\kappa_2 \theta x(1,t) x_\xi(1,t)| + |\kappa_2\theta q x^2(0,t)| &\le \\ \frac{1}{2}\theta\kappa_2(1+2q+q^2) &x^2(1,t) \\ +\kappa_2 \left(\frac{\gamma_4}{2} + \frac{1}{4\gamma_5}\theta q^2 + \theta q \frac{1}{2\gamma_6}\right) &\|x_\xi(\cdot,t)\|_0^2 \\ +\frac{1}{\gamma_4}\kappa_2 \|x_t(\cdot,t)\|_0^2 + \kappa_2 \left(\frac{c}{\gamma_4} + \frac{\gamma_5}{4}\theta q^2 + \theta q\frac{\gamma_6}{2}\right) &\|x(\cdot,t)\|_0^2 \end{aligned} \tag{1.112}$$

Substituting (1.107)-(1.112) into (1.106) yields the next estimation

$$\dot{V}_3(t) \leq -\theta \left[\lambda_2 - M - \kappa_1 \frac{\sqrt{2R}}{\theta}\right] |x_t(1,t)|$$
$$-\theta \left[W_2 - |q| \left(1 + \frac{1}{2}\kappa_1\right)\right] x_t^2(1,t)$$
$$-\theta \left[1 - \frac{|q|}{\gamma_1} - \kappa_1 \frac{|q|}{2\gamma_3}\right] \|x_{t\xi}(\cdot,t)\|_0^2$$
$$-\left[c\left(1 - \frac{\kappa_1}{2\gamma_2}\right) - |q|\theta\left(\gamma_1 + \kappa_1\frac{\gamma_3}{2}\right) - \frac{\kappa_2}{\gamma_4}\right] \|x_t(\cdot,t)\|_0^2$$
$$-\kappa_1\theta\left[\lambda_1 - \lambda_2 - M\right] |x(1,t)|$$
$$-\theta\left[\kappa_1\left(W_1 - \frac{1}{2}|q| - \frac{c\gamma_2}{2\theta}\right) - \frac{\kappa_2}{2}(1 + 2|q| + q^2)\right] x^2(1,t)$$
$$-\kappa_2\left(\theta - \frac{\gamma_4}{2} - \frac{1}{4\gamma_5}\theta q^2 - \theta|q|\frac{1}{2\gamma_6}\right) \|x_\xi(\cdot,t)\|_0^2$$
$$-\kappa_2\left[c\left(1 - \frac{1}{\gamma_4}\right) - \frac{\gamma_5}{4}\theta q^2 - \theta|q|\frac{\gamma_6}{2}\right] \|x(\cdot,t)\|_0^2$$

All coefficients between the square brackets in the above equation must be positive, which can be guaranteed by suitable choice of the design parameters. Inequalities (1.113)-(1.119) provide for the positivity of the first seven coefficients in the given order.

$$\lambda_2 > M, \qquad \kappa_1 < \frac{\theta(\lambda_2 - M)}{\sqrt{2R}} \tag{1.113}$$

$$W_2 > Q, \qquad \kappa_1 < \frac{2(W_2 - |q|)}{|q|} \tag{1.114}$$

$$\gamma_2 > |q|, \qquad \kappa_1 < \frac{2\gamma_3(\gamma_1 - |q|)}{|q|\gamma_1} \tag{1.115}$$

$$c > \Theta_M Q^2, \qquad \kappa_1 < \frac{2(c - |q|\theta\gamma_1)}{\kappa_1\left(\frac{c}{\gamma_2 + |q|\theta\gamma_1}\right)} \tag{1.116}$$

$$\lambda_1 - \lambda_2 > M \tag{1.117}$$

$$W_1 > \frac{1}{2}Q + \frac{cQ}{2\Theta_m}, \quad \kappa_2 < \frac{2\kappa_1\left(W_1 - \frac{1}{2}|q| - \frac{c}{2\theta}\gamma_2\right)}{1 + 2|q| + q^2} \tag{1.118}$$

$$\gamma_4 < 2\theta \qquad \gamma_5 = \gamma_6 = \bar{\gamma}, \qquad \bar{\gamma} > \frac{\theta\left(\frac{q^2}{4} + \frac{1}{2}|q|\right)}{\theta - \frac{\gamma_4}{2}} \tag{1.119}$$

Guaranteeing the positivity of the last coefficient requires more computations. According to (1.119), we select $\gamma_4 = \theta$ ad the corresponding bound on $\bar{\gamma}$ becomes $\bar{\gamma} > \left(\frac{q^2}{2} + |q|\right)$. The last coefficient will be positive if the next relation holds

$$c\left(1 - \frac{1}{\theta}\right) > \frac{\gamma_5}{4}\theta q^2 + \theta|q|\frac{\gamma_6}{2} \tag{1.120}$$

Considering (1.119) we manipulate (1.120) as

$$c > \frac{\theta^2 \left(\frac{q^2}{2} + |q|\right)^2}{2(\theta - 1)} \tag{1.121}$$

The relation (1.121) will be fulfilled if the parameter c satisfies the next inequality

$$c > \frac{\Theta_M^2 \left(\frac{Q^2}{2} + Q\right)^2}{2(\Theta_m - 1)} \tag{1.122}$$

Define $\Psi(x, x_t) = |x(1,t)| + x^2(1,t) + \|x(\cdot,t)\|_0^2 + \|x_t(\cdot,t)\|_0^2$. Under the above restrictions (which, considered together, form the conditions (1.85)-(1.86)) it can be found positive constants ρ_1, ρ_2 and ρ_3, with $\rho_1 < \rho_2$, such that $\rho_1\Psi(x, x_t) \leq V_3(t) \leq \rho_2\Psi(x, x_t)$ and $\dot{V}_3(t) \leq -\rho_3\Psi(x, x_t)$, which imply that $V_3(t)$ exponentially tends to zero. As a result, the exponential stability of the extended state vector (x, x_t) in the space $L_2(0,1) \times L_2(0,1)$ is proven. Since the considered functional is radially unbounded the **global** exponential stability in the corresponding space is then concluded. The Theorem is proven. □

Remark 4. *Within the present paper, the simultaneous compensation of the destabilizing boundary condition and of the matched disturbance is achieved at the price of two additional requirement concerning the reaction coefficient (which must be large enough) and the diffusivity (which is required to be strictly larger than one.)*

1.2.4 Simulation results

Consider the perturbed reaction diffusion equation (1.75)- (1.77), with diffusivity parameter $\theta = 2$, reaction coefficient $c = 15$ and the parameter q appearing in the unstable boundary condition given by $q = -0.5$. The disturbance $\psi(t)$ is set to $\psi(t) = 3 + 5\mathtt{sin}(t)$. The magnitude of the disturbance time derivative $\psi_t(t)$ can be upper-estimated as $M = 5$, according to (1.79). The initial conditions have been set to $x(\xi, 0) = 12sin(\pi\xi) + 6sin(3\pi\xi)$. The bounds on the uncertain q and θ parameters mentioned in the Assumption 4 are set as $Q = 1$, $\Theta_m = 1.5$, $\Theta_M = 2.5$.

Figure 1.7 depicts the spatiotemporal profile of the solution $x(\xi, t)$ in the open loop case (i.e., with $u(t) = 0$). The effect of the unrejected disturbance is clearly visible in the form of persistent oscillations.Controller (1.84) has been implemented with the parameters $\lambda_1 = 11.5$, $\lambda_2 = 5.5$, $W_1 = 6.2$, $W_2 = 1.1$, which are selected in accordance with (1.65). Parameter c also fulfills the condition in (1.65). Figure 1.8 depicts the spatiotemporal profile of the solution $x(\xi, t)$ in the closed loop test with the controller activated. It can be seen that as expected the closed loop trajectory is quickly regulated to the zero value.

1.2.5 Conclusions

The combined twisting/PD algorithm has been shown to be capable of regulating uncertain and perturbed wave and reaction-diffusion processes. As far as the second-order

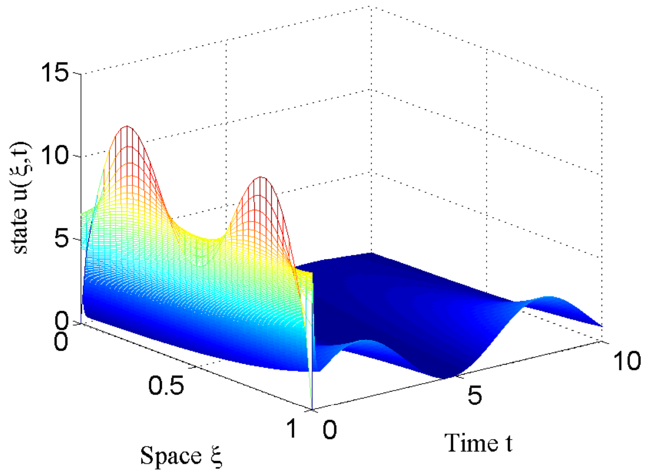

Figure 1.7: Solution $x(\xi, t)$ in the open loop test.

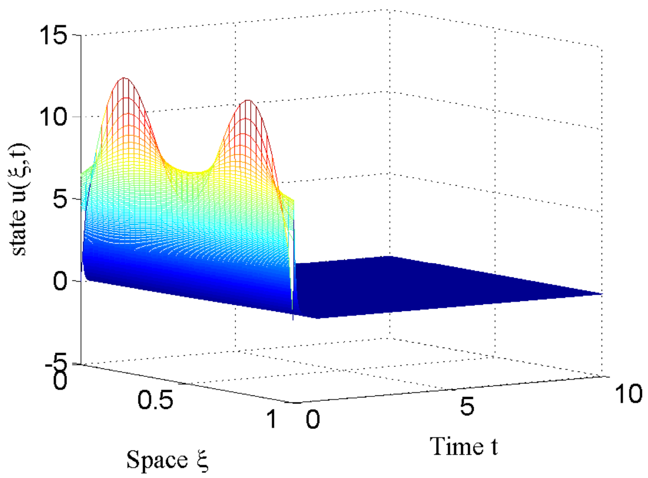

Figure 1.8: Solution $x(\xi, t)$ in the closed loop test.

sliding mode boundary control techniques are concerned, these are the unique contributions currently available in the field. We do not only recall the available results and methods but we do also provide some novelty in the reaction-diffusion process part. Future work will the devoted ro relax the additional restrictions (see Remark 2) which have emerged concerning the reaction and diffusivity coefficients.

Chapter 2

Coupled Reaction-Diffusion Equation

2.1 Boundary stabilization of coupled reaction–diffusion processes with constant parameters

The problem of boundary stabilization is considered for some classes of coupled parabolic linear PDEs of the reaction-diffusion type. With reference to n coupled equations, each one equipped with a scalar boundary control input, a state feedback law is designed with actuation at only one end of the domain, and exponential stability of the closed-loop system is proven. The treatment is addressed separately for the case in which all processes have the same diffusivity and for the more challenging scenario where each process has its own diffusivity and a different solution approach has to be taken. The backstepping method is used for controller design, and, particularly, the kernel matrix of the transformation is derived in explicit form of series of Bessel-like matrix functions by using the method of successive approximations to solve the corresponding PDE. Thus, the proposed control laws become available in explicit form. Additionally, the stabilization of an underactuated system of two coupled reaction-diffusion processes is tackled under the restriction that only a scalar boundary input is available. Capabilities of the proposed synthesis and its effectiveness are supported by numerical studies made for three coupled systems with distinct diffusivity parameters and for underactuated linearized dimensionless temperature-concentration dynamics of a tubular chemical reactor, controlled through a boundary at low fluid superficial velocities when convection terms become negligible.

2.1.1 Introduction

The problem of boundary stabilization is considered for some classes of coupled linear parabolic Partial Differential Equations (PDEs) in a finite spatial domain $x \in [0, 1]$. Particularly, by exploiting the so-called "backstepping" approach [13], [67], this work is devoted to "approximation-free" control synthesis not relying on any discretization or finite-dimensional approximation.

The backstepping-based boundary control problem for scalar heat processes was studied, e.g., in [17], [67]. Several classes of scalar wave processes were studied, e.g., in [14], [58], whereas complex-valued, PDEs such as the Schrodinger equation were also dealt with by means of such an approach [16]. Synergies between the backstepping methodology and the flatness-based approach were studied in [18], [19] with reference to the case of spatially- and time- varying coefficients and covering spatial domains of dimension 2 and higher. In particular, in the latter situation conditions on the target system arise that somewhat resemble those considered in the remainder of the present paper. The backstepping methodology was also applied to observer design for linear parabolic PDEs with non constant coefficients in one- and multi-dimensional spatial domains [43] and [11].

More recently, high-dimensional systems of coupled PDEs are being considered in the backstepping-based boundary control setting. The most intensive efforts of current literature are however oriented towards coupled hyperbolic processes of the transport-type [5, 8, 37, 29, 30]. The state feedback design in [29], which admits stabilization of 2×2 linear heterodirectional hyperbolic systems, was extended in [8] to a particular type of 3×3 linear systems, arising in modeling of multiphase flow, and to the quasilinear case in [30]. In [5], a 2×2 linear hyperbolic system was stabilized by a single boundary control input, with the additional feature that an unmatched disturbance, generated by an *a-priori* known exosystem, is rejected. In [37], a system of $n + 1$ coupled first-order hyperbolic linear PDEs with a single boundary input was studied.

In a recent publication [28], two parabolic reaction-diffusion processes coupled through the corresponding boundary conditions were dealt with. The stabilization of the coupled equations is reformulated in terms of the stabilization problem for a unique process, with piecewise continuous diffusivity and (space-dependent) reaction coefficient, which can be viewed as the "cascade" between the two original systems. The problem is solved by using a unique control input acting only at a boundary. A non conventional backstepping approach with a discontinuous kernel function was employed under a certain inequality constraint involving the diffusivity parameters of the two systems and the corresponding lengths of their spatial domains.

Some specific results concerning the backstepping based boundary stabilization of parabolic coupled PDEs have additionally been presented in the literature [6, 31, 32, 33]. In [6], the Ginzburg-Landau equations, which represent a 2×2 system with equal diffusion coefficients when the imaginary and real parts are expanded, was dealt with. In [31], the linearized 2×2 model of thermal-fluid convection, which entails very dissimilar diffusivity parameters, has been treated by using a singular perturbations approach combined with backstepping and Fourier series expansion. In [32], an observer that estimates the velocity, pressure, electric potential and current fields in a Hartmann flow was presented where the observer gains were designed using multi-dimensional backstepping. In [33], the boundary stabilization of the linearized model of an incompressible magnetohydrodynamic flow in an infinite rectangular 3D channel, also recognized as Hartmann flow, was achieved by reducing the original system to a set of coupled diffusion equations with the same diffusivity parameter and by applying backstepping.

It is of interest to note that the multidimensional transformation considered in the present work generalizes the bi-dimensional backstepping transformation used in [6]. Apart from this, the set of linear coupled kernel PDEs that was derived in [32, 33] for the magnetohydrodynamic channel flow is another inspiration for the present investigation. An additional interesting feature of backstepping, which further motivates our work, is that it admits an easy synergic integration with robust control paradigms such as the sliding mode control methodology (see, e.g., [10]).

Thus motivated, the primary concern of this work is to extend the backstepping synthesis developed in [67], where stabilizing boundary controllers were designed for scalar unstable reaction-diffusion processes. Here, a generalization is provided by considering a set of n reaction-diffusion processes, which are coupled through the corresponding reaction terms. The motivation behind the present investigation comes from chemical processes [20] where coupled temperature-concentration parabolic PDEs occur to describe the process dynamics.

A constructive synthesis procedure, with all boundary controllers given in explicit form, presents the main contribution of the paper to the existing literature. As shown in the paper, this generalization is far from being trivial because the underlying backstepping-based treatment gives rise to more complex development of finding out an explicit solution in the form of Bessel-like matrix series.

The present treatment addresses, side by side, two distinct situations which require quite different solution approaches to be adopted. First, the case where all processes have the same diffusivity ("equi-diffusivity" case, recently announced in [7]) is attacked, and then the more challenging scenario where each process possesses its own diffusivity ("distinct-diffusivity" case) is treated. Under the requirement that the considered multi-dimensional process is fully actuated by a set of n boundary control inputs acting on each subsystem, all these approaches are shown to exponentially stabilize the controlled system with an arbitrarily fast convergence rate.

Apart from this, the stabilization problem of an underactuated system of 2 coupled reaction-diffusion processes, which is relevant to regulation of tubular chemical reactors [20], is addressed under the restriction that only a unique scalar boundary input is available whereas the overall system features a certain minimum-phase property and it meets an additional restriction in the form of a suitable inequality involving both the plant and controller parameters. Exponential stability of the closed loop system is achieved in this case as well, but unlike the previously developed approaches the associated convergence rate cannot be made arbitrarily fast anymore.

The structure of the paper is as follows. In Section 2, the problem statement is presented and the underlying backstepping transformation is introduced. In Section 3, the "equi-diffusivity" scenario is investigated. Explicit solution of the kernel PDE is given for both the direct and inverse transformations, and the resulting boundary control design is presented. In Section 4, the "distinct-diffusivity" case is dealt with, which involves a simplified backstepping transformation defined by a scalar kernel function rather than a matrix one. Section 5 investigates the stabilization problem of an underactuated system of 2 coupled reaction-diffusion processes where only a unique scalar manipulable boundary input is available. Section 6 presents some simulation results. Finally, Section 7 collects concluding remarks and features future perspectives of this research.

Notation

The notation used throughout is fairly standard. $L_2(0,1)$ stands for the Hilbert space of square integrable scalar functions $z(\zeta)$ on $(0,1)$ with the corresponding norm

$$\|z(\cdot)\|_2 = \sqrt{\int_0^1 z^2(\zeta)d\zeta}. \tag{2.1}$$

Also, the notation

$$[L_2(0,1)]^n = \underbrace{L_2(0,1) \times L_2(0,1) \times \ldots \times L_2(0,1)}_{n \ \texttt{times}} \quad \text{and}$$

$$\|Z(\cdot)\|_{2,n} = \sqrt{\sum_{i=1}^{n} \|z_i(\cdot)\|_2^2} \tag{2.2}$$

is adopted for the corresponding norm of a generic vector function

$$Z(\zeta) = [z_1(\zeta), z_2(\zeta), .., z_n(\zeta)] \in [L_2(0,1)]^n \tag{2.3}$$

$J_1(\cdot)$ and $J_2(\cdot)$ ($I_1(\cdot)$ and $I_2(\cdot)$) stand for the first and second order (modified) Bessel functions of the first kind.

With reference to a generic real-valued square matrix A of dimension n, $S[A]$ denotes its symmetric part $S[A] = (A + A^T)/2$, and $\sigma_i(A)$ $(i = 1, 2, ..., n)$ the corresponding eigenvalues. Provided that A is also symmetric and positive definite, $\sigma_m(A)$ and $\sigma_M(A)$ denote respectively the smallest and largest eigenvalues of A, i.e., $\sigma_m(A) = \min_{1\le i\le n} \sigma_i(A)$, $\sigma_M(A) = \max_{1\le i\le n} \sigma_i(A)$. Finally, $I_{n\times n}$ stands for the identity matrix of dimension n.

2.1.2 Problem formulation and backstepping transformation

A n-dimensional system of coupled reaction-diffusion processes is under investigation. Throughout, it is governed by the parabolic PDE

$$Q_t(x,t) \;=\; \Theta Q_{xx}(x,t) + \Lambda Q(x,t) \tag{2.4}$$

and equipped with Neumann-type boundary conditions

$$\begin{aligned} Q_x(0,t) &= 0, && (2.5)\\ Q_x(1,t) &= U(t), && (2.6) \end{aligned}$$

where $Q(x,t) = [q_1(x,t),\ q_2(x,t), \ldots, q_n(x,t)]^T \in [L_2(0,1)]^n$ is the vector collecting the state of all systems, $U(t) = [u_1(t),\ u_2(t), \ldots, u_n(t)]^T \in \Re^n$ is the vector collecting all the manipulable boundary control signals, $\Theta \in \Re^{n\times n}$ is the diagonal diffusivity matrix of the form $\Theta = diag(\theta_i)$, with $\theta_i > 0\ \forall i = 1, 2, ..., n$, $\Lambda \in \Re^{n\times n}$ is a real-valued square matrix whose elements are denoted as λ_{ij}, with $i, j = 1, 2, \ldots, n$.

The open-loop system (2.4)-(2.6) (with $U(t) = 0$) possesses arbitrarily many unstable eigenvalues when the matrix $S[\Lambda]$ has sufficiently large positive eigenvalues. Since the term $\Lambda Q(x,t)$ is the source of instability, the natural objective for a boundary feedback is to "reshape" (or cancel) this term by reversing its effect into a stabilizing one. Thus motivated, our objective is to exponentially stabilize system (2.4)-(2.6) by using an invertible backstepping transformation

$$Z(x,t) = Q(x,t) - \int_0^x K(x,y)Q(y,t)dy \tag{2.7}$$

with a $n \times n$ kernel matrix function $K(x,y)$. The entries $k_{ij}(x,y)$ $(i, j = 1, 2, \ldots, n)$ of $K(x,y)$ are selected in such a manner that the underlying closed-loop system is transformed into the target one

$$\begin{aligned} Z_t(x,t) &= \Theta Z_{xx}(x,t) - CZ(x,t), && (2.8)\\ Z_x(0,t) &= 0, && (2.9)\\ Z_x(1,t) &= 0, && (2.10) \end{aligned}$$

written in terms of the state vector $Z(x,t) = [z_1(x,t), z_2(x,t), \ldots, z_n(x,t)]^T \in [L_2(0,1)]^n$. The exponential stability of the target system (2.8)-(2.10) is then ensured with an arbitrarily fast convergence rate by an appropriate choice of the real-valued square matrix $C \in \Re^{n\times n}$ with entries c_{ij}, $i,j = 1,2,\ldots,n$.

The PDE governing the kernel matrix function $K(x,y)$ is now derived through the standard procedure adopted in the backstepping design [13]. By applying the Leibnitz differentiation rule to (2.7), spatial derivatives $Z_x(x,t)$ and $Z_{xx}(x,t)$ are readily developed as a straightforward matrix generalization of corresponding well-known scalar counterparts. Furthermore, using (2.4) and applying recursively integration by parts, the time derivative $Z_t(x,t)$ is derived as well. Combining such expressions, and performing rather lengthy but straightforward computations (see [7] for more detailed derivations), yield

$$\begin{aligned}
&Z_t(x,t) - \Theta Z_{xx}(x,t) + CZ(x,t)\\
&= \left[\Lambda + C + K_y(x,x)\Theta + \Theta K_x(x,x) + \Theta \tfrac{d}{dx}K(x,x)\right]\\
&\times Q(x,t) + \textstyle\int_0^x \left[\Theta K_{xx}(x,y) - K_{yy}(x,y)\Theta - K(x,y)\Lambda\right.\\
&\left.-CK(x,y)\right] Q(y,t)dy + \left[\Theta K(x,x) - K(x,x)\Theta\right] Q_x(x,t)\\
&+K(x,0)\Theta Q_x(0,t) - K_y(x,0)\Theta Q(0,t).
\end{aligned} \tag{2.11}$$

Clearly, the target system's PDE (2.8) requires that the right hand side of (2.11) has to be identically zero. Employing the homogeneous BC (2.5), this leads to the following relations

$$\Theta K_{xx}(x,y) - K_{yy}(x,y)\Theta - K(x,y)\Lambda - CK(x,y) = 0, \tag{2.12}$$

$$\Lambda + C + K_y(x,x)\Theta + \Theta K_x(x,x) + \Theta \frac{d}{dx}K(x,x) = 0, \tag{2.13}$$

$$\Theta K(x,x) - K(x,x)\Theta = 0, \tag{2.14}$$

$$K_y(x,0)\Theta = 0. \tag{2.15}$$

The main critical feature of (2.12)-(2.15) is in the presence of relation (2.14). While being identically satisfied in the scalar case when $n = 1$ [67], this relation is in general contradictive, and there are two options to fulfill (2.14). One of these options is to impose the constraint that all the coupled processes possess the same diffusivity value $\theta_i = \theta$, $i = 1,2,...,n$, so that

$$\Theta = \theta I_{n\times n}. \tag{2.16}$$

An alternative option is to enforce the next constraint on the form of the kernel matrix

$$K(x,y) = k(x,y)I_{n\times n}. \tag{2.17}$$

Assumption (2.17) greatly simplifies the complexity of the underlying backstepping transformation, which is determined by a scalar function. This simplification, however, will also bring some constraint on the choice of the matrix C when the relation (2.17) is in force. Solution of the kernel PDE (2.12), (2.13), (2.15) under the additional constraints (2.16) or (2.17) will be addressed in Sections 2.1.3 and 2.1.4.

Stability of the target system dynamics

The following result is in force.

Theorem 11. *Consider the target system* (2.8)-(2.10)*. If the matrix* $S[C]$ *is positive definite then system* (2.8)-(2.10) *is exponentially stable in the space* $[L_2(0,1)]^n$ *with the convergence rate specified by*

$$\|Z(\cdot,t)\|_{2,n} \leq \|Z(\cdot,0)\|_{2,n} e^{-\sigma_m(S[C])t}. \tag{2.18}$$

Proof. The detailed proof can be found in [7]. ■ □

2.1.3 Stabilization in the "equi-diffusivity" case

Boundary stabilization of system (2.4)-(2.6) under the constraint (2.16) is addressed by following the previously introduced backstepping design [7] with the corresponding treatment being included in the present work for the sake of completeness.

Explicit solution of the relevant kernel boundary-value problem

Specializing system (2.12), (2.13), (2.15) in light of the actual form (2.16) of the diffusivity matrix Θ yields

$$K_{xx}(x,y) - K_{yy}(x,y) = \frac{1}{\theta}K(x,y)\Lambda + \frac{1}{\theta}CK(x,y), \tag{2.19}$$

$$\Lambda + C + 2\theta\frac{d}{dx}K(x,x) = 0, \tag{2.20}$$

$$K_y(x,0) = 0. \tag{2.21}$$

Integrating (2.20) with respect to x gives $K(x,x) = -\frac{1}{2\theta}(\Lambda + C)x + K(0,0)$. Substituting the boundary conditions (2.5) and (2.9) into the relation $Z_x(0,t) = Q_x(0,t) - K(0,0)Q(0,t)$, which is obtained by spatial differentiation of (2.7) at $x = 0$, one derives that

$$K(0,0) = 0. \tag{2.22}$$

Hence, relation (2.20) is replaced by

$$K(x,x) = -\frac{1}{2\theta}(\Lambda + C)x. \tag{2.23}$$

The following result is in order.

Theorem 12. *The problem* (2.19), (2.21), (2.23) *possesses a solution*

$$K(x,y) = -\sum_{j=0}^{\infty} \frac{(x^2-y^2)^j(2x)}{j!(j+1)!}\left(\frac{1}{4\theta}\right)^{j+1}\left[\sum_{i=0}^{j}\binom{j}{i}C^i(\Lambda + C)\Lambda^{j-i}\right] \tag{2.24}$$

which is of class C^∞ *in the domain* $0 \leq y \leq x \leq 1$.

Proof. The detailed proof is presented in [7]. ■ □

Remark 5. *If the condition $\mathit{\Lambda}C = C\mathit{\Lambda}$ holds, then* (2.24) *simplifies to*

$$K(x,y) \quad = \quad -\sum_{j=0}^{\infty} \frac{(x^2-y^2)^j(2x)}{j!(j+1)!} \left[\frac{\mathit{\Lambda}+C}{4\theta}\right]^{j+1}. \tag{2.25}$$

In the scalar case $n = 1$, relation (2.25) *specifies to that obtained in [67].*

Remark 6. *Uniqueness of the solution* (2.24) *to the kernel PDE* (2.19)*,* (2.21)*,* (2.23) *can be proven following the same steps as, e.g., in [10, Lemma 2.1]. The complete treatment is , however, beyond the scope of the present paper as it does not impact the underlying closed-loop stability result, and it is skipped for brevity.*

Finally, let us show that the transformation (2.7) is invertible, and its inverse is representable in the form

$$Q(x,t) = Z(x,t) + \int_0^x L(x,y)Z(y,t)dy. \tag{2.26}$$

By performing analogous developments as those made for the derivation of the gain kernel PDE (2.19), (2.21), (2.23), the next PDE is obtained

$$L_{xx}(x,y) - L_{yy}(x,y) \quad = \quad -\frac{1}{\theta}L(x,y)C - \frac{1}{\theta}\Lambda L(x,y), \tag{2.27}$$

$$L(x,x) \quad = \quad -\frac{1}{2\theta}\left(\Lambda + C\right)x, \tag{2.28}$$

$$L_y(x,0) \quad = \quad 0, \tag{2.29}$$

governing $L(x,y)$. By comparison between (2.19), (2.21), (2.23) and (2.27)-(2.29) one immediately notice that in this case $L(x,y) = -K(x,y)$ when Λ and C are replaced by $-\Lambda$ and $-C$. To reproduce the latter conclusion it suffices to explicitly denote the dependence of the solutions $L(x,y) = L(x,y;\Lambda,C)$ and $K(x,y) = K(x,y;\Lambda,C)$ on Λ and C and verify that the substitution $L(x,y;\Lambda,C) = -K(x,y;-\Lambda,-C)$ transfers (2.27)-(2.29) into (2.19), (2.21), (2.23).

Boundary controller design

The next result specifies the proposed boundary control design and summarizes the first stability result of this paper.

Theorem 13. *Let matrix C be selected in such a manner that $S[C]$ is positive definite whereas $\sigma_m(S[C])$ is arbitrarily large. Then, the boundary control input*

$$\begin{aligned} U(t) &= -\frac{1}{2\theta}\left(\Lambda + C\right)Q(1,t) + \int_0^1 K_x(1,y)Q(y,t)dy, &(2.30)\\ K_x(1,y) &= -\sum_{j=0}^{\infty}\left[\frac{2(1-y^2)^j + 4j(1-y^2)^{j-1}}{j!(j+1)!}\right]\\ &\times \left(\frac{1}{4\theta}\right)^{j+1}\left[\sum_{i=0}^{j}\binom{j}{i}C^i\left(\Lambda + C\right)\Lambda^{j-i}\right], &(2.31)\end{aligned}$$

exponentially stabilizes system (2.4)-(2.6) in the space $[L_2(0,1)]^n$ with an arbitrarily fast convergence rate in accordance with

$$\|Q(\cdot,t)\|_{2,n} \leq A\|Q(\cdot,0)\|_{2,n}e^{-\sigma_m(S[C])t}, \tag{2.32}$$

where A is a positive constant independent of $Q(x,0)$.

Proof. The backstepping transformation (2.7), (2.24) was derived to map system (2.4)-(2.6) into the target dynamics governed by (2.8). It remains to prove that the homogenous BCs (2.9)-(2.10) hold as well. Spatial differentiation of (2.7) at $x=0$ and $x=1$ yields

$$\begin{aligned} Z_x(0,t) &= Q_x(0,t) - K(0,0)Q(0,t)\\ Z_x(1,t) &= Q_x(1,t) - K(1,1)Q(1,t) - \int_0^1 K_x(1,y)Q(y,t)dy.\end{aligned}$$

The boundary conditions (2.5) and (2.6) and relation (2.23), coupled together, ensure that $K(0,0)=0$ and $K(1,1) = -\frac{1}{2\theta}\left(\Lambda + C\right)$, thereby yielding

$$\begin{aligned} Z_x(0,t) &= 0\\ Z_x(1,t) &= U(t) + \frac{1}{2\theta}\left(\Lambda + C\right)Q(1,t) - \int_0^1 K_x(1,y)Q(y,t)dy.\end{aligned}$$

Thus, the boundary control input vector (2.30)-(2.31), where the kernel spatial derivative $K_x(1,y)$ is obtained by differentiating (2.24) with respect to x at $x=1$, results in the target dynamics (2.8)-(2.10) with homogeneous BCs.

Recall that the exponential stability of (2.8)-(2.10) was guaranteed by Theorem 11 provided that $S[C]$ is positive definite. With this in mind, it is followed [67] to derive analogous convergence properties for the original system (2.4)-(2.6) as well. The estimates $\|K(x,y)\| \leq Me^{2Mx}$ and $\|L(x,y)\| \leq Me^{2Mx}$ are established for some positive constant M by generalizing [67] where the scalar counterparts of such estimates were obtained. A straightforward generalization of [67, Th 4] yields that the above two upper estimates, coupled together, establish the equivalence of norms of $Z(x,t)$ and $Q(x,t)$ in $[L_2(0,1)]^n$ thereby ensuring that there exists a positive constant A independent of $Q(\xi,0)$ such that (2.32) straightforwardly follows from (2.18). This completes the proof of Theorem 13. ■ □

2.1.4 Stabilization in the distinct diffusivity case

In the present section, boundary stabilization of system (2.4)-(2.6) is addressed by following the previously introduced backstepping design specified with (2.17). Relation (2.16) is no longer in force, and now all processes possess their own distinct diffusivity parameter. As noted in Section 2, constraint (2.17) has to be brought into play in order to ensure that the stabilization problem is solvable through the backstepping route.

Let us now specialize system (2.12), (2.13), (2.15) by considering the constraint (2.17) on the kernel matrix:

$$\begin{aligned}
(k_{xx}(x,y) - k_{yy}(x,y))\Theta &= k(x,y)(\Lambda + C) && (2.33)\\
\Lambda + C + 2\frac{d}{dx}k(x,x)\Theta &= 0 && (2.34)\\
k_y(x,0) &= 0. && (2.35)
\end{aligned}$$

Being represented in the component-wise form, relation (2.33) gives rise to n independent scalar PDEs of the form

$$k_{xx}(x,y) - k_{yy}(x,y) = k(x,y)\left(\frac{\lambda_{ii} + c_{ii}}{\theta_i}\right), \; i = 1, 2, .., n \tag{2.36}$$

and to the constraints

$$\lambda_{ij} + c_{ij} = 0, \quad i, j = 1, 2, ..., n, \quad i \neq j. \tag{2.37}$$

In turns, relation (2.34), represented in the component-wise form, results in the same constraints (2.37) and additionally imposes the next scalar relations

$$\frac{d}{dx}k(x,x) = \frac{1}{2}\left(\frac{\lambda_{ii} + c_{ii}}{\theta_i}\right), \quad i = 1, 2, ..., n. \tag{2.38}$$

It is clear that a solution may only exist if the constants $\frac{\lambda_{ii}+c_{ii}}{\theta_i}$ in the right hand sides of (2.36) and (2.38) possess the same value for all $i = 1, 2, ..., n$. Therefore, the next constraints

$$\begin{aligned}
c_{ii} &= \gamma^*\theta_i - \lambda_{ii}, \quad i = 1, 2, ..., n, && (2.39)\\
c_{ij} &= -\lambda_{ij}, \quad i, j = 1, 2, ..., n, \quad i \neq j, && (2.40)
\end{aligned}$$

on the elements of the matrix C must be imposed with an arbitrary constant γ^*, thereby yielding the kernel PDE

$$\begin{aligned}
k_{xx}(x,y) - k_{yy}(x,y) = \gamma^* k(x,y), && (2.41)\\
k_y(x,0) = 0 && (2.42)\\
\frac{d}{dx}k(x,x) = -\frac{\gamma^*}{2}. && (2.43)
\end{aligned}$$

Integrating (2.43) with respect to x gives the relation $k(x,x) = -\frac{\gamma^*}{2}x + k(0,0)$ whereas the additional relation $k(0,0) = 0$ is deduced by specifying the derivation of formula (2.22) to the current case.

System (2.41)-(2.43) can thus be specified to the boundary-value problem

$$
\begin{aligned}
k_{xx}(x,y) - k_{yy}(x,y) &= \gamma^* k(x,y), && (2.44)\\
k_y(x,0) &= 0 && (2.45)\\
k(x,x) &= -\frac{\gamma^*}{2}x, && (2.46)
\end{aligned}
$$

whose explicit solution

$$
k(x,y) = -\gamma^* x \frac{I_1(\sqrt{\gamma^*(x^2-y^2)})}{\sqrt{\gamma^*(x^2-y^2)}} \tag{2.47}
$$

is extracted from [67]. By making lengthy but straightforward computations, the kernel PDE of the inverse transformation can be derived as follows:

$$
\begin{aligned}
l_{xx}(x,y) - l_{yy}(x,y) &= -\gamma^* l(x,y), && (2.48)\\
l_y(x,0) &= 0, && (2.49)\\
l(x,x) &= -\frac{\gamma^*}{2}x, && (2.50)
\end{aligned}
$$

whose explicit solution is also drawn from [67] in the form

$$
l(x,y) = -\gamma^* x \frac{J_1(\sqrt{\gamma^*(x^2-y^2)})}{\sqrt{\gamma^*(x^2-y^2)}}. \tag{2.51}
$$

Controller design

Clearly, relations (2.39)-(2.40) require the γ^*-dependent matrix C to be selected in the form

$$
C = -\Lambda + \gamma^*\Theta. \tag{2.52}
$$

The next condition ensures that matrix $S[C]$ is positive definite.

Condition 1. *The scalar parameter κ and the design parameter γ^* are respectively chosen according to*

$$
\begin{aligned}
\kappa &> \max_{1\le i\le n} |\sigma_i(-S[\Lambda])|, && (2.53)\\
\gamma^* &> \frac{\sigma_M(-S[\Lambda] + \kappa I_{n\times n}) + \kappa}{\sigma_m(\Theta)}, \quad \sigma_m(\Theta) = \min_{1\le i\le n} \theta_i. && (2.54)
\end{aligned}
$$

The proposed boundary control design is specified for the distinct diffusivity case as follows

Theorem 14. *Let matrix C be selected according to* (2.52) *and let Condition 1 be satisfied. Then, the boundary control input*

$$U(t) = -\frac{\gamma^*}{2}Q(1,t) + \int_0^1 k_x(1,y)Q(y,t)dy, \qquad (2.55)$$

$$k_x(1,y) = -\gamma^* \frac{I_1(\sqrt{\gamma^*(1-y^2)})}{\sqrt{\gamma^*(1-y^2)}} - \gamma^* \frac{I_2(\sqrt{\gamma^*(1-y^2)})}{1-y^2}, \qquad (2.56)$$

exponentially stabilizes system (2.4)-(2.6) in the space $[L_2(0,1)]^n$ with an arbitrarily fast convergence rate

$$\|Q(\cdot,t)\|_{2,n} \leq A\|Q(\cdot,0)\|_{2,n}e^{-\sigma_m(S[C])t}, \qquad (2.57)$$

where A is a positive constant independent of $Q(\xi,0)$.

Proof. Noticing that $k(1,1) = -\frac{\gamma^*}{2}$ by virtue of (2.46), the form of the chosen boundary feedback control is justified by following the same line of reasoning used in the beginning of the proof of Theorem 13. The stability properties of the target dynamics (2.8)-(2.10) are established in Theorem 11, that requires $S[C]$ to be positive definite. Now let us show that selecting the matrix C as in (2.52), with the scalar parameter γ^* chosen according to (2.53)-(2.54), ensures that $S[C]$ is positive definite and $\sigma_m(S[C])$ is arbitrarily large.

Since Θ is a diagonal matrix, and γ^* is a scalar, it follows from (2.52) that $S[C] = -S[\Lambda]+\gamma^*\Theta$. Matrix $S[C]$ is positive definite iff the quadratic form $p^TS[C]p$ takes positive value for every nontrivial real-valued column vector p of dimension n. The quadratic form $p^TS[C]p$ can be expanded as follows by adding and subtracting to $S[C]$ the dummy quantity $\kappa I_{n\times n}$

$$\begin{aligned} p^TS[C]p &= p^T\left(-S[\Lambda] + \gamma^*\Theta + \kappa I_{n\times n} - \kappa I_{n\times n}\right)p \\ &= p^T\left(-S[\Lambda] + \kappa I_{n\times n}\right)p + \gamma^* p^T\Theta p - \kappa p^Tp. \end{aligned} \qquad (2.58)$$

It is well-known that adding $\kappa I_{n\times n}$ to any matrix shifts the corresponding eigenvalues by κ, which results in the eigenvalues of matrix $-S[\Lambda] + \kappa I_{n\times n}$ to be located at $k + \sigma_i(-S[\Lambda])$, $i = 1,2,...,n$. Therefore, condition (2.53) guarantees that the symmetric matrix $-S[\Lambda] + \kappa I_{n\times n}$ is positive definite. In light of this, the estimate

$$p^TS[C]p \geq \left[-\sigma_M(-S[\Lambda] + \kappa I_{n\times n}) + \gamma^*\sigma_m(\Theta) - \kappa\right]p^Tp \qquad (2.59)$$

can be derived from (2.58) by exploiting well-known properties of quadratic norms.

By taking into account that $\sigma_i(\Theta) = \theta_i$, it follows from (2.54) that the right hand side of (2.59) is strictly positive, thus ensuring that matrix $S[C]$ is positive definite. Since (2.59) holds for an arbitrary nontrivial $p \in \Re^n$, and its right hand side grows unbounded with increasing γ^*, one concludes that the smallest eigenvalue $\sigma_m(S[C])$ of $S[C]$ can be made arbitrarily large. Thus, the exponential stability of the target system's dynamics (2.8)-(2.10) is established with an arbitrarily fast convergence rate in accordance with Theorem 11.

The rest of the proof follows [67] to derive analogous convergence properties for the original system (2.4)-(2.6) as well. As shown in [67, Th.2, Th.3], both the kernel functions (2.47) and (2.51) are bounded according to the estimates $|k(x,y)| \leq Me^{2Mx}$ and $|l(x,y)| \leq Me^{2Mx}$ where M is a positive constant. [67, Th.4] states that those two upper-bounds, coupled together, establish the equivalence between norms of $Z(x,t)$ and $Q(x,t)$ in $[L_2(0,1)]^n$ which means that there exists a positive constant A independent of $Q(\xi,0)$ such that the estimate (2.57) is in force as a direct consequence of (2.18). Theorem 14 is thus proved. ■ □

2.1.5 Underactuated boundary stabilization of two coupled distinct diffusion processes

Let us now consider a 2-dimensional system of coupled reaction-diffusion processes

$$q_{1t}(x,t) = \theta_1 q_{1xx}(x,t) + \lambda_{11} q_1(x,t) + \lambda_{12} q_2(x,t), \quad (2.60)$$
$$q_{2t}(x,t) = \theta_2 q_{2xx}(x,t) + \lambda_{21} q_1(x,t) + \lambda_{22} q_2(x,t), \quad (2.61)$$

equipped with Neumann-type boundary conditions

$$q_{1x}(0,t) = q_{2x}(0,t) = 0, \quad (2.62)$$
$$q_{1x}(1,t) = u_1(t), \quad q_{2x}(1,t) = 0, \quad (2.63)$$

where $q_i(x,t) \in L_2(0,1)$, $i = 1,2$, are the state variables and $u_1(t)$ is the manipulable boundary input acting on the q_1-subsystem only. To add practical value to the present investigation it is worth noticing that such a system represents linearized dimensionless dynamics of a tubular chemical reactor controlled through a boundary at low fluid superficial velocities when convection terms become negligible (cf. that of [20]). Thus interpreted, the meaning of the two state variables becomes normalized temperature and reactant concentration, respectively.

