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STABILITY AND SUNSPOTS IN ENDOGENOUS GROWTH MODELS

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Preface

This thesis is about the concept of Sunspot Equilibria.

In the first part, an historical evolution is proposed through the studies of the major researchers on the topic: from the forerunners (Samuelson, and Azariadis, in primis), to Cass and Shell, the founders of the study of sunspot as it is considered today, up to the researchers that expanded the concept adapting it to today's interpretation (Peck and Farmer, among others).

The second and third parts describe the concept of sunspot in a particular class of endogenous growth two-sector-models associated to forms of market imperfections (externalities), pointing out the mechanism that leads to the existence of sunspots and Hopf bifurctions.

What emerges is that the concept of sunspot can be understood as a sort of microfoundation of the macroeconomy: sunspots are a microeconomic way to show that there are macroeconomic equilibria of underemployment.

Therefore, there is a space for the State's intervention in the economy which, eliminating restrictions to market participation and other limiting conditions, allows to move from Pareto-optimal situations in a dynamic sense to Pareto - optimal situations in the traditional sense.

This aspect can also be seen from the point of view recalled in the models presented: the possibility of Hopf bifurcations shows there is a non-negligible possibility for the system to be taken out from a low-consumption, low-production-equilibrium, pushing the economy toward higher levels of consumption, production, employment.

CHAPTER 1

Sunspot equilibria in models of general equilibrium

1. The long journey toward the concept of "Sunspot"

The first economist and econometrician to use the term "sunspot" was W. S. Jevons: in his works he tried to discover the causes of business cycles that led to fluctuations in prices. Studying meteorology, he put forward the hypothesis that solar activity (*sunspot*) affects the climate and, in particular, the rain: in Jevons' vision, this randomness of rain would affect the conditions of production in agriculture and thus the well-being of economic subjects.

Sunspot activity which Jevons refers to is then essentially characterized by "intrinsic" uncertainty, which affects the so-called fundamentals of the economy: tastes, production technology and endowments of each subject.

Modern studies on sunspot equilibria, instead, focus on extrinsic uncertainty and, precisely, the term sunspot refers to any activity of extrinsic uncertainty, which is not transmitted through the economic fundamentals, but involves all phenomena that can, nonetheless, influence the choices of economic agents.

The extrinsic uncertainty is also known as market uncertainty: an economy is a social system in which the individuals can not be sure of the behavior of others and therefore, trying to optimize their own actions, each agent must try to predict actions of the other economic agents. Since each agent is uncertain about the actions of the others, he is necessarily uncertain also on the economic results of the actions taken. The extrinsic uncertainty affects the behavior of economic agents, creating distortions that affect the allocations and, ultimately, equilibrium prices. The sunspot equilibrium can thus be defined as a "distorted" equilibrium [24], that is an equilibrium in which the first theorem of welfare no longer holds: according to the theorem, every market equilibrium corresponds to a Pareto-optimal situation and viceversa.

The concept of sunspot equilibrium, as commonly understood today, is the result of a long process of study and research.

In the beginning, Samuelson [21] allowed the formalization of economic models (with overlapping generations) in which, unlike the Arrow - Debreu general equilibrium, there can be multiple steady states which are indeterminate and even not Pareto − optimal.

Samuelson dealt with the problem of multiplicity and indeterminacy of equilibrium. Other papers, including the one by Gale [14], showed that the particular assumptions placed at the base of a economic system such as, for example, the presence of a single good for period, the absence of production, etc. bring to indeterminate equilibrium. However, indeterminate steady states are "robust", that is the occurrence of small changes in preference or endowments will not affect their occurrence [27].

Azariadis [1] showed the possibility of "a paradoxical behavior called extrinsic uncertainty" in the overlapping generations models with rational expectations in which stochastic dynamics might trigger fluctuations and cycles in some level of activities.

One of the first attempts to insert the uncertainty created by an unknown future into a model was the one by Shell (1977); in his paper based on the overlapping generations economies with infinite horizon, the only stochastic feature is represented by the level of sunspot activity which has no effect on economic fundamentals. It is shown that there is an equilibrium in which rational individuals believe that the general price level is influenced by the level of sunspot activity, and these expectations are self-fulfilling.

The work of Cass-Shell [9] was the completion of all these lines of research: their main contribution was that sunspots and extrinsic uncertainty can have effects, despite the presence of rational expectations in the real world; this paper represented a whole new and fundamental approach to extrinsic uncertainty.

2. The new approach to extrinsic uncertainty: the Cass-Shell model

Keynes (and many Keynesians, later) sustained that the volatility of the investment is, at least in part, based on market psychology. The Cass - Shell model analyzes the market equilibrium. The equilibrium in an economy with complete markets of securities and goods and in which consumers share the same probability expectations is equivalent to the equilibrium of certainty of the traditional general equilibrium model, in which sunspots cannot have importance: with no restrictions on the participation to the markets, the sunspot are not relevant. If, however, the subjective probabilities differ among consumers, then the sunspot tend to have importance.

2.1. Formal definition of sunspot. The model presented by Cass-Shell is a simple overlapping generations model: there is a simple exchange economy with a defined time horizon, two goods, two states of nature and a finite number of consumers endowed with goods that are independent of the state of nature. Given that uncertainty has no effect on the fundamentals of the economy, it is purely extrinsic uncertainty and simply considered as *sunspot activity*. A strong version of rational expectations is adopted: consumers have the same beliefs about sunspot activity. This allows the interpretation that subjective probabilities are equal to the objective probabilities.

The exchange process must take place in time: individuals can operate exchanges but they must be concluded when one of parties is alive. The markets are assumed complete in all goods but consumers are naturally limited in their participation to the markets that gather before their birth. Economists generally agree on the fact that the general equilibrium model of Arrow - Debreu is particularly versatile. There is, however, a fundamental and significant aspect of the current dynamic economies that is not reflected in the traditional Arrow - Debreu structure: in the real world, the same process of market exchange takes place over time. The exchange may include promises delivering assets in the future, under certain circumstances, but each of the parties in a exchange must be alive at the date in which it takes place. Even if the subjects alive at "current time" may know the prices that will prevail in the future, they simply can not exchange with individuals whose birth dates are in the future.

Even in a world where the dates of birth and death vary between individuals, a person can readily conceive the existence of a complete set of markets. What can not be imagined, however, it is that there could be an unlimited participation in these markets. At any given time, some of the potential "players" have already left the "scene", while others are yet to enter. In Cass - Shell model, consumers are placed in the flow of time and then it is describes the market structure for the economy; it is assumed that there are two generations: consumers in the G^0 generation were born at the beginning of the time horizon and live to the end of it; consumers of the G^1 generation were born after those of G^0 but, like those of G^0, live until the end of time.

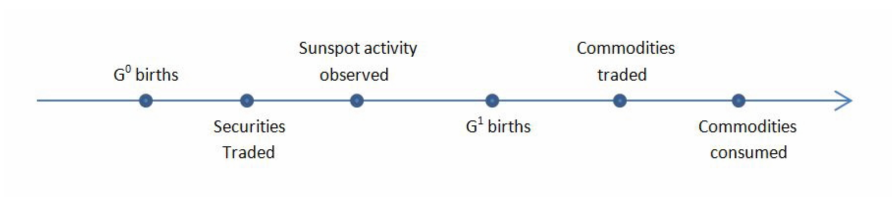

Fig. 1: temporal line of events

Consumers G^0 were born before the sunspot activity occurs and they can exchange with each other in the securities market that are contingent to the occurence of the extrinsic random variable, the sunspot activity; they can also trade with each other and with the G^1 members on the local market for goods, which opens after the observation of sunspot. On the other hand, consumers of G^1 are being born after the extent of sunspot activity is known. They can trade with each other and with the G^0 consumers on the local market for goods but, of course, they cannot trade in the securities market, which is required to meet before their birth. Therefore, participation in the market for goods is not restricted, while participation in the securities market is necessarily restricted.

This form of market imperfection will lead to an important result: the possibility that extrinsic uncertainty can have effects of allocations of equilibrium.

Cass - Shell consider also the following elements:

a) there are two standard goods: $i = 1, 2$;

b) two states of nature[1] are possible: $s = \alpha, \beta$;

c) $x_h^i(s)$ indicates the vector of in state s consumption for the consumer h :

$$\left[x_h^1(s);\ x_h^2(s)\right]$$

d) x_h indicates the vector of prospective consumption[2]:

$$[x_h(\alpha);\ x_h(\beta)] = \left[x_h^1(\alpha),\ x_h^2(\alpha); x_h^1(\beta),\ x_h^2(\beta)\right]$$

[1] A state of nature is a complete description of the environment, from the beginning to the end of the economic system

[2] The term "perspective" reflects the fact that consumes are related to the state of nature that occurs

e) consumer h has prospective goods indicated by ω_h, strictly positive vector

$$\omega_h = [\omega_h(\alpha);\ \omega_h(\beta)] = \left[\omega_h^1(\alpha),\ \omega_h^2(\alpha);\ \omega_h^1(\beta),\ \omega_h^2(\beta)\right]$$

f) the preferences of h are described by the utility function $u_h(\omega_h)$, which is defined according to his prospective consumption plans.

In this simple economy there is no production: fundamentals are represented by endowments and preferences. It is assumed that the uncertainty is purely extrinsic, which allows to consider the random variable s as sunspot activity, indicating the state sunspot with α and the state not-sunspot with β. The endowments are not affected the activity sunspot, namely:

$$\omega_h(\alpha) = \omega_h(\beta) \qquad \forall h \tag{2.1}$$

It is assumed that consumer behavior is based on a utility function of Von Neumann - Morgenstern type[3]. The consumer h believes that "sunspots" occur with probability $\pi_h(\alpha)$ and, therefore, that non-sunspot occur with probability $\pi_h(\beta) = 1 - \pi_h(\alpha)$. Then preferences are represented by the familiar principle of expected utility, i.e.:

$$u_h\left[x_h(\alpha);\ x_h(\beta)\right] = \pi_h(\alpha)v_h(\alpha)\left[x_h(\alpha)\right] + \pi_h(\beta)v_h(\beta)\left[x_h(\beta)\right] \qquad \forall h \tag{2.2}$$

The v_h functions, synthesizing consumer h tastes, are supposed monotone, strictly increasing and strictly concave: it implies that h is strictly risk averse. The preferences are obviously independent of the sunspot, given that the only effect of s on v_h is through its effect on allocating $x_h(s)$. There is not, instead, a direct effect of sunspot activity on consumers' welfare.

The Von Neumann - Morgenstern utility function and the condition $\omega_h(\alpha) = \omega_h(\beta)$ are in themselves sufficient to qualify the sunspot activity to be due to extrinsic uncertainty. Since it is assumed that expectations about sunspot activity are common to all consumers, it is:

$$\pi_h(\alpha) = \pi(\alpha) \qquad \pi_h(\beta) = \pi(\beta) = 1 - \pi(\alpha)$$

which can be interpreted as a strong version of the hypothesis of rational expectations.

At this point, it can be given a formal definition of sunspot: the extrinsic uncertainty, that is the sunspot activity, affects the allocation of resources if for some consumer h it is:

$$\pi_h(\alpha) \neq \pi_h(\beta)$$

i.e., his plan of activity of equilibrium depends on the state of nature. In this case, the sunspot activity is economically significant. On the contrary, if it is:

[3]The Von Neumann - Morgenstern utility function is a real-valued function u, defined on Y, set of all prospects of an agent h; it has the two properties of the keeping of the order (i.e. it respects the order of preference that an entity establishes between two prospects) and of linearity (i.e. the total utility is the sum of utility functions relative to each statement prospect). It does not imply that the economic entity is aware he is using an utility function for his decisions but, simply, that the person behaves like a expected utility maximizer.

$$\pi_h(\alpha) = \pi_h(\beta)$$

all consumer allocations are independent from the state of nature and the sunspot activity is economically irrelevant.

2.2. Pareto-optimality and dynamic Pareto-optimality. The sunspots can also be seen from the point of view of welfare for the economic agents.

In infinite horizon models with overlapping generations of finitely lived consumers, but without uncertainty, the two fundamental theorems of welfare can be expressed not in terms of efficient or Pareto-optimal allocations, but rather in terms of "weakly efficient" Pareto-optimal. The crucial element that distinguishes the welfare problem is that, for a given risk, some people literally have to live with it, while for others (those born after the risk has disappeared) it's just a historical datum, though important.

Then, two standard levels of welfare are defined: the traditional Pareto criterion and a criterion of dynamic (or weak) Pareto. An allocation is dynamic Pareto-optimal if, at least, another allocation can be improved upon without worsening some other's allocations.

Of course, every Pareto optimal allocation is dynamic Pareto-optimal, but the opposite it is not true. The following two propositions are the formal outcome on the analysis of welfare: the sunspot equilibria are indicated by dynamically Pareto-optimal allocations; non-sunspot equilibria are indicated by Pareto-optimal allocations and this follows from the equivalence between non-sunspot equilibria and the traditional certainty equilibria. Therefore, the Pareto-optimal allocations form a subset of the dynamic Pareto - optimal allocations.

This can be graphically represented: Consider the figure 2, where the economy is composed by two only consumers, A and B, which consume one only good.

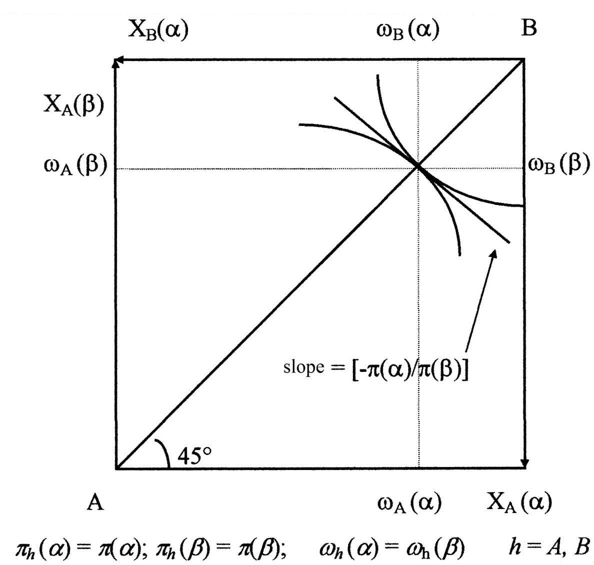

Fig. 2: sunspots have no relevance

The Edgeworth Box shows the measures of good consumption for each of the two consumers on the horizontal axis in the state α, on the vertical axis in the state β. Because uncertainty is purely extrinsic and then aggregated resources are independent of the state of nature, the box is a square. Also, because the individual allocations are independent of the state of nature, the vector of the endowment lies on the diagonal. In this case the competitive equilibrium always exists and we can distinguish two cases:

a) in the first, consumers have the same probability expectations about the occurring of states α and β: then the tangency of the indifference curves and the competitive equilibrium of contingent securities takes place only on the diagonal, so the sunspots have no relevance. In fact, as can be seen in fig. 2, the endowments of the two consumer are identical, since the extrinsic uncertainty exerts no influence on them. Moreover, the indifference curves of the two consumers are tangent to each other and the point of tangency is on the diagonal: it means that the in the tangent point, the slope of the two curves is common to the two agents (allocations on the diagonal of the square give the tangency indifference curve). So, with the assumption of strict concavity (that is to strict risk-aversion), every allocation out of the diagonal is Pareto-dominated by some allocation on the diagonal and, therefore, given that the allocations lie on the diagonal, there is no acceptable mutual trade: the sunspot cannot matter and there are no allocations such that $\frac{p(\alpha)}{\pi(\alpha)} \neq \frac{p(\beta)}{\pi(\beta)}$, implying $x_h(\alpha) \neq x_h(\beta)$ for h in $G^0 = H$.

