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String-brane scattering at high energy

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Presentata da: Cristina Monni
Coordinatore Dottorato: Prof. Gianluca Usai
Relatore: Prof. Giuseppe D’Appollonio

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Abstract

The main purpose of this Thesis is to provide an analysis for tree-level amplitudes involving massive string states in a D-brane background. String-brane interactions provide an ideal framework to study the dynamics of the massive states of the string spectrum in a non-trivial background. The infinite tower of massive modes in the string spectrum is essential for the perturbative finiteness of the string amplitudes.

The high-energy limit of string theory is a promising regime to study the relation between string amplitudes and gravitational phenomena.

We present here an analysis of tree-level amplitudes for processes in which an NS-NS string state from the leading Regge trajectory scatters from a D-brane into another state from the leading Regge trajectory, in general of a different mass, at high energies and small scattering angles. This is done by using world-sheet OPE methods and effective vertex operators. We find that this class of processes has a universal dependence on the energy of the projectile and that the tree-level amplitudes are in agreement with the eikonal operator form of the string S-matrix.

To my family,
my lifeline.

See, I'm the only one who sees the whole picture. That's what they mean by genius.

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The last three years as a PhD student had a heavy impact on my life, thougts and actions. Experiences allowed me to grow and to understand that one never stops improving him/herself. The closing of this cycle is the beginning of a new route.

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Contents

Introduction

Perturbative string theory allows the evaluation of finite and unitary scattering amplitudes, including the exchange of a massless particle of spin 2. Its massless spectrum and the low energy couplings coincide with those of effective supergravity theories. The full string theory spectrum is however much richer. It comprises an infinite tower of massive modes which are essential for the perturbative finiteness of the string amplitudes and other peculiar features of the theory.

String theory thus provides a perturbative description of quantum gravity and as such it is one of the main tools at our disposal to learn about physics at the Planck scale where gravitational interactions become dominant. We would also like to study in the context of string theory those processes which seem to defy a consistent unification of gravitational physics and quantum mechanics, in particular the gravitational collapse, the formation of black holes and the fate of matter falling into a black hole.

The high-energy limit of string theory is a promising regime to study the relation between string amplitudes and gravitational phenomena. Early investigations into the high-energy behaviour of scattering amplitudes in string theory considered two principal regimes, that of fixed scattering angle [40, 42, 43] and that of fixed momentum transfer [7–9]. In the former it was discovered that at each genus the high-energy behaviour of the string scattering amplitudes is dominated by a saddle point in the moduli space of the Riemann surfaces and corresponds to a semiclassical worldsheet in spacetime.

The latter investigations considered graviton-graviton scattering in the limit of large center of mass energy and fixed momentum transfer, referred to as the Regge limit. Several interesting features for different values of the transferred momentum t were identified. For small t, corresponding to large distance processes, it was found that elastic scattering dominates and this is mediated by the long range exchange of particles lying on the Regge trajectory of the graviton, called "reggeized gravitons". For larger values of t, a semiclassical eikonal description dominates: the strings begin to feel the influence of a curved background and this reveals effects attributable to classical gravity. If the value of t is increased further, absorptive processes begin to contribute significantly due to the production of inelastic and diffractive states. Beyond this point the analysis should be extended into the regime of small but fixed scattering angle.

The eikonal amplitudes closely correspond to the semiclassical picture of one string scattering in the classical metric of the other. For a very energetic string this is the Aichelburg-Sexl metric [5]. The combined classical metric of the pair of colliding strings is that of colliding Aichelburg-Sexl metrics, which was shown in [31, 63] to form a closed trapped surface, and thus a black hole, at impact parameters of order $R_S(E)$. However, before concluding that black holes form in the collision of a pair of strings, one needs to check that there are not other effects which could intervene and change qualitatively the dynamics. An important example of this kind of effects is the tidal excitation of massive string modes.

More recently there has been renewed interest in the high-energy behaviour of scattering amplitudes in string theory [11, 27, 61]. In the paper by D'Appollonio, Di Vecchia, Russo and Veneziano (DDRV) [27]

the methods employed by Amati, Ciafaloni and Veneziano (ACV) in [8] were used to examine the high-energy scattering of a graviton from a stack of N parallel Dp-branes in a flat background. Significantly, this process involves the interaction between perturbative and non-perturbative objects for which we have a microscopic description, whereas previously only interactions between perturbative states had been analysed. As in the earlier investigations the limit of large energies is examined in a number of kinematic regions for which both the impact parameter and radius of curvature produced by the D-branes are larger than the string length. For the largest impact parameters one finds a region of perturbative scattering leading to vanishingly small deflection angles, and as the impact parameter is reduced one encounters both classical corrections and corrections due to the nonzero string length. The classical counterpart of the AS metric in the case of the D-branes, is the extremal p-brane metric.
As the impact parameter approaches the scale of curvature of the p-brane spacetime, classically the incoming string should be absorbed by the brane. This would be a very interesting region for further study.

It is of great interest to better understand high-energy string scattering. A great variety of novel phenomena such as excited string states, as well as string-scale non-localities, arise at a threshold energy near the string scale. Also in the high-energy regime far beyond the string scale one can find interesting physics. For example, one has sufficient energy to excite a macroscopic string of length $l \sim E/M^2$. These observations have motivated proposal of a string uncertainty principle [9] and were suggested to play a role in resolving the information paradox. Particularly important is to understand the role of effects due to string excitation, such as long strings, and effects that are more easily described as being essentially gravitational. Massive string excitations and their interactions will be the central topic of this Thesis.

This Thesis is organised as follows. We briefly review in Chapter 1 the most important features of field theories at high energy, introducing the eikonal approximation and, following 't Hooft [55], its application to high-energy collisions in quantum gravity. This will be a useful introduction to the string computation, and we will come back to the gravitational eikonal to compare it to the string in the following Chapters.

In order to compute high energy scattering amplitudes in string theory, in Chapter 2 we will introduce the spectrum of the NS-NS sector of the Type II theory and then we will explain how to compute correlators on the simplest Riemann surfaces: the sphere and the disk for tree-level diagrams, the torus and the annulus for one-loop computations in flat space and in the presence of D-branes, respectively. Then we will motivate the insertion of a D-brane background in the high energy scattering analysis by reviewing the description of D-branes both as defects in spacetime on which open strings can end and as solutions to the low-energy effective actions of Type II string theories.

The issue of Planckian energy scattering is entirely taken into account in Chapter 3, where we perform the asymptotic limit of tree-level and one-loop four-point amplitudes for massless states as was done in [7, 8]. It is shown that the amplitudes in the Regge limit can be resummed to all orders in the string loop parameter. The resulting amplitude has an operator eikonal form. This operator will be analysed in detail in different kinematic regions.

After reviewing the work of ACV, we briefly analyse fixed angle scattering as another important kinematic regime at Planckian energies. Then we introduce the massive string spectrum emphasizing its relevance in high-energy scattering processes. The states we will be using in the amplitudes of Chapter 5 are introduced, especially those lying on the leading Regge trajectory of the NS-NS sector of the Type II superstring, that is, the states with the maximum spin possible for their mass.

The setting of Amati, Ciafaloni and Veneziano is then completed with the addition of a D-brane background in Chapter 4, and again the amplitudes are shown to resum in an eikonal form. In this case the classical picture equivalent to the D-brane background is the propagation of the string in the extremal p-brane metric. Again classical parameters such as the deflection angle and inelastic excitations can be computed. The latter are however taken into account by string excitations, which can be evaluated through the eikonal operator. It turns out that only the excitations transverse to the D-brane world-volume are nonzero.

This issue leads us to investigate the importance of string excitations, and motivated by this in Chapter 5 we study the interactions of a massive closed superstring with a stack of Dp-branes. Such interactions will in general include inelastic processes in which the string is excited/decays into a state of a different mass. Since we expect scattering at large energies to be particularly simple for the leading Regge trajectory, we will take such states as our two external strings with momenta p_1, p_2 such that $\alpha' p_1^2 = -4n$ and $\alpha' p_2^2 = -4n'$ for positive integers n, n'. The presence of the D-branes will break the $SO(1,9)$ invariance to $SO(1,p) \times SO(9-p)$ and momentum will only be conserved parallel to the worldvolume of the D-branes. This implies that the invariant quantities we will primarily consider shall be composed of the momentum flowing parallel to the D-branes, $s = -{p_1}_{\parallel}^2$, and the momentum transferred to the D-branes, $t = -(p_1 + p_2)^2$. We will be interested in the limit $\alpha' s \to \infty$ while t is kept fixed. In order to obtain high-energy scattering amplitudes in the most efficient manner we will make use of OPE methods to construct effective vertex operators as pioneered in [3, 4] and more recently used in [21, 33]. Insertion of these effective vertices onto the upper-half of the complex plane will yield the tree-level amplitudes for these processes which show identical Regge behaviour to that exhibited in the graviton - graviton scattering. This behaviour is then universal for the states on the leading Regge trajectory. We find that the eikonal operator takes into account inelastic string excitations, and confirm that the longitudinal degrees of freedom are not excited at tree level. It could be interesting then to investigate if these excitations show up at one-loop order, and if they are in agreement with the semiclassical picture. We will discuss other possible applications of our work in the Conclusions.

1

High energy behaviour of perturbative field theory

The main topic of this Thesis is the behaviour of the string scattering amplitudes at high-energy and fixed impact parameter. An important role in this context is played by the eikonal approximation. In this chapter we will review the application of this method to the scattering of two particles with trans-planckian energies, which is dominated by the exchange of gravitons. In chapter 3 we will consider the same process in the context of string theory.

Historically the eikonal approximation was developed in geometrical optics in order to study the propagation of light rays according to Fermat principle. Through the work of Hamilton it deeply influenced the development of classical and quantum mechanics. In scattering theory the eikonal approximation is particularly useful for processes involving particles with large center of mass energy. It is based on the assumption that a very energetic particle will be hardly deflected from its path by the interaction with a potential $V(x)$. The idea is then to expand the amplitude in a series around the undeflected trajectory of the particle and so it is a semiclassical limit. The first term of the series is

$$\mathcal{A}^{(0)}(\theta,\phi) = \frac{k}{2\pi i}\int db\, b\int_0^{2\pi} d\phi(e^{-i\chi(\vec{b})} - 1), \tag{1.1}$$

where $\vec{b}$ is the impact parameter, k the momentum of the particle and the phase is given by

$$\chi(\vec{b}) = -\frac{k}{2E}\int_{-\infty}^{+\infty} dz V(\vec{b}, z). \tag{1.2}$$

In the sixties and the seventies many results were derived concerning the high-energy behaviour of the scattering amplitudes of a relativistic quantum field theory. A very detailed analysis was performed in a series of classical papers by H. Cheng and T. T. Wu and their collaborators [24]. They studied the asymptotic behaviour of the Feynman graphs contributing order by order in perturbation theory to a two-particle scattering process. A particular class of diagrams is given by the so-called ladder diagrams. They are characterized by the fact that the large momenta carried in the center of mass frame by the two colliding particles flow through the diagram along two uninterrupted lines. Other particles are exchanged only between these two lines and they carry soft momenta. It was soon realized by Itzykson and Abarbanel [1] that the sum of the ladder diagrams could be represented in impact-parameter space in an eikonal form very similar to the non-relativistic expression. The eikonal phase is in fact given by the Fourier transform of the tree-level amplitude. As an example, the eikonal T-matrix for electron-electron scattering in QED is

$$\mathcal{T}_E(s,t) = -2is\frac{\delta_{\lambda_1\lambda_2}\delta_{\lambda_1'\lambda_2'}}{4m^2}\int d^2b\, e^{i\vec{b}\cdot(\vec{p}_3-\vec{p}_1)}\left\{\exp\left[-ie^2\int\frac{d^2q}{(2\pi)^2}\frac{e^{-i\vec{q}\cdot\vec{b}}}{|\vec{q}|^2+\mu^2}\right]-1\right\}, \tag{1.3}$$

where λ_i are the particle helicities and μ^2 an infrared cutoff for the photon propagator. In the limit $\mu \to 0$ one recovers, apart form the logarithmically divergent phase typical of Coulomb scattering,

$$\mathcal{T}_E(s,t) = -\frac{\pi s}{m^2}\delta_{\lambda_1\lambda_2}\delta_{\lambda_1'\lambda_2'}\left(\frac{e^2}{4\pi}\right)\frac{\Gamma\left(1+\frac{ie^2}{4}\right)}{\Gamma\left(1-\frac{ie^2}{4}\right)}\left(\frac{-t}{2}\right)^{-[1+(ie^2/4\pi)]}. \tag{1.4}$$

The relativistic eikonal approximation rises two main questions. The first one is whether it always resums correctly the ladder graphs. The answer is negative for models where the exchanged particles are scalars, as shown by Tiktopoulos and Treiman [59] and by Eichten and Jackiw [32]. If the exchanged particles have higher spin, and in particular if they are gauge bosons or gravitons, the eikonal approximation does resum the ladder graphs. The second question is whether the sum of the ladder graphs captures the asymptotic behaviour of the amplitudes. Again this is not the case. For instance in QED at order e^8 the ladder diagrams are dominated at high energy by diagrams describing the creation of a particle-antiparticle pair in the t-channel. These amplitudes were investigated in detail both in abelian and non-abelian gauge theories and led to important developments such as the idea of gluon reggeization and the BFKL pomeron in perturbative QCD.

A remarkable feature of the eikonal approximation is that it can be easily extended to the case in which the exchanged particle is a graviton. In this case the colliding particles have simultaneously large energy and large ”gravitational” charge, since this coincides with the energy. The ladder diagrams should then give the correct high-energy behaviour at small momentum transfer. This fact was exploited by 't Hooft [55] to give a formula for the scattering amplitude between two very energetic quanta in a theory of quantum gravity. Its derivation was based on an external field approximation which consists in solving the wave equation of one particle in the background generated by the other. This external field approximation is equivalent to the leading eikonal obtained by the resummation of the ladder graphs. This calculation was performed by Muzinich and Soldate [49] and later by Kabat and Ortiz [47] using the linearized Einstein action. Since Einstein theory is a non-renormalizable field theory, these papers are based on the assumption that whatever mechanism regulates the ultraviolet of the theory it does not affect the high energy behaviour given by the sum of the ladder graphs (which do not involve graviton loops).

The natural context to analyse the scattering of two very energetic quanta is string theory. The string length l_s regularizes the ultraviolet divergences and the high-energy limit of the amplitudes can be studied in a reliable way. This program was carried out in a series of paper by Amati, Ciafaloni and Veneziano (ACV) [7, 8]. They showed that at high-energy and fixed impact parameter the leading contributions to the scattering amplitude of two strings can be resummed into an eikonal operator form. The eikonal operator provides a firmer basis for the field theory results, which can be recovered in the limit $l_s \to 0$. It also takes into account in a unitary way several inelastic channels which describe the excitation of the massive modes of the colliding strings.

1.1 Eikonal approximation in gravity

The eikonal approximation can help also to probe the leading behaviour of high energy forward scattering in gravity, thus giving some insight on the analysis of Planckian and trans-Planckian scattering in quantum gravity. The massless limit of the eikonal is equivalent to the semiclassical computation done by t'Hooft [55–57] and the topological field theory of H. and E. Verlinde [62]. In this section we will review the computation done by t'Hooft for the scattering of a particle off a schock-wave metric of the Aichelburg-Sexl type.

1.1.1 Scattering of an energetic particle off a static one (t'Hooft semiclassical computation)

We consider the scattering of two particles carrying kinetic energies which are large compared to the Planck mass M_P. The dominant interaction is then provided by gravity. We can study the process in the frame where the incoming particle (particle 1) is at rest and the second particle (particle 2) moves nearly at the speed of light. Particle 2 moves towards particle 1 along the following trajectory

$$\begin{pmatrix} x^{(2)} \\ y^{(2)} \end{pmatrix} = \tilde{x} = 0, \quad z^{(2)} = -t^{(2)} \tag{1.5}$$

and with energy

$$p_0^{(2)} = -p_3^{(2)} = O(1/Gm_1), \tag{1.6}$$

where G is Newton constant. Since particle 2 is extremely energetic, we cannot neglect its gravitational field. This is a shock wave first derived by Aichelburg and Sexl [5].

The effect of the passage of this wave is to modify the wavefunction of particle 1 by a shift of the coordinates. The shock wave is obtained by gluing two distinct regions of spacetime, $R_+^4(t > -z)$ and $R_-^4(t < -z)$, along the null plane $z = -t$ as follows

$$\begin{aligned} \tilde{x}_+ &= \tilde{x}_-, \\ z_+ &= z_- + 2Gp_0^{(2)} \log(\tilde{x}^2/C), \\ t_+ &= t_- - 2Gp_0^{(2)} \log(\tilde{x}^2/C), \end{aligned} \tag{1.7}$$

where C is a constant. The wavefunction of particle 1 in the portion of spacetime R_-^4 is

$$\psi_-^{(1)} = \exp(i\tilde{p}^{(1)}\tilde{x}_- + ip_3^{(1)}z - ip_0^{(1)}t) = \exp(i\tilde{p}^{(1)}\tilde{x}_- - ip_+^{(1)}u - ip_-^{(1)}v), \tag{1.8}$$

where we introduced the two light-cone coordinates $u = (t - z)/2$ and $v = (t + z)/2$. Immediately after the scattering (at $v = 0$) the wave function in $R_{(+)}^4$ is

$$\psi_{(+)}^{(1)} = \exp(i\tilde{p}^{(1)} - ip_{(+)}^{(1)}[u + 2Gp_0^{(2)} \log(\tilde{x}^2/C)]) \,. \tag{1.9}$$

The scattering amplitude can be read from its expansion in plane waves

$$\psi^{(1)}_{(+)} = \int A(k_+, \tilde{k}) dk_+ d^2\tilde{k} \exp(i\tilde{k} \cdot \tilde{x} - ik_+ u - ik_- v), \tag{1.10}$$

where the light-cone momenta satisfy

$$k_+ k_- = (\tilde{k}^2 + m^{(1)2}) \tag{1.11}$$

and

$$\begin{aligned} A(k_+, \tilde{k}) &= \delta(k_+ - p_+^{(1)}) \frac{1}{(2\pi)^2} \int d^2\tilde{x} \exp[(\tilde{p}^{(1)} - \tilde{k}^{(1)}) \cdot \tilde{x} + 2iGp_+^{(1)} p_0^{(2)} \log(\tilde{x}^2/C)] \\ &= \frac{1}{4\pi} \delta(k_+ - p_+^{(1)}) \frac{\Gamma(1 - iGs)}{\Gamma(iGs)} \left(\frac{4}{(\tilde{p}^{(1)} - k^{(1)})^2} \right)^{1 - iGs}, \end{aligned} \tag{1.12}$$

where we used (1.6) to write $2iGp_+^{(1)} p_0^{(2)} = -2G(p^{(1)} \cdot p^{(2)}) = Gs$. So the elastic amplitude for particle 1 is given by

$$_{out}\langle k^{(1)} | p^{(1)} \rangle_{in} = \frac{k_+}{4\pi k_0} \delta(k_+ - p_+^{(1)}) \frac{\Gamma(1 - iGs)}{\Gamma(iGs)} \left(\frac{4}{(\tilde{p}^{(1)} - k^{(1)})^2} \right)^{1 - iGs}, \tag{1.13}$$

where we used $dk_+ d\tilde{k} = (k_+/k_0) dk_3 d\tilde{k}$.
Introducing the Mandelstam variable $t = -(\tilde{p}^{(1)} - k^{(1)})^2$ we get

$$\mathcal{A}(s, t) = \frac{\Gamma(1 - iGs)}{4\pi\Gamma(iGs)} \left(\frac{4}{-t} \right)^{1 - iGs} . \tag{1.14}$$

2

Scattering amplitudes in String Theory

In this Chapter we review the techniques that have been developed to compute string scattering amplitudes in the background of a collection of D-branes. We will consider Type II strings and in Section 2.1 we will first introduce their spectrum and low-energy effective actions.
String amplitudes are given by correlation functions of vertex operators, which identify the asymptotic states, integrated over the moduli space of Riemann surfaces. It is then useful to provide a brief derivation of some correlators on the simplest Riemann surfaces with and without boundaries, as done in Section 2.2. First we will consider the sphere and the torus: correlators on these surfaces will be used in Chapter 3, where we compute tree-level and one-loop four-graviton amplitudes in order to study their behaviour at high energies in the Regge limit and to resum the leading contributions to all orders in perturbation theory. These computations will lay the foundations for the main topic of this thesis, the study of the scattering of massless and massive strings from a collection of N D-branes. The presence of the D-branes is taken into account in the perturbative string series by the inclusion of surfaces with boundaries ending on the brane worldvolume. These surfaces describe the interaction of the closed string with the brane and, as we will discuss in more detail in the following chapters, the resummation of all the surfaces with boundaries but no handles should correspond to the semiclassical propagation of the string in the curved spacetime sourced by the D-branes.

Scattering amplitudes in a D-brane background are given at tree level by correlators on the disk and at one loop by correlators on the annulus. We will introduce these correlators in Subsection 2.2.2 and in Subsection 2.2.3 respectively. We will also provide a short review of the Jacobi Theta Functions and their properties in Section 2.2.3, with particular attention for their asymptotic behaviour since it will be essential in order to derive the high-energy limit of one-loop amplitudes both in Chapter 3 and in Chapter 4. As a first example of a torus amplitude in superstring theory, we will compute the vacuum partition function in Section 2.3, where we will also explain how to sum over the different spin structures.

In Section 2.4 we will introduce the D-branes as spacetime defects where open strings can end. We will consider the dynamical degrees of freedom of the D-branes and their low-energy effective action. Then we will briefly review the computation of the D-brane tension and charge, showing that D-branes are half-BPS states.

D-branes will be the main focus of our analysis in Chapter 4, where we will study string-brane scattering amplitudes and compare them with the propagation of a closed string probe in a curved p-brane background. In order to clarify this point we will explain the relation between p-branes and Dp-branes in Section 2.7 where we introduce p-branes as static solutions of the Type II effective actions.

2.1 Type II string theories

2.1.1 Spectrum

The closed string spectrum is the product of two copies of the open string spectrum, with right and left-moving levels matched. Performing the same GSO projection on both sides gives the type IIB string, while performing opposite GSO projections gives the type IIA string. The massless sectors is for the two cases

Type IIA: $(\mathbf{8v} \oplus \mathbf{8s}) \otimes (\mathbf{8v} \oplus \mathbf{8c})$ **Type IIB:** $(\mathbf{8v} \oplus \mathbf{8s}) \otimes (\mathbf{8v} \otimes \mathbf{8s})$,

where **8v**, **8s** and **8c** are respectively the vector (coming from the NS sector) and the two spinor representations (coming from the R sector) of the little Lorentz group $SO(8)$. Expanding the tensor products in the NS-NS sector we obtain

$$(\mathbf{8v} \oplus \mathbf{8v}) = \Phi \oplus B_{\mu\nu} \oplus G_{\mu\nu} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}. \tag{2.1}$$

In the R-R sector, the IIA and IIB spectra are respectively

$$\begin{aligned} \mathbf{8s} \otimes \mathbf{8c} &= [1] \oplus [3] = \mathbf{8v} \oplus \mathbf{56}_t \\ \mathbf{8s} \otimes \mathbf{8s} &= [0] \oplus [2] \oplus [4]+ = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_+. \end{aligned} \tag{2.2}$$

Here $[n]$ denotes the antisymmetric tensor representation of $SO(8)$ of degree n, and $[4]_+$ a self-dual representation. Note that the representations $[n]$ and $[8-n]$ are related by contraction with the 8-dimensional ϵ-tensor. The NS-NS and R-R spectra together with the fermionic degrees of freedom of the NS-R and R-NS sectors give the spectrum of d = 10 IIA (nonchiral) and IIB (chiral) supergravity; we will write these effective supergravity actions in next Subsection. Now we will provide, for future use, more details on the bosonic sector of the two theories. Common to both type IIA and IIB are the NS-NS sector fields

$$\Phi, G_{\mu\nu}, B_{\mu\nu}. \tag{2.3}$$

The first two are the dilaton and the metric tensor respectively, the latter is a rank-two antisymmetric tensor potential, which couples electrically to the fundamental closed string by

$$\frac{1}{2\pi\alpha'} \int_{\mathcal{M}_2} B_{(2)}, \tag{2.4}$$

where $\mathcal{M}_2$ is the world sheet, with coordinates $\xi_a, a = 1, 2$, $B_{(2)} = B_{ab} d\xi_a d\xi_b$, and B_{ab} is the pullback of $B_{\mu\nu}$. By ten dimensional Hodge duality, we can also construct a six form potential $B_{(6)}$, by the relation $dB_{(6)} = *\, dB_{(2)}$. There is a natural electric coupling

$$\frac{1}{(2\pi\alpha')^5} \int_{\mathcal{M}_6} B_{(6)} \tag{2.5}$$

to the world-volume $\mathcal{M}_6$ of a five dimensional extended object. This NS-NS charged object, which is commonly called the "NS5-brane" is the magnetic dual of the fundamental string. In the R-R sector there are additional antisymmetric form potentials:
type IIA : $C_{(1)},\ C_{(3)},\ C_{(5)},\ C_{(7)}$
type IIB : $C_{(0)},\ C_{(2)},\ C_{(4)},\ C_{(6)},\ C_{(8)}$, where in each case the last two forms are Hodge duals of the first two, and $C_{(4)}$ is self-dual.
As we shall discuss when we will introduce D-branes, we expect that there should be p-dimensional extended sources which couple to these fields via an electric coupling of the form:

$$Q_p \int_{\mathcal{M}_{p+1}} C_{(p+1)}, \tag{2.6}$$

where $\mathcal{M}_{p+1}$ is the $p+1$-dimensional worldvolume of the source.

One of the most striking and far reaching results of modern string theory was the identification of the sources of the R-R fields with the D-branes. They carry a charge μ_p which is the smallest allowed by consistency, suggesting that they are the basic sources from which all R-R charged objects may be constructed. So we see that type IIA contains a D0-brane and its magnetic dual, a D6-brane, and a D2-brane and its magnetic cousin, a D4-brane. The last even brane is a ten-dimensional domain wall type solution, the D8-brane, which belongs to the type IIA theory. Meanwhile, type IIB has a string-like D1-brane, which is dual to a D5-brane. There is a self-dual D3-brane, and there is an instanton which is the D(-1)-brane, and its Hodge dual, the D7-brane. To complete the list of odd branes, we note that there is a spacetime filling D9-brane which pertains to the type IB theory.

The NS-R and R-NS sectors are given by the products

$$\mathbf{8v} \otimes \mathbf{8c} = \mathbf{8s} \oplus \mathbf{56c}$$
$$\mathbf{8v} \oplus \mathbf{8s} = \mathbf{8c} \oplus \mathbf{56s}.$$

The **56s,c** are gravitinos. Their vertex operators are made roughly by tensoring a NS field ψ^μ with a vertex operator $V_\alpha = e^{-\varphi/2}\mathbf{S}_\alpha$, where the latter is a spin field. The full gravitino vertex operators, which have one vector and one spinor index, are two fields of weight (0, 1) and (1, 0), respectively, depending upon whether ψ^μ comes from the left or right. These are therefore holomorphic and antiholomorphic worldsheet currents, and the symmetry associated to them in spacetime is the supersymmetry. In the IIA theory the two gravitinos (and supercharges) have opposite chirality, and in the IIB the same.

2.1.2 Low-energy effective actions

The dynamics of the massless string modes can be summarised in terms of a low energy effective supergravity action.
The bosonic part of the low energy action for the type IIA string theory in ten dimensions is given by

$$\begin{aligned} S_{IIA} = \frac{1}{2\kappa_0^2} \int d^{10}x(-G)^{1/2} \Big\{ e^{-2\Phi} \Big[R + 4(\nabla\phi)^2 - \frac{1}{12}(H^{(3)})^2 \Big] \\ - \frac{1}{2}(G^{(2)})^2 - \frac{1}{48}(G^{(4)})^2 \Big\} - \frac{1}{4\kappa_0^2} \int B^{(2)} dC^{(3)} dC^{(3)}. \end{aligned} \tag{2.7}$$

$G_{\mu\nu}$ is the metric in the string frame, Φ is the dilaton, $H^{(3)} = dB^{(2)}$ is the field strength of the NS-NS two-form, while the Ramond-Ramond field strengths are $G^{(2)} = dC^{(1)}$ and $G^{(4)} = dC^{(3)} + H^{(3)} \wedge C^{(1)}$. The bosonic part of the type IIB string is:

$$S_{IIB} = \frac{1}{2\kappa_0^2} \int d^{10}x(-G)^{1/2} \left\{ e^{-2\Phi} \left[R + 4(\nabla\phi)^2 - \frac{1}{12}(H^{(3)})^2 \right] \right. \tag{2.8}$$

$$\left. -\frac{1}{2}\Big(G^{(3)} + C^{(0)}H^{(3)}\Big)^2 - \frac{1}{2}\left(dC^{(0)^2} \frac{1}{48}(G^{(5)} \right)^2 \right\}$$

$$\frac{1}{4\kappa_0^2} \int \left(C^{(4)} + \frac{1}{2}B^{(2)}C^{(2)} \right) G^{(3)}H^{(3)}. \tag{2.9}$$

Now, $G_{(3)} = dC_{(2)}$ and $G_{(5)} = dC_{(4)} + H_{(3)}C_{(2)}$ are R-R field strengths, and $C_{(0)}$ is the R-R scalar. The ten-dimensional Newton constant is given by

$$2\kappa^2 = 2\kappa_0^2 g_s^2 = 16\pi G_N = (2\pi)^7 \alpha'^4 g_s^2, \tag{2.10}$$

where g_s is the string coupling, the asymptotic value of the dilaton at infinity: $g_s \equiv e^{\Phi_0}$.

2.2 Green functions

In the following we will introduce the closed string correlators on Riemann surfaces for the Type II theories. We will start by illustrating the sphere and the disk, which give tree-level amplitudes, and then the torus and the annulus, used in one-loop amplitudes. These correlation functions are simply the Green functions of the Laplacian operator in two dimensions, so we will derive these Green functions for all the Riemann surfaces we listed before.

2.2.1 The sphere

The bosonic and fermionic string operators X^μ and ψ^μ on the sphere are free fields, so they satisfy the Laplace equation

$$\Box X^\mu(\sigma) = 0, \tag{2.11}$$

whose Green function is the logarithm. Taking into account the normalisation of the string action we have

$$\langle X^\mu(z_1, \bar{z}_1) X^\nu(z_2, \bar{z}_2) \rangle = -\frac{\alpha'}{2} \eta^{\mu\nu} \ln |z_1 - z_2|^2. \tag{2.12}$$

Writing separately the holomorphic and anti-holomorphic parts of the worldsheet fields, the correlators for bosons, fermions and scalars ϕ on the sphere that we will need are the following:

$$\begin{aligned}
\langle X^\mu(z) X^\nu(w) \rangle &= -\frac{\alpha'}{2} \eta^{\mu\nu} \log(z - w), \\
\langle \partial X^\mu(z) X^\nu(w) \rangle &= -\frac{\alpha'}{2} \frac{\eta^{\mu\nu}}{z - w}, \\
\langle \partial X^\mu(z) \partial X^\nu(w) \rangle &= \frac{\alpha'}{2} \frac{\eta^{\mu\nu}}{(z - w)^2}, \\
\langle \psi^\mu(z) \psi^\nu(w) \rangle &= \frac{\eta^{\mu\nu}}{z - w},
\end{aligned}$$

$$\langle \partial\psi^\mu(z)\psi^\nu(w)\rangle = -\frac{\eta^{\mu\nu}}{(z-w)^2}, \tag{2.13}$$

$$\langle \partial\psi^\mu(z)\partial\psi^\nu(w)\rangle = \frac{\eta^{\mu\nu}}{(z-w)^3}, \tag{2.14}$$

$$\langle \phi(z)\phi(w)\rangle = -\log(z-w). \tag{2.15}$$

2.2.2 The disk

The two-point functions on the disk can be derived starting from the two-point functions on the sphere, restricting z to the upper-half of the complex plane and using the method of images:

$$\langle X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2)\rangle = -\frac{\alpha'}{2}\ln|z_1-z_2|^2 - \frac{\alpha'}{2}\ln|z_1-\bar{z}_2|^2. \tag{2.16}$$

In this Thesis we will make extensive use of correlators on the disk. The correlators between holomorphic fields are the same as on the sphere, (2.13), but due to the boundary conditions on the disk there are also non-zero correlators between the holomorphic and the anti-holomorphic part of the same string field:

$$\langle X^\mu(z)X^\nu(\bar{z})\rangle = -\frac{\alpha'}{2}\eta^{\mu\nu}\log(z-\bar{z}), \tag{2.17}$$

$$\langle \partial X^\mu(z)X^\nu(\bar{z})\rangle = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{z-\bar{z}},$$

$$\langle \partial X^\mu(z)\partial X^\nu(\bar{z})\rangle = \frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-\bar{z})^2},$$

$$\langle \psi^\mu(z)\psi^\nu(\bar{z})\rangle = \frac{\eta^{\mu\nu}}{z-\bar{z}},$$

$$\langle \partial\psi^\mu(\bar{z})\psi^\nu(w)\rangle = -\frac{\eta^{\mu\nu}}{(z-\bar{z})^2}, \tag{2.18}$$

$$\langle \partial\psi^\mu(\bar{z})\partial\psi^\nu(w)\rangle = \frac{\eta^{\mu\nu}}{(z-\bar{z})^3}, \tag{2.19}$$

$$\langle \phi(z)\phi(\bar{z})\rangle = -\log(z-\bar{z}). \tag{2.20}$$

2.2.3 The torus

The Feynman graph describing the propagation of a closed string in time which goes back to its initial state is the two-dimensional torus, (see Fig. 2.1). The torus can be represented as the quotient of the complex plane by the identification $z \sim z + 2\pi n\tau + 2\pi n$, $\tau \in \mathbb{H}$, $n,\ m \in \mathbb{Z}$ (see Fig. 2.2). The parameter τ is the modulus of the torus. Two tori are conformally equivalent if their moduli are related by the following transformations:

$$T : \tau \mapsto \tau + 1, \quad S : \tau \mapsto -\frac{1}{\tau}. \tag{2.21}$$

The transformations S and T satisfy the relations

$$S^2 = (ST)^3 = 1, \tag{2.22}$$

and generate the modular group $SL(2,\mathbb{Z})$. Its action on τ is given by

$$\tau \mapsto \frac{a\tau+b}{c\tau+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \tag{2.23}$$

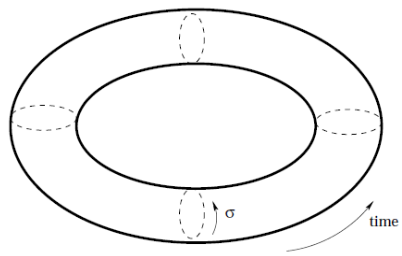

Figure 2.1: The one-loop closed string vacuum diagram is a two-dimensional torus.

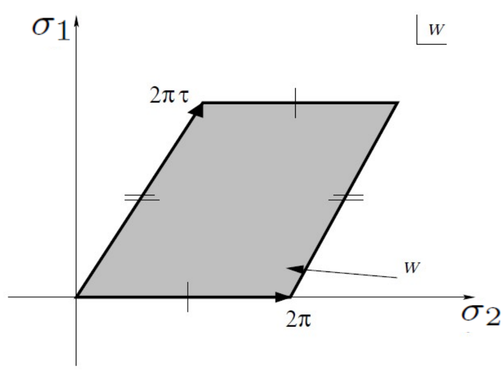

Figure 2.2: The parallelogram representation of the torus. Opposite edges of the parallelogram are periodically identified, i.e. $0 \leqslant \sigma_1 < 2\pi, 0 \leqslant \sigma_2 < 2\pi$. The torus is the represented as the complex w-plane modulo from the identifications $w \sim w + 2\pi n$ and $w \sim w + 2\pi m\tau$ for all integers n and m.

where the $SL(2,\mathbb{Z})$ condition restricts $a,b,c,d \in \mathbb{Z}$ with $ad-bc=1$. The sum over inequivalent tori can then be restricted to a fundamental modular domain F in the upper half of the complex plane, since any point outside F can be mapped into it by a modular transformation.
In Fig. 2.3 we depict a convenient choice for the fundamental region.

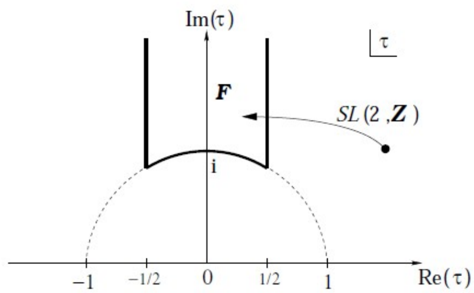

Figure 2.3: A fundamental modular domain F for the torus. The interior of this region is invariant under the transformations (2.21). Any point outside F can be mapped into it by an $SL(2,\mathbb{Z})$ transformation (2.23).

The Green function on the torus is

$$G_{T^2} = \langle X(w,\bar{w})X(w',\bar{w}')\rangle = -\frac{\alpha'}{2}\ln\left|\theta_1\left(\frac{w-w'}{2\pi},\tau\right)\right|^2 + \frac{\alpha'}{4\pi\tau_2}[Im(w-w')]^2. \tag{2.24}$$

The expectation value of a product of vertex operators is

$$\langle\prod_{i=1}^{n} : e^{ik_i\cdot X(z_i,\bar{z}_i)} :\rangle = iC_{T^2}(\tau)(2\pi)^d\delta^d\left(\sum_i k_i\right)$$
$$\times\prod_{i<j}\left|\frac{2\pi}{\partial_\nu\theta_1(0,\tau)}\theta_1\left(\frac{w_{ij}}{2\pi},\tau\right)\exp\left[-\frac{(Imw_{ij})^2}{4\pi\tau_2}\right]\right|^{\alpha' k_i\cdot k_j} \tag{2.25}$$

The fermion correlator on the torus is

$$\langle\psi(z)\psi(w)\rangle = P_F(s;z,w) \equiv \frac{i}{2}\frac{\theta_s(z-w,\tau)}{\theta_1(z-w,\tau)}\frac{\theta_1'(0,\tau)}{\theta_s(0,\tau)}, \tag{2.26}$$

where $s=2,3,4$. For two-dimensional Majorana spinors

$$\Psi(z,\bar{z}) = \begin{pmatrix}\psi(z)\\ \widetilde{\psi}(\bar{z})\end{pmatrix} \tag{2.27}$$

the propagator on the torus reads

$$\langle\Psi(z,\bar{z})\Psi(w,\bar{w})\rangle_{T^2} = P_F(s;z,w)\left(\frac{1+\gamma^3}{2}\right) + \bar{P}_F(\bar{s};\bar{z},\bar{w})\left(\frac{1-\gamma^3}{2}\right), \tag{2.28}$$

where $\gamma^3 = diag(1,1)$ and $s,\ \bar{s}$ are the even spin structures of the left and right components.