In contrast to the investigation of Section 2.1.4, where independent boundary actuation of each subsystem was available, the present system of two coupled diffusion processes is underactuated by a unique boundary control input applied to subsystem (2.60). It it easy to check that system (2.60)-(2.63) can be rewritten in the form (2.4)-(2.6) where $Q(x,t) = [q_1(x,t),\ q_2(x,t)]^T$, $U(t) = [u_1(t), 0]^T$, and

$$\Theta = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}. \quad (2.64)$$

The next "minimum phase" assumption is imposed on the system to ensure that the q_2 subsystem (2.61) of (2.60)-(2.63) is asymptotically stable when $q_1(x,t) = 0$.

Assumption 5. *The parameter λ_{22} is negative.*

Our objective is to exponentially stabilize system (2.60)-(2.63) by applying the invertible backstepping transformation (2.7) specified with

$$K(x,y) = \begin{bmatrix} k(x,y) & 0 \\ 0 & 0 \end{bmatrix}. \quad (2.65)$$

It follows from (2.7) and (2.65) that

$$
\begin{aligned}
z_1(x,t) &= q_1(x,t) - \int_0^x k(x,y) q_1(y,t) dy, && (2.66)\\
z_2(x,t) &= q_2(x,t), && (2.67)
\end{aligned}
$$

i.e., the second state variable of the target dynamics is the same as that of the original system (2.60)-(2.63).

The main difference from the developments of the previous sections comes from the fact that relation (2.12) will be now in general impossible to fulfill. As a consequence, the target system dynamics will contain an additional integral term in contrast to (2.8), and it will take the form of a Partial Integro-Differential Equation (PIDE). It is worth to remark that the presence of extra integral terms in the target system is not unusual in backstepping designs when dealing with terms that cannot be compensated otherwise (see e.g. [37], [30], [34]).

The next lemma presents the derivation of the target system dynamics in the present underactuated scenario.

Lemma 6. *The backstepping transformation* (2.7), (2.65), *where* $k(x,y)$ *is the solution* (2.47) *to the boundary-value problem* (2.44)-(2.46), *transfers system* (2.60)-(2.63) *into the target system dynamics*

$$
\begin{aligned}
Z_t(x,t) &= \Theta Z_{xx}(x,t) - CZ(x,t)\\
&+ \int_0^x \begin{bmatrix} -\lambda_{12} k(x,y) z_2(y,t) \\ \lambda_{21} l(x,y) z_1(y,t) \end{bmatrix} dy, \qquad (2.68)
\end{aligned}
$$

where $Z(x,t) = [z_1(x,t),\ z_2(x,t)]^T \in [L_2(0,1)]^2$ *is the corresponding state vector,* $C = C(\gamma^*) = \{c_{ij}\} \in \Re^{2\times 2}$ *is the* γ^**-dependent real-valued matrix given by*

$$
C(\gamma^*) = -\Lambda + \gamma^* \begin{bmatrix} \theta_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\lambda_{11} + \gamma^*\theta_1 & -\lambda_{12} \\ -\lambda_{21} & -\lambda_{22} \end{bmatrix}, \qquad (2.69)
$$

$\gamma^* \in \Re$ *is an adjustable design parameter and* $l(x,y)$ *is the solution* (2.51) *to the boundary value problem* (2.48)- (2.50)

Proof. To support the derivation of (2.68), the previous multidimensional matrix-based treatment is kept to take advantage of the computations previously made. Particularly, relation (2.11) is still valid and the kernel conditions (2.13)-(2.15) are going to be considered and specialized to the current scenario. As for relation (2.12), it will be now in general impossible to fulfill and a new form of it, with the right-hand side not being identically zero anymore, will be derived and employed within the present proof.

Relations (2.13) and (2.65) yield

$$
\begin{aligned}
\frac{d}{dx} k(x,x) &= \left(\frac{\lambda_{11} + c_{11}}{2\theta_1} \right), && (2.70)\\
\lambda_{12} + c_{12} &= 0, \quad \lambda_{21} + c_{21} = 0, \quad \lambda_{22} + c_{22} = 0. && (2.71)
\end{aligned}
$$

The following relation

$$c_{11} = \gamma^* \theta_1 - \lambda_{11}, \tag{2.72}$$

which involves an arbitrary constant γ^*, must then be enforced. By inspection, relations (2.71) -(2.72) result in the constrained form (2.69) of the γ^*-dependent matrix $C(\gamma^*)$ with a unique free parameter $\gamma^* \in \Re$ which is available for design.

The "critical" relation (2.14) is automatically satisfied due to (2.65), and relation (2.15) yields (2.45).

By taking into account the constraint (2.65) on the kernel matrix one derives that

$$\begin{aligned} \Theta K_{xx}(x,y) &- K_{yy}(x,y)\Theta - K(x,y)\Lambda - CK(x,y) \\ &= \begin{bmatrix} \theta_1 k_{xx}(\cdot) - \theta_1 k_{yy}(\cdot) - (\lambda_{11} + c_{11})k(\cdot) & -\lambda_{12}k(\cdot) \\ -c_{21}k(\cdot) & 0 \end{bmatrix}. \end{aligned} \tag{2.73}$$

Zeroing the first diagonal element in the right hand side of (2.73) yields the scalar PDE (2.44) After employing simple manipulations, analogous to those made in Section 2.1.4, and considering as well (2.70) and (2.72), the kernel boundary-value PDE problem (2.44)-(2.46) is thus verified for the kernel function $k(x, y)$ so that while being a solution of (2.44)-(2.46), it is given by (2.47).

Zeroing the off-diagonal elements in the right hand side of (2.73) requires that both the coefficients λ_{12} and c_{21} should be identically zero (and, by (2.71), the same for λ_{21}). This would clearly trivialize the underlying stabilization problem (see Remark 7). Therefore, as apparent from (2.11), there will be an additional entry in the target dynamics in contrast to (2.8)-(2.10) since the right hand side of (2.12) cannot be made identically zero anymore. By considering (2.11) along with relations (2.73), (2.44) and (2.71), it follows that

$$\begin{aligned} Z_t(x,t) &= \Theta Z_{xx}(x,t) - CZ(x,t) + \int_0^x \begin{bmatrix} 0 & -\lambda_{12}k(x,y) \\ \lambda_{21}k(x,y) & 0 \end{bmatrix} Q(y,t)dy \\ &= \Theta Z_{xx}(x,t) - CZ(x,t) + \int_0^x \begin{bmatrix} -\lambda_{12}k(x,y)q_2(y,t) \\ \lambda_{21}k(x,y)q_1(y,t) \end{bmatrix} dy. \end{aligned} \tag{2.74}$$

To rewrite (2.74) entirely in terms of Z-coordinates, the identity

$$\int_0^x k(x,y)q_1(y,t)dy = \int_0^x l(x,y)z_1(y,t)dy \tag{2.75}$$

is employed. Relation (2.75) is derived by summing (2.66) and the associated inverse transformation

$$q_1(x,t) = z_1(x,t) + \int_0^x l(x,y)z_1(y,t)dy \tag{2.76}$$

and canceling the identical terms in the resulting equality. Substituting (2.75) into the last term of (2.74) the target dynamics PIDE (2.68) is obtained. Lemma 6 is proved. □
□

Remark 7. *It has been demonstrated within the proof of Lemma 6 that in order to obtain a target system dynamics equivalent to* (2.8)-(2.10) *both the coupling coefficients* λ_{12} *and* λ_{21} *must be zero, i.e., the original system* (2.60)-(2.63) *should already be decoupled. Clearly, this would have trivialized the underlying result, which is why such restriction has not been made and the more involved target dynamics PIDE* (2.68) *has been brought into play.*

The subsequent synthesis involves the next condition, which ensures the asymptotic stability of the target system dynamics. This condition relies on the feasibility problem of seeking a solution to a nonlinear inequality subject to a positive definiteness constraint on a certain parameter-dependent matrix.

Condition 2. *The nonlinear inequality*

$$\sigma_m(S[C(\gamma^*)]) > \bar{\lambda}_M \gamma^* e^{2\gamma^*} \tag{2.77}$$

with $\bar{\lambda}_M = \max\{|\lambda_{12}|, |\lambda_{21}|\}$ *possesses a solution* γ^* *such that the symmetric* γ^**-dependent matrix*

$$S[C(\gamma^*)] = \begin{bmatrix} -\lambda_{11} + \gamma^*\theta_1 & -\frac{\lambda_{12}+\lambda_{21}}{2} \\ -\frac{\lambda_{12}+\lambda_{21}}{2} & -\lambda_{22} \end{bmatrix} \tag{2.78}$$

is positive definite.

It is worth noticing that the smallest (real, and positive) eigenvalue $\sigma_m(S[C(\gamma^*)])$ of matrix $S[C(\gamma^*)]$ in Condition 2 admits the explicit representation

$$\sigma_m(S[C(\gamma^*]))) = \frac{T}{2} - \sqrt{\frac{T^2}{4} - D}, \tag{2.79}$$

where

$$T = -\lambda_{11} + \gamma^*\theta_1 - \lambda_{22}, \tag{2.80}$$

$$D = -\lambda_{22}\left(-\lambda_{11} + \gamma^*\theta_1\right) - \frac{(\lambda_{12}+\lambda_{21})^2}{4}, \tag{2.81}$$

are, respectively, the trace and determinant of $S[C(\gamma^*)]$.

Controller design

The next result specifies the proposed boundary control design for the distinct diffusivity case with $n = 2$ and a scalar input only.

Theorem 15. *Consider system* (2.60)-(2.63) *with Assumption 5 and let Condition 2 hold. Then, the boundary control input*

$$u_1(t) = -\frac{\gamma^*}{2} q_1(1,t) + \int_0^1 k_x(1,y) q_1(y,t)dy, \tag{2.82}$$

$$k_x(1,y) = -\gamma^* \frac{I_1(\sqrt{\gamma^*(1-y^2)})}{\sqrt{\gamma^*(1-y^2)}} - \gamma^* \frac{I_2(\sqrt{\gamma^*(1-y^2)})}{1-y^2}, \tag{2.83}$$

exponentially stabilizes system (2.60)-(2.63) *in the space* $[L_2(0,1)]^2$ *with the convergence rate given by*

$$\|Q(\cdot,t)\|_{2,2} \leq A\|Q(\cdot,0)\|_{2,2}e^{-M(\gamma^*)t}, \tag{2.84}$$

where A is a positive constant independent of $Q(x,0)$ and

$$M(\gamma^*) = \sigma_m(S[C(\gamma^*)]) - \bar{\lambda}_M\gamma^* e^{2\gamma^*} \tag{2.85}$$

Proof. The form of the proposed boundary feedback control is justified by following the same line of reasoning as that made in the beginning of the proof of Theorem 13. It guarantees that the target dynamics PIDE (2.68) is actually equipped with the homogeneous BCs

$$Z_x(0,t) = Z_x(1,t) = 0, \tag{2.86}$$

The asymptotic stability of the target system dynamics PIDE (2.68), specified with the BCs (2.86), is investigated by means of the candidate Lyapunov function

$$V(t) = \frac{1}{2}\int_0^1 Z^T(x,t)Z(x,t)dx = \frac{1}{2}||Z(\cdot,t)||_{2,2}^2, \tag{2.87}$$

whose time derivative along the solutions of (2.68), (2.86) takes the form

$$\begin{aligned}
\dot{V}(t) &= \int_0^1 Z^T(x,t)Z_t(x,t)dx \\
&= \int_0^1 Z^T(x,t)\Theta Z_{xx}(x,t)dx - \int_0^1 Z^T(x,t)CZ(x,t)dx \\
&+ \int_0^1 Z^T(x,t)\left(\int_0^x \begin{bmatrix} -\lambda_{12}k(x,y)z_2(y,t) \\ \lambda_{21}l(x,y)z_1(y,t) \end{bmatrix} dy\right) dx.
\end{aligned} \tag{2.88}$$

The first two terms in the right hand side of (2.88) can be estimated as follows (cf. [7, Th. 2]):

$$\int_0^1 Z^T(x,t)\Theta Z_{xx}(x,t)dx \leq -\sigma_m(\Theta)||Z_\xi(\cdot,t)||_{2,2}^2, \tag{2.89}$$

$$\int_0^1 Z^T(x,t)CZ(x,t)dx \leq -\sigma_m(S[C(\gamma^*)])||Z(\cdot,t)||_{2,2}^2, \tag{2.90}$$

where $\sigma_m(\Theta) = min\{\theta_1,\theta_2\}$. By construction, both $\sigma_m(\Theta)$ and $\sigma_m(S[C(\gamma^*)])$ are strictly positive quantities. To estimate the third term in the right hand side of (2.88), which is sign-indefinite, the relations

$$|k(x,y)| \leq He^{2Hx}, \quad |l(x,y)| \leq He^{2Hx}, \tag{2.91}$$

$$H = \gamma^* = \frac{\lambda_{11}+c_{11}}{\theta_1}, \tag{2.92}$$

established in [67, Th.2, Th.3], are subsequently exploited. Within the (x, y) domain of interest (for which $0 \leq x \leq 1$) the worst case value $x = 1$ can be considered in (2.91), i.e.:

$$|k(x,y)| \leq \gamma^* e^{2\gamma^*}, \quad |l(x,y)| \leq \gamma^* e^{2\gamma^*}. \tag{2.93}$$

The third term of (2.88) is expanded as follows:

$$\begin{aligned}
&\int_0^1 Z^T(x,t) \left(\int_0^x \begin{bmatrix} -\lambda_{12} k(x,y) z_2(y,t) \\ \lambda_{21} l(x,y) z_1(y,t) \end{bmatrix} dy \right) dx \\
= \;& \lambda_{21} \int_0^1 z_2(x,t) \left(\int_0^x l(x,y) z_1(y,t) dy \right) dx \\
- \;& \lambda_{12} \int_0^1 z_1(x,t) \left(\int_0^x k(x,y) z_2(y,t) dy \right) dx.
\end{aligned} \tag{2.94}$$

By virtue of (2.93), the magnitude of the first term in the right hand side of (2.94) can be estimated by means of the next chain of inequalities

$$\begin{aligned}
&\left| \lambda_{21} \int_0^1 z_2(x,t) \left(\int_0^x l(x,y) z_1(y,t) dy \right) dx \right| \\
\leq \;& |\lambda_{21}| \left| \int_0^1 |z_2(x,t)| \left(\int_0^x |l(x,y)||z_1(y,t)| dy \right) dx \right| \\
\leq \;& |\lambda_{21}| \gamma^* e^{2\gamma^*} \left| \int_0^1 |z_2(x,t)| \left(\int_0^x |z_1(y,t)| dy \right) dx \right| \\
\leq \;& |\lambda_{21}| \gamma^* e^{2\gamma^*} \left| \int_0^1 |z_2(x,t)| \left(\int_0^1 |z_1(y,t)| dy \right) dx \right|.
\end{aligned} \tag{2.95}$$

Using the triangle and Holder inequalities, the integrand in the last row of (2.95) is manipulated to

$$\begin{aligned}
&|z_2(x,t)| \left(\int_0^1 |z_1(y,t)| dy \right) \leq \frac{1}{2} \left[z_2^2(x,t) \right. \\
&\left. + \left(\int_0^1 |z_1(y,t)| dy \right)^2 \right] \leq \frac{1}{2} \left[z_2^2(x,t) + \|z_1(\cdot,t)\|_2^2 \right].
\end{aligned} \tag{2.96}$$

Substituting (2.96) into (2.95) one concludes that

$$\begin{aligned}
&\left| \lambda_{21} \int_0^1 z_2(x,t) \left(\int_0^x l(x,y) z_1(y,t) dy \right) dx \right| \\
\leq \;& \frac{1}{2} |\lambda_{21}| \gamma^* e^{2\gamma^*} \int_0^1 \left[z_2^2(x,t) + \|z_1(\cdot,t)\|_2^2 \right] dx \\
= \;& \frac{1}{2} |\lambda_{21}| \gamma^* e^{2\gamma^*} \left(\|z_1(\cdot,t)\|_2^2 + \|z_2(\cdot,t)\|_2^2 \right).
\end{aligned} \tag{2.97}$$

By performing analogous manipulations, the last term in the right hand side of (2.94) is straightforwardly shown to obey the estimate

$$\begin{aligned}
&\left|\lambda_{12}\int_0^1 z_1(x,t)\left(\int_0^x k(x,y)z_2(y,t)dy\right)dx\right| \\
\leq\;& \frac{1}{2}\left|\lambda_{12}\right|\gamma^* e^{2\gamma^*}\left(\|z_1(\cdot,t)\|_2^2+\|z_2(\cdot,t)\|_2^2\right). \qquad (2.98)
\end{aligned}$$

Combining (2.97) and (2.98) yields

$$\begin{aligned}
&\left|\int_0^1 Z^T(x,t)\left(\int_0^x \begin{bmatrix} -\lambda_{12}k(x,y)z_2(y,t) \\ \lambda_{21}l(x,y)z_1(y,t)\end{bmatrix} dy\right)dx\right| \\
\leq\;& \bar{\lambda}_M\gamma^* e^{2\gamma^*}\left(\|z_1(\cdot,t)\|_2^2+\|z_2(\cdot,t)\|_2^2\right) \\
=\;& \bar{\lambda}_M\gamma^* e^{2\gamma^*}\|Z(\cdot)\|_{2,2}^2, \qquad (2.99)
\end{aligned}$$

where $\bar{\lambda}_M=\max\{|\lambda_{12}|,|\lambda_{21}|\}$. Therefore, combining (2.99), (2.89) and (2.90), one further elaborates (2.88) by getting the next final estimate of $\dot{V}(t)$:

$$\begin{aligned}
&\dot{V}(t)\leq -\sigma_m(\Theta)||Z_\xi(\cdot,t)||_{2,2}^2-(\sigma_m(S[C(\gamma^*)]) \\
&-\bar{\lambda}_M\gamma^* e^{2\gamma^*}\big)\|Z(\cdot)\|_{2,2}^2\leq -2M(\gamma^*)V(t), \qquad (2.100)
\end{aligned}$$

where $M(\gamma^*)$ is given in (2.85). Thus, under condition (2.77) (which implies that $M(\gamma^*)>0$), the exponential stability of the target dynamics (2.68), (2.86) is concluded. Following [67], analogous exponential convergence properties, as specified in (2.84), are ensured for the original system (2.60)-(2.63) as well, according to the supporting arguments given in the concluding part of the proof of Theorem 14. Theorem 15 is proven. ■ □

Remark 8. *The stabilization result just demonstrated relies on the nonlinear inequality* (2.77) *to possess a feasible solution. The feasibility of such a solution, which critically affects the subsequent stability analysis, intrinsically depends on the plant parameters and there exist some actual plants for which no constant γ^*, satisfying* (2.77)*, can be found. However, the numerical evidences of Subsection 2.1.6 show that the proposed synthesis can be applied to successfully stabilize a physically relevant class of underactuated coupled reaction-diffusion processes. It is also worth to stress that Condition* (2.77) *is only sufficient for an underactuated boundary stabilizing synthesis to exist due to heavily conservative estimations made within the Lyapunov based convergence proof. Finally, it should be pointed out that the developments of the Section 2.1.5 don't really hinge on having constant coefficients and may be likely extended to more general scenarios where the coefficients of* (2.60)-(2.61) *are spatially and/or time varying.*

2.1.6 Simulation results

To support the theory developed, capabilities of the the proposed boundary synthesis are tested in simulation runs. First, the boundary stabilization of three coupled PDEs with distinct diffusivity parameters is treated, and then the underactuated boundary stabilization of two coupled processes is dealt with. To solve the closed-loop PDEs a standard finite-difference approximation method is used in all simulations by discretizing the spatial solution domain $x \in [0,1]$ into a finite number of N uniformly spaced solution nodes $x_i = ih$, $h = 1/(N+1)$, $i = 1, 2, ..., N$. The value $N = 40$ is set and the resulting discretized system of ODEs is then solved in the Matlab-Simulink environment by using the fixed-step Runge-Kutta method with the fixed step $T_s = 10^{-4}$.

TEST 1: fully actuated case

System (2.4)-(2.6) of three ($n = 3$) coupled reaction-diffusion processes, specified with the parameters

$$\Theta = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 5 & 1 \end{bmatrix}, \tag{2.101}$$

is first considered for simulation purposes. The initial conditions are set as $q_1(x,0) = q_3(x,0) = 2+2cos(\pi x)$, $q_2(x,0) = 5cos(\pi x)$. Matrix Λ possesses a real positive eigenvalue and the system in the open-loop (i.e., with $u_1(t) = u_2(t) = u_3(t) = 0$) is unstable, as displayed in the Figure 2.1 which shows the diverging temporal evolutions of the state norms $\|q_1(\cdot,t)\|_2$, $\|q_2(\cdot,t)\|_2$ and $\|q_3(\cdot,t)\|_2$. The boundary controller (2.55)-(2.56) is implemented by selecting the parameter $\gamma^* = 5$ as prescribed in Condition 1 to fulfill the requirement $S[C] > 0$, where C is given in (2.52). The converging spatiotemporal evolutions of the states in the closed-loop is shown in Figure 2.2 as well as the associated norm $\|Q(\cdot,t)\|_{2,3}$ is. As expected, this associated norm monotonically tends to zero. Figure 2.3 displays the time histories of the three control inputs $u_i(t)$ ($i = 1, 2, 3$) showing the initial peaking, and subsequent convergence to zero, which are typical for the backstepping design.

TEST 2: underactuated case

Next, the underactuated system (2.60)-(2.61), specified with the parameters $\theta_1 = 9$, $\theta_2 = 1$, $\lambda_{11} = 3$, $\lambda_{12} = \lambda_{21} = 1$ and $\lambda_{22} = -5$, is under numerical study. The initial conditions are set as $q_1(x,0) = 2 + 2cos(\pi x)$, $q_2(x,0) = 5cos(\pi x)$. The considered system in the open-loop (i.e., with $u_1(t) = 0$) is unstable since the Λ matrix possesses a positive eigenvalue. The unstable behaviour of the open-loop plant is displayed in the Figure 2.4, which shows the diverging spatiotemporal evolutions of the states $q_1(x,t)$ and $q_2(x,t)$.

Clearly, Assumption 1 holds true, and the boundary controller (2.82)-(2.83) is implemented by selecting the parameter $\gamma^* = 0.7$. With the adopted choice of γ^* it turn out that $\sigma_m(S[C(\gamma^*)]) = 2.58$, whereas the right hand side of (2.77) takes the value 2.52, hence Condition 2 is satisfied thereby ensuring that the closed-loop system meets desired exponential stability properties according to Theorem 15.

Figure 2.5 shows the resulting stable spatiotemporal evolutions of the state variables $q_1(x,t)$ and $q_2(t)$ in the closed-loop, which both vanish in L_2 norm as shown in the Figure 2.6. The time evolution of the boundary control input $u_1(t)$ is displayed in the Figure 2.7.

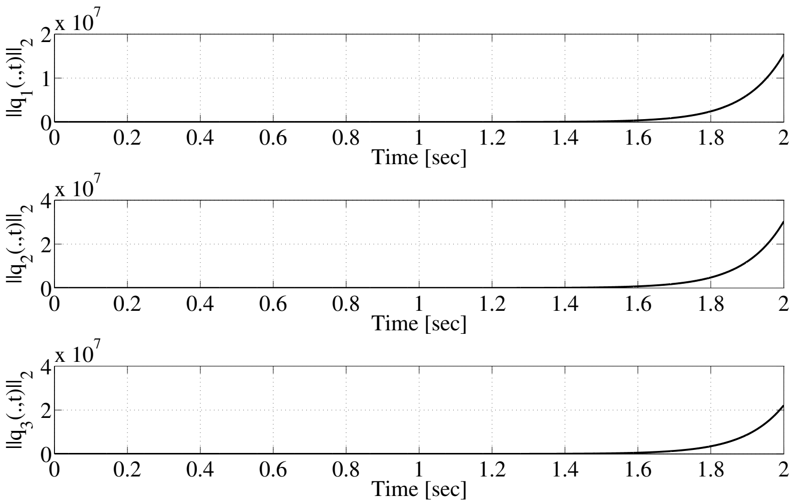

Figure 2.1: TEST 1. Temporal evolution of the norms $\|q_i(\cdot,t)\|_2$, $i = 1, 2, 3$, in the open loop test.

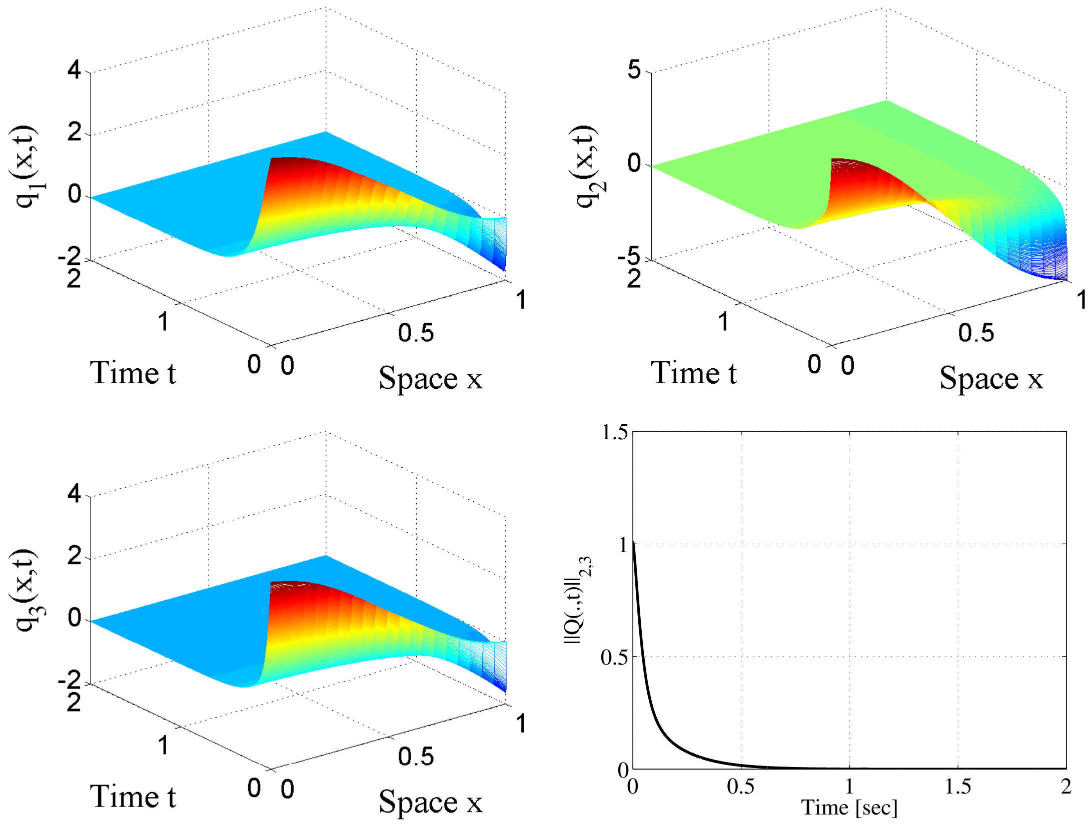

Figure 2.2: TEST 1. Spatiotemporal evolution of the states $q_i(x,t)$, $i = 1, 2, 3$, in the closed-loop test and (bottom-right) time profile of the corresponding norm $\|Q(\cdot,t)\|_{2,3}$

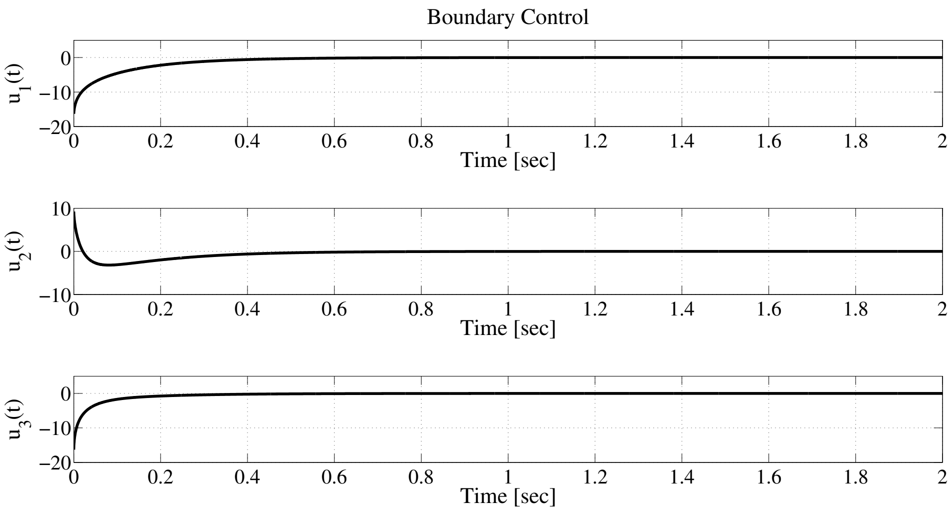

Figure 2.3: TEST 1. Temporal evolution of the boundary controls $u_i(t)$, $i = 1, 2, 3$.

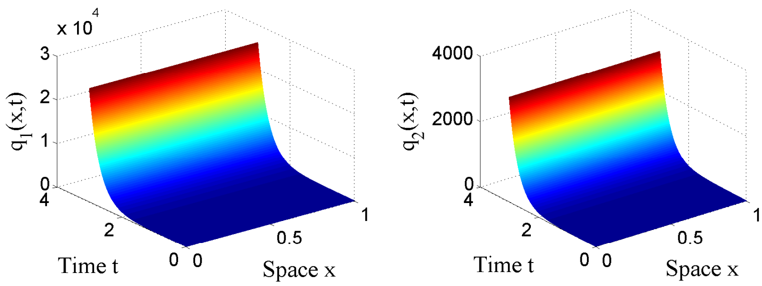

Figure 2.4: TEST 2. Spatiotemporal evolution of $q_1(x, t)$ and $q_2(x, t)$ in the open loop.

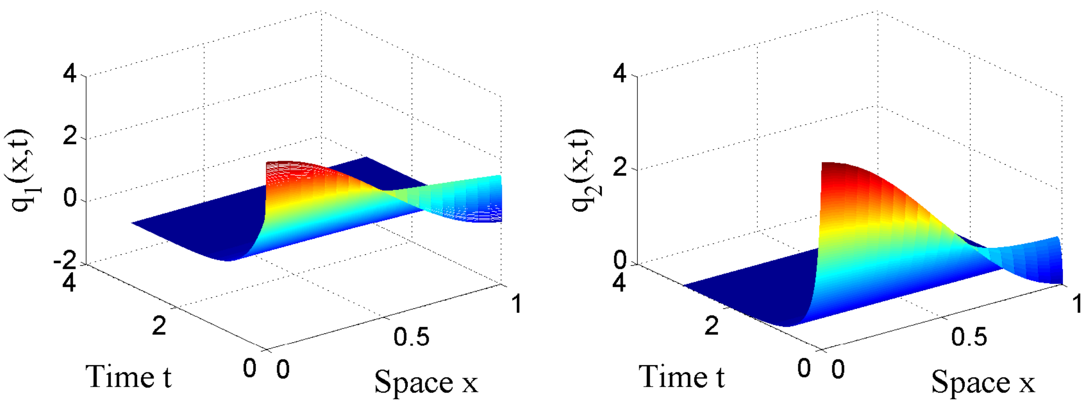

Figure 2.5: TEST 2. Spatiotemporal evolution of $q_1(x, t)$ and $q_2(x, t)$in the closed-loop test.

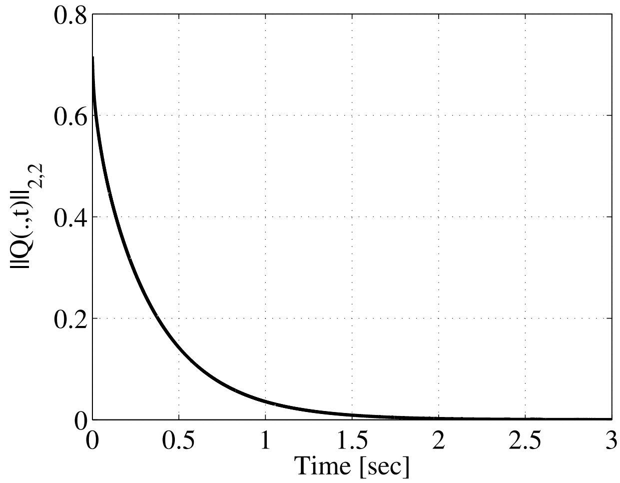

Figure 2.6: TEST 2. Vector norm $\|Q(\cdot)\|_{2,2}$ in the closed-loop test.

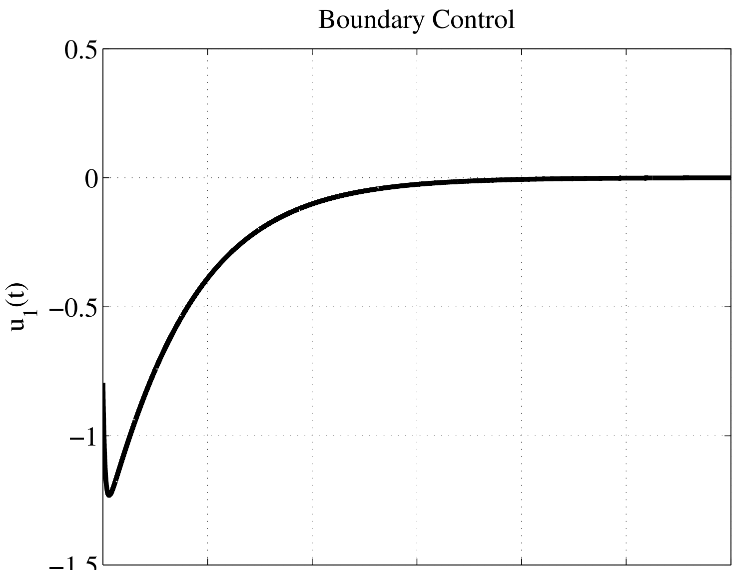

Figure 2.7: TEST 2. Time evolution of the boundary control input $u_1(t)$.

2.1.7 Conclusions

The backstepping-based boundary stabilization of certain classes of unstable coupled parabolic linear PDEs was tackled, and explicit state feedback boundary controllers were derived to attain the exponential decay of the closed-loop system in the state space $[L_2(0,1)]^n$. These results provide a non trivial multidimensional counterpart to the "scalar" ($n = 1$) treatment previously developed in [67]. Addressing the observer-based output feedback design, dealing with spatially-dependent parameters, and including the convection terms in the coupled PDEs, are among the most interesting lines of future related investigations. It is also of interest to deepen the present investigation on the underactuated case where only one scalar manipulable input variable is available, by generalizing the 2-dimensional problem statement, studied in the present work, towards higher dimensional scenarios. Additionally, integration with other design methodologies such as the sliding mode approaches, will be pursued as well to enhance the underlying robustness features. Particularly, recent investigations of [55]-[25] are hoped to complement the presented approaches by integrating them with suitably designed second-order sliding mode based boundary controllers in order to deal with the control of perturbed coupled PDEs.

2.2 Boundary stabilization of coupled reaction-diffusion equations having the same diffusivity parameters

We consider the problem of boundary stabilization for a system of n coupled parabolic linear PDEs. Particularly, we design a state feedback law with actuation on only one end of the domain and prove exponential stability of the closed-loop system with an arbitrarily fast convergence rate. The backstepping method is used for controller design, and the transformation kernel matrix is derived in explicit form by using the method of successive approximations to solve the corresponding PDE. Thus, the suggested control law is also made available in explicit form. Simulation results support the effectiveness of the suggested design.

2.2.1 Introduction

We investigate the boundary stabilization of a class of coupled linear parabolic Partial Differential Equations (PDEs) in a finite spatial domain $x \in [0, 1]$. Particularly, by exploiting the so-called "backstepping" approach [13, 67], we do focus on "approximation-free" control design not relying on any discretization or finite-dimensional approximation. The backstepping-based boundary control problem for several classes of wave processes was studied, e.g., in [58, 14], while heat processes were studied, e.g., in [50, 17, 67]. More involved, complex-valued, PDEs such as the Schrodinger equation were also dealt with by means of such an approach [16]. A cascade of two parabolic reaction-diffusion processes was dealt with in [28] by using a unique control input acting only at a boundary of one side.

More recently, high-dimensional systems of coupled PDEs are being considered in the backstepping-based boundary control setting. The most intensive efforts of current literature appear however to be oriented towards coupled hyperbolic processes of the transport-type [37, 29, 8, 5, 30].

The state feedback design in [29], which allows stabilization of 2×2 linear heterodirectional1 hyperbolic systems, was extended in [8] to a particular type of 3×3 linear systems, arising in modeling of multiphase flow, and to the quasilinear case in [30]. In [5], a 2×2 linear hyperbolic system was stabilized by a single boundary control input, with the additional feature that an unmatched disturbance, generated by a known exosystem, is rejected. In [37] a system of $n + 1$ coupled first-order hyperbolic linear PDEs with a single boundary input was studied. Some specific result concerning the backstepping based boundary stabilization of parabolic coupled PDEs has been presented in the literature. In [75] the linearized 2×2 model of thermal-fluid convection, which entails very dissimilar diffusivity parameters, has been treated by using a singular perturbations approach combined with backstepping and Fourier series expansion. In [6] the Ginzburg-Landau equations, which represent a 2×2 system with equal diffusion coefficients when the imaginary and real parts are expanded, was dealt wit, while in [33] the boundary stabilization of the linearized model of an incompressible magnetohydrodynamic flow in an infinite rectangular 3D channel, known as Hartmann flow, was attacked.

The task of the present paper is to generalize some results presented in [67], where an explicit boundary controller was developed to stabilize a scalar unstable reaction diffusion equation. Here we provide a generalization to the multidimensional case, by considering a set of n reaction diffusion processes coupled through the corresponding reaction terms. The motivation to this investigation comes from chemical processes [20] where such equations occur to describe system dynamics, e.g., coupled temperature-concentration parabolic PDEs.

As shown in the paper, this generalization is far from being trivial because the underlying backstepping-based treatment gives rise to more complex development of finding out an explicit solution and, furthermore, it turns out to be unfeasible in the general case where each process possesses its own diffusivity parameter. In this paper we therefore address the simplified case where all processes have the same diffusivity value, and we postpone the more general case for further investigations, which requires some constraint on the target system (see Remark 1).

An additional interesting feature of backstepping is that it allows an easy synergic integration with robust control paradigms such as the sliding mode control methodology (see e.g. [10]) to enhance the robustness features of the overall scheme by providing the capability of completely rejecting the effect of persistent matching disturbances which are not required to be generated by a known exosystem. In fact, following our recent lines of investigation [55, 56, 25, 24], it is our purpose for next research to complement the presented scheme by integrating it with suitably designed second-order sliding mode based boundary controllers in order to deal with the control of perturbed coupled PDEs.

The structure of the paper is as follows. After introducing in the next subsection some useful notation, in Section II we state the problem under investigation and we introduce the underlying backstepping transformation. In Section III the solution of the kernel PDE is tackled for both the direct and inverse transformations. In Section IV the the proposed boundary control design and main stability result of this paper are drawn. Section V presents the simulation results and Section VI gives some concluding remarks and future perspectives of this research.

Notation

The notation used throughout is fairly standard. $L_2(0,1)$ stands for the Hilbert space of square integrable scalar functions $z(\zeta)$ on $(0,1)$ and the corresponding norm

$$\|z(\cdot)\|_2 = \sqrt{\int_0^1 z^2(\zeta) d\zeta}. \tag{2.102}$$

Throughout the paper we shall also utilize the notation

$$[L_2(0,1)]^n = \underbrace{L_2(0,1) \times L_2(0,1) \times \ldots \times L_2(0,1)}_{n \ \mathtt{times}}, \tag{2.103}$$

and

$$\|Z(\cdot)\|_{2,n} = \sqrt{\sum_{i=1}^{n} \|z_i(\cdot)\|_2^2} \tag{2.104}$$

for the corresponding norm of a generic vector function $Z(\zeta) = [z_1(\zeta), z_2(\zeta), ..., z_n(\zeta)] \in [L_2(0,1)]^n$.

2.2.2 Problem formulation and backstepping transformation

We consider a n-dimensional system of coupled reaction-diffusion processes, equipped with Neumann-type boundary conditions, governed by the next PDE

$$\begin{aligned} Q_t(x,t) &= \theta Q_{xx}(x,t) + \Lambda Q(x,t) && (2.105)\\ Q_x(0,t) &= 0, && (2.106)\\ Q_x(1,t) &= U(t) && (2.107) \end{aligned}$$

where

$$Q(x,t) = [q_1(x,t),\ q_2(x,t), \ldots, q_n(x,t)]^T \in [L_2(0,1)]^n \tag{2.108}$$

is the vector collecting the state of all systems,

$$U(t) = [u_1(t),\ u_2(t), \ldots, u_n(t)]^T \in \Re^n \tag{2.109}$$

is the vector collecting all the manipulable boundary control signals, $\Lambda = \{\lambda_{ij}\} \in \Re^{n\times n}$ is a real-valued square matrix, and $\theta \in \Re^+$ is a positive scalar. The open-loop system (2.105)-(2.107) (with $U(t) = 0$) possesses arbitrarily many unstable eigenvalues when the matrix Λ is positive definite with sufficiently large eigenvalues. Since the term $\Lambda Q(x,t)$ is the source of instability, the natural objective for a boundary feedback is to "reshape" (or cancel) this term by reversing its effect into a stabilizing one. Following this line of reasoning, our objective is to exponentially stabilize system (2.105)-(2.107) by transforming it into the target system

$$\begin{aligned} Z_t(x,t) &= \theta Z_{xx}(x,t) - CZ(x,t) && (2.110)\\ Z_x(0,t) &= 0, && (2.111)\\ Z_x(1,t) &= 0 && (2.112) \end{aligned}$$

where $Z(x,t) = [z_1(x,t),\ z_2(x,t), \ldots, z_n(x,t)]^T \in [L_2(0,1)]^n$ is the corresponding state vector and $C = \{c_{ij}\} \in \Re^{n,n}$ is an arbitrarily chosen real-valued square matrix, by means of an invertible backstepping transformation

$$Z(x,t) = Q(x,t) - \int_0^x K(x,y)Q(y,t)dy \tag{2.113}$$

where $K(x,y)$ is a $n\times n$ matrix function whose elements are denoted as $k_{ij}(x,y)$, with $i,j = 1,2,\ldots,n$. The exponential stability properties of the target system, whose convergence rate can be made arbitrarily fast by a suitable choice of the matrix C, are investigated in detail later in Theorem 18. Following the usual backstepping design, we now derive and solve the PDE governing the kernel matrix function $K(x,y)$.