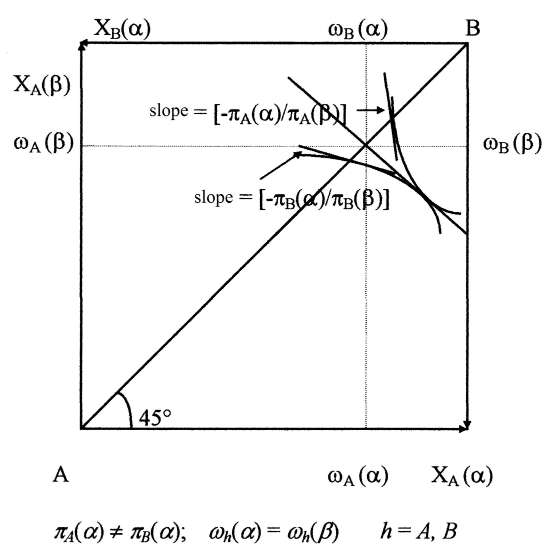

Fig. 3: sunspots do have relevance

b) in the second, consumers have different expectations about the probability of (α) and (β) and this results in a strong motivation to exchange the contingent securities. Endowments always lie on the diagonal but the indifference curves are not tangent on it; it will exist a competitive equilibrium of contingent securities outside of the diagonal, so that the sunspots must have relevance. In fact, as can be seen from figure 3, when the probability expectations differ among individuals, the exstrinsic uncertainty plays a role: the slopes of the two indifference curves measured on the diagonal, i.e. the vector of the endowments, are different from each other: $\frac{p(\alpha)}{\pi(\alpha)} \neq \frac{p(\beta)}{\pi(\beta)}$.

3. Developments in the study of extrinsic uncertainty

Research has established the possibility of indeterminacy of equilibrium and sunspot equilibria in a wide variety of economic models. The main contribution of the analysis of Cass-Shell is the fact that in a world with sunspot, the extrinsic uncertainty can have real effects, despite the presence of rational expectations. Sunspots are compatible with the individual optimization, with self-fulfilling expectations and compensations of competitive markets.

As noted by Cass-Shell, equilibrium indeterminacy and the possibility of sunspot can be seen as elements that reflect a certain degree of market incompleteness or obstacles at the entrance, but also the presence of externalities or some degree of increasing returns.

In what follows, some types of these models are analyzed.

3.1. Stationary Sunspot Equilibria (SSE). Among the authors who have continued in the footsteps of Cass-Shell, Peck [20] is to be counted. In his work he not only deals with the definition of extrinsic uncertainty and proposes maximization problems, but also studies the conditions of existence of sunspot equilibria and, above all, defines the important concept of stationary sunspot equilibrium (SSE).

In each period $t = 1, 2, 3, ...$, a consumer was born, who lives for that period and for the next. The consumer in each generation is indicated by an index linked to his date of birth. The economy starts in period 1 and continues indefinitely. In each period, there is a single perishable good and fiat currency completely durable. Consumption of the consumer t in period s is indicated by x_t^s. The economy is pure exchange, without production; the endowment of the consumer t in period s is ω_t^s $(s = t, t+1)$.

For consumer t, it is respectively:

$$(x_t^t,\ x_t^{t+1}) \in \mathbb{R}_{++}^2 \qquad and \qquad (\omega_t^t,\ \omega_t^{t+1}) \in \mathbb{R}_{++}^2$$

Every consumer has a utility function $u^t(x_t^t,\ x_t^{t+1})$ that is assumed to be strictly monotone, strictly concave and always differentiable. Each consumer pays a tax τ_t, given in fiat money units: when τ_t is positive the consumer is taxed, when τ_t is negative the consumer is given a transfer. Finally, it can be defined p^t (the price of consumption of period t) and p^m (the price of money, with the normalization $p^1 = 1$, so that there is a system of current prices).

Under perfect foresight conditions , in which consumers know the future prices with certainty, the maximization problems are:

a) for the consumer 0:

$$\begin{array}{ll} max & u^0(x_0^1) \\ sub & p^1 x_0^1 = p^1 \omega_0^1 - p^m \tau_0 \\ & x_0^1 > 0 \end{array}$$

b) for the consumer t:

$$\begin{array}{ll} max & u^t(x_t^t,\ x_t^{t+1}) \\ sub & p^t x_t^t + p^{t+1} x_t^{t+1} = p^t \omega_t^t + p^{t+1} \omega_t^{t+1} - p^m \tau_t \\ & x_t^t,\ x_t^{t+1} > 0 \end{array}$$

The market clearing requires that:

$$x_t^t + x_{t-1}^t = \omega_t^t + \omega_{t-1}^t \qquad (t = 1, 2, ...)$$

The only uncertainty considered takes the form of sunspots, with a new realization in each period beginning from period 1. The set of possible types of sunspot in period t can be indicated as:

$$\Omega_t = (1, 2, 3, ..., n)$$

At this point, it can be defined with B^t the set of events that occur in period t. The stochastic process that generates the sunspot is described by a probability value P on the space (Ω, B). Consequently, the probability of the realization of the sunspots, given the time series of previous achievements, is well-defined: if s^t is the time series of sunspot in period t and $s^{t+1} \in \Omega_{t+1}$ is a particular type of sunspot for the period $t+1$, then $\pi(s^t, s^{t+1})$ represents the probability of type s^{t+1} in period $t+1$, conditioned by the particular time series s^t.

At the beginning of each period, the type of sunspot is revealed to everybody and therefore the market for goods is opened. The realization of the sunspot in period 1, that is s^1, is therefore known at the beginning. Young consumers know all the prices, quantities and realizations of sunspot included the first period, but they do not know the future. The consumption decisions of the period t by the consumer t are to be based only on the information available at the time, so x_t^t is measurable in the set B^{t+1}. Consumers in their last period simply spill their remaining wealth on the market, so x_t^{t+1} is measurable in the set B^{t+1}. Consumers know the whole stochastic process that generates the sunspots. At this point, the rational expectations equilibrium for this economy is defined in "traditional" terms: it is that equilibrium for which the markets clear and consumers maximize the expected utility.

Now the following definition can be given: a rational expectations equilibrium for $(u, \omega, \tau, \Omega, B, P)$ is a set of p^m prices $(1, p^2, p^3 ...)$ and consumption $(x_0^1, x_1^1, x_1^2, x_2^2, x_2^3 ...)$ that satisfy the conditions:

i) $\forall t \geq 1, \quad \left(x_t^t, x_t^{t-1}, p^t\right)$ are measurable functions in B^t, the space of the state of nature
ii) $x_0^1 - \omega_0^1 = -p^m \tau_0$
iii) $\forall t \geq 1,\ \forall s^t \in \Omega_i,\ x_t^t$ and x_t^{t-1} solve the maximum program:

$$\begin{array}{ll} max & \pi(s^t,\ s^{t+1}) u^t(x_t^t,\ x_t^{t+1}) \\ sub & p^t x_t^t + p^{t+1} x_t^{t+1} = p^t \omega_t^t + p^{t+1} \omega_t^{t+1} - p^m \tau_t \\ & x_t^t,\ x_t^{t+1} > 0 \end{array}$$

iv) $x_t^t + x_{t-1}^t = \omega_t^t + \omega_{t-1}^t$ condition of market clearing

The definition of sunspot balance is in the "tradition" of the Cass-Shell model: a sunspot equilibrium is a rational expectations equilibrium in which, for given t and s^t, there exist $(\alpha, \beta) \in \Omega_{t+1}$ such that

$$x_t^{t+1}(\alpha) \neq x_t^{t+1}(\beta)$$

and then the sunspots have relevance for some consumer t. Furthermore, a rational expectations equilibrium that is not a sunspot equilibrium is a non-sunspot equilibrium.

The market structure in this model is interpreted as a sequence of spot markets connected to each other by means of the currency and, therefore, goods are exchanged for money (goods that allow to balance in presence of any event of the next period). There are markets of contingent securities as in the equilibrium model by Arrow and as in Cass - Shell, and it is possible to normalize the price of the currency in each local market. This normalization has the interesting detail that all the budgetary constraints a consumer faces in each spot market can be expressed as a single equation *iii*). Another interesting character is the simple relationship between a non-sunspot equilibrium and the corresponding equilibrium for the economy of certainty: if (p^*, p^{m*}) is an equilibrium of certainty for the economy, then the corresponding non-sunspot equilibrium is $p^m = p^{m*}$ and $p^t(s^t) = p^{t*}$, $\forall t,\ \forall s^t$.

When there are multiple perfect foresight equilibria, as often happens, it is easy to build a sunspot equilibrium: it is needed to let all prices and quantities follow a path of perfect foresight if $s^1 = 1$ and another path of perfect foresight if $s^2 = 2$. This form in some trivial way to sunspot equilibrium (i.e. a randomization on the equilibrium of perfect foresight) was treated also in the model of Cass - Shell and is included in the definition of sunspot equilibrium given above, both for mathematical reasons and also for an economic one: because of the multiplicity of perfect foresight equilibria, an initial condition determined outside of the model sets the price of money and prices of goods. However, given that the initial condition is not determined by economic fundamentals, it may be useful to think of it as caused by sunspot. When the realization of sunspot at the beginning of the period 1 affects the price of the currency, the interpretation is that sunspots are determining the initial condition. When the initial condition is considered set independently of sunspot, it can be fixed $\pi_{s^1} = 1$ for $s^1 = 1$, without loss of generality. In order to understand how the sunspots affect prices, just such initial condition is considered: it sets the price of money and consumption in the first period and then, in other words, establishes an expectation of what will probably be the prices next year. Although there is a unique path of perfect foresight that makes this expectation, there are infinitely many pairs of different prices, with associated probabilities, for which the action of the current period is rational. So, there may be a few stochastic process by which the price of the next period will be either the first or the second of the pair of prices, depending on the realization that the event will occur in that period. But why the younger generation should allow the sunspot to influence their demands and then prices? Because different realizations establish different expectations about the prices of the next period and this causes different behaviors by the expected utility maximizers.

At this point, before defining the concept of stationary sunspot equilibrium, it is necessary to characterize some additional assumptions:

a) the utility function u is continuously differentiable, strictly monotonic and strictly concave;

b) endowments ω are constant;

c) $\forall t,\ \forall s^t$, there are at least two realizations α and β such that $\pi_\alpha > 0$ and $\pi_\beta > 0$;

d) the indifference curve of ω has a slope $-1 < m < 0$ at the point of the endowment (that is, the intersection on the diagonal);

e) the consumption of the first and second period are complementary.

Under these assumptions, there is a stationary sunspot equilibrium (SSE), for a given Markov process[4], in which consumption is a function of only the current realization. In fact, the conditions d) and e) ensure that the supply curve is backward bending and has a slope of -1 to 0 at the intersection with the diagonal. On the diagonal, two points (a, a) and (b, b) with $a < b$ are chosen, near the intersection with the diagonal but on opposite sides of the intersection; considering the square defined by the vertices $(a, a), (a, b), (b, b), (b, a)$, it is (a, b) above the supply curve and (b, a) under such curve (see fig. 4).

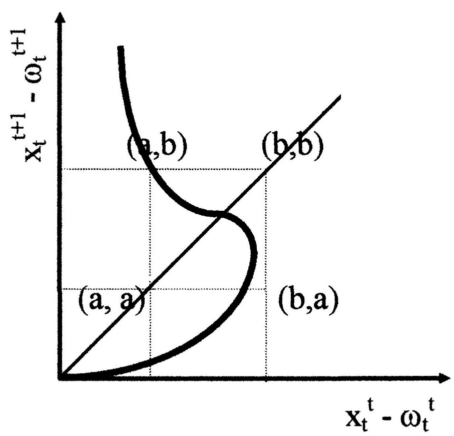

Figure 4: stationary sunspot equilibrium.

The equilibrium is defined as follows:

$p^1 = 1 \qquad p^m = a \qquad s_1 = \alpha$

$\frac{p^t}{p^{t+1}} = 1$ when $s_t = s_{t+1} = \alpha$

$\frac{p^t}{p^{t+1}} = \frac{b}{a}$ when $s_t = \alpha \qquad s_{t+1} = \beta$

$\frac{p^t}{p^{t+1}} = 1$ when $s_t = s_{t+1} = \beta$

$\frac{p^t}{p^{t+1}} = \frac{a}{b}$ when $s_t = \beta \qquad s_{t+1} = \alpha$

$x_t^t(\alpha) = \omega_t^t - a \quad ; \quad x_{t-1}^t(\alpha) = \omega_t^t + a$

$x_t^t(\beta) = \omega_t^t - b \quad ; \quad x_{t-1}^t(\beta) = \omega_t^t + b$

The equilibrium is completed by the construction of stationary transition matrix:

[4]It is a method of description of the states of a decision-making process, through a transition matrix whose elements represent the probability values. For $i = a, b$ and $j = a, b$, an element π_{ij} of this matrix indicates the probability that the sunspot activity is i in the next period if it is j in the current one.

$$\Pi = \begin{vmatrix} \pi_{aa} & \pi_{ab} \\ \pi_{ba} & \pi_{bb} \end{vmatrix}$$

In other words, the SSE may be defined as the equilibrium of rational expectations that is perfectly correlated with extraneous events or factors outside of fundamental of any individual. SSE's are interesting because they allow to understand how the set of equilibria is enlarged by the sunspot assumption, given that the event to be considered is characterized by two values: either sunspot activity or absence of sunspot activity. The occurrence of these events is, of course, governed by the Markov process. Simply, a SSE is a rational expectations equilibrium in which the forecast is ratified by the current price behavior.

3.2. The link between sunspot and economic cycle. One of the most important developments related to the theory of sunspot equilibria is on the trend of the business cycle. Azariadis [3] tried to clarify the relationship between sunspot equilibria and economic cycles and, more precisely, has tried to characterize completely a limited class of equilibria sunspot (stationary SE of order 2, that is, with two possible events or states of nature) in a simple model with overlapping generations of identical families that consume a single good produced.

The concept of sunspot equilibrium is of central importance to a full understanding of the rational expectations equilibrium as an equilibrium construction; the aim of the Azariadis model is to contribute to the clarification of this construction and especially of rational expectations equilibria in nonlinear dynamic economies: in fact, the study of rational expectations equilibria in linear systems has brought remarkable fruits, but the study of the non-linear economies appears even more interesting and, of course, it is bound to be more complex.

The sunspot phenomena are significant when stable expectations are supported as a long-run equilibrium of an economy in an open time-horizon, as the one considered by Azariadis: a simple OLG model in which the perfect foresight equilibria are well understood, and this understanding includes periodic equilibria.

The stationariness in a broad sense is important for two reasons: because it is likely that stable expectations are the asymptotic result of many well-defined learning processes and because the understanding of the SSE is a prerequisite to understand the dynamic sunspot phenomena.

The first result formally expresses a direct connection between sunspot and cycles. Then, the model gives a sufficient condition for the existence of sunspot equilibria. This condition, which is based on stochastic characteristics of extrinsic uncertainty and on the form of the savings function, describes a class of economies in which there exist sunspot equilibria. The structure used is the OG model with fiat money and production, a simple reinterpretation of the pure exchange model examined by Samuelson, Gale and others. The same condition implies the existence of periodic order-2 equilibria . The reasons for this connection are clarified further, describing how the stationary equilibria bifurcate in sunspot equilibria.