Theta functions

In this section we briefly review the basic definitions and properties of theta functions, since they are fundamental in the computation of correlators on the torus and the annulus, and their asymptotic behaviours will be essential in order to perform the high-energy limit of such amplitudes in the following Chapters.
We will consider Riemann's theta function $\theta(z,\tau)$ for $z \in \mathbb{C}$, $\tau \in \mathbb{H}$, the upper-half plane, also known as θ_{00}, and its three variants $\theta_{11}, \theta_{10}, \theta_{11}$. We show how these functions can be used to embed the 2-dimensional torus in complex projective 3-space, and how the equations for the image curve can be found. We then prove the functional equation with respect to $SL(2,\mathbb{Z})$ and show how the moduli space of 1-dimensional tori is given by an algebraic curve. After this, we will review two arithmetic applications of theta series: some famous combinatorial identities that follow from the product expansion and Jacobi formula for the number of representations of a positive integer as the sum of four squares. The theta function can be defined in the form of a series which is absolutely and uniformly convergent in compact sets [48],

$$\theta(\nu,\tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2\pi i n \nu), \tag{2.29}$$

where $\nu \in \mathbb{C}$, $\tau \in \mathbb{H}$, or as an infinite product

$$\begin{aligned} \theta(\nu,\tau) &= \prod_{m=1}^{\infty} (1-q^m)(1+zq^{m-1/2})(1+z^{-1}q^{m-1/2}), \\ q = e^{2\pi i \tau}, &\quad z = e^{2\pi i \nu}. \end{aligned} \tag{2.30}$$

The θ function has interesting periodicity properties. The series (2.29) can be thought of as the Fourier series for a function of ν, periodic with respect to $\nu \mapsto \nu + 1$,

$$\theta(\nu,\tau) = \sum_{n\in\mathbb{Z}} a_n(\tau) \exp(2\pi i n \nu), \quad a_n(\tau) = \exp(\pi i n^2 \tau), \tag{2.31}$$

showing

$$\theta(\nu+1,\tau) = \theta(\nu,\tau). \tag{2.32}$$

From (2.29) it follows immediately

$$\theta(\nu+\tau,\tau) = \exp(-\pi i \tau - 2\pi i \nu)\theta(\nu,\tau). \tag{2.33}$$

If we restrict the variables ν, τ to the case of $\nu = x \in \mathbb{R}$ and $\tau = it,\ t \in \mathbb{R}$, then

$$\theta(x,it) = \sum_{n\in\mathbb{Z}} \exp(-\pi n^2 t) \exp(2\pi i n x) = 1 + 2\sum_{n\in\mathbb{Z}} \exp(-\pi n^2 t) \exp(2\pi n x), \tag{2.34}$$

showing that θ is a real-valued function of two real variables. It has the periodicity in x:

$$\theta(x+1,it) = \theta(x,t), \tag{2.35}$$

and it is the fundamental periodic solution to the heat equation

$$\frac{\partial\theta(x,it)}{\partial t} = \frac{1}{4\pi}\frac{\partial^2\theta(x,it)}{\partial x^2}. \tag{2.36}$$

Under modular transformations

$$\begin{aligned}\theta(\nu, \tau+1) &= \theta(\nu+1/2, \tau),\\ \theta(\nu/\tau, -1/\tau) &= (-i\tau)^{1/2}\exp(\pi i\nu^2/\tau)\theta(\nu,\tau).\end{aligned} \tag{2.37}$$

Other common notations are

$$\begin{aligned}\theta_{00}(\nu,\tau) = \theta_3(\nu,\tau) &= \sum_{n=-\infty}^{\infty} q^{n^2/2} z^n,\\ \theta_{01}(\nu,\tau) = \theta_4(\nu,\tau) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} z^n,\\ \theta_{10}(\nu,\tau) = \theta_2(\nu,\tau) &= \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2} z^{n-1/2},\\ \theta_{11}(\nu,\tau) = -\theta_1(\nu,\tau) &= -i\sum_{n=-\infty}^{\infty} (-1)^n q^{(n-1/2)^2/2} z^{n-1/2}.\end{aligned} \tag{2.38}$$

They also have the following infinite product representations

$$\begin{aligned}\theta_{00}(\nu,\tau) &= \prod_{m=1}^{\infty}(1-q^m)(1+zq^{m-1/2})(1+z^{-1}q^{m-1/2}),\\ \theta_{01}(\nu,\tau) &= \prod_{m=1}^{\infty}(1-q^m)(1+zq^{m-1/2})(1-z^{-1}q^{m-1/2}),\\ \theta_{10}(\nu,\tau) &= 2\exp(\pi i\tau/4)\cos\pi\nu\prod_{m=1}^{\infty}(1-q^m)(1+zq^m)(1+z^{-1}q^m),\\ \theta_{11}(\nu,\tau) &= \prod_{m=1}^{\infty} -2\exp(\pi i\tau/4)\sin\pi\nu(1-q^m)(1-q^m)(1-zq^m)(1-z^{-1}q^m).\end{aligned} \tag{2.39}$$

Their modular transformations are

$$\begin{aligned}\theta_{00}(\nu,\tau+1) &= \theta_{01}(\nu,\tau),\\ \theta_{01}(\nu,\tau+1) &= \theta_{00}(\nu,\tau),\\ \theta_{10}(\nu,\tau+1) &= \exp(\pi i/4)\theta_{10}(\nu,\tau),\\ \theta_{11}(\nu,\tau+1) &= \exp(\pi i/4)\theta_{11}(\nu,\tau).\end{aligned} \tag{2.40}$$

The theta function satisfies the so-called Jacobi "abstruse identity", a special case of Riemann identity:

$$\theta_{00}^4(0,\tau) - \theta_{01}^4(0,\tau) - \theta_{10}^4(0,\tau) = 0. \tag{2.41}$$

Also $\theta_{11}(0,\tau) = 0$. We will also use the Dedekind eta function

$$\eta(\tau) = q^{1/24}\prod_{m=1}^{\infty}(1-q^m) = \left[\frac{\partial_\nu\theta_{11}(0,\tau)}{-2\pi}\right]^{1/3}, \tag{2.42}$$

with modular transformations

$$\begin{aligned}\eta(\tau+1) &= \exp(i\pi/12)\eta(\tau),\\ \eta(-1/\tau) &= (-i\tau)^{1/2}\eta(\tau).\end{aligned} \tag{2.43}$$

2.2.4 The annulus and the method of images

The annulus can be defined as the quotient of the torus under the following involution

$$I_A(z) = 1 - \bar{z}, \tag{2.44}$$

where $\tau = \tau_1 + i\tau_2$ is the modular parameter of the defining torus. The fundamental cells of the involution can be chosen as follows:

$$A : z \in [0, 1/2] \times [0, \tau_2]. \tag{2.45}$$

The thick lines in Fig. 2.4 are the open string boundaries, corresponding to the loci of fixed points. The

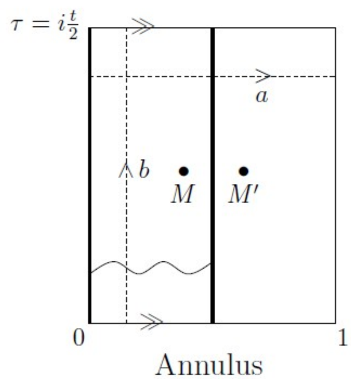

Figure 2.4: Covering tori and fundamental cells for the annulus A. The cycles are represented by dashed lines. The points M' and M are related by the involution. The loci of fixed points drawn in thick are open-string boundaries.

modular parameter for the covering torus is $\tau = it/2$. The bosonic correlators can be expressed in terms of the propagator of the torus G_{T^2} (2.24) by symmetrizing under the involution (2.44):

$$\begin{aligned}\langle X(z)X(w)\rangle &= \frac{1}{2}\left[G_{T^2}(z,w) + G_{T^2}(z, I_A(w)) + G_{T^2}(I_A(z), w) + G_{T^2}(I_A(z), I_A(w))\right] \\ &= G_{T^2}(z,w) + G_{T^2}(z, I_A(w)).\end{aligned} \tag{2.46}$$

We will also use the following correlators:

$$\langle \partial X(z)\partial X(w)\rangle = \partial_z\partial_w G_{T^2}(z,w;\tau), \tag{2.47}$$

$$\langle \partial X(z)\bar{\partial} X(w)\rangle = -\frac{\alpha'}{8\pi\tau_2}, \tag{2.48}$$

The involution for the fermions exchanges left and right components, up to a sign. One consistent choice is [12]:

$$I_A^2 = I_A^3 = I_A^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.49}$$

Symmetrizing the torus propagator under the involution one finds

$$\langle\psi(z)\psi(w)\rangle_A = P_F(s;z,w) \tag{2.50}$$
$$\langle\psi(z)\widetilde{\psi}(w)\rangle_A = P_F(s;z,I_A(w)) \tag{2.51}$$
$$\langle\widetilde{\psi}(\bar{z})\widetilde{\psi}(\bar{w})\rangle_A = \bar{P}_F(\bar{s};\bar{z},\bar{w}) \tag{2.52}$$

2.3 Example: one-loop vacuum amplitude

We will compute explicitly the one-loop closed superstring vacuum diagram, in order to illustrate some general features of superstring theory we will be using in the following Chapters. We start with the bosonic string: consider the string propagation on the torus as depicted in Fig. 2.2. We consider a fixed point on the string lying horizontally in the complex w-plane, which propagates upwards for a time $\sigma^1 = 2\pi Im(\tau) = 2\pi\tau_2$. Then it shifts from this point to the right by $\sigma^2 = 2\pi Re(\tau) = 2\pi\tau_1$. Time translations are generated by the worldsheet Hamiltonian

$$H = L_0 + \tilde{L}_0 - 2, \tag{2.53}$$

while shifts along the string are generated by the worldsheet momentum

$$p = L_0 - \tilde{L}_0. \tag{2.54}$$

The bosonic vacuum path integral is thus given by

$$Z_{bos}(\tau) \equiv \mathrm{Tr}\left(e^{-2\pi\tau_2 H}e^{2\pi i\tau_1 p}\right) = \mathrm{Tr}\left(q^{L_0-1}\bar{q}^{\tilde{L}_0-1}\right), \qquad q \equiv e^{2\pi i\tau}, \tag{2.55}$$

where the trace is taken over all discrete oscillator states and an integration over the zero-mode momenta k^μ is understood. The modular invariant partition function is given by

$$Z^X_{bos}(\tau) = \frac{1}{(4\pi^2\alpha'\tau_2)^{13}}\left|\eta(\tau)\right|^{-52}. \tag{2.56}$$

Including the ghost contribution $Z_{ghost} = |\eta(\tau)|^4$ and integrating over the fundamental domain we get the full vacuum amplitude for the bosinic string:

$$\mathcal{Z}_{bos} = (4\pi^2\alpha')^{-13}\int_F d^2\tau\frac{1}{\tau_2^{12}}\left|\eta(\tau)\right|^{-48}, \tag{2.57}$$

where we used the definition of the Dedekind eta function (2.42). The vacuum amplitude is divergent in the bosonic string due to the presence of a tachyon in the spectrum. This problem is solved by the superstring.

2.3.1 Fermionic one-loop vacuum amplitude

The quantization of fermionic fields on the torus requires the specification of a spin structure, which is a choice of periodic or anti-periodic boundary conditions for the two non-trivial cycles of the torus:

$$\sigma^1 \mapsto \sigma^1 + 2\pi, \qquad \sigma^2 \mapsto \sigma^2 + 2\pi. \tag{2.58}$$

There are four possible spin structures, which we denote by

$$(+,+),(+,-),(-,+),(-,-) \tag{2.59}$$

For the moment we focus our attention only on the right-moving sector. The NS (resp. R) sector corresponds to anti-periodic (resp. periodic) boundary conditions along the string in the σ^2 direction. In order to implement periodicity in the time direction σ^1, we insert the fermion number operator $(-1)^F$ in the traces defining the partition function. Because of the fermionic anticommutation property, its insertion in the trace flips the boundary condition around σ^1, and without it the boundary condition is anti-periodic, in accordance with the standard path integral formulation of finite temperature quantum field theory. The four fermionic spin structure contributions to the path integral are thereby given as

$$(-,+) \equiv \mathrm{Tr}_R\left(e^{-2\pi\tau_2 H}\right), \tag{2.60}$$

$$(+,+) \equiv \mathrm{Tr}_R\left((-1)^F e^{2\pi\tau_2 H}\right), \tag{2.61}$$

$$(-,-) \equiv \mathrm{Tr}_{NS}\left(e^{-2\pi\tau_2 H}\right), \tag{2.62}$$

$$(+,-) \equiv \mathrm{Tr}_{NS}\left((-1)^F e^{-2\pi\tau_2 H}\right), \tag{2.63}$$

where the subscripts on the traces indicate in which sector of the fermionic Hilbert space they are evaluated. It can be shown that the $(+,+)$ structure is modular invariant and indeed the corresponding amplitude (2.61) vanishes, because it contains a Grassmann integral over the constant fermionic zero-modes but the Hamiltonian L_0 is independent from these modes.
This is the only amplitude which contains such fermionic zero-modes. For the remaining spin structures, it follows from the modular transformations acting on the spin structures that their unique modular invariant combination is given up to an overall constant by

$$(-,-) - (+,-) - (-,+). \tag{2.64}$$

This sum is given by the following,

$$Z_{RNS} = \frac{1}{2}\,\mathrm{Tr}_{NS}\left[\left(1-(-1)^F\right) q^{L_0-\frac{1}{2}}\right] - \frac{1}{2}\,\mathrm{Tr}_R\left(q^{L_0}\right). \tag{2.65}$$

where we substituted (2.60), (2.62), (2.63).
In the first term of (2.65) there is the GSO projection operator, so using the vanishing of the $(+,+)$ structure we can write (2.65) as a trace over the full right-moving fermionic Hilbert space as

$$Z_{RNS} = \mathrm{Tr}_{NS\oplus R}\left(P_{GSO} q^{L_0-a}\right), \tag{2.66}$$

where $q = e^{i\pi\tau}$ and a is the normal ordering constant, $a = 0$ for Ramond and $a = \frac{1}{2}$ for NS. Thus the GSO projection also ensures the stability of the supersymmetric vacuum.
The total superstring amplitude in $d = 10$ spacetime dimensions is given by the product

$$\int_F d^2\tau Z_{RNS}\, q^{1/3}\eta(\tau)^{-8}, \tag{2.67}$$

where the power of 8 comes from the $d-2$ transverse oscillators, along with their left-moving counterparts. It turns out that Z_{RNS} vanishes identically. To show this we need to evaluate the traces in (2.65), and to do this we use the relation

$$\mathrm{Tr}\left(q^{\sum_{n=1}^{\infty}\psi_{-n}\psi_n}\right) = \prod_{n=1}^{\infty}(1+q^n). \tag{2.68}$$

So we can evaluate the following traces:

$$
\begin{aligned}
(-,+) &= 16q^{2/3}\prod_{n=1}^{\infty}(1+q^{2n})^8,\\
(+,-) &= q^{2/3}\prod_{n=1}^{\infty}(1-q^{2n-1})^8,\\
(-,-) &= q^{-1/3}\prod_{n=1}^{\infty}(1+q^{2n-1})^8.
\end{aligned}
\tag{2.69}
$$

Then using the Jacobi abstruse identity (2.41) we find that the combination (2.64) vanishes, since at each level there is an equal number of bosonic and fermionic degrees of freedom, coming respectively from the NS and the R sector. The spacetime bosons from the NS sector and the spacetime fermions coming from teh R sector contribute the same way in equal numbers (but with opposite signs due to statistics). This is a consequence of spacetime supersymmetry in Type II string theories.

2.4 D-branes

D-branes are string solutions for which we can provide a microscopic perturbative description. The addition of a D-brane to a string vacuum configuration corresponds to the inclusion of an open string sector. Inequivalent choices of the superconformal boundary conditions satisfied by the sigma model give rise to different D-branes. In the following we will only consider the simplest examples, D-branes in flat Minkowski spacetime or in toroidal compactifications.

2.4.1 T-duality for open strings

The way D-branes were originally discovered [26] was through the study of the $R \to 0$ limit of bosonic open strings, with the usual Neumann boundary conditions, compactified on a circle, $X^{25} \sim X^{25} + 2\pi R$. Open strings cannot wind around the periodic dimension, so when $R \to 0$ the states with nonzero momentum go to infinite mass, but in contrast to what happens in the closed string spectrum, no new continuum of states appears. The behaviour is as in field theory: the compactified dimension disappears, leaving a theory in $d-1$ spacetime dimensions. The seeming paradox arises when one remembers that theories with open strings always have closed strings as well, so that in the $R \to 0$ limit the closed strings live in d spacetime dimensions but the open strings only in $d-1$. The open string endpoints are restricted to a $d-1$ dimensional hyperplane, as follows from the T-duality transformation. The Neumann condition $\partial_n X^{25}$ for the original coordinate becomes $\partial_t X'^{25}$ for the dual coordinate. This is the Dirichlet condition: the X^{25} coordinate of the endpoint is fixed, so the endpoint is constrained to lie on a hyperplane. In fact, all endpoints are constrained to lie on the same hyperplane. To see this, integrate

$$
\begin{aligned}
X'^{25}(\pi) - X'^{25}(0) &= \int_0^{\pi} d\sigma \partial_\sigma X'^{25} = i\int_0^{\pi} d\sigma \partial_\tau X^{25}\\
&= 2\pi\alpha' p^{25} = \frac{2\pi\alpha' n}{R} = 2\pi n R'.
\end{aligned}
\tag{2.70}
$$

That is, the coordinates X'^{25} at the two ends are equal up to an integral multiple of the periodicity of the dual dimension, corresponding to a string that winds. For two different open strings one can carry out the

same argument on a path connecting any two endpoints, so all endpoints lie on the same hyperplane. The ends are still free to move in the other $d-2=p$ spatial dimensions.
The Dp-brane is a $p+1$-dimensional hypersurface in flat ten-dimensional spacetime on which open strings can end, and it is defined by choosing Dirichlet boundary conditions for the directions transverse to the brane and Neumann boundary conditions for the directions parallel to it:

$$\partial_\sigma X^\mu\big|_{\sigma=0,\pi}=0, \quad \mu=0,1,...,p, \tag{2.71}$$

$$\delta X^\mu\big|_{\sigma=0,\pi}=0, \quad \mu=p+1,...,9. \tag{2.72}$$

The positions of the Dp-brane is fixed at the boundary coordinates $X^{p+1},...X^9$ in spacetime, and this corresponds perturbatively to a particular string background. The strings are then free to move along the $p+1$-dimensional hypersurface defined by the Neumann directions. In a seminal paper [52] Polchinski showed that the Dp-branes in Type II string theories are the objects that couple minimally to the R-R gauge potentials $C^{(p+1)}$. On the other hand, all the perturbative states have zero R-R charge. Since in type IIA or IIB strings we can have respectively odd and even-dimensional R-R forms, we will also have different dimensional branes. In type IIA the Dp-branes exist for even values of p, $p=0,2,4,6,8$. The D0-brane for example is a D-particle, while the D8-brane describes a domain wall in ten dimensional spacetime. Type IIB Dp-branes exist for odd values of p, $p=-1,1,3,5,7,9$. The -1 brane is called a D-instanton, and for $p=1$ we have the D-string. Spacetime filling D-branes like the D9-branes can be consistently introduced in perturbative string theory only together with additional non-perturbative objects called orientifold planes. The resulting string modes are the Type I models which contain open and closed unoriented strings.
D-branes are dynamical p-dimensional objects. Their small fluctuations correspond to the open string modes living on their worldvolume. Let us consider briefly the massless spectrum, for instance the massless $SO(8)$ vector A_μ. Because the endpoints of the strings are tied to the worldvolume, these massless fields can be interpreted in terms of a low-energy field theory on the D-brane worldvolume. Precisely, a ten-dimensional gauge field $A_\mu(X),\ =0,1,...,9$ will split into components $A_a, a=0,1,...,p$ corresponding to a $U(1)$ gauge field on the Dp-brane, and components $\Phi_m, m=p+1,...,9$ which are scalar fields describing the fluctuations of the Dp-brane in the $9-p$ transverse directions. In the following Sections we will show that the low-energy dynamics of such a configuration is governed by a supersymmetric gauge theory, which can be obtained from the dimensional reduction of the maximally supersymmetric Yang-Mills theory in ten spacetime dimensions.

2.4.2 Gauge symmetry and Chan-Paton factors

Let us now describe a collection of N D-branes. In order to do that let us go back to open strings in ten dimensions and let us include $U(N)$ Chan-Paton factors at the string endpoints. If we compactify the X^9 direction of spacetime, $X^9\sim X^9+2\pi R$, we can study the effect of a constant background abelian gauge field

$$A_\mu=\delta_{\mu,9}\begin{pmatrix}\frac{\theta_1}{2\pi R} & & 0\\ & \ddots & \\ 0 & & \frac{\theta_N}{2\pi R}\end{pmatrix}, \tag{2.73}$$

where $\theta_i,\ i=1,...,N$ are constants. The introduction of this electromagnetic background generically breaks the Chan-Paton gauge symmetry as $U(N)\rightarrow U(1)^N$, since A_μ is invariant only under an abelian

subgroup of $U(N)$.
The background is pure gauge and locally trivial; it has however non-trivial local effects. We can write

$$A_9 = -i\Lambda^{-1}\partial_9\Lambda, \quad \Lambda(X) = \begin{pmatrix} e^{i\theta_1 x^9/2\pi R} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_N x^9/2\pi R} \end{pmatrix}, \tag{2.74}$$

with a gauge parameter $\Lambda(X^9)$ which transforms non-trivially under the translation $X^9 \mapsto X^9 + 2\pi R$:

$$\Lambda(X^9 + 2\pi R) = W \cdot \Lambda(X), \quad W = \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ 0 & & e^{i\theta_N} \end{pmatrix}. \tag{2.75}$$

The symmetry breaking mechanism induced by the Wilson line W has a very natural interpretation in the T-dual theory in terms of D-branes. The string momenta along the X^9 direction are fractional, meaning that the fields in the T-dual description have fractional winding numbers. This implies that the two open string endpoints no longer lie on the same hyperplane in spacetime. To quantify these last remarks we consider a Chan-Paton wavefunction $|k, ij\rangle$: the state i attached to an end of the open string will acquire a phase factor $e^{i\theta_i X^9/2\pi R}$ due to the gauge transformation (2.74), while the state j will have $e^{i\theta_j X^9/2\pi R}$. The total open string wavefunction will therefore gauge transform to $|k, ij\rangle \cdot e^{i(\theta_i - \theta_j)X^9/2\pi R}$, and so under a periodic translation $X^9 \mapsto X^9 + 2\pi R$ it will acquire a Wilson line factor given by

$$|k, ij\rangle \mapsto e^{i(\theta_i - \theta_j)}\, |k, ij\rangle\,. \tag{2.76}$$

In the following we will interpret the original ten-dimensional open strings as lying on N D9-branes which fill the spacetime. In this picture the string endpoints can sit anywhere in spacetime and correspond to ordinary Chan-Paton factors. Compactifying $9-p$ coordinates X^m, $m = p+1, ..., 9$ the open string endpoints are confined to N Dp-brane hyperplanes of dimension $p+1$. This is a consequence of the corresponding T-duality transformation which maps the Neumann boundary conditions to the Dirichlet ones, with all other $X^a, a = 0, 1, ..., p$ still obeying Neumann boundary conditions. Since T-duality interchanges Neumann and Dirichlet boundary conditions [50], a T-duality transformation along a direction parallel to a Dp-brane produces a D$(p-1)$-brane, while T-duality applied to a direction perpendicular to a Dp-brane yields a D$(p+1)$-brane.
To show the dynamics of D-branes we start by considering the 1+8 dimensional massless spectrum, interpreted in the T-dual string theory, where only the coordinate field X^9 is T-dualised. With N denoting the occupation number of a Chan-Paton state $|k, ij\rangle$, the mass-shell relation reads

$$m_{ij}^2 = (p_{ij}^9)^2 + \frac{1}{\alpha'}\big(\mathcal{N} - 1\big) = \frac{L_{ij}^2}{(2\pi\alpha')^2} + \frac{1}{\alpha'}\big(\mathcal{N} - 1\big), \tag{2.77}$$

where

$$L_{ij} = \big|2\pi n + (\theta_i - \theta_j)\big| R' \tag{2.78}$$

is the minimum length of an open string which winds n times between hyperplanes i and j and R is the compactification radius. To examine the massless states, we set $n = 0$, since it costs energy to wind, and $\mathcal{N} = 1$. Then the string tension T contributes to the energy of a stretched string in such a way that the mass is proportional to the distance L_{ij} between hyperplanes i and j,

$$m_{ij}^{(0)} = \frac{R'}{2\pi\alpha'}|\theta_i - \theta_j| = T \cdot L_{ij}. \tag{2.79}$$

Thus, generically massless states only arise for non-winding open strings whose ends lie on the same D-brane $i = j$. There are two such types of states/vertex operators that may be characterised as follows:

- $\alpha_{-1}^{\mu} |k, ii\rangle$, $\quad V = \partial_{\parallel} X^{\mu}$: these states correspond to a gauge field $A_{\mu}(\xi^a)$ on the D-brane, with $p+1$ coordinates tangent to the hyperplane, where $\mu, a = 0, 1, ..., p$ and $\xi^{\mu} = X^{\mu}$ are the coordinates on the D-brane worldvolume.

- $\alpha_{-1}^{m} |k, ii\rangle$, $\quad V = \partial_{\parallel} X^{m} = \partial_{\perp} X'^{m}$: these states correspond to scalar fields $\Phi^m(\xi^a)$, $m = p + 1, ..., 9$, which originate from the gauge fields in the compact dimensions of the original string theory, and which give the transverse position of the D-brane in the T-dual description. They describe the shape of the D-brane as it is embedded in spacetime, analogously to the string embedding coordinates $X^{\mu}(\tau, \sigma)$.

We conclude that if none of the D-branes coincide, $\theta_i \neq \theta_j$ for $i \neq j$, then there is a single massless vector state $\alpha^M |k, ij\rangle$ associated to each individual D-brane for $i = 1, ..., N$. Together, these states describe a gauge theory with abelian gauge group $U(1)^N$, which is the generic unbroken symmetry group of the problem. When $k \leqslant N$ D-branes coincide: $\theta_1 = \theta_2 = ... = \theta_k = \theta$, it follows from (2.79) that $m_{ij}^{(0)} = 0$ for $1 \leqslant i, j \leqslant k$. Thus new massless states appear in the spectrum of the open string theory, because strings which are stretched between these branes can now attain a vanishing length. In all there are k^2 massless vector states which, by the transformation properties of Chan-Paton wavefunctions, form the adjoint representation of a $U(k)$ gauge group. In the original open string theory, the coincident position limit $\theta_i \neq \theta_j$ for $i \neq j$ corresponds to the Wilson line

$$W = \begin{pmatrix} e^{i\theta}\mathbf{1}_k & & & 0 \\ & e^{i\theta_{k+1}} & & \\ & & \ddots & \\ 0 & & & e^{\theta_N} \end{pmatrix}. \tag{2.80}$$

$$\tag{2.81}$$

The background field configuration leaves unbroken a $U(k) \subset U(N)$ subgroup, acting on the upper left $k \times k$ block of (2.80). Thus the D-brane worldvolume now carries a $U(k)$ gauge field $\alpha_{-1}^{\mu} |k, ij\rangle \leftrightarrow A^{\mu}(\xi^a)_{ij}$; and, at the same time, a set of k^2 massless scalar fields $\alpha_{-1}^{\mu} |k, ij\rangle \leftrightarrow \Psi^m(\xi^a)_{ij}$, where $i, j = 1, ..., k$. The geometrical implications of this non-abelian $U(k)$ symmetry are rather profound. The k D-brane positions in spacetime are promoted to a matrix $\Psi^m(\xi^a)$ in the adjoint representation of the unbroken $U(k)$ gauge group. This reflects the fact that the T-dual string theory rewrites the $R \ll l_s$ limit of the original open string theory in terms of non-commuting, matrix-valued spacetime coordinates.

Generally, the gauge symmetry of the theory is broken to the subgroup of $U(N)$ which commutes with the Wilson line W. But if all N D-branes coincide, then W belongs to the center of the Chan-Paton gauge group, and we recover the original $U(N)$ gauge symmetry. The $U(N)$ symmetry is broken when some (or all) of the D-branes separate, leaving a set of massive fields, with mass equal to that of the stretched open strings. We will see precisely how this works dynamically in the next section, where we introduce the action for the D-branes.

2.5 D-brane dynamics

2.5.1 The Dirac Born-Infeld action and SYM

We have argued that D-branes have dynamical gauge fields living on their worldvolumes. The action describing the low-energy dynamics of D-branes in Type II superstring theory governs the worldvolume dynamics of the gauge fields, the transverse scalar fields and their fermionic partners. It contains the couplings of the worldsheet fields with the bulk supergravity fields, in particular with the Ramond-Ramond form potentials which couple electrically to the D-brane worldvolume. Let us consider first the massles NS-NS sector of the closed string: the spacetime metric $g_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$ and the dilaton Φ, along with the gauge field A_μ and its field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. To produce an action which is invariant under the gauge transformations of the antisymmetric tensor field $B_{\mu\nu}$, we consider the following combination of the abelian field strength $F_{\mu\nu}$ and the B field:

$$\mathcal{F}_{\mu\nu} \equiv 2\pi\alpha' F_{\mu\nu} - B_{\mu\nu}. \tag{2.82}$$

Part of the effective action for a D-brane in a background spacetime is the following Dirac-Born-Infeld action [50].

$$S_{DBI} = -T_p \int d^{p+1}\xi e^{-\Phi} \sqrt{-\det(\gamma_{ab} + B_{ab} + 2\pi\alpha' F_{ab})}, \tag{2.83}$$

where $e^{-\Phi} = \frac{1}{g_s}$, g_{ab} and B_{ab} are the pull-backs of the spacetime supergravity fields to the Dp-brane worldvolume and T_p is the D-brane tension

$$T_p = \frac{1}{\sqrt{\alpha'}} \frac{1}{(2\pi\sqrt{\alpha'})^p}. \tag{2.84}$$

The Dirac-Born-Infeld action is perturbative but exact to all orders in α', and it contains contributions from all massive and massless string states.
In a flat background the equations of motion which follow from (2.83) are

$$\left(\frac{1}{\mathbf{1} - (2\pi\alpha' F)^2}\right)^{\nu\lambda} \partial_\nu F_{\lambda\mu} = 0, \tag{2.85}$$

and they reduce to Maxwell equations in the limit $\alpha' \to 0$ that decouples all the massive string modes.

Let us now expand the flat space ($g_{\mu\nu} = \eta_{\mu\nu}, B_{\mu\nu} = 0$) DBI action (2.83) for slowly-varying fields up to terms of quartic order, F^4, $(\partial X)^4$. This is equivalent to taking the field theory limit $\alpha' \to 0$, which is defined precisely by keeping only degrees of freedom of energy $E \ll \frac{1}{\sqrt{\alpha'}}$, that are observable at length scales $L \gg l_s$. In this limit, the infinite tower of massive string states decouples, because such states have masses $m \sim \frac{1}{\sqrt{\alpha'}} \to \infty$. Using the formula $\det(A) = e^{\mathrm{Tr}\ln(A)}$, the Dirac-Born-Infeld action can be written as

$$S_{DBI} = \frac{T_p}{g_s} V_{p+1} - \frac{T_p (2\pi\alpha')^2}{4 g_s} \int d^{p+1}\xi \left(F_{ab}F^{ab} + \frac{2}{(2\pi\alpha')^2}\partial_a X^m \partial^a X_m\right) + \mathcal{O}(F^4), \tag{2.86}$$

where V_{p+1} is the (regulated) p-brane worldvolume. This is the action for a $U(1)$ gauge theory in $p+1$ dimensions with $9-p$ real scalar fields X^m. But (2.86) is just the action that would result from the dimensional reduction of $U(1)$ Yang-Mills gauge theory (electrodynamics) in ten spacetime dimensions, which is

$$S_{YM} = \frac{1}{4g_{YM}^2} \int d^{10}x F_{\mu\nu}F^{\mu\nu}. \tag{2.87}$$

Indeed, the ten dimensional gauge theory action (8.24) reduces to the expansion (2.86) of the Dp-brane worldvolume action (up to an irrelevant constant) if we take the fields A^a and $A^m = \frac{1}{2\pi\alpha'} X^m$ to depend only on the $p+1$ brane coordinates ξ^a, and be independent on the transverse coordinates $X^{p+1}, ..., X^9$. This requires the identification of the Yang-Mills coupling constant (electric charge) g_{YM} as

$$g_{YM}^2 = g_s T_p^{-1}(2\pi\alpha')^{-2} = \frac{g_s}{\sqrt{\alpha'}}(2\pi\sqrt{\alpha'})^{p-2}. \tag{2.88}$$

For multiple D-branes, there is not a simple effective action which can be given in closed form like the abelian Dirac-Born-Infeld action. In general, the low-energy dynamics of N parallel, coincident Dp-branes in flat space is described in the static gauge by the dimensional reduction to $p+1$ dimensions of $\mathcal{N} = 1$ supersymmetric Yang-Mills theory with gauge group $U(N)$ in ten spacetime dimensions.
The various fields on the brane representing the collective motions, A^a and X^m, become matrices valued in the adjoint representation of the gauge group. As a consequence the spacetime background fields which couple to the branes should be promoted to functions of the brane matrix coordinates. Furthermore, in the Abelian case we have used the partial derivatives $\partial_a X^\mu$ to pull back the spacetime indices μ to the worldvolume indices a, e.g., writing for a field F_μ, $F_a = F_\mu \partial_a X^\mu$ and so on. To make this gauge covariant in the non-Abelian case, the pull back should be performed with the covariant derivative: $F_a = F_\mu \mathcal{D}_a X^\mu = F_\mu(\partial_a X^\mu + [A_a, X^\mu])$.
We also need a prescription to perform the trace over the matrix coordinates in order to get a gauge invariant action. A prescription consistent with various studies of scattering amplitudes is to use the symmetric trace, denoted STr. The resulting action has the following form

$$S_p = -T_p \int d^{p+1}\xi e^{-\Phi} \mathcal{L}, \quad \text{where} \tag{2.89}$$

$$\mathcal{L} = \text{STr}\left(\det^{1/2}\left[E_{ab} + E_{ai}(Q^{-1} - \delta)^{ij} E_{jb} + 2\pi\alpha' F_{ab}\right] \det^{1/2}[Q^i_j]\right), \tag{2.90}$$

where $Q^i_j = \delta^i_j + 2\pi i\alpha'[\Phi^i, \Phi^k]E_{kj}$, and we have raised indices with E^{ij}.
Expanding the action (2.89) to second order in the gauge field, and noting that

$$\det[Q^i_j] = 1 - \frac{(2\pi\alpha')^2}{4}[\Phi^i, \Phi^j][\Phi^i, \Phi^j] + ..., \tag{2.91}$$

we can write the leading order action (2.89) as follows

$$S_p = -\frac{T_p(2\pi\alpha')^2}{4} \int d^{p+1}\xi e^{-\Phi}\, \text{Tr}\left(F^{ab}F_{ab} + 2\mathcal{D}_a\Phi^i \mathcal{D}_a \Phi^i + [\Phi^i, \Phi^j]^2\right). \tag{2.92}$$

This is the dimensional reduction of the d-dimensional Yang-Mills term, displaying the non-trivial commutator for the adjoint scalars. A more detailed discussion can be found for example in [35].

2.5.2 BPS states

D-branes are non-perturbative states of the Type II superstring which carry non-zero RR charge. While the Type II vacuum is invariant under the $\mathcal{N} = 2$ supersymmetry algebra in ten dimensions, the presence of a collection of N parallel Dp-branes breaks half of the supersymmetry charges. The resulting configuration is then invariant under the $\mathcal{N} = 1$ supersymmetry algebra in ten dimensions [46] and is an example of an half BPS (Bogomolny-Prasad-Sommerfeld) state [53].

In general the BPS states of a supersymmetric theory are states that preserve a fraction of the total number of supersymmetry charges. They can carry several conserved charges which are related by the supersymmetry algebra to their mass. This is in particular the case for D-branes. As we will see in the following studying the one-point functions of the bulk field on the brane worldvolume, the D-brane tension is proportional to the Ramond-Ramond charge. The fact that the D-branes are half-BPS states simplifies the derivation of their effective description in the low-energy effective action as solitonic solutions of the classical supergravity equations of motion. It is also worth mentioning that BPS bound states of D-branes can be analysed using the low-energy supersymmetric Yang-Mills theory on the worldvolume and their energies can be matched with the energy of the corresponding field configuration

$$E_{YM} = \frac{\pi^2\alpha' T_p}{g_s}\int d^{p+1}\xi\,\mathrm{Tr}(F_{ab}F^{ab}). \tag{2.93}$$

The relation between the mass and the charge of a half-BPS state is protected by non-renormalization theorems and cannot be modified by any perturbative or non-perturbative effect.

2.6 D-brane charge and tension

In the following we will compute the charge and the tension of a D-brane using the boundary state formalism (see for instance [22] or the review [30]). The boundary state $|B\rangle$ is a BRST invariant functional which implements a superconformal boundary condition on the closed string Hilbert space. It was originally introduced in the study of the factorization of the one-loop open string diagrams in the closed channel. The boundary state can be constructed by solving the equations which relate the left and the right-moving fields in the presence of a boundary. It can be expressed as an infinite superposition of closed string states with coefficients related to the one-point functions of the corresponding vertex operators on the disk. These one-point functions give the coupling between the bulk fields and the brane worldvolume.

For instance the D-brane tension is given by the coupling between the D-brane and the graviton, and the D-brane charge by the coupling between the D-brane and the R-R $(p+1)$-form potential. The simplest way to derive these couplings is to factorize on the massless states the annulus vacuum amplitude which describes the exchange of a closed string between two D-branes, as depicted in Fig. 2.6.

The D-brane tension can be computed either in the open string channel as a loop of open strings [50] or in the closed string channel as a tree-level amplitude of closed strings propagating between two boundary states. The two computations are related by a modular transformation which exchanges the two directions on the worldsheet.

The boundary state for a D-brane located at y is given by

$$|B\rangle = \mathcal{N}\delta^{(d_\perp)}(q-y)\exp\Big[-\sum_{n=1}^{\infty} a_n^{\mu\dagger}\mathcal{S}_{\mu\nu}\tilde{a}_n^{\nu\dagger}\Big]\,|0;\,k=0\rangle\,, \tag{2.94}$$

where the delta function is over the $d_\perp = d-p-1$ directions transverse to the D-brane and $a_n^{\mu\dagger}$, $\tilde{a}_n^{\nu\dagger}$ are the left and right-moving string oscillators. The matrix

$$\mathcal{S}_{\mu\nu} = (\eta_{\alpha\beta},\,\delta_{ij}) \tag{2.95}$$

implements the Dirichlet boundary conditions, due to to the presence of the D-brane, by performing a T-duality transformation on the last $d_\perp$ directions. The normalization constant $\mathcal{N}$ can be fixed by comparing

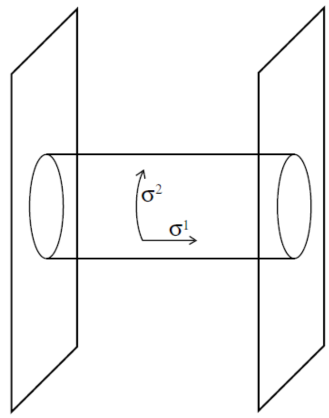

Figure 2.5: The exchange of a closed string between two D-branes can also be interpreted as a one-loop open string vacuum amplitude.

the amplitude evaluated in the closed channel with the amplitude evaluated in the open channel and it is given by

$$\mathcal{N} = \sqrt{\frac{2}{\alpha'\pi}}(2\pi\sqrt{2\alpha'})^{-d/2}(2\pi\sqrt{\alpha'})^{d_\perp}. \tag{2.96}$$

We parametrise the worldsheet with

$$0 < \sigma_1 < \pi, \quad 0 < \sigma_2 < 2\pi t, \quad 0 < t < \infty, \tag{2.97}$$

where t is the modulus of the annulus. A horizontal time-slicing gives an open string loop where σ_2 is the worldsheet time, while a vertical time-slice gives a closed string tree-level propagation with σ_1 as the time. If Y^μ is the separation between the branes, the one-loop amplitude is given by the Coleman-Weinberg formula, a sum of the zero-point energies of all the modes [53]:

$$\mathcal{A} = V_{p+1}\int\frac{d^{p+1}k}{(2\pi)^{p+1}}\int_0^\infty\frac{dt}{2t}\sum_I e^{-2\pi\alpha' t(k^2+M_I^2)}, \tag{2.98}$$

where the sum is over the string spectrum and the momentum k is restricted to the brane world-volume. The trace is taken in the NS and R sectors with the appropriate GSO projection. The amplitude is

$$\mathcal{A} = \int_0^\infty\frac{dt}{2t}\,\mathrm{Tr}_{NS\bigoplus R}\left(\frac{1+(-1)^F}{2}e^{-2\pi t L_0}\right). \tag{2.99}$$

Performing the traces over the open superstring spectrum gives

$$\mathcal{A} = 2V_{p+1}\int\frac{dt}{2t}(8\pi^2\alpha' t)^{-(p+1)/2}e^{-t\frac{Y^2}{2\pi\alpha'}}\,\frac{1}{2}\left(\frac{2\pi}{\theta_1'(0,\tau)}\right)^4\left(-\theta_2^4+\theta_2^4-\theta_4^4\right), \tag{2.100}$$

where $q = e^{-2\pi t}$. The three terms in brackets come from the open string R sector with $1/2$ in the trace, from the NS sector with $1/2$ in the trace, and the NS sector with $1/2(-1)^F$ in the trace.
The R sector with $1/2(-1)^F$ gives no net contribution, and the three terms sum to zero by Jacobi abstruse

identity, as we would expect from the computation of a supersymmetric vacuum diagram.
From the closed string point of view the vanishing of the amplitude is due to the fact that there is no net force between parallel D-branes of the same dimensionality. The attraction due to the exchange of NS-NS fields is balanced by the repulsion caused by the exchange of R-R fields.
We can read the tension and the charge of the brane from the $t \to 0$ limit of the previous amplitude. Comparing the result with the normalization of the corresponding effective field theory diagrams and using

$$2\kappa^2 = 16\pi G_N = (2\pi)^7 \alpha'^4 g_s^2 \tag{2.101}$$

one finds

$$\mu_p = (2\pi)^{-p} \alpha'^{-\frac{p+1}{2}}, \quad \tau_p = g_s^{-1} \mu_p. \tag{2.102}$$

This formula for the tension gives for the D1-brane

$$\tau_1 = \frac{1}{2\pi\alpha' g_s}, \tag{2.103}$$

which sets the ratios of the tension of the fundamental string, $\tau_1^F \equiv T = \frac{1}{2\pi\alpha'}$, and the D-string to be simply the string coupling g_s.
If the D-branes are not parallel, they feel a net force since the cancellation is no longer exact. The extreme case is obtained when one of the D-branes is rotated by π since the coupling to the dilaton and the graviton are unchanged but the coupling to the R-R tensor changes sign. This configuration is in fact equivalent to a system of branes and anti-branes which is not static. The two objects attract each other and eventually annihilate. The instability of the system is reflected in the presence of a mode in the spectrum which becomes tachyonic when the separation of the branes becomes of the order of the string length.