Spatial derivatives $Z_x(x,t)$ and $Z_{xx}(x,t)$ take the form (the Leibnitz differentiation rule is used):

$$Z_x(x,t) = Q_x(x,t) - K(x,x)Q(x,t) - \int_0^x K_x(x,y)Q(y,t)dy \tag{2.114}$$

$$\begin{aligned} Z_{xx}(x,t) &= Q_{xx}(x,t) - \left[\frac{d}{dx}K(x,x)\right]Q(x,t) \\ &- K(x,x)Q_x(x,t) - K_x(x,x)Q(x,t) \\ &- \int_0^x K_{xx}(x,y)Q(y,t)dy \end{aligned} \tag{2.115}$$

where

$$\begin{gathered} \tfrac{d}{dx}K(x,x) = K_x(x,x) + K_y(x,x) \\ K_x(x,x) = K_x(x,y)\big|_{y=x}, \quad K_y(x,y) = K_y(x,y)\big|_{y=x} \end{gathered} \tag{2.116}$$

Using (2.105), and applying recursively integration by parts, the time derivative $Z_t(x,t)$ is given by

$$\begin{aligned} Z_t(x,t) &= Q_t(x,t) - \int_0^x K(x,y)Q_t(y,t)dy \\ &= \theta Q_{xx}(x,t) + \Lambda Q(x,t) - \theta K(x,x)Q_x(x,t) \\ &+ \theta K(x,0)Q_x(0,t) + \theta K_y(x,x)Q(x,t) \\ &- \theta K_y(x,0)Q(0,t) - \theta\int_0^x K_{yy}(x,y)Q(y,t)dy \\ &- \int_0^x K(x,y)\Lambda Q(y,t)dy \end{aligned} \tag{2.117}$$

Combining (2.113), (2.115), (2.117) and performing lengthy but straightforward computations, yield

$$\begin{aligned} & Z_t(x,t) - \theta Z_{xx}(x,t) + CZ(x,t) = \\ & \left[\Lambda + C + \theta\left(K_y(x,x) + K_x(x,x) + \frac{d}{dx}K(x,x)\right)\right]Q(x,t) \\ & + \int_0^x \left[\theta\left(K_{xx}(x,y) - K_{yy}(x,y)\right) - K(x,y)\Lambda - CK(x,y)\right] \times \\ & \times Q(y,t)dy + \theta K(x,0)Q_x(0,t) - \theta K_y(x,0)Q(0,t) \end{aligned} \tag{2.118}$$

Clearly, the target system's equation (2.110) implies that the right hand side of (2.118) has to be identically zero. Considering the homogeneous BC (2.106), this leads to the next relations

$$
\begin{aligned}
K_{xx}(x,y) - K_{yy}(x,y) &= \frac{1}{\theta}K(x,y)\Lambda + \frac{1}{\theta}CK(x,y) && (2.119)\\
\Lambda + C &+ 2\theta\frac{d}{dx}K(x,x) = 0 && (2.120)\\
K_y(x,0) &= 0 && (2.121)
\end{aligned}
$$

Integrating (2.120) with respect to x gives $K(x,x) = -\frac{1}{2\theta}(\Lambda + C)\,x + K(0,0)$, where $K(0,0)$ is obtained by substituting the boundary conditions (2.106) and (2.111) into the next relation, which is derived by specifying (2.114) with $x = 0$

$$
Z_x(0,t) = Q_x(0,t) - K(0,0)Q(0,t) \to K(0,0) = 0. \tag{2.122}
$$

Hence, system (2.119)-(2.121) becomes

$$
\begin{aligned}
K_{xx}(x,y) - K_{yy}(x,y) &= \frac{1}{\theta}K(x,y)\Lambda + \frac{1}{\theta}CK(x,y) && (2.123)\\
K(x,x) &= -\frac{1}{2\theta}(\Lambda + C)\,x && (2.124)\\
K_y(x,0) &= 0 && (2.125)
\end{aligned}
$$

We will show that (2.123)-(2.125) define a well posed system of PDEs, and we shall derive the corresponding solution in explicit form.

Remark 9. *We are confining the present paper to the case in which all the coupled PDEs* (2.105) *have the same diffusivity parameter θ. The reason is that in the more general case where each process has its own diffusivity θ_i, ($i = 1, 2, ..., n$), the corresponding "generalized" version*

$$
\begin{aligned}
\Theta K_{xx}(x,y) - K_{yy}(x,y)\Theta = K(x,y)\Lambda + CK(x,y) && (2.126)\\
\Lambda + C + K_y(x,x)\Theta + \Theta K_x(x,x) + \Theta\frac{d}{dx}K(x,x) = 0 && (2.127)\\
K_y(x,0)\Theta = 0 && (2.128)\\
\Theta K(x,x) = K(x,x)\Theta && (2.129)
\end{aligned}
$$

of (2.123)-(2.125)*, where $\Theta = diag(\theta_i)$, sets an overdetermined PDE without solution, unless specific constraints on the matrix C and on the form of the kernel matrix $K(x,y)$ are met. This topic will be addressed in more detail in next works.*

2.2.3 Solution of the kernel PDE (2.123)-(2.125)

The following result is in order.

Theorem 16. *The problem (2.123)-(2.125) possesses the explicit solution* (2.161) *that is infinitely times continuously differentiable in* $0 \leq y \leq x \leq 1$.

Proof. Following [67], the existence of a solution to problem (2.123)-(2.125) can be proved by transforming it into an integral equation using the variable change

$$\xi = x + y, \qquad \eta = x - y. \tag{2.130}$$

Denoting

$$G(\xi, \eta) = K(x, y) = K\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) \tag{2.131}$$

we have the next relations

$$\begin{aligned} K_x &= G_\xi + G_\eta & (2.132)\\ K_{xx} &= G_{\xi\xi} + 2G_{\xi\eta} + G_{\eta\eta} & (2.133)\\ K_y &= G_\xi - G_\eta & (2.134)\\ K_{yy} &= G_{\xi\xi} - 2G_{\xi\eta} + G_{\eta\eta} & (2.135)\end{aligned}$$

Thus, the gain kernel PDE in the new coordinates becomes

$$\begin{aligned} G_{\xi\eta}(\xi, \eta) &= \frac{1}{4\theta} G(\xi, \eta)\Lambda + \frac{1}{4\theta} C G(\xi, \eta) & (2.136)\\ G(\xi, 0) &= -\frac{1}{4\theta}(\Lambda + C)\,\xi & (2.137)\\ G_\xi(\xi, \xi) &= G_\eta(\xi, \xi) & (2.138)\end{aligned}$$

Integrating (2.136) with respect to η from 0 to η, and considering the relation $G_\xi(\xi, 0) = -\frac{1}{4\theta}(\mathit{\Lambda} + C)$, which directly derives from (2.137), we get:

$$G_\xi(\xi, \eta) = -\frac{1}{4\theta}(\mathit{\Lambda} + C) + \frac{1}{4\theta}\int_0^\eta \left[G(\xi, s)\mathit{\Lambda} + CG(\xi, s)\right] ds \tag{2.139}$$

Integrating (2.139) with respect to ξ from η to ξ yields:

$$\begin{aligned} \int_\eta^\xi G_\tau(\tau, \eta) d\tau &= \int_\eta^\xi -\frac{1}{4\theta}(\mathit{\Lambda} + C) d\tau \\ &+ \frac{1}{4\theta}\int_\eta^\xi \left\{\int_0^\eta \left[G(\tau, s)\mathit{\Lambda} + CG(\tau, s)\right] ds\right\} d\tau & (2.140)\end{aligned}$$

which can be further manipulated as follows

$$G(\xi, \eta) \;-\; G(\eta, \eta) = -\frac{1}{4\theta}(\mathit{\Lambda} + C)(\xi - \eta) + \frac{1}{4\theta}\int_\eta^\xi \left\{\int_0^\eta \left[G(\tau, s)\mathit{\Lambda} + CG(\tau, s)\right] ds\right\} d\tau \tag{2.141}$$

We are now going to derive an explicit form of $G(\eta, \eta)$. We use (2.138) to write

$$\frac{d}{d\xi} G(\xi, \xi) = G_\xi(\xi, \xi) + G_\eta(\xi, \xi) = 2G_\xi(\xi, \xi) \tag{2.142}$$

Using (2.139) with $\eta = \xi$ we can write (2.142) in the form of differential equation for $G(\xi, \xi)$

$$\frac{d}{d\xi} G(\xi, \xi) = -\frac{1}{2\theta}(\Lambda + C) + \frac{1}{2\theta} \int_0^{\xi} [G(\xi, s)\Lambda + CG(\xi, s)]\, ds \tag{2.143}$$

Integrating both sides of (2.143) with respect to ξ, and then making the substitution $\xi = \eta$, we finally obtain

$$G(\eta, \eta) = -\frac{1}{2\theta}(\Lambda + C)\eta + \frac{1}{2\theta} \int_0^{\eta} \left\{ \int_0^{\tau} [G(\tau, s)\Lambda + CG(\tau, s)]\, ds \right\} d\tau \tag{2.144}$$

Substituting (2.144) into (2.141) we obtain an integral equation for $G(\xi, \eta)$

$$\begin{aligned} G(\xi, \eta) &= -\frac{1}{4\theta}(\Lambda + C)\eta - \frac{1}{4\theta}(\Lambda + C)\xi \\ &+ \frac{1}{2\theta} \int_0^{\eta} \left\{ \int_0^{\tau} [G(\tau, s)\Lambda + CG(\tau, s)]\, ds \right\} d\tau \\ &+ \frac{1}{4\theta} \int_{\eta}^{\xi} \left\{ \int_0^{\eta} [G(\tau, s)\Lambda + CG(\tau, s)]\, ds \right\} d\tau \end{aligned} \tag{2.145}$$

We now use the method of successive approximations to show that equation (2.145) has a continuous and smooth solution. Let us start with an initial guess:

$$G^0(\xi, \eta) = 0 \tag{2.146}$$

and set-up the recursive formula for (2.145) as follows:

$$\begin{aligned} G^{n+1}(\xi, \eta) &= -\frac{1}{4\theta}(\Lambda + C)(\xi + \eta) + \frac{1}{2\theta} \int_0^{\eta} \left\{ \int_0^{\tau} [G^n(\tau, s)\Lambda + CG^n(\tau, s)]\, ds \right\} d\tau \\ &+ \frac{1}{4\theta} \int_{\eta}^{\xi} \left\{ \int_0^{\eta} [G^n(\tau, s)\Lambda + CG^n(\tau, s)]\, ds \right\} d\tau \end{aligned} \tag{2.147}$$

If this recursion converges, we can write the solution $G(\xi, \eta)$ as

$$G(\xi, \eta) = \lim_{n \to \infty} G^n(\xi, \eta) \tag{2.148}$$

Let us denote the difference between two consecutive terms as

$$\Delta G^n(\xi, \eta) = G^{n+1}(\xi, \eta) - G^n(\xi, \eta) \tag{2.149}$$

Then, the next recursion is correspondingly obtained by (2.146)-(2.147)

$$\Delta G^0(\xi, \eta) = G^1(\xi, \eta) = -\frac{1}{4\theta}(\Lambda + C)(\xi + \eta) \tag{2.150}$$

$$\begin{aligned} \Delta G^{n+1}(\xi, \eta) &= \frac{1}{2\theta} \int_0^{\eta} \left\{ \int_0^{\tau} [\Delta G^n(\tau, s)\Lambda + C\Delta G^n(\tau, s)]\, ds \right\} d\tau \\ &+ \frac{1}{4\theta} \int_{\eta}^{\xi} \left\{ \int_0^{\eta} [\Delta G^n(\tau, s)\Lambda + C\Delta G^n(\tau, s)]\, ds \right\} d\tau \end{aligned} \tag{2.151}$$

and (2.148) can be alternatively written as

$$G(\xi,\eta)=\sum_{n=0}^{\infty}\Delta G^{n}(\xi,\eta) \tag{2.152}$$

Since variables ξ and η lie in the bounded domain $0\le\eta\le\xi\le 2$, one can readily show by (2.150) that

$$\|\Delta G^{0}(\xi,\eta)\|\le\frac{1}{\theta}(\|\Lambda\|+\|C\|)=M \tag{2.153}$$

Suppose that

$$\|\Delta G^{n}(\xi,\eta)\|\le M^{n+1}\frac{(\xi+\eta)^{n}}{n!} \tag{2.154}$$

Then, by (2.151), (2.153) and (2.154) we can derive the next estimate

$$\begin{aligned}\|\Delta G^{n+1}(\xi,\eta)\| &\le \frac{1}{4\theta}(\|\Lambda\|+\|C\|)\frac{M^{n+1}}{n!}\left|2\int_0^{\eta}\int_0^{\tau}(\tau+s)^n ds d\tau\right.\\ &\quad\left.+\int_{\eta}^{\xi}\int_0^{\eta}(\tau+s)^n ds d\tau\right|\\ &= \frac{1}{4}\frac{M^{n+2}}{n!}\left|2\int_0^{\eta}\int_0^{\tau}(\tau+s)^n ds d\tau\right.\\ &\quad\left.+\int_{\eta}^{\xi}\int_0^{\eta}(\tau+s)^n ds d\tau\right|\end{aligned} \tag{2.155}$$

It is readily shown (cfr. [10], eq. (2.14)) that the next estimate

$$\left|2\int_0^{\eta}\int_0^{\tau}(\tau+s)^n ds d\tau+\int_{\eta}^{\xi}\int_0^{\eta}(\tau+s)^n ds d\tau\right|\le 4\frac{(\xi+\eta)^{n+1}}{(n+1)} \tag{2.156}$$

holds. Therefore, combining (2.155) and (2.156) one gets

$$\|\Delta G^{n+1}(\xi,\eta)\|\le M^{n+2}\frac{(\xi+\eta)^{n+1}}{(n+1)!} \tag{2.157}$$

By mathematical induction, (2.157) is true for all $n>0$. It then follows from the Weierstrass M-test that the series (2.152) converges absolutely and uniformly in $0\le\eta\le\xi\le 2$.

Computing $\Delta G^{n}(\xi,\eta)$ from (2.151) starting with (2.150) we have that

$$\Delta G^{1}(\xi,\eta)=-\frac{\xi^2\eta+\xi\eta^2}{2}\left(\frac{1}{4\theta}\right)^2[(\Lambda+C)\Lambda+C(\Lambda+C)] \tag{2.158}$$

and, iterating the computations, we can thus observe the pattern which leads to the following formula:

$$\Delta G^{n}(\xi,\eta) = -\frac{(\xi\eta)^n(\xi+\eta)}{n!(n+1)!}\left(\frac{1}{4\theta}\right)^{n+1}\left[\sum_{i=0}^{n}\binom{n}{i}C^{i}\left(\Lambda+C\right)\Lambda^{n-i}\right] \tag{2.159}$$

The solution to the integral equation (2.145) is therefore given by the next (absolutely and uniformly converging) series expansion:

$$G(\xi,\eta) \quad = \quad -\sum_{n=0}^{\infty} \frac{(\xi\eta)^n(\xi+\eta)}{n!(n+1)!} \left(\frac{1}{4\theta}\right)^{n+1} \left[\sum_{i=0}^{n} \binom{n}{i} C^i \left(\mathit{\Lambda}+C\right) \mathit{\Lambda}^{n-i}\right] \tag{2.160}$$

Returning to the original x, y variables we get the next series form for the Kernel matrix $K(x,y)$ which solves kernel PDE (2.123)-(2.125)

$$K(x,y) \quad = \quad -\sum_{n=0}^{\infty} \frac{(x^2-y^2)^n(2x)}{n!(n+1)!} \left(\frac{1}{4\theta}\right)^{n+1} \left[\sum_{i=0}^{n} \binom{n}{i} C^i \left(\mathit{\Lambda}+C\right) \mathit{\Lambda}^{n-i}\right] \tag{2.161}$$

Direct inspection reveals that (2.161) is infinitely times continuously differentiable. Theorem 16 is proven □

Remark 10. *If the condition $\mathit{\Lambda}C = C\mathit{\Lambda}$ holds, then the next simplified form of* (2.161) *is obtained*

$$K(x,y) \quad = \quad -\sum_{n=0}^{\infty} \frac{(x^2-y^2)^n(2x)}{n!(n+1)!} \left[\frac{\mathit{\Lambda}+C}{4\theta}\right]^{n+1} \tag{2.162}$$

which appears interestingly rather similar to the well known solution presented in [67] for the scalar case ($n = 1$).

Remark 11. *Uniqueness of the solution can be proven following the same steps as those made, e.g., in Lemma 2.1 of [10]. The complete treatment, which appears beyond the scope of the present paper, will be fully addressed in our future work.*

Inverse transformation

In order to prove stability we need to show that the transformation (2.113) is invertible. Let us write the inverse transformation in the form

$$Q(x,t) = Z(x,t) + \int_0^x L(x,y)Z(y,t)dy \tag{2.163}$$

By performing analogous developments as those made for the derivation of the gain kernel PDE (2.123)-(2.125), we obtain the next PDE governing $L(x,y)$

$$L_{xx}(x,y) - L_{yy}(x,y) \quad = \quad -\frac{1}{\theta}L(x,y)C - \frac{1}{\theta}\Lambda L(x,y) \tag{2.164}$$

$$L(x,x) \quad = \quad -\frac{1}{2\theta}\left(\Lambda + C\right)x \tag{2.165}$$

$$L_y(x,0) \quad = \quad 0 \tag{2.166}$$

By direct comparison between (2.123)-(2.125) and (2.164)-(2.166) one immediately notice that in this case $L(x,y) = -K(x,y)$ when Λ and C are replaced by $-\Lambda$ and $-C$. We then immediately obtain from (2.161) the corresponding explicit solution in the form

$$L(x,y) \;=\; -\sum_{n=0}^{\infty} \frac{(x^2-y^2)^n(2x)}{n!(n+1)!}\left(\frac{1}{4\theta}\right)^{n+1}\left[\sum_{i=0}^{n}\binom{n}{i}(-C)^i\,(\mathit{\Lambda}+C)\,(-\mathit{\Lambda})^{n-i}\right] \tag{2.167}$$

2.2.4 Main result

We begin by stating a preliminary result establishing the stability features of the target dynamics (2.110)-(2.112). The following result is in force.

Theorem 17. *Consider the target system* (2.110)-(2.112). *If the matrix C is such that its symmetric part $C_s = (C + C^T)/2$ is positive definite then system* (2.110)-(2.112) *is exponentially stable in the space $[L_2(0,1)]^n$ with the convergence rate specified by*

$$\|Z(\cdot,t)\|_{2,n} \le \|Z(\cdot,0)\|_{2,n}e^{-\sigma_1(C_s)t} \tag{2.168}$$

where $\sigma_1(C_s)$ is the smallest eigenvalue of C_s.

Proof. Consider the Lyapunov function $V(t) = \frac{1}{2}\int_0^1 Z^T(\xi,t)Z(\xi,t)d\xi = \frac{1}{2}\|Z(\cdot,t)\|^2_{2,n}$. The corresponding time derivative along the solutions of (2.110)-(2.112) is given by

$$\dot{V}(t) \;=\; \int_0^1 Z^T(\xi,t)\Theta Z_{xx}(\xi,t)d\xi - \int_0^1 Z^T(\xi,t)CZ(\xi,t)d\xi \tag{2.169}$$

Integration by parts taking into account (2.111) and (2.112), and exploiting the diagonal form of matrix Θ yield

$$\begin{aligned}\int_0^1 Z^T(\xi,t)\Theta Z_{xx}(\xi,t)d\xi &= Z^T(\chi,t)\Theta Z_x(\chi,t)\big|_{\chi=0}^{\chi=1}\\ &- \int_0^1 Z_x^T(\xi,t)\Theta Z_x(\xi,t)d\xi \le -\theta_m\|Z_x(\cdot,t)\|^2_{2,n}\end{aligned} \tag{2.170}$$

where $\theta_m = \min_{1\le i\le n}\theta_i > 0$. Since the smallest eigenvalue $\sigma_1(C_s)$ of the symmetric matrix $C_s = (C + C^T)/2$ is assumed to be positive then exploiting the trivial inequality $Z^T(\xi,t)CZ(\xi,t) \ge \sigma_1(C_s)^T Z(\xi,t)Z(\xi,t)$ and employing (2.170), one can easily manipulate (2.169) to derive

$$\dot{V}(t) \;\le\; -\theta_m\|Z_\xi(\cdot,t)\|^2_{2,n} - 2\sigma_1(C_s)V(t) \le -2\sigma_1(C_s)V(t)$$

thereby concluding the exponential stability of the target system in the space $[L_2(0,1)]^n$ with a convergence rate, obeying the estimate (2.168). Theorem 17 is proved. □

The next Theorem specifies the proposed boundary control design and summarizes the main stability result of this paper.

Theorem 18. *The boundary control input*

$$U(t) = -\frac{1}{2\theta}(\Lambda + C)\,Q(1,t) + \int_0^1 K_x(1,y)Q(y,t)dy \tag{2.171}$$

$$\begin{aligned} K_x(1,y) = &-\sum_{n=0}^{\infty}\left[\frac{2(1-y^2)^n + 4n(1-y^2)^{n-1}}{n!(n+1)!}\right]\times \\ &\times\left(\frac{1}{4\theta}\right)^{n+1}\left[\sum_{i=0}^{n}\binom{n}{i}C^i\,(\varLambda + C)\,\varLambda^{n-i}\right] \end{aligned} \tag{2.172}$$

where matrix C is selected such that its symmetric part $C_s = (C + C^T)/2$ is positive definite, stabilizes exponentially system (2.105)-(2.107) in the space $[L_2(0,1)]^n$ with an arbitrarily fast convergence rate in accordance with

$$\|Q(\cdot,t)\|_{2,n} \leq A\|Q(\cdot,0)\|_{2,n}e^{-\sigma_1(C_s)t} \tag{2.173}$$

where $\sigma_1(C_s)$ is the smallest eigenvalues of matrix C_s and A is a positive constant independent of $Q(\xi,0)$.

Proof. The developments of Section 1, along with Theorem 1, show that the backstepping transformation (2.113), (2.161) maps system (2.105)-(2.107) into the target dynamics in which the PDE (2.110) holds. ¿From (2.114) it follows that

$$Z_x(0,t) = Q_x(0,t) - K(0,0)Q(0,t) \tag{2.174}$$

$$Z_x(1,t) = Q_x(1,t) - K(1,1)Q(1,t) - \int_0^1 K_x(1,y)Q(y,t)dy \tag{2.175}$$

Considering the boundary conditions (2.106) and (2.107) along with relation (2.124) , which implies that $K(0,0) = 0$ and $K(1,1) = -\frac{1}{2\theta}(\Lambda + C)$, one has that

$$Z_x(0,t) = 0 \tag{2.176}$$

$$Z_x(1,t) = U(t) + \frac{1}{2\theta}(\Lambda + C)\,Q(1,t) - \int_0^1 K_x(1,y)Q(y,t)dy \tag{2.177}$$

Thus, with the boundary control input vector selected as in (2.171)-(2.172), where the form of the kernel $K_x(1,y)$ is readily obtained by differentiating (2.161) with respect to x at $x = 1$, the target dynamics (2.110)-(2.112) with homogeneous BCs is obtained, whose asymptotic stability features were demonstrated in Theorem 17. In particular, according to (2.168), the corresponding convergence rate can be made arbitrarily fast by a proper selection of the C matrix.

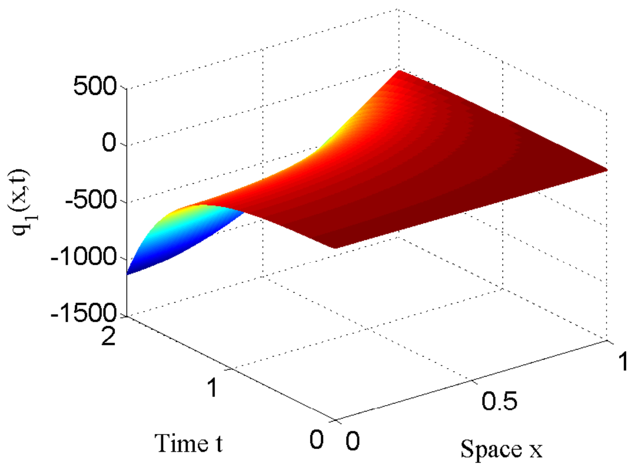

Figure 2.8: Spatiotemporal evolution of $q_1(x,t)$ in the open loop.

¿From now on, we follow [67] to derive analogous convergence properties for the original system (2.105)-(2.107) as well. Observing that $\xi + \eta = x$, it is easy to derive from (2.152)-(2.154) that $\|K(x,y)\| \leq Me^{2Mx}$, and the same bound can be derived for the norm of $L(x,y)$ as well, i.e. $\|L(x,y)\| \leq Me^{2Mx}$. A straightforward generalization of [67, Th 4] yields that those two boundedness relations, coupled together, establish the equivalence of norms of $Z(x,t)$ and $Q(x,t)$ in $[L_2(0,1)]^n$ which means that there exist a positive constant A independent of $Q(\xi,0)$ such that the estimate (2.173) is in force as a direct consequence of (2.168). Theorem 18 is proven. $\square$

2.2.5 Simulation results

To validate the proposed boundary control scheme, an instance of system (2.105)-(2.107) with $n = 2$ coupled reaction-diffusion processes has been considered for simulation purposes, with parameters

$$\theta = 1, \qquad \Lambda = \begin{bmatrix} -5 & 10 \\ 7 & -3 \end{bmatrix} \tag{2.178}$$

The initial conditions are set as $q_1(x,0) = q_2(x,0) = 10cos(\pi x)$. For solving the closed-loop PDE, a standard finite-difference approximation method is used by discretizing the spatial solution domain $x \in [0,1]$ into a finite number of N uniformly spaced solution nodes $x_i = ih$, $h = 1/(N+1)$, $i = 1,2,...,N$. The value $N = 40$ has been used. The resulting 40-th order discretized system is then solved by fixed-step Euler method with step $T_s = 10^{-4}$

The open-loop unstable behaviour of the uncontrolled plant (i.e., with $U(t) = [0,0]^T$) is displayed in the Figures 2.8 and 2.10, which show the diverging spatiotemporal evolution of the states $q_1(x,t)$ and $q_2(t)$.

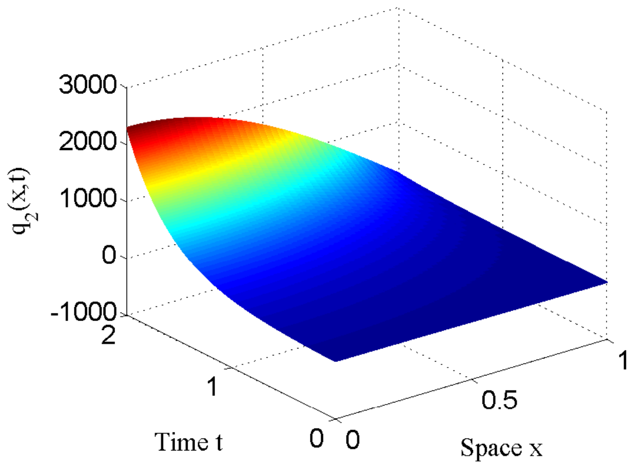

Figure 2.9: Spatiotemporal evolution of $q_2(x, t)$ in the open loop.

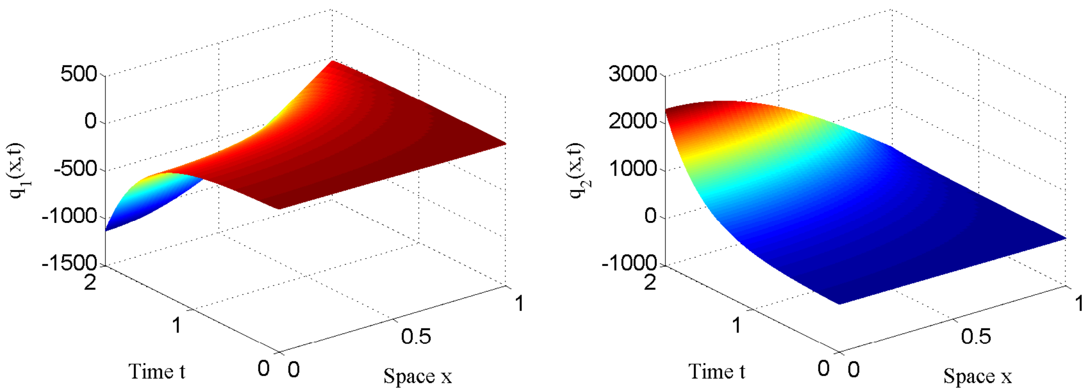

Figure 2.10: Spatiotemporal evolution of $q_1(x, t)$ (left plot) $q_2(x, t)$ (right plot) in the open loop.

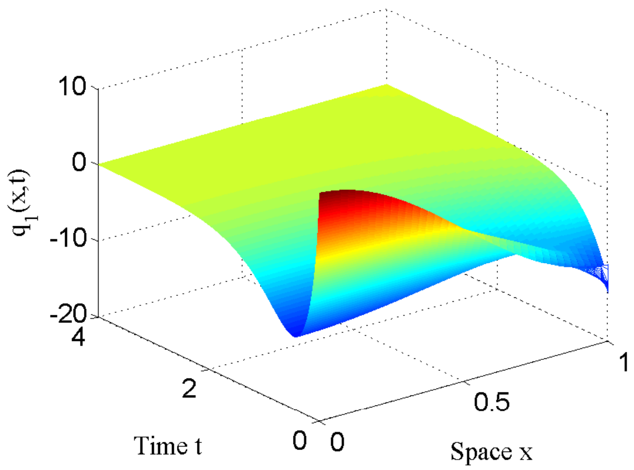

Figure 2.11: Spatiotemporal evolution of $q_1(x,t)$ in the closed loop.

The boundary controller (2.171) has been implemented by selecting the next matrix

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{2.179}$$

which gives the target system desired exponential stability properties. Figures 2.11 and 2.12 show the stable spatiotemporal evolutions of the state variables $q_1(x,t)$ and $q_2(t)$, which both vanishes in L_2 norm as shown in the Figure 2.13. The initial and long-term evolutions of the boundary control inputs $u_1(t)$ and $u_2(t)$ are displayed in the Figure 2.14.

2.2.6 Conclusions

The backstepping based boundary stabilization of a system of n coupled parabolic linear PDEs has been tackled, and an explicit state feedback controller has been derived which allows to enforce an arbitrarily fast exponential decay of the state in the space $[L_2(0,1)]^n$. The extension to the case of different diffusivity parameters, the observer-based output-fedback design, and considering spatially-dependent parameters, are among the most interesting lines of future related investigations. Additionally, integration with other design methodologies such as the (second-order) sliding mode approach, will be pursued as well to enhance the underlying robustness features.

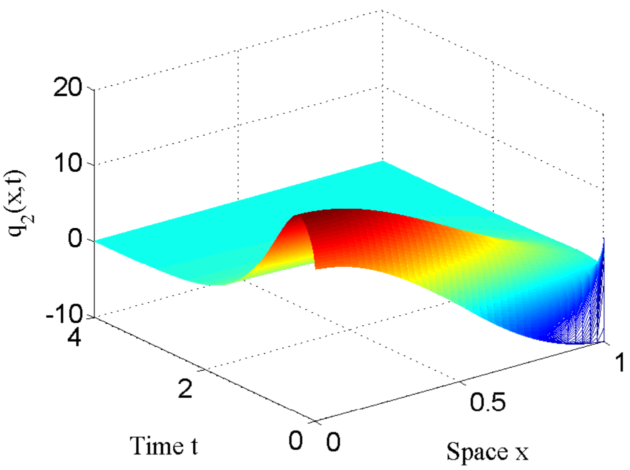

Figure 2.12: Spatiotemporal evolution of $q_2(x,t)$ in the closed loop.

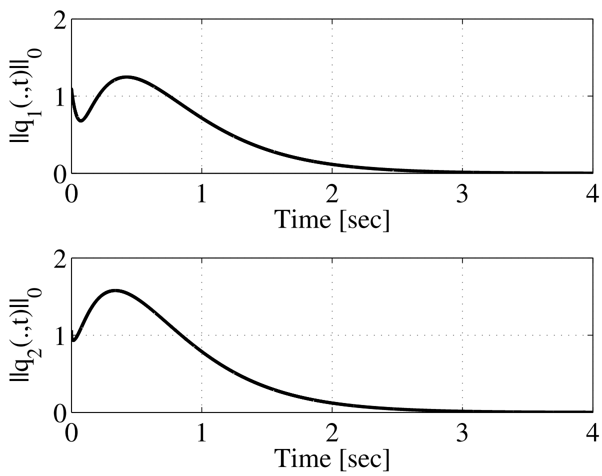

Figure 2.13: L_2 norms $\|q_1(\cdot,t)\|_0$ and $\|q_2(\cdot,t)\|_0$ in the closed loop test.

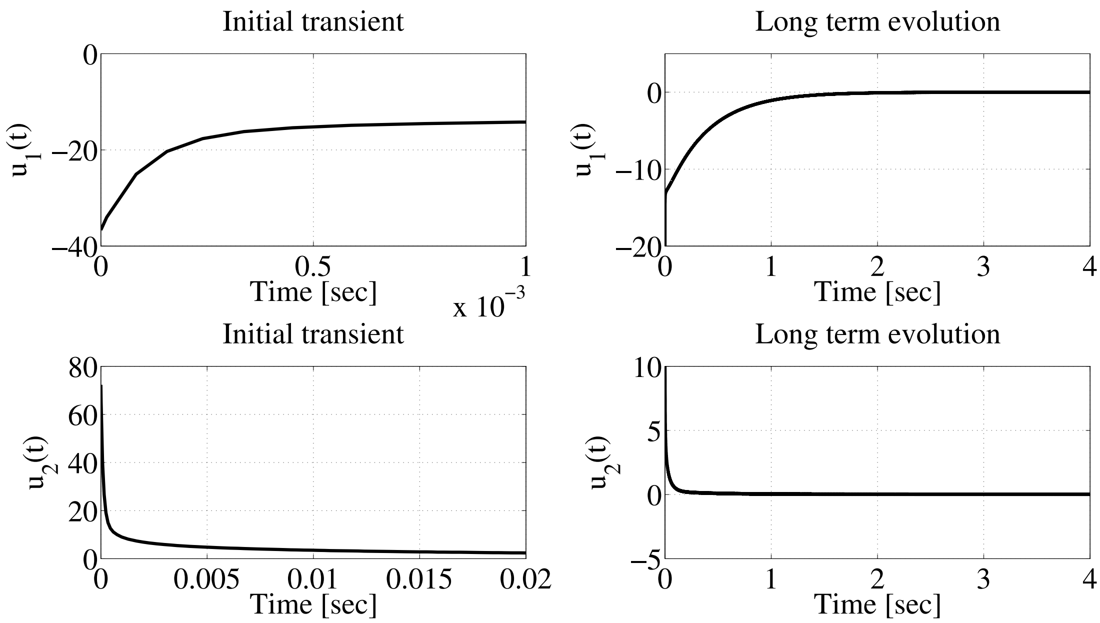

Figure 2.14: Time evolution of the boundary control inputs $u_1(t)$ and $u_2(t)$ 'in the closed loop test.

2.3 Backstepping observer design for a class of coupled reaction-diffusion PDEs

The state observation problem is tackled for a system of n coupled reaction-diffusion PDEs, possessing the same diffusivity parameter and equipped with boundary sensing devices. Particularly, a backstepping-based observer is designed and the exponential stability of the error system is proved with an arbitrarily fast convergence rate. The transformation kernel matrix is derived in the explicit form by using the method of successive approximations, thereby yielding the observer gains in the explicit form, too. Simulation results support the effectiveness of the suggested design.

2.3.1 Introduction

Model-based control and advanced process monitoring of Distributed-Parameter Systems (DPSs), governed by Partial Differential Equations (PDEs), tipically require full state information. However, the available measurements of DPS' are typically located on the boundary of the spatial domain, that motivates the need of the state observer [46, 70].

For linear infinite dimensional systems the Luenberger observer theory was established by replacing matrices with linear operators [69, 72, 70], and the observer design was confined to determining a gain operator that stabilizes the associated error dynamics. In contrast to finite dimensional systems, finding such a gain operator is not trivial even numerically because operators were not generally represented with a finite number of parameters.

Design methods, which are not relying on any discretization or finite-dimensional approximation (thereby preserving the infinite-dimensional representation of the system during the entire design process) and which are yielding the observer gains in the explicit form have only recently been investigated. In this context, the backstepping method appears to be a particularly effective systematic design approach which can be applied for a broad class of systems governed by PDEs [13, 43]. Basically,the backstepping approach relies on the application of an invertible Volterra integral transformation mapping a predefined exponentially stable target system into the observer error dynamics.

For systems governed by parabolic PDEs defined on a 1-dimensional (1D) spatial domain, a systematic observer design approach using boundary sensing is introduced in [43]. Recently, the backstepping-based observer design was presented in [44] for reaction diffusion processes with spatially varying reaction coefficient and a certain weighted average of the state over the spatial domain as measured output. In [71, 11], backstepping-based observer design was addressed for reaction-diffusion processes evolving in multidimensional spatial domains. In [40], the backstepping based design for parabolic processes was applied by adopting a nonconventional target system for the error dynamics, embedding certain discontinuous output injection terms.

More recently, high-dimensional systems of *coupled* PDEs were considered in the backstepping-based boundary control and observer design settings. The most intensive efforts of the current literature seem however to be oriented towards coupled hyperbolic processes of the transport-type [37, 74, 29, 5, 41]. In [5], a 2×2 linear hyperbolic system was stabilized by a single observer-based boundary control input, with an additional feature that an unmatched disturbance, generated by an *a-priori* known exosystem, was rejected. Both the controller and the observer were designed by following the backstepping approach. In [74] a state estimator in a semi-infinite 3-dimensional (3D) domain is presented for a coupled model of magnetohydrodynamic flow, and Fourier transform methods were applied to put the system in a form, where the 1D backstepping method is applicable. In [41], a backstepping-based observer was designed for a system of two diffusion-convection-reaction processes coupled through the corresponding boundary conditions. In [29], a 2×2 system of coupled linear heterodirectional hyperbolic systems was stabilized by a backstepping-based observer-controller under some boundedness restriction on the spatially dependent coupling coefficients. In [37], observer-controller design was studied for a system of $n + 1$ coupled first-order heterodirectional hyperbolic linear PDEs (n of which featured rightward convecting transport, and one leftward) with a single boundary input. Some specific results concerning the backstepping based output feedback boundary stabilization of parabolic coupled PDEs have been presented in the literature. In [31] the controller/observer design for the linearized 2×2 model of thermal-fluid convection has been treated.

In this work, the observer design is developed for a class of n coupled diffusion-reaction PDEs in the 1D spatial domain $x \in [0, 1]$. The task of the present paper is to generalize some results presented in [43], where explicit backstepping observers were developed for a scalar unstable reaction diffusion equation. Here a generalization is made for a set of n reaction diffusion processes, which are coupled through the corresponding reaction terms. The motivation to this investigation comes from chemical processes [20] where coupled temperature-concentration parabolic PDEs were involved to describe system dynamics. This generalization is shown to be far from being trivial because the underlying backstepping-based treatment gives rise to more complex development of finding out an explicit form of the observer gains in the form of matrix Bessel series, and, furthermore, it turns out to be unfeasible in the general case where each process possesses its own diffusivity parameter. In this work we therefore address the simplified case where all processes possess the same diffusivity value, and we postpone the more general case for further investigations (see Remark 1). The present paper can be considered as the observer design counterpart of our recent work [7], where the stabilizing boundary controller design problem was addressed for a similar class of systems differing only in the boundary conditions from that considered in the present work. Subsequently, in [35], the stabilizing boundary control design problem in the general case of different diffusivity parameters was addressed and solved. In [73], the observer design for a class of coupled reaction diffusion equations equipped with different boundary conditions than those considered in the present work was tackled in the framework of an output-feedback controller design problem. In spite of the similar setting, the present scenario of Neumann BC in the uncontrolled side $x = 0$ of the spatial domain yields a significantly different solution as compared to that obtained in [73] for the case of Dirichlet BC.

Particularly, in the present context two output injections are needed in the observer dynamics (one distributed along the spatial domain, and another one located at the uncontrolled boundary) whereas in [73] the latter was not applied.

The structure of the paper is as follows. After introducing some useful notation in the next subsection, Section II states the problem to be investigated and introduces the proposed observer structure with the underlying backstepping transformation and (matrix) kernel PDE. In Section III, the explicit solution of the kernel PDE is derived. In Section IV, the proposed observer design is summarized and the main result of this paper is presented. Section V discusses supporting simulation results, and Section VI collects some concluding remarks and future perspectives of this research.

Notation

The notation used throughout is fairly standard. $L_2(0,1)$ stands for the Hilbert space of square integrable scalar functions $z(\zeta)$ on $(0,1)$ with the corresponding norm

$$\|z(\cdot)\|_2 = \sqrt{\int_0^1 z^2(\zeta)d\zeta}. \tag{2.180}$$

Throughout the paper the notation

$$[L_2(0,1)]^n = \underbrace{L_2(0,1) \times L_2(0,1) \times \ldots \times L_2(0,1)}_{n \texttt{ times}} \tag{2.181}$$

is also utilized and

$$\|Z(\cdot)\|_{2,n} = \sqrt{\sum_{i=1}^{n} \|z_i(\cdot)\|_2^2} \tag{2.182}$$

stands for the corresponding norm of a generic vector function $Z(\zeta) = [z_1(\zeta), z_2(\zeta), ..., z_n(\zeta)] \in [L_2(0,1)]^n$.