These results lead to investigate in greater depth the relationship between sunspot equilibria and cycles. The result obtained by Azariadis is surprisingly strong: the order-2 cycles exist if and only if there exist the order-2 sunspot equilibria.

Time extends from one to infinity; at discrete points in time ($t = 1, 2, ...$) appears a generation of given measure of identical individuals, who lives for two periods ("young" age and age "old") and dies in $t + 2$. Consumption occurs only in the age "old", while production takes place only in the "young". Each member of the generation t is endowed with $e_1 > 0$ divisible leisure units in youth and $e_2 > 0$ units of a single perishable good in old age. The only exception to this pattern is the first generation, which was born "old" at the time $t = 1$: every member of it has e_2 units of the consumption good and a unit of fiat currency, that is an inherently useless which is the only store of value in the economy.

Each member of the younger generation can use a technology with constant returns of scale to transform $0 < n < e_1$ units of his free time in$y \leq n$ good units of perishable good to buy the currency reserve and finance the excess consumption of e_2, in old age. The entire stock of assets is then held by the old, since it has positive value. All individuals are price takers and have perfect foresight about future prices.

The utility of an individual born at time t depends on several factors: first, from leisure to which he renounces of the time t or, equivalently, the amount of goods he offers, y_t; second, from his consumption c_{t+1} at time $t + 1$. The utility function denoted by $u(c_{t+1}, y_t)$ is assumed monotone, completely twice differentiable and strictly concave. For all this work it is assumed that consumption and leisure are normal goods and that young would choose positive savings when facing a zero real interest rate.

The excess demand for consumer goods from the community in period t is the sum $(x_t - y_t)$ of the demand excess of the old (x_t) and youth ($-y_t$). In this simple model x_t necessarily equals the purchasing power ($1/p_t$) of existing cash balances, so that the excess of aggregate demand can be defined as:

$$D\left(p_t,\ p_{t+1}\right) = \frac{1}{p_t} - s\left(\frac{p_t}{p_{t+1}}\right)$$

where the savings function of the representative family is:

$$s(R) = max\ u(e_2 + Ry,\ y) \qquad (0 \leq y \leq e_1)$$

(where R is the wage in real terms).

A competitive equilibrium is associated with a sequence of non-negative prices $(p_t)_{t=1}^{\infty}$ that satisfies the condition:

$$D\left(p_t,\ p_{t+1}\right) = 0 \qquad \forall t$$

The competitive equilibrium is equivalently associated with a sequence $(m_t)_{t=1}^{\infty}$ of real money balances that satisfies the condition:

$$D\left(1/m_t,\ 1/m_{t+1}\right) = 0 \qquad \forall t$$

($m = 1/p_t$ by definition).

The search for equilibrium with perfect foresight is equivalent to solve the difference equation:

$$D\left(p_t,\ p_{t+1}\right) = 0$$

One solution has the form $m_{t+1} = \varphi(m_t)$, where φ is a known function.

Of particular interest is the concept of periodicity. It is called periodic competitive equilibrium of order$-k$ (or $k - cycle$) the sequence:

$$(p_t)_{t=1}^{\infty}$$

(if $p_t = p_{t+k},\ t = 0, 1, 2...,\ k \geq 2$)

An important feature of the competitive equilibria (in general) and of those periodic (in particular) is that, if the stationary monetary equilibrium, is locally stable then there exists a *cycle* $-$ 2 (that is, a cycle of period-2).

The following figure illustrates the above:

- in a), the competitive equilibrium is the sequence $(p_1,\ p_2,\ p_3, ...)$;

- the b), the cycle-2 is the sequence of alternate prices: $\left(\frac{1}{\widehat{m}}, \frac{1}{f(\widehat{m})}, \frac{1}{\widehat{m}}, ...\right)$

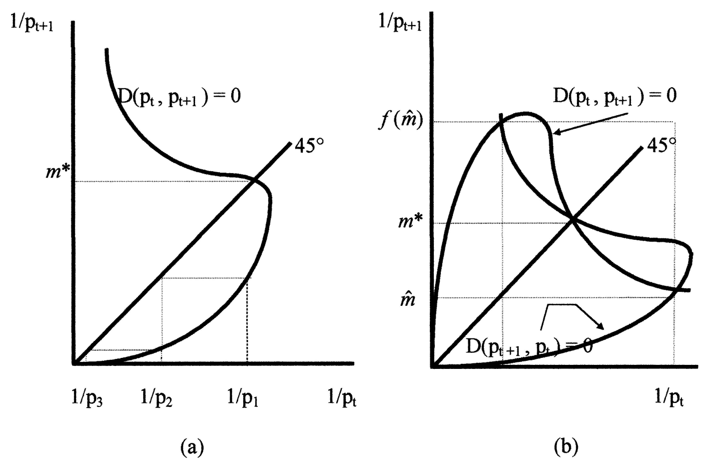

Fig. 5: sunspot equilibrium and cycle-2

The sunspot equilibria are equilibria of rational expectations that are perfectly correlated with extraneous events or factors outside of preferences, endowments and production sets and sets of every individual. The equilibria of this type are not necessarily stationary, but in this work is concerned only on stationary sunspots, for which to understand how the set of equilibria is expanded by the sunspot hypothesis.

The event that is considered now is characterized by two values: either sunspot activity (a) or absence of sunspot activity (b). The occurrence of a and b is governed by a Markov process with the following stationary transition probability matrix:

$$\Pi = \begin{vmatrix} \pi_{aa} & \pi_{ab} \\ \pi_{ba} & \pi_{bb} \end{vmatrix}$$

For $(i = a, b)$ and $(j = a, b)$, an element of this matrix denotes the probability that the sunspot activity will be i tomorrow, given j today.

Now, it is assumes that all agents in the economy believe in a perfect and stationary correlation of future prices with the sunspot activity; in other words, individuals expect that future prices are:

$$p = \varphi(i) \qquad (i = a, b)$$

if i occurs tomorrow. In other words, a stationary sunspot equilibrium is a rational expectations equilibrium in which the prediction is validated by the current price behavior.

Before proceeding to define the stationary sunspot equilibria, some useful properties of the $z-$saving function (the counterpart of rational expectations conditions of perfect foresight s saving function) are to be indicated. Consider the function:

$$z = z(R, \pi)$$

It is single-valued, continuous and such that:

a) $z(R, 0) = s(R) \quad \forall R$

b) $z(1, \pi) = s(1) \quad \forall \pi$

c) $z(R, \pi)$ lies between $s(R)$ and $s(1)$ $\forall R, \pi$

d) $z(R, \widehat{\pi})$ lies between $z(R, 0)$ and $z(R, \pi)$, if $\widehat{\pi} < \pi$

Possessing one only value and continuity of z descend from the strict concavity and continuity of maximization function of the consumer with respect to y. This feature becomes useful once the first order conditions are written and differentiated with respect to π: the key result is that z is a simple deformation of s, which coincides when $\pi = 0$.

Now, it is defined $\eta(R, \pi)$ as the elasticity of saving with respect to wages, in conditions of random expectations on real wage R, evaluated in (R, π). Then it will be valid the relationship:

$$\eta(1, \pi) = (1 - \pi)\varepsilon(1) \qquad \forall \pi$$

where $\varepsilon(1)$ is the corresponding elasticity of the saving, in conditions of perfect foresight.

Having defined the function z, stationary sunspot equilibrium can be now formally defined. A balance stationary sunspot is a quadruple $(p_a, p_b, \pi_{aa}, \pi_{bb})$ of positive numbers such that:

a) π_{aa}, π_{bb} lie in the open interval $(0, 1)$;

b) $p_a \neq p_b$

c) the demand excess of the consumer good is zero in any current state, that is:

i) $D^a = \frac{1}{p_a} - z\left(\frac{p_a}{p_b}, \pi_{aa}\right) = 0$

ii) $D^b = \frac{1}{p_b} - z\left(\frac{p_b}{p_a}, \pi_{bb}\right) = 0$

A SSE $(p_a, p_b, \pi_{aa}, \pi_{bb})$ is a stationary sunspot equilibrium with respect to an exogenous matrix Π if the numbers π_{aa}, π_{bb} in the definition are the elements of the diagonal of the matrix Π. This definition is in accordance with that informal one given previously. If the event a (respectively b) occurs in the current period, p_a (respectively p_b) is the equilibrium price according to the equations (i) and (ii). Expectations for which $p_a = \varphi(a)$ and $p_b = \varphi(b)$ are then self-fulfilling.

Furthermore, the definition requires that is ($p_a \neq p_b$, with $0 < \pi_{aa} < 1$ and $0 < \pi_{bb} < 1$). If it is ($p_a = p_b$) the SSE degenerates into a stationary equilibrium of the type "golden rule". Another type of degeneration is obtained when certain transitions are eliminated from the transition matrix. In particular, if $\pi_{aa} = \pi_{bb} = 0$, the occurrence of the event a (or respectively b) today, ensures the occurrence of b (or respectively a) tomorrow. In other words, the equilibrium prices $p_a,\ p_b$ necessarily follow one another. The SSE then degenerates into a cycle-2, as can be seen from the equations (i) and (ii): the cycle-2 therefore appears as a limiting sunspot equilibrium associated with a degenerate 2×2 matrix that has zero in diagonal.

As a direct consequence it can be expressed the following important result, referred to as Theorem of equilibrium in the neighborhood of a cycle-2: in an economy that admits a periodic equilibrium of order-2, there is generally a neighborhood $\nu(\overline{\Pi})$ of the matrix (2×2) such that there exists an stationary sunspot equilibrium with respect to any Π in $\nu(\overline{\Pi})$.

If an equilibrium of order-2 them satisfies the conclusions of the theorem stated above, it is called "regular".

To study the existence of SSE, it is set $w = p_a/p_b$ and defined the following single-valued function:

$$F(w, \pi_{aa}, \pi_{bb}) = wz(w, \pi_{aa}) - z(1/w, \pi_{bb})$$

A SSE exists if and only if F has a positive root $w \neq 1$ for some $\pi_{aa} \in (0, 1)$ and $\pi_{bb} \in (0, 1)$. This is because each SSE that satisfies the relations (i) and (ii) for some $p_a \neq p_b$ also satisfies:

$$\frac{1}{w} = \frac{z(w, \pi_{aa})}{(1/w, \pi_{bb})}$$

and therefore also the $F(\bullet) = 0$. Moreover, for each positive root $w \neq 1$, there can be found two $p_a,\ p_b$ positive numbers such that the relations (i) and (ii) hold true.

The function $F(w, \pi_{aa}, \pi_{bb})$ is continuous for all (w, π_{aa}, π_{bb}) with $w > 0$. It has the following properties for each (π_{aa}, π_{bb}):

a) $F(1, \pi_{aa}, \pi_{bb}) = 0$

b) $F \to \infty$ for $w \to \infty$

c) for w sufficiently small, $F(w, \pi_{aa}, \pi_{bb}) < 0$

d) if w is a root of $F(w, \pi_{aa}, \pi_{bb})$, then also $1/w$ is a root of $F(w, \pi_{aa}, \pi_{bb})$

After defining these properties, there is an answer to two related questions: first, what it can be said about the 2×2 matrix of transition probabilities for which exists a SSE ? second, can sunspot equilibria be found in the neighborhood of a stationary equilibrium of perfect foresight?

Evaluating with $w = 1$ the derivative of the function F with respect to w, it is:

$$\partial_w F(1, \pi_{aa}, \pi_{bb}) = s(1)\left[1 + \eta(1, \pi_{aa}) + (1, \pi_{bb})\right]$$

This relationship combined with the one about $\eta(R, \pi)$ allows to write:

$$\partial_w F(1, \pi_{aa}, \pi_{bb}) < 0 \qquad if \quad [2 - \pi_{aa} - \pi_{bb}]\, \varepsilon(1) < -1$$

A direct implication of this relationship is the following theorem: assuming that the utility function satisfies the regularity assumptions regarding differntiability, concavity and behavior at the limit, then a sufficient condition for the existence of a sunspot equilibrium with respect to a given Markov matrix Π of transition probabilities is:

$$\varepsilon(1) < 0 \qquad and \qquad \pi_{aa} + \pi_{bb} < 2 - \frac{1}{|\varepsilon(1)|}$$

In fact, if $\varepsilon(1) \geq 0$, this would violate the theorem just stated and also the condition $F(1, \pi_{aa}, \pi_{bb}) = 0$. The inequality $\partial_w F < 0$ is sufficient to ensure that F has two roots other than $w = 1$. This condition, given $\pi_{aa}, \pi_{bb} > 0$, is applied directly to the existence of cycles-2. This theorem identifies a subset of the set of all of the transition probability matrices for the which there exist SSE. This subset can be represented in the following figure:

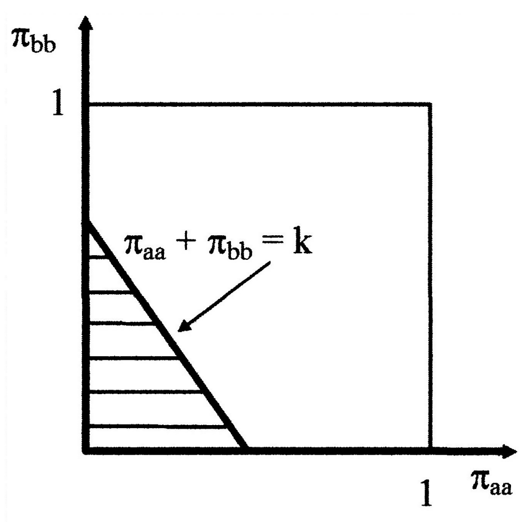

Fig.6: subset of cycles-2

The figure indicates the shaded area with $k = 2 - \frac{1}{|\varepsilon(1)|}$. The unit square indicates, instead, the whole set of matrices for which there exist the cycles-2.

To understand some of the properties of the theorem, the sunspot equilibria in the neighborhood of the line $(\pi_{aa} + \pi_{bb} = k)$ in Fig. 8 are considered. When $[\pi_{aa} + \pi_{bb}]$ decreases passing through k, it is found that $\partial_w F$ is strictly positive, then it becomes infinitesimal and then becomes negative. The passing through zero of the derivative of F is what mathematicians call bifurcation. It is shown that, given a path P on the plane $(\pi_{aa},\ \pi_{bb})$ that transversely intersects the line $(\pi_{aa} + \pi_{bb} = k)$ at a given point C, the graph of $w = p_a/p_b$ as a function of the $x-$coordinate of P has a point of equilibrium before C and three points of equilibrium after C.

Finally Azariadis states that the existence of sunspot equilibria in the neighborhood of periodic cycles is potentially a very general property, which remains valid for $n-$dimensional systems and of any order$-k$ cycles.

3.3. Sunspot in not-overlapping-generation models. Advances in understanding of indeterminacy of equilibria and dynamics of economic systems were in direct proportion to the progress made in the mathematical methods. In this regard, also other types of models (not OLG but which have enabled significant advances in understanding the working of the economy) deserve mention.

The models with infinitely-life economic agents have been used, among other things, for the study of equilibrium indeterminacy of economic systems, in the presence of elements such as monopolistic competition or externalities or even increasing returns. The conclusions which these models typically reach relates primarily to the fact that the extrinsic uncertainty must be taken seriously, because it can generate fluctuations driven by expectations: in fact, it is known that the type of differential equations that characterize the equilibrium conditions of a competitive economy with infinite horizon can have solutions in which the endogenous variables fluctuate in response to sunspot variables, that is random events, which have nothing to do with economic fundamentals and therefore does not directly affect the equilibrium conditions. It is possible to see these sunspot equilibria as a representation of a real phenomenon, that is economic fluctuations not caused by exogenous shocks to fundamentals but rather by revisions of agents' expectations in response to some event, for which those expectations become self-fulfilling.