2.7 *p*-branes in Type II string theories

In this section we will introduce the p-brane metric which will be used in this Thesis, especially in Chapter 4. The type II effective actions have static solutions which are charged under the R-R gauge fields and invariant under the $p+1$-dimensional Poincaré group. The solutions charged under $A_{(p+1)}$ are called black p-branes. One solution in the family is a half-BPS state and is called the extremal p-brane solution. For $0 \leqslant p \leqslant 7$ the solution has the following form in the string frame:

$$\begin{aligned} ds^2_{Str} &= H^{-1/2}(r)\Big(- dt^2 + (dx^1)^2 + ... + (dx^p)^2\Big) + H^{1/2}(r)\Big((dx^{p+1})^2 + ... + (dx^9)^2\Big), \\ F_{(p+2)} &= dH^{-1}(r) \wedge dt \wedge dx^1 \wedge ... \wedge dx^p, \\ e^{-2\Phi} &= H^{(p-3)/2}(r), \end{aligned} \tag{2.104}$$

where

$$r^2 = \Big((x^{p+1})^2 + ... + (x^9)^2\Big) \tag{2.105}$$

and $H(r)$ is a harmonic function of the transverse coordinates $(x^{p+1}, ..., x^9)$:

$$\Delta^\perp H = \sum_{i=p+1}^{9} \partial_i \partial_i H = 0. \tag{2.106}$$

By requiring that the solution becomes asymptotically flat at transverse infinity and normalizing the metric such that it approaches the standard Minkowski metric we get

$$H(r) = 1 + \frac{Q_p}{r^{7-p}}. \tag{2.107}$$

Q_p measures the flux of the R-R field strength at infinity. A convenient way to parametrise it is the following:

$$Q_p = N c_p, \quad c_p = \frac{(2\pi)^{7-p}}{(7-p)\Omega_{8-p}} (\alpha')^{\frac{7-p}{2}} g_s, \tag{2.108}$$

where c_p is the fundamental quantum of R-R p-brane charge and N an integer which counts the number of branes. Ω_n is the volume of the n-dimensional unit sphere,

$$\Omega_n = \frac{2\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}. \tag{2.109}$$

In order to review the most important features of the metric (2.104), which can be seen as a generalization of the extreme Reissner-Nordstrom solution of four-dimensional Einstein-Maxwell theory, we rewrite it in the Einstein frame

$$ds^2_{Einst} = -H^{\frac{p-7}{8}}(r)\Big(-dt^2 + (dx^1)^2 + ... + (dx^p)^2\Big) + H^{\frac{p+1}{8}}(r)\Big(((dx)^{p+1})^2 + ... + (dx^9)^2\Big). \tag{2.110}$$

The directions $t, x_1, ..., x_p$ are called longitudinal or world-volume directions, the others transverse directions. Since the solution has translational invariance, it has infinite mass unless the worldvolume directions are compactified on a torus. However, the tension T_p (the energy per unit volume) is finite. Since the solution becomes asymptotically flat in the transverse directions, the tension can be defined by a generalization of the ADM construction of general relativity. The curvature of the extremal p-brane spacetime can be expressed in terms of the following length scale

$$R_p^{d-3-p} = \frac{16\pi G_N^{(d)} T_p}{(d-2)\Omega_{d-2-p}}. \tag{2.111}$$

For $r \to 0$ the solution (2.104) has a null singularity. The only exception is the 3-brane which is regular. The p-brane (2.104) is the extremal limit of a more general black p-brane solution, which has a time-like singularity along a p-dimensional surface and a regular event horizon. In the extremal limit, the singularity and the even horizon coincide. This behaviour is similar to the Reissner-Nordstrom black hole.
Black p-brane solutions satisfy the Bogomol'nyi bound which relates their tension T_p and charge Q_p as follows:

$$T_p \geqslant \hat{Q}_p, \tag{2.112}$$

where $\hat{Q}_p$ is a redefined charge which has the dimension of a tension. This inequality guarantees the existence of an event horizon hiding the singularity, just as for charged black holes. The extremal solution has a multicentered generalization. When replacing $H(r)$ by

$$H(\mathbf{x}_\perp) = 1 + \sum_{i=1}^{N} \frac{|Q_p^{(i)}|}{|\mathbf{x}_\perp - \mathbf{x}_\perp^{(i)}|^{7-p}}, \tag{2.113}$$

one still has a static solution, provided that all the charges $Q_p^{(i)}$ have the same sign. Here $\mathbf{x}_\perp = (x^{p+1}, ..., x^9)$ and $\mathbf{x}_\perp^{(i)}$ is the position of the i-th p-brane. It is remarkable that the solution is static for arbitrary positions

$\mathbf{x}_\perp^{(i)}$, because this implies that the gravitational attraction and the 'electrostatic' repulsion cancel precisely. The remarkable properties of these solutions are due to the fact that the extremal p-brane spacetime is a half-BPS solution of Type II supergravity. The maximal number of Killing spinors is equal to the number of supersymmetry charges, which is 32 in Type II theory. Solutions with the maximal number of Killing spinors are invariant under all supersymmetry transformations. They are the analogues of maximally symmetric spaces in Riemannian geometry, which by definition have as many isometries as flat space.
The maximally supersymmetric solutions of Type IIB supergravity in ten-dimensions are for instance flat ten-dimensional Minkowski space, $AdS_5 \times S^5$ and a special pp-wave solution which is the Penrose limit of $AdS_5 \times S^5$. The p-brane solution (2.104) has 16 Killing spinors and is an half-BPS state. BPS states have the minimal tension possible for their charge and are absolutely stable. This minimization of energy also accounts for the existence of static multicentered solutions.

The extremal p-brane solutions provide an effective gravitational description of the D-branes.

Since p-branes are extended supergravity solutions with non-trivial spacetime metric, whereas D-branes are string solitons in flat space-time, we should of course be more precise in what we mean by the previous identification. We have seen that both kinds of objects have the same charge and tension, the same global symmetries and are both half-BPS. They seem to represent the same BPS state of the theory but in different regions of the parameter space. A description in terms of N D-branes works within string perturbation theory. In the presence of D-branes the effective open string coupling is Ng_s instead of g_s, since in a background with D-branes every boundary component can end on each of the N D-branes and therefore g_s always occurs multiplied by N. The perturbative string description is then reliable if

$$Ng_s \ll 1. \tag{2.114}$$

Using the brane radius (2.111) we see that this is equivalent to

$$R_p \ll \sqrt{\alpha'}, \tag{2.115}$$

which means that the gravitational scale is much smaller then the string scale. In this region of the parameter space the effective supergravity description will be unreliable since it will receive important higher-derivative corrections. On the other hand, the extremal p-branes are solutions of the type II effective actions where we neglect α' corrections, which become relevant only when the curvature, measured in string units, becomes large. The condition for having small curvature in string units is

$$R_p \gg \sqrt{\alpha'}, \tag{2.116}$$

or, equivalently

$$Ng_s \gg 1, \tag{2.117}$$

which is opposite to (2.114), (2.115). The p-brane solution should then capture the strong coupling limit of the perturbative string description of a collection of N parallel D-branes. One can interpolate between the two regimes changing the string coupling g_s while keeping the charge Nc_p fixed. In general it is not clear that one can believe in the results of such interpolations, but in our case we know that the p-brane/D-brane is the object of minimal tension for the given charge. There is no mechanism compatible with supersymmetry through which this state could decay or become a non-BPS state and several checks have been performed of this correspondence between D-branes and extremal p-branes.

A very direct connection between the classical D-brane solution and its microscopic string description is provided by the boundary state introduced in Section 2.6. It was shown in [29] that the long-distance behavior of the fields of the supergravity solution can be reproduced by studying the emission of massless string states from the boundary state.

3

High energy scattering in massless String Theory

In this Chapter we will discuss the high-energy limit of string scattering in different kinematical regimes, and review the trans-Planckian S-matrix analysis of Amati, Ciafaloni and Veneziano (ACV) [7–10]. Our main motivation is the fact that at high energy, where gravitational effects dominate, string theory allows for a consistent evaluation of scattering amplitudes involving graviton exchange.

We will study the scattering of two massless strings (e.g. two gravitons) at center of mass energy

$$2E = \sqrt{s} \gg M_{Planck} \tag{3.1}$$

and impact parameter b. We consider Type II strings compactified to d-dimensions on a $(10-d)$-dimensional torus. Let us recall that in string theory there is a fundamental length scale, the string length l_s

$$l_s = \sqrt{\alpha'}. \tag{3.2}$$

The Planck length λ_P, which is the quantum gravity scale, is related to l_s by

$$\lambda_P^{d-2} = G_N \propto \frac{g l_s^8}{V}, \tag{3.3}$$

where G_N is the Newton constant, V the volume of the torus and $g \ll 1$ is the string loop expansion parameter, assumed to be small.

Another relevant scale is the gravitational length scale associated with the total energy in the center of mass frame, the Schwarzschild radius

$$R_s^{d-3} = 2G_N\sqrt{s}. \tag{3.4}$$

The ratio

$$\left(\frac{R_s}{l_P}\right)^{d-3} = \sqrt{s\,R_s}\,, \tag{3.5}$$

shows that in our high-energy regime R_s is much larger than λ_P. However from the ratio

$$\left(\frac{R_s}{l_s}\right)^{d-3} = g\sqrt{s\,l_s}\left(\frac{l_s}{V}\right)^{10-d}, \tag{3.6}$$

we see that R_s can be larger or smaller than the string length.

So far, we have highlighted three fundamental length scales: b, R_s, or l_s and we can find three distinct regimes according to which one of the three exceeds the other two.

If $b \gg R, l_s$ one deals with *small deflection-angle scattering*. This is well described by a leading eikonal approximation with small string-size and classical corrections corresponding to the expansion parameters l_s/b and R_s/b, respectively. The former can be interpreted as string excitations due to the tidal forces induced on each string by the effective shock-wave metric produced by the other string, which is of the Aichelburgh-Sexl type. This is the regime we will analyse in this Chapter.

In the regime $l_s \gg b, R_s$, also investigated through *fixed-angle scattering* [40–43], the classical gravitational collapse conditions are never met, because the threshold for black hole formation is given by $E_{threshold} \sim M_s g^{-2} \sim M_P g^{-1}$. However, by performing the limit $l_s \to R_s > b$, it can be found that a black-hole-like behaviour can occur even below the expected threshold for the actual black hole production. The analysis of this regime suggested the generalized uncertainty relation [9]

$$l_s < \frac{1}{\Delta p} + \alpha' \Delta p < \Delta x, \tag{3.7}$$

according to which the minimal scale which can be probed by perturbative strings is l_s.

By contrast, for $R_s > l_s$, new semiclassical gravitational phenomena take place. They extend beyond the impact parameter at which string fluctuations, including those due to diffractive excitations, are large. The interesting region is the one in which b approaches R_s from above and possibly goes below it, a situation in which, classically, a gravitational collapse would take place. This has been studied for example in [11].

In this Chapter we will review only the scattering at large impact parameters $b > R_s > l_s$ and show that the resummed multi-loop S-matrix exponentiates in terms of an eikonal function of order $G_N s$ which can be expanded in powers of R_s/b. The results at leading order can be interpreted as the propagation of one string in the effective metric produced by the other. If string effects are neglected, one recovers the classical 't Hooft result reviewed in Chapter 1.

This Chapter is organised as follows: first we will introduce the eikonal approximation in string theory in Section 3.1, which leads to the description of high-energy string-string scattering, following the work of ACV. We start by computing the four-point closed string amplitude at tree-level and one-loop in Section 3.2. Then in Section 3.3 we will resum the leading high-energy contributions to all orders in the string coupling, and note that it can be written in an eikonal form. In Section 3.4 we will analyse the behaviour of the resummed amplitude in different regions of impact parameter space, and finally we show that the resummed amplitude does not violate unitarity in Section 3.5. A discussion on these results is given in Section 3.6.

We will also investigate another interesting high-energy regime, that of fixed angle scattering, by computing the four-tachyon amplitude. To conclude this Chapter, in Section 3.8 we point out the importance of the massive string states. The inelastic channels corresponding to their excitation become in fact dominant at impact parameters much larger than the string scale, as clearly shown by the eikonal operator. Therefore we review the density of string states, the Hagedorn phase transitions and the instability of massive states. At the end we will get the vertex operators for the massive states lying on the leading Regge trajectory of the NS-NS sector of the superstring, which will be useful in the following Chapters.

3.1 Eikonal approximation in string theory

For large s and small momentum transfer a closed string tree amplitude is dominated by the exchange of "Reggeized" gravitons, i.e, soft particles in the Regge trajectory of the graviton:

$$\mathcal{M} \sim \beta_g(t) s^{2+\alpha' t} + O(s^{1-\alpha' t}) \tag{3.8}$$

where $\beta_g(t) \xrightarrow{t\to 0} \frac{1}{t}$, and also at higher orders the amplitudes are dominated by multiple soft-graviton exchange. In this context the eikonal approximation can be easily applied to string theory, since it allows to resum the exchange of soft particles order by order in perturbation theory.
To illustrate the eikonal approximation in string theory, we consider the scattering of two scalars (dilatons) by multiple Reggeized graviton exchange in the limit of large s and fixed momentum transfer q. We note that in the center of mass frame the momentum transfer is transverse, $q^2 = -q_\perp^2$.
The N_{th} term in the eikonal series is [49]

$$\mathcal{M}_N = -2is\frac{i^N}{N!}(\kappa^2 s)^N (2\pi)^{(2-d)(N-1)} \int dk_\perp(1)...dk_\perp(N)\delta^{D-2}\left(\vec{q}_\perp - \sum_{j=1}^{N} k_\perp(j)\right)\prod_{l=1}^{N}\frac{F(k^2(l),s)}{k_\perp^2(l)}, \tag{3.9}$$

where κ is the gravitational coupling and a minimal form for F is $F(k_\perp^2, s) = s^{-\alpha' k_\perp^2}$. It is a form factor for the tree-level Reggeized graviton exchange at small momentum transfer. In general, F can be a four-point tree amplitude containing the graviton Regge trajectory. The dominant contributions come from soft graviton lines, this is why Reggeized graviton exchange is the most relevant process. So the total momentum transfer to a heavy line at any point is assumed to be small; this approximation is valid at fixed order. Moreover, graphs with interaction between soft gravitons are suppressed since their self-interaction is weak.
After summing over N and switching to the impact parameter representation, the relativistic eikonal amplitude reads

$$\mathcal{M}(s, q^2) = -2is\int d^{d-2}x_\perp e^{iq_\perp\cdot x_\perp}\left(e^{ik^2 sA(|k_\perp|,s)} - 1\right), \tag{3.10}$$

where A is the Fourier transform of $F(k_\perp^2, s)/k_\perp^2$. An important application of the eikonal form of the amplitude in string theory has been given by Amati, Ciafaloni and Veneziano in the context of four-graviton scattering at Planckian energies and small momentum transfer. In the following, we will illustrate their work as an introduction to the main topic of this Thesis.

In a series of papers [7–9], Amati, Ciafaloni and Veneziano introduced a technique to resum Type II superstring loop amplitudes for high s and small t in flat spacetime. Their result for the resummed amplitude was an operator eikonal form, whose expansion showed classical terms in agreement with General Relativity, and quantum string corrections. The loop amplitudes can be obtained by the usual string multi-loop expansion or by the equivalent Regge-Gribov method [39], consisting in cutting and gluing tree amplitudes. In [8] for example, it was shown that at one-loop level the two methods give the same result, and that the Regge-Gribov technique can be extended to any number of loops, leading to the resummation of the whole series in the string loop parameter.

The asymptotic form of the graviton-graviton loop amplitude is given by the exchange of two massless particles in the leading Regge trajectory of the graviton. Its angular momentum representation shows a fixed pole whose residue can be defined as the two gravi-Reggeon vertex, i.e, the operator describing the exchange of gravitons in the leading Regge trajectory.
Since we want to resum to all orders in string loops, we will keep the string loop expansion parameter small

$$G_N \alpha'^{1-\frac{d}{2}} \ll 1, \tag{3.11}$$

where D is the number of space-time dimensions and G_N is Newton constant, related to α', the Planck mass M_P and the D-dimensional gauge coupling g by the following relation:

$$\frac{g^2}{4\pi}\alpha' = 4G_N = 4M_P^{2-d}. \tag{3.12}$$

The kinematic regime of interest is the Regge regime, high center of mass energy

$$\alpha' s \gg (M_P\sqrt{\alpha'})^{d-2} \gg 1 \tag{3.13}$$

and small momentum transfer

$$|t| < \frac{1}{\alpha'}, \quad |t| < M_c^2, \tag{3.14}$$

where M_c is the compactifiction radius, so that the compactified momentum excitations are negligible.
In impact parameter space these conditions are more relaxed. We also introduce the Schwarzschild radius

$$R_s^{d-3}(E) = 2G_N E \tag{3.15}$$

which is the relevant length associated with an energy E, to state the conditions in impact parameter space:

$$b > R_s(E), \quad b > l_s = \sqrt{2\alpha'}, \tag{3.16}$$

l_s being the fundamental quantum string length.

3.2 Four-graviton loop amplitude

In the following we will set $\alpha' = 2$. The four graviton tree amplitude in the Regge limit is given by

$$\begin{aligned} A_{tree}(ab \to cd) &= \epsilon_a \cdot \epsilon_d \, \epsilon_b \cdot \epsilon_c \, a_{tree}(s,t), \\ a_{tree} &\sim 2g^2 \frac{\Gamma\left(-\frac{t}{2}\right)}{\Gamma\left(1+\frac{t}{2}\right)} \left(\frac{s}{2}\right)^{2+t} e^{-it\frac{\pi}{2}} = \beta(t) s^{\alpha(t)}, \end{aligned} \tag{3.17}$$

where $\alpha(t) = 2 + t$ is the leading Regge trajectory of the graviton.
Since the amplitude (3.17) in impact parameter space violates partial wave unitarity, loop corrections are important.
We will compute the asymptotic form of the loop amplitude to show that it is given by the double graviton Regge exchange, as depicted in Fig. 3.1.

We already computed one-loop amplitudes in Type II string theory in Chapter 2, so we will skip the details of the measure of the moduli and the fermionic spin structures. As introduced in Chapter 2, we will consider a torus with modular parameter τ , obtained identifying points in the complex plane according to

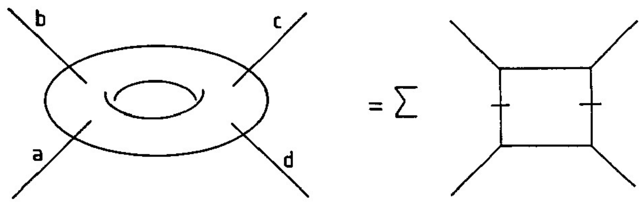

Figure 3.1: The loop amplitude for four-graviton scattering is given by the double graviton Regge exchange.

$z \sim z + 2\pi \sim z + 2\pi\tau$. We will be using the variable ν related to z by $z = 2\pi\nu$. The asymptotic behaviour of loop amplitudes can be evaluated by saddle-point methods as discussed in [6], where it is shown that the dominant contributions to the scattering amplitude come from the region of large $Im(\tau)$.
The amplitude is given by

$$\mathcal{A}_4 = 2g^2\mathcal{K}\,\frac{g_{10}^2}{(4\pi)^2}\int_F \frac{d^2\tau}{(Im\tau)^5}F_2(\tau, R_{c,i})\int \prod_{i=a,b,c} d^2\nu_i \prod_{i<j}\chi_{ij}^{2k_i\cdot k_j}, \tag{3.18}$$

where the dominant term in the kinematical factor is $\mathcal{K} \sim \left(\frac{s}{2}\right)^4 \epsilon_a\cdot\epsilon_d\,\epsilon_b\cdot\epsilon_c$ and F_2 is the compactification factor for closed strings. For toroidal compactification it depends on the radii $R_{c,i}$. Since we consider $t \ll R_{c,i}^{-2}$, the details of compactification do not enter. In this loop integral the fundamental region F is

$$\begin{aligned} &|Re\,\tau| < \frac{1}{2}, \quad |Re\,\nu_i| < \frac{1}{2}, \quad i = a, b, c \\ &0 \leqslant Im\,\nu_i \leqslant Im\,\tau, \quad 1 \leqslant Im\,\tau < \infty, \end{aligned} \tag{3.19}$$

but the relevant limit in parameter space is $Im\tau \to \infty$, so for F_2 we find [8]

$$g_{10}^2 F_2(\tau, R_{c,i}) \xrightarrow{Im\tau\to\infty} g^2(8\pi^2 Im\tau)^{5-D/2}, \tag{3.20}$$

The correlation functions on the torus $\chi_{ij} = \chi\left(e^{2\pi i(\nu_i-\nu_j)}, e^{2\pi i\tau}\right)$ were given in Section 2.2.3:

$$\chi\left(e^{2\pi i\nu},\, e^{2\pi i\tau}\right) = 2\pi\left|\frac{\theta_1(\nu,\tau)}{\theta_1'(0,\tau)}\right|\exp\left(-\frac{\pi(Im\nu)^2}{Im\tau}\right), \tag{3.21}$$

which in the relevant limit $\tau \to \infty,\ \frac{\nu}{\tau} > 0$ fixed, yields the following asymptotic behaviour:

$$\chi\left(e^{2\pi i\nu},\, e^{2\pi i\tau}\right) \sim \exp\left[-\frac{\pi(Im\nu)^2}{Im\tau} + Re\left(i\pi\nu + e^{2i\pi\nu} + e^{2\pi i(\tau-\nu)}\right)\right] \tag{3.22}$$

The amplitude on the mass shell $(s + t + u = 0)$ is convergent for imaginary s, so we have to perform the $s \to i\infty$ limit.

The correlators in (3.18) are given by

$$f = \left(\frac{\chi_{ab}\chi_{cd}}{\chi_{ac}\chi_{bd}}\right)^{-s}\left(\frac{\chi_{bc}\chi_{ad}}{\chi_{ac}\chi_{bd}}\right)^{-t}, \quad \nu_d = \tau. \tag{3.23}$$

From (3.22) we can deduce that the relevant integration region for $s \to i\infty$ is around the points

$$Im(\nu_b - \nu_c) = 0,\ Im(\tau - \nu_a) = 0 \tag{3.24}$$

and the region $Im\tau = O(\log s)$. So we first integrate the asymptotic form of χ in (3.22) in this region:

$$\begin{aligned}
&\int dIm(\tau-\nu_a)\, dIm(\nu_b-\nu_c)\chi \sim \\
&\int dIm(\tau-\nu_a)\, dIm(\nu_b-\nu_c)\exp\left[-\frac{2\pi s}{Im\tau} Im(\tau-\nu_a)\, dIm(\nu_b-\nu_c)\right] \xrightarrow{s\to i\infty} \frac{2i}{s} Im\tau. \qquad (3.25)
\end{aligned}$$

Once we integrate those variables out, we can integrate the function f by setting $Im(\tau-\nu_a) = Im(\nu_b - \nu_c) = 0$. To perform the remaining integrals we introduce the new variables a, b, c and x as follows

$$\begin{aligned}
Re\,\nu_a &= Re\,\tau - (b+c), \\
Re\,\nu_b &= Re\,\tau - (a+c), \\
Re\,\nu_c &= Re\,\tau - (a+b), \\
x &= \frac{Im(\nu_b+\nu_c)}{2Im\,\tau}. \qquad (3.26)
\end{aligned}$$

Substituting into (3.23) and setting $Im(\tau-\nu_a) = Im(\nu_b-\nu_c) = 0$ we find

$$\begin{aligned}
f &= [4\sigma_+\sigma_- \exp(2\pi x(1-x)Im\,\tau)]^{-t} \\
&\quad \times \exp\left[s\Big(e^{-2\pi x Im\tau} cos(2\pi a) + e^{-2\pi(1-x)Im\tau}\cos 2\pi(Re\,\tau - a)\Big)4\sigma_+\sigma_-\right], \\
\sigma_\pm &= |\sin\pi(b\pm c)| \qquad (3.27)
\end{aligned}$$

The integrations over $Re\,\tau$ and a give two Bessel functions

$$B(a,b) = 2\int_0^{\pi/2} dx \sin^{2\alpha-1} x \cos^{2\beta-1} x. \qquad (3.28)$$

Restoring the numerical factors and the kinematic term the amplitude can be written as follows:

$$\mathcal{A}^{(h=1)} \xrightarrow{s\to i\infty} g^4 \epsilon_a\cdot\epsilon_d\,\epsilon_b\cdot\epsilon_c\, i\left(\frac{s}{2}\right)^3 I, \qquad (3.29)$$

$$\begin{aligned}
I = \int_0^1 dx \int_0^\infty dIm\tau (Im\tau)^{\frac{4-d}{2}} &\int_0^1 \frac{d\sigma_+ d\sigma_- (4\sigma_+\sigma_+)^{-t}}{[(1-\sigma_+^2)(1-\sigma_-^2)]^{1/2}} \frac{2^{2-d/2}}{\pi(2\pi)^{d-4}} e^{2\pi t x(1-x)Im\tau} \\
&\times J_0(4is\sigma_+\sigma_- e^{-2\pi x Im\tau})\, J_0(4is\sigma_+\sigma_- e^{-2\pi(1-x)Im\tau}). \qquad (3.30)
\end{aligned}$$

We follow the derivation of [8], writing the amplitude in a form involving powers of s, so that it is easier to interpret the Regge cut.

The integrations over $\sigma_\pm$ and $Im\,\tau$ yield Bessel functions in the form of series expansions, so we can use the identity

$$\int_0^1 dx \frac{\pi^{\frac{d-2}{2}}\Gamma\left(\frac{6-d}{2}\right)}{[q^2 x(1-x) + 2nx + 2m(1-x)]^{\frac{6-d}{2}}} = \int \frac{d^{d-2}\bar{k}}{\left[2n + \left(\frac{1}{2}\bar{q}+\bar{k}\right)^2\right]\left[2m + \left(\frac{1}{2}\bar{q}-\bar{k}\right)^2\right]}, \qquad (3.31)$$

where x plays the role of a Feynman parameter for a diagram with two propagators and an integrated momentum over $d-2$ dimensions:

$$I = \frac{1}{2}\int \frac{d\bar{k}^{d-2}}{(2\pi)^{d-2}} \sum_{n,m} \frac{(-1)^{n+m}\left(\frac{is}{2}\right)^{2(n+m)}}{(n!)^2(m!)^2} \frac{4^{2n+2m-t} B^2\left(1-\frac{t}{2}+m+n, \frac{1}{2}\right)}{\pi^2\left[2n+\left(\frac{1}{2}\bar{q}+\bar{k}\right)^2\right]\left[2m+\left(\frac{1}{2}\bar{q}-\bar{k}\right)^2\right]}. \qquad (3.32)$$

In order to evaluate the double sum we rewrite it as a Cauchy integral and then deform the contour integration, to pick the poles at $2n = -\left(\frac{1}{2}\bar{q} + \bar{k}\right) = t_1$ and $2m = -\left(\frac{1}{2}\bar{q} - \bar{k}\right) = t_2$ and keeping only the singularity giving the leading power in s.
For a single sum

$$\sum_{m=0}^{\infty} \frac{1}{m!} \frac{f(m)s^m}{m+t} = -\int_{\mathcal{C}_1} \frac{dz}{2\pi i} \frac{f(z)s^z}{z+t} e^{-i\pi z}\Gamma(-z) \sim s^{-t} f(-t)\Gamma(t)e^{i\pi t}, \tag{3.33}$$

so the amplitude yields

$$\begin{aligned} \mathcal{A}^{(h=1)} \xrightarrow{s\to\infty} & \, g^4 \epsilon_a \cdot \epsilon_d \, \epsilon_b \cdot \epsilon_c \, i \left(\frac{s}{2}\right)^3 \frac{1}{2} \int \frac{d^{d-2}\bar{k}}{(2\pi)^{d-2}} \left(\frac{se^{-i\frac{\pi}{2}}}{2}\right)^{t_1+t_2} \\ & \times \frac{\Gamma\left(-\frac{t_1}{2}\right)}{\Gamma\left(+\frac{t_1}{2}\right)} \frac{\Gamma\left(-\frac{t_2}{2}\right)}{\Gamma\left(+\frac{t_2}{2}\right)} \frac{\Gamma^2(1+t_1+t_2-t)}{\Gamma^4\left(1+\frac{t_1+t_2-t}{2}\right)}, \end{aligned} \tag{3.34}$$

where we used

$$\frac{\Gamma(z+1/2)\Gamma(1/2)}{\Gamma(z+1)} = \pi 2^{-2z} \frac{\Gamma(2z+1)}{\Gamma^2(z+1)}. \tag{3.35}$$

The amplitude can be written in terms of two gravi-Reggeon exchange and its factorized form is explicit:

$$\mathcal{A}^{(h=1)} \xrightarrow{s\to\infty} \epsilon_a \cdot \epsilon_d \, \epsilon_b \cdot \epsilon_c \, \left(\frac{i}{2s}\right) \int \frac{d^{d-2}\bar{q}_1 d^{d-2}\bar{q}_2}{2(2\pi)^{d-2}} \delta^{d-2}(\bar{q} - \bar{q}_1 - \bar{q}_2) \tag{3.36}$$

$$\times a_{tree}(s, \bar{q}_1) \, a_{tree}(s, \bar{q}_2) [V_2(\bar{q}_1, \bar{q}_2)]^2, \tag{3.37}$$

where a_{tree} is given in (3.17).

3.3 Generalization to *N* loops and resummation in an eikonal form

We have shown that the leading contribution of the four-graviton amplitude at one loop is given by the two gravi-Reggeons cut and it can be expressed by the square of the two gravi-Reggeon vertex (3.36).
The h-loop amplitude will thus be given by the N-gravi-Reggeon cut, and it can be obtained in terms of N gravi-Reggeon vertices $V_N(q_1, ..., q_N)$, with $N = h+1$. N vertices can be obtained from $2N+2$ graviton tree amplitudes by the OPE of N pairs of graviton emission operators.
The N gravi-Reggeon vertex will be given by

$$\begin{aligned} V_N(q_1...q_N) &= \int \prod_{J=1}^{N-1} \frac{dM_J^2}{2\pi i} a_{\alpha'_1...\alpha'_N}(M_1^2...M_{N-1}^2;\, q_1...q_N) \\ &= \int_0^{2\pi} \prod_{J=1}^{N} \frac{d\sigma_J}{2\pi i} \langle 0| \prod_{J=1}^{N} :e^{iq_J\cdot X(\sigma)}: |0\rangle \\ &= \int_0^{2\pi} \prod_{J=1}^{N} \frac{d\sigma_J}{2\pi i} \prod_{1\leqslant j<l\leqslant N} |e^{i\sigma_j} - e^{i\sigma_l}|^{2q_j\cdot q_l}. \end{aligned} \tag{3.38}$$

So the leading asymptotic expression for the h-loop contribution to the four-graviton amplitude is

$$\begin{aligned} A^{(h)}(s,q) \xrightarrow{s\to\infty} & \, \epsilon_a \cdot \epsilon_d \, \epsilon_b \cdot \epsilon_c \left(\frac{i}{2s}\right)^h \int \frac{d^{d-2}q_1...d^{d-2}q_{h+1}}{(h+1)!(2\pi)^{(d-2)h}} \delta^{(d-2)}\left(q - \sum_i q_i\right) \\ & \times \prod_{j=1}^{h+1} a_{tree}(s,q_j) \int \prod_{j=1}^{h+1} \frac{d\sigma_j d\sigma'_j}{(2\pi)^2} \langle 0| \prod_j :e^{i\tilde{q}_j(X(\sigma_j)-X(\sigma'_j))}: |0\rangle \end{aligned} \tag{3.39}$$

Now we can show that the amplitude resums in an eikonal fashion. If we define

$$\frac{1}{s}A(s,q) = \epsilon_a \cdot \epsilon_b\, \epsilon_b \cdot \epsilon_c\, 4 \int d^{D-2}b\, e^{i\tilde{q}\cdot\tilde{b}} a(s,b), \tag{3.40}$$

we get

$$a^{(h)}(s,b) = \frac{(2i)^h}{(h+1)!} \langle 0| \left[\delta(s,b;\ X,X')\right]^{h+1} |0\rangle\,, \tag{3.41}$$

where δ is an operator functional on the string field $X(\sigma)$:

$$\delta = \frac{1}{4}\int \frac{d^{d-2}q}{(2\pi)^{d-2}} \frac{1}{s} a_{tree}(s,q) \int \frac{d\sigma d\sigma'}{(2\pi)^2} : e^{i\tilde{q}\cdot(\tilde{b}+X(\sigma)-X'(\sigma'))} : . \tag{3.42}$$

The sum over loops yields the final amplitude in an operator eikonal form:

$$a(s,b) = \sum_{h=0}^{\infty} a^{(h)}(s,b) = \langle 0\big| \frac{1}{2i}(\exp[2i\delta(s,b;\ X,X')] - 1)\big|0\rangle \tag{3.43}$$

The phase δ of the eikonal shows that the collision process can be interpreted as a rescattering series at displaced b, $b+X+X'$. In the following chapters we will analyse the importance of string excitations, which is one of the main topics of this thesis. Now we will start the treatment of the physical content of (3.43) by neglecting string excitations, i.e, setting $X=0$ in the eikonal phase:

$$\begin{aligned} \delta(b,s) = a_{tree}(b,s) &= \frac{g^2 s}{16\pi^{-2+d/2}} b^{4-d} \int^{\frac{b^2}{4(Y+i\pi/2)}} dt\, t^{\frac{d}{2}-3} e^{-t} \\ &\xrightarrow{b \gg 2\sqrt{Y}} \left(\frac{b_c}{b}\right)^{d-4} + 2\frac{\pi g^2 s}{32(\pi Y)^{-1+d/2}} \exp\left(-\frac{b^2}{4Y}\right), \end{aligned} \tag{3.44}$$

where $Y=\log s$, $b_c^{d-4} = \frac{g^2 s}{8\pi\Omega_{d-4}}$ and $\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$ is the solid angle in d dimensions. We can note that δ has an IR divergence for $d=4$. Then we can distinguish two regimes:

- For $b < b_I = 2Y$, $Im\delta$ is large and $\exp(2i\delta) \to 0$. This describes an absorptive black disk expanding logaritmically with the energy. The cross section in this region is indeed inelastic,

$$\sigma_{in}(s) \sim \frac{2}{D-2}\Omega_{d-2}(2\log s)^{d-2}. \tag{3.45}$$

- For $b > 2Y$, $Im\ \delta$ dies out exponentially and so δ is mostly real. $Re\delta$ decreases as a power, with weight given by b_c, so only for $b > b_c$ the perturbative expansion in powers of s is justified.
- In the remaining region $2Y < b < b_c$ where the eikonal is still valid, the amplitude $a(b,s)$ has a large real phase and so it shows an oscillatory behaviour.

3.4 Asymptotic result of the eikonal amplitude

In order to analyse the asymptotic behaviour of the eikonal amplitude (3.43), we consider its Hankel transform:

$$\begin{aligned} \frac{1}{s}a(s,q) \sim{}& 2i\left(\frac{2\pi b_I}{q}\right)^{\frac{d}{2}-1} J_{\frac{d}{2}-1}(b_I q) e^{2i(b_c/b_I)^{d-4}} \\ &+ \frac{g^2 s}{q^2} 2^{\frac{d}{2}-2}\Gamma\left(\frac{d}{2}-1\right) \int_{b_I q}^{\infty} dx\, x^{2-\frac{d}{2}} J_{\frac{d}{2}-1}(x) e^{2i(b_c q/x)^{d-4}}. \end{aligned} \tag{3.46}$$

The first term of (3.46) shows the black disk diffractive pattern, while the second term summarises the contributions for $b > b_I$. In momentum space we find

- For $q < \frac{1}{b_c} \sim (g^2 s)^{\frac{1}{4-d}}$, so for very small $q^2 = -t$, the loop expansion of the eikonal controls the $t = 0$ singularities. The amplitude at one loop is
$$\frac{1}{s} a(s,q) \xrightarrow{t\to 0} \frac{g^2 s}{|t|} + i \cos t \left[\frac{(g^2 s)^2}{(d-4)(6-d)} |t|^{\frac{d}{2}-3} + \frac{(g^2 s)^{\frac{d-2}{d-4}}}{(d-4)(d-6)} \right], \tag{3.47}$$
showing a graviton pole with unrenormalized residue. For $d < 6$ the imaginary part of the amplitude has an IR singularity and so it yields an infinite elastic cross-section, while for $d > 6$ the elastic cross-section is finite, $\sigma_{el} \sim s^{(d-2)/(d-4)}$ and violates the Froissart bound, due to the exchange of a massless particle.

- For $\frac{1}{b_c} \ll q \lesssim 1$ the second term has a large phase with a saddle point in b, typical of a classic scattering process:
$$q = \frac{g^2 s}{2 b^{d-3} \Omega_{d-2}} = \frac{8\pi G_N (k_a \cdot k_b)}{b^{d-3} \Omega_{d-2}}. \tag{3.48}$$
The amplitude evaluated in the saddle point
$$\frac{1}{s} a(s,b)|_{eik} \sim \frac{g^2 s}{|t|} e^{i\phi_d} \left(\frac{g^2 s |t|^{\frac{d}{2}-2}}{2\Omega_{d-2}} \right)^{\frac{4-d}{2d-6}} \left(\frac{i}{2} \right)^{\frac{4-d}{2}} \frac{\Gamma\left(\frac{d-2}{2}\right)}{\sqrt{d-3}}, \tag{3.49}$$
$$\phi_d = \left(\frac{g^2 s |t|^{\frac{d}{2}-1}}{2\Omega_{d-2}} \right)^{\frac{1}{d-3}} \frac{d-3}{d-4}, \tag{3.50}$$
gives a Rutherford cross-section $d\sigma_{el} = d^{-d} b(q)$. This saddle point also dominates the eikonal amplitude at small but fixed deflection angle
$$\vartheta = \frac{q}{k} = \frac{2q}{\sqrt{s}}, \tag{3.51}$$
if the stationary value b in the saddle point satisfies
$$b^{d-3} = \frac{8\pi G_N \sqrt{s}}{\Omega_{d-2} \vartheta} \gg b_I. \tag{3.52}$$
This angle is typical of the scattering of a massless particle at impact parameter b off a mass $M = \sqrt{s}$ to first non-trivial order in the Schwarzschild metric. It equals the angle obtained classically from the effective potential
$$V_{eff}(b,z) = \frac{\sqrt{s}}{2} \left(\frac{r_s}{r} \right)^{d-3} \frac{1}{2} \left(1 + \frac{z^2}{r^2} \right), \tag{3.53}$$
where $r^2 = b^2 + z^2$ and $r_s^{d-3} = \frac{8\pi G_N \sqrt{s}}{(d-2)\Omega_{d-1}}$ is the Schwarzschild radius in d spacetime dimensions.

One can conclude that the high energy scattering of two massless gravitons in the eikonal region is equivalent to the classical scattering in a non-trivial metric obtained from the Einstein equation, to first order in the metric. Higher order contributions are important to restore unitarity, as we will show in the next Section. This happens at large b, where the elastic scattering occurs.

For small impact parameters instead inelastic scattering dominates, corresponding to an expanding black disk. As we will show in Chapter 4, this picture could be modified by string corrections due to massive excitations to order $\left(\frac{\sqrt{\alpha'}}{b}\right)^4$ and higher to both $Im\delta$ and $Re\,\delta$. These corrections have the effect to turn the black disk into a grey one.

3.5 Unitarity

Unitarity of the amplitude depends on the value of the impact parameter. For $b > O(\log s)$ the operator δ is Hermitian and unitarity is satisfied by the diffractive intermediate states only.

When δ is not Hermitian, the discontinuity can be given in terms of a cut Reggeon. This corresponds to inelastic string states whose masses sum up to $\sqrt{s}$. In the case of an h-loop diagram there are N_c cut Reggeons, with N_c running from 0 to $h+1$, and the s-channel cuts corresponding to masses with sum $O(\sqrt{s})$ can be related to two-body diffractive cuts having small masses, which follow the eikonal and so are much more under control. This can be done by generalizing the AGK rules to our specific case. So we can write the amplitude discontinuity as a sum over N_c:

$$\frac{2}{i}(a_N - a_N^\dagger) = \sum_{N_c} S^\dagger_{N_-} \frac{\left[\frac{2}{i}(\delta-\delta^\dagger)\right]^{N_c}}{N_c!} S_{N_+} \delta_{N,N_++N_-+N_c}, \tag{3.54}$$

where S_{N+} or $S^\dagger_{N_-}$ are the rescattering operators with N_+ (N_-) exchanged Reggeons to the right (left) of the disc cut. The sum runs over the non-negative values of N_c, N_+, N_-, satisfying $N_c + N_+ + N_- = N$ except for $N_c = N_+ = 0$ ($N_- = N$) and the symmetric value, which correspond to "disconnected" terms. A review of the formalism of the AGK rules can be found in Appendix A.
In our case we can set operatorially

$$2ia_N = S_N = \frac{(2i\delta)^N}{N!}, \tag{3.55}$$

which is identically satisfied. Since δ and $\delta^\dagger$ are functionals of the $\tau = 0$ field $X(\sigma)$, they commute and so we get

$$0 = \left[2i\delta - 2i\delta^\dagger + \frac{2}{i}(\delta-\delta^\dagger)\right]^N = \sum_{N_c+N_++N_-=N} \frac{(-2i\delta^\dagger)^{N_-}}{N_-!} \frac{(-2i\delta)^{N_+}}{N_+!} \frac{\left[\frac{2}{i}(\delta-\delta^\dagger)\right]^{N_c}}{N_c!} \tag{3.56}$$

Summing and isolating the $N_c = 0$ term we get

$$1 - S^\dagger S = \sum_{N_c=1}^{\infty} S^\dagger \left[\frac{2}{i}(\delta-\delta^\dagger)\right]^{N_c} \frac{S}{N_c!}, \tag{3.57}$$

where we used $2ia_N = S_N = \frac{(2i\delta)^N}{N!}$. We have obtained the unitary defect in the space of diffractive states as a sum of non-negative inelastic terms, provided that $Im\delta$ is semi-positive definite. In our case it is positive definite since δ depends on the Hankel transform of the tree amplitude $a_{tree}(s,b)$ which has a positive imaginary part. So the eikonal phase δ satisfies inelastic s-channel unitarity in the sense described above.