With reference to a generic real-valued symmetric matrix W of dimension n, $\sigma_1(W)$ denotes the smallest eigenvalue of W. Finally, $I_{n\times n}$ stands for the identity matrix of dimension n.

2.3.2 Problem formulation and backstepping transformation

The following n-dimensional system of coupled reaction-diffusion processes, equipped with Neumann-type boundary conditions and governed by the boundary-value problem

$$Q_t(x,t) = \theta Q_{xx}(x,t) + \Lambda Q(x,t) \tag{2.183}$$
$$Q_x(0,t) = 0 \tag{2.184}$$
$$Q(1,t) = U(t) \tag{2.185}$$

is under study. Hereinafter,

$$Q(x,t) = [q_1(x,t),\ q_2(x,t), \ldots, q_n(x,t)]^T \in [L_2(0,1)]^n \tag{2.186}$$

is the vector collecting the state of all systems,

$$U(t) = [u_1(t),\ u_2(t), \dots, u_n(t)]^T \in \Re^n \tag{2.187}$$

is the boundary input vector, $\Lambda = \{\lambda_{ij}\} \in \Re^{n\times n}$ is a real-valued square matrix, and $\theta \in \Re^+$ is a positive scalar. The open-loop system (2.183)-(2.185) (with $U(t) = 0$) may possess arbitrarily many unstable eigenvalues when the symmetric part $(\Lambda + \Lambda^T)/2$ of matrix Λ possesses sufficiently large positive eigenvalues. For system (2.183)-(2.185) of n coupled reaction-diffusion processes, the following observer

$$\hat{Q}_t(x,t) = \theta\hat{Q}_{xx}(x,t) + \Lambda\hat{Q}(x,t) + G(x)\left[Q(0,t) - \hat{Q}(0,t)\right] \tag{2.188}$$

$$\hat{Q}_x(0,t) = M\left[Q(0,t) - \hat{Q}(0,t)\right] \tag{2.189}$$

$$\hat{Q}(1,t) = U(t) \tag{2.190}$$

is proposed with $G(x)$ being a n-th order square matrix of observer gain functions, and $M \in \Re^{n,n}$ being a square matrix of constant observer gains. The error variable

$$\tilde{Q}(x,t) = Q(x,t) - \hat{Q}(x,t) \tag{2.191}$$

is then governed by the error system

$$\tilde{Q}_t(x,t) = \theta\tilde{Q}_{xx}(x,t) + \Lambda\tilde{Q}(x,t) - G(x)\tilde{Q}(0,t) \tag{2.192}$$

$$\tilde{Q}_x(0,t) = -M\tilde{Q}(0,t) \tag{2.193}$$

$$\tilde{Q}(1,t) = 0 \tag{2.194}$$

To design the observer gains $G(x)$ and M, the backstepping approach is involved to find out an invertible transformation

$$\tilde{Q}(x,t) = \tilde{Z}(x,t) - \int_0^x P(x,y)\tilde{Z}(y,t)dy \tag{2.195}$$

where $P(x,y)$ is a $n \times n$ matrix kernel function whose elements are denoted as $p_{ij}(x,y)$, $i,j = 1,2,\dots,n$, which maps the error system (2.192)-(2.194) into the exponentially stable[1] target error dynamics

$$\tilde{Z}_t(x,t) = \theta\tilde{Z}_{xx}(x,t) - \bar{C}\tilde{Z}(x,t) \tag{2.196}$$

$$\tilde{Z}_x(0,t) = 0 \tag{2.197}$$

$$\tilde{Z}(1,t) = 0 \tag{2.198}$$

The following lemma is in order

[1]The exponential stability properties of the target error system (2.196)-(2.198) will be investigated in Theorem 20.

Lemma 7. *The error system* (2.192)-(2.194) *is transferred by* (2.195) *into the target error dynamics* (2.196)-(2.198) *provided that the design terms M and $G(x)$ are selected as*

$$G(x) = \theta P_y(x,0) \tag{2.199}$$
$$M = P(0,0) \tag{2.200}$$

where $P(x,y)$ is a solution to the kernel PDE

$$\begin{aligned} P_{xx}(x,y) - P_{yy}(x,y) &= -\frac{1}{\theta}\left[P(x,y)\bar{C} + \Lambda P(x,y)\right] && (2.201)\\ P(x,x) &= \frac{\Lambda + \bar{C}}{2\theta}(x-1) && (2.202)\\ P(1,y) &= 0 && (2.203)\end{aligned}$$

Proof. Employing the Leibnitz differentiation rule, the spatial differentiation of (2.195) results in

$$\begin{aligned} \tilde{Q}_x(x,t) &= \tilde{Z}_x(x,t) - P(x,x)\tilde{Z}(x,t)\\ &- \int_0^x P_x(x,y)\tilde{Z}(y,t)dy, && (2.204)\\ \tilde{Q}_{xx}(x,t) &= \tilde{Z}_{xx}(x,t) - \left[\frac{d}{dx}P(x,x)\right]\tilde{Z}(x,t) - P(x,x)\tilde{Z}_x(x,t) - P_x(x,x)\tilde{Z}(x,t)\\ &- \int_0^x P_{xx}(x,y)\tilde{Z}(y,t)dy. && (2.205)\end{aligned}$$

In turn, the temporal differentiation of (2.195), and recursive integration by parts, yields

$$\begin{aligned} \tilde{Q}_t(x,t) &= \tilde{Z}_t(x,t) - \int_0^x P(x,y)\tilde{Z}_t(y,t)dy\\ &= \tilde{Z}_t(x,t) - P(x,x)\theta\tilde{Z}_x(x,t)\\ &+ \theta P(x,0)\tilde{Z}_x(0,t) + \theta P_y(x,x)\tilde{Z}(x,t)\\ &- \theta P_y(x,0)\tilde{Z}(0,t) - \theta\int_0^x P_{yy}(x,y)\tilde{Z}(y,t)dy\\ &+ \int_0^x P(x,y)\bar{C}\tilde{Z}(y,t)dy. && (2.206)\end{aligned}$$

By evaluating (2.195) at $x = 0$ and $x = 1$, and considering (2.198), one derives that

$$\begin{aligned} \tilde{Q}(0,t) &= \tilde{Z}(0,t), && (2.207)\\ \tilde{Q}(1,t) &= -\int_0^1 P(1,y)\tilde{Z}(y,t)dy. && (2.208)\end{aligned}$$

Substituting (2.195), (2.197) and (2.205)-(2.207) into (2.192), and performing lengthy but straightforward computations, yields

$$
\begin{aligned}
&\tilde{Z}_t(x,t) - \theta \tilde{Z}_{xx}(x,t) + \overline{C}\tilde{Z}(x,t) = -\left\{\theta\left[\frac{d}{dx}P(x,x)\right]\right. \\
&\quad \left. + \theta P_y(x,x) + \theta P_x(x,x) - \Lambda - \overline{C}\right\}\tilde{Z}(x,t) \\
+ &\quad [P(x,x)\theta - \theta P(x,x)]\,\tilde{Z}_x(x,t) \\
+ &\quad [\theta P_y(x,0) - G(x)]\,\tilde{Z}(0,t) \\
+ &\quad \int_0^x [\theta P_{yy}(x,y) - \theta P_{xx}(x,y) \\
&\quad -P(x,y)\bar{C} - \Lambda P(x,y)\big]\,\tilde{Z}(y,t)dy. \qquad (2.209)
\end{aligned}
$$

By evaluating (2.204) at $x = 0$, and considering (2.197), it follows that

$$\tilde{Q}_x(0,t) = -P(0,0)\tilde{Z}(0,t). \qquad (2.210)$$

Substituting (2.210) and (2.207)-(2.208) into (2.193) and (2.194) one derives the conditions

$$
\begin{aligned}
[M - P(0,0)]\tilde{Z}(0,t) &= 0, &\qquad (2.211)\\
\int_0^1 P(1,y)\tilde{Z}(y,t)dy &= 0. &\qquad (2.212)
\end{aligned}
$$

Clearly, to obtain the target error PDE (2.196) the right hand side of (2.209) should be identically zero. To meet this requirement, it suffices to employ relations (2.211)-(2.212), and exploit the identity $\frac{d}{dx}P(x,x) = P_x(x,x) + P_y(x,x)$, thereby obtaining both the kernel boundary value problem

$$
\begin{aligned}
\theta\left(P_{xx}(x,y) - P_{yy}(x,y)\right) &= -P(x,y)\bar{C} - \Lambda P(x,y) \\
& &\qquad (2.213)\\
2\theta\frac{d}{dx}P(x,x) &= \Lambda + \bar{C} &\qquad (2.214)\\
P(1,y) &= 0 &\qquad (2.215)
\end{aligned}
$$

and the observer gain design conditions in the form of (2.199)-(2.200). Integrating (2.214) with respect to x and considering (2.200) results in

$$
\begin{aligned}
P(x,x) &= \frac{1}{2\theta}\left(\Lambda + \bar{C}\right)x + P(0,0) \\
&= \frac{1}{2\theta}\left(\Lambda + \bar{C}\right)x + M \qquad (2.216)
\end{aligned}
$$

Evaluating (2.216) at $x = 1$ yields

$$P(1,1) = \frac{1}{2\theta}\left(\Lambda + \bar{C}\right) + M \qquad (2.217)$$

By evaluating (2.215) at $y = 1$ it is concluded that $P(1,1) = 0$, thus getting from (2.217) that

$$M = -\frac{1}{2\theta}\left(\Lambda + \bar{C}\right) \tag{2.218}$$

Considering (2.216) and (2.218), one finally rewrites (2.213)-(2.215) in the form of (2.201)-(2.203). Lemma 7 is proven. □

Remark 12. *The present paper is confined to the case in which all the coupled PDEs* (2.183) *possess the same diffusivity parameter θ. The reason behind is that in the more general case where each process has its own diffusivity θ_i, ($i = 1, 2, ..., n$), the corresponding "generalized" version*

$$\begin{aligned}
\Theta\left(P_{xx}(x,y) - P_{yy}(x,y)\right) &= -P(x,y)\bar{C} \\
&- \Lambda P(x,y) && (2.219)\\
\Theta\frac{d}{dx}P(x,x) + \Theta P_x(x,x) + \Theta P_y(x,x) &= \Lambda + \bar{C} && (2.220)\\
P(x,x)\Theta &= \Theta P(x,x) && (2.221)\\
P(1,y) &= 0 && (2.222)
\end{aligned}$$

of (2.201)-(2.203), *where $\Theta = diag(\theta_i)$, sets an overdetermined boundary value problem that has no solution, unless specific constraints are imposed on the matrix $\bar{C}$ and on the form of the kernel matrix $P(x,y)$. This topic calls for further investigation and will be published elsewhere.*

2.3.3 Solving the kernel PDE (2.201)-(2.203)

For later use, the following result is reproduced

Theorem 19. *Problem* (2.201)-(2.203) *possesses a solution*

$$\begin{aligned}
P(x,y) &= -\sum_{n=0}^{\infty}\frac{2(1-x)((1-y)^2-(1-x)^2)^n}{n!(n+1)!} \\
&\times \left(\frac{1}{4\theta}\right)^{n+1}\left[\sum_{i=0}^{n}\binom{n}{i}\mathit{\Lambda}^i(\mathit{\Lambda}+\bar{C})\bar{C}^{n-i}\right]
\end{aligned} \tag{2.223}$$

which is of class C^∞ in the domain $0 \le y \le x \le 1$.

Proof. By the invertible change of variables

$$\bar{x} = 1 - y \quad \bar{y} = 1 - x$$

one transforms (2.201)-(2.203) into

$$\begin{aligned}
\bar{P}_{\bar{x}\bar{x}}(\bar{x},\bar{y}) - \bar{P}_{\bar{y}\bar{y}}(\bar{x},\bar{y}) &= \frac{1}{\theta}\left[\bar{P}(\bar{x},\bar{y})\bar{C} + \Lambda\bar{P}(\bar{x},\bar{y})\right] && (2.224)\\
\bar{P}(\bar{x},\bar{x}) &= -\frac{\Lambda+\bar{C}}{2\theta}\bar{x} && (2.225)\\
\bar{P}(\bar{x},0) &= 0 && (2.226)
\end{aligned}$$

Following [67], the existence of a solution to problem (2.224)-(2.226) can be shown by transforming it into an integral equation using the change of the variables

$$\xi = x + y, \qquad \eta = x - y. \tag{2.227}$$

Setting

$$H(\xi,\eta) = \bar{P}(x,y) = \bar{P}\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) \tag{2.228}$$

the relations

$$\begin{aligned}
\bar{P}_x &= H_\xi + H_\eta, \quad \bar{P}_{xx} = H_{\xi\xi} + 2H_{\xi\eta} + H_{\eta\eta} && (2.229)\\
\bar{P}_y &= H_\xi - H_\eta, \quad \bar{P}_{yy} = H_{\xi\xi} - 2H_{\xi\eta} + H_{\eta\eta} && (2.230)
\end{aligned}$$

are obtained, and the matrix kernel boundary-value problem (2.224)-(2.226), written in the new coordinates, takes the form

$$\begin{aligned}
H_{\xi\eta}(\xi,\eta) &= \frac{1}{4\theta}H(\xi,\eta)\bar{C} + \frac{1}{4\theta}\Lambda H(\xi,\eta) && (2.231)\\
H(\xi,0) &= -\frac{1}{4\theta}\left(\Lambda+\bar{C}\right)\xi && (2.232)\\
H(\xi,\xi) &= 0. && (2.233)
\end{aligned}$$

Integrating (2.231) with respect to η from 0 to η, and considering the relation $H_\xi(\xi,0) = -\frac{1}{4\theta}(\mathit{\Lambda}+\bar{C})$, which follows from (2.232), one obtains

$$H_\xi(\xi,\eta) = -\frac{1}{4\theta}(\mathit{\Lambda}+\bar{C}) + \frac{1}{4\theta}\int_0^\eta \left[H(\xi,s)\bar{C} + \mathit{\Lambda}H(\xi,s)\right] ds. \tag{2.234}$$

Integrating (2.234) with respect to ξ from η to ξ yields

$$\begin{aligned}
\int_\eta^\xi H_\tau(\tau,\eta)d\tau = \int_\eta^\xi -\frac{1}{4\theta}(\mathit{\Lambda}+\bar{C})d\tau\\
+\frac{1}{4\theta}\int_\eta^\xi \left\{\int_0^\eta \left[H(\tau,s)\bar{C} + \mathit{\Lambda}H(\tau,s)\right] ds\right\} d\tau && (2.235)
\end{aligned}$$

which can further be manipulated to

$$\begin{aligned}
H(\xi,\eta) - H(\eta,\eta) = -\frac{1}{4\theta}(\mathit{\Lambda}+\bar{C})(\xi-\eta)\\
+\frac{1}{4\theta}\int_\eta^\xi \left\{\int_0^\eta \left[H(\tau,s)\bar{C} + \mathit{\Lambda}H(\tau,s)\right] ds\right\} d\tau. && (2.236)
\end{aligned}$$

An explicit form of $H(\eta, \eta)$ is subsequently derived. For this purpose, (2.233) is used to obtain

$$H(\eta, \eta) = 0. \tag{2.237}$$

By substituting (2.237) into (2.236) one derives an integral equation for $H(\xi, \eta)$:

$$H(\xi, \eta) = -\frac{1}{4\theta}(\mathit{\Lambda} + \bar{C})(\xi - \eta) + \frac{1}{4\theta}\int_{\eta}^{\xi}\left\{\int_{0}^{\eta}\left[H(\tau, s)\bar{C} + \mathit{\Lambda} H(\tau, s)\right] ds\right\} d\tau. \tag{2.238}$$

The method of successive approximations is then applied to show that equation (2.238) has a smooth solution. Let us start with an initial approximation

$$H^0(\xi, \eta) \quad = \quad 0 \tag{2.239}$$

and set-up the recursive formula for (2.238) as follows

$$\begin{aligned} H^{n+1}(\xi, \eta) &= -\frac{1}{4\theta}(\mathit{\Lambda} + \bar{C})(\xi - \eta) \\ &+\frac{1}{4\theta}\int_{\eta}^{\xi}\left\{\int_{0}^{\eta}\left[H^n(\tau, s)\bar{C} + \mathit{\Lambda} H^n(\tau, s)\right] ds\right\} d\tau. \end{aligned} \tag{2.240}$$

Provided that this recursion converges, the solution $H(\xi, \eta)$ can be represented as

$$H(\xi, \eta) = \lim_{n\to\infty} H^n(\xi, \eta). \tag{2.241}$$

Let

$$\Delta H^n(\xi, \eta) \quad = \quad H^{n+1}(\xi, \eta) - H^n(\xi, \eta) \tag{2.242}$$

stand for the difference between two consecutive terms. Then, the recursion

$$\Delta H^0(\xi, \eta) \quad = \quad H^1(\xi, \eta) = -\frac{1}{4\theta}(\mathit{\Lambda} + \bar{C})(\xi - \eta) \tag{2.243}$$

$$\Delta H^{n+1}(\xi, \eta) \quad = \quad \frac{1}{4\theta}\int_{\eta}^{\xi}\left\{\int_{0}^{\eta}\left[\Delta H^n(\tau, s)\bar{C} + \mathit{\Lambda}\Delta H^n(\tau, s)\right] ds\right\} d\tau \tag{2.244}$$

is correspondingly concluded from (2.239)-(2.240), and (2.241) is alternatively represented as

$$H(\xi, \eta) = \sum_{n=0}^{\infty} \Delta H^n(\xi, \eta). \tag{2.245}$$

Since variables ξ and η lie in the bounded domain $0 \le \eta \le \xi \le 2$, one can apply (2.243) to show that

$$\|\Delta H^0(\xi, \eta)\| \le \frac{1}{\theta}(\|\mathit{\Lambda}\| + \|\bar{C}\|) = N. \tag{2.246}$$

In order to apply the mathematical induction method suppose that

$$\|\Delta H^n(\xi,\eta)\| \leq N^{n+1}\frac{(\xi+\eta)^n}{n!}. \tag{2.247}$$

Then, by employing (2.244), (2.246) and (2.247) one arrives at

$$\begin{aligned}\|\Delta H^{n+1}(\xi,\eta)\| &\leq \frac{1}{4\theta}(\|\Lambda\|+\|\bar{C}\|)\frac{N^{n+1}}{n!} \\ &\times \left|2\int_0^\eta\int_0^\tau(\tau+s)^n ds d\tau + \int_\eta^\xi\int_0^\eta(\tau+s)^n ds d\tau\right| \\ &= \frac{N^{n+2}}{4n!}\left|2\int_0^\eta\int_0^\tau(\tau+s)^n ds d\tau + \int_\eta^\xi\int_0^\eta(\tau+s)^n ds d\tau\right|. \end{aligned} \tag{2.248}$$

It is readily shown (cf. [10], eq. (2.14)) that the next estimate

$$\left|2\int_0^\eta\int_0^\tau(\tau+s)^n ds d\tau + \int_\eta^\xi\int_0^\eta(\tau+s)^n ds d\tau\right| \leq 4\frac{(\xi+\eta)^{n+1}}{(n+1)} \tag{2.249}$$

holds. Therefore, combining (2.248) and (2.249) one gets

$$\|\Delta H^{n+1}(\xi,\eta)\| \leq N^{n+2}\frac{(\xi+\eta)^{n+1}}{(n+1)!}. \tag{2.250}$$

Thus, by mathematical induction, (2.250) holds for all $n \geq 0$. It then follows from the Weierstrass M-test that the series (2.245) converges absolutely and uniformly in $0 \leq \eta \leq \xi \leq 2$. By (2.243)-(2.244), it follows that

$$\Delta H^1(\xi,\eta) = -\frac{\xi^2\eta+\xi\eta^2}{2}\left(\frac{1}{4\theta}\right)^2\left[(\Lambda+\bar{C})\Lambda+\bar{C}(\Lambda+\bar{C})\right] \tag{2.251}$$

Iterating on the computations, one observes the pattern which leads to the following formula

$$\Delta H^n(\xi,\eta) = -\frac{(\xi\eta)^n(\xi-\eta)}{n!(n+1)!}\left(\frac{1}{4\theta}\right)^{n+1}\left[\sum_{i=0}^{n}\binom{n}{i}\Lambda^i\left(\Lambda+\bar{C}\right)\bar{C}^{n-i}\right]. \tag{2.252}$$

The solution to the integral equation (2.238) is therefore given by the next series expansion

$$H(\xi,\eta) = -\sum_{n=0}^{\infty}\frac{(\xi\eta)^n(\xi-\eta)}{n!(n+1)!}\left(\frac{1}{4\theta}\right)^{n+1}\left[\sum_{i=0}^{n}\binom{n}{i}\Lambda^i\left(\Lambda+\bar{C}\right)\bar{C}^{n-i}\right] \tag{2.253}$$

which is absolutely and uniformly converging.

Converting (2.253) into the original x, y variables, one obtains the series expansion (2.223) for the Kernel matrix $P(x,y)$ which solves the kernel boundary-value problem (2.224)-(2.226). Straightforward inspection reveals that (2.223) is infinitely times continuously differentiable. Returning back to the original (x,y) variables, one obtains (2.223). Theorem 19 is thus proven. □

Inverse transformation

Transformation (2.195) is a matrix Volterra integral equation of the second type. Since $P(x, y)$ is continuous by Theorem 19, there exists a continuous inverse kernel $L(x, y)$ (see, e.g., [37, 45] for the scalar case which is straightforwardly extended to the present vector case) such that

$$\tilde{Q}(x,t) = \tilde{Z}(x,t) + \int_0^x L(x,y)\tilde{Z}(y,t)dy \tag{2.254}$$

implicitly defined on $T = \{(x, y) \in R^2 : 0 \leq y \leq x \leq 1\}$ by

$$L(x,y) = P(x,y) + \int_y^x L(x,s)P(s,y)ds \tag{2.255}$$

Relation (2.255) can in fact be easily derived by substituting (2.195) into (2.254) and performing straightforward manipulations of the resulting integral equation. The method of successive approximations can be then applied to show that (2.255) gives rise to a unique $R(x, y)$, which has as much regularity as $P(x, y)$ has. Detailed computations, which follow similar steps as those carried out in the proof of Theorem 19, are skipped for brevity.

2.3.4 Main result

Taking advantage of the explicit solution (2.223) to the kernel boundary-value problem (2.201)-(2.203), the explicit representation

$$M = -\frac{\Lambda + \bar{C}}{2\theta} \tag{2.256}$$

$$G(x) = \theta \sum_{n=0}^{\infty} \frac{4n(1-x)(2x-x^2)^{n-1}}{n!(n+1)!} \left(\frac{1}{4\theta}\right)^{n+1} \left[\sum_{i=0}^{n} \binom{n}{i} \Lambda^i(\Lambda + \bar{C})\bar{C}^{n-i}\right] \tag{2.257}$$

of the observer gains is straightforwardly derived by specifying (2.199)-(2.200) accordingly.

The stability features of the target error dynamics (2.196)-(2.198) are going to be studied. The following result is in force.

Theorem 20. *If the design matrix $\bar{C}$ is selected such that its symmetric part $\bar{C}_s = (\bar{C} + \bar{C}^T)/2$ is positive definite then system* (2.196)-(2.198) *is exponentially stable in the space $[L_2(0, 1)]^n$ with the convergence rate specified by*

$$\|\tilde{Z}(\cdot,t)\|_{2,n} \leq \|\tilde{Z}(\cdot,0)\|_{2,n} e^{-\sigma_1(\bar{C}_s)t} \tag{2.258}$$

Proof. Consider the Lyapunov function $V(t) = \frac{1}{2}\int_0^1 \tilde{Z}^T(\xi,t)\tilde{Z}(\xi,t)d\xi = \frac{1}{2}\|\tilde{Z}(\cdot,t)\|_{2,n}^2$. The corresponding time derivative along the solutions of (2.196)-(2.198) is given by

$$\dot{V}(t) = \int_0^1 \tilde{Z}^T(\xi,t)\Theta\tilde{Z}_{xx}(\xi,t)d\xi - \int_0^1 \tilde{Z}^T(\xi,t)\bar{C}\tilde{Z}(\xi,t)d\xi \tag{2.259}$$

Integrating by parts taking into account the BCs (2.197) and (2.198), and exploiting the diagonal form of matrix Θ, yield

$$\int_0^1 \tilde{Z}^T(\xi,t)\Theta\tilde{Z}_{xx}(\xi,t)d\xi = \tilde{Z}^T(\chi,t)\Theta\tilde{Z}_x(\chi,t)\Big|_{\chi=0}^{\chi=1}$$
$$-\int_0^1 \tilde{Z}_x^T(\xi,t)\Theta\tilde{Z}_x(\xi,t)d\xi \le -\theta_m\|\tilde{Z}_x(\cdot,t)\|_{2,n}^2 \tag{2.260}$$

where $\theta_m = \min_{1\le i\le n}\theta_i > 0$. Since $\sigma_1(\bar{C}_s)$ is assumed to be positive then exploiting the trivial inequality $\tilde{Z}^T(\xi,t)\bar{C}\tilde{Z}(\xi,t) \ge \sigma_1(\bar{C}_s)^T\tilde{Z}(\xi,t)\tilde{Z}(\xi,t)$ and employing (2.260), one manipulates (2.259) to derive

$$\begin{aligned}\dot{V}(t) &\le -\theta_m\|\tilde{Z}_\xi(\cdot,t)\|_{2,n}^2 - 2\sigma_1(\bar{C}_s)V(t) \\ &\le -2\sigma_1(\bar{C}_s)V(t)\end{aligned} \tag{2.261}$$

thereby concluding the exponential stability of the target error dynamics in the space $[L_2(0,1)]^n$ with a convergence rate obeying the estimate (2.258). Theorem 20 is proven. □

The next Theorem specifies the proposed observer design and summarizes the main result of this paper.

Theorem 21. *The observer* (2.188)-(2.190)*, with gains M and $G(x)$ set as in* (2.256)-(2.257) *and with matrix $\bar{C}$ being selected such that its symmetric part $\bar{C}_s = (\bar{C}+\bar{C}^T)/2$ is positive definite, reconstructs the state of system (2.183)-(2.185) with an arbitrarily fast convergence rate in accordance with*

$$\|\tilde{Q}(\cdot,t)\|_{2,n} \le A\|\tilde{Q}(\cdot,0)\|_{2,n}e^{-\sigma_1(C_s)t} \tag{2.262}$$

where A is a positive constant independent of $\tilde{Q}(\xi,0)$.

Proof. In Lemma 7 and Theorem 19, it was shown that the error system (2.192)-(2.194) is transferred, by means of (2.195), into the target error dynamics (2.196)-(2.198) provided that the gains M and $G(x)$ are selected as in (2.199)-(2.200) where the solution $P(x,y)$ to the kernel PDE (2.201)-(2.203) is given by (2.223). Specifying (2.199)-(2.200) in light of the actual form of the solution (2.223) it straightforwardly results in (2.256) and (2.257), where $P(0,0)$ is derived by specifying (2.202) at $x=0$ and $P_y(x,0)$ is readily obtained by differentiating (2.223) with respect to y at $y=0$.

The asymptotic stability features of (2.196)-(2.198), subject to the design requirement that the arbitrary design parameter $\bar{C}_s = (\bar{C}+\bar{C}^T)/2$ is positive definite, were demonstrated in Theorem 20. In particular, according to (2.258), the corresponding convergence rate can be made arbitrarily fast by a proper selection of the $\bar{C}$ matrix.

From now on, we follow [67] to derive analogous convergence properties for the original system (2.183)-(2.185) as well. Observing that $\xi+\eta = x$, one derives from (2.245)-(2.247) that $\|P(x,y)\| \le Ne^{2Nx}$, and the same bound can be derived for the norm of the inverse transformation kernel matrix $L(x,y)$ as well, i.e. $\|L(x,y)\| \leq Ne^{2Nx}$. A straightforward generalization of [67, Th 4] yields that those two boundedness relations, coupled together, establish the equivalence of norms of $\tilde{Z}(x,t)$ and $\tilde{Q}(x,t)$ in $[L_2(0,1)]^n$ which means that there exist a positive constant A independent of $\tilde{Q}(\xi,0)$ such that the estimate (2.262) is in force as a direct consequence of (2.258). Theorem 21 is proven. $\square$

2.3.5 Simulation results

Academic example

To validate the proposed observer, system (2.183)-(2.185) of coupled reaction-diffusion processes is specified for simulation purposes with $n = 3$ and with parameters

$$\theta = 2, \qquad \Lambda = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 5 & 1 \end{bmatrix} \tag{2.263}$$

The initial conditions are set to $q_1(x,0) = q_2(x,0) = q_3(x,0) = 2sin(\pi x) + 2sin(3\pi x)$. For solving the underlying PDEs, a standard finite-difference approximation method is used by discretizing the spatial solution domain $x \in [0,1]$ into a finite number of N uniformly spaced solution nodes $x_i = ih$, $h = 1/(N+1)$, $i = 1,2,...,N$. The value $N = 40$ is then used. The resulting 40-th order discretized system is subsequently solved by fixed-step Runge-Kutta ODE4 method with step $T_s = 10^{-4}$.

The unstable behaviour of the plant subject to the open-loop input vector $U(t) = [5sint, 10sin2t, 15sin3t]^T$ is displayed in the Figure 2.15, which for certainty shows the diverging spatiotemporal evolution of the states $q_1(x,t)$ and $q_3(x,t)$.

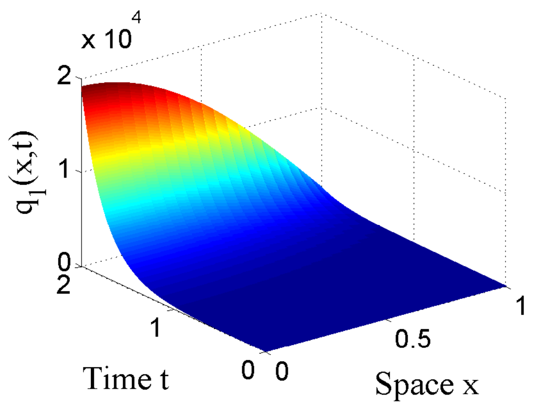

Figure 2.15: Spatiotemporal evolution of $q_1(x,t)$ (left plot) and $q_3(x,t)$ (right plot).

The observer (2.188)-(2.190), (2.256)-(2.257) has been implemented by selecting the design matrix $\bar{C} = 10I_{3\times3}$, and by specifying the initial conditions at $\hat{q}_1(x,0) = \hat{q}_2(x,0) = \hat{q}_3(x,0) = 0$. Figure 2.16 displays the spatiotemporal evolution of the observed states $\hat{q}_1(x,t)$ and $\hat{q}_3(x,t)$, which clearly mimic the corresponding actual states. Figure 2.18 shows the temporal evolution of the norm $\|\tilde{Q}(\cdot,t)\|_{2,3}$, which tends to zero exponentially, thus confirming the correct functioning of the proposed observer and supporting the theoretical analysis.

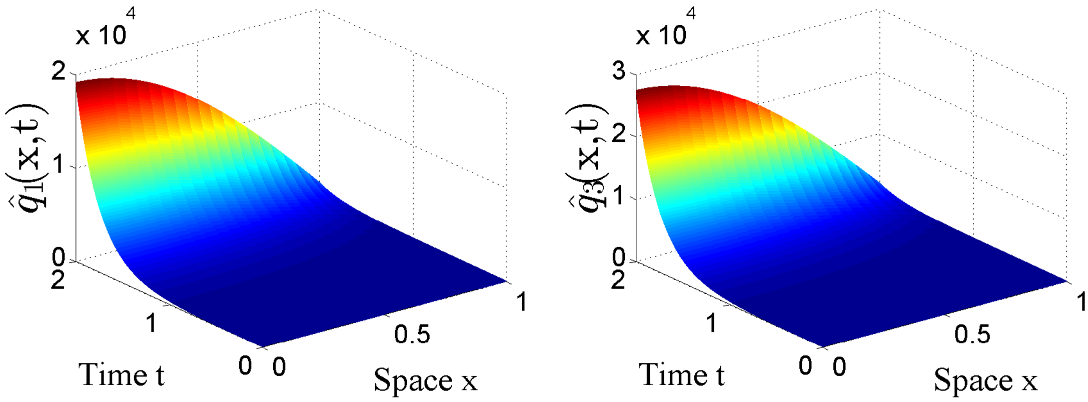

Figure 2.16: Spatiotemporal evolution of $\hat{q}_1(x,t)$ (left plot) and $\hat{q}_3(x,t)$ (right plot).

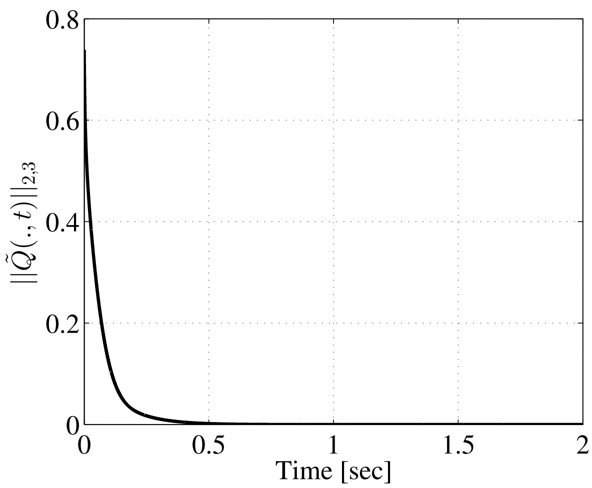

Figure 2.17: Temporal evolution of the norm $\|\tilde{Q}(\cdot,t)\|_{2,3}$.

Application example

To provide a more valuable validation of the proposed scheme, we consider the coupled temperature-concentration dynamics of a Chemical Tubular Reactor (CTR) at low fluid superficial velocities, when convection terms become negligible, dealt with in [20]. After a suitable transformation, the next dimensionless model was derived

$$
\begin{aligned}
\frac{\partial x_1}{\partial t} &= D_1 \frac{\partial^2 x_1}{\partial \xi^2} + k_0 \delta (1 - x_2) e^{-\frac{\gamma}{1+x_1}} && (2.264) \\
\frac{\partial x_2}{\partial t} &= D_2 \frac{\partial^2 x_2}{\partial \xi^2} + k_0 (1 - x_2) e^{-\frac{\gamma}{1+x_1}} && (2.265) \\
x_{1\xi}(0,t) &= x_{2\xi}(0,t) = 0 && (2.266) \\
x_1(1,t) &= u_1(t), && (2.267) \\
x_2(1,t) &= u_2(t), && (2.268)
\end{aligned}
$$

where the states x_1 and x_2 denote the normalized temperature and concentration, respectively, and the underlying physical parameters take the values

$$
\begin{aligned}
D_1 &= D_2 = 0.167, \quad \delta = 0.5 && (2.269) \\
k_0 &= 2.426 \cdot 10^7 \quad \gamma = 20 && (2.270)
\end{aligned}
$$

Its linearization around the constant profiles

$$
\begin{aligned}
x_1^*(\xi,t) &= 0.1 && (2.271) \\
x_2^*(\xi,t) &= 0.98 && (2.272)
\end{aligned}
$$

give rise to the model (2.183) - (2.185) with the following diffusivity and reaction parameters

$$
\theta = 0.167, \qquad \Lambda = \begin{bmatrix} 1.018 & 0.154 \\ 2.037 & 0.308 \end{bmatrix} \qquad (2.273)
$$

The open-loop control input $U(t) = [5sint, 10sin2t]^T$ was selected. The plant ICs are set to $x_1(x,0) = x_2(x,0) = 2sin(\pi\xi) + 2sin(3\pi\xi)$. The unstable open-loop behaviour of the plant state $x_2(\xi,t)$ is displayed in the Figure 2.18-left. The observer (2.188)-(2.190), (2.256)-(2.257) has been implemented by selecting the design matrix $\bar{C} = 20I_{2\times2}$, and by specifying the ICs $\hat{x}_1(\xi,0) = \hat{x}_2(\xi,0) = 0$. Figure 2.18-right shows that the observer is able to correctly reconstruct the unstable profile of the plant state $x_2(\xi,t)$. Figure 2.19 shows the temporal evolution of the norm $\|\tilde{Q}(\cdot,t)\|_{2,2}$, which confirms the correct functioning of the observer for the estimation of the state variable $x_2(\xi,t)$, too.

2.3.6 Conclusions

The backstepping-based anti-collocated observer design of a system of n coupled parabolic linear PDEs has been tackled, and an explicit representation of the underlying observer gains has been derived which allows one to enforce an arbitrarily fast exponential decay of the observation error dynamics in the space $[L_2(0,1)]^n$. The extension to the case of different diffusivities and spatially-dependent parameters, and the observer-based output-feedback design of a stabilizing controller $U(t)$, are among the most interesting future lines of related investigations that will be pursued in our future work.

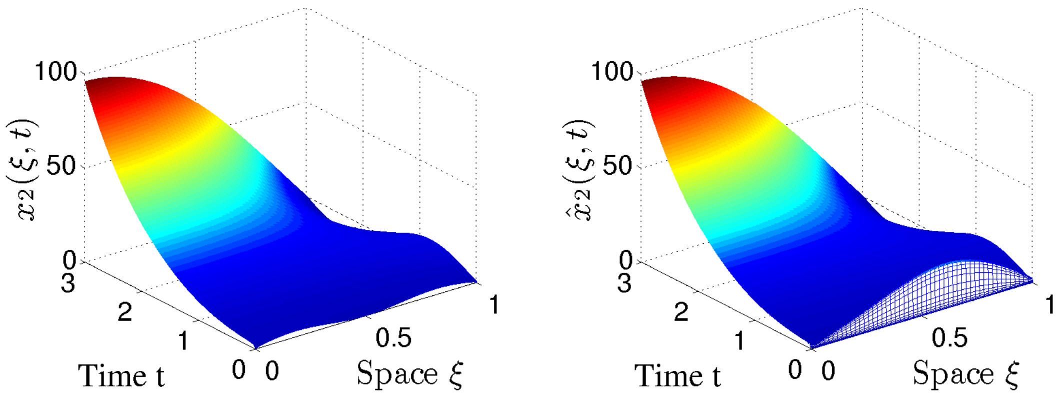

Figure 2.18: Spatiotemporal evolution of $x_2(\xi, t)$ (left plot) and $\hat{x}_2(\xi, t)$ (right plot).

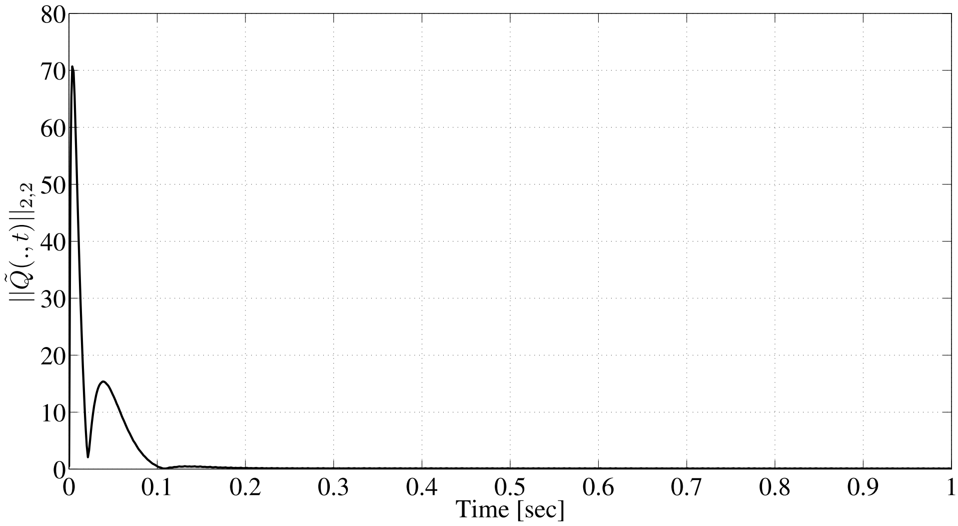

Figure 2.19: Temporal evolution of the norm $\|\tilde{Q}(\cdot, t)\|_{2,2}$.

2.4 Output feedback boundary stabilization of coupled reaction-diffusion PDE

The problem of output feedback boundary stabilization is considered for n coupled plants, distributed over the one-dimensional spatial domain $[0, 1]$ where they are governed by linear reaction-diffusion Partial Differential Equations (PDEs). All plants are equipped with its own scalar boundary control input, acting at one end of the domain. First, a state-feedback law is designed to exponentially stabilize the closed-loop system with an arbitrarily fast convergence rate. Then, collocated and anti-collocated observers are designed, using a single boundary measurement for each plant. The exponential convergence of the observed state towards the actual one is demonstrated for both observers, with a convergence rate that can be made as fast as desired. Finally, the state-feedback controller and the selected, either collocated or anti-collocated, observer are coupled together to yield an output-feedback stabilizing controller. The distinct treatments are proposed separately for the case in which all processes have the same diffusivity and for the more challenging scenario where each process has its own diffusivity. The backstepping method is used for both controller and observer designs, and, particularly, the kernel matrices of the underlying transformations are derived in analytical form by using the method of successive approximations to solve the corresponding kernel PDEs. Thus, the resulting control laws and observers become available in explicit form. Capabilities of the proposed synthesis and its effectiveness are supported by a numerical study made for three coupled systems with distinct diffusivity parameters.

2.4.1 Introduction

Reaction-diffusion equations are parabolic Partial Differential Equations (PDEs) which often occur in practice, e.g., to model the concentration of one or more substances, distributed in space, under the influence of different phenomena such as local chemical reactions, in which the substances are transformed into each other, and diffusion, which causes the substances to spread out over a surface in space. Certainly, reaction-diffusion PDEs are not confined to chemical applications (see e.g. [20]), but they also describe dynamical processes of non-chemical nature, with examples being found in thermodynamics, biology, geology, physics, ecology, etc. (see e.g. [60, 61]).

In the present work, the problem of output feedback boundary stabilization is considered for coupled linear reaction-diffusion PDEs with Dirichlet boundary conditions provided that only boundary flows are available for measurements. The adopted treatment does not rely on any discretization or finite-dimensional approximation of the underlying PDEs and it preserves the infinite-dimensional structure of the system during the entire design process. The proposed output feedback synthesis is based on the so-called "backstepping" approach [13]. Basically, the backstepping approach deals with an invertible Volterra integral transformation, mapping the system dynamics onto a predefined exponentially stable target dynamics. Backstepping is a versatile and powerful approach to boundary control and observer design, applicable to a broad spectrum of linear PDEs, and under certain circumstances controllers and observers are derived in explicit forms

[13].