Another very important type is the "one-step-forward-looking" structure: in it, the state of equilibrium at time t only depends on expectations, common among agents, of the value of future state variables. It is a very simple structure, whose time horizon is limited to a period which, however, has enabled the achievement of important results: one of the first was the so-called local stationary sunspot equilibrium, i.e. a sunspot equilibrium whose support is in the neighborhood the steady state of the economy. This has also allowed to deepen the connection between the existence of local stationary sunspot equilibria and the indeterminacy of steady state, a theme appeared since the early works of Azariadis (1981) on self-fulfilling expectations in the structure of the models with overlapping generations.

Later, it has been possible to apply to one-step-forward-looking models tools such as continuous time, the study of the global dynamics (and not just the local one), the application of the results of the studies on deterministic chaos. With these elements it has been possible to give an alternative approach, in particular, in the study of the economic system growth: in fact many causal mechanisms had already been identified as human capital, research and innovation, externalities, but it had not been reached a satisfactory description of the fluctuations in growth rates. It has been shown that the fluctuations in the growth rates will occur through a mechanism of self-fulfilling expectations, expectations already incorporated in economy through the theory of sunspot equilibria. The one-step-forward-looking models made it possible to incorporate the concept of sunspot equilibrium in a continuous time frame. Such a possibility has been suggested first by Woodford and other authors: the most interesting character of the use of the continuous-time methods is the possibility that it brings a full understanding of the properties of the global dynamics of these stochastic equilibria, not limiting it to the local dynamic study. Today there are a number of results in new literature of growth that have established the possibility of multiple paths of stationary growth or cyclical growth paths, and then, in general, the strong indeterminacy in the long run. Finally, another result due to one-step-forward-looking models is the possibility of application of the non-stationary nature of the economic environment: first, it significantly broadens the class of economies compatible with the existence of sunspot equilibria; second, this non-stationarity allows to study both the economic cycles and the economy's growth into a unified structure.

4. Further contributions

4.1. A geometric approach. The paper by Gaetano Bloise [8], in the tradition of the work by Grandmont [15] and Woodford [28], aims to present a geometric method useful for the study of sunspot equilibria in nonlinear multidimensional economic models with no predetermined variables. This approach allows to characterize the support of a sunspot equilibrium by its invariance properties in the underlying deterministic dynamics. It provides a complete description of stochastic endogenous fluctuations around a stationary state.

He considers the simplest reduced form model allowing the study of sunspot equilibria in a nonlinear dynamic economy, assuming the temporary excess demand only depends on current prices and on the expectation over prices in the next period and, in addition, is separable in these two arguments. Such a formulation represents the smallest departure from linear rational expectation models which allows to encompass nonlinearities. However, it is undoubtedly restrictive and is mainly motivated by the need for technical tractability. So, clear results are obtained at the cost of limited applicability.

The work focuses on first-order sunspot equilibria: in the current period, prices provide all the information about the stochastic distribution of prices in the next period. This notion of equilibrium is more restrictive than the one proposed by Woodford and can be reconciled better with the problem of coordination of beliefs. In approaching the issue of existence of sunspot equilibria, the argument is divided into two parts. First, pursuing the line of research initiated by Grandmont et al. [15], it is shown that, if a given set is invariant for the deterministic equilibrium map, then it is possible to construct persistent equilibrium fluctuations taking place on this set and not vanishing in the long run. Second, a suitable invariant set can be found close to the steady state equilibrium whenever it is (locally) indeterminate.

The analysis leads to two main general conclusions. One is that nonlinear models admit stationary first-order sunspot equilibria whenever indeterminacy occurs; the second is that, under some slightly more restrictive assumptions, it is possible to show the existence of stochastic equilibria persisting on a full dimensional set even when indeterminacy is lower dimensional.

Considered an infinite horizon economy with no intrinsic uncertainty on fundamentals, equilibrium is a time-homogenous Markov process, where the current state is a sufficient statistic for the future evolution of the system. Uncertainty is created by self-fulfilling revisions of individuals' expectations.

An economy is described by a state space $E = R^N$, a pair of diffeomorphisms (v, w) from E onto itself and a temporary equilibrium relation[5]. Equilibrium

[5] The vector space E (as well as any of its subsets) is endowed with the norm topology. It is in fact sufficient to require that the pair of diffeomorphisms (v, w) is only locally defined on open sets.

processes will take place in the state space, so that a state should be thought of as a complete description of all relevant current variables. The two mappings (v, w) are intended to capture all the fundamentals of the economy and to impose constraints on the transition law for the state variable at equilibrium. A temporary equilibrium is a pair (x, μ), where x is a state and μ is a probability measure on the state space with compact support, satisfying the condition

$$v(x) - \int_E w(y)\, d\mu(y) = 0$$

This restriction embodies short-run equilibrium constraints: if (x, μ) solves this equation, then agents have no incentive to take actions different from those prescribed at x when the expectation (commonly held by all agents) of the distribution at the next period state is given by μ; so, the equation might be thought of as a no-arbitrage requirement or, possibly, as relating the marginal utility of consumption today and the expected marginal utility of consumption tomorrow.

The paper directs to time-homogenous Markov equilibria. Given a (nonempty) compact subset X of E, a Markov process on the support X is a measurable transition map ϕ, from X into $P(X)$[6]. An invariant (probability) measure for a Markov process (ϕ, X) is a probability measure v in $P(X)$ such that, for all measurable subsets B of X,

$$v(B) = \int_X \phi_x(B)\, dv(x)$$

So a definition of *Sunspot Equilibrium* can be given: a sunspot equilibrium on the support X is a Markov process (ϕ, X) such that, for each x in X, the pair (x, ϕ_x) is a temporary equilibrium; that is, for each x in X,

$$v(x) - \int_X w(y)\, d\phi_x(y) = 0$$

Thus, a sunspot equilibrium is a time-homogeneous stochastic process over some set X such that for each value x of the state variable there is a (conditional) probability measure ϕ_x over X that justifies x: if agents commonly believe that the

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Given a closed subset E' of E, we use $b(E')$ to denote the σ-field of Borel subsets of E'; for any closed subset E'' of E', we have that

$$b(E'') = \{B \subseteq E'' : B \in b(E')\}$$

The set of all probability measures on $(E', b(E'))$ (or, simply, on E') is denoted by $P(E')$; for any closed subset E'' of E', we have that

$$P(E'' = \{\mu \in P(E') : \mu(E'') = 1\}$$

Given a closed subset E' of E, the space $P(E')$, endowed with the topology of weak convergence, is a metrizable topological space; for any closed subset E'' of E', the weak convergence topology on $P(E'')$ is the relativization of the weak convergence topology on $P(E')$ to $P(E'')$. Finally, for any pair of closed subsets (E', E'') of E, any continuous mapping $h : E' \to E''$, and any probability measure μ in $P(E')$, μh^{-1} is the probability measure in $P(E'')$ defined by $\mu h^{-1}(B) = \mu(h^{-1}(B))$ for all measurable subsets B of E''.

[6]For convenience, we write (ϕ, X) interchangeably with $\phi : X \to P(X)$ and ϕ_x interchangeably with $\phi(x)$.

future state variable is distributed according to ϕ_x, then x clears markets in the current period. Additionally, this agents' belief coincides with the true (conditional) probability distribution on the future state variable. Note that, whenever a sunspot equilibrium is truly stochastic, probability distributions only reflect an uncertainty originating in the beliefs of agents, rather than in some intrinsic fluctuation of fundamentals.

Deterministic equilibria represent economic situations which do not involve any uncertainty. To examine such equilibria, it is useful to introduce the diffeomorphism g from E onto itself defined by the composition $\left(w^{-1} \circ v\right)$. A deterministic equilibrium is then any (nonempty) compact subset X of E such that $g(X) \subseteq X$: for each initial condition x_0 in X, the sequence of states generated by iterating g on x_0 is a sequence of temporary equilibria with perfect foresight which remains in the support X forever. A steady state equilibrium, in turn, is a fixed point of the map g. Following traditional terminology in the literature on sunspot equilibria in sequential economies, a steady state is (locally) indeterminate whenever there is a continuum of other (nonstationary) deterministic equilibria arbitrarily close to this steady state.

Then, the definition of *indeterminacy* is given: a steady state equilibrium a is said to be (locally) indeterminate whenever, for any open neighborhood U of a, there exists a deterministic equilibrium on an uncountable support X contained in U. It is a well-known result that verifying whether a steady state is indeterminate requires the study of the linear operator $D_a g$: the steady state is (locally) indeterminate whenever $D_a g$ has at least one eigenvalue inside the unit circle.

A deterministic equilibrium is consequently identified with a (nonempty) compact subset X of E such that $f(X) \subseteq X$ and, without loss of generality, we can assume that the origin of E is a steady state equilibrium (thus, a fixed point of f). Clearly, indeterminacy of this steady state is unaffected by our transformation of the state space: the steady state is (locally) indeterminate whenever $D_0 f$ has at least one eigenvalue inside the unit circle. From now on, the map f (jointly with the equilibrium restriction (2.3)) is the only object we need to check for the occurrence of sunspot equilibria. It is referred to as the deterministic equilibrium map.

The investigation in Bloise's paper has shed some additional light on the occurrence of sunspot equilibria in sequential economies. Whenever the steady state equilibrium is indeterminate, sunspot equilibria exist close to such a steady state. These equilibria are such that the current state of the economy gives all the information about the distribution of the next period state. Moreover, under some assumptions, in a nonlinear economy, there exist stationary, persistent equilibrium fluctuations around the steady state taking place on a full-dimensional set and admitting an invariant measure absolutely continuous with respect to the Lebesgue measure. More, the existence of sunspot equilibria in sequential economies with an indeterminate steady state is a well-established result in the literature. However, from this geometric approach some conclusions which have been neglected by the traditional linear approximation method can be drawn. It is, for instance, now clearer that, in nonlinear models, one should not expect sunspot equilibria to take place on the stable manifold. Even though the paper considered a very specific form of the temporary equilibrium relation, results pose a serious question on the empirical implications of sunspot equilibria. In linear models, all (bounded) stochastic processes must take place on the stable linear manifold, thus implying that a strong linear correlation among, say, prices must exist. In nonlinear models, on the contrary, such equilibria might not involve any linear restriction and, in general, may persist on a support that is qualitatively different from the linear stable manifold, so that little predictions can be proposed. Among many others, a serious drawback of this analysis is that it does not account for predetermined variables.

4.2. The concept of global sunspot. The work of Roger Farmer [13] can be considered one of the most important contributions on the sunspots literature. He studied in particular the aspect of financial crises as global sunspots, continuing the research on financial markets begun in 1998 with Jess Benhabib.

In his paper, Farmer constructs a heterogenous agent general equilibrium model to explain asset prices. In this model, asset price fluctuations are caused by random shocks to the price level that reallocate consumption across two kinds of people. Asset prices are volatile and price dividend ratios are persistent even though there is no fundamental uncertainty and financial markets are sequentially complete. Following David Cass and Karl Shell, he refers to the random variables that drive equilibria as "sunspots".

Farmer's work differs in three ways from standard asset pricing models: first, it allows for birth and death (by exploiting Blanchard's concept of perpetual youth); second, there are two types of people that differ in the rate at which they discount the future; third, there is an asset, government debt, denominated in currency (dollars).

The model has no fundamental uncertainty of any kind and a set of perfect foresight equilibria that are solutions to a difference equation which converges to a unique steady state.

The results in Farmer's paper rely on all three of these pieces; perpetual youth, multiple types and nominal debt. The initial price level is indeterminate and, because of this fact, the initial price level is indeterminate: so, there is more than one solution to the difference equation, each of which is an equilibrium, and each of which begins at a different initial point.

Then the author exploits the indeterminacy of the set of the perfect foresight equilibria to construct a rational expectations equilibrium in which uncertainty is nonfundamental. The people in the model come to believe that the future price level is a random variable, driven by a sunspot, and they write financial contracts contingent on its realization. But the unborn cannot participate in the financial markets that open before they are born. As a consequence, sunspot shocks reallocate resources between people of different generations. Most sunspot models add a shock to the perfect foresight equilibria of a model that has been linearized around an indeterminate steady state. This method may be used to generate local sunspot equilibria, but there is no guarantee that the sunspot solutions of a linear approximation are close to the equilibria of the original model once the variance of the shocks becomes large.

In this paper, Farmer takes advantage of the nonlinear nature of the solution to compute global sunspot equilibria. Although the model represents an endowment economy, the framework provided can easily be extended to allow for production by adding capital and a labor market. If the explanation for asset price volatility is accepted, models that build on this framework have the potential to unify macroeconomics with finance theory in a simple way.

In this model people have infinite horizons but finite lives. These people survive from one period to the next with an age-invariant probability. There are two types of people, one of which is more patient than the other. There is no production, and each type is endowed with a single commodity in every period. In the absence of money, the unique equilibrium of this model is characterized by a difference equation in a single state variable that converges to a unique steady state. The initial condition of this difference equation is the net indebtedness in the first period, of patient to impatient types.

In a steady state equilibrium, patient people consume less than their endowment when young and more than their endowment when old: impatient people consume more than their endowment when young and less than their endowment when old. In the steady state, there is an exponential age distribution of each type.

A government is added to this model: it consists of a treasury and a central bank. The treasury issues currency (dollar) denominated debt and, although money is used as a unit of account, no agent in the model holds money. Each period, the treasury raises lump-sum taxes that it uses to pay the interest on its debt and to roll over the principal. The central bank fixes the nominal interest rate at a constant.

The model possesses a set of perfect foresight equilibria that are solutions to a first order difference equation. There is more than one perfect foresight equilibrium because the initial price level is a free variable. This fact is used to construct a rational expectations equilibrium in which the price level is random. In this equilibrium, price-level fluctuations reallocate the tax burden of government debt between current and future generations. These fluctuations exist as equilibria, even in the presence of a complete set of state-contingent securities, because the unborn cannot insure against the state of the world into which they are born.

Among the many possible models of sunspot equilibria, this one combines standard assumptions about preferences with a plausible demographic structure to generate equilibria, driven by self-fulfilling prophecies, that exhibit many of the features that we see in real world asset markets. Farmer characterizes equilibria of the model, after a series of assumption:

(1) about people, apples and trees; there are two types of people, each type endowed with one unit of a unique perishable commodity in every period in which is alive - called "apple". The wealth of a person in the year of his birth is equal to the discounted present value of his "apples"; so, this is called a tree. People have logarithmic preferences, with two discount factors: the first is bigger than the second, reflecting the fact that people in the first group are more patient than the second. People of each type die with a given probability and, when a person dies, he is replaced by a new person of the same type. The population is a constant, of measure 1.

(2) about uncertainty; uncertainty in period t is indexed by a random variable S_t with compact support $\mathbf{S}$, so that $S_t \in \mathbf{S}$. A $\tau-$period sequence S_t^τ is a $\tau-$period history with root S_t:

$$S_t^\tau = \{S_t, S_{t+1}, ..., S_\tau\}$$

The root is the initial date-state pair and a history S_t^τ is a $(\tau\text{-}t)-$dimensional random variable with support $\mathbf{S}_t^{\tau-t}$.

(3) about the asset markets; asset markets are sequentially complete; three assets are actively traded: Arrow securities, government debt and trees.