3.6 Discussion

In the series of papers by Amati, Ciafaloni and Veneziano it was shown that an eikonal resummation of the high-energy limit of the S-matrix leads to results that are fully consistent with general relativity if the string size can be neglected. The effective geometry at leading order is the Aichelburg-Sexl (AS) shock-wave metric [5] produced by an energetic point- like massless particle, which we introduced in Chapter 1. This interpretation unfortunately fails beyond the leading-eikonal (or small deflection angle) approximation

[10]. String effects prevent gravitational collapse when the Schwarzschild radius of the would-be back hole is smaller than the string length parameter l_s. The string theory eikonal differs from the gravitational one in the presence of inelastic excitations due to string effects.

These excitations are due to a superposition of massive states, and it could be a difficult task to distinguish which states are actually excited. One way to understand this is to verify the eikonal for massive string scattering, as will be done in Chapter 5. Before computing the amplitudes we should introduce the massive string spectrum, in particular in Section 3.8 we will review the massive spectrum of the NS-NS sector of Type II superstring. We will show that only the leading Regge trajectory excitations are expected to dominate high-energy scattering amplitudes, and we will demonstrate this in Chapter 5, by explicitly computing the asymptotic limit of the inelastic amplitudes between any massive states.

The amplitudes we will compute in the following Chapters are similar to those computed by ACV, but with the addition of N parallel Dp-branes in Minkowski spacetime. From the point of view of perturbative string theory the presence of a collection of Dp-branes is entirely taken into account by the addition of an open string sector with suitable boundary conditions and does not require any modification of the background. On the other hand, from the point of view of the low-energy effective field theory the Dp-branes are a massive charged state and their presence will necessarily result in a curved spacetime. In this perspective, by evaluating string-brane scattering amplitudes in flat space, we can get some information about the dynamics of strings in an effective curved spacetime.

A different feature of the D-brane set-up is that partial-wave unitarity is not broken at any order. If we keep gN fixed (and thus let the number of branes N large), contributions with many open string loops (boundaries) are not suppressed, while in ACV papers it was shown that in the high-energy regime the closed string scattering amplitude grows with the energy in such a way that partial-wave unitarity breaks down at any finite order. With the presence of D-branes, the large contributions to the S-matrix from diagrams with a different number of boundaries combine together to give a unitary eikonal form for the resummed amplitude.

Moreover, in the original ACV-type collision the resummation of the closed string loops in the string-string scattering leads to an effective geometry that depends non-trivially upon the energies of the two colliding quanta. The problem of string scattering from a stack of D-branes is much easier since it is similar to that of scattering in an external field with the closed string acting as a probe and not back- reacting on the geometry, so the metric that emerges from the resummation of open string loops that we will consider in Chapter 4 depends only on the D-brane system.

Finally, we note that in our approach the existence of a non-trivial background is taken into account by the inclusion of surfaces with boundaries in the perturbative series, rather than by a direct modification of the couplings in the closed string sigma model. Since the string-brane scattering amplitude is evaluated using this microscopic definition of the system, in principle we can also analyse the background generated by N Dp-branes in regions where the effective description is not reliable.

3.7 High energy and fixed angle scattering

In this Section we review another important high-energy limit of string scattering amplitudes at high energy, the fixed-angle one. This limit was extensively studied in a series of papers by Gross and Mende [42, 43], where they showed that the sum over Riemann surfaces is dominated by a saddle point and in the high

energy, fixed-angle limit the amplitudes show an exponential decay in the momentum much faster than the fall-off as powers of momentum typical of field theory.
The motivation for studying high energy, fixed angle amplitudes is that of probing the short-distance structure of string theory, which can be regarded as the $\alpha' \to \infty$ limit, the opposite of the field theory limit $\alpha' \to 0$ which we will also consider in this Thesis. While the classic limit can be analysed in different ways, the short distance structure in string theory can be studied only using the strings themselves as external probes, and to avoid divergences one must consider only on-shell scattering amplitudes. So one effective way to investigate short-distance string theory is to compute the asymptotic behaviour at high energy of on-shell string scattering amplitudes in perturbation theory with a specific background, since this is the only way we know to evaluate string amplitudes.
Another background different from flat space is that given by D-branes, which are also non-perturbative objects, so string-brane scattering at high energy and fixed angle can provide an extension of the work of Gross and Mende. This limit will be analysed in Chapter 4, while in this Section we will introduce four-point scattering amplitudes in flat spacetime.

The exponential behaviour emerging from the saddle point in the integration ever Riemann surfaces is universal, and indicates that a particular Riemann surface, or a particular point in the moduli space of all Riemann surfaces dominates in the limit of high energy and fixed angle. This point is always far from the boundary of moduli space, and so any possible infrared divergence, which always arises from surfaces at the boundary, is avoided. This is not valid instead in the case of fixed momentum transfer, the limit analysed by ACV and reviewed above, since there the saddle point apporaches the boundary.

In the following we will evaluate the dominant saddle points for tree-level and one-loop four-point scattering amplitudes and then generalise the result to all orders.

We can start with the four-tachyon amplitude in closed bosonic string, the Virasoro-Shapiro amplitude [50]

$$\mathcal{A}_{tree} = \frac{8\pi i g_c^2}{\alpha'}(2\pi)^{26}\delta^{26}\Big(\sum_i k_i\Big)\int_{\mathbb{C}} d^2 z_4 |z_4|^{-\alpha' u/2-4}|1-z_4|^{-\alpha' t/2-4}, \tag{3.58}$$

where the integral runs over the complex plane $\mathbb{C}$ and we set $z_1 = 0,\ z_2 = 1,\ z_3 = \infty$. We used the usual Mandelstam variables where

$$s + t + u = -\frac{16}{\alpha'}. \tag{3.59}$$

In the limit $s \to \infty$ with fixed angle ϑ, one has also t/s fixed, since

$$\sin^2(\vartheta/2) = -\frac{t}{s+32} \sim -\frac{t}{s}, \tag{3.60}$$

and in this limit the exponential is large and the integral is dominated by the stationary points of the exponent. For $s,\ t,\ u \gg 1$ the saddle point is given by

$$\frac{\partial}{\partial z}\Big(u \ln |z_4|^2 + t \ln |1-z_4|^2\Big) = \frac{u}{z_4} + \frac{t}{z_4 - 1} = 0, \tag{3.61}$$

with solution $z_4 = -u/s$. Since $s+t+u = O(1)$, the saddle point approximations then gives the amplitude

$$S \sim \exp\Big[-\alpha' s (s \ln s + t \ln t + u \ln u)/2\Big]. \tag{3.62}$$

This exponential behaviour is common also to superstring scattering amplitudes: for example the four-point amplitudes for massless particles in Type II superstring theory have the same dependence on the

Mandelstam variables as the Virasoro-Shapiro amplitude, apart from normalization and kinematic factors. This indicates that the saddle point evaluation has a general validity. The result (3.62) is equal to that obtained using Stirling approximation of the Gamma function,

$$A_{tree} \sim \exp\left\{\alpha'\left(\frac{s}{2}+16\right)\left[\sin^2\frac{\vartheta}{2}\ln\sin^2\frac{\vartheta}{2}+\cos^2\frac{\vartheta}{2}\ln\cos^2\frac{\vartheta}{2}\right]\right\}. \tag{3.63}$$

The saddle-point of the one-loop amplitude gives, as we would expect, the half of the tree-level result (3.62). The four-tachyon amplitude at one loop is given by

$$\mathcal{A}_{one-loop} = \frac{4g^4}{\pi^{26}}\int d^2\tau \prod_{i=1}^{3} d^2\nu_i \left(Im\tau^{-13}\left|\frac{\theta_1'}{\pi}\right|^{-16}\right)\prod_{i<j}\chi(\nu_i-\nu_j,\tau)^{\alpha' p_i\cdot p_j}, \tag{3.64}$$

where χ is related to the Green's function for the torus of Chapter 2,

$$-\ln\chi(\nu_i-\nu_j,\tau) = G_{T^2}(\nu_i,\nu_j) \sim= -\frac{\alpha'}{2}\ln\left|\theta_1\left(\frac{\nu_i-\nu_j}{2\pi},\tau\right)\right|^2 + \frac{\alpha'}{4\pi\tau_2}[Im(\nu_i-\nu_j')]^2. \tag{3.65}$$

and $Im\tau = \tau_2$. In order to evaluate the saddle point, we have to place the insertion points symmetrically. Following [42], we set $\nu_i = 1/2$, $\nu_2 = \tau/2$, $\nu_3 = (\tau+1)/2$, $\nu_4 = 0$. At these values the function χ has a simpler form,

$$\chi\left(\frac{1}{2},\tau\right) = \alpha'\pi\left|\frac{\theta_2(0,\tau)}{\theta_1'(0,\tau)}\right|, \quad \chi\left(\frac{\tau}{2},\tau\right) = \alpha'\pi\left|\frac{\theta_4(0,\tau)}{\theta_1'(0,\tau)}\right|, \quad \chi\left(\frac{\tau+1}{2},\tau\right) = \alpha'\pi\left|\frac{\theta_3(0,\tau)}{\theta_1'(0,\tau)}\right|. \tag{3.66}$$

Then the amplitude takes the form of

$$\mathcal{A}_{one-loop} \sim e^{-\alpha'\varepsilon_2}, \tag{3.67}$$

with

$$\begin{aligned}\varepsilon_2 &= (s+16)\ln|\theta_3(0,\tau)| + (t+16)\ln|\theta_2(0,\tau)| + (u+16)\ln|\theta_4(0,\tau)| + \ln|\theta_1'|^{-8}\\ &= \frac{1}{4}\left[(t+16)\ln\left|\frac{\theta_2^4(0,\tau)}{\theta_3^4(0,\tau)}\right| + (u+16)\ln\left|\frac{\theta_4^4(0,\tau)}{\theta_3^4(0,\tau)}\right|\right] + \ln|\theta_3|^8 + \ln|\theta_1'|^{-8}.\end{aligned} \tag{3.68}$$

The dominant terms for large $s,\ t,\ u$ are those in brackets, so we vary them with respect to τ to determine the saddle point equation for τ. We have to perform the derivatives of theta functions, so we will be using the following relations among theta functions:

$$\frac{\partial\theta_\alpha(\nu,\tau)}{\partial\tau} = \frac{1}{4\pi i}\frac{\partial^2\theta_\alpha(\nu,\tau)}{\partial\nu^2} \tag{3.69}$$

$$\theta_3^4 = \theta_2^4 + \theta_4^4, \tag{3.70}$$

$$\frac{\theta_3''}{\theta_3} = \frac{\theta_2''}{\theta_2}\left(\frac{\theta_2^4}{\theta_3^4}\right) + \frac{\theta_4''}{\theta_4}\left(\frac{\theta_4^4}{\theta_3^4}\right), \tag{3.71}$$

$$\frac{\theta_2''}{\theta_2} = \frac{\theta_3''}{\theta_3} - \pi^2\theta_4^4 = \frac{\theta_4''}{\theta_4} - \pi^2\theta_3^4, \tag{3.72}$$

$$\theta_\alpha'(0,\tau) = \theta_\alpha'''(0,\tau) = 0 \text{ for } \alpha = 2,3,4, \text{ since } \theta_\alpha \text{ is an even function of } \nu. \tag{3.73}$$

Using the above identities we find that the derivative of ε with respect to τ vanishes when

$$\begin{aligned}\frac{\theta_2^4}{\theta_3^4} &= -\frac{t+16}{s} \sim \sin^2(\vartheta/2),\\ \frac{\theta_4^4}{\theta_3^4} &= -\frac{u+16}{s} \sim \cos^2(\vartheta/2).\end{aligned} \tag{3.74}$$

Evaluating the exponent ε at the saddle point, we get

$$\mathcal{A}_{one-loop} \sim \exp(\varepsilon_2(\nu_i, \tau)) = \frac{\alpha'}{2} \exp\Big[- \frac{\alpha' s}{2}(s \ln s + t \ln t + u \ln u)/2 \Big], \tag{3.75}$$

which is the half of the result on the sphere.

As can be seen from the one-loop example, it is clear that at general orders in the string loop expansion the dominant term in the amplitude is

$$\exp\left[- \sum_{i<j} p_i \cdot p_j F'(z_i, z_j) \right], \tag{3.76}$$

so to generalise the saddle point method to all orders we need to find the saddle point of the exponent with respect to both the vertex operator positions and the moduli of the Riemann surface. This resembles an electrostatic problem, where the exponent is the energy of n point charges p_i^μ on the Riemann surface and μ labels d different electrostatic fields. To find the dominant saddle point at each genus, Gross and Mende considered the Riemann surface

$$z_2^N = \frac{(z_1 - a_1)(z_1 - a_2)}{(z_1 - a_3)(z_1 - a_4)}, \tag{3.77}$$

which is a subspace of $S^2 \times S^2$. It can be regarded as N sheets glued together at branch cuts, and it has genus $N - 1$. Considering the four-point amplitude, if the charges are placed at branch cuts a_i, then the amplitude is stationary with respect to their positions, i.e, $\frac{\partial z_2}{\partial z_1} = \infty$. Moreover there is the following symmetry

$$z_2 \mapsto \exp(2\pi i/N) z_2, \tag{3.78}$$

requiring that the vertical component of the gradient has to vanish. Then one has only to extremise with respect to the positions a_i, and to do this we simply recall that the electric field in this case is the same as that on the sphere with charges at $z_i = a_i$ but divided between N sheets, so the energy is equal to that on the sphere, but multiplied by $N \cdot N^{-2} = N^{-1}$. Möbius invariance helps to fix three coordinates for example to $0, 1, \infty$ and then only the derivative with respect to the fourth coordinate is left. At genus g one then finds

$$\mathcal{A}_g \sim \exp\left[\frac{-\alpha'(s \ln s + t \ln t + u \ln u)}{2(g+1)} \right]. \tag{3.79}$$

This result holds also for excited string states, since the amplitude is still dominated by (3.76), so it could be interesting to extend the asymptotic limit of massive string-brane scattering amplitudes of Chapter 5, but the high-energy fixed angle limit of those amplitudes is outside the scope of this Thesis.

3.8 Massive string states

A basic property of superstring theory is the presence of an infinite tower of massive states, with an exponential degeneracy at high mass level. Massive excitations are a key feature for the stability of string theory, and for what concerns our work we will focus on the influence they exert in the context of high-energy scattering. In this Section we review some important features such as the density of states of string theory and the issue of the decay of massive strings states into lighter ones.

We also would like to emphasize that the analysis of the massive string spectrum and of the high energy regime of string theory can improve our understanding of higher spin gauge theory. It has been argued that the masses of the string states arise from some sort of generalised Higgs mechanism for the spontaneous breaking of higher spin gauge symmetries. Therefore, investigating higher-spin dynamics can help to better understand string theory and, vice versa, a closer look at string theory at high energies or in the $\alpha' \to \infty$ limit can provide important clues on higher-spin dynamics. Some examples of interactions between higher spin fields were introduced in [34] and some recent discussions can be found in [16, 54, 58]. At the end of this Section we will review the massive spectrum of the NS-NS sector of Type II superstring, focusing on the leading Regge trajectory, whose vertex operators will be used in Chapter 5.

3.8.1 Density of states and phase transitions

At one loop level the free energy of a thermal gas of strings is given by the vacuum amplitude in $R^{d-1} \times S^1$, where S^1 is the Euclidean time direction and for bosonic strings $d = 26$. The one loop contribution to the free energy can thus be represented as an integral over the conventional fundamental domain.
The one loop free energy of bosonic string theory diverges for the presence of the tachyon, as we have shown in Section 2.3, but even if the contribution of the usual tachyon is disregarded, there is still divergence at a finite temperature, the *Hagedorn temperature* T_H. As discussed also in [13], the fact that the latter is the temperature at which a certain mode has $M^2 = 0$, makes it clear that it is associated with a phase transition. The Hagedorn transition is a first order transition that occurs at a critical temperature $T_{crit} < T_H$ with a very large latent heat. We will show in detail how this works.
For a general CFT the partition function can be written as follows,

$$Z(\tau) = \sum_i q^{h_j - c/24} \bar{q}^{\,\tilde{h}_i - \tilde{c}/24} (-1)^{F_i}, \tag{3.80}$$

where i runs over all states of the CFT and F_i is again the world-sheet fermion number. Modular invariance under $\tau \to \tau + 1$ requires $h_i - \tilde{h}_i - \frac{1}{24}(c - \tilde{c})$ to be and integer and $(c - \tilde{c})$ to be a multiple of 24. For general operators then the spin must be an integer,

$$(h_i - \tilde{h}_i) \in \mathbb{Z}. \tag{3.81}$$

Modular invariance under $\tau \to -1/\tau$ imposes another constraint on the spectrum. If we consider the partition function for $\tau = il$ as $l \to 0$, the factor $q = \exp(-2\pi l)$ tends to one, so the partition function is determined by the density of states at high weight. By modular invariance, this is equal to the partition function at $\tau = i/l$, but in this case $q = \exp(-2\pi l) \xrightarrow{l \to 0} 0$, so the sum is dominated by the lowest weight state. This is the vacuum state, with $L_0 = \tilde{L}_0 = 0$, and it gives

$$Z(il) \sim \exp\left[\frac{\pi(c + \tilde{c})}{12l}\right]. \tag{3.82}$$

Thus the density of states at high weight is determined by the central charge. From the generic behaviour of the partition function at high weight h, eq. (3.82), and the relation $M^2 = 4h/\alpha'$, for a state of mass M, the density of string states $n(M)$ satisfies

$$\int_0^\infty dM\, n(M) \exp(-\alpha' \pi M^2 l) \sim \exp(-4\pi/l), \tag{3.83}$$

implying $n(M) \sim \exp(4\pi M\sqrt{\alpha'})$. The thermal partition function of a single string is then

$$\int_0^\infty dM \exp(4\pi M\sqrt{\alpha'}) \exp(-M/T). \tag{3.84}$$

Because the density of states grows exponentially, the partition function (3.84) diverges for temperatures $T \gg T_H$,

$$T_H = \frac{1}{4\pi\alpha'^{1/2}}, \tag{3.85}$$

where T_H is the Hagedorn temperature.
At low temperature the energy dominates, but at high temperature the entropy dominates and it is favorable to produce strings of arbitrary length. In an ensemble of such strings the density will become large, and the interactions will be important near the transition.
This divergence suggests a phase transition, similar to that found in the strong interaction, where there is a similar growth in the density of states of highly excited hadrons, and the phase transition occurs from a low temperature hadron phase to a high temperature quark-gluon phase.
The thermal partition function of a bosonic field of mass M in d dimensions is

$$F(T, M^2) = -\int_0^\infty \frac{dt}{t}(2\pi t)^{-d/2}\sum_{r=1}^{\infty}\exp\left(-\frac{M^2 t}{2} - \frac{r^2}{2T^2 t}\right). \tag{3.86}$$

In the second form, one can readily sum over the string spectrum. Following the derivation of the vacuum amplitudes in Section 2.3, the free energy density in 26 flat dimensions is

$$F(T) = -\int_R \frac{d\tau d\bar{\tau}}{4\tau_2}(4\pi^2\alpha'\tau_2)^{-13}|\eta(\tau)|^{-48}\sum_{r=1}^{\infty}\exp\left(-\frac{r^2}{4\pi T^2\alpha'\tau_2}\right). \tag{3.87}$$

The integral runs over the full region

$$R: \quad \tau_1 > 0, \quad |\tau_2| \leq \frac{1}{2}. \tag{3.88}$$

There is a possible divergence as $\tau_2 \to 0$; the integrand is largest when the real part of τ is zero, so let $\tau = i\tau_2$. From the modular transformation of the eta function (2.43), the asymptotic behaviour of the eta function in this limit is

$$\eta(i\tau_2) = \eta(i/\tau_2)\tau_2^{-1/2} \sim \exp(\pi/12\tau_2) \text{ as } \tau_2 \to \infty. \tag{3.89}$$

The τ integral thus converges for $T < T_H$ and diverges for $T > T_H$.
T-duality should relate the high and low temperature behaviours, and in fact the Hagedorn temperature is just the self-dual point. The same occurs in field theory, where the partition function is given by a Euclidean path integral, with time having period $1/T$.
Let we define

$$\tilde{F}(T) = F(T) + \rho_0, \tag{3.90}$$

the thermal part (3.87) of the free energy plus the vacuum part (2.57). The logarithm of the path integral is $-\tilde{F}(T)/T$, so T-duality implies

$$\frac{1}{T}\tilde{F}(T) = \frac{T}{T_H^2}\tilde{F}(T_H^2/T). \tag{3.91}$$

This relates the partition function below the Hagedorn temperature to that above it. The Hagedorn divergence for $T > T_H$ is a reflection of the tachyon divergence in the low temperature theory, but a similar formal result holds for the supersymmetric string, which has no tachyonic divergence. For the partition function we obtain

$$\tilde{F}(T) \to \frac{T^2}{T_H^2}\rho_0 \tag{3.92}$$

as $T \to \infty$. This is a measure of the number of high energy degrees of freedom, which are very few. A single quantum field in d spacetime dimensions has a free energy of order T^d, so the high energy limit of string theory looks like a field theory in two spacetime dimensions.
For supersymmetric string theories, there is no zero-temperature tachyon, but it can be shown that a winding mode becomes tachyonic at the Hagedorn temperature, thus causing a divergence of the free energy at one loop.

Even for superstrings the Hagedorn transition is first order and there is a genus zero contribution to the free energy above the critical temperature. Most of the qualitative discussion of bosonic strings carries over to supersymmetric string theories. In [13] it has been shown that the Hagedorn temperature for Type II superstrings is again the temperature at which a certain mode becomes tachyonic. It could hardly be otherwise, since the one loop modular integral could not diverge unless a tachyon appears.

3.8.2 Instability of massive states

In Type II superstring theory in flat, ten-dimensional non-compact spacetime, all massive strings are generally expected to be unstable and decay into lighter particles.

As explained in [46], for small string coupling the dominant elementary process is the decay into two particles. If these particles are massive, then each of them will subsequently decay into two lighter particles, and the process ends when only massless particles remain.

A general formula for the decay can be obtained from a one-loop calculation of the mass-shift. The inverse lifetime $\mathcal{T}^{-1}$ of a massive state of mass M is then given by

$$\mathcal{T}^{-1} = \Gamma = \frac{Im\Delta M^2}{2M}. \tag{3.93}$$

where ΔM^2 represents the radiative correction to the mass. At one loop it receives a contribution from the two-particle intermediate states, and thus Γ is the lifetime for decaying into two particles. One can obtain the one-loop expression for ΔM^2 starting from the zero and one loop expressions for the four graviton amplitudes, as was done in [45].
$Im(\Delta M^2)$ can be written as a sum over the contributions of different channels, according to the masses $M_1,\ M_2$ of the particles of the decay product. However, most of these processes are exponentially suppressed, except for a line in the space $M_1,\ M_2$, where the decay rate has a power-like dependence on the coupling.
According to their decay properties, the quantum states of the Type II string spectrum can be divided into two classes: those for which the dominant decay channel is by massless emission and those for which the dominant decay channel is by emission of massive particles. Massless emission is the dominant channel whenever the quantum state corresponds to large closed strings which cannot break during the evolution. For these states, the decay channels into massive states are exponentially suppressed. In [25] it was found

$$\Gamma_{massive} = O(e^{-cM^2}), \tag{3.94}$$

where the value of c depends on the specific state. The lifetime is given by $\mathcal{T}^{(i)} = \Gamma^{-1}_{massless}$.

An example of the second class of states is the folded closed string corresponding to the quantum state with maximum angular momentum: the points of the string are always in contact with the other side so the string can break at any moment. For a detailed discussion see [25].

3.8.3 Massive spectrum of the NS-NS superstring on the leading Regge trajectory

In order to compute high-energy massive string amplitudes, we will need the vertex operator for massive string states.

In this section we analyse the massive spectrum of the NS-NS sector of the Type II superstring, focusing on the highest spin states at a given mass, i.e, the leading Regge trajectory. As an illustrative example, we will focus on the BRST quantization of the first massive level. We will be using the metric with positive signature, $\eta = diag(-,+,+,...,+)$, and we will follow the conventions used in [50, 51]. The mass shell condition for the n^{th} level of the NS-NS sector of the superstring after the GSO projection is given by

$$\alpha' M^2 = 4n, \tag{3.95}$$

with $n = 0, 1, 2....$ In the following we will write the closed string state as the product of two copies of the open string sector, so the states we will consider carry momentum $p^\mu = 2k^\mu$ where k will represent the momentum of the open string vertex. Henceforth, when a result is stated for holomorphic fields, it is with the understanding that an analogous result will hold for the antiholomorphic quantities.

For the first massive level the mass shell condition (3.95) with $n = 1$ is satisfied by the following states in the -1 picture:

$$|\phi_\varepsilon\rangle = \varepsilon_{\mu\nu}\alpha^\mu_{-1}\psi^\nu_{-\frac{1}{2}}\,|0;k\rangle\,, \tag{3.96a}$$

$$|\phi_A\rangle = \mathcal{A}_{\mu\nu\rho}\psi^\mu_{-\frac{1}{2}}\psi^\nu_{-\frac{1}{2}}\psi^\rho_{-\frac{1}{2}}\,|0;k\rangle\,, \tag{3.96b}$$

$$|\phi_B\rangle = B_\mu\psi^\mu_{-\frac{3}{2}}\,|0;k\rangle\,. \tag{3.96c}$$

To find the physical states, we write the most general state at this level as a linear combination:

$$|\phi\rangle = \left(\varepsilon_{\mu\nu}\alpha^\mu_{-1}\psi^\nu_{-\frac{1}{2}} + \mathcal{A}_{\mu\nu\rho}\psi^\mu_{-\frac{1}{2}}\psi^\nu_{-\frac{1}{2}}\psi^\rho_{-\frac{1}{2}} + B_\nu\psi^\nu_{-\frac{3}{2}}\right)|0;k\rangle \tag{3.97}$$

The corresponding vertex operator may be written with superghost charge -1 as follows:

$$V_{-1} = ce^{-\varphi}\left(\frac{i}{\sqrt{2\alpha'}}\varepsilon_{\mu\nu}\partial X^\mu\psi^\nu + \mathcal{A}_{\mu\nu\rho}\psi^\mu\psi^\nu\psi^\rho + B_\nu\partial\psi^\nu\right)e^{ik\cdot X}. \tag{3.98}$$

Here we have introduced one of the reparameterisation ghosts $c(z)$, of the bc system, along with the field $\varphi(z)$ which together with anticommuting bosonic fields η, ξ form the bosonisation of the superghost fields. In order for this state to be physical it must be invariant under BRST transformations, these being generated by the operator

$$Q_B = \frac{1}{2\pi i}\oint_C (dz j_B - d\bar{z}\tilde{j}_B), \tag{3.99}$$

where the BRST current is given in terms of a matter current and a ghost current, $j_B = j_m + j_g$, which may be written as

$$j_m = -c\Big(\frac{1}{\alpha'}\partial X^\mu\partial X_\mu + \frac{1}{2}\psi^\mu\partial\psi_\mu\Big) + i\sqrt{\frac{2}{\alpha'}}e^\varphi\eta\psi^\mu\partial X_\mu, \tag{3.100a}$$

$$j_g = bc\partial c + \frac{3}{4}\partial c\partial\varphi + c\partial\xi\eta - \frac{1}{2}c\partial\varphi\partial\varphi - \frac{1}{4}c\partial\partial\varphi + be^{2\varphi}\partial\eta\eta. \tag{3.100b}$$

These all have obvious antiholomorphic counterparts. In fact, the state generated by (3.98) does not satisfy the condition $\{Q_B, V_{-1}\} = 0$ and so we add to our vertex operator a contribution from an exact state of the form $Q_B|\chi\rangle$, whose vertex operator is in the -1 picture, to render the sum physical.

If we use as a candidate the state

$$\mathcal{U} = e^{-2\varphi}\partial\xi c\left(\lambda_{[\mu\nu]}\psi^\mu\psi^\nu + \frac{i}{\sqrt{2\alpha'}}\omega_\mu\partial X^\mu\right)e^{ikX} + e^{-2\varphi}\partial^2\xi c\Lambda e^{ikX}, \tag{3.101}$$

then we are adding to our vertex (3.98) the following operator:

$$\begin{aligned}\delta\mathcal{V} = Q_B\,\mathcal{U} = e^{-\varphi}c\Bigg\{&\sqrt{2\alpha'}i\Lambda\eta_{\mu\nu}\partial X^\mu\partial X^\nu + \left(\frac{i}{\sqrt{2\alpha'}}\omega_\mu - \sqrt{2\alpha'}\Lambda k_\mu\right)\partial\psi^\mu \\ &-\left(i\sqrt{\frac{2}{\alpha'}}\lambda_{[\mu\nu]} - \frac{1}{2}\sqrt{\frac{\alpha'}{2}}k_{[\mu}\omega_{\nu]}\right)\partial X^{[\mu}\psi^{\nu]} \\ &-\sqrt{\frac{\alpha'}{2}}k_{(\mu}\omega_{\nu)}\partial X^{(\mu}\psi^{\nu)} + k_{[\mu}\lambda_{\nu\rho]}\psi^\mu\psi^\nu\psi^\rho\Bigg\}e^{ikX},\end{aligned} \tag{3.102}$$

so that we find agreement with the results reported in [16]. In particular, as shown in [16], we can gauge away the scalar, the antisymmetric rank-2 tensor and the vector. We are left with two physical states:

- $\varepsilon_{(\mu\nu)}\alpha^\mu_{-1}\psi^\nu_{-\frac{1}{2}}\left|0;0\right\rangle$ has a polarization $\varepsilon_{(\mu\nu)}$ which is a completely symmetric, traceless, tensor invariant under the little group $SO(9)$, so it carries to 44 degrees of freedom. This is the state belonging to the open leading Regge trajectory which we will be using in our amplitude computations;

- The state $\mathcal{A}_{[\mu\nu\rho]}\psi^\mu\psi^\nu\psi^\rho$ has a polarization $\mathcal{A}_{[\mu\nu\rho]}$ which is a three-form of $SO(9)$, corresponding to 84 degrees of freedom.

The requirement of BRST invariance for these states brings the constraints

$$k^\mu\mathcal{G}_{(\mu\nu)} = 0, \quad k^\mu\mathcal{A}_{[\mu\nu\rho]} = 0. \tag{3.103}$$

Together, these two states have 128 degrees of freedom, which is the full bosonic content of the left sector of the first massive level, as explained in [38].

As we consider higher and higher levels, it can be a difficult task to find the BRST invariant vertex operators,since the number of states increases dramatically. Nevertheless, at each level one could consider the states in the light-cone gauge, and these can always be rearranged into irreducible representations of the little group of $SO(9)$, which describe the BRST-invariant states. This has been shown in [16] for the first two levels.

In the computation of our amplitudes we will use vertex operators for a state from the leading Regge trajectory at the generic mass level n. To do this we will generalise the result for $n = 1$. We start by constructing the vertex for an open string state which is totally symmetric in its polarisation. In general the tensor product of two such states will give a closed string state in a reducible representation containing physical states of the same mass but different spin, however, by symmetrising over all indices of the polarisation one is left with a state of spin $J = (\alpha' M^2 + 4)/2$. This is the maximum possible spin in the Type II string spectrum for a fixed mass level and it is the set of all such states that comprises the leading Regge trajectory. This can be seen pictorially in figure 3.2. Written in terms of oscillators, such an open string state will have the form $\left|\phi_n\right\rangle = \varepsilon_{\mu_1\ldots\mu_n\alpha}\prod_{i=1}^n\alpha^{\mu_i}_{-1}\psi^\alpha_{-\frac{1}{2}}\left|0;k\right\rangle$, and this will give the following closed string state,

$$\left|\phi_n\right\rangle^{closed} = \epsilon_{\mu_1\ldots\mu_n\alpha\,\nu_1\ldots\nu_n\beta}\left[\prod_{i=1}^n\alpha^{\mu_i}_{-1}\widetilde{\alpha}^{\nu_i}_{-1}\right]\psi^\alpha_{-\frac{1}{2}}\widetilde{\psi}^\beta_{-\frac{1}{2}}\left|0;p\right\rangle, \tag{3.104}$$

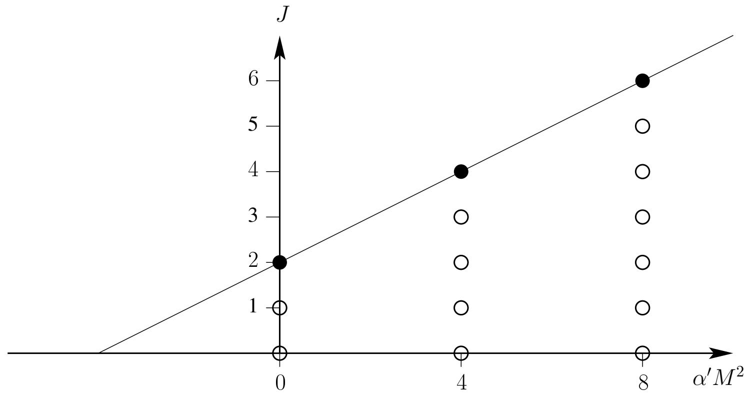

Figure 3.2: Diagrammatic depiction of the Type II string spectrum indicating the mass and spin of each physical state, represented by circles. Filled circles show states on the leading Regge trajectory.

where $\epsilon_{\mu_1\ldots\mu_n\alpha\nu_1\ldots n_n\beta} = \varepsilon_{\mu_1\ldots\mu_n\alpha} \otimes \widetilde{\varepsilon}_{\nu_1\ldots\nu_n\beta}$.

By virtue of the state-operator isomorphism we obtain from the state above a vertex operator of superghost charge (-1,-1),

$$W^{(n)}_{(-1,-1)}(k,z,\bar{z}) = \frac{\kappa}{2\pi}\epsilon_{\mu_1\ldots\mu_n\alpha\,\nu_1\ldots\nu_n\beta} V^{\mu_1\ldots\mu_n\alpha}_{-1}(k,z)\widetilde{V}^{\nu_1\ldots\nu_n\beta}_{-1}(k,\bar{z}) \tag{3.105}$$

with

$$V^{\mu_1\ldots\mu_n\alpha}_{-1}(k,z) = \frac{1}{\sqrt{n!}}\left(\frac{i}{\sqrt{2\alpha'}}\right)^n e^{-\varphi(z)}\left(\prod_{i=1}^{n}\partial X^{\mu_i}\right)\psi^\alpha e^{ik\cdot X(z)}. \tag{3.106}$$

Above we have written $X(z)$ for the open string field, obtained by decomposing the complete string coordinate field into $X(z,\bar{z}) = (X(z) + \widetilde{X}(\bar{z}))/2$, and as previously stated we make use of the open string momentum k for convenience while the physical momentum of the string is $p = 2k$. Since we wish to compute amplitudes on a world-sheet with the topology of the disc the vertices which make up this amplitude should have a total superghost charge of -2, consequently we will need a vertex operator of the state above in the $(0,0)$ picture,

$$W^{(n)}_{(0,0)}(k,z,\bar{z}) = -\frac{\kappa}{\pi\alpha'}\epsilon_{\mu_1\ldots\mu_n\alpha\,\nu_1\ldots\nu_n\beta} V^{\mu_1\ldots\mu_n\alpha}_{0}(k,z)\widetilde{V}^{\nu_1\ldots\nu_n\beta}_{0}(k,\bar{z}), \tag{3.107}$$

where

$$V^{\mu_1\ldots\mu_n\alpha}_{0}(k,z) = \frac{1}{\sqrt{n!}}\left(\frac{i}{\sqrt{2\alpha'}}\right)^{n+1}\left(\partial X^{\mu_n}\partial X^\alpha - 2i\alpha' k\cdot\psi\partial X^{\mu_n}\psi^\alpha - n2\alpha'\partial\psi^{\mu_n}\psi^\alpha\right)$$
$$\left(\prod_{i=1}^{n-1}\partial X^{\mu_i}\right)e^{ik\cdot X(z)}. \tag{3.108}$$

where $\kappa^2 = 2^6\pi^7\alpha'^4 g^2$ is the gravitational coupling constant in ten dimensions and we follow the conventions used in [51]. We have normalised these states presented here such that the two-point function on the sphere for two states having the same spin is $2\alpha' g^2\,\mathrm{Tr}(\epsilon_1\cdot\epsilon_2)$. Finally, we should make one point; strictly speaking, the vertices given in equations (3.108) and (3.106) can only be applied to the cases $n > 0$, however we shall extend their use to the case $n = 0$ with the understanding that when one encounters a product of the form $\prod_{i=1}^{0} a_i$ we are to replace it by unity.

4
String-brane scattering at high energy

The string-brane system is an ideal framework to understand the way in which string scattering amplitudes evaluated in flat space can provide information about the dynamics in an effective curved spacetime. Dp-branes arise from classical solutions of the low-energy string effective action coupled to graviton, dilaton and $(p+1)$-form R-R potential, and since their tension is proportional to the inverse of the string coupling constant, they correspond to non-perturbative states of string theory.
In this chapter we will analyze the scattering of a closed string off a stack of N parallel Dp-branes in Minkowski spacetime at high energy, in order to get some insight on the dynamics of strings in an effective curved space-time. Scattering amplitudes of closed strings off a stack of N Dp-branes can be computed in perturbation theory summing over Riemann surfaces with an arbitrary number of boundaries and handles.
In this Section we will probe the N Dp-brane system with high energy massless closed strings, in the high energy limit where gravity dominates. Since we want to neglect non-planar diagrams, which get suppressed by powers of the string coupling g, we will work in the regime of small string coupling, $g \ll 1$.

Moreover, we want to isolate the backreaction of the branes on the geometry of spacetime, so we will consider the limit of fixed open string coupling $\lambda = gN$, while $g \to 0$ and $N \to \infty$. In this limit only diagrams with an arbitrary number of boundaries and no handles are dominant in the sum, and the resummed amplitude, which has an operator eikonal form, can be interpreted in terms of a closed string propagation in the effective spacetime geometry of the Dp-brane system, which we will describe in section (4.3).
This Chapter is organised as follows: we begin by computing string-brane scattering amplitudes in Section 4.1, focusing on the NS-NS sector, and in particular on the graviton. We first illustrate the kinematics for the scattering of a massless string off a D-brane, introducing suitable boundary conditions imposed by the breaking of Lorentz invariance. Then we compute the tree-level amplitude on the disk and the one-loop amplitude, which is on the annulus, showing that the latter can be factorised with the exchange of two Reggeized gravitons. We then generalise to N loops and show that the amplitude resums in an operator

eikonal form in Section 4.1.4. In Section 4.2 we compute the string corrections to the leading eikonal. Then in Section 4.3 we will compare the results we have obtained from string computations in flat spacetime to what we would expect from the propagation of point-like or string-like objects in the p-brane background metric, introduced in Chapter 2. In particular we will compute the deflection angle and the tidal excitations at leading order.

4.1 Scattering of massless closed strings off D-branes

As noted by Amati, Ciafaloni and Veneziano in their series of papers [7, 8], even if the deflection angle ϑ is kept fixed and t becomes large, the scattering is dominated by the exchange of many gravitons, each one of momentum of order $R_p^{-1} \gg \sqrt{-t}$. This justifies the Regge approximation order by order in the loop expansion.

Another limit we will use is $R_p > l_s$, where string-brane interactions are dominated by gravity effects, so we expect a relation with the effective geometry of the Dp-brane. Since the small angle condition requires $b \gg R_p$, our scattering amplitudes are evaluated at impact parameters much larger than the string scale, $b \gg l_s$.

The upper limit on the energy E can be made arbitrarily large for sufficiently large N, and so the two-point amplitude can be written as a sum of amplitudes with a different number of boundaries,

$$\mathcal{A}(p_1, \varepsilon_1; p_2, \varepsilon_2) = i(2\pi)^{p+1}\delta^{p+1}(p_{1\|}, p_{2\|}) \sum_{h=1}^{\infty} \mathcal{A}_h(p_1, \varepsilon_1; p_2, \varepsilon_2), \tag{4.1}$$

where h denotes the number of boundaries. In the Regge limit each amplitude is given by

$$\mathcal{A}_h(p_1, \varepsilon_1; p_2, \varepsilon_2) \sim \mathrm{Tr}(\varepsilon_1\varepsilon_2)\mathcal{A}_h(s, t), \tag{4.2}$$

where now $\mathcal{A}_h$ contains the leading dependence of s at high energy. Thus we can write the T-matrix related to the S-matrix by $S = 1 + iT$:

$$\begin{aligned} T(p_1, \varepsilon_1; p_2, \varepsilon_2) &\sim (2\pi)^{p+1}\delta^{p+1}(p_{1\|}, p_{2\|})\, \mathrm{Tr}(\varepsilon_1\varepsilon_2) \sum_{h=1}^{\infty} \mathcal{A}_h \\ &= (2\pi)^{p+1}\delta^{p+1}(p_{1\|}, p_{2\|})\, \mathrm{Tr}(\varepsilon_1\varepsilon_2) \frac{\mathcal{A}(s,t)}{2E}, \end{aligned} \tag{4.3}$$

where we normalise each asymptotic state with a factor $\sqrt{2E}$.

The first term in (4.1) is the disk amplitude, the second one is the annulus amplitude. As shown in [27], it is possible to obtain the leading behaviour of the full perturbative series, and the resummed amplitude takes an operator eikonal form in impact parameter space. This operator allows to study both classical gravity effects and string corrections to the classical dynamics.