Related literature

The backstepping-based boundary control of scalar reaction-diffusion processes was studied, e.g., in [17], [67] whereas scalar wave processes were studied, e.g., in [14], [58]. Complex-valued PDEs such as the Schrodinger equation were dealt with by means of such an approach [16]. Synergies between the backstepping methodology and the flatness approach were exploited in [18], [19] to control parabolic PDEs with spatially and time-varying coefficients in spatial domains of dimension 2 and higher. In addition, an interesting feature of backstepping is that it admits a synergic integration with robust control paradigms such as the sliding mode control methodology (see, e.g., [10]).

The implementation of backstepping controllers usually requires the full state information. From the practical standpoint, the available measurements of Distributed Parameter Systems (DPSs) are typically located at the boundary of the spatial domain, that motivates the need of the state observer design [46, 70]. For linear infinite-dimensional systems, the Luenberger observer theory was established by replacing matrices with linear operators [69, 72, 70], and the observer design was confined to determining a gain operator that stabilizes the associated observation error dynamics. In contrast to finite-dimensional systems, finding such a gain operator was not trivial even numerically because operators were not generally represented with a finite number of parameters.

Observer design methods that would be capable of yielding the observer gains in the analytical form have only recently been investigated. In this context, the backstepping method appears to be a particularly effective systematic observer design approach [13, **?**]. For scalar systems governed by parabolic PDEs defined on a 1-dimensional (1D) spatial domain, a systematic observer design approach, using boundary sensing, is introduced in [**?**]. Recently, the backstepping-based observer design was presented in [44] for reaction-diffusion processes with spatially-varying reaction coefficients while measuring a certain integral average value of the state of the plant. In [71, 11], backstepping-based observer design was addressed for reaction-diffusion processes evolving in multi-dimensional spatial domains.

More recently, high-dimensional systems of *coupled* PDEs were considered in the backstepping boundary control and observer design settings. The most intensive efforts of current literature were oriented towards coupled hyperbolic processes of the transport-type [5, 8, 37, 29, 30]. In [5], a 2×2 linear hyperbolic system was stabilized by a scalar observer-based output-feedback boundary control input, with an additional feature that an unmatched disturbance, generated by an *a-priori* known exosystem, was rejected. In [29], a 2×2 system of coupled linear heterodirectional hyperbolic equations was stabilized by observer-based output feedback. The underlying design was extended in [8] to a particular type of 3×3 linear systems, arising in modeling of multiphase flow, and to the quasilinear case in [30]. In [37], backstepping observer-based output-feedback design was presented for a system of $n+1$ coupled first-order linear heterodirectional hyperbolic PDEs (n of which featured rightward convecting transport, and one leftward) with a single boundary input.

Some specific results on the backstepping based boundary stabilization of parabolic coupled PDEs have additionally been presented in the literature [6, 28, 75, 32, 33]. In [28], two parabolic reaction-diffusion processes, coupled through the corresponding boundary conditions, were dealt with. The stabilization of the coupled equations was reformulated in terms of the stabilization problem for a unique process, which possessed piecewise-continuous diffusivity and (space-dependent) reaction coefficient and which was viewed as the “cascade” between the two original systems. The problem was then solved by using a scalar boundary control input and by employing a non conventional backstepping approach with a discontinuous kernel function. In [6], the Ginzburg-Landau equation with the imaginary and real parts expanded, thus being specified to a 2×2 parabolic system with equal diffusion coefficients, was dealt with. In [75], the linearized 2×2 model of thermal-fluid convection was treated by using a singular perturbations approach combined with backstepping and Fourier series expansion. In [33], the boundary stabilization of the linearized model of an incompressible magnetohydrodynamic flow in an infinite rectangular 3D channel, also recognized as Hartmann flow, was achieved by reducing the original system to a set of coupled diffusion equations with the same diffusivity parameter and by applying backstepping. In [32], an observer that estimated the velocity, pressure, electric potential and current fields in a Hartmann flow was presented where the observer gains were designed using multi-dimensional backstepping. In [41], a backstepping observer was designed for a system of two diffusion-convection-reaction processes coupled through the corresponding boundary conditions.

The recent authors‘ work [35], which appeared to be more closely related to the present investigation, dealt with the state-feedback controller design for coupled reaction-diffusion processes equipped with Neumann (rather than Dirichlet) boundary conditions. The same publication also addressed a state-feedback stabilization problem for two coupled reaction-diffusion processes, which were underactuated by a scalar boundary input applied just to one of the processes.

Results and contributions of the paper

Thus motivated, the primary concern of this work is to extend the backstepping synthesis developed in [**?**], where explicit stabilizing output-feedback boundary controllers were designed for scalar unstable reaction-diffusion processes with constant parameters. Here, a generalization is provided by considering a set of n reaction-diffusion processes, which are coupled through the corresponding reaction terms.

A constructive observer-based output-feedback synthesis procedure, with all controllers and observers given in explicit form, presents the main contribution of this work to the existing literature. As shown in the paper, this generalization is far from being trivial because the underlying backstepping-based treatment gives rise to more complex development of finding out an analytical solution in the form of Bessel-like matrix series.

The present treatment addresses, side by side, two distinct situations which require quite different solution approaches to be adopted. First, the case where all processes have the same diffusivity parameter (“equi-diffusivity” case) is attacked, and then the more challenging situation where each process possesses its own diffusivity (“distinct-diffusivity” case) is treated.

Under the requirement that the considered multi-dimensional process is fully actuated by a set of n boundary control inputs acting on each subsystem, all these approaches are shown to exponentially stabilize the controlled system with an arbitrarily fast convergence rate. Particularly, in the present paper output-feedback stabilizing controllers using both collocated and anti-collocated observers are presented.

Organization

The structure of the paper is as follows. In Section 2.4.2, the problem statement is presented along with the associated assumptions. In Section 2.4.3, the state-feedback controller synthesis is developed. Sections 2.4.4 and 2.4.5 present, respectively, the anti-collocated and collocated observer designs. Section 2.4.6 develops the output-feedback controller design by providing a demonstration of the stable coupling between the designed controllers and observers. Section 2.4.8 discusses some simulation results. Finally, Section 2.4.9 collects concluding remarks and features future perspectives of this research.

Notation

$L_2(0,1)$ stands for the Hilbert space of square integrable scalar functions $z(\zeta)$ on the domain $(0,1)$ with the corresponding L_2-norm

$$\|z(\cdot)\|_2 = \sqrt{\int_0^1 z^2(\zeta)d\zeta}. \tag{2.274}$$

$\mathrm{H}^\ell(0,1)$, with $\ell = 0,1,2,\ldots$, denotes the Sobolev space of absolutely continuous scalar functions $z(\zeta)$ on the domain $(0,1)$, with square integrable derivatives $z^{(k)}(\varsigma)$ up to order ℓ and the corresponding H^ℓ-norm

$$\|z(\cdot)\|_{\mathrm{H}^\ell} = \sqrt{\sum_{k=0}^{\ell} \|z^{(k)}(\cdot)\|_2^2}. \tag{2.275}$$

Also, the notations

$$L_2^n = \underbrace{L_2(0,1) \times L_2(0,1) \times \ldots \times L_2(0,1)}_{n \ \texttt{times}}$$

and

$$H^{\ell,n} = \underbrace{H^\ell(0,1) \times H^\ell(0,1) \times \ldots \times H^\ell(0,1)}_{n \ \texttt{times}}$$

are utilized and

$$\|Z(\cdot)\|_{2,n} = \sqrt{\sum_{i=1}^{n} \|z_i(\cdot)\|_2^2}, \tag{2.276}$$

$$\|W(\cdot)\|_{H^{\ell,n}} = \sqrt{\sum_{i=1}^{n} \|w_i(\cdot)\|_{H^\ell}^2} \tag{2.277}$$

stand, respectively, for the L_2-norm of a vector function $Z(\zeta) = [z_1(\zeta), z_2(\zeta), ..., z_n(\zeta)] \in L_2^n$ and for the H^ℓ-norm of a vector function $W(\zeta) = [w_1(\zeta), w_2(\zeta), ..., w_n(\zeta)] \in H^{\ell,n}$.

Throughout, $I_1(\cdot)$ stands for the first order modified Bessel functions of the first kind, and $\mathcal{T}$ denotes the domain

$$\mathcal{T} = \{(x, y) \in \mathbb{R}^2 : 0 \le y \le x \le 1\}. \tag{2.278}$$

Given a generic real-valued square matrix A, the symbol $S[A]$ denotes its symmetric part $S[A] = (A + A^T)/2$. Provided that A is symmetric, the inequality $A > 0$ means that it is positive definite. Just in case, $\sigma_m(A)$ denotes the smallest eigenvalue of A.

Given a real-valued square matrix function $M(x)$ of order n, whose entries $m_{ij}(x)$ are defined on a set X, its $\mathcal{C}^0(X)$-norm is determined by

$$\|M(x)\|_{\mathcal{C}^0(X)} = \max_{i,j=1,2,...,n} \sup_{x\in X} |m_{ij}(x)|. \tag{2.279}$$

Finally, $I_{m\times m}$ stands for the identity matrix of dimension m.

For later use, an instrumental lemma is presented.

Lemma 8. *(cf. [55, Lemma 2]) Let $b(\zeta) \in \mathrm{L}_2(0, 1)$. Then, the following inequality*

$$\left[\int_0^1 |b(\zeta)| d\zeta\right]^2 \le \|b(\cdot)\|_2^2 \tag{2.280}$$

holds.

2.4.2 Problem statement

A system of n coupled reaction-diffusion processes, governed by the reaction-diffusion vector PDE

$$Q_t(x, t) = \Theta Q_{xx}(x, t) + \Lambda Q(x, t), \tag{2.281}$$

which is equipped with the Dirichlet-type Boundary Conditions (BCs)

$$Q(0, t) = 0, \tag{2.282}$$

$$Q(1, t) = U(t), \tag{2.283}$$

and subject to the Initial Condition (IC)

$$Q(x, 0) = Q_0(x) \in H^{4,n}, \tag{2.284}$$

is under investigation. Hereinafter,

$$Q(x, t) = [q_1(x, t),\ q_2(x, t), \ldots, q_n(x, t)]^T \in H^{4,n} \tag{2.285}$$

is the state vector,

$$U(t) = [u_1(t),\ u_2(t), \ldots, u_n(t)]^T \in \mathbb{R}^n \tag{2.286}$$

is the boundary control vector, $\Lambda \in \mathbb{R}^{n\times n}$ is a real-valued square matrix, and $\Theta = diag(\theta_i)$ is the diagonal diffusivity matrix with $\theta_i > 0$ for all $i = 1, 2, ..., n$.

To avoid imposing a restrictive compatibility condition on the initial function (2.284) to satisfy the BCs (2.282)-(2.283), solutions of the Boundary Value Problem (BVP) (2.281)-(2.283) (as well as solutions of any BVP to be used in the sequel) are viewed in the weak sense throughout as those to the variational problem of finding a function $Q(x,t) \in H^{1,n}$ subject to the BCs (2.282)-(2.283) such that

$$\begin{aligned}\int_0^1 W^T(\xi)Q_t(\xi,t)d\xi &= W^T(1)Q_\xi(1,t) - W^T(0)Q_\xi(0,t) - \int_0^1 W_\xi^T(\xi)Q_\xi(\xi,t)d\xi \\ &+ \int_0^1 W^T(\xi)\Lambda Q(\xi,t)d\xi \end{aligned}\tag{2.287}$$

for any $t > 0$ and for any $W(\cdot) \in H^{1,n}$. Such a solution of (2.287), satisfying (2.282)-(2.283), is further referred to as a weak solution of the BVP (2.281)-(2.283) that has become standard in the literature.

If confined to a linear feedback input $U(\cdot)$, the closed-loop system (2.281)-(2.284) is well-known[2] to possess a unique weak solution of class $H^{\ell,n}$ with an arbitrarily large integer ℓ provided that the initial state is of the same class. For technical reasons, the weak solutions of (2.281)-(2.283) are required to evolve in the state space $H^{4,n}$ to guarantee that the corresponding second order spatial derivative evolves in the state space $H^{2,n}$. Due to this, the IC (2.284) has been pre-specified to belong to $H^{4,n}$.

The open-loop system (2.281)-(2.284) (with $U(t) = 0$) possesses arbitrarily many unstable eigenvalues whenever $S[\Lambda]$ has positive and sufficiently large eigenvalues. Since the term $\Lambda Q(x,t)$ is the source of such an instability, the problem then arises to exponentially stabilize the closed-loop system by "reshaping" this term via reversing its effect into a stabilizing one. This problem will be addressed under two distinct scenarios:

i.) *anti-collocated measurement setup*, where the only measurement of the flow $Q_x(0,t)$ is available at the uncontrolled boundary;

ii.) *collocated measurement setup*, where sensing of $Q_x(1,t)$ is available at the controlled boundary only.

To facilitate exposition the treatment is first addressed by deriving a stabilizing control law using the state-feedback. Then the corresponding collocated and anti-collocated observers are designed. Finally, feeding the proposed state feedback controller with the state of either observer, running in parallel, yields the output-feedback stabilizing control laws.

[2]See, e.g., [76] for the Fourier representation of such a solution similar to (2.296) used in the proof of Theorem 22.

2.4.3 State-feedback controller design

The rationale of the backstepping state-feedback boundary control design is to exponentially stabilize system (2.281)-(2.283) by exploiting an invertible transformation

$$Z(x,t) = Q(x,t) - \int_0^x K(x,y)Q(y,t)dy \tag{2.288}$$

with a $n \times n$ kernel matrix function $K(x,y)$. An appropriate choice of the kernel $K(x,y)$ and that of the state-feedback input vector U allows one to transform the underlying closed-loop system into the target system

$$Z_t(x,t) = \Theta Z_{xx}(x,t) - CZ(x,t), \tag{2.289}$$
$$Z(0,t) = 0, \tag{2.290}$$
$$Z(1,t) = 0, \tag{2.291}$$

written in terms of the state vector $Z(x,t) = [z_1(x,t),\, z_2(x,t), \dots, z_n(x,t)]^T$ and where $C \in \mathbb{R}^{n\times n}$ is a design matrix parameter, subject to the IC

$$Z(x,0) = Q_0(x) - \int_0^x K(x,y)Q_0(y)dy, \tag{2.292}$$

which follows from (2.284) and (2.288). To ensure that an arbitrary weak solution of the target system BVP (2.289)-(2.292) evolves in the same state space $H^{4,n}$ it suffices to assume that the kernel matrix function $K(x,y)$ is smooth enough in its domain $\mathcal{T}$ defined in (2.278). The validity of this assumption is subsequently verified when the analytical representation of $K(x,y)$ is derived.

With the above consideration in mind, the exponential stability of the target system (2.289)-(2.292) is then ensured with an arbitrarily fast convergence rate by an appropriate choice of the real-valued square matrix $C \in \mathbb{R}^{n\times n}$. The following result is in order.

Theorem 22. *Let matrix C be such that $S[C] > 0$. Then, system* (2.289)-(2.292) *is exponentially stable in the space $H^{2,n}$ with the decay rate $\sigma_m(S[C])$ according to*

$$\|Z(\cdot,t)\|_{H^{2,n}} \le \|Z(\cdot,0)\|_{H^{2,n}} e^{-\sigma_m(S[C])t}. \tag{2.293}$$

Additionally, the following point-wise estimates

$$\max_{x\in[0,1]} |z_i(x,t)| \le \sqrt{2}\|Z(\cdot,0)\|_{H^{2,n}} e^{-\sigma_m(S[C])t}, \quad i = 1,2,...,n \tag{2.294}$$
$$\max_{x\in[0,1]} |z_{ix}(x,t)| \le \sqrt{2}\|Z(\cdot,0)\|_{H^{2,n}} e^{-\sigma_m(S[C])t}, \quad i = 1,2,...,n \tag{2.295}$$

are in force, where $z_{ix}(x,t)$ denotes the i-th element of $Z_x(x,t)$.

Proof. To begin with, let us note that under the conditions of the theorem a weak solution $Z(x,t)$ of (2.289)-(2.292) admits a Fourier representation

$$Z(x,t) = \sum_{k=1}^{\infty} Z_k(t)\sin(\pi k x), \tag{2.296}$$

where $Z_k(t),\ k=1,2,\ldots$, is a solution of the ODE $\dot{Z}_k = -[(\pi k)^2\Theta + C]Z_k$ (see, e.g., [76, 77] for details). It is then straightforward to verify that the spatial derivatives $Z_x(x,t)$ and $Z_{xx}(x,t)$ constitute weak solutions of the BVPs

$$\begin{aligned} Z_{tx}(x,t) &= \Theta Z_{xxx}(x,t) - CZ_x(x,t), && (2.297)\\ Z_{txx}(x,t) &= \Theta Z_{xxxx}(x,t) - CZ_{xx}(x,t), && (2.298)\\ Z_{xx}(0,t) &= Z_{xx}(1,t) = 0, && (2.299)\end{aligned}$$

inherited from (2.289)-(2.291). Remarkably, the same BCs (2.299) are of Neumann type for the PDE (2.297) in Z_x, and of Dirichlet type for the PDE (2.298) in Z_{xx}.

Taking this into account, let us now consider the Lyapunov function

$$\begin{aligned} V(t) &= \frac{1}{2}\|Z(\cdot,t)\|^2_{H^{2,n}} = \frac{1}{2}\int_0^1 Z^T(\xi,t)Z(\xi,t)d\xi + \frac{1}{2}\int_0^1 Z_\xi^T(\xi,t)Z_\xi(\xi,t)d\xi \\ &+ \frac{1}{2}\int_0^1 Z_{\xi\xi}^T(\xi,t)Z_{\xi\xi}(\xi,t)d\xi. && (2.300)\end{aligned}$$

In light of (2.297)-(2.298), the corresponding time derivative of the Lyapunov function (2.300) along the solutions of (2.289)-(2.291) and (2.297)-(2.299) is given by

$$\begin{aligned} \dot{V}(t) = &\int_0^1 Z^T(\xi,t)\Theta Z_{\xi\xi}(\xi,t)d\xi - \int_0^1 Z^T(\xi,t)CZ(\xi,t)d\xi + \int_0^1 Z_\xi^T(\xi,t)\Theta Z_{\xi\xi\xi}(\xi,t)d\xi \\ -&\int_0^1 Z_\xi^T(\xi,t)CZ_\xi(\xi,t)d\xi + \int_0^1 Z_{\xi\xi}^T(\xi,t)\Theta Z_{\xi\xi\xi\xi}(\xi,t)d\xi - \int_0^1 Z_{\xi\xi}^T(\xi,t)CZ_{\xi\xi}(\xi,t)d\xi. \end{aligned} \quad (2.301)$$

The first integral term in the right hand side of equality (2.301), being integrated by parts, is estimated as

$$\begin{aligned} \int_0^1 Z^T(\xi,t)\Theta Z_{\xi\xi}(\xi,t)d\xi &= Z^T(\chi,t)\Theta Z_x(\chi,t)\big|_{\chi=0}^{\chi=1} - \int_0^1 Z_\xi^T(\xi,t)\Theta Z_\xi(\xi,t)d\xi \\ &\le -\theta_m\|Z_x(\cdot,t)\|^2_{2,n}, && (2.302)\end{aligned}$$

where relations (2.290), (2.291) and the diagonal form of matrix Θ have been taken into account, and the notation $\theta_m = \min_{1\le i\le n}\theta_i > 0$ has been used. Following the same route, the third and fifth integral terms in the right hand side of (2.301) are estimated as

$$\begin{aligned} \int_0^1 Z_\xi^T(\xi,t)\Theta Z_{\xi\xi\xi}(\xi,t)d\xi &= Z_x^T(\chi,t)\Theta Z_{xx}(\chi,t)\big|_{\chi=0}^{\chi=1} - \int_0^1 Z_{\xi\xi}^T(\xi,t)\Theta Z_{\xi\xi}(\xi,t)d\xi \\ &\le -\theta_m\|Z_{xx}(\cdot,t)\|^2_{2,n}, && (2.303)\\ \int_0^1 Z_{\xi\xi}^T(\xi,t)\Theta Z_{\xi\xi\xi\xi}(\xi,t)d\xi &= Z_{xx}^T(\chi,t)\Theta Z_{xxx}(\chi,t)\big|_{\chi=0}^{\chi=1} - \int_0^1 Z_{\xi\xi\xi}^T(\xi,t)\Theta Z_{\xi\xi\xi}(\xi,t)d\xi \\ &\le -\theta_m\|Z_{xxx}(\cdot,t)\|^2_{2,n}, && (2.304)\end{aligned}$$

where the BCs (2.299) have been used. To manage the remaining integral terms in the right hand side of (2.301), the well-known property

$$\zeta^T C\zeta \geq \sigma_m(S[C])\zeta^T\zeta \tag{2.305}$$

of the quadratic form $\zeta^T C\zeta$ is exploited with the matrix C, whose symmetric part is positive definite by assumption, and an arbitrary n-dimensional vector ζ. Substituting (2.302)-(2.304) into (2.301), one readily obtains

$$\dot{V}(t) \;\leq\; -\theta_m\|Z_\xi(\cdot,t)\|^2_{H^{2,n}} - 2\sigma_m(S[C])V(t) \leq -2\sigma_m(S[C])V(t) \tag{2.306}$$

by applying straightforward manipulations, made according to (2.305). By definition of the Lyapunov function (2.300), relation (2.306) ensures the exponential stability of the target system (2.289)-(2.291) in the space $H^{2,n}$ with the decay rate obeying the estimate (2.293).

It remains to establish the point-wise estimates (2.294) and (2.295). For this purpose, let us note that relation (2.293) remains in force in the component-wise form

$$\|z_i(\cdot,t)\|_{H^2} \leq \|Z(\cdot,0)\|_{H^{2,n}}e^{-\sigma_m(S[C])t}, \quad i=1,2,\ldots,n \tag{2.307}$$

and due to the trivial inequalities $\|z_i(\cdot,t)\|_2 \leq \|z_i(\cdot,t)\|_{H^2}$, $\|z_{ix}(\cdot,t)\|_2 \leq \|z_i(\cdot,t)\|_{H^2}$, the next estimates

$$\|z_i(\cdot,t)\|_2 \leq \|Z(\cdot,0)\|_{H^{2,n}}e^{-\sigma_m(S[C])t},\ \|z_{ix}(\cdot,t)\|_2 \leq \|Z(\cdot,0)\|_{H^{2,n}}e^{-\sigma_m(S[C])t} \tag{2.308}$$

are in force as well. The point-wise estimate (2.294) is then trivially derived from that obtained by employing Agmon's inequality and utilizing the estimates (2.308):

$$\max_{x\in[0,1]} z_i^2(x,t) \leq 2\|z_i(\cdot,t)\|_2\|z_{ix}(\cdot,t)\|_2 \leq 2\|Z(\cdot,0)\|^2_{H^{2,n}}e^{-2\sigma_m(S[C])t},\ i=1,2,\ldots,n. \tag{2.309}$$

To prove (2.295) let us consider an arbitrary *constant* $\bar{x}\in[0,1]$ and write down the trivial relation

$$z_{ix}(\bar{x},t) \;=\; z_{ix}(x,t) - \int_{\bar{x}}^{x} z_{i\xi\xi}(\xi,t)d\xi, \quad \bar{x}\in[0,1], \quad i=1,2,...,n \tag{2.310}$$

where $z_{ix}(\cdot)$ and $z_{ixx}(\cdot)$ denote the i-th element of vectors $Z_x(\cdot)$ and $Z_{xx}(\cdot)$. Squaring both sides of (2.310) and applying the triangle inequality yield

$$z_{ix}^2(\bar{x},t) \;\leq\; 2z_{ix}^2(x,t) + 2\left[\int_{\bar{x}}^{x} z_{i\xi\xi}(\xi,t)d\xi\right]^2, \quad \bar{x}\in[0,1], \quad i=1,2,...,n. \tag{2.311}$$

By virtue of Lemma 8, specified with $b(\cdot)=z_{i\xi\xi}(\cdot)$, the chain of inequalities

$$\begin{aligned} z_{ix}^2(\bar{x},t) \;&\leq\; 2z_{ix}^2(x,t) + 2\left[\int_0^1 |z_{i\xi\xi}(\xi,t)|d\xi\right]^2 \\ &\leq\; 2z_{ix}^2(x,t) + 2\|z_{ixx}(\cdot,t)\|_2^2, \quad \bar{x}\in[0,1], \quad i=1,2,...,n \end{aligned} \tag{2.312}$$

is derived from (2.311). Then by integrating both sides of (2.312) with respect to the spatial variable x from 0 to 1 and by exploiting relation (2.307), one gets

$$z_{ix}^2(\bar{x},t) \le 2\|z_{ix}(\cdot,t)\|_2^2 + 2\|z_{ixx}(\cdot,t)\|_2^2 \le 2\|z_i(\cdot,t)\|_{H^2} \le 2\|Z(\cdot,0)\|_{H^{2,n}}^2 e^{-2\sigma_m(S[C])t}$$
$$\bar{x} \in [0,1], \quad i = 1,2,...,n. \tag{2.313}$$

By noticing that $\bar{x}$ is an arbitrary constant in the interval [0,1], the point-wise estimate (2.295) is straightforwardly concluded from (2.313). The proof of Theorem 22 is thus completed. □

Remark 13. *It should be pointed out that relations* (2.294) *and* (2.295) *do not truly establish the exponential point-wise decay of* $z_i(x,t)$ *and* $z_{ix}(x,t)$ *due to the fact that* $\|Z(\cdot,0)\|_{H^{2,n}}$*, rather than* $|z_i(x,0)|$ *and, respectively,* $|z_{ix}(x,0)|$*, appears in the corresponding right-hand sides of these relations. However, such "quasi-exponential" decays prove to be suitable for establishing the exponential stability of the original system (2.281)-(2.283) in the space* $H^{2,n}$ *under the output-feedback boundary controller to subsequently be designed.*

The BVP governing the kernel matrix function $K(x,y)$ is now derived through the standard procedure adopted in the backstepping design [13]. Next developments closely follow our recent works [7, 35], where the same analysis were conducted for coupled reaction diffusion equations equipped with Neumann rather than Dirichlet BCs.

By applying the Leibnitz differentiation rule to (2.288), spatial derivatives $Z_x(x,t)$ and $Z_{xx}(x,t)$ are readily developed as a straightforward matrix generalization of corresponding well-known scalar counterparts. Furthermore, using (2.281) and applying recursively integration by parts, the time derivative $Z_t(x,t)$ is derived as well. Combining such expressions and performing rather lengthy but straightforward computations (see [7] for more detailed derivations) yield

$$\begin{aligned}
& Z_t(x,t) - \Theta Z_{xx}(x,t) + CZ(x,t) \\
= & \left[\Lambda + C + K_y(x,x)\Theta + \Theta K_x(x,x) + \Theta\frac{d}{dx}K(x,x)\right] Q(x,t) \\
+ & \int_0^x \left[\Theta K_{xx}(x,y) - K_{yy}(x,y)\Theta - K(x,y)\Lambda - CK(x,y)\right] Q(y,t)dy \\
+ & \left[\Theta K(x,x) - K(x,x)\Theta\right] Q_x(x,t) + K(x,0)\Theta Q_x(0,t) - K_y(x,0)\Theta Q(0,t)
\end{aligned} \tag{2.314}$$

Clearly, the target system PDE (2.289) requires the right hand side of (2.314) to be identically zero. Considering the homogeneous BC (2.282), this leads to the next relations

$$\Theta K_{xx}(x,y) - K_{yy}(x,y)\Theta = K(x,y)\Lambda + CK(x,y), \tag{2.315}$$
$$\Lambda + C + K_y(x,x)\Theta + \Theta K_x(x,x) + \Theta\frac{d}{dx}K(x,x) = 0, \tag{2.316}$$
$$\Theta K(x,x) - K(x,x)\Theta = 0, \tag{2.317}$$
$$K(x,0) = 0. \tag{2.318}$$

As in the Neumann BCs case [7], the main critical feature of (2.315)-(2.318) is in the presence of relation (2.317). While being identically satisfied in the scalar case ($n = 1$) [67], this relation is generally contradictive, and there are two options to fulfill (2.317). One of these options is to impose the constraint that all the coupled processes possess the same diffusivity value θ, i.e.,

$$\Theta = \theta I_{n\times n}. \tag{2.319}$$

An alternative option is to enforce the next constraint

$$K(x,y) = k(x,y) I_{n\times n} \tag{2.320}$$

on the form of the kernel matrix. Assumption (2.320) greatly simplifies the complexity of the underlying backstepping transformation, which is simply determined by a scalar function. This simplification, however, will also bring some constraint on the choice of the matrix C which is no longer an arbitrary design parameter when the relation (2.320) is in force. The above arguments motivate the need of treating separately the equi-diffusivity case, where constraint (2.319) is in force, and the distinct diffusivity case where the kernel matrix is subject to the constraint (2.320).

Equi-diffusivity case

Specializing system (2.315), (2.316), (2.318) in light of the equi-diffusivity constraint (2.319) and exploiting the identity $\frac{d}{dx}K(x,x) = K_x(x,x) + K_y(x,x)$ yield the BVP

$$K_{xx}(x,y) - K_{yy}(x,y) = \frac{1}{\theta}K(x,y)\Lambda + \frac{1}{\theta}CK(x,y), \tag{2.321}$$

$$\Lambda + C + 2\theta\frac{d}{dx}K(x,x) = 0, \tag{2.322}$$

$$K(x,0) = 0. \tag{2.323}$$

Integrating (2.322) with respect to x gives $K(x,x) = -\frac{1}{2\theta}(\Lambda + C)x + K(0,0)$. It follows from (2.323) that $K(0,0) = 0$, hence relation (2.322) is replaced by

$$K(x,x) = -\frac{1}{2\theta}(\Lambda + C)x. \tag{2.324}$$

The following result is in order.

Theorem 23. *The boundary-value problem* (2.321), (2.323), (2.324) *possesses a solution*

$$K(x,y) = -\sum_{j=0}^{\infty} \frac{(x^2-y^2)^j(2y)}{j!(j+1)!}\left(\frac{1}{4\theta}\right)^{j+1}\left[\sum_{i=0}^{j}\binom{j}{i}C^i\,(\mathit{\Lambda}+C)\,\mathit{\Lambda}^{j-i}\right] \tag{2.325}$$

which is of class $\mathcal{C}^\infty$ *in the domain* $\mathcal{T}$ *defined in* (2.278).

Proof. The proof of the present theorem is very similar to that of [7, Th. 1], where the Neumann BCs were in play. Therefore, only a sketch of the proof is given.

Inspired from [67], the substitution

$$\xi = x + y, \qquad \eta = x - y, \tag{2.326}$$

of the independent variables is adopted to represent the BVP (2.321), (2.323), (2.324) in terms of

$$G(\xi,\eta) = K(x,y) = K\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) \tag{2.327}$$

as follows

$$G_{\xi\eta}(\xi,\eta) = \frac{1}{4\theta}G(\xi,\eta)\Lambda + \frac{1}{4\theta}CG(\xi,\eta), \tag{2.328}$$

$$G(\xi,0) = -\frac{1}{4\theta}\left(\Lambda + C\right)\xi, \tag{2.329}$$

$$G(\xi,\xi) = 0. \tag{2.330}$$

Relation (2.328), being integrated first with respect to η from 0 to η and then with respect to ξ from η to ξ, results in the following integral equation

$$G(\xi,\eta) = -\frac{1}{4\theta}(\mathit{\Lambda} + C)\left(\xi - \eta\right) + \frac{1}{4\theta}\int_\eta^\xi \left\{\int_0^\eta \left[G(\tau,s)\mathit{\Lambda} + CG(\tau,s)\right] ds\right\} d\tau. \tag{2.331}$$

By applying the method of successive approximations it yields that equation (2.331) possesses a solution expressed in the form

$$G(\xi,\eta) = \sum_{j=0}^{\infty} \Delta G^j(\xi,\eta), \tag{2.332}$$

where $\Delta G^j(\xi,\eta)$ satisfies the recursion

$$\Delta G^0(\xi,\eta) = -\frac{1}{4\theta}(\mathit{\Lambda} + C)(\xi - \eta), \tag{2.333}$$

$$\Delta G^{j+1}(\xi,\eta) = \frac{1}{4\theta}\int_\eta^\xi \left\{\int_0^\eta \left[\Delta G^j(\tau,s)\mathit{\Lambda} + C\Delta G^j(\tau,s)\right] ds\right\} d\tau,\ j = 0,1,\dots, \tag{2.334}$$

and the absolute and uniform convergence of the series (2.332)-(2.334) in the domain $0 \le \eta \le \xi \le 2$ is guaranteed by the Weierstrass M-test. Successively computing (2.334) for $j = 0,1,\dots$, with the initial term $\Delta G^0(\xi,\eta)$, given by (2.333), one observes the pattern leading to

$$\Delta G^j(\xi,\eta) = -\frac{(\xi\eta)^j(\xi-\eta)}{j!(j+1)!}\left(\frac{1}{4\theta}\right)^{j+1}\left[\sum_{i=0}^{j}\binom{j}{i} C^i\left(\mathit{\Lambda} + C\right)\mathit{\Lambda}^{j-i}\right]. \tag{2.335}$$

A solution to the integral equation (2.331) is thus given by the absolutely and uniformly converging series (2.332), composed of continuous generic terms (2.335), and it is therefore a continuous function.

By substituting the change of variables (2.326) into (2.332), (2.335) one returns back to the original variables x and y, and according to (2.327) one obtains the series form (2.325) of the kernel matrix $K(x, y)$ which solves the BVP (2.321), (2.323), (2.324).

To complete the proof it remains to note that being given by the integral equality (2.331), the continuous function $G(\xi, \eta)$ is at least twice continuously differentiable in the domain $0 \leq \eta \leq \xi \leq 2$. Moreover, by iterating on the successive differentiation of (2.331), one concludes that $G(\xi, \eta)$ is of class $\mathcal{C}^\infty$ in its domain. By virtue of (2.327), the solution (2.325) is thus shown to be of class $\mathcal{C}^\infty(\mathcal{T})$. This concludes the proof of Theorem 23. □

Remark 14. *Uniqueness of a solution to some BVPs, similar to* (2.321), (2.323), (2.324), *has been addressed in the literature (see, e.g., [67, 10]). This valuable issue does not, however, affect the underlying synthesis and it therefore remains beyond the scope of the paper.*

The designed state-feedback boundary controller for the equi-diffusivity case takes the form

$$\begin{aligned} U(t) &= \int_0^1 K(1, y) Q(y, t) dy, && (2.336) \\ K(1, y) &= -\sum_{n=0}^{\infty} \left[\frac{2y(1-y^2)^n}{n!(n+1)!} \right] \left(\frac{1}{4\theta} \right)^{n+1} \left[\sum_{i=0}^{n} \binom{n}{i} C^i \left(\Lambda + C \right) \Lambda^{n-i} \right]. && (2.337) \end{aligned}$$

The following result is in order:

Theorem 24. *Let matrix C be selected in such a manner that $S[C] > 0$ whereas $\sigma_m(S[C])$ is arbitrarily large. Then, the boundary control input* (2.336)-(2.337) *exponentially stabilizes system (2.281)-(2.283) in the space $H^{2,n}$ with the corresponding norm obeying the estimate*

$$\|Q(\cdot, t)\|_{H^{2,n}} \leq a \|Q(\cdot, 0)\|_{H^{2,n}} e^{-\sigma_m(S[C])t}, \tag{2.338}$$

where a is a positive constant independent of $Q(x, 0)$.

Proof. See Subsection 2.4.3 □

Inverse transformation and stability issues Relevant results, concerning the invertibility of the backstepping transformation (2.288) and the smoothness of the inverse kernel matrix, are collected in this subsubsection to be used in the proof of Theorem 24.

Transformation (2.288) is a matrix Volterra integral equation. We look for an inverse transformation in the form

$$Q(x, t) = Z(x, t) + \int_0^x L(x, y) Z(y, t) dy. \tag{2.339}$$

Existence and smoothness properties of $L(x, y)$ are investigated in the following lemma (see, e.g., [37, 45] for the scalar case which is going to be extended to the present vector case).

The following lemma is in order.

Lemma 9. *There exist a kernel matrix $L(x, y)$, of class $\mathcal{C}^\infty(\mathcal{T})$ with the domain $\mathcal{T}$ specified in* (2.278)*, such that the inverse transformation* (2.339) *of* (2.288) *is in force.*

Proof. Substituting (2.288) into (2.339) and performing straightforward manipulations, one derives the integral equation

$$L(x, y) = K(x, y) + \int_y^x L(x, s)K(s, y)ds, \tag{2.340}$$

that implicitly defines the inverse kernel matrix $L(x, y)$ on $\mathcal{T}$. The method of successive approximations is going to be applied to show that a smooth solution to (2.340) exists. Let us start with the initial guess $L^0(x, y) = 0$ and construct the recursive formula

$$L^{j+1}(x, y) = K(x, y) + \int_y^x L^j(x, s)K(s, y)ds, \quad j = 0, 1, 2, \dots \tag{2.341}$$

Let us denote the difference between two consecutive terms as

$$\Delta L^j(x, y) = L^{j+1}(x, y) - L^j(x, y), \quad j = 0, 1, 2, \dots \tag{2.342}$$

Then, the next recursion is obtained by (2.341)

$$\Delta L^0(x, y) = L^1(x, y) = K(x, y), \tag{2.343}$$
$$\Delta L^{j+1}(x, y) = \int_y^x \Delta L^j(x, s)K(s, y)ds, \quad j = 0, 1, 2, \dots \tag{2.344}$$

If the recursion (2.343)-(2.344) converges, a solution $L(x, y)$ to (2.340) takes the form

$$L(x, y) = \sum_{j=0}^{\infty} \Delta L^j(x, y). \tag{2.345}$$

The kernel matrix $K(x, y)$ is continuous (cf. Theorem 23), hence its $\mathcal{C}^0$-norm (2.279) admits a uniform upperbound in the compact set $\mathcal{T}$. It means that there exists a positive constant M such that

$$\|\Delta L^0(x, y)\|_{\mathcal{C}^0(\mathcal{T})} = \|K(x, y)\|_{\mathcal{C}^0(\mathcal{T})} \le M. \tag{2.346}$$

Suppose that

$$\|\Delta L^j(x, y)\|_{\mathcal{C}^0(\mathcal{T})} \le M^{j+1}\frac{x^j}{j!}. \tag{2.347}$$

Then, by (2.344), (2.346) and (2.347) one derives the next estimate

$$\begin{aligned}\|\Delta L^{j+1}(x,y)\|_{\mathcal{C}^0(\mathcal{T})} &\leq \left|\int_y^x \|\Delta L^{j+1}(x,s)\|_{\mathcal{C}^0(\mathcal{T})}\|K(s,y)\|_{\mathcal{C}^0(\mathcal{T})}ds\right| \\ &\leq \frac{M^{j+2}}{j!}x^j\left|\int_y^x ds\right| = \frac{M^{j+2}}{j!}x^j\,|x-y|\,. \end{aligned} \tag{2.348}$$

Due to the inequalities $0 \leq y \leq x$, which come from the domain definition (2.278), the following estimate

$$x^j\,|x-y| \leq x^{j+1}, \quad (x,y) \in \mathcal{T}, \tag{2.349}$$

is in force. Therefore, combining (2.348) and (2.349), one gets

$$\|\Delta L^{j+1}(x,y)\| \leq M^{j+2}\frac{x^{j+1}}{(j+1)!}. \tag{2.350}$$

By mathematical induction, (2.350) is true for all $j \geq 0$. It then follows from the Weierstrass M-test that the series (2.345) converges absolutely and uniformly in $\mathcal{T}$. Thus, (2.343)-(2.345) is a solution to (2.340) and it thereby implements the inverse transformation (2.339).

Due to the fact that $K(x,y)$ is of class $\mathcal{C}^\infty(\mathcal{T})$, the integral equation (2.340) allows one to conclude that the function $L(x,y)$ is continuous and at least one time continuously differentiable in the domain $\mathcal{T}$. By iterating on the successive differentiation of (2.340), one further derives that $L(x,y)$ is of class $\mathcal{C}^\infty$ in its domain. Lemma 9 is proven.

□

By generalizing [12, Th 2.3], it is now proved that the properties $K(x,y) \in \mathcal{C}^\infty(\mathcal{T})$ and $L(x,y) \in \mathcal{C}^\infty(\mathcal{T})$ result in the equivalence of the norms of $Z(x,t)$ and $Q(x,t)$ in $H^{2,n}$.