(4) about government; government consists of a central bank and a treasury; the treasury issues dollar denominated one-period debt and faces a budget constraint. The central bank sets the gross interest rate equal to a constant; a monetary policy rule of this kind is called passive. The treasury issues sufficient nominal debt to roll over its existing debt, net of tax revenues. A fiscal policy of this kind is called active.

Now, an equilibrium is a possibly stochastic sequence that satisfies a pair of stochastic difference equations.

At first, only perfect foresight equilibria are taken into consideration. The equilibria are characterized by non-stochastic sequences. For a calibrated version of the model, there is a single steady state solution to the equations and this steady state is a saddle. The saddle path, also called the stable manifold, is a one-dimensional manifold of points with the property that trajectories that begin on this manifold converge to the steady state (in the words of Guckenheimer and Holmes).

Saddle-path is usually associated with uniqueness of equilibrium. In this model, although there is a unique stable saddle path, the initial value of outstanding government debt depends on the initial price level (and this can take on a continuum of values in an open set). Because of the dependence of the initial condition on the dollar price of apples, this model is associated with a continuum of perfect foresight equilibria. An equilibrium is characterized by a sequence that begins at an arbitrary point on the stable manifold and converges to the unique steady state over time.

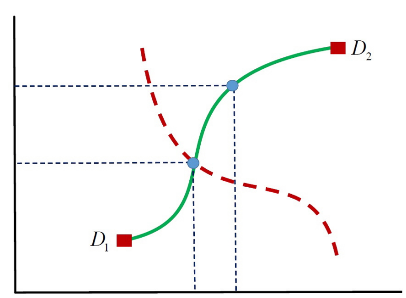

Fig.5: set of perfect foresight equilibria

The stable manifold is a set $D = [D_1, D_2]$ with the property that every point that begins on this manifold follows a first order difference equation that converges to the steady state (the dashed curve is the unstable manifold).

The model proceeds with the construction of a set of rational expectations equilibria, by the randomization over the perfect foresight equilibria of the underlying model. In these equilibria, people form self-fulfilling beliefs about the distribution of future prices.

In the finite Arrow-Debreu model there is, generically, a finite odd number of equilibria and new stochastic equilibria cannot be constructed by randomizing across the existing perfect foresight equilibria, because of the first welfare theorem which asserts that every competitive equilibrium is Pareto optimal. This result breaks down when there is incomplete participation in asset markets as a consequence of overlapping generations: so, it possible to construct randomizations across the perfect foresight equilibria of the model, that are themselves equilibria.

Because there are multiple perfect foresight equilibria, there are multiple possible values of prices. In a stationary environment, people come to understand that the future price is a random variable and they form beliefs that are indexed to an observable shock: This shock is a sunspot that is unrelated to fundamentals.

Farmer makes a very bright example of what makes agents coordinate beliefs on a sunspot equilibrium supposing that two agents A and B believe the writing of an economic journalist who makes accurate prediction of asset prices. This journalist predicts either a 10% rise or a 10% fall in the price of trees.

The 2 agents, wishing to insure against wealth fluctuations, use the articles to write a contract: in the event of a rise agent A agrees, in advance, that he will transfer wealth to B. In the event of a fall, the transfer is in the other direction. These contracts have the effect of ensuring that the journalist's predictions are self-fulfilling. How can that be an equilibrium? There are three groups of people involved in any potential trade: patient agents alive today, impatient agents alive today and agents of both types who will be born tomorrow. Fluctuations in the price of trees cause a wealth redistribution from the newly born to the existing generations. This wealth redistribution operates by a transfer of tax obligations to or from the unborn. Because the existing agents have different propensities to consume out of wealth, they choose to change their net obligations to each other in different ways depending on whether the transfer from the unborn is positive or negative. In a rational expectations equilibrium, the different behaviors of the 2 agents are self-fulfilling.

Farmer, then, constructs a sunspot equilibrium with two future states, the 10% increase and the 10% fall in the price (after this, he generalizes the construction to many possible states).

This is the essential and characterizing aspect of Farmer's paper: the construction of global sunspot equilibria. Farmer introduces a new method for computing sunspot-driven rational expectations equilibria. The usual method of computing sunspot equilibria proceeds by linearizing a dynamic stochastic general equilibrium model around an indeterminate steady state and adding random shocks to the resulting linear system. This method produces a valid approximation to the equilibria of a non-linear model but the accuracy of the approximation decreases as the variance of the shocks becomes larger. He shows how to construct a higher order global approximation that remains valid for persistant and "larger" shocks.

In fact, Farmer's work explains three asset pricing puzzles that are difficult to reconcile with the now standard representative agent approach to macroeconomics: i) asset prices are volatile and persistent and price-dividend ratios are predictable; ii) long-lived risky assets earn 5% more on average than short term government debt; iii) the volatility of asset prices changes through time. So the author argues that all of these "problems" are caused by the simple fact that no one can buy insurance over the state of the world he was born into. Then, a series of simulations and graphs are presented: these simulations give a great insight into the explanations of the three feeatures. To construct global sunspot equilibria, the pricing kernel is mapped into the interval $[0, 1]$: the pricing kernel (m') is a random variable with the property that the price of any asset can be computed as the expected value of its product with m'. It is assumed that, for any value of m (the marginal rate of substitution of a person alive in two consecutive periods), the variable m' has a distribution with mean $f(m)$, where $m' = f(m)$ is the stable manifold of the map. The assumption implies that in any given period, people believe that m' is a random variable with support D for every value of $m \in D$. This means that, however well the economy is doing today, there is always positive probability that the next period will be associated with an extreme value in which the discount factor is at its upper or lower bound.

By modeling m' as distributed random variable, it is possible to capture in a parsimonious way, the idea that people believe that equilibria will be selected by the psychology of market participants. Because people are assumed to be risk averse, they would always prefer the mean of a gamble to the gamble itself. And, in the case of sunspot fluctuations, that mean is available.

Farmer arrives to a conclusion common to the sunspot literature: asset price fluctuations cause Pareto inefficient reallocations of wealth between current and future generations and these reallocations lead to substantial fluctuations in welfare. So there is room for policy makers to stabilize asset prices through monetary and fiscal interventions that, in turn lead to welfare improving.

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CHAPTER 2

Sunspots in endogenous growth two sector models

1. Introduction

The problem of the indeterminacy and sunspot equilibrium in economic financial models has been analysed by many authors in recent times: among them, Nishimura, Shigoka, Yano (2006), and Benhabib, Nishimura, Shigoka (2008), Slobodyan (2009).

Many papers reported the occurrence of the stochastic behaviour (sunspots) also in presence of individual optimization, self-fulfilling expectations and compensations of competitive markets. We remember that a phenomenon is called sunspot when the fundamental characteristics of an economy are deterministic but the economic agents believe nevertheless that equilibrium dynamics is affected by random factors apparently irrelevant to the fundamental characteristics (Nishimura, Shigoka, Yano 2006).

In this paper, we consider the mechanism that leads to the existence of sunspots close to the indeterminate equilibrium (the Hopf orbit) in a class of endogenous growth two-sector-models with externality. Nishimura, and Shigoka (2006) consider the reduced form of the Lucas and Romer models, as a continous deterministic three-dimensional non linear differential system, with one pre-determined variable (a combination of the state variables) and two non-predetermined variables (related with the control variables). They constructed a stationary sunspot equilibrium near the stable Hopf cycle that emerges from the unique equilibrium point adding a Wiener variable to the non-predetermined variables; in their formulation, the cycle represents a compact solution of the stochastic process associated with the deterministic model.

Benhabib, Nishimura, Shigoka (2008) prove the existence of a sunspot equilibrium that comes from a Hopf cycle or a homoclinic orbit in a continuous time model of economic growth with positive externalities and with variable capacity utilization; the model has one only predetermined variable (the state variable) and one non-predetermined variable (the control variable). In their model, the positive externality produces the existence of multiple equilibria. Through dynamical analysis they show that the equilibrium is globally indeterminate in the periodic orbit; moreover, there exists a sunspot equilibrium with a support located in the bounded region enclosed by either a homoclinic orbit or a periodic orbit, such that each sample path does not converge to any specific point and continues to fluctuate without decaying asymptotically. As in the previous model, the stochastic formulation comes from adding a white noise to the non-predetermined variable.

Slobodyan (1999) treated the indeterminacy in a deterministic continuous-time model with infinitely lived agents, one predetermined variable (the state variable) and one non-predetermined variable (the control variable); the model is characterized by increasing social returns to scale due to externality in the production function of which the agents are assumed to be unaware. There are two steady states: one has zero capital and zero consumption (the origin), while the other is characterized by positive levels of both capital and consumption. For some parameter values, both steady states are indeterminate, and the whole state space is separated into two regions of attraction of the steady states. The basin of attraction of the origin can be regarded as a development trap. Also in this case, the indeterminacy allows for the existence of sunspot equilibria related with the non-predetermined variable. Slobodyan (2009) studied the possibility of "rescuing" an economy from a development trap through sunspot-driven self-fulfilling expectations.

In this work, we construct sunspot equilibria in a deterministic general class of endogenous growth two sector models with externalities in the line of Mulligan and Sala-i-Martin (1993), Venturi (2014) and compare the dynamical situations arising from the different applications: the Lucas model, the Romer model and a disposable - resource endogenous growth two-sector model (Bella 2010).

Following Mulligan and Sala-i-Martin (1993), the endogeneous growth two-sector model is presented in the reduced form; it means considering a three dimensional deterministic continous non - linear differential system, whose steady states correspond to the Balanced Growth Path (BGP), that is all variables grow at a constant rate.

Starting from a three-dimensional standard reduced deterministic model that admits stable cycles (with one predetermined variable and two non-predetermined variables), the model can be reformulated adding a stochastic term to the non-predetermined variables (as a white noise) and transforming it into a stochastic model that leads to indeterminacy, multiple steady stades and bifurcations. If for a given endogenous growth model, there exists a continuum of equilibria in a small neighborhood of a BGP that continues to stay in this neighborhood, it is said that equilibrium is locally indeterminate. If there exists a continuum of equilibria outside a small neighborhood of a BGP, it is said that equilibrium is globally indeterminate (i.e. Hopf cycle or homoclinic orbit; see Mattana, Nishimura, Shigoka 2008).

Following Slobodyan 2009, in our formulation, the stochastic approch suggests a way out from the cycle trap only for disposable resource application (the model has multiple equilibrium points); in the other examples (Lucas, Romer) the model has one only equilibrium point and we have no way out when the initial conditions start inside the stable bounded cycle: this is the so-called poverty development trap.

The paper[1] is organized as follows: the second section analyzes the general economic model; the third introduces the Hopf orbit; the fourth section compares Lucas, Romer and the natural resource model (Bella 2010) and presents some numerical simulations; in the last we show the results and the economic implications of existence of sunspot equilibrium in a natural resource system with externalities.

[1]A short version of this paper is published as Venturi B., Pirisinu A. – "Bifurcation and sunspots in continuous time optimal model with externalities" in *Mola, Conversano, Vichi (Eds.) – Classification, (Big) Data Analysis and Statistical Learning, pag. 233/240 – Springer Nature, Zurich, 2018* – ISBN (978)-3-319-55707-6

2. The economic general model

Now, we consider the deterministic economic general model (Mulligan, Sala-i-Martin 1993, Venturi 2014) that deals with the maximization of an objective function

$$\underset{c(t),\ u(t)}{Max} \int_0^{\infty} U(c) e^{-\rho t} dt \tag{2.1}$$

subject to:

$$\dot{k} = A\left[(r(t)^{\alpha_r} u(t)^{\alpha_u}\right]\left[v(t)^{\alpha_\nu} k(t)^{\alpha_k}\right] \hat{r}(t)^{\alpha_{\hat{r}}} k(t)^{\alpha_{\hat{k}}} - \tau_k k(t) - c(t)$$

$$\dot{r} = B\left[r(t)^{\beta_r}\left((1-u(t))^{\beta_u}\right]\left[(1-v(t))^{\beta_\nu} k(t)^{\beta_k}\right] \hat{r}(t)^{\beta_{\hat{r}}} k(t)^{\beta_{\hat{k}}} - \tau_r r(t)$$

$$k(0) = k_0$$

$$r(0) = r_0$$

where

$$U(c) = \frac{c^{1-\sigma} - 1}{1-\sigma} \tag{2.2}$$

is a standard utility function, c is per-capita consumption, ρ is a positive discount rate and σ is the inverse of the intertemporal elasticity of substitution.

The constraints are two equations related with the growth process of the analyzed economic system.

Notation is as follows: k is a physical capital and r could represent the human capital (see the Lucas model 1998), the knowledge (the Romer model, 1990) or a natural resource (see Bella, 2010); individuals have a fixed endowment of time, normalized to unity at each point in time, which is allocated to physical and to the other capital sector (respectively: human, knowledge or natural resource); α_k and α_r being the private share of physical and the other capital in the output sector, β_k and β_r being the corresponding shares in the second sector, u and v are the fractions of r and k capital used in the final output sector at instant t; conversely, $(1-u)$ and $(1-v)$ are the fractions used in the second sector; the parameters A and B represent the level of the technology in each sector; τ is a depreciation rate; $\alpha_{\hat{k}}$ is a positive externality parameter in the production of physical capital; $\alpha_{\hat{r}}$ is a positive externality parameter in the production of the second sector

The equalities $\alpha_k + \alpha_r = 1$ and $\beta_k + \beta_r = 1$ ensure that there are constant returns to scale at the private level. At the social level, however, there may be increasing, constant or decreasing returns depending on the signs of the externality parameters.

All other parameters $\pi = (\alpha_k, \alpha_{\hat{k}}, \alpha_r, \alpha_{\hat{r}}, \beta_k, \beta_{\hat{k}}, \beta_r, \beta_{\hat{r}}, \sigma, \gamma, \delta, \rho)$ lie inside the following set:

$\Pi \subset (0,1) \times (0,1) \times (0,1) \times (0,1) \times (0,1) \times (0,1) \times (0,1) \times (0,1) \times \mathbb{R}_+^4$.

The representative agent's problem (2.1) - (2.2) is solved by defining the current value Hamiltonian:

$$\begin{aligned} H \quad = \quad & \frac{c^{1-\sigma}-1}{1-\sigma} + \lambda_1(A((r(t)^{\alpha_r} u(t)^{\alpha_u})(v(t)^{\alpha_\nu} k(t)^{\alpha_k})\hat{r}(t)^{\alpha_{\hat{r}}} k(t)^{\alpha_{\hat{k}}} - \tau_k k(t) - c(t)) + \\ & + \lambda_2 B(r(t)^{\beta_r}(1-u(t))^{\beta_u})((1-v(t))^{\beta_\nu} k(t)^{\beta_k} \hat{r}(t)^{\beta_{\hat{r}}} k(t)^{\beta_{\hat{k}}} - \tau_r r(t)) \end{aligned} \tag{2.3}$$

where λ_1 and λ_2 are co-state variables which can be interpreted as shadow prices of the accumulation of the state variables k and r. The solution candidate comes from the first-order necessary conditions (for an interior solution) obtained by means of the Pontryagin Maximum Principle with the usual transversality condition

$$\lim_{t\to\infty} \left[e^{-\rho t}\left(\lambda_1 k + \lambda_2 r\right)\right] = 0 \tag{2.4}$$

Only the competitive equilibrium solution is considered (as well known, it follows from the presence of the externality that the competitive solution differs from the planner's solution[2]).