In the following we will compute the disk and annulus amplitude and then illustrate both classical and stringy effects, following the derivation of [27].

4.1.1 Kinematics of a string scattering from a Dp-brane

Here we illustrate the kinematics appropriate to tree-level interactions between a single closed string and a Dp-brane. Throughout this work we will use a flat metric of positive signature, $\eta = diag(-, +, +, ..., +)$.

In the following computations we will consider massless states with polarizations ε_i and momentum p_i such that $\alpha' M_i^2 = 0$. We can decompose the momenta into vectors which are parallel and orthogonal to the directions in which the D-brane is extended, that is

$$p_i^\mu = p_{i\|}^\mu + p_{i\perp}^\mu. \tag{4.4}$$

The mass of a D-brane scales as $1/g$ and so to leading order in the perturbative expansion it is infinitely massive. Consequently we may neglect its recoil and momentum is only conserved in the $p+1$ directions parallel to the D-brane,

$$p_{1\|}^\mu + p_{2\|}^\mu = 0. \tag{4.5}$$

For brevity we denote $p_{1\|}^\mu = -p_{2\|}^\mu = p_\|^\mu$. Using the mass-shell condition we may relate this quantity to the $(9-p)$ orthogonal components of momentum,

$$p_{1\perp}^2 = -\, p_\|^2 \tag{4.6a}$$

$$p_{2\perp}^2 = -\, p_\|^2. \tag{4.6b}$$

The vectors $p_{i\perp}$ are by definition space-like[1] so we can infer that $p_\|$ is necessarily time-like. Considering this we define the following kinematic invariants for use as the parameters in our computations,

$$s \equiv -p_\|^2, \tag{4.7a}$$

$$t \equiv -(p_1 + p_2)^2, \tag{4.7b}$$

$$E \equiv \sqrt{s}. \tag{4.7c}$$

After a Lorentz transformation to a frame in which the spatial components of p_1 are nonzero only in the orthogonal directions, the quantity E will be equal to the energy of the string.
It can be useful to express t in terms of s and the relative angle ϑ between $p_{1\perp}^\mu$ and $p_{2\perp}^\mu$,

$$\begin{aligned} t &= -(p_1 + p_2)^2 \\ &= -(p_{1\perp} + p_{2\perp})^2 \\ &= -2s\cos^2\frac{\vartheta}{2}. \end{aligned} \tag{4.8}$$

The invariants s corresponds to the energy of the incoming string, while t is the momentum transferred to the brane. The impact parameter vector b lies in the direction of $-(p_1 + p_2)$, which is approximately orthogonal to the direction $(p_1 - p_2)$ for small deflection angles, and so these two directions define the collision plane. We define the bosonic string coordinates by splitting holomorphic and anti-holomorphic components

$$X^\mu(z, \bar{z}) = \frac{X^\mu(z) + \bar{X}^\mu(\bar{z})}{2}, \tag{4.9}$$

where

$$X^\mu(z) = x^\mu - i2\alpha' \frac{p^\mu}{2} \ln z + i\sqrt{\alpha'} \sum_{m\neq 0} \frac{1}{m} \frac{\alpha'^\mu_m}{z^\mu}, \tag{4.10}$$

and similarly for the antiholomorphic field, so that the holomorphic and antiholomorphic fields carry only half of the physical momentum: $p_i^\mu = 2k_i^\mu$.

[1] ignoring the specific case of a D-instanton.

We are interested in a scattering process in the presence of D-branes which break Lorentz invariance; this changes the kinematics and the boundary conditions, imposing Neumann boundary conditions on the direction parallel to the brane, Dirichlet boundary conditions on the directions orthogonal to it. For the Neumann conditions we have on the real axis

$$\begin{aligned} X(z) &= \bar{X}(\bar{z}) \\ \psi(z) &= \widetilde{\psi}(z), \end{aligned} \tag{4.11}$$

while for the Dirichlet conditions we require

$$\begin{aligned} X(z) &= -\bar{X}(\bar{z}) \\ \psi(z) &= -\widetilde{\psi}(z), \end{aligned} \tag{4.12}$$

thus the holorphic and antiholomorphic fields get mixed and the result is that we have nonzero correlations functions between the two.
The correlators among holomorphic fields in the upper half of the complex plane are

$$\begin{aligned} \langle X^\mu(z) X^\nu(w) \rangle &= -2\alpha' \eta^{\mu\nu} \log(z-w), \\ \langle \psi^\mu(z) \psi^\nu(w) \rangle &= -2\alpha' \frac{\eta^{\mu\nu}}{z-w}, \\ \langle \varphi(z) \varphi(w) \rangle &= -\log(z-w), \end{aligned} \tag{4.13}$$

and the antiholomorphic fields have analogous correlators.
To implement the new boundary conditions on the correlators between the holomorphic and antiholomorphic fields it is convenient to use the doubling trick. As reviewed in [36] in the context of D-brane physics, this simplifies the treatment by substituting anti-holomorphic fields $\bar{X}(\bar{z})$, $\widetilde{\varphi}(\bar{z})$ which are functions of an anti-holomorphic variable, with holomorphic fields depending on $\bar{z}$ treated as an independent holomorphic variable. This is equivalent to sending $X^\mu(\bar{z}) \to D^\mu{}_\nu X^\nu(\bar{z})$, $\widetilde{\psi}^\mu(\bar{z}) \to D^\mu{}_\nu \psi^\nu(\bar{z})$ and $\widetilde{\varphi}(\bar{z}) \to \varphi(\bar{z})$ for correlation functions evaluated on $\mathbb{H}_+$, where $D = (\eta_{p+1}, -1_{9-p})$. Having employed the doubling trick we need only compute correlators between the holomorphic fields and antiholomorphic fields in terms of the correlators between the holomorphic fields,

$$\begin{aligned} \langle X^\mu(z) \bar{X}^\nu(\bar{w}) \rangle &= -2\alpha' D^{\mu\nu} \log(z-\bar{w}), \\ \langle \psi^\mu(z) \widetilde{\psi}^\nu(\bar{w}) \rangle &= \frac{D^{\mu\nu}}{z-\bar{w}}, \\ \langle \varphi(z) \widetilde{\varphi}(\bar{w}) \rangle &= -\log(z-\bar{w}). \end{aligned} \tag{4.14}$$

There are several kinematic factors which appear in the following calculation which depend on the momenta carried by the holomorphic and antiholomorphic fields; we would like to be able to express these in terms of the variables above. From the definition of $D^\mu{}_\nu$ it can be seen that the momenta satisfy $(D\cdot k)^\mu = k^\mu_\parallel - k^\mu_\perp$ and so, with the aid of the conservation of momentum, one can deduce the following identities,

$$\begin{aligned} k_1 \cdot k_2 &= -\frac{t}{8}, \\ k_1 \cdot D \cdot k_1 &= -\frac{s}{2}, \\ k_2 \cdot D \cdot k_2 &= -\frac{s}{2}, \\ k_1 \cdot D \cdot k_2 &= \frac{s}{2} + \frac{t}{8}. \end{aligned} \tag{4.15}$$

4.1.2 Disk amplitude

In this Section we will compute the exact disk amplitude in order to extract the high-energy behaviour. In Chapter 5 we will show that the asymptotic behaviour in the Regge regime can be extracted in a straightforward way using OPE techniques.

For the process we are interested in the amplitude is computed using two closed string vertex operators inserted on the upper-half of the complex plane, with Neumann boundary conditions applied on the real axis for the $p+1$ directions parallel to the D-branes world-volume, and Dirichlet boundary conditions applied for the $9-p$ directions orthogonal to the D-brane. We can write this as

$$A = \mathcal{N} \int_{\mathbb{H}_+} d^2 z_1 d^2 z_2 V_{CKG} \langle V_1(k_1, z_1, \bar{z}_1) V_2(k_2, z_2, \bar{z}_2) \rangle. \tag{4.16}$$

the integrals range over the upper half of the complex plane and V_{CKG} is the Conformal Killing volume of the residual $SL(2,R)$ symmetry. The vertex operators can be decomposed in their holomorphic and anti-holomorphic components:

$$\begin{aligned} V_1(k_1, z_1, \bar{z}_1) &= \varepsilon_{1\mu\nu} : V_1^\mu(k_1, z_1) :: \tilde{V}_{-1}^\nu(k_1, \bar{z}_1) :, \\ V_2(k_2, z_2, \bar{z}_2) &= \varepsilon_{2\mu\nu} : V_0^\mu(k_2, z_2) :: \tilde{V}_0^\nu(k_2, \bar{z}_2) :, \end{aligned} \tag{4.17}$$

The normalization constant $\mathcal{N}$ in (4.16) is formed from the product of the normalization of a vertex in the $(0,0)$ picture, $\kappa/(\pi\alpha')$, the normalization of a vertex in the $(-1,-1)$ picture, $\kappa/(2\pi)$ and the topological factor for a disc amplitude $C_{D_2} = \alpha'(2\pi)^2\, T_p/\kappa$, where κ is the gravitational coupling constant in ten dimensions and T_p is the coupling for closed string states to a Dp-brane. Overall this gives the normalization

$$\mathcal{N} = \frac{\kappa T_p}{2} = \frac{7-p}{4} R_p^{7-p} \frac{2\pi^{\frac{9-p}{2}}}{\Gamma\left(\frac{9-p}{2}\right)}, \tag{4.18}$$

where R_p represents a characteristic size for the stack of D-branes and is related to the t'Hooft coupling, $\lambda = gN$ as follows,

$$R_p^{7-p} = gN \frac{(2\pi\sqrt{\alpha'})^{7-p}}{(7-p)\Omega_{8-p}}, \qquad \Omega_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}. \tag{4.19}$$

The subscripts 0, 1 represent the superghost charge carried by the vertex operators which, due to superdiffeomorphism invariance, must sum to 2 on the upper half of the complex plane; the holomorphic components above can thus be written as

$$V_1^\mu(k_1, z_1) = \frac{\kappa}{2\pi} \frac{1}{\sqrt{\alpha'}} e^{-\varphi(z_1)} \psi^\mu(z_1) e^{ik_1 \cdot X(z_1)}, \tag{4.20}$$

$$V_0^\mu(k_2, z_2) = -\frac{\kappa}{\pi\alpha'} \frac{1}{\sqrt{\alpha'}} (\partial X^\mu(z_2) + ik_2 \cdot \psi(z_2)\psi^\mu(z_2)) e^{ik_1 \cdot X(z_2)}, \tag{4.21}$$

The antiholomorphic components take the same form but with the fields $X(z), \psi(z)$ and $\varphi(z)$ replaced by their antiholomorphic counterparts $\tilde{X}(z)$, $\tilde{\psi}$ (z) and $\tilde{\varphi}(z)$.
To implement the new boundary conditions implied by the presence of the D-branes, we can rewrite the vertex operators in (4.17) in terms of (4.14):

$$\begin{aligned} V_1(k_1, z_1, \bar{z}_1) &= \varepsilon_{1\mu\lambda} D^\lambda{}_\nu : V_1^\mu(k_1, z_1) :: \tilde{V}_{-1}^\nu(k_1, \bar{z}_1) :, \\ V_2(k_2, z_2, \bar{z}_2) &= \varepsilon_{2\mu\lambda} D^\lambda{}_\nu : V_0^\mu(k_2, z_2) :: \tilde{V}_0^\nu(k_2, \bar{z}_2) : . \end{aligned} \tag{4.22}$$

By using equations (4.14), (4.17) and the kinematic identities (4.15), we have the correlator in (4.16):

$$\begin{aligned}\langle V_1(k_1,z_1,\bar{z}_1)V_2(k_2,z_2,\bar{z}_2)\rangle &= \left(\frac{(z_1-\bar{z}_2)(\bar{z}_1-z_2)}{(z_1-z_2)(\bar{z}_1-\bar{z}_2)}\right)^{\alpha'\frac{t}{4}}\left(\frac{(z_1-\bar{z}_1)(z_2-\bar{z}_2)}{(z_1-\bar{z}_2)(\bar{z}_1-z_2)}\right)^{-\alpha' s/2}\\ \times\Bigg[\frac{\mathrm{Tr}(\varepsilon_1 D)\,\mathrm{Tr}(\varepsilon_2 D)(-\alpha' s/2-1)}{(z_1-\bar{z}_1)^2(z_2-\bar{z}_2)^2} &+ \frac{1}{(z_1-z_2)(\bar{z}_1-\bar{z}_2)(z_1-\bar{z}_2)(\bar{z}_1-z_2)}2\alpha' C_1\\ &+\frac{1}{(z_1-\bar{z}_1)(z_2-\bar{z}_2)(z_1-\bar{z}_2)(\bar{z}_1-\bar{z}_2)}2\alpha' C_2\Bigg]\,,\end{aligned} \tag{4.23}$$

where

$$\begin{aligned}C_1 &= \mathrm{Tr}(\varepsilon_1\cdot D)k_1\varepsilon_2 k_1 - k_1\varepsilon_2 D\varepsilon_1 k_2 - k_1\varepsilon_2\varepsilon_1 D k_1 - k_1\varepsilon_2\varepsilon_1 D k_1\\ &-\frac{1}{2}(k_1\varepsilon_2\varepsilon_1 k_2 + k_1\varepsilon_2\varepsilon_1 k_2) - \frac{s}{8}\,\mathrm{Tr}(\varepsilon_1\varepsilon_2) + \{1\to 2\}\,, \end{aligned}\tag{4.24}$$

$$\begin{aligned}C_2 &= \mathrm{Tr}(\varepsilon_1 D)(k_1\varepsilon_2 D k_2 + k_2 D\varepsilon_2 k_1 + k_2 D\varepsilon_2 D k_2) + k_1 D\varepsilon_1 D\varepsilon_2 D k_2\\ &-\frac{1}{2}(k_1 D\varepsilon_1\varepsilon_2 D k_2 + k_1 D\varepsilon_1\varepsilon_2 D k_2) - \frac{s}{8}\,\mathrm{Tr}(\varepsilon_1 D\varepsilon_2 D)\\ &\frac{s}{8}\,\mathrm{Tr}(\varepsilon_1\varepsilon_2) + \{1\to 2\}\,.\end{aligned}\tag{4.25}$$

To fix the $SL(2,\mathbb{R})$ invariance of the amplitude we introduce the $SL(2,\mathbb{R})$ invariant variable

$$\omega = \frac{(z_1-\bar{z}_1)(z_2-\bar{z}_2)}{(z_1-z_2)(\bar{z}_1-\bar{z}_2)} \tag{4.26}$$

and we use the measure $d^2z_1 d^2z_2 = d\omega(z_1-z_2)^2(\bar{z}_1-\bar{z}_2)^2$ to account for the ghost zero modes and to write the integral in terms of ω. The result is

$$\mathcal{A}(k_i,z_i) = \int_{-\infty}^{0} d\omega\left(\frac{\omega}{1-\omega}\right)^{-\alpha' s/2}(1-\omega)^{\alpha'\frac{t}{4}}\left[\frac{\mathrm{Tr}(\varepsilon_1 D)\,\mathrm{Tr}(\varepsilon_2 D)(-\alpha' s/2-1)}{\omega^2}+\frac{2\alpha' C_1}{1-\omega}+\frac{\alpha' C_2}{\omega(1-\omega)}\right]. \tag{4.27}$$

Changing again variable to $x=\omega/(1-\omega)$ we find that the amplitude yields a beta function whose arguments are the Lorentz invariant variables s and t:

$$\mathcal{A} = 2\alpha'^2\frac{\Gamma(-\alpha' s/2)\Gamma(-\alpha' t/4)}{\Gamma(1-\alpha' s/2-\alpha' t/4)}\left(-\frac{s}{2}c_1+\frac{t}{4}c_2\right), \tag{4.28}$$

where $c_1 = C_1$ and $c_2 = C_2 - \mathrm{Tr}(\varepsilon_1 D)\,\mathrm{Tr}(\varepsilon_2 D)(-s/4-t/8)$.
As was pointed out in [36], the disk amplitude (4.28) contains two sets of poles corresponding to closed strings in the t-channel for $\alpha' m^2 = 4n$ and to open strings in the s-channel for $\alpha' m^2 = n$, where $n = 0,1,2....$

It is worth mentioning here that the polarisations appearing in the scattering amplitude (4.28) are the same of the long-range background fields for the D-brane, and they correspond to those of extremally charged p-brane solutions of the low energy effective action, as was derived in [36].

We can compute the high energy limit of the disk amplitude (4.28) using Stirling's approximation for the Gamma function for large $|z|$,

$$\Gamma(z)\sim\sqrt{2\pi}z^{z-1/2}e^{-z}. \tag{4.29}$$

We will consider the most important high-energy regimes, fixed-angle and Regge.

Fixed-angle regime. In the case of fixed-angle scattering we perform the limit $s \to \infty$ and express t in terms of s and ϑ using the last equation of (4.8). Using Stirling approximation we find that the amplitude in this regime yields

$$\mathcal{A}(s,\vartheta) \sim \sqrt{\frac{4\pi}{\alpha' s}} \frac{4\alpha'}{\sin\vartheta} e^{-i\alpha' \frac{s}{2}\cos\frac{\vartheta}{2}} \exp\left[\alpha' \frac{s}{2}\left(\cos^2\frac{\vartheta}{2}\ln\left|cos^2\frac{\vartheta}{2}\right| + \sin^2\frac{\vartheta}{2}\ln\left|\sin^2\frac{\vartheta}{2}\right|\right)\right] \times \left(c_1 + c_2\cos^2\frac{\vartheta}{2}\right). \quad (4.30)$$

Here we performed a Wick rotation of the argument of the Gamma function, since Stirling approximations holds only for positive arguments of the Gamma function.

Regge Regime. In the case of large s and fixed t we perform the same operation, but keeping in mind that in this regime t is not necessarily large. Moreover, in the Regge limit the leading kinematical factor is $\mathrm{Tr}(\varepsilon_1\varepsilon_2)(\alpha' s)^2$. We find

$$\mathcal{A}(s,t) \sim -\frac{\pi^{\frac{9-p}{2}} R_p^{7-p}}{\Gamma\left(\frac{7-p}{2}\right)} \mathrm{Tr}(\varepsilon_1\varepsilon_2)\Gamma(-\alpha' t/4) e^{-i\pi\alpha'\frac{t}{4}} (\alpha' s)^{1+\alpha'\frac{t}{4}}. \quad (4.31)$$

The amplitude in the Regge limit shows that the scattering process is dominated by the exchange in the t-channel of the leading Regge trajectory of the graviton. The imaginary part of the amplitude describes inelastic processes where the incoming closed string excites the open strings attached to the D-brane world-volume. As the energy increases, unitarity breaks down but it is restored taking into account multi-reggeon exchanges, described in our case by diagrams with a large number of boundaries.

4.1.3 Annulus amplitude

The second term in the sum (4.1) is given by correlation functions of the worldsheet theory integrated over the moduli space of a Riemann surface with two boundaries, the annulus. The modular parameter is only one and it is purely imaginary: using the notation of [27] we will denote it by $\tau_{op} = i\tau_2$, where τ_{op} refers to the fact that in the open string channel the amplitude can be interpreted as a one-loop diagram or annulus diagram if we take the worldsheet time direction parallel to the two boundaries; or else if we take the orthogonal time direction the amplitude is a tree-level diagram of closed strings, the cylinder diagram. The two descriptions are dual since we can relate the modular parameters of the open and closed channels as follows:

$$\tau_{op} \mapsto \tau_{cl} = -\frac{1}{\tau_{op}}. \quad (4.32)$$

As shown in [27], the dominant contributions to the scattering amplitude come from the region of large $\lambda = Im(\tau_{cl})$, for this reason it is more convenient to represent the amplitude in the closed string channel, with two closed string vertices and two boundary states, as explained in Section 2.6:

$$\mathcal{A}_2(k_1\varepsilon_1; k_2,\varepsilon_2) = \mathcal{N}\int_0^\infty \frac{d\tau_2}{2\tau_2}\frac{1}{2}\sum_{\alpha.\beta} c_{\alpha\beta} Z_{\alpha\beta}(\tau)\int dz_1 \int dz_2 \langle V_0(k_1,\varepsilon_1)V_0(k_2,\varepsilon_2)\rangle_{A,\alpha,\beta}, \quad (4.33)$$

where $\tau = i\tau_2$ and the vertex operators are both in the zero picture. The functions $Z_{\alpha\beta}$ represent the contribution to the vacuum amplitude of the spin structure $(\alpha;\beta)$ with $\alpha;\beta = 0,1$

$$Z_{\alpha\beta}(\tau) = \frac{1}{(8\pi^2\alpha'\tau_2)^{\frac{p+1}{2}}}\frac{1}{\eta^8(\tau)}\left[\frac{\theta_{\alpha\beta}(0,\tau)}{\eta(\tau)}\right]^4. \quad (4.34)$$

The sum over the four spin structures implements the GSO projection as we already explained in Section 2.3 where we computed one-loop closed superstring vacuum diagram. The correlation function depends on the insertion points of the two vertex operators and on the modulus of the annulus, in total five real parameters which can be reduced to four by translation invariance on the cylinder. The correlation functions on the annulus can be derived from the corresponding ones on the torus using the method of images, explained in section 2.2.4. Consider a torus with modular parameter τ , obtained identifying points in the complex plane according to $z \sim z + 2\pi \sim z + 2\pi\tau$. The annulus is the fundamental region with respect to the involution $I : z \mapsto I(z) = 2\pi\bar{z}$. It is useful to introduce a new complex coordinate $\nu = \rho + i\omega$ related to z by $z = 2\pi\nu$.
The range of the modular parameter τ_2 and of the insertion points ν_a and ν_b,

$$0 \leqslant \tau_2 < \infty, \qquad 0 \leqslant \rho_a, \rho_b \leqslant \frac{1}{2}, \qquad 0 \leqslant \omega_a, \omega_b \leqslant \tau_2, \tag{4.35}$$

define the domain P of integration for the integral in the annulus amplitude:

$$\int_P d\mu \equiv \int_0^\infty \frac{d\tau_2}{2\tau_2} \int dz_a \int dz_b = 4(2\pi)^4 \int_0^\infty \frac{d\tau_2}{2\tau_2} \int_0^{1/2} d\rho_b \int_0^{\tau_2} d\omega_a \int_0^{\tau_2} d\omega_b. \tag{4.36}$$

The leading term in the Regge limit is the one with the highest power of s and it is proportional to $\mathrm{Tr}(\varepsilon_1 \cdot \varepsilon_2)$. In order to get a contribution of this form we have to consider the contractions of fields in the term of the annulus amplitude which contains eight fermions. In flat space this is the only relevant one, since all terms with less than eight fermions vanish. Schematically the form of this term is

$$[Dk_1 \widetilde{\psi}\widetilde{\psi}^\nu](I(\nu_1)[k_1\psi\psi^\mu(\nu_1)[k_2\psi\psi^\sigma](\nu_b)[Dk_2\widetilde{\psi}\widetilde{\psi}^\kappa](I(\nu_a)). \tag{4.37}$$

Since we are looking for a term proportional to $\varepsilon_1 \cdot \varepsilon_2$, we should first contract the couples of fermions $(\mu\sigma)$ and $(\nu\kappa)$. Then the contractions (12)(34) give a term proportional to $4s^2$, the contractions (13)(24) give a term proportional to $\left(2s + \frac{t}{2}\right)^2$ and finally the contractions (14)(23) give a term proportional to t^2, subleading in the Regge limit.
The expressions resulting from these contractions simplify once multiplied by the vacuum factors $Z_{\alpha\beta}(\tau)$. Now we can sum over the spin structures and using of the Riemann identity

$$\frac{1}{2} \sum_{\alpha,\beta} c_{\alpha\beta} \prod_{i=1}^{4} \theta_{\alpha\beta}(\nu_i, \tau) = \prod_{i=1}^{4} \theta_1(\tilde{\nu}_i, \tau), \tag{4.38}$$

where $\alpha, \beta = 0, 1$ denote the spin structures and

$$c_{\alpha\beta} = e^{i\pi(\alpha+\beta+\alpha\beta)} \tag{4.39}$$

implements the GSO projection and we have $\theta_{00} = \theta_3,\ \theta_{10} = \theta_2,\ \theta_{01} = \theta_4,\ \theta_{11} = \theta_1$,

$$\begin{aligned} 2\tilde{\nu}_1 &= \nu_1 + \nu_2 + \nu_3 + \nu_4, \qquad & 2\tilde{\nu}_2 &= \nu_1 - \nu_2 - \nu_3 + \nu_4, \\ 2\tilde{\nu}_3 &= \nu_1 + \nu_2 - \nu_3 - \nu_4, \qquad & 2\tilde{\nu}_4 &= \nu_1 - \nu_2 + \nu_3 - \nu_4. \end{aligned} \tag{4.40}$$

After we perform the sum, the contractions (12)(34) and (13)(24) both give

$$\frac{1}{2} \sum_{\alpha,\beta} c_{\alpha\beta} Z_{\alpha\beta} F_{\alpha\beta}(\nu_1, \nu_2) F_{\alpha\beta}(I(\nu_1), I(\nu_2)) F_{\alpha\beta}(\nu_2, I(\nu_2)) F_{\alpha\beta}(I(\nu_1), \nu_1) = \frac{1}{2(8\pi^2\alpha'\tau_2)^{\frac{p+1}{2}}}. \tag{4.41}$$

The same is true for all the other contractions in the computation of the full annulus amplitude. In total there are 60 terms proportional to the previous constant up to a sign resulting from the number of permutations of the fermion fields and a product of four metric tensors $\eta_{\mu\nu}$, one for each pair of indices. We can now analyse the asymptotic behaviour of the amplitude, considering that the correlator of the two exponentials contains the leading dependence on the external momenta and so it controls the asymptotic form of the amplitude:

$$\mathcal{A}_2(k_1,\varepsilon_1;k_2,\varepsilon_2) \sim \mathrm{Tr}(\varepsilon_1\varepsilon_2)\frac{(gN)^2(\alpha' s)^2}{(8\pi\alpha')^{\frac{p+1}{2}}}\int_P d\mu\,\tau_2^{-\frac{p+1}{2}}\langle e^{ik_1\cdot X(\nu_1)}e^{ik_2\cdot X(\nu_2)}\rangle_A. \tag{4.42}$$

We are considering the full closed string operators, while for the tree-level computation we splitted in holomorphic and antiholomorphic components. The only difference is that now we will have to use self-correlators like those shown in section 2.2.2.

If we define

$$\chi(\nu_1,\nu_2,\tau) = \frac{2\pi}{\theta_1'(0,\tau)}\theta_1(\nu_1-\nu_2,\tau)e^{-\frac{\pi}{\tau_2}(\omega_1-\omega_2)^2}, \tag{4.43}$$

the correlators of the exponentials become

$$E(s,t) \equiv \langle e^{ik_1\cdot X(\nu_1)}e^{ik_2\cdot X(\nu_2)}\rangle_A = e^{-\alpha' sV_s - \frac{\alpha' t}{4}V_t}, \tag{4.44}$$

where

$$\begin{aligned} V_s &= \ln\left[\frac{\chi(\nu_1,-\bar\nu_1,\tau)\chi(\nu_2,-\bar\nu_2,\tau)}{\chi(\nu_1,-\bar\nu_1,\tau)\chi(-\bar\nu_2,-\nu_2,\tau)}\right], \\ V_t &= \ln\left[\frac{\chi(\nu_1,\bar\nu_2,\tau)\chi(\bar\nu_1,-\bar\nu_2,\tau)}{\chi(\nu_1-,\bar\nu_2,\tau)\chi(-\bar\nu_1,-\nu_2,\tau)}\right]. \end{aligned} \tag{4.45}$$

In terms of θ_1 we have

$$V_s = \frac{2\pi}{\tau_2}\omega_{12}^2 + \ln\left[\frac{\theta_1(2\rho_1,i\tau_2)\theta_1(2\rho_2,i\tau_2)}{\theta_1(\zeta+i\omega_{12},i\tau_2)\theta_1(\zeta-i\omega_{12},i\tau_2)}\right],$$

where

$$\zeta = \rho_1+\rho_2, \qquad \rho_{12} = \rho_1-\rho_2, \qquad \omega_{12} = \omega_1-\omega_2. \tag{4.46}$$

It is important to stress that the integrand does not depend on $\sigma = \omega_1+\omega_2$ as a consequence of translation invariance on the cylinder. The problem has now been reduced to the study of the asymptotic behaviour of the integral

$$\int_P d\mu\,\tau_2^{-\frac{p+1}{2}}E(s,t). \tag{4.47}$$

Generically a critical point x_c contributes to the amplitude a term proportional to $e^{-sV_s(x_c)}$ but when $V_s(x_c) = 0$ we can have terms with a power-like dependence on s. Each critical point corresponds to a specific degeneration limit of the annulus with two punctures, so we should consider the behaviour of the integrand near critical points in the interior and on the boundary of the integration domain. We can show that there are no critical points in the interior, while on the boundary of the domain there is a critical surface Σ_1 parametrised by (ζ,σ,τ_2) when

$$\rho_{12} = \omega_{12} = 0, \tag{4.48}$$

and a critical surface Σ_2 parametrised by $(\zeta,\sigma,\omega_{12})$ when $\tau_2 \to 0$. On both critical surfaces V_s vanishes. It is Σ_2 that provides the leading term. The limit $\tau_2 \to 0$ represents the ultraviolet region from the open string perspective, which is better analyzed after a modular transformation from the open to the closed channel

$$\lambda = \frac{1}{\tau_2}. \tag{4.49}$$

The limit $\tau_2 \to 0$ becomes $\lambda \to \infty$, which represents the infrared region from the closed string perspective; in this limit the surface becomes an infinitely long cylinder. The amplitude then factorizes in the product of the amplitude for the emission of two reggeized gravitons and the contribution of the two infinite tubes ending on the two boundaries. The higher order terms $\mathcal{A}_h$, $h \geqslant 3$ behave similarly and we can resum the series in an operator eikonal form which allows the study of classical gravity and string effects.

To estimate the asymptotic behaviour of $E(s,t)$ we need the expansion of $\theta_1(\nu,\tau)$ in the limit $\tau_2 \to \infty$ with $\omega/\tau_2 > 0$. This can be obtained from the following representation of the logarithm of the theta function

$$\ln \frac{\pi\theta_1(\nu,\tau)}{\theta_1'(0,\tau)} = \ln \sin \pi\nu + 4\sum_{m=1}^{\infty} \frac{q^{2m}}{1-q^{2m}} \frac{\sin^2 m\pi\nu}{m}, \tag{4.50}$$

where $q = e^{2\pi i\tau}$. So the leading terms come from the logarithm and from the first term in the sum and give

$$\ln \frac{\pi\theta_1(\nu,\tau)}{\theta_1'(0,\tau)} \sim \ln \frac{i}{2} - i\pi\nu - e^{2\pi i\nu} - e^{2\pi i(\tau-\nu)}. \tag{4.51}$$

From the previous expression we obtain

$$\ln \left| \frac{\pi\theta_1(\nu,\tau)}{\theta_1'(0,\tau)} \right| e^{-\frac{\pi\omega^2}{\tau_2}} = -\frac{\pi\omega^2}{\tau_2} + \mathrm{Re}\left[\ln \frac{\pi\theta_1(\nu,\tau)}{\theta_1'(0,\tau)}\right] \sim -\ln 2 - \frac{\pi\omega^2}{\tau_2} - \mathrm{Re}[i\pi\nu - e^{2i\pi\nu} + e^{2i\pi(\tau-\nu)}], \tag{4.52}$$

which is useful in the study of the asymptotic behaviour of the four-point function on the torus.
The other case of interest to us is

$$\ln \frac{\pi\theta_1(i\lambda\nu, i\lambda)}{\theta_1'(0,i\lambda)} \sim \ln \frac{i}{2} + \pi\lambda\nu - e^{-2\pi\lambda\nu} - e^{-2\pi\lambda(1-\nu)}, \tag{4.53}$$

for large positive λ and finite values of ρ.
Using the second modular transformation of (2.37), we can write

$$\frac{2\pi\theta_1(\nu, i\tau_2)}{\theta_1'(0,i\tau_2)} e^{-\frac{\pi\omega^2}{\tau_2}} = \frac{2\pi i\theta_1(i\lambda\nu, i\lambda)}{\lambda\theta_1'(0,i\lambda)} e^{-\pi\lambda(\rho^2+2i\rho\omega)}, \quad \lambda = \frac{1}{\tau_2}. \tag{4.54}$$

So we can write V_s and V_t in terms of λ:

$$\begin{aligned} V_s &= -2\pi\lambda\rho_{12}^2 + \ln\left[\frac{\theta_1(2i\lambda\rho_1, i\lambda)\theta_1(2i\lambda\rho_2, i\lambda)}{\theta_1(\lambda\omega_{12} + i\lambda\zeta, i\lambda)\theta_1(\lambda\omega_{12} - i\lambda\zeta, i\lambda)}\right], \\ V_t &= \ln\left[\frac{\theta_1(\lambda\omega_{12} + i\lambda\rho_{12}, i\lambda)\theta_1(\lambda\omega_{12} - i\lambda\rho_{12}, i\lambda)}{\theta_1(\lambda\omega_{12} + i\lambda\zeta, i\lambda)\theta(\lambda\omega_{12} - i\lambda\zeta, i\lambda)}\right]. \end{aligned} \tag{4.55}$$

In the limit $\lambda \to \infty$ we have

$$V_s \sim -2\pi\lambda\rho_{12}^2 - e^{-4\pi\lambda\rho_1} - e^{-4\pi\lambda\rho_2} - e^{-2\pi\lambda(1-2\rho_1)} - e^{-2\pi\lambda(1-2\rho_2)} + 2\cos 2\pi\lambda\omega_{12}\left(e^{-2\pi\lambda\zeta} + e^{-2\pi\lambda(1-\zeta)}\right). \tag{4.56}$$

For $\lambda \to \infty$ with $\rho_{12} = 0$ we find a family of critical points parametrised by ω_1, ω_2 and ζ which provides the dominant contribution to the integral.
Leaving aside for the moment the prefactor

$$\mathrm{Tr}(\varepsilon_1\varepsilon_2) \frac{(gN)^2(\alpha's)^2}{(8\pi\alpha')^{\frac{p+1}{2}}}, \tag{4.57}$$

the integral to evaluate is

$$I = 32\pi^4 \int_0^\infty d\lambda \int_0^{1/2} d\rho_1 \int_0^{1/2} d\rho_2 \int_0^{1/\lambda} d\omega_1 \int_0^{1/\lambda} d\omega_2 \lambda^{\frac{p-1}{2}} E(s,t). \tag{4.58}$$

By changing variables to ζ and ρ_{12} we notice that the integral in ρ_{12} is Gaussian and is dominated by $\rho_{12} = 0$. Expanding V_s and V_t around this point we have

$$\begin{aligned} V_s &\sim -4\sin^2 \pi\lambda\omega_{12} \left(e^{-2\pi\lambda\zeta} + e^{-2\pi\lambda(1-\zeta)}\right), \\ V_t &\sim -2\pi\lambda\zeta(1-\zeta) + \ln[4\sin^2 \pi\lambda\omega_{12}], \end{aligned} \tag{4.59}$$

and the amplitude is then approximated by

$$I \sim \frac{16\pi^4 i}{\sqrt{2\lambda\alpha' s}} \int_0^\infty d\lambda \int_0^1 d\zeta \int_0^{1/\lambda} d\omega_{12} \lambda^{\frac{p-1}{2}} E(s,t). \tag{4.60}$$

Now we change variables to ω_{12} and $\sigma = \omega_1 + \omega_2$. The integrand does not depend on σ and can be rewritten as

$$I \sim \frac{16\pi^4 i}{\lambda\sqrt{2\lambda\alpha' s}} \int_0^\infty d\lambda \int_0^1 d\zeta \int_0^{1/\lambda} d\omega_{12} \lambda^{\frac{p-1}{2}} E(s,t), \tag{4.61}$$

and the asymptotic form of $E(s,t)$ is

$$E(s,t) \sim (4\sin^2 \pi\lambda\omega_{12})^{-\frac{\alpha' t}{4}} e^{2\pi\lambda\zeta(1-\zeta)\frac{\alpha' t}{4}} \exp\left\{4\alpha' s \sin^2 \pi\lambda\omega_{12}\left(e^{-2\pi\lambda\zeta} + e^{-2\pi\lambda(1-\zeta)}\right)\right\}. \tag{4.62}$$

Expanding the exponential

$$\exp\left\{4\alpha' s \sin^2 \pi\lambda\omega_{12}(e^{-2\pi\lambda\zeta} + e^{-2\pi\lambda(1-\zeta)})\right\} = \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{1}{n!m!} (4\alpha' s \sin^2 \pi\lambda\omega_{12})^{n+m} e^{-2n\pi\lambda\zeta} e^{-2m\pi\lambda(1-\zeta)}, \tag{4.63}$$

we can perform the integral in ω_{12} setting $x = \pi\lambda\omega_{12}$ and using the usual expression for the Beta function

$$B(a,b) = 2\int_o^{\pi/2} dx \sin^{2a-1} x \cos^{2b-1} x. \tag{4.64}$$

The integral in λ can then be evaluated using the integral representation of the gamma function and the result reads

$$I \sim \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{16\pi^3 i}{\sqrt{2\alpha' s}} \int_0^1 d\zeta \frac{1}{n!m!} \frac{2^{2(n+m)} - \frac{\alpha' t}{2}(\alpha' s)^{n+m} \Gamma\left(\frac{p-4}{2}\right) B\left(\frac{1}{2} + n + m - \frac{\alpha' t}{4}, \frac{1}{2}\right)}{\pi^{\frac{p-4}{2}} \left[2n\zeta + 2m(1-\zeta) - \zeta(1-\zeta)\frac{\alpha' t}{2}\right]^{\frac{p-4}{2}}}. \tag{4.65}$$

Now we use the identity already seen in Section 3.2,

$$\int_0^1 d\zeta \frac{(2\pi\alpha')\Gamma\left(\frac{p-4}{2}\right)}{\left[2n\zeta + 2m(1-\zeta) - \zeta(1-\zeta)\frac{\alpha' t}{2}\right]^{\frac{p-4}{2}}} = \int \frac{d^{8-p}k}{(2\pi)^{8-p}} \frac{1}{\left[2n - \frac{\alpha' t_1}{2}\right]\left[2m - \frac{\alpha' t_2}{2}\right]}, \tag{4.66}$$

where

$$t_1 \equiv -\left(\frac{q}{2} + k\right)^2, \qquad t_1 \equiv -\left(\frac{q}{2} + k\right)^2, \qquad q^2 = -t. \tag{4.67}$$

The result is

$$I \sim \frac{16\pi i}{\sqrt{2\alpha' s}} \left(\frac{2}{\alpha'}\right)^{\frac{p-8}{2}} \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{1}{n!m!} \int d^{8-p}k \frac{B\left(\frac{1}{2} + n + m - \frac{\alpha' t}{4}, \frac{1}{2}\right)}{\left[2n - \frac{\alpha' t_1}{2}\right]\left[2m - \frac{\alpha' t_2}{2}\right]} 2^{2(n+m) - \frac{\alpha' t}{2}} (\alpha' s)^{n+m}. \tag{4.68}$$

In order to evaluate the double sum we rewrite it as a Cauchy integral and then deform the contour of integration, keeping only the singularity giving the leading power in s, like we did in the derivation of the four-graviton loop amplitude, (3.33). Indeed the integration is similar to the one carried out in Section 3.2. We obtain

$$I \sim \frac{4\pi i}{\sqrt{2\alpha' s}}\left(\frac{2}{\alpha'}\right)^{\frac{p-8}{2}} \int d^{8-p}k e^{-i\pi\frac{\alpha' t}{4}(t_1+t_2)}\Gamma\left(-\frac{\alpha' t_1}{4}\right)\Gamma\left(-\frac{\alpha' t_2}{4}\right) \times B\left(\frac{1}{2}+\frac{\alpha'}{4}(t_1+t_2-t),\frac{1}{2}\right)2^{\frac{\alpha'}{2}(t_1+t_2-t)}(\alpha' s)^{t_1+t_2}. \tag{4.69}$$

Using

$$\frac{\Gamma(z+1/2)\Gamma(1/2)}{\Gamma(z+1)} = \pi 2^{-2z}\frac{\Gamma(2z+1)}{\Gamma^2(z+1)}, \tag{4.70}$$

and reintroducing the prefactor (4.57) the final result is

$$\mathcal{A}_2(k_1,\varepsilon_1;k_2,\varepsilon_2) \sim \mathrm{Tr}(\varepsilon_1\varepsilon_2)(gN)^2\frac{4\pi^2 i}{(8\pi^2\alpha')^{\frac{p+1}{2}}}\left(\frac{2}{\alpha'}\right)^{\frac{p-8}{2}}\frac{1}{\sqrt{2\alpha' s}}\int d^{8-p}k e^{-\pi\frac{\alpha'}{4}(t_1+t_2)} \times\Gamma\left(-\frac{\alpha' t_1}{4}\right)\Gamma\left(-\frac{\alpha' t_1}{4}\right)\frac{\Gamma\left(1+\frac{\alpha'}{2}(t_1+t_2-t)\right)}{\Gamma^2\left(1+\frac{\alpha'}{4}(t_1+t_2-t)\right)}(\alpha' s)^{2+\frac{\alpha'}{4}(t_1+t_2)}. \tag{4.71}$$

Since

$$\left(\frac{2}{\alpha'}\right)^{\frac{p-8}{2}}\frac{2\sqrt{2}c_p^2\alpha'^{3/2}(2\pi)^{10-p}}{(8\pi^2\alpha')^{\frac{p+1}{2}}} = 1, \tag{4.72}$$

The amplitude can be written in a factorised form as the product of the vertex for the emission of two reggeized gravitons and two disk propagators which encode the boundary conditions:

$$\mathcal{A}_2(k_1,\varepsilon_1;k_2,\varepsilon_2) \sim \mathrm{Tr}(\varepsilon_1\varepsilon_2)\frac{1}{2}\frac{i}{2\sqrt{s}}\int\frac{d^{8-p}k}{(2\pi)^{8-p}}a_{D_p}(s,t_1)a_{D_p}(s,t_2)V_2(t_1,t_2,t), \tag{4.73}$$

where we find the vertex for the emission of two reggeized gravitons derived by Amati, Ciafaloni and Veneziano and introduced in Section 3.2,

$$V_2(t_1,t_2,t) = \frac{\Gamma\left(1+\frac{\alpha'}{2}(t_1+t_2-t)\right)}{\Gamma^2\left(1+\frac{\alpha'}{4}(t_1+t_2-t)\right)}. \tag{4.74}$$

The amplitude (4.73) has simple poles at

$$t-t_1-t_2 = \frac{2}{\alpha'}(1+l), \qquad l\in\mathbb{N}. \tag{4.75}$$

The singularity with $l=0$ however is cancelled in the complete amplitude by the subleading pole in (4.68). The singularities with $l\neq 0$ should be cancelled by a similar mechanism involving terms with lower powers of $\alpha' s$ that were not included in our approximation.