Lemma 10. *Consider the direct and inverse backstepping transformations* (2.288) *and* (2.339) *with the associated kernel matrices* $K(x,y), L(x,y) \in \mathcal{C}^\infty(\mathcal{T})$ *on the domain* $\mathcal{T}$ *defined in* (2.278). *Then, there are positive constants* b_1 *and* b_2 *such that*

$$\|Q(\cdot,t)\|_{H^{2,n}} \leq b_1\|Z(\cdot,t)\|_{H^{2,n}}, \tag{2.351}$$

$$\|Z(\cdot,t)\|_{H^{2,n}} \leq b_2\|Q(\cdot,t)\|_{H^{2,n}}. \tag{2.352}$$

Proof. To begin with, one notices that properties $K(x,y) \in \mathcal{C}^\infty(\mathcal{T})$ and $L(x,y) \in \mathcal{C}^\infty(\mathcal{T})$ guarantee the existence of positive constants $M_1, M_2, ..., M_8$ such that

$$\|K(x,y)\|_{\mathcal{C}^0(\mathcal{T})} \leq M_1, \qquad \|L(x,y)\|_{\mathcal{C}^0(\mathcal{T})} \leq M_2, \tag{2.353}$$

$$\|K_x(x,y)\|_{\mathcal{C}^0(\mathcal{T})} \leq M_3, \qquad \|L_x(x,y)\|_{\mathcal{C}^0(\mathcal{T})} \leq M_4, \tag{2.354}$$

$$\|K_y(x,y)\|_{\mathcal{C}^0(\mathcal{T})} \leq M_5, \qquad \|L_y(x,y)\|_{\mathcal{C}^0(\mathcal{T})} \leq M_6, \tag{2.355}$$

$$\|K_{xx}(x,y)\|_{\mathcal{C}^0(\mathcal{T})} \leq M_7, \qquad \|L_{xx}(x,y)\|_{\mathcal{C}^0(\mathcal{T})} \leq M_8. \tag{2.356}$$

From relation (2.339) one concludes, after straightforward manipulations (similar to those of [12, Th 2.3]), that

$$\|Q(\cdot,t)\|_{2,n} \leq \|Z(\cdot,t)\|_{2,n} + \|L(x,y)\|_{\mathcal{C}^0(\mathcal{T})}\|Z(\cdot,t)\|_{2,n} \leq (1+M_2)\|Z(\cdot,t)\|_{2,n}. \quad (2.357)$$

Spatial derivatives $Q_x(x,t)$ and $Q_{xx}(x,t)$ are computed by iteratively applying the Leibnitz differentiation rule to (2.339). It yields

$$Q_x(x,t) = Z_x(x,t) + L(x,x)Z(x,t) + \int_0^x L_x(x,y)Z(y,t)dy, \quad (2.358)$$

$$\begin{aligned} Q_{xx}(x,t) &= Z_{xx}(x,t) + \left[\frac{d}{dx}L(x,x)\right]Z(x,t) + L(x,x)Z_x(x,t) + L_x(x,x)Z(x,t) \\ &+ \int_0^x L_{xx}(x,y)Z(y,t)dy. \end{aligned} \quad (2.359)$$

It is concluded from (2.358), (2.353) and (2.354) that

$$\begin{aligned} \|Q_x(\cdot,t)\|_{2,n} &\leq \|Z_x(\cdot,t)\|_{2,n} + \|L(x,x)\|_{\mathcal{C}^0(\mathcal{T})}\|Z(\cdot,t)\|_{2,n} + \|L_x(x,y)\|_{\mathcal{C}^0(\mathcal{T})}\|Z(\cdot,t)\|_{2,n} \\ &\leq \|Z_x(\cdot,t)\|_{2,n} + (M_2+M_4)\|Z(\cdot,t)\|_{2,n}. \end{aligned} \quad (2.360)$$

By applying a similar estimation to (2.359), and noticing that by (2.354) and (2.355) the relation

$$\begin{aligned} \left\|\frac{d}{dx}L(x,x)\right\|_{\mathcal{C}^0(\mathcal{T})} &= \|L_x(x,x)+L_y(x,x)\|_{\mathcal{C}^0(\mathcal{T})} \leq \|L_x(x,x)\|_{\mathcal{C}^0(\mathcal{T})} + \|L_y(x,x)\|_{\mathcal{C}^0(\mathcal{T})} \\ &\leq M_4 + M_6 \end{aligned} \quad (2.361)$$

is in force, one straightforwardly concludes that

$$\|Q_{xx}(\cdot,t)\|_{2,n} \leq \|Z_{xx}(\cdot,t)\|_{2,n} + M_2\|Z_x(\cdot,t)\|_{2,n} + (2M_4+M_6+M_8)\|Z(\cdot,t)\|_{2,n}. \quad (2.362)$$

From (2.357), (2.360) and (2.362) one gets (2.351) with the constant $b_1 = 1 + 2M_2 + 3M_4 + M_6 + M_8\}$. Relation (2.352) is obtained by applying the similar analysis starting from relation (2.288). This concludes the proof of Lemma 10.

□

Proof of Theorem 24 The backstepping transformation (2.288), (2.325) was derived to map system (2.281)-(2.283) into the target dynamics governed by the PDE (2.289). It remains to prove that the homogenous BCs (2.290)-(2.291) hold as well. By specifying (2.288) with $x=0$ and $x=1$, and considering (2.282) and (2.283), yield

$$\begin{aligned} Z(0,t) &= Q(0,t) = 0, & (2.363) \\ Z(1,t) &= Q(1,t) - \int_0^1 K(1,y)Q(y,t)dy = U(t) - \int_0^1 K(1,y)Q(y,t)dy. & (2.364) \end{aligned}$$

Thus, the boundary control input vector (2.336)-(2.337), where the kernel $K(1, y)$ is readily obtained by specifying (2.325) for $x = 1$, results in the target BVP (2.289)-(2.291) with homogeneous BCs. The exponential stability of (2.289)-(2.291) in the space $H^{2,n}$ was established in Theorem 22 provided that $S[C] > 0$. Particularly, relation (2.293) was proven. Coupling (2.293) and (2.351), one derives that

$$\|Q(\cdot, t)\|_{H^{2,n}} \leq b_1 \|Z(\cdot, 0)\|_{H^{2,n}} e^{-\sigma_m(S[C])t}. \tag{2.365}$$

Specifying (2.352) with $t = 0$ and substituting the resulting relation in (2.365), one obtains (2.338) with the constant $a = b_1 b_2$ which is independent of $Q(x, 0)$. This completes the proof of Theorem 24.

Distinct diffusivity case

In the present subsection, boundary stabilization of system (2.281)-(2.283) with distinct diffusivity parameters is addressed by following the previously introduced backstepping design, specified with (2.320). Specializing system (2.315), (2.316), (2.318) in view of the constraint (2.320) on the kernel matrix yields

$$(k_{xx}(x, y) - k_{yy}(x, y))\Theta = k(x, y)(\Lambda + C), \tag{2.366}$$

$$2\frac{d}{dx}k(x, x)\Theta = -(\Lambda + C), \tag{2.367}$$

$$k(x, 0) = 0. \tag{2.368}$$

By following [35, Sect. 4], where system (2.281), equipped with homogeneous Neumann BCs, was under investigation, one concludes that to guarantee the solvability of (2.366)-(2.368) the matrix C has to be selected in the constrained form

$$C = -\Lambda + \gamma^* \Theta, \tag{2.369}$$

where γ^* is a scalar parameter.

Remark 15. *It was proven in [35, Th. 4] that one can always select the parameter γ^* in* (2.369) *large enough such that $S[C] > 0$ and $\sigma_m(S[C])$ is arbitrarily large.*

Substituting (2.369) into (2.366) and (2.367), and performing straightforward manipulations, the kernel function $k(x, y)$ proves to be a solution to the following BVP

$$k_{xx}(x, y) - k_{yy}(x, y) = \gamma^* k(x, y), \tag{2.370}$$

$$k(x, x) = = -\frac{\gamma^*}{2}x, \tag{2.371}$$

$$k(x, 0) = 0, \tag{2.372}$$

whose explicit solution

$$k(x, y) = -\gamma^* y \frac{I_1(\sqrt{\gamma^*(x^2 - y^2)})}{\sqrt{\gamma^*(x^2 - y^2)}} \tag{2.373}$$

is extracted from [13].

Remark 16. *The BVP* (2.370)-(2.372) *is a particular case of* (2.321)*,* (2.323)*,* (2.324)*. It thus follows from Theorem 23 that* $k(x, y)$ *is of class* $\mathcal{C}^\infty(\mathcal{T})$ *with* $\mathcal{T}$ *defined in* (2.278)*. Clearly, the inverse transformation of* (2.288)*,* (2.320) *takes the form* (2.339) *specified with* $L(x, y) = l(x, y)I_{n\times n}$*. By Lemma 9 one concludes that* $l(x, y)$ *is of class* $\mathcal{C}^\infty(\mathcal{T})$*, too.*

The next theorem specifies the proposed state-feedback boundary control design for the distinct diffusivity case.

Theorem 25. *Let matrix* C *be selected according to* (2.369) *with sufficiently large parameter* $\gamma^* > 0$ *to ensure that* $S[C] > 0$ *and* $\sigma_m(S[C])$ *is arbitrarily large. Then, the boundary control input*

$$U(t) = \int_0^1 k(1, y)Q(y, t)dy, \qquad k(1, y) = -\gamma^* y \frac{I_1(\sqrt{\gamma^*(1 - y^2)})}{\sqrt{\gamma^*(1 - y^2)}} \tag{2.374}$$

exponentially stabilizes system (2.281)-(2.283) in the space $H^{2,n}$ *with decay rate*

$$\|Q(\cdot, t)\|_{H^{2,n}} \leq a\|Q(\cdot, 0)\|_{H^{2,n}} e^{-\sigma_m(S[C])t}, \tag{2.375}$$

where a *is a positive constant independent of* $Q(x, 0)$*.*

Proof. The form (2.374) of the chosen boundary feedback control is readily justified by specifying (2.363) and (2.364) with (2.320) and noticing that $k(1, y)$ is obtained by specifying (2.373) with $x = 1$. Thus, with the feedback law (2.374) system (2.281)-(2.283) is transferred by (2.288), (2.320) into the target dynamics (2.289)-(2.291) with the matrix C given by (2.369). According to Remark 15, one can always select the parameter γ^* large enough such that $S[C] > 0$ and $\sigma_m(S[C])$ is arbitrarily large. Provided that $S[C] > 0$, the stability of the target dynamics (2.289)-(2.291) in the space $H^{2,n}$ was established in Theorem 22. Noticing that Lemma 10 is still in force due to Remark 16, the rest of the proof follows that of Theorem 24. The stability of the original system (2.281)-(2.283) is then established in the space $H^{2,n}$ by employing (2.375) with the same constant $a = b_1 b_2$. Theorem 25 is thus proved. □

2.4.4 Observer design for the anti-collocated measurement setup

For system (2.281)-(2.283) of n coupled reaction-diffusion processes with the boundary flow $Q_x(0, t)$ being the only available measurement, the state observer

$$\begin{aligned}
\hat{Q}_t(x, t) &= \Theta\hat{Q}_{xx}(x, t) + \Lambda\hat{Q}(x, t) + G(x)\left[Q_x(0, t) - \hat{Q}_x(0, t)\right], && (2.376)\\
\hat{Q}(0, t) &= 0, && (2.377)\\
\hat{Q}(1, t) &= U(t), && (2.378)\\
\hat{Q}(x, 0) &= \hat{Q}_0(x) \in H^{4,n} && (2.379)
\end{aligned}$$

is proposed, where $\hat{Q}(x, t)$ is the observed state and $G(x)$ is a square matrix of spatially-dependent observer gains to subsequently be designed. The observer is equipped with analogous BCs as those of the original system (2.282)-(2.283). The meaning of the BVP (2.376)-(2.379) is viewed in the weak sense as the system (2.281)-(2.284) is. In order to ensure that the weak solutions of (2.376)-(2.379) evolve in the state space $H^{4,n}$ the IC (2.379) is pre-specified to belong to $H^{4,n}$.

Introduce the estimation error variable

$$\tilde{Q}(x,t) = Q(x,t) - \hat{Q}(x,t), \tag{2.380}$$

and consider the associated BVP

$$\begin{aligned} \tilde{Q}_t(x,t) &= \Theta\tilde{Q}_{xx}(x,t) + \Lambda\tilde{Q}(x,t) - G(x)\tilde{Q}_x(0,t), && (2.381)\\ \tilde{Q}(0,t) &= 0, && (2.382)\\ \tilde{Q}(1,t) &= 0, && (2.383)\\ \tilde{Q}(x,0) &= Q_0(x) - \hat{Q}_0(x), && (2.384)\end{aligned}$$

which is readily derived from (2.281)-(2.284) and (2.376)-(2.379).

To design the observer gain matrix $G(x)$ the backstepping approach is involved for finding out the conditions under which an invertible transformation

$$\tilde{Q}(x,t) = \tilde{Z}(x,t) - \int_0^x P(x,y)\tilde{Z}(y,t)dy, \tag{2.385}$$

with a $n \times n$ matrix kernel function $P(x,y)$, maps the error system (2.381)-(2.383) into the exponentially stable target error BVP

$$\begin{aligned} \tilde{Z}_t(x,t) &= \Theta\tilde{Z}_{xx}(x,t) - \bar{C}\tilde{Z}(x,t), && (2.386)\\ \tilde{Z}(0,t) &= 0, && (2.387)\\ \tilde{Z}(1,t) &= 0. && (2.388)\end{aligned}$$

To derive the corresponding IC, the inverse transformation of (2.385) comes into play, which takes the form

$$\tilde{Z}(x,t) = \tilde{Q}(x,t) + \int_0^x R(x,y)\tilde{Q}(y,t)dy. \tag{2.389}$$

Specifying (2.389) with $t = 0$, it yields

$$\tilde{Z}(x,0) = \tilde{Q}(x,0) + \int_0^x R(x,y)\tilde{Q}(y,0)dy \in H^{4,n}, \tag{2.390}$$

which complements the boundary value problem (2.386)-(2.388).

The meaning of (2.386)-(2.388), (2.390) is also viewed in the weak sense. In the sequel, the BVP, governing the kernel matrix $P(x,y)$, and the tuning rule of selecting the observer gain matrix $G(x)$ are derived.

Spatial differentiation of (2.385) yields

$$\tilde{Q}_x(x,t) = \tilde{Z}_x(x,t) - P(x,x)\tilde{Z}(x,t) - \int_0^x P_x(x,y)\tilde{Z}(y,t)dy. \tag{2.391}$$

By specifying (2.385) and (2.391) with $x = 0$, and substituting (2.387) in the resulting relations, one arrives at

$$\begin{aligned}
\tilde{Q}(0,t) &= \tilde{Z}(0,t) = 0, &(2.392)\\
\tilde{Q}_x(0,t) &= \tilde{Z}_x(0,t) - P(0,0)\tilde{Z}(0,t) = \tilde{Z}_x(0,t). &(2.393)
\end{aligned}$$

Specifying (2.385) with $x = 1$, substituting the resulting expression in (2.383), and imposing the BC (2.388), the relation

$$\int_0^1 P(1,y)\tilde{Z}(y,t)dy = 0 \tag{2.394}$$

is obtained to derive the BC

$$P(1,y) = 0. \tag{2.395}$$

By differentiating (2.391) with respect to x, the second-order spatial derivative $\tilde{Q}_{xx}(x,t)$ is readily developed (all spatial differentiations involve the use of the Leibnitz differentiation rule). Differentiating (2.385) in time, substituting (2.386) in the resulting relation, and applying recursively integration by parts, one readily obtains the time derivative $\tilde{Q}_t(x,t)$ as well. Substituting (2.385), (2.392), (2.393) and the obtained expressions of $\tilde{Q}_{xx}(x,t)$ and $\tilde{Q}_t(x,t)$ into (2.381) and performing lengthy but straightforward computations yield

$$\begin{aligned}
&\tilde{Z}_t(x,t) - \Theta\tilde{Z}_{xx}(x,t) + \bar{C}Z(x,t) = \\
&[\Theta P(x,x) - P(x,x)\Theta]\,\tilde{Z}_x(x,t) - [G(x) + P(x,0)\Theta]\,\tilde{Z}_x(0,t) \\
&- \left\{\Theta\left[\frac{d}{dx}P(x,x)\right] + P_y(x,x)\Theta + \Theta P_x(x,x) - \Lambda - \bar{C}\right\}\tilde{Z}(x,t) \\
&- \int_0^x \left[\Theta P_{xx}(x,y) - P_{yy}(x,y)\Theta + P(x,y)\bar{C} + \Lambda P(x,y)\right]\tilde{Z}(y,t)dy
\end{aligned} \tag{2.396}$$

To meet the PDE (2.386) the right-hand side of (2.396) should be identically zero. From this requirement, coupled to the BC (2.395), the BVP

$$\begin{aligned}
\Theta P_{xx}(x,y) - P_{yy}(x,y)\Theta &= -P(x,y)\bar{C} - \Lambda P(x,y), &(2.397)\\
\Theta\frac{d}{dx}P(x,x) + \Theta P_x(x,x) &+ P_y(x,x)\Theta = \Lambda + \bar{C}, &(2.398)\\
P(x,x)\Theta &= \Theta P(x,x), &(2.399)\\
P(1,y) &= 0, &(2.400)
\end{aligned}$$

governing the kernel matrix $P(x,y)$ is derived, and the observer gain tuning condition is obtained in the form

$$G(x) = -P(x,0)\Theta. \tag{2.401}$$

Similarly to system (2.315)-(2.318), that was derived in the state-feedback controller design, due to relation (2.399) the boundary value problem (2.397)-(2.400) is overdetermined and it has no solution unless either the equi-diffusivity constraint (2.319) is met or, alternatively, the relation

$$P(x, y) = p(x, y) I_{n\times n} \tag{2.402}$$

is enforced in analogy to (2.320). Thus, duality between the controller and observer designs is in force, and the observer treatment is then separately studied for the equi-diffusivity and distinct-diffusivity scenarios.

Equi-diffusivity case

Specializing system (2.397)-(2.400) with the equi-diffusivity constraint (2.319) and exploiting the identity $\frac{d}{dx}P(x, x) = P_x(x, x) + P_y(x, x)$ yield the BVP

$$\begin{aligned} P_{xx}(x, y) - P_{yy}(x, y) &= -\frac{1}{\theta}\left[P(x, y)\bar{C} + \Lambda P(x, y)\right], && (2.403)\\ 2\theta \frac{d}{dx} P(x, x) &= \Lambda + \bar{C}, && (2.404)\\ P(1, y) &= 0, && (2.405) \end{aligned}$$

whereas the tuning condition (2.401) simplifies as

$$G(x) = -\theta P(x, 0). \tag{2.406}$$

Integrating (2.404) with respect to x gives

$$P(x, x) = \frac{1}{2\theta}\left(\Lambda + \bar{C}\right) x + P(0, 0). \tag{2.407}$$

Evaluating (2.407) at $x = 1$ yields

$$P(1, 1) = \frac{1}{2\theta}\left(\Lambda + \bar{C}\right) + P(0, 0). \tag{2.408}$$

On the other hand, by evaluating (2.405) at $y = 1$ it is concluded that $P(1, 1) = 0$, thereby obtaining

$$P(0, 0) = -\frac{1}{2\theta}\left(\Lambda + \bar{C}\right). \tag{2.409}$$

In light of (2.407) and (2.409) one thus rewrites (2.403)-(2.405) as

$$\begin{aligned} P_{xx}(x, y) - P_{yy}(x, y) &= -\frac{1}{\theta}\left[P(x, y)\bar{C} + \Lambda P(x, y)\right], && (2.410)\\ P(x, x) &= \frac{\Lambda + \bar{C}}{2\theta}(x - 1), && (2.411)\\ P(1, y) &= 0. && (2.412) \end{aligned}$$

Conditions (2.410)-(2.412) form a well-posed BVP which admits an analytical solution. The following result is in order.

Theorem 26. *The boundary-value problem* (2.410)-(2.412) *possesses a solution*

$$P(x,y) = -\sum_{j=0}^{\infty} \frac{2(1-x)((1-y)^2-(1-x)^2)^j}{j!(j+1)!} \left(\frac{1}{4\theta}\right)^{j+1} \left[\sum_{i=0}^{j} \binom{j}{i} \mathit{\Lambda}^i(\mathit{\Lambda}+\bar{C})\bar{C}^{j-i}\right] \tag{2.413}$$

which is infinitely times continuously differentiable in the domain $\mathcal{T}$ *defined in* (2.278).

Proof. By making the invertible change of variables

$$\bar{x} = 1-y, \qquad \bar{y} = 1-x, \tag{2.414}$$

one transforms the boundary-value problem (2.410)-(2.412) into

$$\begin{aligned}
\bar{P}_{\bar{x}\bar{x}}(\bar{x},\bar{y}) - \bar{P}_{\bar{y}\bar{y}}(\bar{x},\bar{y}) &= \frac{1}{\theta}\left[\bar{P}(\bar{x},\bar{y})\bar{C} + \Lambda\bar{P}(\bar{x},\bar{y})\right], && (2.415)\\
\bar{P}(\bar{x},\bar{x}) &= -\frac{\Lambda+\bar{C}}{2\theta}\bar{x}, && (2.416)\\
\bar{P}(\bar{x},0) &= 0. && (2.417)
\end{aligned}$$

By direct comparison between (2.415)-(2.417) and (2.321), (2.323)-(2.324) one immediately notices that $\bar{P}(\bar{x},\bar{y}) = K(x,y)$ when Λ and C are respectively replaced by $\bar{C}$ and $\mathit{\Lambda}$. Thus, from (2.325), one obtains the solution of (2.415)-(2.417) in the form

$$\bar{P}(\bar{x},\bar{y}) = -\sum_{j=0}^{\infty} \frac{(\bar{x}^2-\bar{y}^2)^j(2\bar{y})}{j!(j+1)!} \left(\frac{1}{4\theta}\right)^{j+1} \left[\sum_{i=0}^{j} \binom{j}{i} \mathit{\Lambda}^i\left(\mathit{\Lambda}+\bar{C}\right)\bar{C}^{j-i}\right]. \tag{2.418}$$

By substituting the change of variables (2.414) into (2.418) one returns back to the original variables x and y, thereby getting the series expansion (2.413) for the Kernel matrix $P(x,y)$ which solves the BVP (2.410)-(2.412). Clearly, due to the smooth change of coordinates (2.414) the solution $P(x,y)$ inherits the smoothness properties of $K(x,y)$ to be of class $\mathcal{C}^\infty(\mathcal{T})$. Theorem 26 is thus proven. □

The representation

$$G(x) = \theta\sum_{j=0}^{\infty} \frac{2(1-x)(1-(1-x)^2)^j}{j!(j+1)!} \left(\frac{1}{4\theta}\right)^{j+1} \left[\sum_{i=0}^{j} \binom{j}{i} \mathit{\Lambda}^i(\mathit{\Lambda}+\bar{C})\bar{C}^{j-i}\right] \tag{2.419}$$

of the observer gain matrix is straightforwardly derived by specifying (2.406) with the solution (2.413) to the boundary-value problem (2.410)-(2.412).

The next theorem summarizes the proposed anti-collocated observer design for the equi-diffusivity case.

Theorem 27. *Let matrix $\bar{C}$ be selected such that $S[\bar{C}] > 0$ and $\sigma_m(S[\bar{C}])$ is arbitrarily large. Then, the observer* (2.376)-(2.378), (2.419) *reconstructs the state of system (2.281)-(2.283),* (2.319) *with the associated error decay rate obeying the estimate*

$$\|\tilde{Q}(\cdot,t)\|_{H^{2,n}} \leq b\|\tilde{Q}(\cdot,0)\|_{H^{2,n}} e^{-\sigma_m(S[\bar{C}])t}, \tag{2.420}$$

where b is a positive constant independent of $\tilde{Q}(\xi,0)$.

Proof. It was shown in the present section that the backstepping transformation (2.385), (2.413) transfers the error system (2.381)-(2.384) into the exponentially stable target error dynamics (2.386)-(2.388), (2.390), provided that the observer gain $G(x)$ is selected as in (2.419). By straightforwardly specifying relation (2.293) with the state $\tilde{Z}(x,t)$ of the target error dynamics one obtains the estimate

$$\|\tilde{Z}(\cdot,t)\|_{H^{2,n}} \leq \|\tilde{Z}(\cdot,0)\|_{H^{2,n}} e^{-\sigma_m(S[\bar{C}])t}. \tag{2.421}$$

Owing on the smoothness properties of $P(x,y)$, established in Theorem 26, and applying Lemma 9 one concludes that the kernel matrix $R(x,y)$ is of class $\mathcal{C}^\infty(\mathcal{T})$ as well. Thus, Lemma 10 is straightforwardly reformulated with reference to the direct and inverse backstepping transformations (2.385) and (2.389) along with the associated smooth kernel matrices $P(x,y)$ and $R(x,y)$. Particularly, relations

$$\|\tilde{Q}(\cdot,t)\|_{H^{2,n}} \leq c_1\|\tilde{Z}(\cdot,t)\|_{H^{2,n}}, \tag{2.422}$$
$$\|\tilde{Z}(\cdot,t)\|_{H^{2,n}} \leq c_2\|\tilde{Q}(\cdot,t)\|_{H^{2,n}}, \tag{2.423}$$

readily follow from (2.351)-(2.352) for some positive constants c_1 and c_2. Coupling together (2.421) and (2.422), one derives that

$$\|\tilde{Q}(\cdot,t)\|_{H^{2,n}} \leq c_1\|\tilde{Z}(\cdot,0)\|_{H^{2,n}} e^{-\sigma_m(S[\bar{C}])t}. \tag{2.424}$$

Finally, specifying (2.423) with $t = 0$, and substituting the resulting relation in (2.424), one obtains (2.420) with the constant $b = c_1c_2$ which is independent on $\tilde{Q}(x,0)$. Theorem 27 is proven. □

Distinct diffusivity case

In the present subsection, the anti-collocated observer design is addressed by dispensing with the equi-diffusivity requirement (2.319) (i.e., all processes possess their own distinct diffusivity parameters) and by introducing an extra constraint (2.402) on the kernel matrix $P(x,y)$ of the backstepping transformation (2.385).

By specializing the BVP (2.397)-(2.400) with the constraint (2.402), and applying the identity $\frac{d}{dx}p(x,x) = p_x(x,x) + p_y(x,x)$, one obtains

$$\begin{aligned} (p_{xx}(x,y) - p_{yy}(x,y))\,\Theta &= -p(x,y)(\Lambda + \bar{C}), && (2.425)\\ 2\frac{d}{dx}p(x,x)\Theta &= \Lambda + \bar{C}, && (2.426)\\ p(1,y) &= 0, && (2.427) \end{aligned}$$

whereas the observer gain (2.401) specializes to

$$G(x) \quad = \quad -\Theta p(x,0). \tag{2.428}$$

The BVP (2.425)-(2.427) shares the same structure of (2.366)-(2.368). Thus, its solvability is addressed by following [35, Sect. 4] thereby arriving in analogy with (2.369) to the constrained form

$$\bar{C} = -\Lambda + \bar{\gamma}^*\Theta \tag{2.429}$$

of the matrix $\bar{C}$ in the target error dynamics (2.386)-(2.388), where $\bar{\gamma}^* \in \mathbb{R}$ is a design parameter. Substituting (2.429) into (2.425) and (2.426) it yields the scalar BVP

$$p_{xx}(x,y) - p_{yy}(x,y) \quad = \quad -\bar{\gamma}^* p(x,y), \tag{2.430}$$

$$\frac{d}{dx}p(x,x) \quad = \quad \frac{\bar{\gamma}^*}{2}, \tag{2.431}$$

$$p(1,y) \quad = \quad 0. \tag{2.432}$$

Integrating (2.431) with respect to x gives the relation $p(x,x) = \frac{\bar{\gamma}^*}{2}x + p(0,0)$ whereas another relation $p(0,0) = -\frac{\bar{\gamma}^*}{2}$ is deduced from (2.432) by noticing that $p(1,1) = 0$.

System (2.430)-(2.432) can thus be specified to the BVP

$$p_{xx}(x,y) - p_{yy}(x,y) \quad = \quad -\bar{\gamma}^* p(x,y), \tag{2.433}$$

$$p(x,x) \quad = \quad \frac{\bar{\gamma}^*}{2}(x-1), \tag{2.434}$$

$$p(1,y) \quad = \quad 0, \tag{2.435}$$

whose solution

$$p(x,y) = -\bar{\gamma}^*(1-x)\frac{I_1(\sqrt{\bar{\gamma}^*(2x - x^2 + y^2 - 2y)})}{\sqrt{\bar{\gamma}^*(2x - x^2 + y^2 - 2y)}} \tag{2.436}$$

is well-known from [13]. The representation

$$G(x) \quad = \quad \Theta\bar{\gamma}^*(1-x)\frac{I_1(\sqrt{\bar{\gamma}^* x(2-x)})}{\sqrt{\bar{\gamma}^* x(2-x)}} \tag{2.437}$$

of the observer gain is straightforwardly derived by specifying (2.428) with the solution (2.436) to the BVP (2.433)-(2.435) evaluated in $y = 0$.

The next theorem specifies the proposed anti-collocated observer design for the distinct diffusivity case.

Theorem 28. *Let the constant $\bar{\gamma}^*$ be chosen large enough such that $S[\bar{C}] > 0$ and $\sigma_m(S[\bar{C}])$ is arbitrarily large where $\bar{C}$ is given in* (2.429). *Then, the observer* (2.376)-(2.378), (2.437) *reconstructs the state of system (2.281)-(2.283) with the observation error decay obeying the estimate*

$$\|\tilde{Q}(\cdot,t)\|_{H^{2,n}} \leq b\|\tilde{Q}(\cdot,0)\|_{H^{2,n}} e^{-\sigma_m(S[\bar{C}])t}, \tag{2.438}$$

with a positive constant b, independent of $\tilde{Q}(x,0)$.

Proof. The backstepping transformation (2.385), (2.402), specified with (2.436), transfers the error system (2.381)-(2.384) into the target error dynamics (2.386)-(2.388), (2.390), where $\bar{C}$ is given by (2.429), and the observer gain $G(x)$ is selected as in (2.437). According to Remark 15, one can always select the parameter $\bar{\gamma}^*$ large enough such that $S[\bar{C}] > 0$ and $\sigma_m(S[\bar{C}])$ is arbitrarily large. Thus, exponential stability of the target error dynamics is in force with an arbitrarily fast convergence rate according to estimate (2.421). Since the BVP (2.433)-(2.435) is a particular instance of (2.410)-(2.412), its solution (2.402) is guaranteed by Theorem 26 to be of class $\mathcal{C}^\infty(\mathcal{T})$. Clearly, the inverse backstepping transformation takes the form (2.389), specified with $R(x,y) = r(x,y)I_{n\times n}$, and by a straightforward extension of Lemma 9, one concludes that $r(x,y)$ is of class $\mathcal{C}^\infty(\mathcal{T})$, too. The rest of the proof follows the same steps used in the proof of Theorem 27. Particularly, relations (2.422)-(2.423) are shown to be in force for some positive constants c_1 and c_2. These relations, along with (2.421), result in the estimate (2.438). This concludes the proof of Theorem 28. □

2.4.5 Observer design for the collocated measurement setup

In the present section the state observer design for system (2.281)-(2.284) is addressed and solved under the assumption that only the boundary flow $Q_x(1,t)$ at the controlled side of the spatial domain is available for measurements. The design closely follows that of the Section 2.4.4, with a slightly different form of the backstepping transformations used. All similar developments to those of the anti-collocated scenario will be skipped. The proposed collocated observer takes the form

$$\begin{aligned}
\hat{Q}_t(x,t) &= \Theta\hat{Q}_{xx}(x,t) + \Lambda\hat{Q}(x,t) + G(x)\left[Q_x(1,t) - \hat{Q}_x(1,t)\right], && (2.439)\\
\hat{Q}(0,t) &= 0, && (2.440)\\
\hat{Q}(1,t) &= U(t), && (2.441)\\
\hat{Q}(x,0) &= \hat{Q}_0(x) \in H^{4,n}, && (2.442)
\end{aligned}$$

where $G(x)$ is a square matrix of observer gain functions to subsequently be designed.

The observation error variable (2.380) is governed by the BVP

$$\begin{aligned}
\tilde{Q}_t(x,t) &= \Theta\tilde{Q}_{xx}(x,t) + \Lambda\tilde{Q}(x,t) - G(x)\tilde{Q}_x(1,t), && (2.443)\\
\tilde{Q}(0,t) &= 0, && (2.444)\\
\tilde{Q}(1,t) &= 0, && (2.445)\\
\tilde{Q}(x,0) &= Q_0(x) - \hat{Q}_0(x) \in H^{4,n}. && (2.446)
\end{aligned}$$

To design the observer gain $G(x)$ extra conditions are to be involved under which an invertible transformation

$$\tilde{Q}(x,t) = \tilde{Z}(x,t) - \int_x^1 P(x,y)\tilde{Z}(y,t)dy \qquad (2.447)$$

maps the error BVP (2.443)-(2.446) into the exponentially stable target error dynamics (2.386)-(2.388). The IC (2.388) is rewritten as

$$\tilde{Z}(x,0)=\tilde{Q}(x,0)+\int_x^1 R(x,y)\tilde{Q}(y,0)dy, \tag{2.448}$$

where $R(x,y)$ is the kernel matrix of the inverse transformation

$$\tilde{Z}(x,t)=\tilde{Q}(x,t)+\int_x^1 R(x,y)\tilde{Q}(y,t)dy. \tag{2.449}$$

Note that the integration interval adopted in (2.447) and (2.449) is different from that of (2.385) and (2.389), which constitutes the main observer design difference between the anti-collocated and collocated case. Due to this difference, the domain of the kernel matrices $P(x,y)$ and $R(x,y)$ is actually given by the set

$$\mathcal{T}_1=\{(x,y)\in\mathbb{R}^2: 0\le x\le y\le 1\} \tag{2.450}$$

which is symmetrical to the domain (2.278), considered in the previous sections. Apart from these minor differences, the subsequent treatment follows the same line of reasoning used before.

As assumed throughout, the meaning of the BVPs (2.439)-(2.442), (2.443)-(2.446), and that of (2.386)-(2.388), (2.448) are viewed in the weak sense and the weak solutions $\hat{Q}_(x,t)$, $\tilde{Q}(x,t)$, $\tilde{Z}(x,t)$ are required to evolve in the state space $H^{4,n}$. Due to this, the corresponding ICs (2.442), (2.446) and (2.390) are pre-specified to belong to $H^{4,n}$.

Similar developments to those of Section 2.4.4, which are skipped for brevity, yield the following BVP

$$\begin{aligned}\Theta P_{xx}(x,y)-P_{yy}(x,y)\Theta &= -P(x,y)\bar{C}-\Lambda P(x,y), && (2.451)\\ \Theta\frac{d}{dx}P(x,x)+\Theta P_x(x,x) &+ P_y(x,x)\Theta=-\Lambda-\bar{C}, && (2.452)\\ P(x,x)\Theta &= \Theta P(x,x), && (2.453)\\ P(0,y) &= 0, && (2.454)\end{aligned}$$

governing the kernel matrix $P(x,y)$, and the observer gain tuning condition

$$G(x) = P(x,1)\Theta \tag{2.455}$$

is involved. Due to relation (2.453), the BVP (2.451)-(2.454) admits a solution iff either the equi-diffusivity constraint (2.319) holds or relation (2.402) is enforced. This is in analogy to the BVP (2.315)-(2.318), that was involved in the state-feedback controller design, and in analogy to the BVP (2.397)-(2.400), that was employed in the anti-collocated observer design. These two separate scenarios are investigated independently.

Equi-diffusivity case

Specializing system (2.451)-(2.454) with the equi-diffusivity constraint (2.319) and exploiting the identity $\frac{d}{dx}P(x,x) = P_x(x,x) + P_y(x,x)$ yield after straightforward manipulations

$$P_{xx}(x,y) - P_{yy}(x,y) = -\frac{1}{\theta}\left[P(x,y)\bar{C} + \Lambda P(x,y)\right], \tag{2.456}$$

$$P(x,x) = -\frac{\Lambda + \bar{C}}{2\theta}x, \tag{2.457}$$

$$P(0,y) = 0. \tag{2.458}$$

Conditions (2.456)-(2.458) form a well-posed BVP which admits an analytical solution as shown in the following theorem.

Theorem 29. *The BVP* (2.456)-(2.458) *possesses a solution*

$$P(x,y) = -\sum_{j=0}^{\infty} \frac{2x(y^2-x^2)^j}{j!(j+1)!}\left(\frac{1}{4\theta}\right)^{j+1}\left[\sum_{i=0}^{j}\binom{j}{i}\mathit{\Lambda}^i(\mathit{\Lambda}+\bar{C})\bar{C}^{j-i}\right] \tag{2.459}$$

which is infinitely times continuously differentiable in the domain (2.450).

Proof. By making the invertible change of variables

$$\bar{x} = y, \qquad \bar{y} = x, \tag{2.460}$$

one transforms (2.456)-(2.458) into

$$\bar{P}_{\bar{x}\bar{x}}(\bar{x},\bar{y}) - \bar{P}_{\bar{y}\bar{y}}(\bar{x},\bar{y}) = \frac{1}{\theta}\left[\bar{P}(\bar{x},\bar{y})\bar{C} + \Lambda\bar{P}(\bar{x},\bar{y})\right], \tag{2.461}$$

$$\bar{P}(\bar{y},\bar{y}) = -\frac{\Lambda + \bar{C}}{2\theta}\bar{y}, \tag{2.462}$$

$$\bar{P}(\bar{x},0) = 0. \tag{2.463}$$

The BC (2.462) can be rewritten in the equivalent form

$$\bar{P}(\bar{x},\bar{x}) = -\frac{\Lambda + \bar{C}}{2\theta}\bar{x}. \tag{2.464}$$

Substituting $\bar{C}$ and $\mathit{\Lambda}$ into the BVP (2.321), (2.323), (2.324) for $\mathit{\Lambda}$ and C, respectively, one arrives at the BVP (2.461), (2.463), (2.464), thereby establishing the relation $\bar{P}(\bar{x},\bar{y}) = K(x,y)$ between the solutions of these BVPs. With this in mind, the solution representation (2.325) allows one to reproduce the solution of (2.461), (2.463), (2.464) in the form

$$\bar{P}(\bar{x},\bar{y}) = -\sum_{j=0}^{\infty} \frac{(\bar{x}^2-\bar{y}^2)^j(2\bar{y})}{j!(j+1)!}\left(\frac{1}{4\theta}\right)^{j+1}\left[\sum_{i=0}^{j}\binom{j}{i}\mathit{\Lambda}^i\left(\mathit{\Lambda}+\bar{C}\right)\bar{C}^{j-i}\right] \tag{2.465}$$

By substituting the change of variables (2.460) into (2.465) one returns back to the original variables x and y to obtain the series expansion (2.459) of the kernel matrix $P(x,y)$ which solves the BVP (2.456)-(2.458). Due to the smooth change of coordinates (2.460) the solution $P(x,y)$ inherits the smoothness properties of $K(x,y)$, and therefore it proves to be of class $\mathcal{C}^\infty(\mathcal{T}_1)$. Theorem 29 is thus proven $\square$

The observer gain representation

$$G(x) = P(x,1)\theta = -\theta \sum_{n=0}^{\infty} \frac{2x(1-x^2)^n}{n!(n+1)!} \left(\frac{1}{4\theta}\right)^{n+1} \left[\sum_{i=0}^{n} \binom{n}{i} \Lambda^i (\Lambda + \bar{C}) \bar{C}^{n-i}\right] \quad (2.466)$$

is straightforwardly derived by specifying (2.455) according to the solution representation (2.459) for the BVP (2.456)-(2.458).

The next theorem summarizes the proposed anti-collocated observer design for the equi-diffusivity case.

Theorem 30. *Let matrix $\bar{C}$ be selected such that $S[\bar{C}] > 0$ and $\sigma_m(S[\bar{C}])$ is arbitrarily large. Then, the observer* (2.439) -(2.441), (2.466) *reconstructs the state of system (2.281)-(2.283),* (2.319) *with decay rate specified by* (2.420)*, where b is a positive constant independent of $\tilde{Q}(\xi, 0)$.*

Proof. The proof is identical to that of Theorem 27. □

Distinct diffusivity case

In the present subsection, the collocated observer design is addressed in the distinct diffusivity scenario. The content of this section, being similar to that of Section 2.4.4, is not accompanied with design details as they can straightforwardly be derived from the corresponding anti-collocated design.

The observer gain takes the form

$$G(x) = -\bar{\gamma}^* x \frac{I_1(\sqrt{\bar{\gamma}^*(1-x^2)})}{\sqrt{\bar{\gamma}^*(1-x^2)}} \Theta, \quad (2.467)$$

and the next result is in force.

Theorem 31. *Let the constant $\bar{\gamma}^*$ be chosen large enough to ensure that $S[\bar{C}] > 0$ with $\bar{C}$ given in* (2.429) *and $\sigma_m(S[\bar{C}])$ is arbitrarily large. Then, the observer* (2.439)-(2.441), (2.467) *reconstructs the state of system (2.281)-(2.283) with the observation error decay obeying the estimate* (2.438)*, where b is a positive constant independent of $\tilde{Q}(x, 0)$.*

Proof. The proof is identical to that of Theorem 28. □

2.4.6 Output-feedback stabilization

In this section, the anti-collocated and collocated backstepping observers of Sections 2.4.4 and 2.4.5 are combined with their natural dual backstepping controllers of Section 2.4.3 to present the output-feedback exponential stabilization of system (2.281)-(2.284).

2.4.7 Anti-collocated measurement setup

The following result is in order

Theorem 32. *Consider system (2.281)-(2.284) driven by the controller*

$$U(t) = \int_0^1 K(1, y)\hat{Q}(y, t)dy \tag{2.468}$$

and fed by observer (2.376)-(2.378), (2.419). *Let the matrices C and $\bar{C}$ be selected such that $S[C] > 0$ and $S[\bar{C}] > 0$, and let $K(1, y)$ be given by* (2.337). *Then, the closed-loop system (2.281)-(2.284),*(2.376)-(2.378), (2.419), (2.468) *is exponentially stable in the space $H^{2,n} \times H^{2,n}$.*

Proof. Lengthy but straightforward manipulations show that the backstepping transformation

$$\hat{Z}(x, t) = \hat{Q}(x, t) - \int_0^x K(x, y)\hat{Q}(y, t)dy \tag{2.469}$$

maps the observer dynamics (2.376)-(2.378) into the system

$$\begin{aligned}
\hat{Z}_t(x, t) &= \Theta\hat{Z}_{xx}(x, t) - \bar{C}\hat{Z}(x, t) + F_1(x)\tilde{Z}_x(0, t), && (2.470)\\
\hat{Z}(0, t) &= 0, && (2.471)\\
\hat{Z}(1, t) &= 0, && (2.472)
\end{aligned}$$

with

$$F_1(x) = \left[G(x) - \int_0^x K(x, y)G(y)dy\right]. \tag{2.473}$$

The $\tilde{Z}(x, t)$-system, governed by (2.386)-(2.388), is exponentially stable in the space $H^{2,n}$ as well as the homogeneous part of the $\hat{Z}(x, t)$-system (2.470)-(2.472) is if considered separately with the external term $\tilde{Z}_x(0, t)$ deliberately set to zero. Following [**?**, Sect. 5.1], one notices that the interconnection of the two systems in the ($\hat{Z}$, $\tilde{Z}$) coordinates is in cascade form, and it was shown in Theorem 22 that all entries of the forcing term $\tilde{Z}_x(0, t)$ escape "quasi-exponentially" to zero according to (2.295) (see Remark 13). Owing on the boundedness and smoothness of $G(x)$ and $K(x, y)$ in the corresponding domains, the function $F_1(x)$ is bounded and smooth in its domain $0 \le x \le 1$, too. Thus, the combined ($\hat{Z}$, $\tilde{Z}$)-system straightforwardly proves to be exponentially stable in the space $H^{2,n} \times H^{2,n}$. As a result, the ($\hat{Q}$, $\tilde{Q}$)-system is exponentially stable in the same space since it is related to ($\hat{Z}$, $\tilde{Z}$) by the invertible coordinate transformations (2.385) and (2.469) whose kernel matrix gains $P(x, y)$ and $K(x, y)$, along with the corresponding inverse transformation matrices $R(x, y)$ and $L(x, y)$, belong to $\mathcal{C}^\infty(\mathcal{T})$, where $\mathcal{T}$ is defined in (2.278). Indeed, a straightforward generalization of Lemma 10 shows that these smoothness properties guarantee the equivalence between norms of ($\hat{Z}$, $\tilde{Z}$) and ($\hat{Q}$, $\tilde{Q}$) in the space $H^{2,n} \times H^{2,n}$, directly ensuring the exponential stability of the closed-loop system of interest in the space $H^{2,n} \times H^{2,n}$. Theorem 32 is proven. □

The proof of the stable coupling of the controller to the anti-collocated observer, designed in the distinct-diffusivity case, follows the same line of reasoning and it is therefore omitted.