After eliminating $v(t)$ the rest of the first order conditions and accumulation constraints entail four first order non linear differential equations in four variables: two control variables (c and u) and two state variables (k and r). The solution of this autonomous system is called a Balanced Growth Path (BGP) if it entails a set of functions of time solving the optimal control problem (2.1)-(2.4) such that k, r and c grow at a constant rate and u is constant.

With a change of variable, in standard way, (since k, r and c grow at a constant rate and u is a constant in the BGP), a system of four first order ordinary differential equations in c, u, k and r is transformed into a system of three first order ordinary differential equations with two non-predetermined variables (the control variables) and one predetermined (a linear combination of the state variables).

Setting $A = B = 1$ and

$$x_1 = kr^{\frac{\alpha_{\hat{r}}}{(\alpha_{\hat{r}}-1)}} \quad ; \quad x_2 = u; \quad x_3 = \frac{c}{k} \tag{2.5}$$

it is:

$$\begin{aligned} \dot{x}_1 &= \phi_1(x_1, x_2, x_3) \\ \dot{x}_2 &= \phi_2(x_1, x_2, x_3) \\ \dot{x}_3 &= \phi_3(x_1, x_2, x_3) \end{aligned} \tag{2.6}$$

where the $\phi_i \in R^3$, are countinuos and derivable nonlinear functions,which depend of the parameters $(\alpha_k, \alpha_{\hat{k}}, \alpha_r, \alpha_{\hat{r}}, \beta_k, \beta_{\hat{k}}, \beta_r, \beta_{\hat{r}}, \sigma, \gamma, \delta, \rho)$ of the model, and $\phi_i : U \times R^3 \longrightarrow R^3$ with $U \subset R$, an open subset, and $i = 1, 2, 3$.

3. The emergence of a Hopf orbit in the general model.

As well known a stationary (equilibrium) point of system (2.6) is any solution of

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[2]The planner's solution involves a choice of k, r, c, u, and r_a which maximizes the control optimal model (2.1) and to $r = r_a$ for all t.

In the other hand the path for r coincides with the given path r_a in the competive solution then the system is in equilibrium (see Lucas 1990, Mattana and Venturi, 1999).

The equilibrium solution taking r_a as exogenously determined.

$$\begin{aligned} \dot{x}_1 &= \phi_1(x_1, x_2, x_3) = 0 \\ \dot{x}_2 &= \phi_2(x_1, x_2, x_3) = 0 \\ \dot{x}_3 &= \phi_3(x_1, x_2, x_3) = 0 \end{aligned} \tag{3.1}$$

Assuming the existence of at least one solution, at some point $P^*(x_1^*, x_2^*, x_3^*)$ the local dynamical properties of (2.6) are described in terms of the Jacobian matrix of (2.6), $J(P)$,with $J(P^*) = J^*$, for brevity.

LEMMA 1. *Let* $\pi \in \widehat{\Pi} \subseteq \Pi$ *be. In* $\widehat{\Pi}$ *there is at least one value* $\sigma = \sigma_c$, *(*σ *is the bifurcation parameter*[3]*) such that the Jacobian matrix* J^* *has a pair of purely imaginary roots and a real root different from zero.*

PROOF. By Routh-Hurvitz's criterion we can state that J^* can have one (real) eigenvalue $\lambda_1 = r$ and two complex eigenvalues $\lambda_{2/3} = p \pm qi$ whose real parts can be either positive or negative. The real part of the two complex conjugate roots is a countinous function (a four order polinomial) $G(\sigma)$:

$$G(\sigma) = -B(J^*)Tr(J^*) + Det(J^*) \tag{3.2}$$

that changes sign in $\widehat{\Pi}$ when the parameter σ varies.

In fact, since the real parts of the complex conjugate roots vary continuously with respect to σ, there must exist at least one value $\sigma = \sigma_c$ such that $G(\sigma) = 0$. When this occurs, by Vieta's theorem, J^* has a simple pair of purely imaginary eigenvalues. The solutions of characteristic polinomial

$$-\lambda^3 + Tr(J^*)\lambda^2 - B(J^*)\lambda + Det(J^*) = 0$$

for $\sigma = \sigma_c$ becomes:

$$\lambda_1 = Tr(J^*) \text{ and } \lambda_{2/3} = \pm i\sqrt{B(J)}$$

Q.E.D. □

LEMMA 2. *If* $\pi \in \widehat{\Pi}$, *the derivative of the real part of the complex conjugate eigenvalues with respect to* σ, *evaluated at* $\sigma = \sigma_c$, *is always different from zero.*

PROOF. We have only to verify that the following derivative $\frac{d}{d\sigma}G(\sigma)$ is different from zero in $\widehat{\Pi}$. □

THEOREM 1. *The system (2.6) undergoes at hopf bifurcations in* $\widehat{\Pi}$ *for* $\sigma = \sigma_c$.

PROOF. It follows directly from lemmas 1 and 2 that the assumptions of Hopf Bifurcations theorem are satisfied. □

As well known the study of the stability of the emerging orbits on the center manifold[4], can be performed by calculating the sign of a coefficient q depending on second and third order derivatives of the non-linear part of the system written in normal form.

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[3]The dynamical characteristic of the Jacobian matrix suggests which bifurcation parameter to choose. We choose the parameter σ.

[4]It is a manifold associated with the complex conjugate roots with real part zero.

If $q > 0$ (resp. $q < 0$) then the closed orbits Hopf-bifurcating from the steady state $P_c^*(x_1^*, x_2^*, x_3^*)$ are attracting or *super-critical* (resp. repelling or *sub-critical*) on the center manifold.

4. Stochastic dynamics

Now, it can be built a stochastic system that has sunspot equilibria as solutions: the costruction is very similar to that reported in T. Shigoka (1994) and J. Benhabib, K. Nishimura, and T. Shigoka (2006), J. Benhabib, K. Nishimura and Y. Mitra, (2008).

The system (2.6) includes one predetermined variable x_1 and two non-predetermined variables x_2 and x_3, in vectorial form :

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3)^T = \phi_i \ (x_1, x_2, x_3) \tag{4.1}$$

with $\phi_i : U \times R^3 \longrightarrow R^3$, $U \subset R$, $i = 1, 2, 3$.

If the deterministic dynamic given by (2.7) satisfies the hypothesis of the theorem 1, in other words the parameters set of the model belongs to $\widehat{\Pi}$, then (2.6) has a period solution Γ [5].

The equilibrium is globally indeterminate in the interior of the bounded region enclosed by Γ.

We remember that a probability space is a triple $(\Omega, B_{R^3}, P_{R^3})$ where: Ω denotes the space of events, B is the set of possible outcomes of a random process; B is a family of subsets of Ω that, from a mathematical point of view, represents a σ-algebra[6].

The σ-algebra can be interpreted as information (on the properties of the events)[7].

Now a "noise" (a Wiener process) is added in the equations related with the control variables of the optimal choice problem.

Let $s_t(\omega) = (\omega, t)$ be a random variable irrelevant to fundamental characteristic of the optimal economy, it means that doesn't affect preferences , technology and endowment (i.e. sunspot). We assume that a set of sunspot variable $\{s_t(\omega)\}_{t\ \geq 0}$ is generated by a two-state continuous-time Markov process with stationary transition probabilities and that $s_t : \Omega \longrightarrow \{1, 2\}$ for each $t \geq 0$. Let $[\{s_t(\omega)\}_{t \geq 0}, (\Omega,\ B_{R^3},\ P_{R^3})]$ be a continuous time stochastic process [8], where $\omega \in \Omega$, B_Ω is a σ-field in Ω, and P_Ω is a probability measure.The probability space is a complete measure space and the stochastic process is separable.

Let $(R^3_{+++},\ B_{R^3_{+++}},\ P_{R^3_{+++}})$ be a probability space on the open subset R^3_{+++} of R^3 where $B_{R^3_{+++}}$ denotes the Borel σ-field in R^3_{++}. Let (Φ, B, P) be the product probability space of $(R^3_{+++},\ B_{R^3_{+++}},\ P_{R^3_{+++}})$ and $(\Omega,\ B_\Omega,\ P_\Omega)$, that is $(R^3_{+++} \times \Omega,$

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[5] The Hopf cycle is an invariant set inside a two dimensional manifold: the center manifold.

[6] A σ - algebra differs from an algebra, for the property that the union of infinite elements of the family must belong to the set.

[7] The smallest σ- algebra that can be built with subsets of real numbers is represented by open intervals that are called Borel sets and indicated with B. So, it can be concluded that a probability space is a triple $(\Omega,\ F,\ Pr)$ where Ω denotes the space of events and F a family of subsets of Ω.

[8] A stochastic process is a family of random variables.

$B_{R^3_{+++}} \times B_\Omega$, $P_{R^3_{+++}} \times P_\Omega)$. Let $(\Phi,\ B^*,\ P^*)$ be the completion of (Φ, B, P). Let $(x_0^1,\ x_0^2, x_0^3)$ be the value of our model at the time $t = 0$. Now, a point $(x_0^1,$ $x_0^2, x_0^3, \omega)$ in Φ can be denoted as φ: in other words, $\varphi = (x_0^1,\ x_0^2, x_0^3, \omega)$. Let $B_t = B(x_0^1, x_0^2, x_0^3, s_s, s \leq t)$ the smallest σ-field of φ respect to which $(x_0^1,\ x_0^2, x_0^3)\ s_s, s \leq t$ are measurable.

Let $B_t^* = B^*(x_0^1,\ x_0^2, x_0^3, s_s, s \leq t)$ be the σ-field of φ sets which are either B_t sets or which differ from B_t sets by sets of probability zero.

Let E_t the conditional expectation operator relative to B_t^*.

The following equation is a first order condition of some intertemporal optimization problem with market equilibrium conditions incorporated:

$$(\dot{x}_1, E_t(d\dot{x}_2/dt), E_t(d\dot{x}_3/dt) = \phi_i(x_1(\varphi), x_2(\varphi), x_3(\varphi)) \tag{4.2}$$

where $(x_0^1(\varphi),\ x_0^2(\varphi), x_0^3(\varphi)) = (x_0^1,\ x_0^2, x_0^3)$ and $\frac{dx_{1t}}{dt}$, $\frac{dx_{2t}}{dt}$, $\frac{dx_{3t}}{dt}$ are defined as:

$$\frac{dx_t^i}{dt} = \lim_{h \to 0^+} \frac{(x_{t+h}^i - x_t^i)}{h} (i = 1, 2, 3)$$

if the limit exists.

Definition 1. *Suppose that* $\{(x_{1t}(\varphi), x_{2t}(\varphi), x_{3t}(\varphi))\}_{t\ \geq 0}$ *is a solution of the stochastic differential equation (4.2) with* $(x_{1t}(\varphi), x_{2t}(\varphi), x_{3t}(\varphi)) \in R^3_{+++}$ *. If for any pair* $(t > s \geq 0)$,$(x_{1t}(\varphi), x_{2t}(\varphi), x_{3t}(\varphi))$ *is* B_t^**-measurable but non* B_s^**-measurable it constitutes a sunspot equilibrium.*

Theorem 2. *If the deterministic system (4.1) has a Hopf solution or a homoclinic orbit (a cycle), then a sunspot equilibrium (SE) is a solution of the stochastic process* $\{(x_0^1(\varphi), x_0^2(\varphi), x_0^3(\varphi))\}_{t\ \geq 0}$ *with a compact support.*

4.1. The Lucas model. As an application of the general model indicated above, the Lucas model is consedered. In the original optimal control model, the state variables are: k, the physical capital and $r = h$, the human capital; the control variables are: u, the non-leisure time and c, the consumption.

The deterministic reduced form of this model is given by:

$$\begin{aligned} \dot{x}_1 &= Ax_1^\beta x_2^{1-\beta} + \tfrac{\delta(1-\beta+\gamma)}{\beta}(1 - x_2)x_1 - x_3x_1 \\ \dot{x}_2 &= \tfrac{\delta(\beta-\gamma)}{\beta}x_2^2 + \tfrac{\delta(1-\beta+\gamma)}{\beta}x_2 - x_3x_2 \\ \dot{x}_3 &= -\tfrac{\rho}{\sigma}x_3 + A\tfrac{\beta-\sigma}{\sigma}x_1^{\beta-1}x_2^{1-\beta}x_3 + x_3^2 \end{aligned} \tag{4.3}$$

where: $x_1 = h^{\left(\frac{1-\beta+\gamma}{\beta-1}\right)}k$; $x_2 = u$; $x_3 = \frac{c}{k}$; $\gamma = \alpha_{\hat{h}}$.

The variable x_1 is a pre-determined one (it is a combination of the state variables), while x_2 and x_3 are the non - predetermined variables.

The three dimensional deterministic system (4.3) undergoes a stable Hopf bifurcation. In line with Nishimura Shigoka Yano, the model can be written in the form of a stochastic system and it is possible to verify the above theorem 2. Then the system has a compact sunspot equilibrium as a solution of the stochastic process.

The model has one only equilibrium point and there is no way out when the initial conditions start inside the stable bounded cycle or very close to the boundary.

4.2. The modified Romer model. Another application of the generalized model is the modified Romer model. In the original optimal control model, the state variables are: k, the physical capital and $r = A$,where A is the level of knowledge currently available, the human capital (Romer 1990, Slobodyan 2007); the control variables are: H_Y, is the human capital, the skilled labour employed in the final sector; c, the consumption.

The deterministic reduced form of this model is given by:

$$\begin{aligned}
\dot{x}_1 &= x_1^{\gamma} x_2^{\alpha} - \tfrac{\zeta-\gamma}{1-\gamma}\delta(1-x_2)x_1 - x_3 x_1 \qquad (4.4)\\
\dot{x}_2 &= \tfrac{\gamma(\zeta-\gamma)}{\zeta(1-\alpha)} x_1^{\gamma-1} x_2^{\alpha+1} + \tfrac{\delta(\zeta-\gamma-1)}{1-\alpha} x_2 - \tfrac{\delta}{1-\alpha}(1-\zeta+\gamma+\tfrac{\gamma}{\zeta\alpha}(\zeta-\gamma))x_2^2 - \tfrac{\gamma}{1-\alpha} x_2 x_3\\
\dot{x}_3 &= x_3^2 + (\tfrac{\gamma^2}{\sigma\zeta} - 1)x_1^{\gamma-1} x_2^{\alpha} x_3 - \tfrac{\rho}{\sigma} x_3
\end{aligned}$$

where: $x_1 = A^{\left(\frac{\alpha+\beta+\gamma}{\alpha+\beta}\right)} k$; $x_2 = H_Y$; $x_3 = \frac{c}{k}$; $\gamma = \alpha_{\underset{h}{\wedge}}$.

x_1 is a pre-determined variable (it is a combination of the state variables), while x_3 and x_2 are the non pre-determined variables; the parameters $\alpha + \beta + \gamma = 1$. and $\zeta \geq 1$ is a parameter that captures the degree of complementarity between the inputs (the case $\zeta = 1$ corresponds to non complementarity).

The three dimensional deterministic system (4.4) undergoes a stable Hopf bifurcation in a parameter set. In line with Nishimura Shigoka Yano (2006), we can re-write the model in the form of a stochastic system and we can apply the theorem 2. Then the system has a compact sunspot equilibrium as a solution of the stochastic process.

The model has one only equilibrium point and there is no way out when the initial conditions start inside the stable bounded cycle or very close to the boundary.