To all orders in α' the leading annulus contribution can be written in the following form

$$\frac{\mathcal{A}_2^{(3)}}{2E} = \frac{i}{2}\int\frac{d^{8-p}k}{(2\pi)^{8-p}}\frac{\mathcal{A}_1(s,t_1)}{2E}\frac{\mathcal{A}_1(s,t_2)}{2E}V_2(t_1,t_2,t), \tag{4.76}$$

where

$$q^2=-t, \qquad t_1=-k^2, \qquad t_2=-(q-k)^2. \tag{4.77}$$

The superscript of $\mathcal{A}_2^{(3)}$ in (4.76) denotes the contribution to the amplitude that scales as E^3, so the dominant term in the energy. As we will show in the following, there is also a subleading term scaling as E^2.

4.1.4 Resummation of the amplitude

In this section we generalise the results of the previous sections to surfaces with more than two boundaries, using the amplitudes for the emission of h reggeized gravitons, $\mathcal{A}^{\alpha_1\cdots\alpha_h}$ derived in [7, 8] and reported in Chapter 3.

For a generic amplitude $\mathcal{A}_h$ with h boundaries, the term with the highest power of E should correspond to the exchange of h gravi-reggeons and scale as E^{h+1}, as can be inferred for example from (4.76). So we write

$$\frac{\mathcal{A}_h^{(h+1)(s,t)}}{2E} \sim \frac{1}{h!}\frac{i^{h-1}}{(2E)^h}\prod_{i=1}^{h-1}\int \frac{d^{8-p}k_i}{(2\pi)^{8-p}}\mathcal{A}_1(s,t_1)...\mathcal{A}_1(s,t_h)V_h(k_1,k_2,...,k_h), \tag{4.78}$$

where the vertex for the emission of h gravi-reggeons V_h generalises the expression (4.74) for $h=2$ and

$$\begin{aligned} t_i \equiv -k_i^2, \quad i = 1...h \\ \sum_{i=1}^{h} k_i = q \end{aligned} \tag{4.79}$$

To resum the leading contributions we use the representation of V_h in terms of vacuum expectation values of string vertex operators, as derived in Chapter 3:

$$V_h(k_1,...,k_h) = \langle 0|\prod_{i=1}^{h}\int_0^{2\pi}\frac{d\sigma_i}{2\pi} : e^{ik_i\cdot X(\sigma_i)} : |0\rangle, \tag{4.80}$$

where the string fields $X(\sigma)$ were defined in Eq. (4.10). Now, by using the operator form for V_h we rewrite the integrand in (4.78) as a convolution in momentum space and then we diagonalise the full string expression going to impact parameter space,

$$\begin{aligned} i\frac{\mathcal{A}_h^{(h+1)(s,b)}}{2E} &= \int \frac{d^{8-p}q}{(2\pi)^{8-p}}e^{ib\cdot q}\, i\frac{\mathcal{A}_h^{(h+1)(s,t)}}{2E}, \\ &= \frac{i^h}{h!}\langle 0|\prod_{i=1}^{h}\int_0^{2\pi}\frac{d\sigma_i}{2\pi}\int \frac{d^{8-p}k_i}{(2\pi)^{8-p}}\frac{\mathcal{A}_1(s,-k_i^2)}{2E} : e^{ik_i\cdot(b+X(\sigma_i))} : |0\rangle. \end{aligned} \tag{4.81}$$

Summing the contributions we obtain

$$\sum_{h=1}^{\infty}\frac{\mathcal{A}^{(h+1)}(s,b)}{2E} \sim \langle 0|\frac{1}{i}\left[e^{2i\hat{\delta}(s,b)}-1\right]|0\rangle, \tag{4.82}$$

where

$$2\hat{\delta}(s,b) = \int_0^{2\pi}\frac{d\sigma_i}{2\pi}\int \frac{d^{8-p}k_i}{(2\pi)^{8-p}}\frac{\mathcal{A}_1(s,-k^2)}{2E} : e^{ik_i\cdot(b+X(\sigma))} := \int_0^{2\pi}\frac{d\sigma_i}{2\pi}\frac{:\mathcal{A}_1(s,b+X(\sigma)):}{2E}. \tag{4.83}$$

The amplitude then resums in an operator eikonal form.

The eikonal operator (4.83) gives the leading behaviour of string-brane scattering at high energy and in a series expansion in powers of R_p/b. As was discussed in the context of string-string collisions at transplanckian energies [7], string amplitudes in flat space to all order in perturbation theory can be interpreted with the motion of the string in an effective curved background. As was then shown in [27], also the amplitude of a massless closed string on a stack of N Dp-branes can be interpreted as describing the semiclassical

propagation of a closed string in a curved spacetime generated by the D-branes. In the following we can study two main effects which are taken into account by the eikonal operator: the deflection of the trajectory of the projectile and, if we consider that the string is an extended object, the excitation of its internal degrees of freedom. The agreement between the resummation of open string loops in Minkowski space and the quantization of the string in an external metric confirms that, for large R_p, a collection of N coincident Dp-branes is well approximated by the extremal p-brane solution of the supergravity equations of motion. It also shows that, at least in certain limits, our approach provides a quantitative tool to study the classical dynamics of a string in a curved spacetime. The advantage of the present approach, based on the microscopic and manifestly unitary D-brane description, is that it does not rely on the existence of an effective external metric and can be used to analyse other interesting dynamical regimes.
In order to derive from the eikonal operator (4.83) both the deflection angle and the excitation spectrum, we expand the eikonal phase in a power series in the string position operators X^i,

$$2\hat{\delta}(s, b+X) \sim \frac{1}{2E}\left[\mathcal{A}_1(s,b) + \frac{1}{2}\frac{\partial^2 \mathcal{A}_1(s,b)}{\partial b^1 \partial b^j}\overline{X^i X^j} + ...\right] \tag{4.84}$$

assuming $b \gg R_p \gg ls\sqrt{\ln(\alpha' s)}$ and keeping only the first two non-trivial terms. The indices i, j label the $8-p$ directions of the impact parameter space transverse to the brane and to the collision axis and the symbol $\bar{A}$ denotes the average of a local operator $A(\sigma,\tau)$ on the worldsheet,

$$\bar{A} \equiv \frac{1}{2\pi}\int_0^{2\pi} d\sigma : A(\sigma, \tau = 0) : . \tag{4.85}$$

The two terms in (4.84) will give rise respectively to the leading contribution to the deflection angle and to the leading contribution to the tidal excitation of the string modes.

4.2 String corrections to the leading eikonal

The excitations of the internal degrees of freedom of the string are taken into account by the second term in Eq. (4.84), which is of leading order both in R_p/b and in α'/b^2. The higher-derivative terms in the Taylor expansion of the eikonal phase lead to higher string corrections of the same order R_p/b but of higher order in α'/b^2 and are therefore suppressed when $b \gg l_s$. The term with two derivatives provides the leading contribution to the imaginary part of the eikonal phase, which accounts for the excitation of the string under the influence of the long-range gravitational field of the brane. In order to study this effect we derive the explicit form of the eikonal operator in terms of the standard left-moving and right- moving operator modes in the string field expansion (4.10), $\alpha_n^i, \bar{\alpha}_n^i$. We first compute the second derivative term in (4.84), keeping in mind that the eikonal phase depends only on the modulus of the impact parameter:

$$\begin{aligned}\frac{1}{4\sqrt{s}}\frac{\partial^2 \mathcal{A}_1(s,b)}{\partial b^i \partial b^j} &= \frac{1}{4\sqrt{s}}\left[\frac{1}{b}\frac{d\mathcal{A}_1(s,b)}{db}\delta_{ij} + \frac{b_i b_j}{b^2}\frac{d^2\mathcal{A}_1(s,b)}{db^2} - \frac{1}{b}\frac{d\mathcal{A}_1(s,b)}{db}\right], \\ &= Q_\perp\left[\delta_{ij} - \frac{b_i b_j}{b^2}\right] + Q_\parallel \frac{b_i b_j}{b^2},\end{aligned} \tag{4.86}$$

where we splitted the eigenvalues of the matrix of the second derivative to those orthogonal to the impact parameter,

$$Q_\perp(s,b) = \frac{1}{4\sqrt{s}}\frac{1}{b}\frac{d\mathcal{A}_1(s,b)}{db} = -\frac{\sqrt{\pi}}{2}\sqrt{s}\frac{\Gamma\left(\frac{8-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)}\frac{R^{7-p}}{b^{8-p},} \tag{4.87}$$

which are $7-p$ coincident eigenvalues, and one eigenvalue associated to the component parallel to the impact parameter

$$Q_{\|}(s,b) = \frac{1}{4\sqrt{s}}\frac{d^2\mathcal{A}_1(s,b)}{db^2} = -(7-p)Q_{\perp}(s,b). \tag{4.88}$$

In order to rewrite the eikonal operator expansion in a normal ordered form, we express the string field expansion in terms of the operators introduced in [7],

$$T^i_{\pm,n} = \frac{1}{n}\alpha_{\mp n}\bar{\alpha}^i_{\pm n}, \qquad 2T^i_{0,n} = 1 + \frac{1}{n}(\alpha^i_{-n}\alpha^i_n + \bar{\alpha}^i_{-n}\bar{\alpha}^i_n), \tag{4.89}$$

so we get

$$e^{2i\hat{\delta}(s,\mathbf{b})} \sim e^{\frac{i}{2\sqrt{s}}\mathcal{A}_1(s,b)}\prod_{n=1}^{\infty}\prod_{k=p+1}^{8} e^{i\frac{\alpha'}{n}Q_k(2T^k_{0,n}-1-T^k_{+,n}-T^k_{-,n})}, \tag{4.90}$$

where Q_k denotes $Q_\perp$ or $Q_\|$ according to whether the direction k is perpendicular or parallel to the impact parameter. To write (4.90) in a normal-ordered form we use the identity

$$e^{x(2T_0-T_+-T_-)} = e^{\frac{x}{1-x}T_+}e^{-2\ln(1-x)T_0}e^{-\frac{x}{1-x}T_-}, \tag{4.91}$$

and get

$$\begin{aligned} e^{2i\hat{\delta}(s,\mathbf{b})} \sim &\prod_{n=1}^{\infty}\prod_{k=p+1}^{8}\frac{e^{-\frac{i\alpha' Q_k}{n}}}{1-\frac{i\alpha' Q_k}{n}} \\ &\times e^{-\frac{i\alpha' Q_k}{n-i\alpha' Q_k}T^k_{+,n}}e^{-\ln\left(\frac{i\alpha' Q_k}{n}\right)(2T^k_{0,n}-1)}e^{-\frac{i\alpha' Q_k}{n-i\alpha' Q_k}T^k_{-,n}}. \end{aligned} \tag{4.92}$$

The amplitude is given by the vacuum expectation value of the operator (4.92). Using the infinite-product representation of the Gamma function and introducing the Euler-Mascheroni constant γ we get

$$\langle 0|e^{2i\hat{\delta}(s,b)}|0\rangle \sim e^{\frac{i}{2\sqrt{s}}\mathcal{A}_1(s,b)}e^{-i\alpha'\gamma((7-p)Q_\perp+Q_\|)}\Gamma(1-i\alpha' Q_\|)\Gamma^{7-p}(1-i\alpha' Q_\perp), \tag{4.93}$$

showing that for large b the string corrections to the real part of the eikonal phase are negligible.
The most important effect of string corrections is to induce an imaginary part with an absorptive meaning:

$$|\langle 0|e^{2i\hat{\delta}(s,b)}|0\rangle| \sim^{\frac{i}{2\sqrt{s}}Im\mathcal{A}_1(s,b)} \left[\frac{\pi\alpha' Q_\perp}{\sinh\pi\alpha' Q_\perp}\right]^{\frac{7-p}{2}}\left[\frac{\pi\alpha' Q_\|}{\sinh\pi\alpha' Q_\|}\right]^{\frac{1}{2}}. \tag{4.94}$$

The high energy limit of this imaginary part yields

$$|\langle 0|e^{2i\hat{\delta}(s,b)}|0\rangle| \sim^{\frac{i}{2\sqrt{s}}Im\mathcal{A}_1(s,b)} (2\pi\alpha')^{\frac{8-p}{2}}|Q_\perp(s,b)|^{\frac{7-p}{2}}|Q_\|(s,b)|^{\frac{1}{2}}e^{-\frac{\pi}{2}\alpha'[(7-p)Q_\perp(s,b)+|Q_\|(s,b)|]}. \tag{4.95}$$

Substituting the definition of $Q_\|$ in terms of $Q_\perp$, Eq. (4.88):

$$\langle 0|e^{2i\hat{\delta}(s,b)}|0\rangle \sim^{\frac{i}{2\sqrt{s}}Im\mathcal{A}_1(s,b)} \Gamma(1+i\alpha'(7-p)Q_\perp)\Gamma^{7-p}(1+i\alpha' Q_\perp), \tag{4.96}$$

$$\left|\langle 0|e^{2i\hat{\delta}(s,b)}|0\rangle\right| \sim^{\frac{i}{2\sqrt{s}}Im\mathcal{A}_1(s,b)} (2\pi\alpha'|Q_\perp(s,b)|)^{\frac{8-p}{2}}\sqrt{7-p}\,e^{-\pi\alpha'(7-p)|Q_\perp(s,b)|}. \tag{4.97}$$

By analysing (4.97) we find that the absorption of the elastic channel is non negligible for $b \leqslant b_D$, where

$$b_D^{8-p} = \frac{\pi}{2}\alpha'\sqrt{\pi s}(7-p)\frac{\Gamma\left(\frac{8-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)}R_p^{7-p}. \tag{4.98}$$

At large distances this effect is relevant already for $b \gg l_s$ and $b \gg R_p$.
In the next Section we will show that the tidal excitations of the string given by (4.97) agree with the results of a semiclassical computation in the Dp-brane spacetime.

4.3 String and D-brane dynamics

In this Section we will compare the results we have obtained from string computations in flat spacetime to what we would expect from the propagation of point or string-like objects in the non-trivial background (4.99) generated by the D-branes.

In Section 2.4 we introduced the D-branes and their gauge dynamics in spacetime, also relating D-branes with the extremal p-brane metrics in Section 2.7 as solutions to the Type II effective actions which are charged under the R-R potentials.
In the following we will need the extremal p-brane metric in order to analyse the string dynamics in the effective curved background generated by the D-branes. For $p < 7$ and in the string frame the extremal p-brane solution is given by

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{H(r)}}(\eta_{\alpha\beta}dx^\alpha dx^\beta) + \sqrt{H(r)}(\delta_{ij}dx^i dx^j), \\ e^{\phi(x)} &= g[H(r)]^{\frac{3-p}{4}}, \qquad \mathcal{C}_{01\ldots p}(x) = \frac{1}{H(r)} - 1, \end{aligned} \tag{4.99}$$

where the indices $\alpha,\ \beta, \ldots$ run along the Dp-brane world-volume, the indices $i, j \ldots$ indicate the transverse directions and $r^2 = \delta^{ij}x^i x^j$, and

$$H(r) = 1 + \left(\frac{R_p}{r}\right)^{7-p}, \qquad R_p^{7-p} = \frac{gN(2\pi\sqrt{\alpha'})^{7-p}}{(7-p)\Omega_{8-p}}, \qquad \Omega_m = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}. \tag{4.100}$$

In (4.100) g is the dimensionless string coupling constant, N the number of Dp-branes and Ω_n is the volume of the n-dimensional unit sphere. This description is valid as long as the curvature is smaller than the string length: $R_p \ll l_s$.

In the high-energy limit, only gravitons sensitive to the metric part of the background dominate, since gravitational couplings are proportional to the energy, while the dilaton and the RR-form backgrounds would provide subleading corrections which we will not consider. We will focus our attention on the comparison of the deflection angles up to next-to-leading order in the point-particle limit, and tidal excitation of the stringy probes at leading order. As we shall see, there is full agreement on both effects between the string-loop calculations and curved-spacetime expectations.

4.3.1 Classical deflection angle

In order to compute the deflection angle of a massless point-like probe in the metric (4.99) produced by our stack of Dp-branes, we will use a general form of the metric which allows to change conformal frame (e.g. from the string frame to the Einstein frame), as was done in [27],

$$ds^2 = -\alpha(r)dt^2 + \beta(r)(dr^2 + r^2 d\vartheta^2). \tag{4.101}$$

The coordinates involved in the geodesics are only the time t and the spatial coordinates r and ϑ of the plane in which the motion takes place. The differential equation relating ϑ and the radial coordinate can be derived from the invariance of the action under arbitrary reparametrisations of the world-line coordinate u, and from the conservation of the energy E and the angular momentum J,

$$\frac{d\vartheta}{dr} = -\frac{b}{r^2\sqrt{\frac{\beta}{\alpha} - \frac{b^2}{r^2}}} \Leftrightarrow \frac{d\vartheta}{d\rho} = \frac{\hat{b}}{\sqrt{1+\rho^{7-p} - \hat{b}^2\rho^2}}, \tag{4.102}$$

where $b = JE$, $\rho = R_p/r$, $\hat{b} = b/R_p$, and we used the actual form of α/β given in Eq. (4.99).
The trajectory of a probe particle in the metric (4.101) is symmetric around r_*, where $\rho = R_p/r_*$ is the smallest root of the equation $1 + \rho^{7-p} - \hat{b}^2\rho^2 = 0$. The value of the angle ϑ at the turning point r_* is

$$\vartheta(r_*) = \int_\infty^{r_*} \frac{d\vartheta}{dr} = \int_0^{\rho_*} \frac{\hat{b}}{\sqrt{1 + \rho^{7-p} - \hat{b}^2\rho^2}}. \tag{4.103}$$

For the symmetry, the deflection angle Θ_p is given by

$$\Theta_p = 2\vartheta(r_*) - \pi \Rightarrow \Theta_p = 2\int_0^{\rho_*} \frac{\hat{b}}{\sqrt{1 + \rho^{7-p} - \hat{b}^2\rho^2}} - \pi. \tag{4.104}$$

The leading and next-to-leading terms in the large impact parameter expansion for arbitrary p can be computed and read [27]

$$\Theta_p = \sqrt{\pi}\left[\frac{\Gamma\left(\frac{8-p}{2}\right)}{\Gamma\left(\frac{8-p}{2}\right)}\left(\frac{R_p}{b}\right)^{7-p} + \frac{1}{2}\frac{\Gamma\left(\frac{15-2p}{2}\right)}{\Gamma(6-p)}\left(\frac{R_p}{b}\right)^{2(7-p)} + ...\right], \tag{4.105}$$

in perfect agreement with the string calculations in [27].

4.3.2 Tidal excitation of the closed string at leading order

There is much work in the literature (see for example [28, 37, 44, 60]) where string excitations coming from string-string collisions find agreement with the quantization of a closed string in the Aichelburg-Sexl metric. In our case we will compare the excitations Q_k of the closed string found in Section 4.2 with those obtained through quantization of a closed string in the non-trivial metric (4.99). We should note that the string calculation relies on the eikonal-operator formula (4.82), which refers to the leading contribution in R_p/b but is supposed to hold at all orders in α'/b^2. On the other hand, the curved-spacetime calculation we shall present below is limited to small string fluctuations around the point-particle null geodesic while, in principle, it can be extended to higher orders in R_p/b. Our comparison will be made in the overlap of the domains of validity of the two calculations, namely at leading order both in α'/b^2 and in R_p/b. We will find complete agreement between the two calculations despite the difference between techniques we have used, and so this agreement represents a check of the validity of our approach. We will be using the metric [18]

$$ds^2 = \alpha(r)\left(-dt^2 + \sum_{a=1}^{p}(dx^a)^2\right) + \beta(r)(dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\Omega_{7-p}^2), \tag{4.106}$$

where for the D-brane we have $\beta(r) = 1/\alpha(r) = \sqrt{H(r)}$. We want to describe the metric around the null geodesic considered in Section 4.3.1, to do so we consider new coordinates u, v, z, x^a, y^i in which the geodesic correspond to constant u, v, z, x^a, y^i and $u = u(r)$ is an affine parameter along the geodesic:

$$\begin{aligned} dv &= -dt + bd\vartheta + Cdr, \qquad & dz &= d(\vartheta + \bar{\vartheta}(u)), \\ dU &= \pm\tfrac{\beta dr}{C}, & C(r) &= \sqrt{\tfrac{\beta(r)}{\alpha(r)} - \tfrac{b^2}{r^2}}. \end{aligned} \tag{4.107}$$

Here $\bar{\vartheta}(u)$ is the angular coordinate ϑ, evaluated along the null geodesic as in (4.102) and expressed in terms of u via Eq. (4.107). The 7 coordinates x_a, y_i represent the fluctuations orthogonal to the null geodesic: (x_a)

are the fluctuations parallel to the directions of the brane world-volume, (y_i) are along the $(7-p)$ directions which are orthogonal both to the brane and to the plane of the geodesic. The z coordinate is orthogonal to the brane, but lies in the plane of the geodesic. In our conventions the point $u = 0$ corresponds to the turning point r_* and the choice of sign in the equation for u depends on the point of the geodesic we are considering: we choose the minus sign in the approaching region and so parametrise the part of the geodesic from infinity to r_* with the interval $-\infty < u \leqslant 0$; for the remaining part from r_* to infinity we choose the plus sign and so it corresponds to the interval $0 \leqslant u < 1$. In these adapted coordinates the metric takes the form

$$ds^2 = 2dudv - \alpha dv^2 + 2b\alpha dvdz + r^2\alpha C^2 dz^2 + \alpha dx^a dx^a + \beta r^2 \sin^2(z - \bar{\vartheta}) d\Omega^2_{7-p}. \tag{4.108}$$

Now we perform the Penrose limit of this metric, which corresponds to the high-energy limit of the probe. We will focus on a small neighborhood of the null geodesic, in order to do it we expand to second order the dependence on the coordinates transverse to the light-cone and keep only linear terms in v. By doing this we eliminate the terms in dv^2 and $dvdz$. Then we change again the coordinates:

$$\begin{aligned} z &= \frac{\hat{y}^0}{\sqrt{r^2\alpha C^2}}, \qquad y^i = \frac{\hat{y}^i}{\sqrt{\beta}r\sin\bar{\vartheta}}, \qquad x^a = \frac{\hat{x}^a}{\sqrt{\alpha}}, \\ v &= \hat{v} + \frac{1}{2}\left[\sum_{a=1}^{p} \hat{x}_a^2 \partial_u \ln(\sqrt{\alpha}) + \sum_{i=1}^{7-p} \hat{y}_i^2 \partial_u \ln(\sqrt{\beta r \sin\bar{\vartheta}}) + \hat{y}_0^2 \partial_u \ln(\sqrt{r^2\alpha C^2})\right], \end{aligned} \tag{4.109}$$

to write the metric in pp-wave form

$$\begin{aligned} ds^2 &= 2dud\hat{v} + \sum_{a=1}^{p} d\hat{x}_a^2 + \sum_{i=1}^{7-p} d\hat{y}_i^2 + d\hat{y}_0^2 + \mathcal{G}(u, \hat{x}_a, \hat{y}_i, \hat{y}_0) du^2, \\ \mathcal{G} &= \frac{\partial_u^2 \sqrt{\alpha}}{\sqrt{\alpha}} \sum_{a=1}^{p} \hat{x}_a^2 + \frac{\partial_u^2(\sqrt{\beta}r\sin\bar{\vartheta})}{\sqrt{\beta}r\sin\bar{\vartheta}} \sum_{i=1}^{7-p} \hat{y}_i^2 + \frac{\partial_u^2\sqrt{\beta r^2 - b^2\alpha}}{\sqrt{\beta r^2 - b^2\alpha}} \hat{y}_0^2 \\ &\equiv \mathcal{G}_x \sum_{a=1}^{p} \hat{x}_a^2 + \mathcal{G}_y \sum_{i=1}^{7-p} \hat{y}_i^2 + \mathcal{G}_0 \hat{y}_0^2. \end{aligned} \tag{4.110}$$

The bosonic sigma model can be written as follows:

$$S = S_0 - \frac{1}{4\pi\alpha'} \int d\tau \int_0^{2\pi} d\sigma \eta^{\alpha\beta} \partial_\alpha U \partial_\beta U \mathcal{G}(U; X^a, Y^i, Y^0), \tag{4.111}$$

where S_0 is the string action in Minkowski space, $\eta^{\alpha\beta}$ is the worldsheet metric and $\mathcal{G}$ is now a function of the string coordinates. Now we can quantize the string using light-cone gauge to simplify the computation:

$$U(\sigma, \tau) = \alpha' p^u \tau \to \alpha' E \tau. \tag{4.112}$$

Substituting in the action (4.111) we find

$$\begin{aligned} S - S_0 &= \frac{E}{2} \int_0^{2\pi} \frac{d\sigma}{2\pi} \int_{-\infty}^{\infty} du \mathcal{G}(u, X^a(\sigma, u/\alpha' E)), Y^i(\sigma, u/\alpha' E), Y^0(\sigma, u/\alpha' E)) \\ &\to \frac{E}{2} \int_0^{2\pi} \frac{d\sigma}{2\pi} \int_{-\infty}^{\infty} du \left(\mathcal{G}_x(u) \sum_{a=1}^{p} X_a^2(\sigma, 0) + \mathcal{G}_y(u) \sum_{i=1}^{7-p} Y_i^2(\sigma, 0) + \mathcal{G}_0 Y_0^2(\sigma, 0) \right) \\ &\equiv \frac{E}{2} \int_0^{2\pi} \frac{d\sigma}{2\pi} \left(c_x \sum_{a=1}^{p} X_a^2(\sigma, 0) + c_y \sum_{i=1}^{7-p} Y_i^2(\sigma, 0) + c_0 Y_0^2(\sigma, 0) \right), \end{aligned} \tag{4.113}$$

where in the second step we used the high energy limit. From what appears in (4.113) the string gets excited in every direction, but as shown in [27] at leading order in R_p/b only the Dirichlet directions are excited. To show this, we will compute the coefficients c_x, c_y and c_0. We start by writing

$$c_x = 2\int_0^\infty \frac{\partial_u^2\sqrt{\alpha}}{\sqrt{\alpha}}du = 2\int_0^\infty \left[(\partial_u \ln\sqrt{\alpha})^2\right]du. \tag{4.114}$$

In the second integral the first term is proportional to $\left(\frac{R_p}{R}\right)^{2(7-p)}$ and since $\sqrt{\alpha} \sim O(R_p/r)^{7-p}$ it is of higher order in R_p/b. The second term is the total derivative of a function that vanishes on the integration boundaries. To show this we change the derivative with respect to u into a r-derivative using (4.3.2), and then use $C(r_*) = 0$:

$$\partial_u \ln\sqrt{\alpha}\;|_0^\infty = \left[\frac{C}{\beta}\partial_r \ln\sqrt{\alpha}\right]_{r_*}^\infty = 0, \tag{4.115}$$

For c_y we have a similar computation, acting as we did for c_x we can write

$$c_y = \int_0^\infty \left[\left(\partial_u \ln\sqrt{\beta}r\sin\bar{\vartheta}\right)^2 + \partial_u^2\left(\ln\sqrt{\beta}r\sin\bar{\vartheta}\right)\right]du. \tag{4.116}$$

As in the previous case, the first term does not contribute at the order $\left(\frac{R_p}{R}\right)^{7-p}$ since $r\sin\bar{\vartheta} = b + O(R/r))^{7-p}$, while the second term gives

$$c_y \sim 2\partial_u \ln[\sqrt{\beta}r\sin\bar{\vartheta}(u)]|_0^\infty = \left[\frac{C}{\beta}\partial_r \ln[\sqrt{\beta}r\sin\bar{\vartheta}(r)]\right]_{r_*}^\infty. \tag{4.117}$$

Since $C(R_+) = 0$, the non-vanishing contribution comes from the derivative of $\sin\vartheta$. Using (4.102) we get

$$c_y \sim 2\left.\frac{b\cos\vartheta}{r^2\beta\sin\vartheta}\right|_{r=r_*} \tag{4.118}$$

then using the relation between Θ_p and $\vartheta(r_*)$ in (4.104) and keeping terms of order $\left(\frac{R_p}{b}\right)^{7-p}$ we get

$$c_u \sim 2\frac{b\cos\left(\frac{\Theta_p+\pi}{2}\right)}{r_*^2\beta(r_*)\sin\left(\frac{\Theta_p+\pi}{2}\right)} \sim -\frac{\Theta_p}{b}. \tag{4.119}$$

Then using the expansion of Θ_p (4.105) we get the orthogonal excitations $Q_\perp$ of the string computation in Section 4.2,

$$c_y \sim -\frac{\sqrt{\pi}}{b}\frac{\Gamma\left(\frac{8-p}{2}\right)}{\Gamma\left(\frac{8-p}{2}\right)}\left(\frac{R_p}{b}\right)^{7-p} \Rightarrow \frac{E}{2}c_y = Q_\perp(s,b). \tag{4.120}$$

The coefficient c_0 is given by

$$c_0 = \int_{-\infty}^\infty \frac{\partial_u^2\sqrt{\beta r^2 - b^2\alpha}}{\sqrt{\beta r^2 - b^2\alpha}}du = 2\int_{r_*}^\infty \frac{dr}{\sqrt{\beta r^2 - b^2\alpha}}\partial_r\partial_u\sqrt{\beta r^2 - b^2\alpha}. \tag{4.121}$$

We have to perform the derivatives, which now are not subleading as in the computation of the other coefficients,

$$\partial_u\sqrt{\beta r^2 - b^2\alpha} = \frac{\partial r}{\partial u}\partial_r\sqrt{\beta r^2 - b^2\alpha} = -\alpha^{-1/2}\left(1 + \frac{r\partial_r\beta}{2\beta} - \frac{b^2}{2r}\frac{\partial_r\alpha}{\beta}\right). \tag{4.122}$$

Expanding α and β for large r we can obtain the leading terms in $\left(\frac{R_p}{b}\right)^{7-p}$,

$$\alpha \sim 1 + C_\alpha \left(\frac{R_p}{r}\right)^{7-p}, \qquad \beta \sim 1 + C_\beta \left(\frac{R_p}{r}\right)^{7-p}, \tag{4.123}$$

where the coefficients C_α and C_β are arbitrary, but substituting into (4.122) we get

$$\partial_u \sqrt{\beta r^2 - b^2\alpha} = \frac{1}{2}\left(\frac{R_p}{r}\right)^{7-p}\left(-C_\alpha + (p-7)C_\beta - \frac{b^2}{r^2}(p-7)C_\alpha\right) \tag{4.124}$$

Then we perform the r-derivative inside the integral (4.121):

$$\begin{aligned}\partial_r\left[\partial_u \sqrt{\beta r^2 - b^2\alpha} = \frac{1}{2}\left(\frac{R_p}{r}\right)^{7-p}\left(-C_\alpha + (p-7)C_\beta - \frac{b^2}{r^2}(p-7)C_\alpha\right)\right] \\ = \frac{p-7}{2r}\left(\frac{R_p}{r}\right)^{7-p}\left[-C_\alpha + C_\beta(p-7) - \frac{b^2}{r^2}C_\alpha(p-9)\right].\end{aligned} \tag{4.125}$$

Now we substitute in the integral (4.121) and make the following approximations valid at the order we are considering: $r_* \sim b$ in the lower extremum of integration and $\alpha, \beta \sim 1$ in the square root in the denominator. We get two integrals of the form

$$\int_b^\infty dr \frac{r^{1-2\gamma}}{\sqrt{r^2-b^2}} = \frac{\sqrt{\pi}b^{1-2\gamma}}{2}\frac{\Gamma\left(\gamma-\frac{1}{2}\right)}{\Gamma(\gamma)}, \tag{4.126}$$

so we obtain

$$c_0 = \frac{\sqrt{\pi}}{b}(7-p)\left(\frac{R_p}{b}\right)^{7-p}\frac{G\left(\frac{8-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)}(C_\beta - C_\alpha). \tag{4.127}$$

Since $C_\beta - C_\alpha = 1$ both in the string and in the Einstein frame, we reproduce the parallel excitations $Q_\parallel$ of the string computation,

$$\frac{E}{2}c_0 = Q_\parallel(s,b). \tag{4.128}$$

4.4 Discussion

Here we summarise all the different regimes that the eikonal operator allows to describe depending on the scales involved. The most important physical quantities we considered are the energy of the colliding string E, the impact parameter b, the curvature radius R_p of the D-brane background and the string length l_s. We focused mainly on the kinematic region where the impact parameter b is bigger than R_p and the interaction between the string probe and the D-branes is dominated by gravity ($\alpha' s \gg 1$), allowing the comparison with the dynamics of a string propagating in an extremal p-brane background. In the region $R_p < l_s\sqrt{\alpha' s}$ we have shown that string effects become important and so the classical analogy of the propagation in the D-brane metric is no longer reliable. As we lower the impact parameter we find indeed distinct regimes:

- $b \gg R_p, ls$ is the perturbative region of small angle scattering, where a new imaginary part arises describing tidal forces acting on the string and induced by the D-brane. Tidal excitation becomes relevant for $b < b_D$, where the critical value b_D was defined in (4.98).
- $b \sim R_p > l_s$ is the large angle scattering;

- $b < l_s, R_p$: here the gravity description fails since there is the stringy regime, in this region for $R_p < l_s$ the probe gets captured for a critical value of $b = b_c$.

As shown in [27] it is useful to draw an (R_p, b) phase diagram where we mark the different regimes we have studied. Since R_p is independent on the energy, we can describe the various regimes at fixed large energy. In the diagram we keep $b > l_s\sqrt{\ln \alpha' s}$. The straight black line at 45 degrees is a curve of constant deflection angle below which the phenomenon of capture occurs, while the red line delimits the impact parameter b_D below which tidal excitation becomes important. The phase diagram is reported in Fig. 4.1. Inelastic effects are relevant already at large impact parameters, and so it would be interesting to study the string excitations. These arise only along the Dirichlet directions for large b, but at smaller distances one could find nonzero Neumann excitations. The region of massive string excitations is perturbative and so we could also check the eikonal in this region by computing massive states amplitudes. These are the main motivations for the study of high-energy massive string-brane scattering, which we will discuss in Chapter 5.

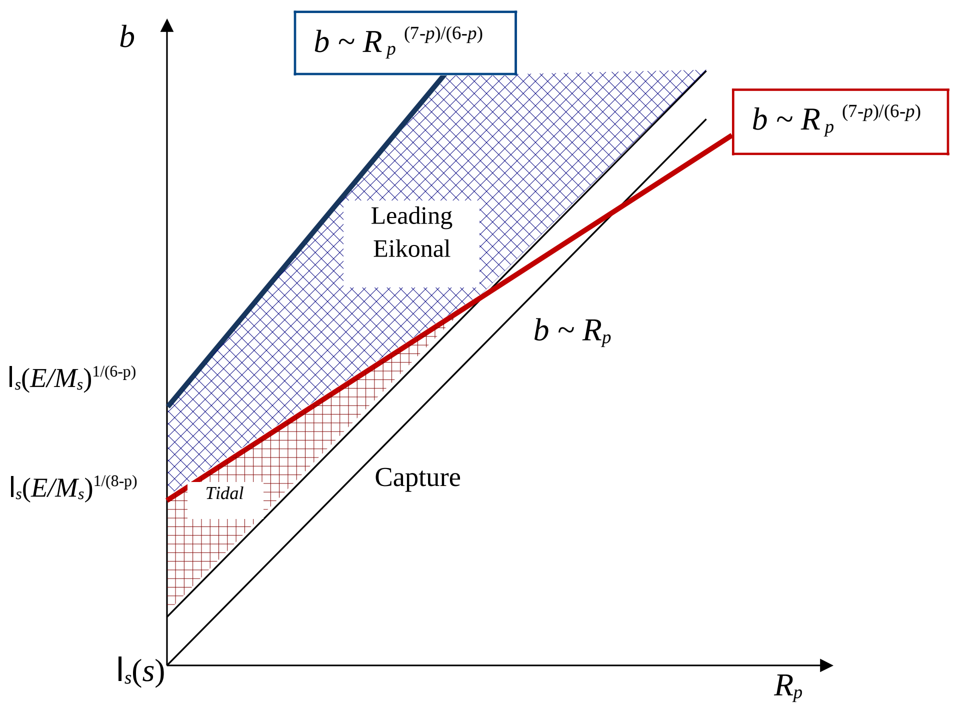

Figure 4.1: The diagram summarises the regimes studied in the scattering of massless closed strings off D-branes. It is a logarithmic plot of the impact parameter b versus R_p, both taken to be bigger than the effective string scale.

5

High energy string-brane scattering for massive string states

In the previous Chapters we have analysed the high-energy scattering of a graviton from a stack of N parallel Dp-branes in a flat background, which has the same features of the string amplitudes described in [8].

In this Chapter we are motivated by these previous works to study the interactions of a massive closed superstring with a stack of N Dp-branes, where the string is excited/decays into a state of a different mass, as was done in [17]. Since we expect scattering at large energies to be particularly simple for the leading Regge trajectory, we will take such states as our two external strings, with momenta p_1, p_2 such that $\alpha' p_1^2 = -4n$ and $\alpha' p_2^2 = -4n'$ for positive integers n, n'. We will be interested in the limit $\alpha' s \to \infty$ while t is kept fixed. In order to obtain high-energy scattering amplitudes in the most efficient manner we will make use of OPE methods to construct effective vertex operators as pioneered in [3, 4] and more recently used in [21, 33]. Insertion of these effective vertices onto the upper-half of the complex plane will yield the tree-level amplitudes of these processes which show identical Regge behaviour to that exhibited in the graviton - graviton scattering. This behaviour is universal for the states on the leading Regge trajectory.

This Chapter is organised as follows: we will first review in Section 5.1 the kinematics for string-brane scattering, but now considering massive states. Then in 5.2 we compute the full tree-level inelastic amplitude for the transitions between a graviton and a state of mass $\alpha' p_2^2 = -4$, and then its asymptotic behaviour. We generalise the result to two generic massive states in Section 5.3, isolating the leading terms at high energy by OPE methods. We can resum the amplitudes with an arbitrary number of boundaries in an eikonal form, as it was done for the massless case in Chapter 4; this is done in Section 5.4. Finally in Section 5.5 we discuss the results obtained for the eikonal resummation.

5.1 Kinematics for massive string scattering from a Dp-brane

In our computations we will consider some initial state with momentum p_1 such that $\alpha' M_1^2 = -\alpha' p_1^2 = 4n$; after interacting with the Dp-brane we are left with another state with momentum p_2 which in general can have a different mass $\alpha' M_2^2 = -\alpha' p_2^2 = 4n'$. Again we decompose these momenta into vectors which are parallel and orthogonal to the directions in which the D-brane is extended, and momentum is only conserved in directions parallel to the D-brane,

$$p^{\mu}_{1\parallel} + p^{\mu}_{2\parallel} = 0. \tag{5.1}$$

The mass-shell conditions give

$$p^2_{1\perp} = -\, p^2_{\parallel} - \frac{4n}{\alpha'}, \tag{5.2a}$$

$$p^2_{2\perp} = -\, p^2_{\parallel} - \frac{4n'}{\alpha'}. \tag{5.2b}$$

In this Chapter we consider massive states scattering at high energy, and we are interested in analysing the Regge regime of such amplitudes, meaning their behaviour for $\alpha' s \to \infty$ and $\alpha' t$ fixed.
In 3.8 we have shown that the leading Regge trajectory of massive string states shows the typical Regge behaviour, but we could in principle explore various kinematic regimes, depending on how large are the masses if compared to the center of mass energy s and the momentum transfer t. In particular, by definition, s and t have the following physical boundaries:

$$max(M_1^2,\, M_2^2) \leq s < \infty, \tag{5.3}$$

$$\left(\sqrt{s - M_1^2} - \sqrt{s - M_2^2}\right)^2 \leq |t| \leq \left(\sqrt{s - M_1^2} + \sqrt{s - M_2^2}\right)^2. \tag{5.4}$$

In this Chapter we will analyze the Regge regime of the disk amplitudes for massive string states from the leading Regge trajectory, meaning their behaviour for $\alpha' s \to \infty$ and $\alpha' t$ fixed, showing that they demonstrate typical Regge behaviour in this limit. More precisely, we will keep the external states fixed while taking the large s limit, which implies that the kinetic energy of the projectile is very large.