Collocated measurement setup

The following theorem is in force

Theorem 33. *Consider system (2.281)-(2.284) driven by the controller*

$$U(t) = \int_0^1 K(1,y)\hat{Q}(y,t)dy \tag{2.474}$$

and fed by observer (2.439)-(2.441), (2.466). *Let the matrices C and $\bar{C}$ be selected such that $S[C] > 0$ and $S[\bar{C}] > 0$, and let $K(1,y)$ be given by* (2.337). *Then, the closed-loop system (2.281)-(2.284),*(2.439)-(2.441), (2.466), (2.474) *is exponentially stable in the space $H^{2,n} \times H^{2,n}$.*

Proof. One shows that the backstepping transformation (2.469) maps the observer dynamics (2.439)-(2.441) into the BVP

$$\begin{aligned}
\hat{Z}_t(x,t) &= \Theta\hat{Z}_{xx}(x,t) - \bar{C}\hat{Z}(x,t) + F_1(x)\tilde{Z}_x(1,t), && (2.475)\\
\hat{Z}(0,t) &= 0, && (2.476)\\
\hat{Z}(1,t) &= 0, && (2.477)
\end{aligned}$$

which only differs from (2.470)-(2.472) in that $\tilde{Z}_x(1,t)$ rather than $\tilde{Z}_x(0,t)$ enters the corresponding PDE as an external input premultiplied by the smooth matrix gain $F_1(x)$. The rest of the proof follows the same steps and reasonings used in the proof of Theorem 32. The proof of Theorem 33 is thus concluded. □

2.4.8 Simulation results

For simulation purposes, system (2.281)-(2.283) is specified with $n = 3$ and with the parameters

$$\Theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} 8 & 9 & 6 \\ 7 & 9 & 5 \\ 7 & 7 & 9 \end{bmatrix} \tag{2.478}$$

to validate the performance of the closed-loop system driven by the proposed output-feedback controllers. The ICs are set to $q_1(x,0) = 2sin(\pi x)$, $q_2(x,0) = sin(2\pi x)$ and $q_3(x,0) = sin(3\pi x)$.

For solving the underlying BVPs, a standard finite-difference approximation method is used by discretizing the spatial solution domain $x \in [0,1]$ into a finite number of N uniformly spaced solution nodes $x_i = ih$, $h = 1/(N+1)$, $i = 1,2,...,N$. The value

$N = 40$ is then used. The resulting 40-th order discretized system is subsequently solved by fixed-step Runge-Kutta ODE4 method with step $T_s = 10^{-4}$.

The unstable behaviour of the plant subject to the open-loop input vector $U(t) = [0, 0, 0]^T$ is displayed in the Figure 2.20, which for certainty shows the diverging spatiotemporal evolution of the states $q_1(x,t)$, $q_2(x,t)$ and $q_3(x,t)$.

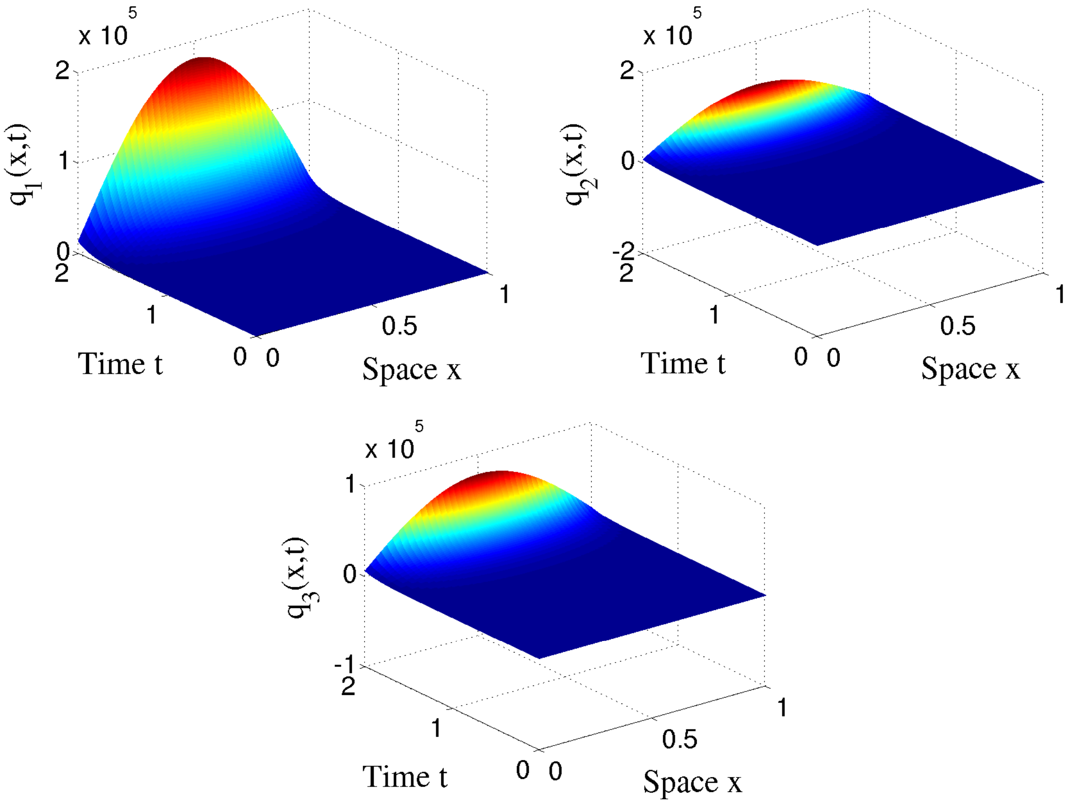

Figure 2.20: Spatiotemporal evolution of $q_1(x,t)$ (left plot) $q_2(x,t)$ (central plot) $q_3(x,t)$ (right plot) in the open-loop test.

The state-feedback boundary controller (2.374) has been implemented, with the underlying backstepping kernel matrix subject to (2.320) according to the distinct-diffusivity solution setup. The design parameter $\gamma^* = 15$ is selected, which yields $\sigma_m[S[C]] = 1.31$ thereby imposing desired exponential stability properties on the target system. Figure 2.21 shows the resulting spatiotemporal evolutions of the state variables $q_1(x,t)$, $q_2(x,t)$and $q_3(x,t)$, which vanish in H^2-norm as shown in the figure 2.22. The time evolutions of the boundary control inputs $u_1(t)$, $u_2(t)$ and $u_3(t)$ are displayed in the Figure 2.23.

The output-feedback stabilizer (2.468) using the anti-collocated observer (2.376)-(2.378), (2.437) has been implemented by selecting the design parameters $\gamma^* = 15$ and $\bar{\gamma}^* = 25$, resulting in $\sigma_m(S[\bar{C}]) = 13.36$, and by specifying the observer initial conditions as $\hat{q}_1(x,0) = \hat{q}_2(x,0) = \hat{q}_3(x,0) = 0$. Figure 2.24 displays the spatiotemporal evolution of the state variables $q_1(x,t)$, $q_2(x,t)$ and $q_3(x,t)$, which exhibit the vanishing dynamics.

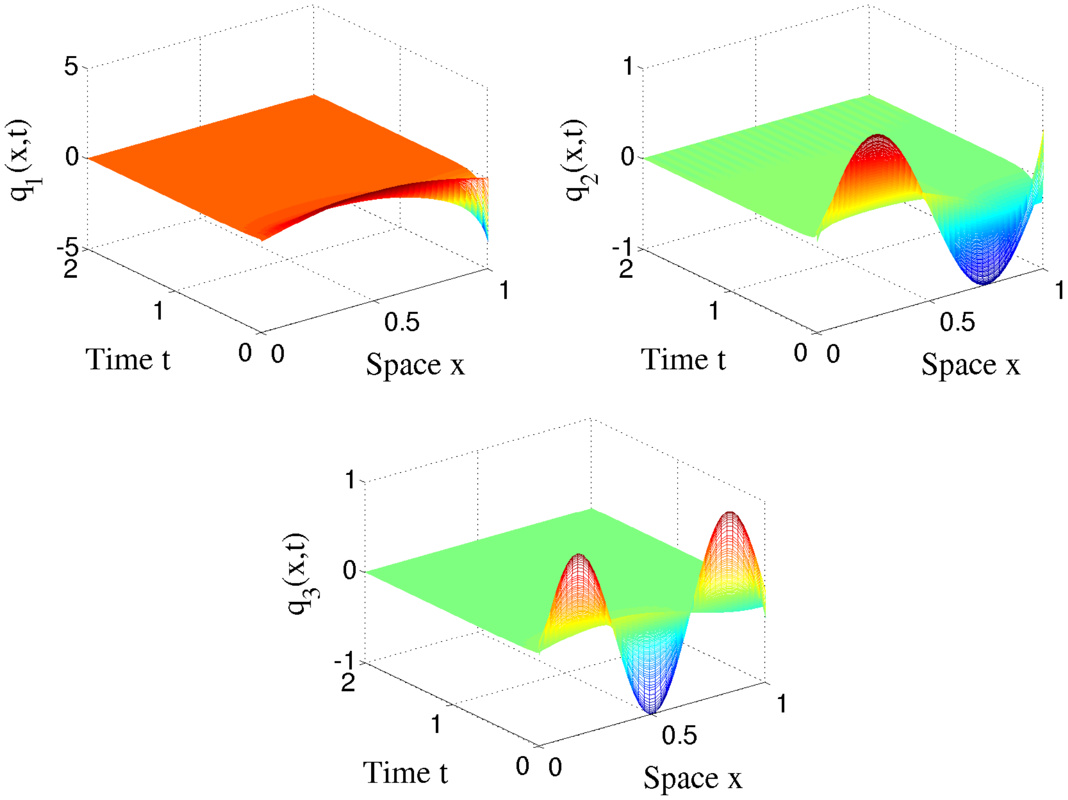

Figure 2.21: Spatiotemporal evolution of $q_1(x,t)$ (left plot) $q_2(x,t)$ (central plot) $q_3(x,t)$ (right plot) in the closed-loop test with the state-feedback controller.

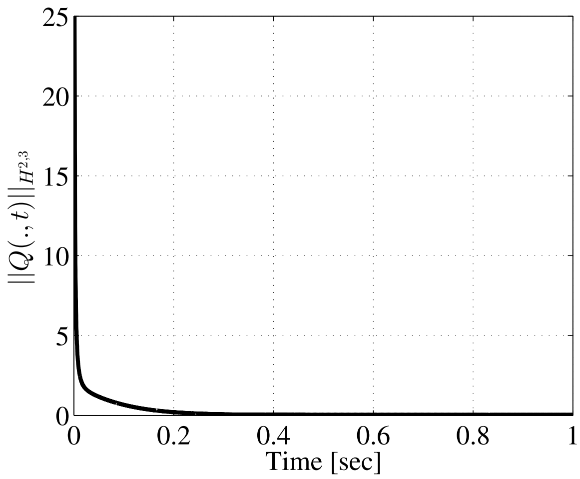

Figure 2.22: $||Q(\cdot,t)||_{H^{2,3}}$ norm in the closed-loop test with the state-feedback controller.

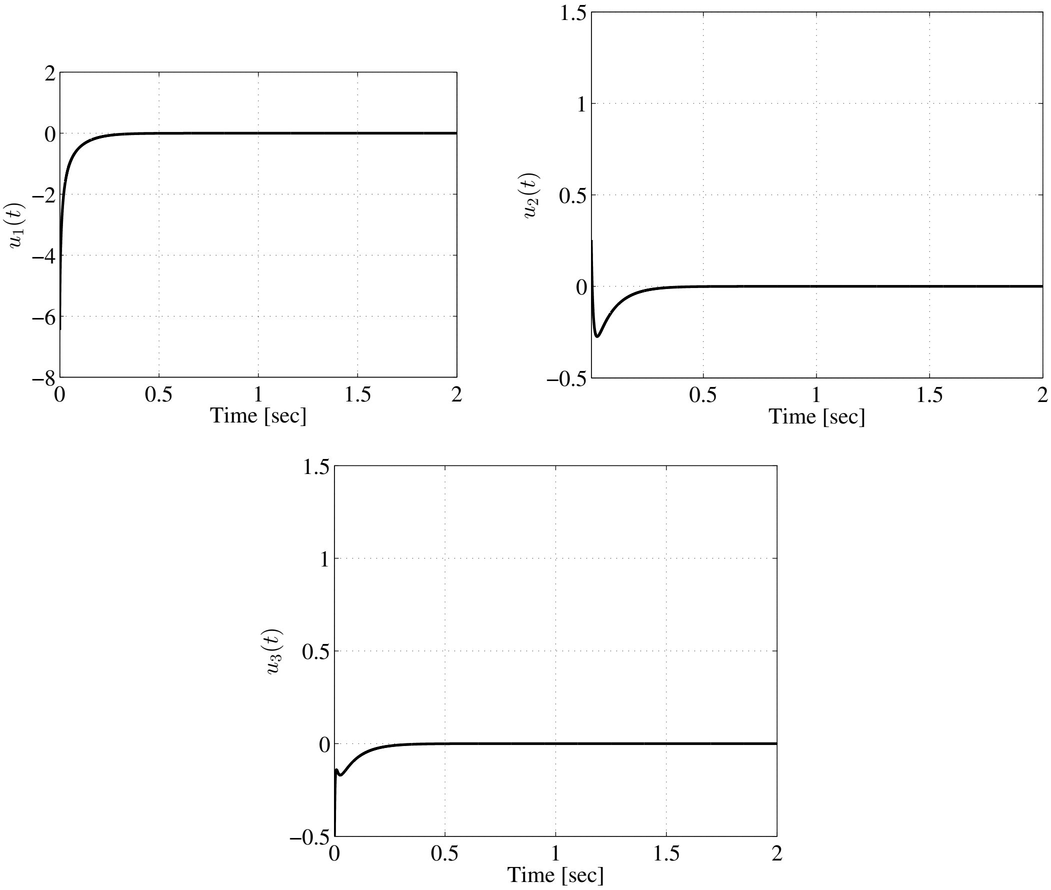

Figure 2.23: Boundary controls $u_1(t)$ (left plot), $u_2(t)$ (central plot), $u_3(t)$ (right plot) in the closed-loop test with the state-feedback controller.

Figure 2.25 shows the superimposed temporal evolutions of the state and observation error norms $\|Q(\cdot,t)\|_{H^{2,3}}$ and $\|\tilde{Q}(\cdot,t)\|_{H^{2,3}}$. The latter tends to zero exponentially, thus confirming the correct functioning of the proposed observer and thereby supporting the theoretical analysis. In addition, Figure 2.25 shows that the observer has a faster convergence than the controlled plant, according to the adopted tuning of the γ^* and $\bar{\gamma}^*$ parameters.

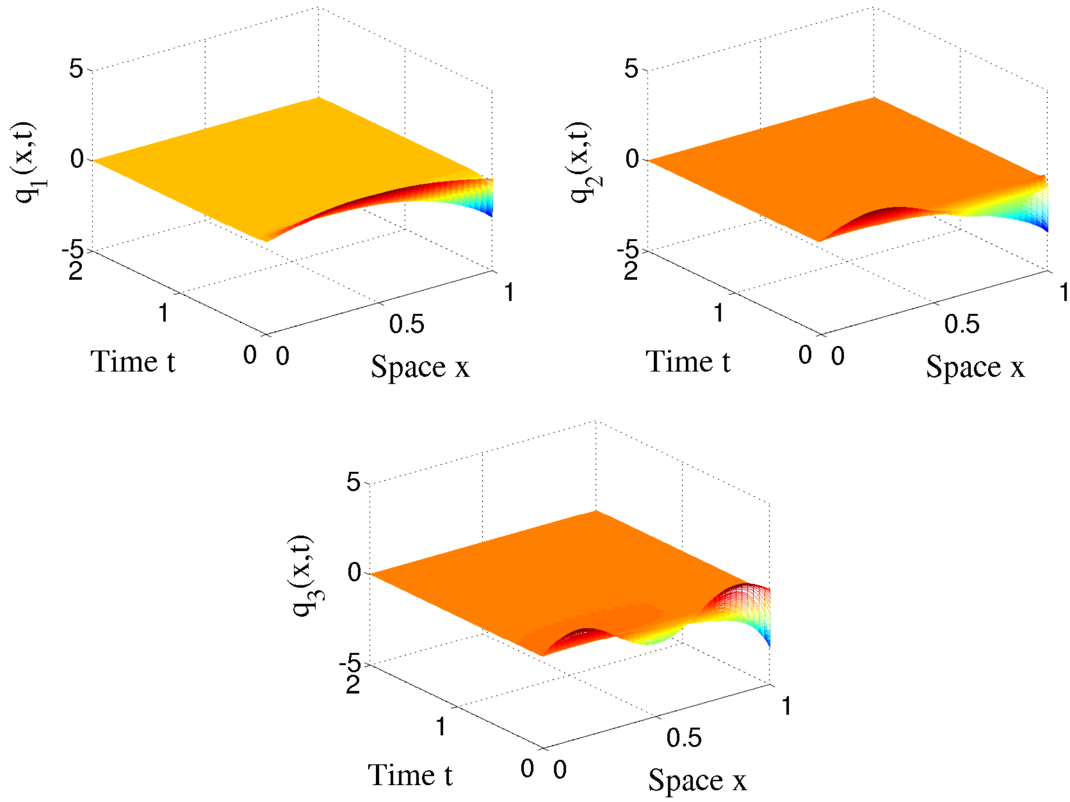

Figure 2.24: Spatiotemporal evolution of the state variables in the closed-loop test with the anti-collocated output-feedback stabilizer: $q_1(x,t)$ (left plot), $q_2(x,t)$ (central plot), $q_3(x,t)$ (right plot).

The output-feedback controller using the collocated observer (2.439)-(2.441), (2.467) has been also tested by simulation runs. The same parameters $\gamma^* = 15$ and $\bar{\gamma}^* = 25$, and observer initial conditions $\hat{q}_1(x,0) = \hat{q}_2(x,0) = \hat{q}_3(x,0) = 0$ of the previous test have been taken. Figure 2.26 displays the spatiotemporal evolution of the state variables $q_1(x,t)$, $q_2(x,t)$ and $q_3(x,t)$. Figure 2.27 depicts the temporal evolution of the norms $\|Q(\cdot,t)\|_{H^{2,3}}$ and $\|\tilde{Q}(\cdot,t)\|_{H^{2,3}}$. It is concluded from this figure that also in this case the observer has a faster convergence than the controlled plant, as expected due to the chosen values of the γ^* and $\bar{\gamma}^*$ parameters.

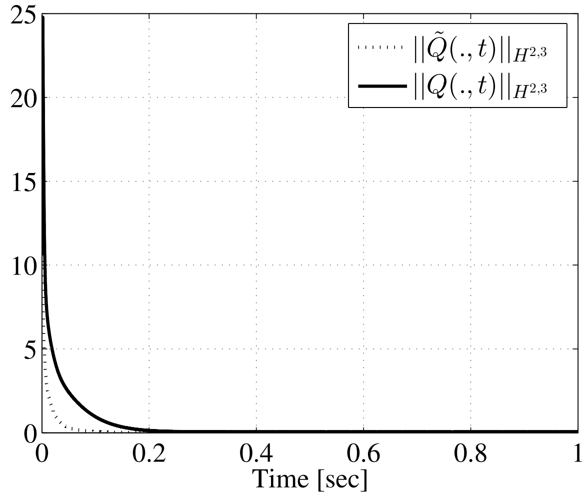

Figure 2.25: Temporal evolution of the norms $\|\tilde{Q}(\cdot,t)\|_{H^{2,3}}$ and $\|Q(\cdot,t)\|_{H^{2,3}}$ with the anti-collocated output-feedback stabilizer.

2.4.9 Conclusions

The observer-based output feedback boundary stabilization of a system of n coupled parabolic linear PDEs has been tackled by exploiting the backstepping approach, and explicit controllers and observers have been derived to enforce an arbitrarily fast exponential decay of the state in the space $H^{2,n}$.

Involving spatially and/or temporally dependent parameters into the proposed synthesis and its extension to broader classes of PDEs (e.g., coupled reaction-diffusion-advection PDEs) are among the most interesting lines of future investigations.

Additionally, integration with other design methodologies, such as the sliding-mode approach, is due to enhance the underlying robustness features. Particularly, recent investigations [21]-[25] are hoped to complement the present approach in order to control uncertain DPS' governed by perturbed coupled PDEs of parabolic type.

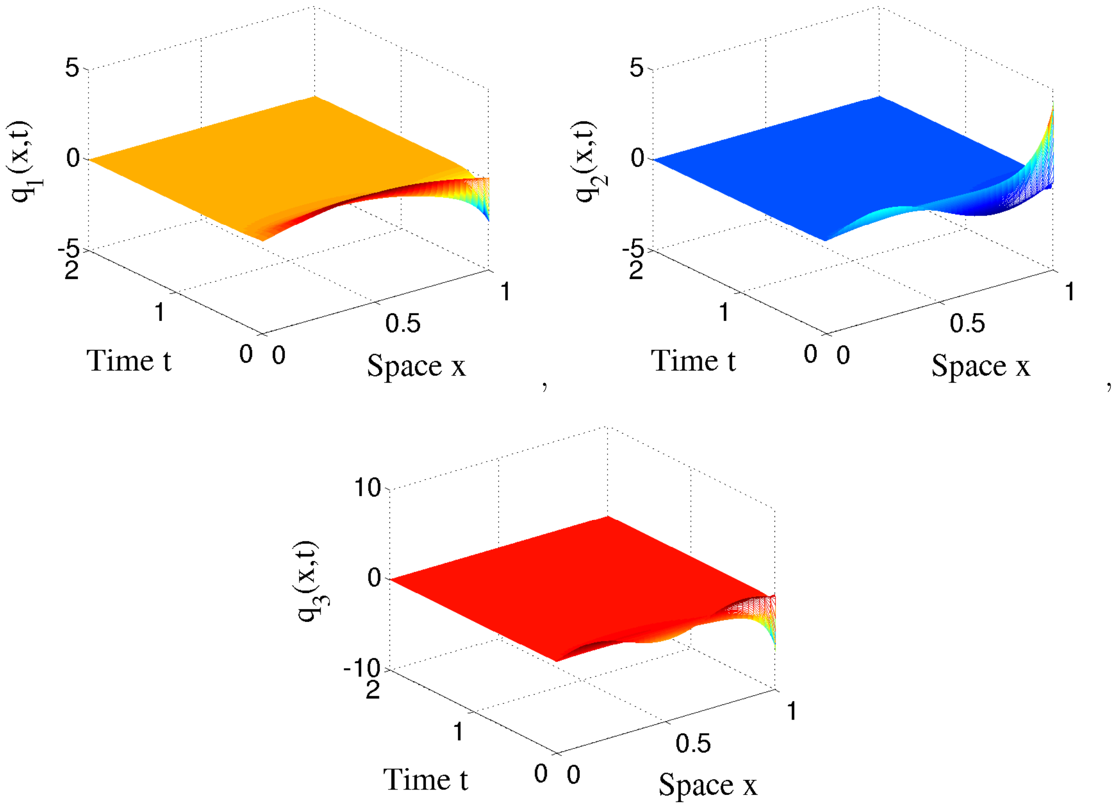

Figure 2.26: Spatiotemporal evolution of the state variables in the closed-loop test with the collocated output-feedback stabilizer: $q_1(x,t)$ (left plot), $q_2(x,t)$ (central plot), $q_3(x,t)$ (right plot).

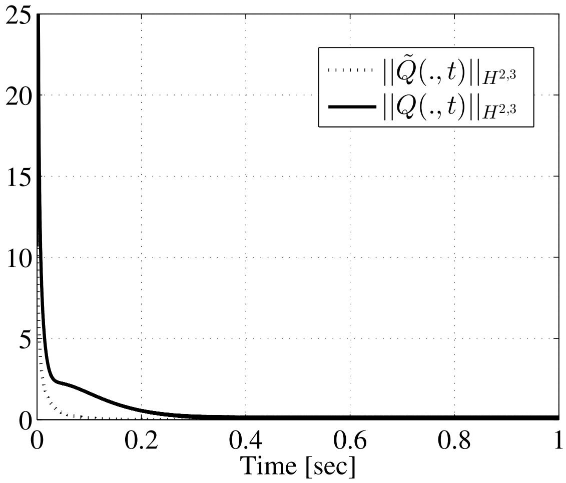

Figure 2.27: Temporal evolution of the norms $\|\tilde{Q}(\cdot,t)\|_{H^{2,3}}$ and $\|Q(\cdot,t)\|_{H^{2,3}}$ with the collocated output-feedback stabilizer.

Chapter 3

Coupled Reaction-Diffusion-Advection Equation

3.1 Boundary stabilization of coupled reaction-advection-diffusion equations having the same diffusivity parameters

We consider the problem of boundary stabilization for a system of n coupled parabolic linear PDEs. Particularly, we design a state feedback law with actuation on only one end of the domain and prove exponential stability of the closed-loop system with an arbitrarily fast convergence rate. The backstepping method is used for controller design, and the transformation kernel matrix is derived in explicit form by using the method of successive approximations to solve the corresponding PDE. Thus, the suggested control law is also made available in explicit form. Simulation results support the effectiveness of the suggested design.

3.1.1 Introduction

We investigate the boundary stabilization of a class of coupled linear parabolic Partial Differential Equations (PDEs) in a finite spatial domain $x \in [0, 1]$ by taking advantage of the so-called "backstepping" approach (see [13, 67]) which does not relies on any discretization or finite-dimensional approximation. Backstepping-based boundary controllers for several classes of reaction-diffusion processes were presented, e.g., in [50, 17, 67]. More involved, complex-valued, PDEs such as the Schrodinger equation were also dealt with by means of such an approach (see [16]). A cascade of two parabolic reaction-diffusion processes was dealt with in [28] by using a unique control input acting only at a boundary of one side.

More recently, high-dimensional systems of coupled PDEs are under investigation in the backstepping-based boundary control setting. The most intensive efforts of the recent literature appear however to be oriented towards coupled hyperbolic processes of the transport-type (see [37, 29, 8, 5, 30]).

The state feedback design in [29], which allows stabilization of 2×2 linear heterodirectional1 hyperbolic systems, was extended in [8] to a particular type of 3×3 linear systems, arising in modeling of multiphase flow, and to the quasilinear case in [30]. In [5], a 2×2 linear hyperbolic system was stabilized by a single boundary control input, with the additional feature that an unmatched disturbance, generated by a known exosystem, is rejected. In [37] a system of $n + 1$ coupled first-order hyperbolic linear PDEs with a single boundary input was studied. Some specific result concerning the backstepping based boundary stabilization of parabolic coupled PDEs has been presented in the literature. In [75] the linearized 2×2 model of thermal-fluid convection, which entails very dissimilar diffusivity parameters, has been treated by using a singular perturbations approach combined with backstepping and Fourier series expansion. In [6] the Ginzburg-Landau equations, which represent a 2×2 system with equal diffusion coefficients when the imaginary and real parts are expanded, was dealt wit, while in [33] the boundary stabilization of the linearized model of an incompressible magnetohydrodynamic flow in an infinite rectangular 3D channel, known as Hartmann flow, was attacked.

The task of the present paper is to generalize some results presented in [67], where an explicit boundary controller was developed to stabilize a scalar unstable reaction diffusion equation. Here we provide a generalization to the multidimensional case, by considering a set of n reaction diffusion processes coupled through the corresponding reaction terms. The motivation to this investigation comes from chemical processes [20] where such equations occur to describe system dynamics, e.g., coupled temperature-concentration parabolic PDEs.

As shown in the paper, this generalization is far from being trivial because the underlying backstepping-based treatment gives rise to more complex development of finding out an explicit solution and, furthermore, it turns out to be unfeasible in the general case where each process possesses its own diffusivity parameter. In this paper we therefore address the simplified case where all processes have the same diffusivity value, and we postpone the more general case for further investigations, which requires some constraint on the target system (see Remark 1).

An additional interesting feature of backstepping is that it allows an easy synergic integration with robust control paradigms such as the sliding mode control methodology (see e.g. [**?**]) to enhance the robustness features of the overall scheme by providing the capability of completely rejecting the effect of persistent matching disturbances which are not required to be generated by a known exosystem. In fact, following our recent lines of investigation [55, 56, 25, 24], it is our purpose for next research to complement the presented scheme by integrating it with suitably designed second-order sliding mode based boundary controllers in order to deal with the control of perturbed coupled PDEs.

The structure of the paper is as follows. After introducing in the next subsection some useful notation, in Section II we state the problem under investigation and we introduce the underlying backstepping transformation. In Section III the solution of the kernel PDE is tackled for both the direct and inverse transformations. In Section IV the the proposed boundary control design and main stability result of this paper are drawn. Section V presents the simulation results and Section VI gives some concluding remarks and future perspectives of this research.

Notation

The notation used throughout is fairly standard. $L_2(0,1)$ stands for the Hilbert space of square integrable scalar functions $z(\zeta)$ on $(0,1)$ and the corresponding norm

$$\|z(\cdot)\|_2 = \sqrt{\int_0^1 z^2(\zeta)d\zeta}. \tag{3.1}$$

Throughout the paper we shall also utilize the notation

$$[L_2(0,1)]^n = \underbrace{L_2(0,1) \times L_2(0,1) \times \ldots \times L_2(0,1)}_{n \ \texttt{times}}, \tag{3.2}$$

and

$$\|Z(\cdot)\|_{2,n} = \sqrt{\sum_{i=1}^{n} \|z_i(\cdot)\|_2^2} \tag{3.3}$$

for the corresponding norm of a generic vector function $Z(\zeta) = [z_1(\zeta), z_2(\zeta), ..., z_n(\zeta)] \in [L_2(0,1)]^n$. I_n denotes the identity matrix of dimension n.

3.1.2 Problem formulation and backstepping transformation

We consider a n-dimensional system of coupled reaction-advection-diffusion processes, equipped with Dirichlet-type boundary conditions, governed by the next vector-valued PDE

$$
\begin{aligned}
Q_t(x,t) &= \theta Q_{xx}(x,t) + DQ_x(x,t) + \Lambda Q(x,t) && (3.4)\\
Q(0,t) &= 0, && (3.5)\\
Q(1,t) &= U(t) && (3.6)
\end{aligned}
$$

where

$$Q(x,t) = [q_1(x,t),\ q_2(x,t), \ldots, q_n(x,t)]^T \in [L_2(0,1)]^n \tag{3.7}$$

is the vector collecting the state of all systems,

$$U(t) = [u_1(t),\ u_2(t), \ldots, u_n(t)]^T \in \Re^n \tag{3.8}$$

is the vector collecting all the manipulable boundary control signals, $\Lambda = \{\lambda_{ij}\} \in \Re^{n\times n}$ is the real-valued "reaction" matrix, $D \in \Re^{n\times n}$ is the diagonal "advection" matrix having the form $D = diag(d_i)$, with $d_i > 0$ $\forall i = 1, 2, ..., n$, and $\theta \in \Re^+$ is a positive scalar. The open-loop system (3.4)-(3.6) (with $U(t) = 0$) possesses arbitrarily many unstable eigenvalues when the matrix Λ is positive definite with sufficiently large eigenvalues.

Remark 17. *Under the restriction*

$$d_1 = d_2 = \ldots = d_n \equiv d, \tag{3.9}$$

the invertible change of variables

$$W(x,t) = Q(x,t)e^{\frac{d}{2\theta}x} \tag{3.10}$$

can be implemented which, after straightforward manipulations, analogous to those made in [13] to address the scalar case when $n = 1$, *yields the advection-free transformed system of coupled PDEs*

$$
\begin{aligned}
W_t(x,t) &= \theta W_{xx}(x,t) + \left[\Lambda - \frac{d^2}{4\theta}I_n\right] W(x,t) && (3.11)\\
W(0,t) &= 0, && (3.12)\\
W(1,t) &= U(t)e^{\frac{d}{2\theta}} && (3.13)
\end{aligned}
$$

whose stabilization can be addressed by following the procedure described in [15]. In the general case where the condition (3.9) *is not fulfilled, such an approach is not feasible and another solution has to be found, which is the main goal of the present paper.*

Here, we exploit the invertible backstepping transformation

$$Z(x,t) = Q(x,t) - \int_0^x K(x,y)Q(y,t)dy \tag{3.14}$$

where $K(x,y)$ is a $n \times n$ matrix function whose elements are denoted as $k_{ij}(x,y)$ $(i,j = 1,2,\ldots,n)$ to exponentially stabilize system (**??**)-(**??**) by transforming it into the target system

$$\begin{aligned} Z_t(x,t) &= \theta Z_{xx}(x,t) + DZ_x(x,t) - CZ(x,t) && (3.15)\\ Z(0,t) &= 0, && (3.16)\\ Z(1,t) &= 0, && (3.17) \end{aligned}$$

where $Z(x,t) = [z_1(x,t),\ z_2(x,t),\ldots,z_n(x,t)]^T \in [L_2(0,1)]^n$ is the corresponding state vector and $C = \{c_{ij}\} \in \Re^{n,n}$ is an arbitrarily chosen real-valued square matrix.

The exponential stability properties of the target system (3.15)-(3.17), whose convergence rate can be made arbitrarily fast by a suitable choice of the matrix C, are investigated later in Theorem 35.

Following the usual backstepping design, we now derive and solve the PDE governing the kernel matrix function $K(x,y)$. Spatial derivatives $Z_x(x,t)$ and $Z_{xx}(x,t)$ take the form (the Leibnitz differentiation rule is used):

$$Z_x(x,t) = Q_x(x,t) - K(x,x)Q(x,t) - \int_0^x K_x(x,y)Q(y,t)dy \tag{3.18}$$

$$\begin{aligned} Z_{xx}(x,t) &= Q_{xx}(x,t) - \left[\frac{d}{dx}K(x,x)\right]Q(x,t)\\ &- K(x,x)Q_x(x,t) - K_x(x,x)Q(x,t)\\ &- \int_0^x K_{xx}(x,y)Q(y,t)dy \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} &\tfrac{d}{dx}K(x,x) = K_x(x,x) + K_y(x,x)\\ &K_x(x,x) = \left.\tfrac{\partial K(x,y)}{\partial x}\right|_{y=x}, \quad K_y(x,x) = \left.\tfrac{\partial K(x,y)}{\partial y}\right|_{y=x}. \end{aligned} \tag{3.20}$$

Using (3.4), and applying recursively integration by parts, the time derivative $Z_t(x,t)$

is obtained in the form

$$\begin{aligned}
Z_t(x,t) = Q_t(x,t) &- \int_0^x K(x,y)Q_t(y,t)dy = \\
\theta Q_{xx}(x,t) &+ DQ_x(x,t) + \Lambda Q(x,t) \\
-K(x,x)\theta Q_x(x,t) &+ K(x,0)\theta Q_x(0,t) \\
+K_y(x,x)\theta Q(x,t) &- K_y(x,0)\theta Q(0,t) \\
-\int_0^x K_{yy}(x,y)\theta Q(y,t)dy &- K(x,x)DQ(x,t) \\
+K(x,0)DQ(0,t) &+ \int_0^x K_y(x,y)DQ(y,t)dy \\
-\int_0^x K(x,y)\Lambda Q(y,t)dy. & \qquad (3.21)
\end{aligned}$$

Combining (3.14), (3.19), (3.21) and performing lengthy but straightforward computations, yield

$$\begin{aligned}
Z_t(x,t) &- \theta Z_{xx}(x,t) - DZ_x(x,t) + CZ(x,t) \\
&= [\Lambda + C + K_y(x,x)\theta + \theta K_x(x,x) \\
&+ \left.\theta\frac{d}{dx}K(x,x) + DK(x,x) - K(x,x)D\right] Q(x,t) \\
&+ \int_0^x [\theta K_{xx}(x,y) - K_{yy}(x,y)\theta - K(x,y)\Lambda \\
&- CK(x,y) + K_y(x,y)D + DK_x(x,y)]\, Q(y,t)dy \\
&+ K(x,0)\Theta Q(0,t) = 0. \qquad (3.22)
\end{aligned}$$

Clearly, the target system's equation (3.15) implies that the right hand side of (**??**) has to be identically zero. Considering the homogeneous BC (3.5), this leads to the next relations

$$\begin{aligned}
K_{xx}(x,y) &- K_{yy}(x,y) + \frac{1}{\theta}D\left(K_x(x,y) + K_y(x,y)\right) = \\
&\frac{1}{\theta}K(x,y)\Lambda + \frac{1}{\theta}CK(x,y) \qquad (3.23) \\
2\theta\frac{d}{dx}K(x,x) &+ DK(x,x) - K(x,x)D = -\left(\Lambda + C\right) \qquad (3.24) \\
K(x,0) &= 0 \qquad (3.25)
\end{aligned}$$

Relation (3.24) is rewritten in expanded form as

$$\begin{aligned}
2\theta\frac{d}{dx}k_{ii}(x,x) &= -(\lambda_{ii} + c_{ii}), \quad i = 1,2,...,n \qquad (3.26) \\
2\theta\frac{d}{dx}k_{ij}(x,x) &+ (d_i - d_j)k_{ij}(x,x) = -(\lambda_{ij} + c_{ij}) \\
&i \neq j, i,j = 1,2,...,n \qquad (3.27)
\end{aligned}$$

By virtue of relations $k_{ij}(0,0) = 0$, which derive from (3.25), the solutions of (3.26), (3.27) can be straightforwardly derived as

$$k_{ii}(x,x) = -\frac{(\lambda_{ii} + c_{ii})}{2\theta}x, \quad i = 1, 2, ..., n \tag{3.28}$$

and

$$k_{ij}(x,x) \;=\; \frac{\lambda_{ij} + c_{ij}}{d_i - d_j}\left(e^{-\frac{d_i - d_j}{2\theta}x} - 1\right), \tag{3.29}$$

respectively. Rewriting (3.24) taking into account (3.28) and (3.29) yields the modified form

$$\begin{aligned} K_{xx}(x,y) \;&-\; K_{yy}(x,y) + \frac{1}{\theta}D\left(K_x(x,y) + K_y(x,y)\right) = \\ &\frac{1}{\theta}K(x,y)\Lambda + \frac{1}{\theta}CK(x,y) \end{aligned} \tag{3.30}$$

$$K(x,x) \;=\; \begin{cases} -\frac{(\lambda_{ii}+c_{ii})}{2\theta}x, & if \quad i = j \\ \frac{\lambda_{ij}+c_{ij}}{d_i - d_j}\left(e^{-\frac{d_i - d_j}{2\theta}x} - 1\right), & if \quad i \neq j \end{cases} \tag{3.31}$$

$$K(x,0) \;=\; 0 \tag{3.32}$$

of (3.23)-(3.25). It should be pointed out that the BC (3.31) are of completely different form as compared to that obtained when $n = 1$. We will show that, in spite of this difference from the scalar case, system (3.30)-(3.32) is well posed and admitting an explicit solution in series form.

Remark 18. *We are confining the present paper to the case in which all the coupled PDEs* (3.4) *have the same diffusivity parameter* θ*. The reason is that in the more general case where each process has its own diffusivity* θ_i*,* $(i = 1, 2, ..., n)$*, the corresponding "generalized" version*

$$\begin{aligned} \Theta K_{xx}(x,y) \;&-\; K_{yy}(x,y)\Theta + K_y(x,y)D + DK_x(x,y) \\ &=\; K(x,y)\Lambda + CK(x,y) \end{aligned} \tag{3.33}$$

$$\begin{aligned} \Lambda + C \;&+\; K_y(x,x)\Theta + \Theta K_x(x,x) + \Theta\frac{d}{dx}K(x,x) \\ &+\; DK(x,x) - K(x,x)D = 0 \end{aligned} \tag{3.34}$$

$$\Theta K(x,x) \;-\; K(x,x)\Theta = 0 \tag{3.35}$$

$$K(x,0) \;=\; 0. \tag{3.36}$$

of (3.23)-(3.25)*, where* $\Theta = diag(\theta_i)$*, sets an overdetermined PDE without solution, unless specific constraints on the matrix* C *and on the form of the kernel matrix* $K(x,y)$ *are met. This topic will be addressed in more detail in next works.*

3.1.3 Solution of the kernel PDE (3.23)-(3.25)

The following result is in order.

Theorem 34. *The problem* (3.30)-(3.32) *admits a unique solution that is twice continuously differentiable in* $0 \le y \le x \le 1$.