4.3. The natural resource model. The natural disposal resource system includes two non-predetermined variables x_1 and x_2 and one, predetermined variable x_3.:

$$\begin{aligned}
\dot{x}_1 &= (-\tfrac{\rho}{\sigma})x_1 + (\tfrac{\beta-\sigma}{\sigma})x_1 - x_1^2\\
\dot{x}_2 &= (\tfrac{\gamma\delta}{\beta})(1-x_2)x_2 + x_1 x_2 \qquad (4.5)\\
\dot{x}_3 &= (\tfrac{\gamma\delta}{\beta})((1-x_2)x_3 + (\beta-1)x_3^2
\end{aligned}$$

where: $x_1 = \frac{c}{k}$; $x_2 = nr$; $x_3 = \frac{y}{k}$.

A stochastic process can be built considering the following equation:

$$(E_t(d\dot{x}_1/dt), E_t(d\dot{x}_2/dt), \dot{x}_3) = \phi_i(x_1(\varphi), x_2(\varphi), x_3(\varphi)) \qquad (4.6)$$

where $(x_0^1(\varphi),\ x_0^2(\varphi), x_0^3(\varphi)) = (x_0^1,\ x_0^2, x_0^3)$ and $\frac{dx_{1t}}{dt}, \frac{dx_{2t}}{dt}, \frac{dx_{3t}}{dt}$ are defined as $\frac{dx_t^i}{dt} = \underset{h\to 0^+}{lim} \frac{(x_{t+h}^i - x_t^i)}{h}$ $(i = 1, 2, 3)$ if the limit exists.

In order to analyze the local dynamical properties of (4.5), first we find the stationary (equilibrium) points of the system (4.5) , which are any solution of:

$$(-\tfrac{\rho}{\sigma})x_1 + (\tfrac{\beta-\sigma}{\sigma})x_1 - x_1^2 = 0$$

$$(\tfrac{\gamma\delta}{\beta})(1-x_2)x_2 + x_1 x_2 = 0$$

$$(\tfrac{\gamma\delta}{\beta})((1-x_2)x_3 + (\beta-1)x_3^2 = 0$$

There are eight steady states values:

1) $P_1^*(0,0,0)$
2) $P_2^*\left(0,0,\left(\frac{\gamma\delta}{\beta(1-\beta)}\right)\right)$
3) $P_3^*(0,1,0)$ (double solution)
4) $P_4^*(\frac{\rho}{\sigma},0,0)$
5) $P_5^*\left(\frac{1}{\sigma}\left(\rho-\frac{\gamma\delta(\beta-\sigma)}{\beta(1-\beta)}\right),0,\left(\frac{\gamma\delta}{\beta(1-\beta)}\right)\right)$
6) $P_6^*(\frac{\rho}{\sigma},1-\frac{(\beta\rho)}{\sigma\gamma\delta},0)$
7) $P_7^*(\frac{\rho(1-\beta)}{\beta(1-\sigma)},1-\frac{\rho(1-\beta)}{\gamma\delta(1-\sigma)},\frac{\rho}{\beta(1-\sigma)})$

It is well-known that many theoretical results related to the system depend upon the eigenvalues of the Jacobian matrix evaluated at the stationary point P_i^* with $i=1,2,3,4,5,6,7$.

The local bifurcation analysis permits to determine the structural stabililty of the solutions of the model. The Jacobian matrix associated with the system (4.5) is:

$$J(P)=\left[\begin{array}{c|c|c} \frac{1}{\sigma}(-\rho+2\sigma x_1+ +(\beta-\sigma)x_3) & 0 & \frac{1}{\sigma}x_1(\beta-\sigma) \\ \hline -x_2 & \frac{1}{\beta}(\gamma\delta(1-2x_2)+ +\beta x_1) & 0 \\ \hline 0 & \frac{1}{\beta}\gamma\delta x_3 & \frac{1}{\beta}\gamma\delta(1-x_2)+ +2(\beta-1)x_3 \end{array}\right] \tag{4.8}$$

Now, considered the function:

$$G(\sigma)=-B(J^*)Tr(J^*)+Det(J^*)$$

(where B is the sum of the principal minors, Tr is the trace and Det is the determinant of the Jacobian matrix). According to the theorem 1 (see section 3), there is a parameter value $\sigma=\sigma_c$ such that, in the steady state, the Jacobian Matrix $J(P^*)$ possesses two complex conjugated roots with real part equal to zero and the real root different from zero: this happens only for the steady state P_7^* (in fact, the Jacobian evaluated in the steady states 1 to 6 becomes triangular, leaving no possibilities of bifurcations). So, given the function:

$$G(\sigma)=-B(J_{P_7^*})Tr(J_{P_7^*})+Det(J_{P_7^*}) \tag{4.9}$$

the invariant elements of the Jacobian are:

$Tr(J_{P_7^*})=\frac{\rho(1-\beta)}{\beta(1-\sigma)}-\frac{\gamma\delta}{\beta}$;
$B(J_{P_7^*})=-\left(\frac{\rho(1-\beta)}{\beta(1-\sigma)}\right)$;
$Det(J_{P_7^*})=\frac{\rho^2\gamma\delta(1-\beta)}{\beta^2\sigma(1-\sigma)}\left(1-\frac{\rho(1-\beta)}{\gamma\delta(1-\sigma)}\right)$.

It is possible to determine two solutions in which the function $G(\sigma)$ vanishes:

$$\begin{aligned} \sigma_c^* &= \beta \\ \sigma_c^{**} &= \frac{\gamma\delta-\rho(1-\beta)}{\gamma\delta} \end{aligned} \tag{4.10}$$

The derivative of the real part of the complex conjugate eigenvalues of $(J_{P_7^*})$ evaluated at $\sigma = \sigma_c^*$ and $\sigma = \sigma_c^{**}$, is always different from zero: then, a family of Hopf bifurcations emerges around the steady state P_7^* of the system (4.5), for $\sigma = \sigma_c^*$ and $\sigma = \sigma_c^{**}$.

Now, we consider some numerical simulations: in particular, consider the following two sets of parameters (see [2]):
set a) $\rho = 0.002,\ \beta = 0.66, \gamma = 2, \delta = 0.04, \sigma_c^* = 0.66;$
set b) $\rho = 0.002, \beta = 0.66, \gamma = 2, \delta = 0.04, \sigma_c^{**} = 0.975.$

We evaluate the growth rate of the economy and get: $\mu = -\frac{\rho}{\sigma - \beta\gamma + \beta^2}$.
In a first simulation, fixed $\beta = 0.66, \gamma = 2$, we consider μ as a function of σ :

$$\mu = f(\sigma) = -\frac{0{,}002}{\sigma - 2 \cdot 0{,}66 + 0{,}66^2}$$

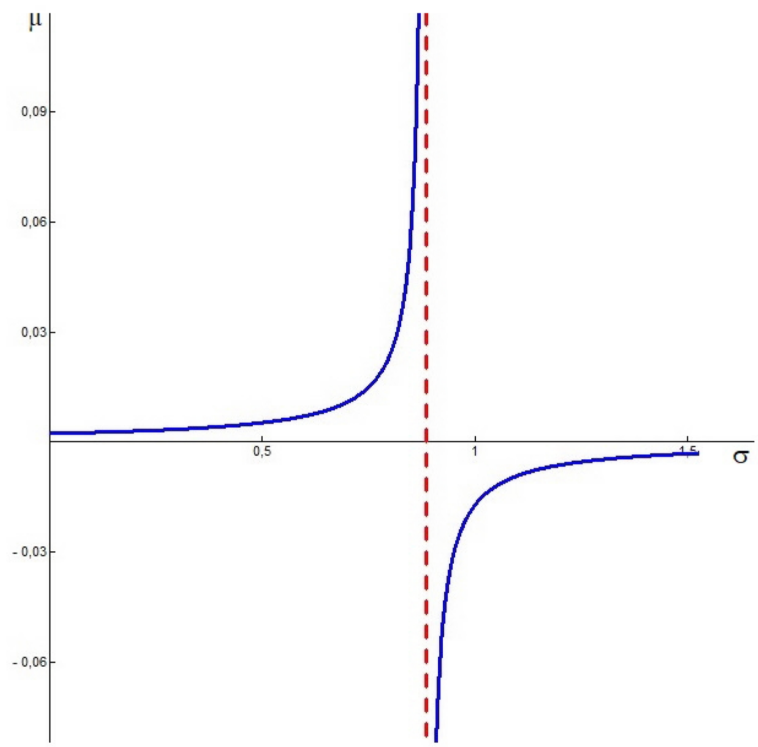

Fig. 1: $\mu = f(\sigma)$

The growth rate is positive for $\sigma < 0.8844$.

A second simulation shows the growth rate as function of β and σ:
$\mu = f(\sigma, \beta) = \frac{0.002}{2\beta - \beta^2 + \sigma}$

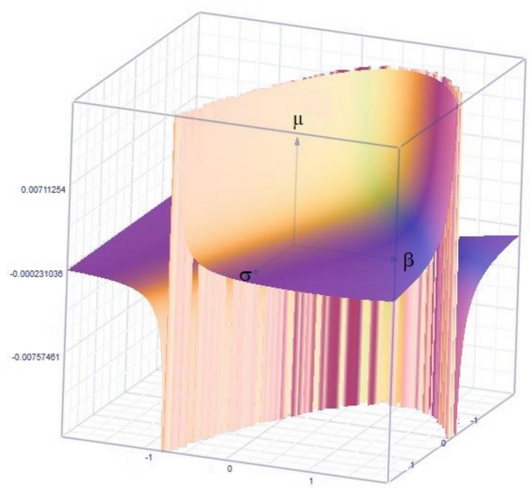

Fig. 2: $\mu = f(\sigma, \beta)$

The last simulation shows the growth rate as function of γ and σ :
$\mu = f(\sigma, \gamma) = \frac{0.002}{0.66\gamma - \sigma - 0.4356}$

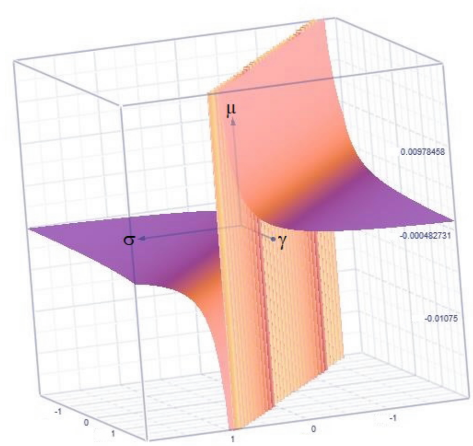

Fig. 3: $\mu = f(\sigma, \gamma)$

The calculation of the stability coefficient q (as in Bella, 2010) gives some results for our parameter sets; for the set a), it is $q = -2.40 \cdot 10^{12} < 0$; it means that bifurcation is super-critical, the steady state is unstable and the periodic orbits are attracting on the center manifold. For the set b), it is $q = 8.37 \cdot 10^{14} > 0$; it means that bifurcation is sub-critical and the periodic orbits are repelling.

The deterministic equilibrium dynamic (4.5) has a family of periodic orbits Γ_{σ_c} emerging from one steady state, with Γ_{σ_c} in the center manifold (a two-dimensional invariant manifold in R^3_{+++}). For some set of parameters in the model, there exists a sunspot equilibrium whose support is located in the bounded region enclosed by the periodic orbit Γ_{σ_c}. Each sample path of the sunspot equilibrium does not converge to any specific point and continues to fluctuate without decaying asymptotically.

For some parameter value, due to pessimistic self-fulfilling expectations, sunspot equilibria exist in some neighbourhood of a steady-state. If the periodic orbit emerging from a steady state is super-critical, there is no way out (Slobodyan, 2007). If the periodic solution is repelling, then there is a possibility of a way out of the orbit and the optimal path can reach another steady state. Such situation can be understood as a poverty or development trap.

5. Conclusions

In the applicative examples here proposed, we analyzed different aspects.

In Lucas model, there are two main causes of endogeneous growth: the first is the accumulation of human capital, that is the endogeneous growth is due to the fact that the factors determining human capital accumulation remain unchanged; the second is the presence of the externality: it is not necessary to have endogenous growth but it works as an incentive to accumulate human capital and not to let it decrease as time passes by.

In his models, Romer (1986) describes a system in which capital that has decreasing returns to scale on the microeconomic level but increasing returns on the macroeconomic level, due to spillovers; so it predicts positive sustained per capita growth. On the other hand, Romer (1990) puts knowledge (and the technological change that is its fruit) at the heart of economic growth: it provides the incentive for capital accumulation and accounts for much of the increase in output per hour worked. In this line, authors like Sala-i-Martin (1997) have shown that the investment share is a robust variable in explaining economic growth.

In the model of natural resource, the endogeneous growth and the chance to escape the development trap are strictly bounded to the expectations: positive or negative outlook can crucially determinate the growth of the economic system.

This positive and statistically significant effect of investment on the growth rate of countries suggests that investment not only affects the stock of physical capital but also increases intangible capital (for example, knowledge) in a way such that the social return to investment is larger than the private return.

For real-world economies, it's important to underline that, in case of endogenous growth, the balanced growth rate crucially depends on the marginal product of physical capital, which varies positively with the stock of knowledge capital. Thus, when the level of knowledge capital plays its important role, economies with a large stock of knowledge may compensate for a large stock of physical capital. Consequently, the growth rate will be higher in those countries in which the stock of knowledge capital is relatively large: this fact can explain high growth rates of Germany and Japan after World War II, for example.

More generally, economies may be both globally and locally indeterminate. Global indeterminacy refers to the balanced growth rate that is obtained in the long run and states that the initial value of consumption crucially determines to which BGP the economy converges and, thus, the long-run balanced growth rate.

Moreover, local indeterminacy around the BGP with the lower growth rate, can be observed if the parameter constellation is such that the trace of the Jacobian matrix is smaller than zero, so that the eigenvalues of the Jacobian matrix have negative real parts. If in that situation a certain parameter is varied, two purely imaginary eigenvalues may be observed that generate a Hopf bifurcation, which leads to stable limit cycles.

Some other authors have studies both development and poverty traps: it is indicated that poverty traps and indeterminacy in macroeconomic models may be caused by the same set of reasons, like externalities or increasing returns to scale. Among many, Slobodyan (1999) tried to understand, in this framework, how important sunspot-driven fluctuations could be for the economy's escape from the poverty trap: for a chosen level of the noise intensity (approximately 14% SD of the log consumption), the probability of escaping the trap is not negligible, only when the initial condition is very close to the trap boundary. The set of those initial conditions is not very large and is restricted to initial level of consumption, within 85% of the level necessary to put the system right on the boundary between the poverty trap and the region of attraction of the positive steady state.

The probability of escape, as expected, increases as expectations become more optimistic: for very optimistic expectations (i.e., initial consumption very close to the boundary) absolute majority of escapes happens very fast. So the economy that starts with a very low initial capital and very pessimistic expectations of future interest rates and wages gets trapped. It will probably continue the downward spiral (the change from "pessimistic" level of consumption to the "optimistic" one may constitute hundreds and thousands percent of the "pessimistic" level). The escape happens if it chosen a random variable with bounded support, as the sunspot variable. The sunspot variable has a natural interpretation of a change in perceived present discounted wealth.

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CHAPTER 3

Sunspots in a optimal resource control model with externalities

1. Introduction

A great path of research in the economic field regards the effects of those phenomena known as externalities, that is the unwanted effects of actions that create a difdference between private and social costs, in Pigou's definition. The externalities, positive or negative, give rise to direct effects on agents and allocations of the ecnomic system, which are not "registered" by the setup of market prices.