As we will show in section 5.3, if we consider Regge kinematics the leading contribution to the amplitudes is captured by the OPE of the vertex operators inserted far from the boundary of the worldsheet. The amplitudes could be dominated by the OPE and show Regge behaviour at high energy also when we consider external states with very large masses, as long as the difference $n' - n$ is negligible compared to $\alpha' s$ so that t/s can be small. If instead we allow a large mass gap between initial and final states and we let it grow with the center of mass energy, $\Delta M \sim E$, then we are in a kinematic regime where $t \sim s$. One could consider the limit $s \sim M^2$. In this regime we expect the leading contribution to the amplitude to be given not by the OPE but by a semiclassical worldsheet corresponding to a saddle-point in the moduli space. The amplitudes should then show an exponential decay in the energy, typical of scattering processes at fixed angle [40–43]. The analysis of the scattering amplitudes in the limit of large masses for the external states is however beyond the scope of this Thesis.

5.2 Disk amplitude

As already noted in Chapter 5, the scattering amplitude for the interaction of two closed string states with a stack of N Dp-branes at tree level is given by the insertion of two closed string vertex operators onto the upper-half of the complex plane with suitable boundary conditions,

$$A_{n,n'} = N \int_{\mathbb{H}_+} \frac{dz_1^2 dz_2^2}{V_{CKG}} \left\langle W^{(n)}_{(0,0)}(k_1, z_1, \bar{z}_1) W^{(n')}_{(-1,-1)}(k_2, z_2, \bar{z}_2) \right\rangle_{\mathbb{H}_+} . \tag{5.5}$$

We have already computed the vertex operators for states belonging to the leading Regge trajectory of the graviton in Section 3.8. Here we have the vertex operator (3.107) for a state with momentum $p_1 = 2k_1$ and mass $M^2 = 4n/\alpha'$ carrying a superghost charge $(0,0)$ and the vertex operator (3.105) for a state with momentum $p_2 = 2k_2$ and mass $M^2 = 4n'/\alpha'$ carrying a superghost charge $(-1,-1)$. In this section we will compute the amplitude involving one graviton and one massive symmetric state at the level $n' = 1$. In doing so we shall see many features which are not only common to the methods introduced in Section (5.3) but also motivate them.

The vertex operators we will be using were derived in Section 3.8.3. For the graviton in the $(0,0)$ picture the vertex is

$$W^{(0)}_{(0,0)}(k_1, z_1, \bar{z_1}) = -\frac{\kappa}{\pi\alpha'} \epsilon_{\mu\nu} V_0^{\mu}(k_1, z_1) \widetilde{V}_0^{\nu}(k_1, \bar{z}_1) \tag{5.6}$$

with

$$V_0^{\mu}(k_1, z_1) = \frac{1}{\sqrt{2\alpha'}} \left(\partial X^{\mu}(z_1) + 2\alpha' k_1 \cdot \psi(z_1)\psi^{\mu}(z_1)\right) e^{ik_1\cdot X(z_1)}, \tag{5.7}$$

and the vertex operator in the $(-1,-1)$ picture for the state at the level $n = 1$ is

$$W^{(1)}_{(-1,-1)}(k_2, z_2, \bar{z_2}) = \frac{\kappa}{2\pi} G_{\rho\sigma\tau\zeta} V_{-1}^{\rho\sigma}(k_2, z_2) \widetilde{V}_{-1}^{\tau\zeta}(k_2, \bar{z}_2) \tag{5.8}$$

with

$$V_{-1}^{\rho\sigma}(k_2, z_2) = \frac{i}{\sqrt{2\alpha'}} e^{-\varphi(z_2)} \partial X^{\rho}(z_2) \psi^{\sigma}(z_2) e^{ik_2\cdot X(z_2)}. \tag{5.9}$$

With the aid of the conservation of momentum, one can deduce the following identities,

$$k_1 \cdot k_2 = \frac{n}{2\alpha'} + \frac{n'}{2\alpha'} - \frac{t}{8}, \tag{5.10}$$

$$k_1 \cdot D \cdot k_1 = -\frac{s}{2} + \frac{n}{\alpha'}, \tag{5.11}$$

$$k_2 \cdot D \cdot k_2 = -\frac{s}{2} + \frac{n'}{\alpha'}, \tag{5.12}$$

$$k_1 \cdot D \cdot k_2 = \frac{s}{2} + \frac{t}{8} - \frac{n}{2\alpha'} - \frac{n'}{2\alpha'}. \tag{5.13}$$

Inserting the vertex operators (5.6-5.9) and the normalization (4.18) into the integral (5.5) we obtain

$$\begin{aligned}
A_{0,1} =& \frac{\kappa T_p}{2} \int_{\mathbb{H}_+} \frac{d^2 z_1 d^2 z_2}{V_{CKG}} \left\langle W^{(0)}_{(0,0)}(k_1, z_1, \bar{z}_1) W^{(1)}_{(-1,-1)}(k_2, z_2, \bar{z}_2) \right\rangle_{\mathbb{H}_+} \\
=& \frac{\kappa T_p}{2} \left(-\frac{1}{(2\alpha')^2}\right) \epsilon_{\mu\lambda} D^{\lambda}{}_{\nu} G_{\rho\sigma\alpha\beta} D^{\alpha}{}_{\tau} D^{\beta}{}_{\zeta} \int_{\mathbb{H}_+} \frac{d^2 z_1 d^2 z_2}{V_{CKG}} \\
& \Big\langle : \left(i\partial X^{\mu}(z_1) + 2\alpha' k_1 \cdot \psi(z_1)\psi^{\mu}(z_1)\right) e^{ik_1\cdot X(z_1)} : \\
& : \left(i\bar{\partial} X^{\nu}(\bar{z}_1) + 2\alpha' k_1 \cdot D \cdot \psi(\bar{z}_1)\psi^{\nu}(\bar{z}_1)\right) e^{ik_1\cdot D\cdot X(\bar{z}_1)} : \\
& : e^{-\varphi(z_2)} \partial X^{\rho}(z_2) \psi^{\sigma}(z_2) e^{ik_2\cdot X(z_2)} :: e^{-\varphi(\bar{z}_2)} \bar{\partial} X^{\tau}(\bar{z}_2) \psi^{\zeta}(\bar{z}_2) e^{ik_2\cdot D\cdot X(\bar{z}_2)} : \Big\rangle
\end{aligned} \tag{5.14}$$

The leading term in the high energy Regge limit is given by the contraction of the two operators quadratic in the fermionic fields. This term is proportional to $2\alpha' \, k_1 \cdot D \cdot k_1 = \alpha' s + 2$ and the overall s factor assures that this term is dominant with respect to all the other contractions in the amplitude.

In order to see that this is the case we introduce again the $SL(2,\mathbb{R})$ invariant variable

$$\omega = \frac{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)}{(z_1 - \bar{z}_2)(\bar{z}_1 - z_2)} \tag{5.15}$$

and use this in writing the correlation function which results from the $e^{ik\cdot X}$ operators

$$\left\langle e^{ik_1\cdot X(z_1)} e^{ik_1\cdot D\cdot X(\bar{z}_1)} e^{ik_2\cdot X(z_2)} e^{ik_2\cdot D\cdot X(\bar{z}_2)} \right\rangle = \omega^{-\alpha' \frac{t}{4}+1} (\omega - 1)^{-\alpha' s} (z_2 - \bar{z}_2)^2, \tag{5.16}$$

then the remaining explicit z-dependence can be combined with that from the other possible contractions in (5.14), together with the appropriate measure $\frac{d^2 z_1 d^2 z_2}{V_{CKG}} = d\omega (z_1 - \bar{z}_2)^2 (\bar{z}_1 - z_2)^2$, to give some multiplicative $SL(2,\mathbb{R})$ invariant function $F(\omega)$ and the amplitude then takes the following schematic form

$$A_{0,1} = \frac{\kappa T_p}{2} \int_0^1 d\omega \, \omega^{-\alpha' \frac{t}{4}+1} (\omega - 1)^{-\alpha' s} F(\omega). \tag{5.17}$$

The behaviour of this integral when we take the limit $\alpha' s \to \infty$ is controlled by the integrand in the neighbourhood of the point $\omega = 0$ and since $F(\omega)$ is, in general, a sum of terms composed of powers of ω, $(1-\omega)$ and their inverse quantities, $A_{0,1}$ itself consists of a sum of integrals of the form shown below,

$$\int_0^1 d\omega \, \omega^{-\alpha' \frac{t}{4}+a} (1-\omega)^{-\alpha' s + b} = \frac{\Gamma\left(-\alpha' \frac{t}{4} + a + 1\right) \Gamma\left(-\alpha' s + b + 1\right)}{\Gamma\left(-\alpha' \frac{t}{4} - \alpha' s + a + b + 2\right)}. \tag{5.18}$$

It can be seen that in the large energy limit this quantity will scale with s as $(\alpha' s)^{-a-1}$ and therefore the dominant contribution to $A_{0,1}$ will come from the term with the lowest value of a.

Having learnt this, one can quickly deduce that we are interested in the terms of $F(\omega)$ which are obtained from the maximum possible number of contractions between the holomorphic and antiholomorphic fields in equation (5.14). In general there are many such terms but, as mentioned below (5.14), those which contain the contraction $k_1 \cdot \psi \; k_1 \cdot D \cdot \psi$ will bring an additional factor $\alpha' s$ and so will ultimately be the leading terms at high energy.

By evaluating the correlation function in equation (5.14), then employing the physical state conditions $\epsilon_{\mu\nu} k_1^{\mu} = G_{\rho\sigma\tau\zeta} k_2^{\rho} = 0$ and momentum conservation $k_1^{\mu} + (D\cdot k_1)^{\mu} + k_2^{\mu} + (D\cdot k_2)^{\mu} = 0$, we can determine the function $F(\omega)$ and perform the integral in equation (5.17). If we take the limit $\alpha' s \to \infty$ we find that only one term dominates in this case, which is proportional to $k_1^{\rho} G_{\rho\sigma\tau\zeta} \, \epsilon^{\sigma\tau} k_1^{\zeta}$. Subsequent application of Stirling's approximation for the Gamma function yields the following form for the amplitude in the high energy limit

$$A_{0,1} \sim \frac{\kappa T_p}{2} e^{-i\pi\alpha' \frac{t}{4}} \Gamma\left(-\tfrac{\alpha' t}{4}\right) (\alpha' s)^{\frac{\alpha' t}{4}+1} \, \frac{\alpha'}{2} q^{\rho} G_{\rho\sigma\tau\zeta} \, \epsilon^{\sigma\tau} q^{\zeta}, \tag{5.19}$$

where we have replaced k_1 and k_2 with the physical momentum transferred $q = 2(k_1 + k_2)$ by virtue of the physical state conditions.

It is instructive to compare this result with the analogous result for the elastic scattering of a graviton by a D-brane 4.31,

$$A_{0,0} \sim \frac{\kappa T_p}{2} e^{-i\pi\alpha' \frac{t}{4}} \Gamma\left(-\tfrac{\alpha' t}{4}\right) (\alpha' s)^{\frac{\alpha' t}{4}+1} \, \epsilon_{1\sigma\tau} \, \epsilon_2^{\sigma\tau}. \tag{5.20}$$

Considered purely as functions of the complex variables s and t, these two equations demonstrate identical behaviour; the physical amplitude is obtained by taking s,t real with large positive s and negative t, but for positive t we see that there are poles corresponding to the exchange of a string of mass $M^2 = t$ with the D-brane and, furthermore, the s-dependence indicates that the exchanged string has spin $J = (\alpha' M^2 + 4)/2$ — it belongs to the Regge trajectory of the graviton. The two amplitudes above only differ in a multiplicative factor containing only the initial and final polarisation states, and the exchanged momentum q. This is a result of the different nature of the two processes, the latter amplitude being for elastic scattering whilst the former corresponds to the excitation of internal degrees of freedom within the string by tidal forces. The contractions between the polarisation tensors are unchanged but we note that the remaining indices are both contracted with a factor of q, a feature which will be seen in the more general analysis which follow in Section 5.3.

In the next Section we shall generalise this result and compute the leading high energy contribution to scattering amplitudes involving two generic states of the leading Regge trajectory, at mass levels n and n' of the spectrum; to do this we shall make use of the OPE for the two vertex operators to isolate those terms in the amplitude which dominate as the vertices are brought together on the world-sheet.

5.3 Computation via OPE methods

To compute the leading terms in the amplitude given by equation (5.5) in the Regge limit, $\alpha' s \to \infty$ and $\alpha' t$ fixed, it is key to note that the integral over the world-sheet in (5.5) is dominated at large $\alpha' s$ by the behaviour of the vertex operators as $z_1 \to z_2$, as exemplified in equation (5.17). Specifically, when working in the Regge regime for scattering processes with two external string states the leading contribution to the integral over the world-sheet can be taken from the OPE of the vertex operators. Subsequent integration over their separation $w = z_1 - z_2$ will give an amplitude with the expected Regge behaviour; however, care must be taken in this process, as we will see, since there exist terms subleading in w which will contribute factors of $\alpha' s$ and by doing so prevent us from neglecting them.

As a result, rather than evaluating the correlator in (5.5), integrating the result and then computing the asymptotic form of this result for large $\alpha' s$ we may instead determine the OPE of the two vertex operators and integrate out the dependence on the seperation $w = z_1 - z_2$. This process yields a quasi-local operator referred to as the Pomeron vertex operator, first introduced in [3] as the reggeon and recently used in [21, 33].

As illustrated in figure 5.1(a), the process of constructing this operator involves taking the limit in which two physical vertices approach one another on a surface topologically equivalent to the infinite cylinder, hence it can be considered to properly describe the t-channel exchange of a closed string. The Pomeron vertex operator itself also satisfies the physical state conditions for any $\alpha' t$, and for $\alpha' t = 4n$ ($n = 0, 1, \ldots$) we can think of it as describing the exchange of a string with mass given by $\alpha' M^2 = 4n$. As explained in [21] the importance of single Pomeron exchange lies in the fact that it dominates scattering amplitudes in the Regge regime both in QCD for $N_c \to \infty$ and in string theory.

In our case the presence of a Dp-brane imposes a world-sheet with a single boundary, as indicated in figure 5.1(b). This can be easily done using boundary state methods of Section 2.6, which were introduced in [23] and are reviewed in [30]. For our purposes we can emulate the effects of the boundary state using the doubling trick, but further applications are allowed using the same vertex.

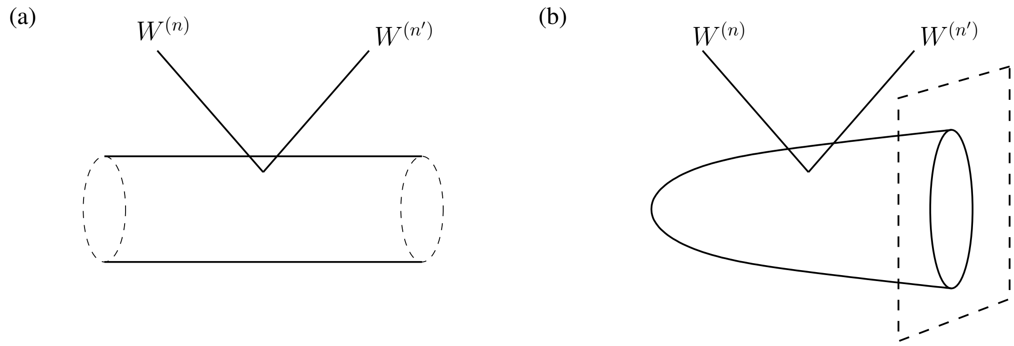

Figure 5.1: (a) The Pomeron vertex operator given by two physical vertices W^n and $W^{n'}$ on the infinite cylinder. (b) The addition of a boundary gives the process in which these states interact with a D-brane.

We will illustrate the use of the Pomeron vertex operator with some specific examples before proceeding to the general case. If we label by (n, n') the amplitude in which a state of mass $\alpha' M_1^2 = 4n$ undergoes a transition to a state of mass $\alpha' M_2^2 = 4n'$ then the examples we shall consider will be $(0, n)$ and $(1, n)$. At the end of this Section we will generalise our results to the case (n, n').

5.3.1 Graviton-massive state transitions

In this case we examine the processes in which a massless state is excited to another of arbitrary mass. The physical polarizations for massive states will be written as tensor products of their holomorphic and anti-holomorphic components,

$$\epsilon_{\mu_1\ldots\mu_n\alpha\,\nu_1\ldots\nu_n\beta} = \varepsilon_{\mu_1\ldots\mu_n\alpha} \otimes \widetilde{\varepsilon}_{\nu_1\ldots\nu_n\beta}, \tag{5.21}$$

$$G_{\mu_1\ldots\mu_n\alpha\,\nu_1\ldots\nu_n\beta} = \mathcal{G}_{\mu_1\ldots\mu_n\alpha} \otimes \widetilde{\mathcal{G}}_{\nu_1\ldots\nu_n\beta}, \tag{5.22}$$

In this Section we compute the OPE of the vertex operators $W^{(0)}_{(0,0)}(k_1, z_1, \bar{z}_1)$ and $W^{(n)}_{(-1,-1)}(k_2, z_2, \bar{z}_2)$ as $z_1 \to z_2$ and $\bar{z}_1 \to \bar{z}_2$; it is possible to immediately identify which contractions will end up giving the leading order contribution to this amplitude. We will need the vertex operators for the graviton

$$\begin{aligned} W^{(0)}_{(0,0)}(k_1, z_1, \bar{z}_1) &= -\frac{\kappa}{\pi\alpha'}\epsilon_{\mu\nu} V_0^\mu(k_1, z_1)\widetilde{V}_0^\nu(k_1, \bar{z}_1), \\ V_0^\mu(k_1, z_1) &= \frac{1}{\sqrt{2\alpha'}}\left(i\partial X^\mu(z_1) + 2\alpha' k_1\cdot\psi(z_1)\psi^\mu(z_1)\right)e^{ik_1\cdot X(z_1)}, \end{aligned} \tag{5.23}$$

and for the state of mass $\alpha' M^2 = 4n'$ in the leading Regge trajectory, (3.105), (3.106):

$$\begin{aligned} W^{(n')}_{(-1,-1)}(k_2, z_2, \bar{z}_2) &= \frac{1}{n'!}\frac{\kappa}{2\pi}\mathcal{G}_{\rho_1\ldots\rho_{n'}\sigma}\widetilde{\mathcal{G}}_{\lambda_1\ldots\lambda_{n'}\gamma}V_{-1}^{\rho_1\ldots\rho_{n'}\sigma}(k_2, z_2)\widetilde{V}_{-1}^{\lambda_1\ldots\lambda_{n'}\gamma}(k_2, \bar{z}_2), \\ V_{-1}^{\rho_1\ldots\rho_{n'}\sigma}(k_2, z_2) &= \left(\frac{i}{\sqrt{2\alpha'}}\right)^{n'} e^{-\varphi(z_2)}\prod_{i=1}^{n'}\partial X^{\rho_i}\psi^\sigma e^{ik_2\cdot X(z_2)}. \end{aligned} \tag{5.24}$$

We can write the resulting OPE in terms of the world-sheet separation $w = z_1 - z_2$ and the point $z = \frac{z_1+z_2}{2}$ which then takes the following form

$$W^{(0)}_{(0,0)}(k_1, z + \tfrac{w}{2}, \bar{z} + \tfrac{\bar{w}}{2})W^{(n')}_{(-1,-1)}(k_2, z - \tfrac{w}{2}, \bar{z} - \tfrac{\bar{w}}{2}) \sim |w|^{-\alpha'\frac{t}{2}+2n'}\mathcal{O}(z, w)\widetilde{\mathcal{O}}(\bar{z}, \bar{w}). \tag{5.25}$$

It is simple to check that the operators $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ are polynomials of at most degree $(n'+1)$ in w^{-1} and $\bar{w}^{-1}$ respectively, with an exponential factor contributing terms subleading in the small w limit, that is

$$\mathcal{O}(z,w) = e^{i\frac{1}{2}(k_1-k_2)\cdot\partial X(z)w} \sum_{p=1}^{n'+1} \frac{\mathcal{O}_p(z)}{w^p}, \quad \widetilde{\mathcal{O}}(\bar{z},\bar{w}) = e^{i\frac{1}{2}(k_1-k_2)\cdot\bar{\partial} X(\bar{z})\bar{w}} \sum_{q=1}^{n'+1} \frac{\widetilde{\mathcal{O}}_q(\bar{z})}{\bar{w}^q}. \tag{5.26}$$

It is necessary to retain these particular subleading terms, as we will see, because contractions between $k_1 \cdot \partial X$ and $k_1 \cdot \bar{\partial} X$ will generate factors of s meaning that these terms are effectively of order $sw \sim 1$. These terms which will generate the Regge behaviour that we expect.

The momentum exchanged between the string and the brane may be written as $q = p_1+p_2 = 2(k_1+k_2)$, hence $-q^2 = t$, and it is also useful to define a vector $\tilde{q} = 2(k_1 - k_2)$. The w integration of the OPE (5.25) over the complex plane can be done using the Pomeron vertex operator introduced in [21, 33]. In the expansions given by equation (5.26) the dominant terms are the most singular ones. Each of these dominant terms give an integral of the following form

$$\int_{\mathbb{C}} d^2w\, |w|^{-\alpha'\frac{t}{2}-2} e^{i\frac{\tilde{q}}{4}\cdot\partial X(z)w} e^{i\frac{\tilde{q}}{4}\cdot\bar{\partial} X(\bar{z})\bar{w}} \tag{5.27}$$

The integration of (5.27) can be done by introducing new variables $u = \frac{\tilde{q}}{4}\cdot\partial X(z)\, w$, $\bar{u} = \frac{\tilde{q}}{4}\cdot\bar{\partial} X(\bar{z})\, \bar{w}$,

$$e^{-i\pi\alpha'\frac{t}{4}} \int_{\mathbb{C}} d^2u\, |u|^{-\alpha'\frac{t}{2}-2}\, e^{i(u+\bar{u})} \left(i\frac{\tilde{q}}{4}\cdot\partial X(z)\right)^{\alpha'\frac{t}{4}} \left(i\frac{\tilde{q}}{4}\cdot\bar{\partial} X(\bar{z})\right)^{\alpha'\frac{t}{4}} \tag{5.28}$$

Then we integrate over the positions $u = re^{i\theta}$ using the following integrals:

$$\int_0^{2\pi} d\theta\, e^{-2ir\cos\theta} = 2\pi\, J_0(2r), \tag{5.29}$$

$$\int_0^{\infty} dr\, r^a\, J_0(2r) = \frac{1}{2}\frac{\Gamma\left(\frac{1+a}{2}\right)}{\Gamma\left(\frac{1-a}{2}\right)}, \tag{5.30}$$

and we get

$$\int_{\mathbb{C}} d^2w\, |w|^{-\alpha'\frac{t}{2}-2} e^{i\frac{\tilde{q}}{4}\cdot\partial X(z)w} e^{i\frac{\tilde{q}}{4}\cdot\bar{\partial} X(\bar{z})\bar{w}} = \Pi(t) \left(i\frac{\tilde{q}}{4}\cdot\partial X(z)\right)^{\alpha'\frac{t}{4}} \left(i\frac{\tilde{q}}{4}\cdot\bar{\partial} X(\bar{z})\right)^{\alpha'\frac{t}{4}} \tag{5.31}$$

where $\Pi(t)$ is the Pomeron propagator and is given by the following

$$\Pi(t) = 2\pi \frac{\Gamma(-\alpha'\frac{t}{4})}{\Gamma(1+\alpha'\frac{t}{4})}\, e^{-i\pi\alpha'\frac{t}{4}}. \tag{5.32}$$

The two-point function is reduced to a one-point function of the effective Pomeron vertex on the disc.

From what we have discussed so far, one can conclude that to leading order in energy the Pomeron vertex operator should take the following form

$$\int_{\mathbb{C}} d^2w\, W^{(0)}_{(0,0)}(k_1, z+\tfrac{w}{2}, \bar{z}+\tfrac{\bar{w}}{2}) W^{(n')}_{(-1,-1)}(k_2, z-\tfrac{w}{2}, \bar{z}-\tfrac{\bar{w}}{2}) \sim -K_{0,n'}(q,\epsilon,G)\Pi(t)\mathcal{O}(z)\widetilde{\mathcal{O}}(\bar{z}), \tag{5.33}$$

where we have the Pomeron vertex operator inserted in a worldsheet with boundaries, the normal-ordered operators

$$\mathcal{O}(z) = \sqrt{2\alpha'}\left(i\frac{\tilde{q}}{4}\cdot\partial X\right)^{\alpha'\frac{t}{4}} k_1\cdot\psi e^{-\varphi}e^{i\frac{q}{2}\cdot X}, \tag{5.34a}$$

$$\widetilde{\mathcal{O}}(\bar{z}) = \sqrt{2\alpha'}\left(i\frac{\tilde{q}}{4}\cdot\bar{\partial} X\right)^{\alpha'\frac{t}{4}} k_1\cdot\widetilde{\psi} e^{-\widetilde{\varphi}}e^{i\frac{q}{2}\cdot X}, \tag{5.34b}$$

and a kinematic function, dependent upon the polarisation tensors and the momentum transferred from the string to the brane q,

$$K_{0,n'}(q,\epsilon,G) = (2\alpha')^{n'}\mathcal{G}_0\cdot\varepsilon_0\,\widetilde{\mathcal{G}}_0\cdot\widetilde{\varepsilon}_0. \tag{5.35}$$

Here we have introduced for later convenience the notation

$$\mathcal{G}_a\cdot\varepsilon_a = \left(\prod_{i=1}^{a}\eta^{\rho_i\mu_i}\right)\left(\prod_{j=a+1}^{n'}k_1^{\rho_j}\right)\left(\prod_{k=a+1}^{n}k_2^{\mu_k}\right)\mathcal{G}_{\rho_1\ldots\rho_{n'}\sigma}\eta^{\sigma\alpha}\varepsilon_{\mu_1\ldots\mu_n\alpha} \tag{5.36}$$

where $a\in\{1,\ldots,\min\{n,n'\}\}$ will count the number of contractions between $\mathcal{G}$ and ε in addition to that arising from the fermionic fields, and an analogous expression holds for the polarisation of the antiholomorphic components. This notation will prove useful since the polarisation tensors for all states on the leading Regge trajectory are symmetric in all holomorphic indices and symmetric in all antiholomorphic indices, as a result the order of contractions with these indices is immaterial; all we need to do is keep track of how many factors of k_1 are contracted with $\mathcal{G}$, $\widetilde{\mathcal{G}}$ and how many factors of k_2 are contracted with ε, $\widetilde{\varepsilon}$. Furthermore, due to the requirement that longitudinal polarisations vanish we find that we may replace all occurrences of k_1 and k_2 in equation (5.36) and its antiholomorphic partner with the transferred momentum $q = 2(k_1+k_2)$.

To get the amplitude, we integrate on the disc the one-point function given by the contraction of the operators $\mathcal{O}(z)$, $\widetilde{\mathcal{O}}(\bar{z})$ in (5.34). To leading order in s, discarding the masses, we obtain:

$$-K_{0,n'}(q,\epsilon,G)\int_{D_2}\frac{dzd\bar{z}}{V_{CKG}}\Pi(t)\langle:\mathcal{O}(z)::\widetilde{\mathcal{O}}(\bar{z}):\rangle_{D_2}\sim -K_{0,n'}(q,\epsilon,G)(\alpha's)^{\alpha'\frac{t}{4}+1} \times\frac{\Pi(t)}{2\pi}\Gamma\left(1+\alpha'\tfrac{t}{4}\right) \tag{5.37}$$

The factor $\frac{1}{2\pi}$ arises from the ratio between the integration over the insertion point of the vertex operator and the volume of the Conformal Killing Group of the disc, $SL(2,\mathbb{R})$.

We obtain an $SL(2,\mathbb{R})$invariant function which gives the leading high-energy behaviour of the amplitude for a graviton scattering from a D-brane into a state on the leading Regge trajectory with a mass n',

$$A_{0,n'}(s,t) = \frac{\kappa T_p}{2}K_{0,n'}(q,\epsilon,G)\Gamma(-\alpha'\tfrac{t}{4})e^{-i\pi\alpha'\frac{t}{4}}\,(\alpha's)^{\alpha'\frac{t}{4}+1}. \tag{5.38}$$

5.3.2 Transitions from the lowest massive state

We next move on to the case of a string with mass $M_1^2 = 4/\alpha'$ interacting with a D-brane to leave a string of mass $M_2^2 = 4n'/\alpha'$. The vertex operators are now given by

$$\begin{aligned} W^{(1)}_{(0,0)}(k_1,z_1,\bar{z}_1) &= -\frac{\kappa}{\pi\alpha'}\varepsilon_{\mu\alpha}\widetilde{\varepsilon}_{\nu\beta}V_0^{\mu\alpha}(k_1,z_1)\widetilde{V}_0^{\nu\beta}(k_1,\bar{z}_1),\\ V_0^{\mu\alpha}(k_1,z_1) &= \frac{i}{2\alpha'}\left(i\partial X^\mu\partial X^\alpha + 2\alpha' k_1\cdot\psi\psi^\alpha\partial X^\mu - i2\alpha'\partial\psi^\mu\psi^\alpha\right)e^{ik_1\cdot X}, \end{aligned} \tag{5.39}$$

and

$$W^{(n')}_{(-1,-1)}(k_2,z_2,\bar z_2)=\frac{\kappa}{2\pi}\mathcal{G}_{\rho_1\ldots\rho_{n'}\sigma}\widetilde{\mathcal{G}}_{\lambda_1\ldots\lambda_{n'}\gamma}V^{\rho_1\ldots\rho_{n'}\sigma}_{-1}(k_2,z_2)\widetilde{V}^{\lambda_1\ldots\lambda_{n'}\gamma}_{-1}(k_2,\bar z_2),$$
$$V^{\rho_1\ldots\rho_{n_2}\sigma}_{-1}(k_2,z_2)=\frac{1}{\sqrt{n'!}}\left(\frac{i}{\sqrt{2\alpha'}}\right)^{n'}e^{-\varphi(z_2)}\prod_{i=1}^{n'}\partial X^{\rho_i}\psi^{\sigma}e^{ik_2\cdot X(z_2)}, \tag{5.40}$$

The methods developed in the previous example need little modification in order to deal with this problem. With them we can easily determine the form of the Pomeron vertex operator and it can be written in the same manner as in the $(0,n')$ case,

$$\int_{\mathbb{C}}d^2w\,W^{(1)}_{(0,0)}(k_1,z+\tfrac{w}{2},\bar z+\tfrac{\bar w}{2})W^{(n')}_{(-1,-1)}(k_2,z-\tfrac{w}{2},\bar z-\tfrac{\bar w}{2})\sim -K_{1,n'}(q,\epsilon,G)\Pi(t)\mathcal{O}(z)\widetilde{\mathcal{O}}(\bar z), \tag{5.41}$$

where this time

$$K_{1,n'}(q,\epsilon,G)=(2\alpha')^{n'+1}\bigg[\mathcal{G}_0\cdot\varepsilon_0\,\widetilde{\varepsilon}_0\cdot\widetilde{\mathcal{G}}_0-\frac{n'}{2\alpha'}\mathcal{G}_1\cdot\varepsilon_1\,\widetilde{\varepsilon}_0\cdot\widetilde{\mathcal{G}}_0$$
$$-\frac{n'}{2\alpha'}\mathcal{G}_0\cdot\varepsilon_0\,\widetilde{\varepsilon}_1\cdot\widetilde{\mathcal{G}}_1+\left(\frac{n'}{2\alpha'}\right)^2\mathcal{G}_1\cdot\varepsilon_1\,\widetilde{\varepsilon}_1\cdot\widetilde{\mathcal{G}}_1\bigg] \tag{5.42}$$

and all other quantities remain as previously defined. Because of this, the resulting amplitude will be identical to that of equation (5.38) other than the form of the kinematic function which is given in (5.42),

$$A_{1,n'}(s,t)=\frac{\kappa T_p}{2}K_{1,n'}(q,\epsilon,G)\,\Gamma(-\alpha'\tfrac{t}{4})e^{-i\pi\alpha'\frac{t}{4}}\,(\alpha's)^{\alpha'\frac{t}{4}+1}. \tag{5.43}$$

Note that we have taken $2\alpha' k_1\cdot D\cdot k_1\sim-\alpha's$ in the large s limit, neglecting a mass term of the order of unity. In the next case we will move on to masses which contribute terms of order n, n' and we reiterate that we shall be considering only those states for which the rest mass contribution to the total energy is negligible.

5.3.3 Transitions within the leading Regge trajectory

Here we return to our original consideration, the process in which a string in a state on the leading Regge trajectory is scattered from a Dp-brane into some other state on the leading Regge trajectory. The vertex operators are those given by equations (3.107) and (3.105), with polarisation tensors $\varepsilon\otimes\widetilde{\varepsilon}$ and $\mathcal{G}\otimes\widetilde{\mathcal{G}}$, respectively, and our methods will imitate those we have seen already. We initially suppose n and n' to be fixed at some finite values, then we have to identify the terms in the OPE of these vertices which will give the leading high energy contributions. These terms can be found performing the OPE between the vertices (3.107) and (3.105), which is given in Appendix B.

Once we have the leading OPE, by integrating out the dependence on their separation we obtain the same general form for the Pomeron vertex operator as in Section 5.3.2,

$$\int_{\mathbb{C}}d^2w\,W^{(n)}_{(0,0)}(k_1,z+\tfrac{w}{2},\bar z+\tfrac{\bar w}{2})W^{(n')}_{(-1,-1)}(k_2,z-\tfrac{w}{2},\bar z-\tfrac{\bar w}{2})\sim -K_{n,n'}(q,\epsilon,G)\Pi(t)\mathcal{O}(z)\widetilde{\mathcal{O}}(\bar z), \tag{5.44}$$

where now we can see the full structure of the kinematic function, which may be written as the product of a contribution from the holomorphic operators with a contribution from the antiholomorphic operators,

$$K_{n,n'}(q,\epsilon,G)=(2\alpha')^{n+n'}\sum_{a,b=0}^{\min\{n,n'\}}(-2\alpha')^{-a-b}C_{n,n'}(a)C_{n,n'}(b)\mathcal{G}_a\cdot\varepsilon_a\,\widetilde{\varepsilon}_b\cdot\widetilde{\mathcal{G}}_b. \tag{5.45}$$

Here $\min\{n,n'\}$ indicates the smallest value from the set $\{n,n'\}$, and the combinatorial factors $C_{n,n'}$ (one each coming from the holomorphic and antiholomorphic contractions) come from the large number of possible contractions in the OPE which lead to the same operator after taking into account the symmetry of the polarisation tensors. These functions are given by

$$C_{n,n'}(p) = \frac{n!n'!}{p!(n-p)!(n'-p)!} \tag{5.46}$$

and this can be deduced in the following manner. If we consider the contribution from the holomorphic operators then $K_{n,n'}$ is determined by all possible contractions amongst

$$\varepsilon_{\mu_1\ldots\mu_n\alpha}\mathcal{G}_{\rho_1\ldots\rho_{n'}\sigma} : \prod_{i=1}^{n} \partial X^{\mu_i} e^{ik_1\cdot X} :: \prod_{i=1}^{n'} \partial X^{\rho_i} e^{ik_2\cdot X} :, \tag{5.47}$$

since ε and $\mathcal{G}$ are symmetric in all indices these contractions can simply be labelled by the number of contractions between ∂X^{μ_i} and ∂X^{ρ_i}, let this be a. This being the case we must count how many ways one can generate a such contractions, first one must choose a operators from a total of n possibilities for which there are $\binom{n}{a} = n!/a!(n-a)!$ different choices. Similarly we must choose a further a operators from a set of n' possibilities giving another factor of $\binom{n'}{a}$. Finally, from this set of $2a$ operators there $a!$ possible ways to contract them in pairs. The product of these numbers gives the total number of different contractions which result in a factor $\mathcal{G}_a \cdot \varepsilon_a$ and is equal to the function $C_{n,n'}(a)$.

With this new form for the kinematic factor K we are finished, the rest of the computation having already been solved in the previous two examples. The final result for the amplitude of a finite mass string scattering from a D-brane is

$$A_{n,n'}(s,t) = \frac{\kappa T_p}{2} K_{n,n'}(q,\epsilon,G)\, \Gamma(-\alpha'\tfrac{t}{4}) e^{-i\pi\alpha'\frac{t}{4}} \,(\alpha' s)^{\alpha'\frac{t}{4}+1}. \tag{5.48}$$

5.4 Comparison with the eikonal analysis

In Chapter 4 we have shown that tree-level amplitudes of closed strings in the background of N Dp-branes exponentiate to give the S-matrix an operator eikonal form at high energy,

$$S(s,\mathbf{b}) = e^{i2\hat{\delta}(s,\mathbf{b})}, \qquad 2\hat{\delta}(s,\mathbf{b}) = \frac{1}{2\sqrt{s}} \int_0^{2\pi} \frac{d\sigma}{2\pi} : \mathcal{A}\left(s, \mathbf{b} + \hat{\mathbf{X}}(\sigma)\right) :, \tag{5.49}$$

thus generalising the field theory result for the S-matrix. The amplitude $\mathcal{A}(s,\mathbf{b})$ is the Fourier transform from the space of transverse momenta q to that of impact parameter b,

$$\mathcal{A}(s,\mathbf{b}) = \int \frac{d^{9-p}q}{(2\pi)^{9-p}}\, A(s,\mathbf{q}) e^{i\mathbf{b}\cdot\mathbf{q}}. \tag{5.50}$$

was obtained from the Regge limit of the disk amplitude for the elastic scattering of a graviton from the Dp-branes after stripping it of its dependence upon the polarisation,

$$A(s,\mathbf{q}) = \frac{\kappa T_p}{2} \Gamma(\alpha' \frac{q^2}{4}) e^{i\pi\alpha'\frac{q^2}{4}} \,(\alpha' s)^{-\alpha'\frac{q^2}{4}+1}. \tag{5.51}$$

The shift of the impact parameter by the string position operator $\hat{X}$, is introduced in order to take into account the finite size of the string and it is in this respect that the eikonal operator of string theory differs from the field theory eikonal. In the case of massive strings scattering, effects due to the string size turn out to be of vital importance since internal degrees of freedom of the string are excited. It is also crucial to note that the eikonal operator acts in our case on the Hilbert space of physical closed string states, but the action on fermionic operators is trivial, as we will show in the following. Moreover, in the background of a Dp-brane the eikonal depends only on bosonic oscillators transverse to the brane. We will show this by deriving the kinematic factor (5.45) as a function of the transverse momentum q only.

The exponential in (5.49) represents a resummation of the perturbative expansion, so tree level diagrams arise from the linear term in this exponential. Keeping this in mind, we can compute the amplitudes for the high-energy limit of tree level processes from matrix elements of the operator $\bar{\mathcal{A}}$, defined as follows:

$$\bar{\mathcal{A}} \equiv \int_0^{2\pi} \frac{d\sigma}{2\pi} : \mathcal{A}\left(s, \mathbf{b} + \mathbf{X}(\sigma)\right) := \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k \mathcal{A}(s,\mathbf{b})}{\partial b^{\mu_1} \dots \partial b^{\mu_k}} \overline{X^{\mu_1} \dots X^{\mu_k}}, \tag{5.52}$$

To do so we will first reproduce the tree-level graviton to graviton scattering amplitude of Chapter 4, before moving on to the more general case, by computing the expectation value of the $\bar{\mathcal{A}}$ operator between initial and final states representing the graviton. These states in the standard oscillator mode representation are

$$|i\rangle = \epsilon_{1\mu\nu} \psi^{\mu}_{-\frac{1}{2}} \widetilde{\psi}^{\nu}_{-\frac{1}{2}} \left|0;0\right\rangle , \tag{5.53a}$$

$$|f\rangle = \epsilon_{2\rho\sigma} \psi^{\rho}_{-\frac{1}{2}} \widetilde{\psi}^{\sigma}_{-\frac{1}{2}} \left|0;0\right\rangle . \tag{5.53b}$$

To evaluate the expectation value of $\bar{\mathcal{A}}$ we simply have to take into account the fermionic operators of (5.53), but they can be considered as vacuum states, so the tree-level graviton to graviton amplitude is thus given by the vacuum expectation value of the operator $\bar{\mathcal{A}}$

$$\langle f| \, \bar{\mathcal{A}} \, |i\rangle = \epsilon_{1\mu\nu} \, \epsilon_{2\rho\sigma} \, \eta^{\mu\rho} \, \eta^{\nu\sigma} \, \langle 0;0| \, \bar{\mathcal{A}} \, |0;0\rangle = \mathrm{Tr}(\epsilon_1^T \, \epsilon_2) \, \mathcal{A}(s,\mathbf{b}). \tag{5.54}$$

It is clear from the definition of $\mathcal{A}(s,\mathbf{b})$ that this will reproduce the expected result in q-space upon a Fourier transformation.