Proof. Following [67], the existence of a solution to problem (3.30)-(3.32) can be proved by transforming it into an integral equation using the variable change

$$\xi = x + y, \qquad \eta = x - y. \tag{3.37}$$

Denoting

$$G(\xi, \eta) = K(x, y) = K\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2}\right) \tag{3.38}$$

and exploiting the trivial relations

$$\begin{aligned} K_x &= G_\xi + G_\eta, & (3.39)\\ K_{xx} &= G_{\xi\xi} + 2G_{\xi\eta} + G_{\eta\eta}, & (3.40)\\ K_y &= G_\xi - G_\eta, & (3.41)\\ K_{yy} &= G_{\xi\xi} - 2G_{\xi\eta} + G_{\eta\eta}, & (3.42)\end{aligned}$$

the gain kernel PDE in the new coordinates becomes

$$\begin{aligned} G_{\xi\eta}(\xi, \eta) &+ \frac{1}{2\theta} D G_\xi(\xi, \eta) = \frac{1}{4\theta} G(\xi, \eta)\Lambda + \frac{1}{4\theta} C G(\xi, \eta) & (3.43)\\ G(\xi, 0) &= \begin{cases} -\frac{(\lambda_{ii} + c_{ii})}{4\theta}\xi, & if \quad i = j \\ \frac{\lambda_{ij} + c_{ij}}{d_i - d_j}\left(e^{-\frac{d_i - d_j}{4\theta}\xi} - 1\right) & if \quad i \ne j \end{cases} & (3.44)\\ G(\xi, \xi) &= G(\eta, \eta) = 0; & (3.45)\end{aligned}$$

Integrating (3.43) with respect to ξ from η to ξ, and considering the relation $G(\eta, \eta) = G_\eta(\eta, \eta) = 0$, which follows from (3.45), we get

$$G_\eta(\xi, \eta) = \frac{1}{4\theta}\int_\eta^\xi \left[G(\tau, \eta)\Lambda + CG(\tau, \eta)\right] d\tau - \frac{1}{2\theta} D G(\xi, \eta). \tag{3.46}$$

Integrating (3.46) with respect to η from 0 to η yields

$$\int_0^\eta G_s(\xi, s) ds = -\frac{1}{2\theta}\int_0^\eta DG(\xi, s) ds + \frac{1}{4\theta}\int_0^\eta \int_\eta^\xi \left[G(\tau, s)\Lambda + CG(\tau, s)\right] d\tau ds \tag{3.47}$$

which can further be manipulated to

$$G(\xi, \eta) = G(\xi, 0) - \frac{1}{2\theta}\int_0^\eta DG(\xi, s) ds + \frac{1}{4\theta}\int_0^\eta \int_\eta^\xi \left[G(\tau, s)\Lambda + CG(\tau, s)\right] d\tau ds \tag{3.48}$$

Now we apply the method of successive approximations to show that equation (3.48) has a continuous and smooth solution. Let us start with an initial approximation

$$G^0(\xi,\eta) \quad = \quad 0 \tag{3.49}$$

and set-up the recursive formula for (3.48) as follows

$$G^{n+1}(\xi,\eta) = G(\xi,0) - \frac{1}{2\theta}\int_0^\eta DG^n(\xi,s)ds + \frac{1}{4\theta}\int_0^\eta\int_\eta^\xi \left[G^n(\tau,s)\Lambda + CG^n(\tau,s)\right]d\tau ds \tag{3.50}$$

Provided that this recursion converges, we can write the solution $G(\xi,\eta)$ as

$$G(\xi,\eta) = \lim_{n\to\infty} G^n(\xi,\eta). \tag{3.51}$$

Let

$$\Delta G^n(\xi,\eta) \quad = \quad G^{n+1}(\xi,\eta) - G^n(\xi,\eta) \tag{3.52}$$

be standing for the difference between two consecutive terms. Solution (3.51) is then alternatively rewritten in the series form

$$G(\xi,\eta) = \sum_{n=0}^{\infty} \Delta G^n(\xi,\eta). \tag{3.53}$$

and by (3.49) and (3.50), the next recursion holds

$$\Delta G^0(\xi,\eta) = G^1(\xi,\eta) = G(\xi,0) \tag{3.54}$$

$$\Delta G^{n+1}(\xi,\eta) = -\frac{1}{2\theta}\int_0^\eta D\Delta G^n(\xi,s)ds +$$

$$\frac{1}{4\theta}\int_0^\eta\int_\eta^\xi \left[\Delta G^n(\tau,s)\Lambda + C\Delta G^n(\tau,s)\right]d\tau ds \tag{3.55}$$

Denote

$$M_1 \quad = \quad \sup_{\xi\in[0,2],i=1,2,\ldots n} \frac{|(\lambda_{ii}+c_{ii})|}{4\theta}\xi \tag{3.56}$$

$$M_2 \quad = \quad \sup_{\xi\in[0,2],i,j=1,2,\ldots n} \left|\frac{\lambda_{ij}+c_{ij}}{d_i-d_j}\left(e^{-\frac{d_i-d_j}{4\theta}\xi}-1\right)\right| \tag{3.57}$$

$$M \quad = \quad M_1 + M_2 + \frac{1}{2\theta}\left(\|\Lambda\| + \|C\| + \|D\|\right) \tag{3.58}$$

Thus, the estimate

$$\|\Delta G^0(\xi,\eta)\| \ \le M \tag{3.59}$$

is clearly in force by virtue of (3.44) and (3.54)

In order to apply the mathematical induction method suppose that

$$\| \Delta G^n(\xi,\eta)\| \ \le M^{n+1}\frac{(\xi+\eta)^n}{n!}. \tag{3.60}$$

Then, by employing (3.55), (3.59) and (3.60) one obtains

$$\begin{aligned}
&\| \Delta G^{n+1}(\xi,\eta)\| \ \le \\
&\frac{1}{4\theta}(\|\Lambda\|+\|C\|)\frac{M^{n+1}}{n!}\left|2\int_0^\eta\int_\eta^\xi(\tau+s)^n d\tau ds\right| + \\
&\frac{1}{2\theta}\| \ D\| \ \frac{M^{n+1}}{n!}\left|\int_0^\eta(\xi+s)^n ds\right| \\
&\le \frac{M^{n+1}}{n!}\left(\frac{1}{4\theta}(\|\Lambda\|+\|C\|)\left|2\int_0^\eta\int_\eta^\xi(\tau+s)^n d\tau ds\right| + \right. \\
&\left.\frac{1}{2\theta}\| \ D\| \ \left|\int_0^\eta(\xi+s)^n ds\right|\right)
\end{aligned} \tag{3.61}$$

It is readily shown (cf. [**?**], eq. (2.14)) that the next estimates

$$\left|\int_0^\eta\int_\eta^\xi(\tau+s)^n d\tau ds\right| \ \le \ \left|\frac{(\xi+\eta)^{n+1}}{n+1}-\frac{(2\eta)^{n+1}}{n+1}\right| \le 2\frac{(\xi+\eta)^{n+1}}{(n+1)} \tag{3.62}$$

$$\left|\int_0^\eta(\xi+s)^n ds\right| \ = \ \frac{(\xi+\eta)^{n+1}}{(n+1)} \tag{3.63}$$

hold. Therefore, combining (3.61) and (3.62)-(3.63) one obtains the chain of inequalities

$$\begin{aligned}
\| \ \Delta G^{n+1}(\xi,\eta)\| \ &\le \ \frac{1}{2\theta}\left(\|\Lambda\|+\|C\|+\|D\|\right)\times \\
\frac{M^{n+1}}{n!}\frac{(\xi+\eta)^{n+1}}{(n+1)} \ &\le \ M^{n+2}\frac{(\xi+\eta)^{n+1}}{(n+1)!}.
\end{aligned} \tag{3.64}$$

By mathematical induction, relation (3.64) holds for all $n>0$. It then follows from the Weierstrass M-test that the series (3.53) converges absolutely and uniformly in $0\le\eta\le\xi\le 2$. Thus, $G(\xi,\eta)$ in (3.53)-(3.55) is a continuous solution of Equation (3.48), and hence is at least twice continuously differentiable in $0\le\eta\le\xi\le 2$ and $|G(\xi,\eta)|\le Mexp(M(\xi+\eta))$. To show that such a solution is unique, we follow the procedure suggested in [**?**, Lemma 2.1]. It suffices to show that the equation

$$G(\xi,\eta) = -\frac{1}{2\theta}\int_0^\eta DG(\xi,s)ds + \frac{1}{4\theta}\int_0^\eta\int_\eta^\xi\left[G(\tau,s)\Lambda + CG(\tau,s)\right]d\tau ds \tag{3.65}$$

has the zero solution only. Define the mapping $F_0:\Omega_0\to\Omega_0$, $\Omega_0=\{G : G(\xi,\eta)$ is continuous in $0\le\eta\le\xi\le 2\}$

$$\begin{aligned}
(F_0G)(\xi,\eta) = -\frac{1}{2\theta}\int_0^\eta DG(\xi,s)ds + \\
\frac{1}{4\theta}\int_0^\eta\int_\eta^\xi\left[G(\tau,s)\Lambda + CG(\tau,s)\right]d\tau ds, \ \forall G\in\Omega_0
\end{aligned} \tag{3.66}$$

Then F_0 is a compact operator on Ω_0. By (3.64), the spectral radius of F_0 is zero. So 0 is the unique spectrum of F_0. Therefore, (3.65) has zero solution only. The proof is complete.

□

Inverse transformation

In order to prove stability we need to show that the transformation (3.14) is invertible. Let us write the inverse transformation in the form

$$Q(x,t) = Z(x,t) + \int_0^x L(x,y)Z(y,t)dy \tag{3.67}$$

By performing analogous developments as those made for the derivation of the gain kernel PDE (3.23)-(3.25), we obtain the next PDE governing $L(x,y)$

$$\begin{aligned} L_{xx}(x,y) - L_{yy}(x,y) &- \frac{1}{\theta}D\left(L_x(x,y) + L_y(x,y)\right) = \\ -\frac{1}{\theta}L(x,y)C &- \frac{1}{\theta}\Lambda L(x,y) \end{aligned} \tag{3.68}$$

$$\begin{aligned} 2\theta\frac{d}{dx}L(x,x) &+ DL(x,y) - L(x,y)D = \\ &- (\Lambda + C) \end{aligned} \tag{3.69}$$

$$L(x,0) = 0 \tag{3.70}$$

By direct comparison between (3.23)-(3.25) and (3.68)-(3.70) one immediately notice that in this case $L(x,y) = -K(x,y)$ when Λ, C and D are replaced by $-\Lambda$, $-C$ and $-D$.

3.1.4 Main result

We begin by stating a preliminary result establishing the stability features of the target dynamics (3.15)-(3.17). The following result is in force.

Theorem 35. *Consider the target system* (3.15)-(3.17)*. If the matrix C is such that its symmetric part $C_s = (C + C^T)/2$ is positive definite then system* (3.15)-(3.17) *is exponentially stable in the space $[L_2(0,1)]^n$ with the convergence rate specified by*

$$\|Z(\cdot,t)\|_{2,n} \leq \|Z(\cdot,0)\|_{2,n}e^{-\sigma_1(C_s)t} \tag{3.71}$$

where $\sigma_1(C_s)$ is the smallest eigenvalue of C_s.

Proof. Consider the Lyapunov function

$$V(t) = \frac{1}{2}\int_0^1 Z^T(\xi,t)Z(\xi,t)d\xi = \frac{1}{2}\|Z(\cdot,t)\|^2_{2,n}. \tag{3.72}$$

The corresponding time derivative along the solutions of (3.15)-(3.17) is given by

$$\begin{aligned}\dot{V}(t) &= \int_0^1 Z^T(\xi,t)\theta Z_{xx}(\xi,t)d\xi + \\ &\int_0^1 Z^T(\xi,t)DZ_x(\xi,t)d\xi - \\ &\int_0^1 Z^T(\xi,t)CZ(\xi,t)d\xi\end{aligned} \tag{3.73}$$

Integration by parts the first integral in (3.73) taking into account (3.16) and (3.17) yield

$$\int_0^1 Z^T(\xi,t)\theta Z_{xx}(\xi,t)d\xi = \theta\; Z^T(\chi,t)Z_x(\chi,t)\big|_{\chi=0}^{\chi=1}$$
$$-\theta\int_0^1 Z_x^T(\xi,t)Z_x(\xi,t)d\xi \le -\theta\|Z_x(\cdot,t)\|_{2,n}^2 \tag{3.74}$$

Integrating by parts the second integral in (3.73) taking into account (3.16) and (3.17) yield

$$\int_0^1 Z^T(\xi,t)DZ_x(\xi,t)d\xi = \frac{1}{2}Z(1,t)^T DZ(1,t) - \frac{1}{2}Z(0,t)^T DZ(0,t) = 0 \tag{3.75}$$

Since the smallest eigenvalue $\sigma_1(C_s)$ of the symmetric matrix $C_s = (C + C^T)/2$ is assumed to be positive then exploiting the trivial inequality $Z^T(\xi,t)CZ(\xi,t) \ge \sigma_1(C_s)^T Z(\xi,t)Z(\xi,t)$
and employing (3.74),(3.75), one can easily manipulate (3.73) to derive

$$\begin{aligned}\dot{V}(t) &\le -\theta\|Z_\xi(\cdot,t)\|_{2,n}^2 - 2\sigma_1(C_s)V(t) \\ &\le -2\sigma_1(C_s)V(t)\end{aligned} \tag{3.76}$$

thereby concluding the exponential stability of the target system in the space $[L_2(0,1)]^n$ with a convergence rate, obeying the estimate (3.71). Theorem 35 is proved. □

The next Theorem specifies the proposed boundary control design and summarizes the main stability result of this paper.

Theorem 36. *The boundary control input*

$$U(t) = \int_0^1 K(1,y)Q(y,t)dy \tag{3.77}$$

where matrix C is selected such that its symmetric part $C_s = (C + C^T)/2$ is positive definite, stabilizes exponentially system (3.4)-(3.6) in the space $[L_2(0,1)]^n$ with an arbitrarily fast convergence rate in accordance with

$$\|Q(\cdot,t)\|_{2,n} \le A\|Q(\cdot,0)\|_{2,n}e^{-\sigma_1(C_s)t} \tag{3.78}$$

where $\sigma_1(C_s)$ is the smallest eigenvalues of matrix C_s and A is a positive constant independent of $Q(\xi,0)$.

Proof. The developments of Section 1, along with Theorem 1, show that the backstepping transformation (3.14) map system (3.4) -(3.6) into the target dynamics in which the PDE (3.15) holds. From (3.14) it follows that

$$Z(0,t) = Q(0,t) \tag{3.79}$$

$$Z(1,t) = Q(1,t) - \int_0^1 K(1,y)Q(y,t)dy \tag{3.80}$$

Considering the boundary conditions (3.5) and (3.6), one has that

$$Z(0,t) = 0 \tag{3.81}$$

$$Z(1,t) = U(t) - \int_0^1 K(1,y)Q(y,t)dy \tag{3.82}$$

Thus, with the boundary control input vector selected as in (3.77), where the form of the kernel $K(1,y)$ is obtained by numerically solve the (3.23) - (3.25), the target dynamics (3.15)-(3.17) with homogeneous BCs is obtained, whose asymptotic stability features were demonstrated in Theorem 35. In particular, according to (3.71), the corresponding convergence rate can be made arbitrarily fast by a proper selection of the C matrix.

From now on, we follow [67] to derive analogous convergence properties for the original system (3.4)-(3.6) as well. Observing that $\xi + \eta = x$, it is easy to derive from (3.53)-(3.60) that $\|K(x,y)\| \leq Me^{2Mx}$, and the same bound can be derived for the norm of $L(x,y)$ as well, i.e. $\|L(x,y)\| \leq Me^{2Mx}$. A straightforward generalization of [67, Th 4] yields that those two boundedness relations, coupled together, establish the equivalence of norms of $Z(x,t)$ and $Q(x,t)$ in $[L_2(0,1)]^n$ which means that there exist a positive constant A independent of $Q(\xi,0)$ such that the estimate (3.78) is in force as a direct consequence of (3.71). Theorem 36 is proven. □

3.1.5 Simulation results

To validate the proposed boundary control scheme, an instance of system (3.4)-(3.6) with $n = 2$ coupled reaction-advection-diffusion processes has been considered for simulation purposes, with parameters

$$\theta = 1, \qquad \Lambda = \begin{bmatrix} -5 & 20 \\ 20 & -5 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \tag{3.83}$$

The initial conditions are set as $q_1(x,0) = q_2(x,0) = 10cos(\pi x)$. For solving the closed-loop PDE, a standard finite-difference approximation method is used by discretizing the spatial solution domain $x \in [0,1]$ into a finite number of N uniformly spaced solution nodes $x_i = ih$, $h = 1/(N+1)$, $i = 1, 2, ..., N$. The value $N = 40$ has been used. The resulting 40-th order discretized system is then solved by fixed-step Euler method with step $T_s = 10^{-4}$

The open-loop unstable behaviour of the uncontrolled plant (i.e., with $U(t) = [0,0]^T$) is displayed in the Figures 3.1 and 3.2, which show the diverging spatiotemporal evolution of the states $q_1(x,t)$ and $q_2(t)$.

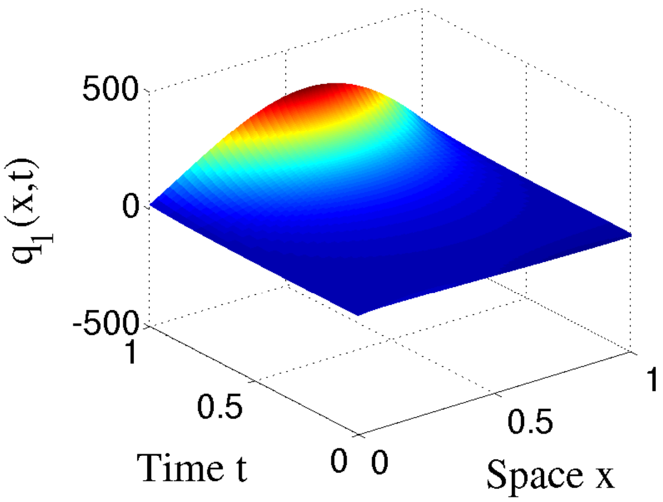

Figure 3.1: Spatiotemporal evolution of $q_1(x,t)$ in the open loop.

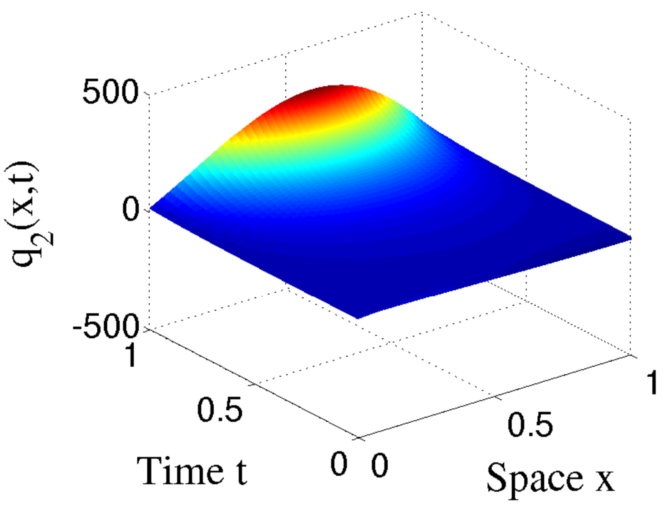

Figure 3.2: Spatiotemporal evolution of $q_2(x,t)$ in the open loop.

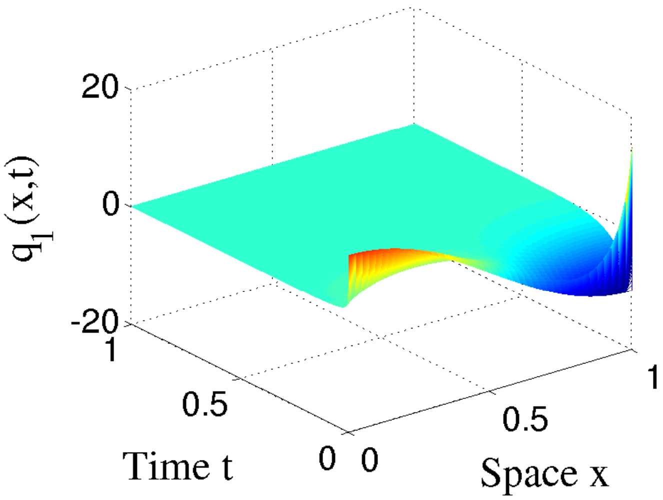

Figure 3.3: Spatiotemporal evolution of $q_1(x,t)$ in the closed loop.

The boundary controller (3.77) has been implemented by selecting the next matrix

$$C = \begin{bmatrix} 15 & 15 \\ 15 & 15 \end{bmatrix} \tag{3.84}$$

which gives the target system desired exponential stability properties. Figures 3.3 and 3.4 show the stable spatiotemporal evolutions of the state variables $q_1(x,t)$ and $q_2(t)$, which both vanishes in L_2 norm as shown in the Figure 3.5. The initial and long-term evolutions of the boundary control inputs $u_1(t)$ and $u_2(t)$ are displayed in the Figure 3.6.

3.1.6 Conclusions

The backstepping based boundary stabilization of a system of n coupled parabolic linear PDEs has been tackled, and a state feedback controller has been derived which allows to enforce an arbitrarily fast exponential decay of the state in the space $[L_2(0,1)]^n$. The extension to the case of different diffusivity parameters, the observer-based output-fedback design, and considering spatially-dependent parameters, are among the most interesting lines of future related investigations. Additionally, integration with other design methodologies such as the (second-order) sliding mode approach, will be pursued as well to enhance the underlying robustness features.

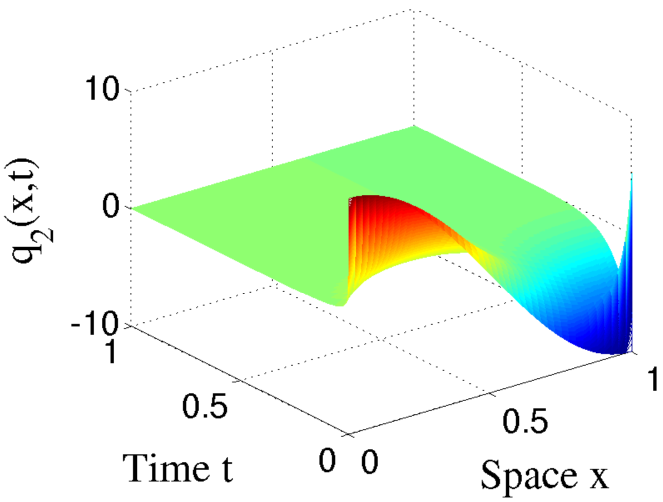

Figure 3.4: Spatiotemporal evolution of $q_2(x, t)$ in the closed loop.

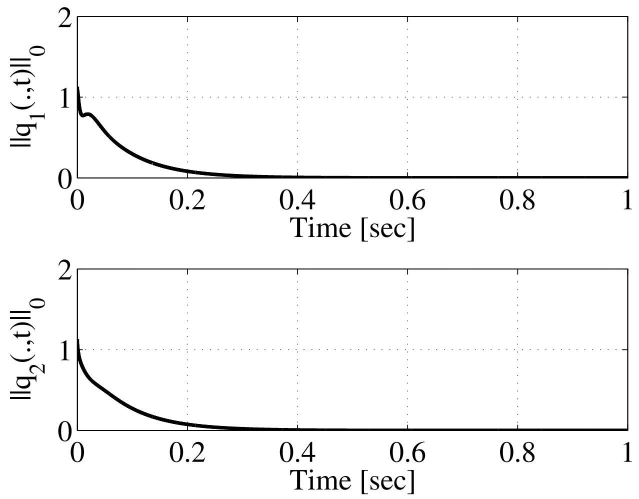

Figure 3.5: L_2 norms $\|q_1(\cdot, t)\|_0$ and $\|q_2(\cdot, t)\|_0$ in the closed loop test.

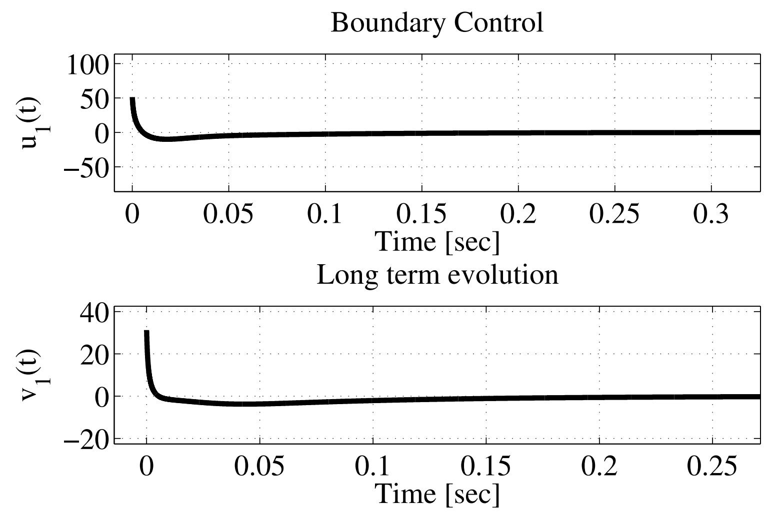

Figure 3.6: Time evolution of the boundary control inputs $u_1(t)$ and $v_1(t)$ 'in the closed loop test.

Chapter 4

Conclusion

In this Thesis, the backstepping based boundary stabilization, state observation, and output feedback boundary control problems are addressed and solved with reference to a class of n coupled parabolic linear reaction-diffusion PDEs. These achievements generalize previous results by Krstic and coworkers that only considered scalar equations. Preliminary results concerning the boundary stabilization of coupled reaction-diffusion-advection equations are also provided, and finally a synergic combination between the backstepping and the sliding mode control methodologies has been devised to provide for the rejection of persistent boundary disturbances in an open-loop unstable scalar reaction-diffusion PDE.

Hereinafter a summary of the main part of the Thesis along with some comments and hints about the potential future research directions for the contribution of each Chapter are discussed below:

• **In Chapter II.1** a class of unstable and perturbed reaction-diffusion processes has been stabilized in the space $L_2(0,1)$ by means of a synergic combination between the infinite-dimensional backstepping and the sliding mode control methodologies. An interesting topic for future generalization of the present result is that of widening the controlled class of systems by including the convection term and by covering the case of spatially and/or temporally varying parameters. Other research directions will be explored as well, namely the use of second-order sliding mode control to alleviate chattering, and the design of robust observers for reconstructing the state $z(\xi,t)$ using boundary sensing.

Also the combined twisting/PD algorithm has been shown to be capable of regulating uncertain and perturbed wave and reaction-diffusion processes. As far as the second-order sliding mode boundary control techniques are concerned, these are the unique contributions currently available in the field. We do not only recall the available results and methods but we do also provide some novelty in the reaction-diffusion process part.

• **In Chapter II.2** the backstepping-based boundary stabilization of certain classes of unstable coupled parabolic linear PDEs was tackled, and explicit state feedback boundary controllers were derived to attain the exponential decay of the closed-loop system in the state space $[L_2(0,1)]^n$. These results provide a non trivial multidimensional counterpart to the "scalar" ($n=1$) treatment previously developed in [67]. It is also of interest to deepen the present investigation on the underactuated case where only one scalar manipulable input variable is available, by generalizing the 2-dimensional problem statement, studied in the present work, towards higher dimensional scenarios. Additionally, integration with other design methodologies such as the sliding mode approaches, will be pursued as well to enhance the underlying robustness features. Particularly, recent investigations of [55]-[25] are hoped to complement the presented approaches by integrating them with suitably designed second-order sliding mode based boundary controllers in order to deal with the control of perturbed coupled PDEs.

The backstepping-based anti-collocated observer design of a system of n coupled parabolic linear PDEs has been tackled, and an explicit representation of the underlying observer gains has been derived which allows one to enforce an arbitrarily fast exponential decay of the observation error dynamics in the space $[L_2(0,1)]^n$.

Finally the observer-based output feedback boundary stabilization of a system of n coupled parabolic linear PDEs has been tackled by exploiting the backstepping approach, and explicit controllers and observers have been derived to enforce an arbitrarily fast exponential decay of the state in the space $H^{2,n}$. Involving spatially and/or temporally dependent parameters into the proposed synthesis and its extension to broader classes of PDEs (e.g., coupled reaction-diffusion-advection PDEs) are among the most interesting lines of future investigations.

• **In Chapter II.3** the backstepping based boundary stabilization of a system of n coupled parabolic linear PDEs has been tackled, and a state feedback controller has been derived which allows to enforce an arbitrarily fast exponential decay of the state in the space $[L_2(0,1)]^n$. The extension to the case of different diffusivity parameters, the observer-based output-fedback design, and considering spatially-dependent parameters, are among the most interesting lines of future related investigations.

Bibliography

[1] A. N. Tikhonov and A. A. Samarskii "Equations of Mathematical Physics. New York: E. Mellen, 2000.

[2] T. I. Seidman, "Two results on exact boundary control of parabolic equations" Appl. Math. Optim., vol. 11, pp. 145–152, 1984.

[3] D. Colton, "The solution of initial-boundary value problems for parabolic equations by the method of integral operators" J. Diff. Equat.,vol. 26, pp. 181–190, 1977.

[4] W. J. Liu, "Boundary feedback stabilization of an unstable heat equation" SIAM J. Control Optim., vol. 42, no. 3, pp. 1033–1043.

[5] O.M. Aamo "Disturbance Rejection in 2 × 2 Linear Hyperbolic Systems" IEEE Trans. Aut. Contr., 58(2013), 5, 1095-1106.

[6] O.M. Aamo, A. Smyshlyaev, and M. Krstic, "Boundary Control Of The Linearized Ginzburg-Landau Model Of Vortex Shedding" SIAM J. Contr. Optimizat., 43(2005), 6, 1953-1971.

[7] A. Baccoli, Y. Orlov, A. Pisano, "On the boundary control of coupled reaction-diffusion equations having the same diffusivity parameters". Proc. 2014 CDC, Los Angeles (US).

[8] F. Di Meglio, R. Vazquez, M. Krstic and N. Petit, "Backstepping stabilization of an underactuated 3 × 3 linear hyperbolic system of fluid flow equations". Proc. 2012 ACC, Montreal, Canada.

[9] F. Di Meglio, R. Vazquez and M. Krstic, "Stabilization of a System of Coupled First-Order Hyperbolic Linear PDEs With a Single Boundary Input" IEEE Trans. Aut. Contr., 58(12)(2013), 3097-3111.

[10] B.-Z. Guo, F.-F. Jin"Sliding mode control and active disturbance rejection control to the stabilization of one-dimensional Schrödinger equation subject to boundary control matched disturbance" Int. J. Rob. Nonlin. Contr., (2013), in press DOI: 10.1002/rnc.2977.

[11] L. Jadachowski, T. Meurer, and A. Kugi, "Backstepping observers for linear PDEs on higher-dimensional spatial domains" Automatica., 51(2015), 85-97.

[12] A. Smyshlyaev and M. Krstic, “Adaptive Control of Parabolic PDEs” Princeton University Press, 2010, ISBN 978-0691142869.

[13] M. Krstic and A. Smyshlyaev, “Boundary Control of PDEs: A Course on Backstepping Designs” SIAM Advances in Design and Control Series, 2008, ISBN 978-0-89871-650-4.

[14] M. Krstic, “Adaptive control of an anti-stable wave PDE” Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis., 17(2010), 853-882.

[15] A. Baccoli, Y. Orlov, and A. Pisano, “Boundary control of coupled reaction-diffusion processes with constant parameters” Automatica, 54(2015), 80-90.

[16] M. Krstic, B.-Z. Guo, and A. Smyshlyaev, ”Boundary controllers and observers for the linearized Schrodinger equation” SIAM Journal of Control and Optimization, 49(2011), 1479-1497.

[17] W. Liu, “Boundary Feedback Stabilization Of An Unstable Heat Equation” SIAM J. Contr. Opt., 42(2003), 3, 1033-1043.

[18] T. Meurer, and A. Kugi “Tracking control for boundary controlled parabolic PDEs with varying parameters: combining backstepping with flatness.” Automatica, 45(2009), 5, 1182-1194.

[19] T. Meurer, “Control of Higher-Dimensional PDEs: Flatness and Backstepping Designs.” Communications and Control Engineering Series, Springer-Verlag, 2012.

[20] Y. Orlov, D. Dochain “Discontinuous feedback stabilization of minimum-phase semilinear infinite-dimensional systems with application to chemical tubular reactor”. IEEE Transactions on Automatic Control, 47(2002), 8, 1293-1304.

[21] Y. Orlov, A. Pisano and E. Usai, ”Continuous state-feedback tracking of an uncertain heat diffusion process”. Systems and Control Letters, 59(2010), 754-759.

[22] Y. Orlov, A. Pisano and E. Usai, ”Tracking Control of the Uncertain Heat and Wave Equation via Power-Fractional and Sliding-Mode Techniques” SIAM J. Contr. Opt., 49(2011), 363-382.

[23] Y. Orlov, A. Pisano and E. Usai, ”Exponential stabilization of the uncertain wave equation via distributed dynamic input extension”. IEEE Trans. Aut. Contr., 56(2011), 212-216.

[24] Y. Orlov, A. Pisano and E. Usai, ”Boundary control and observer design for an uncertain wave process by second-order sliding-mode technique”. Proc. 2013 CDC, Florence, Italy.

[25] A. Pisano, Y. Orlov. “Boundary second-order sliding-mode control of an uncertain heat process with unbounded matched perturbation”, Automatica, 48(2012), 1768-1775.

[26] A. Smyshlyaev, M. Krstic "Boundary control of an anti-stable wave equation with anti-damping on uncontrolled boundary", Syst. Contr. Lett., 58(2009), 617-623.

[27] A. Smyshlyaev, M. Krstic, "Closed-Form Boundary State Feedbacks for a Class of 1-D Partial Integro-Differential Equations" IEEE Trans. Aut. Contr., 49(12)(2004), 2185-2202.

[28] D. Tsubakino, M. Krstic, and Y. Yamashita, "Boundary Control of a Cascade of Two Parabolic PDEs with Different Diffusion Coefficients". Proc. 2013 CDC, Florence, Italy.

[29] R. Vazquez, M. Krstic and J.-M. Coron, "Backstepping Boundary Stabilization and State Estimation of a 2×2 Linear Hyperbolic System". Proc. 2011 joint CDC-ECC, Orlando, Florida.

[30] R. Vazquez, J.-M. Coron, M. Krstic, and G. Bastin, "Local exponential H^2 stabilization of a 2×2 quasilinear hyperbolic system using backstepping", SIAM J. Contr. Opt., 51(2013), 2005–2035.

[31] R. Vazquez, and M. Krstic, "Boundary Observer for Output-Feedback Stabilization of Thermal-Fluid Convection Loop" IEEE Trans. Contr. Syst. Tech., 18(4)(2010), 789-797.

[32] R. Vazquez, E. Schuster, and M. Krstic, "Magnetohydrodynamic state estimation with boundary sensors" Automatica, 44(2008), 2517–2527.

[33] R. Vazquez, E. Schuster, and M. Krstic, "A Closed-Form Full-State Feedback Controller for Stabilization of 3D Magnetohydrodynamic Channel Flow" ASME J. Dyn. Syst. Meas. Trans. Contr., 131(2009), .

[34] R. Vazquez, E. Trelat and J.-M. Coron, "Control for fast and stable laminar-to-high-Reynolds-numbers transfer in a 2D Navier-Stokes channel flow" Discrete and Continuous Dynamical Systems D Series B, 10(2008), 925—956.

[35] A. Baccoli, Y. Orlov, A. Pisano, "Boundary control of coupled reaction-diffusion processes with constant parameters". Automatica, 54(2015), 4, 80-90

[36] R.F. Curtain, and H. Zwart. An introduction to infinite-dimensional linear systems theory. Springer, 1995.

[37] F. Di Meglio, R. Vazquez and M. Krstic, "Stabilization of a System of Coupled First-Order Hyperbolic Linear PDEs With a Single Boundary Input" IEEE Trans. Aut. Contr., 58(12)(2013), 3097-3111.

[38] Z. Hidayat, R. Babuska, B. De Schutter, and A. Nunnez, "Observers for linear distributed-parameter systems: A survey" Proc. 2011 IEEE International Symposium on Robotic and Sensors Environments (ROSE 2011), Montreal, Canada, pp. 166.171, 2011.

[39] L. Jadachowski, T. Meurer, and A. Kugi “State estimation for parabolic PDEs with varying parameters on 3-dimensional spatial domains”. Proc. 18th IFAC World Congress. Milano, Italy, August/September 2011, pp. 13338-13343, 2011.

[40] R. Miranda, I. Chairez, J. Moreno, “Observer design for a class of parabolic PDE via sliding modes and backstepping” Proc. 11th International Workshop on Variable Structure Systems (VSS 2010), Mexico City, June 2010, pp. 215 - 220, 2010.

[41] S. Moura, J. Bendtsen, and V. Ruiz “Observer Design for Boundary Coupled PDEs: Application to Thermostatically Controlled Loads in Smart Grids”, Proc. 52nd Conference on Decision and Control (CDC 2013), pp. 6286 - 6291, Florence, 2013.

[42] T. Lasiecka and R. Triggiani, Control theory for partial differential equations: continuous and approximation theories, I abstract parabolic systems. Cambridge University Press, 2000.

[43] A. Smyshlyaev, M. Krstic “Backstepping observers for a class of parabolic PDEs”, Syst. Contr. Lett., 54(2005), 613-625.

[44] D. Tsubakino, and S. Hara “Backstepping observer design for parabolic PDEs with measurement of weighted spatial averages, Automatica, Volume 53, March 2015, Pages 179-187

[45] R. Vazquez, Boundary control laws and observer design for convective, turbulent and magnetohydrodynamic flows, Ph.D. thesis, Univ. California, San Diego, CA, USA, 2006.

[46] A. Vande Wouwer, and M. Zeitz. “State estimation in distributed parameter systems”. In H. Unbehauen (Ed.), Encyclopedia of life support systems (EOLSS). Oxford, UK: EOLSS Publishers, (Chapter) Control systems, robotics and automation, Article No. 6.43.19.3, 2001.

[47] D. M. Boskovic, M. Krstic, and W. Liu “Boundary Control of an Unstable Heat Equation Via Measurement of Domain-Averaged Temperature” IEEE Trans. Aut. Contr., 46(2001), 2022-2028.

[48] E. Fridman, Y. Orlov “An LMI approach to H_∞ boundary control of semilinear parabolic and hyperbolic systems” Automatica, 45(2009), 2060-2066.

[49] B.-Z. Guo, F.-F. Jin“Sliding Mode and Active Disturbance Rejection Control to Stabilization of One-Dimensional Anti-Stable Wave Equations subject to Disturbance in Boundary Input” IEEE Trans. Aut. Contr, (2013), in press.

[50] M. Krstic, A. Smyshlyaev “Adaptive boundary control for unstable parabolic PDEs. Part I. Lyapunov design” IEEE Trans. Aut. Contr., 53(2008), 1575-1591.

[51] A. Levant, “Sliding order and sliding accuracy in sliding mode control”, Int. J. Contr., 58(1993), 1247-1263.

[52] L. Levaggi, "Sliding modes in banach spaces", Diff. Integr. Equat., 15(2002), 167–189.

[53] O. Morgul, "Stabilization and Disturbance Rejection for the Wave Equation" IEEE Trans. Aut. Contr., 43(1998), 89-95.

[54] Y. Orlov, "Discontinuous unit feedback control of uncertain infinite-dimensional systems" IEEE Trans. Aut. Contr., 45(2000), 834-843.

[55] Y. Orlov, A. Pisano and E. Usai, "Second-order sliding-mode control of the uncertain heat and wave equations" SIAM J. Contr. Opt., 49(2011), 363-382.

[56] Y. Orlov, A. Pisano and E. Usai, "Exponential stabilization of the uncertain wave equation via distributed dynamic input extension". IEEE Trans. Aut. Contr., 56(2011), 212-216.

[57] Y. Orlov, V. I. Utkin "Sliding mode control in infinite-dimensional systems" Automatica, 23(1987), 753-757.

[58] A. Smyshlyaev, M. Krstic "Boundary control of an anti-stable wave equation with anti-damping on uncontrolled boundary", Syst. Contr. Lett., 58(2009), 617-623.

[59] A. Smyshlyaev, M. Krstic, "Explicit State and Output Feedback Boundary Controllers for Partial Differential Equations" Journal of Automatic Control, 13(2)(2003), 1-9.

[60] P. Grindrod, "Patterns and Waves: The Theory and Applications of Reaction-Diffusion Equations" Clarendon Press, 1991.

[61] E.E. Holmes et al. "Partial differential equations in ecology: spatial interactions and population dynamics." Ecology, 75(1)(1994), 17-29

[62] A.F. Filippov, *Differential Equations with Discontinuous Right-Hand Side*, Kluwer,Dordrecht, The Netherlands.

[63] E. Fridman, Y. Orlov "An LMI approach to H_∞ boundary control of semilinear parabolic and hyperbolic systems" Automatica, 45(2009), 2060-2066.

[64] Y. Orlov *Discontinuous Systems Lyapunov Analysis and Robust Synthesis under Uncertainty Conditions*, Communications and Control Engineering Series, Springer Verlag, Berlin, (2009).

[65] A. Pisano, "Adaptive unit-vector control of an uncertain heat diffusion process". Journal of Franklin Institute, 351(4)(2014), 2062-2075.

[66] V.I. Utkin, *Sliding Modes In Control And Optimization*, Springer Verlag,New York, (1983).

[67] A. Smyshlyaev, M. Krstic, "Closed-Form Boundary State Feedbacks for a Class of 1-D Partial Integro-Differential Equations" IEEE Trans. Aut. Contr., 49(12)(2004), 2185-2202.

[68] B.-Z. Guo, F.-F. Jin"Sliding Mode and Active Disturbance Rejection Control to Stabilization of One-Dimensional Anti-Stable Wave Equations subject to Disturbance in Boundary Input" IEEE Trans. Aut. Contr, 58(2013), pp. 1269–1274.

[69] R.F. Curtain, and H. Zwart. An introduction to infinite-dimensional linear systems theory. Springer, 1995.

[70] Z. Hidayat, R. Babuska, B. De Schutter, and A. Nunnez, "Observers for linear distributed-parameter systems: A survey" Proc. 2011 IEEE International Symposium on Robotic and Sensors Environments (ROSE 2011), Montreal, Canada, pp. 166.171, 2011.

[71] L. Jadachowski, T. Meurer, and A. Kugi "State estimation for parabolic PDEs with varying parameters on 3-dimensional spatial domains". Proc. 18th IFAC World Congress. Milano, Italy, August/September 2011, pp. 13338-13343, 2011.

[72] T. Lasiecka and R. Triggiani, Control theory for partial differential equations: continuous and approximation theories, I abstract parabolic systems. Cambridge University Press, 2000.

[73] Y. Orlov, A. Pisano, A. Baccoli, and E. Usai, Output-feedback stabilization of coupled reaction-diffusion processes. SIAM Journal on Control and Optimization. Submitted for pubblication.

[74] R. Vazquez, E. Schuster, and M. Krstic, Magnetohydro-dynamic state estimation with boundary sensors Automatica, 44(10)(2008), pp. 2517-2527.

[75] R. Vazquez, and M. Krstic, "Boundary Observer for Output-Feedback Stabilization of Thermal-Fluid Convection Loop" IEEE Trans. Contr. Syst. Tech., 18(4)(2010), 789-797.

[76] A.G. Butkovskiy, "Green's Functions and Transfer Functions Handbook" Ellis Horwood Lmt., Chichester, 1982.

[77] M.A. Krasnoselskii, P.P. Zabreyko, E.I. Pustylnik, and P.E. Sobolevski, "Integral operators in spaces of summable functions" Noordhoff, Leyden, 1976