Now we will show that the tree-level amplitude for a state of mass $\alpha' M_1^2 = 4n$ to become a state of mass $\alpha' M_2^2 = 4n'$ after scattering from a D-brane in the high-energy limit is given by the expectation value of the operator $\bar{\mathcal{A}}$ between the initial and final states. We will use the standard oscillator modes representation of such states

$$|n\rangle = \varepsilon_{\mu_1 \dots \mu_n \alpha} \, \widetilde{\varepsilon}_{\nu_1 \dots \nu_n \beta} \prod_{i=1}^{n} \alpha_{-1}^{\mu_i} \widetilde{\alpha}_{-1}^{\nu_i} \, \psi^{\alpha}_{-\frac{1}{2}} \widetilde{\psi}^{\beta}_{-\frac{1}{2}} \left|0;0\right\rangle , \tag{5.55a}$$

$$|n'\rangle = \mathcal{G}_{\rho_1 \dots \rho_{n'} \sigma} \, \widetilde{\mathcal{G}}_{\lambda_1 \dots \lambda_{n'} \gamma} \prod_{j=1}^{n'} \alpha_{-1}^{\rho_j} \widetilde{\alpha}_{-1}^{\lambda_j} \psi^{\sigma}_{-\frac{1}{2}} \widetilde{\psi}^{\gamma}_{-\frac{1}{2}} \left|0;0\right\rangle . \tag{5.55b}$$

Since the operator $\bar{\mathcal{A}}$ acts non-trivially only on bosonic oscillators, we first contract fermionic operators in the expectation value

$$\langle n'| \, \bar{\mathcal{A}} \, |n\rangle = K_{\rho_1 \dots \rho_{n'} \lambda_1 \dots \lambda_{n'} \mu_1 \dots \mu_n \nu_1 \dots \nu_n} \, \langle 0;0| \prod_{i=1}^{n'} \alpha_1^{\rho_i} \widetilde{\alpha}_1^{\lambda_i} \, \bar{\mathcal{A}} \prod_{j=1}^{n} \alpha_{-1}^{\mu_j} \widetilde{\alpha}_{-1}^{\nu_j} \left|0;0\right\rangle , \tag{5.56}$$

where the polarisations are contained within the tensor

$$K_{\rho_1 \dots \rho_{n'} \lambda_1 \dots \lambda_{n'} \mu_1 \dots \mu_n \nu_1 \dots \nu_n} = \mathcal{G}_{\rho_1 \dots \rho_{n'} \sigma} \eta^{\sigma\alpha} \varepsilon_{\mu_1 \dots \mu_n \alpha} \, \widetilde{\mathcal{G}}_{\lambda_1 \dots \lambda_{n'} \gamma} \eta^{\gamma\beta} \widetilde{\varepsilon}_{\nu_1 \dots \nu_n \beta}. \tag{5.57}$$

To test that the amplitudes we are considering are well described by the eikonal, we compute the values of equation (5.56) and compare them with result stated in equations (5.45) and (5.48).

In expanding the operator $\bar{\mathcal{A}}$ in oscillator modes it can be seen that very few terms will contribute a nonzero value to this matrix element; these terms can only be composed of the modes $\alpha_{\pm 1}$, $\widetilde{\alpha}_{\pm 1}$, and since they must be normal ordered there can be at most n occurrence of the operators α_1, $\widetilde{\alpha}_1$ and n' occurrences of the operators α_{-1}, $\widetilde{\alpha}_{-1}$. As a result we only expect nonzero contributions from terms of order $2|n'-n|$ through to $2(n'+n)$. Furthermore, the terms in $\bar{\mathcal{A}}$ containing k oscillators are generated by

$$\frac{1}{k!}\frac{\partial^k \mathcal{A}(s,\mathbf{b})}{\partial b^{\mu_1}\dots\partial b^{\mu_k}}\overline{X^{\mu_1}\dots X^{\mu_k}}, \tag{5.58}$$

and the invariance of the partial derivative under a change in order of differentiation will result in the oscillators being symmetric under exchange of their indices, so rather than keeping track of these indices we need only count how many ways we can generate oscillator terms of the form given above.

Using this one can deduce that we can substitute for $\bar{\mathcal{A}}$ in eq. (5.56) the following quantity

$$\sum_{a,b=0}^{\min\{n,n'\}}\frac{(-1)^{n+n'}}{(n-a)!(n-b)!(n'-a)!(n'-b)!}\frac{\partial^{2(n+n'-a-b)}\mathcal{A}(s,\mathbf{b})}{\partial b^{i_1}\dots\partial b^{\ell_{n-b}}}$$
$$\times\,\alpha_{-1}^{i_1}\dots\alpha_{-1}^{i_{n'-a}}\tilde{\alpha}_{-1}^{j_1}\dots\tilde{\alpha}_{-1}^{j_{n'-b}}\alpha_1^{k_1}\dots\alpha_1^{k_{n-a}}\tilde{\alpha}_1^{\ell_1}\dots\tilde{\alpha}_1^{\ell_{n-b}}. \tag{5.59}$$

This has been written such that a and b will be seen to count the number of contractions between the polarisation tensors and they take on the range of values $a,b=0,1,\dots,\min\{n,n'\}$; naturally we must count how many ways these terms may be generated using the symmetry of the partial derivatives in the expansion of $\bar{\mathcal{A}}$. The result of this substitution is the amplitude shown below

$$\mathcal{A}_{n,n'}(s,\mathbf{b}) = K_{\rho_1\dots\rho_{n'}\lambda_1\dots\lambda_{n'}\mu_1\dots\mu_n\nu_1\dots\nu_n}\sum_{a,b=0}^{\min\{n,n'\}}(-1)^{n+n'}C_{n,n'}(a)C_{n,n'}(b)\frac{\partial^{2(n+n'-a-b)}\mathcal{A}(s,\mathbf{b})}{\partial b^{i_1}\dots\partial b^{\ell_{n-b}}}$$
$$\times\,\delta^{\rho_1 i_1}\dots\delta^{\rho_{n'-a}i_{n'-a}}\delta^{k_1\mu_1}\dots\delta^{k_{n-a}\mu_{n-a}}\delta^{\rho_{n'-a+1}\mu_{n'-a+1}}\dots\delta^{\rho_{n'}\mu_n}$$
$$\times\,\delta^{\lambda_1 j_1}\dots\delta^{\lambda_{n'-b}j_{n'-b}}\delta^{\ell_1\nu_1}\dots\delta^{\ell_{n-b}\nu_{n-b}}\delta^{\lambda_{n'-b+1}\nu_{n'-b+1}}\dots\delta^{\lambda_{n'}\nu_n}. \tag{5.60}$$

Performing the Fourier transform we arrive at the expected result

$$A_{n,n'}(s,\mathbf{q}) = K(\mathbf{q},\epsilon,G)A(s,\mathbf{q}), \tag{5.61}$$

where now the kinematic function is that seen in equation (5.45) with $k_1,k_2\to q$.

In general, the index structure in (5.60) will result in a function which appears more complicated than its Fourier transform, however it is interesting to consider this quantity for very large impact parameters. In such a limit b takes non-zero components in just $(8-p)$ of the transverse directions and the function $\mathcal{A}$ takes the form

$$\mathcal{A}(s,\mathbf{b}) \sim s\sqrt{\pi}\frac{\Gamma\left(\frac{6-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)}\frac{R_p^{7-p}}{b^{6-p}} + i\frac{s\pi}{\Gamma\left(\frac{7-p}{2}\right)}\sqrt{\frac{\pi\alpha' s}{\ln\alpha' s}}\left(\frac{R_p}{\sqrt{2\alpha'\ln\alpha' s}}\right)^{7-p}e^{-\frac{b^2}{2\alpha'\ln\alpha' s}}, \tag{5.62}$$

where to reflect the change to coordinate space we choose to express the normalisation of the amplitude in terms of the scale R_p. To be more specific we will examine impact parameters for which $b\gg R_p\gg\sqrt{2\alpha'\ln\alpha' s}$, that is, those much larger than both the effective string length and the characteristic size of the

Dp-branes; this being the case, we shall ignore the imaginary part of $\mathcal{A}(s,\mathbf{b})$ and focus on the real part. The result of these considerations is that the dominant contribution to $\mathcal{A}_{n,n'}$ comes from the term with the least number of derivatives in b. Without loss of generality let us assume that $n \leq n'$ and denote the difference between the two by $N = n' - n$, then this term is given by

$$\mathcal{A}_{n,n+N}(s,\mathbf{b}) \sim s\sqrt{\pi}\frac{\Gamma\left(\frac{6-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)} R_p^{7-p} K_{\rho_1\ldots\rho_{n+N}\lambda_1\ldots\lambda_{n+N}\mu_1\ldots\mu_n\nu_1\ldots\nu_n} (-1)^{2n+N} \frac{(n+N)!}{n!(N!)^2} \frac{\partial^{2N} b^{-(6-p)}}{\partial b^{i_1}\ldots\partial b^{j_N}}$$
$$\times\, \delta^{\rho_1 i_1}\ldots\delta^{\rho_N i_N}\delta^{\rho_{N+1}\mu_1}\ldots\delta^{\rho_{n+N}\mu_n}\delta^{\lambda_1 j_1}\ldots\delta^{\lambda_N j_N}\delta^{\lambda_{N+1}\nu_1}\ldots\delta^{\lambda_{n+N}\nu_n}. \quad (5.63)$$

The derivative above can be decomposed into a product consisting of an overall factor $N!\, b^{-(6+2N-p)}$ multiplied by some tensor which may be determined from the Gegenbauer polynomials in the following way. The Gegenbauer polynomial $C_n^{(\lambda)}(x)$ is given in terms of the hypergeometric functions by

$$C_n^{(\lambda)}(x) = \binom{n+2\lambda-1}{n} {}_2F_1(-n, n+2\lambda; l+\tfrac{1}{2}; \tfrac{1}{2}(1-x)); \quad (5.64)$$

the tensor we are interested in is obtained by taking $C_{2N}^{(6-p)/2}(x)$ and for each term in this polynomial we can attach the appropriate index structure by substituting $b_{i_r}/|b|$ for each factor of x, pairing the remaining indices up with Kronecker deltas and symmetrising the result. Since it will not be required for this analysis we shall make no attempt to write the generic form for this tensor and we shall instead write it as $K(\mathbf{b},\epsilon,G)$ after taking the contractions with $K_{\rho_1\ldots\rho_{n'}\lambda_1\ldots\lambda_{n'}\mu_1\ldots\mu_n\nu_1\ldots\nu_n}$, this function can be thought of as a hyperspherical harmonic. Thus equation (5.63) becomes

$$\mathcal{A}_{n,n+N}(s,\mathbf{b}) \sim s\sqrt{\pi}\frac{\Gamma\left(\frac{6-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)} R_p^{7-p} K(\mathbf{b},\epsilon,G)(-1)^{2n+N}\frac{(n+N)!}{n!N!}\frac{1}{b^{6+2N-p}}. \quad (5.65)$$

From this expression it is seen that tidal excitation of the string begins to contribute significantly to scattering processes for impact parameters $b \leq b_N$ where

$$b_N^{6+2N-p} = s\sqrt{\pi}\frac{\Gamma\left(\frac{6-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)} R_p^{7-p}\frac{(n+N)!}{n!N!}. \quad (5.66)$$

The indication is that if one first considers scattering at impact parameters so large that only elastic scattering is relevant and then begins to reduce this impact parameter then the elastic channel will be gradually absorbed, first by small string transitions and by larger transitions as b is further decreased in size.

5.5 Discussion

In this Chapter we have shown that high-energy and small-angle scattering of bosonic string states on the leading Regge trajectory from a stack of N Dp-branes exhibits universal tree-level behaviour, provided the masses of these states remain finite. In equation (5.48) this behaviour is characterised by the dependence on the square of the momentum flowing parallel to the D-branes, s, which is contained entirely within the factor $(\alpha' s)^{\alpha'\frac{t}{4}+1}$. This indicates that this process is dominated by the exchange of states from the leading Regge trajectory between the string and the D-branes. The mass of the initial and final string states will determine the form of the polarisation function $K_{n,n'}(q,\epsilon,G)$ and influence the dependence of the amplitude upon the transferred momentum q.

We have also shown that the tree-level result (5.48) in the Regge regime can be reproduced by the linear term in the perturbative expansion of the eikonal operator $\hat{\delta}(s, \mathbf{b})$.

As it was already shown in [7, 8], and as outlined in Chapter 3, string corrections to the eikonal operator, given by the expansion in $\sqrt{\alpha'}/b$, are neatly taken into account by a simple shift of the impact parameter in the Fourier transform of the tree-level string amplitude. In Chapter 4 the form of the eikonal operator was determined to all orders in $\sqrt{\alpha'}/b$ and at leading order and in R_p/b. Also the next-to-leading term in R_p/b of the eikonal operator was derived, but not the string corrections to it. It would be interesting to clarify the full structure of the eikonal operator and to understand how the string corrections enter at next-to-leading order in R_p/b. For this purpose it would be useful to move one step further in the eikonal expansion, computing the one-loop amplitude for two massive states.

In this Chapter we kept the external states fixed while taking the high energy limit and therefore we could consistently neglect their masses, $\alpha' s \gg n, n'$. Since the string spectrum contains states with arbitrarily large masses, one could consider a different limit in which also the masses are allowed to be very large. As long as $|n' - n| \ll \alpha^{'} s$, the momentum transferred can be kept finite and the amplitudes could still display Regge behaviour at high energy. For larger differences between the levels we expect that the inelastic amplitudes will decay exponentially with the energy, the behaviour typical of scattering processes at fixed angle.

Conclusions

In this Thesis we have analysed the asymptotic limit of string-string and string-brane scattering amplitudes in the Regge regime, that is, high center of mass energy and small momentum transfer. The most important feature of all these amplitudes is that the resummation to all orders in the string loop parameter can be written in a closed form in terms of the eikonal operator introduced in [7]. We have defined this operator for four-graviton scattering in Chapter 3, where we have also shown that for large impact parameters the same scattering can be interpreted as the incoming string being deflected by the Aichelburg-Sexl metric generated by the other. This is a purely gravitational effect and it is a striking physical application of string theory.

This interpretation fails beyond the leading-eikonal (or small deflection angle) approximation. The reason for this fact is that string theory provides an infinte number of excited states, so in the regime where inelastic excitations due to tidal effects start to be important, the string theory eikonal differs from the gravitational one.

We have extended the analysis of ACV in Chapter 4 by considering the presence of a collection of parallel D-branes in a flat Minkowski background. The presence of a collection of Dp-branes does not require any modification of the background, but since they are massive charged states, their presence results in a curved spacetime. So the evaluation of string-brane scattering amplitudes in flat space can provide some information about the dynamics of strings in an effective curved spacetime.

Keeping gN fixed, and thus letting the number of branes N large, contributions with many open string loops (boundaries) are not suppressed. With the presence of D-branes, the eikonal form of the resummed amplitude is given by the large contributions to the S-matrix from diagrams with a different number of boundaries.

Moreover, in the original ACV-type collision the resummation of the closed string loops in the string-string scattering leads to an effective geometry that depends non-trivially upon the energies of the two colliding particles. The problem of string scattering from a stack of D-branes is much easier than string-string scattering, where the effective geometry depends non-trivially upon the energies of the two colliding particles. The D-brane does not back-react on the geometry, so the metric that emerges from the resummation of open string loops that we considered in Chapter 4 depends only on the D-brane system.

In this case indeed the parameter R_p, which sets the scale of the effective geometry, is independent from E. We have also limited our analysis to the region $R_p, b > l_s(s)$, but it would be interesting to discuss the new phenomena that occur for b or R_p smaller than $l_s(s)$, where the strong gravity regime dominates.

In Chapter 4 we have seen that for very large b, the region of infinitesimal deflection angles, the eikonal phase is small and we are in a perturbative regime.

Below a critical value of $b = b_c$ gravitational effects dominate and the string probe gets captured. The region $b \sim b_c$ is then interesting because if we could describe this phenomenon in string theory, which is a unitary

S-matrix theory, we could shed some light on the relation between quantum mechanics and gravitational physics, for instance on the process of a particle falling into a black hole.
As we lower b we encounter a region in which both kinds of correction (string-size and classical) come together.
When $b \gg R_p$, gravity effects dominate the interaction between the Dp-branes and the string probe, allowing the comparison between the string dynamics in the extremal p-brane background and that resulting from the string scattering amplitudes. There are other interesting regions to consider. The first is the string region $R_p < l_s(s)$, where string corrections to the geometry are important and one expects that the dynamics will be very different from the one predicted by the effective background in eq. (4.99). The second is the region $b \sim R_p > l_s(s)$, where the dynamics should be dominated by strong gravity effects. As we lower the impact parameter to study these two new regions, we should also be able to compare with the analysis of high-energy amplitudes at fixed angle as discussed in [42] and, in the context of D-branes, in [14, 15].
The quantitative analysis of a string-brane scattering process in the string region $R_p < l_s(s)$ and in the strong gravity region $b \sim R_p$ requires some control on both the classical and string corrections to the leading eikonal operator. An example of the possible effects of these corrections is provided by the string excitations along the Dp-brane world-volume, which can be studied by computing amplitudes involving massive string states, the main subject of this Thesis.

In Chapter 5 we have seen that at high energies the small-angle scattering of bosonic string states on the leading Regge trajectory from a stack of N Dp-branes exhibits universal tree-level behaviour, provided the masses of these states remain finite. The tree-level result (5.48) in the Regge regime can be reproduced by the linear term in the perturbative expansion of the eikonal operator $\hat{\delta}(s, \mathbf{b})$.
If the states under consideration are massive, internal degrees of freedom of the string are excited and this is taken into account by shifting the impact parameter by the string position operator. It is key to note that these string corrections are relevant already at large impact parameters. When $b \gg l_s(E), R_p$ we find that although the amplitudes for both elastic scattering and all possible internal excitations are of leading order in R_p/b, they will differ in the order of α'/b^2. This characterises in a quantum mechanical way the intuitive notion that if the string passes closer to the stack of D-branes it should experience stronger tidal effects.
If one first considers scattering at impact parameters so large that only elastic scattering is relevant and then begins to reduce this impact parameter, then the elastic channel will be gradually absorbed, first by small string transitions, and then by larger transitions as b is further decreased in size. It would be interesting to move one step further in the eikonal expansion, computing the one-loop amplitude for two massive states.

It is important to stress that the kinematic regime we have considered is the Regge regime, where $s \gg n, n'$. By computing the complete OPE (see for example the Appendix B), it can be seen that there are many terms of leading order in the OPE which can no longer be neglected as subleading in s — there are now many more relevant channels — and it is difficult to isolate the dominant terms from these analytically. This leads us to argue that the eikonal approximation could not be valid if the mass of the string is of the same order as the center of mass energy, even if one remains within the Regge regime by keeping the transferred momentum small.

From a more general perspective, it would be very interesting to extend the analysis of both string and gravitational corrections to the eikonal operator which are of a higher order in R_p/b and α'/b^2 by computing explicitly the high-energy behaviour of amplitudes with three or more boundaries. This would be a generalisation of the comparison between the eikonal operator, the direct computation of the amplitudes and the geometrical description. One way to do this is to use the sigma model expansion in Fermi coordinates

(see, for example, [18–20]).
Another interesting generalization of our setup is to change the nature of the massive target and possibly also the nature of the asymptotic space by including compact directions. One could consider for example the D1/D5 system as suggested in [27], whose geometries are well-known and have some interesting features such as the breaking of rotational symmetry in the transverse space.

Finally, the study of string amplitudes could clarify some aspects of the holographic correspondence, since the techniques described here could help to compute string correlators in more general curved space-times. This aspect can have important applications in the AdS/CFT correspondence.

A

Abramovsky-Gribov-Kancheli cutting rules

In 1973 Abramovsky, Gribov and Kancheli [2] analysed the relative contributions of processes of the particles produced to the total cross section of multiple Pomeron exchange in Reggeon field theory. They found that different contributions to the imaginary part belong to different cuts across the multi-Pomeron diagrams, and each cut has its own final state characteristics. As a result, the authors found counting rules for final states with different particle multiplicities, and they proved cancellations among rescattering corrections to single-particle and double particle inclusive cross sections.
The assumptions made in the original AGK paper states that the coupling of the Pomerons to the external particle are symmetric under the exchange of the Pomerons (Bose symmetry), and that they remain unchanged if some of the Pomerons are cut.
In terms of the T-matrix, unitarity $(SS^{\dagger})$ may be expressed in the symbolic form,

$$2Im\, T_{ab} = \sum_{c} T_{ac} T_{bc}^{*} \tag{A.1}$$

where a, b label the initial and final states, which could be various types of particles with definite momenta and quantum numbers. The sum implies integration over phase-space for each particle in the channel c and sum over all channels c. If the state a consists of two particles and the state b is chosen equal to a (i.e. forward elastic scattering), one has for the cross-section:

$$\sigma_{el} \sim |T|^{2}, \tag{A.2}$$

$$\sigma_{tot} \sim Im\, T, \quad \textbf{Optical theorem.} \tag{A.3}$$

The rules state that the sum of the diagrams with all possible cuts and any number of cut and uncut Pomerons is equal to twice the discontinuity of the elastic scattering amplitude at zero angle. As an illustrative example, we consider the coupling of four gluons to a proton. The simplest model of a symmetric

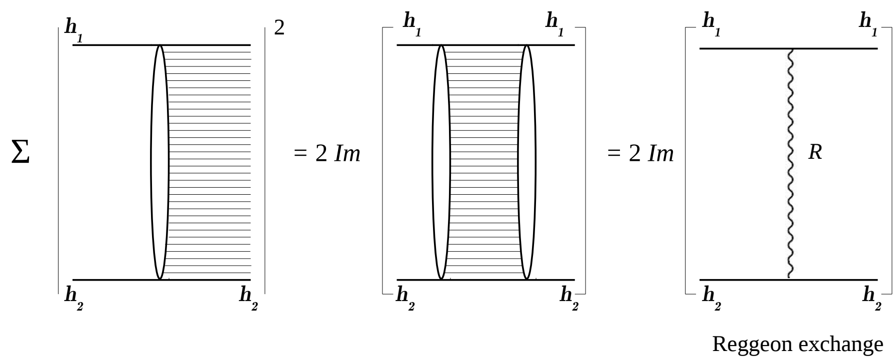

Figure A.1: Diagrammatic expression of the AGK rules.

coupling is a sum of three pieces, each of which contains only the simplest color structure denoted by $\delta_{a_i a_j}$. In order to be able to generalise and to sum over an arbitrary number of gluon chains, it is necessary to be more specific. Inserting the eikonal ansatz

$$\sum_{pairings} \phi^A(k_1, k_2; \omega_{12})\delta_{a_1 a_2} \cdot ... \cdot \phi^A(k_{2n-1}, k_{2n}; \omega_{2n-1,2n})\delta_{a_{2n-1}\, a_{2n}} \tag{A.4}$$

into the hadron - hadron scattering amplitude, using the large color number N_c approximation, and switching to the impact parameter representation, one obtains the following formula for the contribution of k cut gluon ladders:

$$ImA_k = 4s \int d^2 b e^{i\bar{q}\cdot\bar{b}P(s;\bar{b})}, \tag{A.5}$$

where

$$P(s, \bar{b}) = \frac{[\Omega(s;\bar{b})]^k}{k!} e^{-\Omega(s;\bar{b})} \tag{A.6}$$

and Ω stands for the (cut) two-gluon ladder.

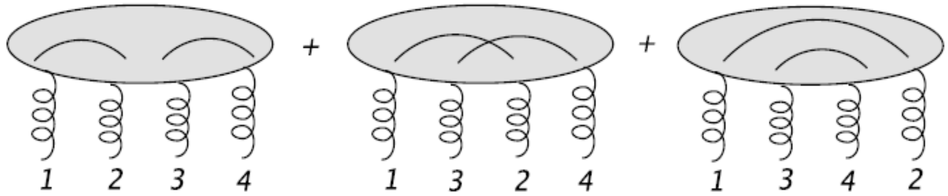

Figure A.2: The symmetric coupling of four gluons to an external particle. The lines inside the blob denote the color connection, e.g. the first term has the color structure $\delta_{a_1 a_2}$.

B
Complete OPE computation

Here we consider the complete OPE computation between the vertices (3.107) and (3.105), in order to isolate the leading terms in the high energy limit. In order to write everything in a more compact form we will require a modification to the notation used until now. For convenience we can make the following substitution for our indices: $\alpha \to \mu_{n+1}$, $\beta \to \nu_{n+1}$, $\sigma \to \rho_{n'+1}$, $\gamma \to \lambda_{n'+1}$. Then we will find the scalar quantity written below in the OPE,

$$[\mathcal{G} \cdot \varepsilon]_a = \prod_{i=1}^{a} \prod_{j=a+1}^{n'+1} \prod_{k=a+1}^{n+1} \eta^{\rho_i \mu_i} k_1^{\rho_j} k_2^{\mu_k} \mathcal{G}_{\rho_1 \ldots \rho_{n'+1}} \varepsilon_{\mu_1 \ldots \mu_{n+1}}, \tag{B.1}$$

along with the vectors

$$[\mathcal{G}_{\rho_{n'+1}} \cdot \varepsilon]_a = \prod_{i=1}^{a} \prod_{j=a+1}^{n'} \prod_{k=a+1}^{n+1} \eta^{\rho_i \mu_i} k_1^{\rho_j} k_2^{\mu_k} \mathcal{G}_{\rho_1 \ldots \rho_{n'+1}} \varepsilon_{\mu_1 \ldots \mu_{n+1}}, \tag{B.2a}$$

$$[\mathcal{G} \cdot \varepsilon_{\mu_{n+1}}]_a = \prod_{i=1}^{a} \prod_{j=a+1}^{n'+1} \prod_{k=a+1}^{n} \eta^{\rho_i \mu_i} k_1^{\rho_j} k_2^{\mu_k} \mathcal{G}_{\rho_1 \ldots \rho_{n'+1}} \varepsilon_{\mu_1 \ldots \mu_{n+1}}. \tag{B.2b}$$

We will denote contractions between these vectors by objects such as $[\mathcal{G} \cdot \varepsilon]_a \cdot [\widetilde{\mathcal{G}} \cdot \widetilde{\varepsilon}]_b$, this signifying the contraction between the μ_{n+1} and $\lambda_{n'+1}$ indices of the ε and $\widetilde{\mathcal{G}}$ tensors.

With this we can write the leading divergence as for $z_1 = z + w/2$ and $z_2 = z - w/2$ for $w \to 0$

$$W^{(m)}_{(0,0)} W^{(n)}_{(-1,-1)} \sim -\frac{(2\alpha')^{n+n'+1}}{|w|^{2(n+n'+1)}} P \widetilde{P}, \tag{B.3}$$

where

$$
\begin{aligned}
P = & \sum_{a=0}^{\min\{n+1,n'\}} (-2\alpha')^{-a} C_{n+1,n'}(a)[\mathcal{G}_\sigma \cdot \varepsilon]_a \mathcal{O}^\sigma \\
& + \sum_{a=0}^{\min\{n,n'\}} (-2\alpha')^{-a} C_{n,n'}(a)[\mathcal{G} \cdot \varepsilon]_{a+1} k_1 \cdot \mathcal{O} \\
& - \sum_{a=0}^{\min\{n,n'\}} (-2\alpha')^{-a} C_{n,n'}(a)[\mathcal{G} \cdot \varepsilon_\alpha]_a \mathcal{O}^\alpha \\
-m & \sum_{a=0}^{\min\{n-1,n'\}} (-2\alpha')^{-a} C_{n-1,n'}(a)[\mathcal{G} \cdot \varepsilon_\alpha]_{a+1} \mathcal{O}^\alpha,
\end{aligned}
$$

and we have the operator

$$
\mathcal{O}^\mu(z,w) = e^{-\varphi(z)} \psi^\mu(z) e^{i\frac{\tilde{q}}{4}\cdot \partial X(z) w + i\frac{q}{2}\cdot X(z)}, \tag{B.4}
$$

whilst obvious antiholomorphic counterparts may be found in $\widetilde{P}$. To obtain the effective vertex we integrate this OPE over a neighbourhood of the origin of w. If this is then inserted into the upper complex plane we will generally find 16 terms which are contained in the function $K_{m,n}$,

$$
\begin{aligned}
A_{m,n} &= \langle \int_{\mathbb{C}} d^2 w \, W^{(n)}_{(0,0)}(k_1, z + \tfrac{w}{2}, \bar{z} + \tfrac{\bar{w}}{2}) W^{(n')}_{(-1,-1)}(k_2, z - \tfrac{w}{2}, \bar{z}_2 - \tfrac{w}{2}) \rangle_{\mathbb{H}_+} \\
&= -2\pi \Gamma(-\alpha' \tfrac{t}{4}) e^{-i\pi\alpha' \frac{t}{4}} \, (\alpha' s)^{\alpha' \frac{t}{4}} (2\alpha')^{n+n'+1} K_{n,n'},
\end{aligned} \tag{B.5}
$$

where

$$
K_{n,n'} = k_1 \cdot D \cdot k_1 K^{(s)}_{n,n'} + n^2 K^{(n^2)}_{n,n'} + n K^{(n)}_{n,n'} + K^{(0)}_{n,n'}. \tag{B.6}
$$

Among these terms, the dominant one used in Chapter 5 is the first one, $K^{(s)}_{n,n'}$ since it has an energy-dependent prefactor $k_1 \cdot D \cdot k_1 \sim \alpha' s$, so all the others are subleading in energy. All the terms in (B.5) are given in the following:

$$
\begin{aligned}
K^{(s)}_{n,n'} = & \sum_{a,b=0}^{\min\{n,n'\}} (-2\alpha')^{-a-b} C_{n,n'}(a) C_{n,n'}(b) [\varepsilon \cdot \mathcal{G}]_{a+1} [\widetilde{\mathcal{G}} \cdot \widetilde{\varepsilon}]_{b+1} \\
K^{(n^2)}_{n,n'} = & \sum_{a=0}^{\min\{n-1,n'\}} \sum_{b=0}^{\min\{n-1,n'\}} (-2\alpha')^{-a-b} C_{n-1,n'}(a) C_{n-1,n'}(b) [\mathcal{G} \cdot \varepsilon]_{a+1} \cdot D [\widetilde{\varepsilon} \cdot \widetilde{\mathcal{G}}]_{b+1} \\
K^{(n)}_{n,n'} = & \sum_{a=0}^{\min\{n+1,n'\}} \sum_{b=0}^{\min\{n-1,n'\}} (-2\alpha')^{-a-b} C_{n+1,n'}(a) C_{n-1,n'}(b) [\varepsilon \cdot \mathcal{G}]_a \cdot D \cdot [\widetilde{\varepsilon} \cdot \widetilde{\mathcal{G}}]_{b+1} \\
& + \sum_{a=0}^{\min\{n,n'\}} \sum_{b=0}^{\min\{n-1,n'\}} (-2\alpha')^{-a-b} C_{n,n'}(a) C_{n-1,n'}(b) [\varepsilon \cdot \mathcal{G}]_{a+1} [\widetilde{\mathcal{G}} \cdot \widetilde{\varepsilon}]_{b+1} \cdot D \cdot k_1 \\
& + \sum_{a=0}^{\min\{n,n'\}} \sum_{b=0}^{\min\{n-1,n'\}} (-2\alpha')^{-a-b} C_{n,n'}(a) C_{n-1,n'}(b) [\mathcal{G} \cdot \varepsilon]_a \cdot D \cdot [\widetilde{\varepsilon} \cdot \widetilde{\mathcal{G}}]_{b+1} \\
& + \sum_{a=0}^{\min\{n-1,n'\}} \sum_{b=0}^{\min\{n+1,n'\}} (-2\alpha')^{-a-b} C_{n-1,n'}(a) C_{n+1,n'}(b) [\mathcal{G} \cdot \varepsilon]_{a+1} \cdot D \cdot [\widetilde{\mathcal{G}} \cdot \widetilde{\varepsilon}]_b
\end{aligned}
$$

$$+\sum_{a=0}^{\min\{n-1,n'\}}\sum_{b=0}^{\min\{n,n'\}}(-2\alpha')^{-a-b}C_{n-1,n'}(a)C_{n,n'}(b)[\mathcal{G}\cdot\varepsilon]_{a+1}\cdot D\cdot k_1[\widetilde{\varepsilon}\cdot\widetilde{\mathcal{G}}]_{b+1}$$

$$+\sum_{a=0}^{\min\{n-1,n'\}}\sum_{b=0}^{\min\{n,n'\}}(-2\alpha')^{-a-b}C_{n-1,n'}(a)C_{n,n'}(b)[\mathcal{G}\cdot\varepsilon]_{a+1}\cdot D[\widetilde{\varepsilon}\cdot\widetilde{\mathcal{G}}]_{b}$$

and

$$\begin{aligned}
K^{(0)}_{n,n'} =& \sum_{a,b=0}^{\min\{n+1,n'\}}(-2\alpha')^{-a-b}C_{n+1,n'}(a)C_{n+1,n'}(b)[\varepsilon\cdot\mathcal{G}]_a\cdot D\cdot[\widetilde{\mathcal{G}}\cdot\widetilde{\varepsilon}]_b\\
&+\sum_{a=0}^{\min\{n+1,n'\}}\sum_{b=0}^{\min\{n,n'\}}(-2\alpha')^{-a-b}C_{n+1,n'}(a)C_{n,n'}(b)k_1\cdot D\cdot[\mathcal{G}\cdot\varepsilon]_a[\widetilde{\mathcal{G}}\cdot\widetilde{\varepsilon}]_{b+1}\\
&+\sum_{a=0}^{\min\{n+1,n'\}}\sum_{b=0}^{\min\{n,n'\}}(-2\alpha')^{-a-b}C_{n+1,n'}(a)C_{n,n'}(b)[\varepsilon\cdot\mathcal{G}]_a\cdot D\cdot[\widetilde{\varepsilon}\cdot\widetilde{\mathcal{G}}]_b\\
&+\sum_{a=0}^{\min\{n,n'\}}\sum_{b=0}^{\min\{n+1,n'\}}(-2\alpha')^{-a-b}C_{n,n'}(a)C_{n+1,n'}(b)[\varepsilon\cdot\mathcal{G}]_{a+1}k_1\cdot D\cdot[\widetilde{\mathcal{G}}\cdot\widetilde{\varepsilon}]_b\\
&+\sum_{a,b=0}^{\min\{n,n'\}}(-2\alpha')^{-a-b}C_{n,n'}(a)C_{n,n'}(b)[\varepsilon\cdot\mathcal{G}]_{a+1}[\widetilde{\mathcal{G}}\cdot\widetilde{\varepsilon}]_b\cdot D\cdot k_1\\
&+\sum_{a=0}^{\min\{n,n'\}}\sum_{b=0}^{\min\{n+1,n'\}}(-2\alpha')^{-a-b}C_{n,n'}(a)C_{n+1,n'}(b)[\mathcal{G}\cdot\varepsilon]_a\cdot D\cdot[\widetilde{\mathcal{G}}\cdot\widetilde{\varepsilon}]_b\\
&+\sum_{a,b=0}^{\min\{n,n'\}}(-2\alpha')^{-a-b}C_{n,n'}(a)C_{n,n'}(b)[\mathcal{G}\cdot\varepsilon]_a\cdot D\cdot k_1[\widetilde{\mathcal{G}}\cdot\widetilde{\varepsilon}]_{b+1}\\
&+\sum_{a,b=0}^{\min\{n,n'\}}(-2\alpha')^{-a-b}C_{n,n'}(a)C_{n,n'}(b)[\mathcal{G}\cdot\varepsilon]_a\cdot D\cdot[\widetilde{\varepsilon}\cdot\widetilde{\mathcal{G}}]_b
\end{aligned}$$

$$\begin{aligned}
K_{n,n'} =& \sum_{a,b=0}^{\min\{n+1,n'\}}(-2\alpha')^{-a-b}C_{n+1,n'}(a)C_{n+1,n'}(b)[\varepsilon\cdot\mathcal{G}]_a\cdot D\cdot[\widetilde{\mathcal{G}}\cdot\widetilde{\varepsilon}]_b\\
&+\sum_{a=0}^{\min\{n+1,n'\}}\sum_{b=0}^{\min\{n,n'\}}(-2\alpha')^{-a-b}C_{n+1,n'}(a)C_{n,n'}(b)k_1\cdot D\cdot[\mathcal{G}\cdot\varepsilon]_a[\widetilde{\mathcal{G}}\cdot\widetilde{\varepsilon}]_{b+1}\\
&+\sum_{a=0}^{\min\{n+1,n'\}}\sum_{b=0}^{\min\{n,n'\}}(-2\alpha')^{-a-b}C_{n+1,n'}(a)C_{n,n'}(b)[\varepsilon\cdot\mathcal{G}]_a\cdot D\cdot[\widetilde{\varepsilon}\cdot\widetilde{\mathcal{G}}]_b\\
&+m\sum_{a=0}^{\min\{n+1,n'\}}\sum_{b=0}^{\min\{n-1,n'\}}(-2\alpha')^{-a-b}C_{n+1,n'}(a)C_{n-1,n'}(b)[\varepsilon\cdot\mathcal{G}]_a\cdot D\cdot[\widetilde{\varepsilon}\cdot\widetilde{\mathcal{G}}]_{b+1}\\
&+\sum_{a=0}^{\min\{n,n'\}}\sum_{b=0}^{\min\{n+1,n'\}}(-2\alpha')^{-a-b}C_{n,n'}(a)C_{n+1,n'}(b)[\varepsilon\cdot\mathcal{G}]_{a+1}k_1\cdot D\cdot[\widetilde{\mathcal{G}}\cdot\widetilde{\varepsilon}]_b
\end{aligned}$$

$$
\begin{aligned}
&+ k_1 \cdot D \cdot k_1 \sum_{a,b=0}^{\min\{n,n'\}} (-2\alpha')^{-a-b} C_{n,n'}(a) C_{n,n'}(b) [\varepsilon \cdot \mathcal{G}]_{a+1} [\widetilde{\mathcal{G}} \cdot \widetilde{\varepsilon}]_{b+1} \\
&+ \sum_{a,b=0}^{\min\{n,n'\}} (-2\alpha')^{-a-b} C_{n,n'}(a) C_{n,n'}(b) [\varepsilon \cdot \mathcal{G}]_{a+1} [\widetilde{\mathcal{G}} \cdot \widetilde{\varepsilon}]_{b} \cdot D \cdot k_1 \\
&+ m \sum_{a=0}^{\min\{n,n'\}} \sum_{b=0}^{\min\{n-1,n'\}} (-2\alpha')^{-a-b} C_{n,n'}(a) C_{n-1,n'}(b) [\varepsilon \cdot \mathcal{G}]_{a+1} [\widetilde{\mathcal{G}} \cdot \widetilde{\varepsilon}]_{b+1} \cdot D \cdot k_1 \\
&+ \sum_{a=0}^{\min\{n,n'\}} \sum_{b=0}^{\min\{n+1,n'\}} (-2\alpha')^{-a-b} C_{n,n'}(a) C_{n+1,n'}(b) [\mathcal{G} \cdot \varepsilon]_{a} \cdot D \cdot [\widetilde{\mathcal{G}} \cdot \widetilde{\varepsilon}]_{b} \\
&+ \sum_{a,b=0}^{\min\{n,n'\}} (-2\alpha')^{-a-b} C_{n,n'}(a) C_{n,n'}(b) [\mathcal{G} \cdot \varepsilon]_{a} \cdot D \cdot k_1 [\widetilde{\mathcal{G}} \cdot \widetilde{\varepsilon}]_{b+1} \\
&+ \sum_{a,b=0}^{\min\{n,n'\}} (-2\alpha')^{-a-b} C_{n,n'}(a) C_{n,n'}(b) [\mathcal{G} \cdot \varepsilon]_{a} \cdot D \cdot [\widetilde{\varepsilon} \cdot \widetilde{\mathcal{G}}]_{b} \\
&+ m \sum_{a=0}^{\min\{n,n'\}} \sum_{b=0}^{\min\{n-1,n'\}} (-2\alpha')^{-a-b} C_{n,n'}(a) C_{n-1,n'}(b) [\mathcal{G} \cdot \varepsilon]_{a} \cdot D \cdot [\widetilde{\varepsilon} \cdot \widetilde{\mathcal{G}}]_{b+1} \\
&+ m \sum_{a=0}^{\min\{n-1,n'\}} \sum_{b=0}^{\min\{n+1,n'\}} (-2\alpha')^{-a-b} C_{n-1,n'}(a) C_{n+1,n'}(b) [\mathcal{G} \cdot \varepsilon]_{a+1} \cdot D \cdot [\widetilde{\mathcal{G}} \cdot \widetilde{\varepsilon}]_{b} \\
&+ m \sum_{a=0}^{\min\{n-1,n'\}} \sum_{b=0}^{\min\{n,n'\}} (-2\alpha')^{-a-b} C_{n-1,n'}(a) C_{n,n'}(b) [\mathcal{G} \cdot \varepsilon]_{a+1} \cdot D \cdot k_1 [\widetilde{\varepsilon} \cdot \widetilde{\mathcal{G}}]_{b+1} \\
&+ m \sum_{a=0}^{\min\{n-1,n'\}} \sum_{b=0}^{\min\{n,n'\}} (-2\alpha')^{-a-b} C_{n-1,n'}(a) C_{n,n'}(b) [\mathcal{G} \cdot \varepsilon]_{a+1} \cdot D [\widetilde{\varepsilon} \cdot \widetilde{\mathcal{G}}]_{b} \\
&+ m^2 \sum_{a=0}^{\min\{n-1,n'\}} \sum_{b=0}^{\min\{n-1,n'\}} (-2\alpha')^{-a-b} C_{n-1,n'}(a) C_{n-1,n'}(b) [\mathcal{G} \cdot \varepsilon]_{a+1} \cdot D [\widetilde{\varepsilon} \cdot \widetilde{\mathcal{G}}]_{b+1}
\end{aligned}
$$

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