This paper analyzes an optimal control model with externalities: we point out the consequences of a higher degree of exploitation of natural resources (as it is for wood, for example, used far beyond the threshold of re-generation). These phenomena can also be seen in the shorter and shorter obsolescence of many products, in particular of electronic devices (computer, moile phones, as well as tv sets): this is sometimes called "programmed obsolescence" and it's a crucial factor in the major consumption of raw materials and in the growing pollution from the electronic waste. In Wirl (2004) such situation of growing pollution leads to highly negative externalities that can be easily understood in terms of higher costs of contrasting pollution and renewing (or preserving) the reserves of natural resources.

He analyzes stable limit cycles as optimal long-run effects of intertemporal policy and finds multiple equilibria separated by a threshold curve in the state plane and a steady state that is between the traditional long run harvesting rule and the maximum sustainable yield.

Antoci Galeotti Russu (2011 JET), following the framework used by Wirl, globally analyze an economic growth model with environmental negative externalities; they find two stationary states: the first is in fact, a poverty trap; the second has saddle-point stability. The model exhibits global indeterminacy, since either the first or the second state can be selected according to agent expectations. More, there is a chance for existence two limit cycles near the two stationary states so that, even starting from the same initial values the economy may approach either of the two equilibrium points. Numerical simulations suggest that, even if the economy starts very close to the saddle-point, locally determinate, it can move quite far away from it, in particular toward the poverty trap.

Bella (2010) studies the presence of closed orbits that signal economic fluctuations around the steady state, due to the exploitation of natural resources; this leads, in turn, to an indeterminate equilibrium and suggests the emergence of a poverty-environment trap, related to the degree and the extent of the resorce utilization.

Kogan (2014) investigates the impact of negative externalities, resulting from the transboundary pollution; to do so, he illustrates a two-player differential game model of pollution, that contemplates possibility of changing of the biosphere from a carbon sink to a carbon source.

Our paper follows these lines of research but imposing a different second constraint, a third-degree function describing the dynamics of the natural resource. The economic reason behind this phenomenon is to be found in the highly rising demand of raw material from industrialized countries. None the less, this leads to an increasing pollution and a quick depletion of reserves.

By using bifurcation theory we prove that the model undergoes a Hopf bifurcations in some parameter set. Following Benhabib and Nishimura (2006) we are able to show that the 3-dimensional, continuous-time, reduced form of this economic system possesses stochastic characteristics which arise from indeterminate equilibrium close to steady state and Hopf cycle (see Chiappori and Guesnerie, 1991, Benhabib, Nishimura, 2006, Slobodyan 2009). In our formulation, the stochastic approach suggests a way out from the cycle trap (the poverty enviroments trap, see Slobodyan 2009).

The paper[1] develops as follows: the second section analyses the economic model; the third and fourth sections regard the steady states and the Hopf bifurcations; the fifth section illustrates the stochastic dynamics of the model; then, the sixth section presents some numerical simulations and the last section is displays the results and the economic implications of existence of sunspots equilibrium in a natural resource system with externalities.

2. The model

We consider a natural resource optimal control model with externalities. It deals with the maximization of a standard utility function with constraints:

$$(P) \qquad \begin{array}{ll} \underset{c(t),\ n(t)}{Max} & \int_0^\infty \frac{c^{1-\sigma}-1}{1-\sigma} e^{-\rho t} dt \\ subject\ to & \dot{k} = Ak^\beta \left(nr\right)^{1-\beta} r_a^\gamma - c \\ & \dot{r} = \delta r \left[1 - (nr)^2\right] \\ and & k(0) = k_0 \\ & r(0) = r_0 \end{array}$$

where: c is per-capita consumption; $\rho \in \mathbb{R}_{++}$ is a positive discount rate; σ is the inverse of the intertemporal elasticity of substitution; k is the physical capital; $A \in \mathbb{R}_{++}$ is a measure of the stock of existing technology (without loss of generality, we put $A = 1$); r is the stock of the renewable natural resource; n is a fraction of r, $n \in [0, 1]$; $\delta \in \mathbb{R}_{++}$ is the growth rate of the variable r; r_a represents an external effect due to the presence of a common pool natural resource; $\gamma \in \mathbb{R}$ is a parameter, the esponent of r_a. The vector of parameters $\theta \equiv (\beta, \delta, \gamma, \rho, \sigma)$ lies inside $\Theta = \mathbb{R}^2_{++} \times \mathbb{R} \times \mathbb{R}^2_{++} \times (0, 1)$.

The state variables of the optimal control model are k and r. The control variables are c and n.

The solution candidates for the problem $\mathcal{P}$ can be obtained by means of the Pontryagin Maximum Principle with the usual transversality condition:

[1]Presented as: Venturi B, Pirisinu A. "Sunspot in a resource optimal control model with externalities" at the XLI AMASES Meeting – Cagliari, Sept. 14-16, 2017

$$\lim_{t\to\infty}\left[e^{-\rho t}\left(\lambda_1 k+\lambda_2 r\right)\right]=0. \tag{2.1}$$

We consider only the competitive equilibrium solution (as well known, it follows from the presence of the externality that the competitive solution differs from the planner's solution[2]) in equilibrium $r=r_a$ for all t.

In standard way, we introduce the discounted Hamiltonian:

$$\mathcal{H}=\frac{c^{1-\sigma}-1}{1-\sigma}+\lambda_1\left[k^{\beta}\left(nr\right)^{1-\beta}r_a^{\gamma}-c\right]+\lambda_2\delta r(1-(nr)^2) \tag{2.2}$$

where λ_1 and λ_2 are the costate variables. The necessary first order conditions are:

$$\frac{\partial\mathcal{H}}{\partial c}=0 \quad c^{-\sigma}=\lambda_1 \tag{2.2.a}$$

$$\frac{\partial\mathcal{H}}{\partial n}=0 \implies 2\delta n^2 r^2\lambda_2=\lambda_1(1-\beta)k^{\beta}n^{1-\beta}r^{1-\beta+\gamma-1} \tag{2.2.b}$$

$$-\frac{\partial\mathcal{H}}{\partial k}=\dot{\lambda}_1-\rho\lambda_1 \Longrightarrow \dot{\lambda}_1=\rho\lambda_1-\lambda_1\beta k^{\beta-1}n^{1-\beta}r^{1-\beta+\gamma} \tag{2.2.c}$$

$$\begin{aligned}-\tfrac{\partial\mathcal{H}}{\partial r}=\dot{\lambda}_2-\rho\lambda_2 \Longrightarrow \dot{\lambda}_2&=\\ =\rho\lambda_2-\lambda_1(1-\beta+\gamma)k^{\beta}n^{1-\beta}r^{-\beta+\gamma}&-\lambda_2\delta\left[1-3\left(rn\right)^2\right]\end{aligned} \tag{2.2.d}$$

Since the (not maximized) Hamiltonian is jointly concave in both its state and control variables, by Mangasarian's condition the first order conditions are also sufficient for the existence of an interior solution of the problem $\mathcal{P}$.

By eliminating the costate variables λ_1 and λ_2, we can re-write the equations of the optimal control problem in terms of a system of four non-linear differential equations in the state (k and r) and control variables (c and n):

$$\begin{aligned}\dot{k}&=k\left(k^{\beta-1}\left(nr\right)^{1-\beta}r^{\gamma}-\tfrac{c}{k}\right)\\ \dot{r}&=r\delta\left[1-(nr)^2\right]\\ \dot{c}&=c\left(-\tfrac{\rho}{\sigma}+\tfrac{\beta}{\sigma}k^{\beta-1}n^{1-\beta}r^{1-\beta+\gamma}\right)\\ \dot{n}&=n\left[-\tfrac{\beta}{(\beta+1)}\tfrac{c}{k}+\tfrac{\delta(-\beta+\gamma-1)}{(\beta+1)}+\delta(nr)^2\left(\tfrac{(1-\beta+\gamma)}{(1-\beta)}\right)\right]\end{aligned} \tag{2.3}$$

We call a Balanced Growth Path (BGP) a solutions of the system (2.3) in which the growth rate of k, r, c and n are constant.

---

[2]The planner's solution involves a choice of k, r, c, n which solves the optimal control problem (2.1) and to $r=r_a$ for all t.

On the other hand, the path of r_a coincides, in equilibrium, with the path of r (see Lucas 1990, Mattana and Venturi, 1999).

The equilibrium solution takes r_a as exogenously determined.

We can write the above conditions in term of variables' growth rates and, by using the following substitution:

$$x_1 = \frac{c}{k};\ x_2 = nr;\ x_3 = \frac{y}{k} \tag{2.4}$$

where $y = k^{\beta}(nr)^{1-\beta} r_a^{\gamma}$ is the production function. We get a deterministic system of three differential equations with two non-predetermined variables and one predetermined variable. At the end with some computation, we find the simpler form:

$$\begin{aligned} \dot{x}_1 &= x_1\left[x_1 + \left(\tfrac{1}{\sigma} - 1\right)x_3 - \tfrac{\rho}{\sigma}\right] \\ \dot{x}_2 &= x_2\left[-\tfrac{\beta}{(\beta+1)}x_1 + \tfrac{\delta\gamma}{1-\beta}x_2^2 + \tfrac{\delta\gamma}{\beta+1}\right] \\ \dot{x}_3 &= x_3\left[\tfrac{1-\beta}{\beta+1}x_1 - (1-\beta)x_3 + \tfrac{2\delta\gamma}{\beta+1}\right] \end{aligned} \tag{2.5}$$

3. Steady states analysis

A stationary (equilibrium) point $P^*$of the system (2.5) is any solution of:

$$\begin{aligned} &x_1\left[x_1 + \left(\tfrac{1}{\sigma} - 1\right)x_3 - \tfrac{\rho}{\sigma}\right] = 0 \\ &x_2\left[-\tfrac{\beta}{(\beta+1)}x_1 + \tfrac{\delta\gamma}{1-\beta}x_2^2 + \tfrac{\delta\gamma}{\beta+1}\right] = 0 \\ &x_3\left[\tfrac{1-\beta}{\beta+1}x_1 - (1-\beta)x_3 + \tfrac{2\delta\gamma}{\beta+1}\right] = 0 \end{aligned} \tag{3.1}$$

We find eight steady states values; the eight solutions are:

(3.2)

$1)P^*1 = (0,0,0)$

$2)P^*2 = \left(0, 0, \frac{2\delta\gamma}{1-\beta^2}\right)$

$3)P^*3 = \left(\frac{\rho}{\sigma}, 0, 0\right)$

$4)P^*4 = \left[\frac{1}{\sigma}\left(\rho - \frac{(1-\sigma)(1-\beta+2\sigma\delta\gamma)}{(1-\beta)(1-\sigma\beta^2)}\right), 0, \frac{1-\beta+2\sigma\delta\gamma}{(1-\beta)(1-\sigma\beta^2)}\right]$

$5-6)P^*5, P^*6 = \left[\frac{\rho}{\sigma}, \pm\sqrt{\frac{1-\beta}{1+\beta}\left(\frac{\beta\rho}{\delta\gamma\sigma} - 1\right)}, 0\right]$

not admissible because $\left(\frac{\beta\rho}{\delta\gamma\sigma} - 1\right) < 0$

$$7-8)P^*7, P^*8 = \left[\begin{array}{c} \frac{1}{\sigma}\left(\rho - \frac{(1-\sigma)(1-\beta+2\sigma\delta\gamma)}{(1-\beta)(1-\sigma\beta^2)}\right), \\ \pm\sqrt{\frac{1-\beta}{1+\beta}\left[\frac{\beta}{\delta\gamma\sigma}\left(\rho - \frac{(1-\sigma)(1-\beta+2\sigma\delta\gamma)}{(1-\beta)(1-\sigma\beta^2)}\right) - 1\right]}, \frac{1-\beta+2\sigma\delta\gamma}{(1-\beta)(1-\sigma\beta^2)} \end{array}\right];$$

not admissible because: $\left[\frac{\beta}{\delta\gamma\sigma}\left(\rho - \frac{(1-\sigma)(1-\beta+2\sigma\delta\gamma)}{(1-\beta)(1-\sigma\beta^2)}\right) - 1\right] < 0$

The admissible solutions are:

(3.3)
$$
\begin{aligned}
&P^*1 = (0,0,0)\\
&P^*2 = \left(0,0,\tfrac{2\delta\gamma}{1-\beta^2}\right)\\
&P^*3 = \left(\tfrac{\rho}{\sigma},0,0\right)\\
&P^*4 = \left[\tfrac{1}{\sigma}\left(\rho-\tfrac{(1-\sigma)(1-\beta+2\sigma\delta\gamma)}{(1-\beta)(1-\sigma\beta^2)}\right),0,\tfrac{1-\beta+2\sigma\delta\gamma}{(1-\beta)(1-\sigma\beta^2)}\right]
\end{aligned}
$$

Let $J = J(P)$ denote the Jacobian matrix J of the system (2.5) evaluated at the point $P(x_1, x_2, x_3)$:

(3.4)
$$
J(P) = \begin{bmatrix}
2x_1 + \frac{1-\sigma}{\sigma}x_3 - \frac{\rho}{\sigma} & 0 & \frac{1-\sigma}{\sigma}x_1\\
-\frac{\beta}{\beta+1}x_2 & -\frac{\beta}{\beta+1}x_1 + \frac{3\gamma\delta}{1-\beta}x_2^2 + \frac{\delta\gamma}{\beta+1} & 0\\
\frac{1-\beta}{\beta+1}x_3 & 0 & \frac{1-\beta}{\beta+1}x_1 - 2(1-\beta)x_3 + \frac{2\delta\gamma}{\beta+1}
\end{bmatrix}
$$

Let $J(P^*i) = J_{P^*i}$ the Jacobian matrix in stationary point $P^* \equiv \left(x_1^*, x_2^*, x_3^*\right)$ for some values of the parameters.

Let furthermore

(3.5)
$$
Det\left(\lambda\mathbf{I}-\mathbf{J}\right) = \lambda^3 - Tr(\mathbf{J})\lambda^2 + B(\mathbf{J})\lambda - Det(\mathbf{J})
$$

be the characteristic polynomial of $\mathbf{J}(P)$**,** where $\mathbf{I}$ is the identity matrix and $Tr(\mathbf{J})$, $Det(\mathbf{J})$ and $B(\mathbf{J})$, are trace, determinant and sum of principal minors of order 2 of $\mathbf{J}$, respectively. It is well-known that many theoretical stability results relating to the system (2.5) depend upon the sign of the eigenvalues λ_i solutions of polynomial characteristic evaluated in the stationary points P_i^* (with $i = 1,2,3,4$).

With the aim to simplify the notations, we re-write the Jacobian $\mathbf{J}(P)$ in the following easy way:

(3.6)
$$
\mathbf{J} = \begin{bmatrix} J_{11} & 0 & J_{13}\\ J_{21} & J_{22} & 0\\ J_{31} & 0 & J_{33}\end{bmatrix}
$$

where:

$$J_{11} = 2x_1 + \frac{1-\sigma}{\sigma}x_3 - \frac{\rho}{\sigma}$$

$$J_{13} = \frac{1-\sigma}{\sigma}x_1$$

$$J_{21} = -\frac{\beta}{\beta+1}x_2$$

$$J_{22} = -\frac{\beta}{\beta+1}x_1 + \frac{3\gamma\delta}{1-\beta}x_2^2 + \frac{\delta\gamma}{\beta+1}$$

$$J_{31} = \frac{1-\beta}{\beta+1}x_3$$

$$J_{33} = \frac{1-\beta}{\beta+1}x_1 - 2(1-\beta)x_3 + \frac{2\delta\gamma}{\beta+1}$$

The invariants of the matrix $\mathbf{J}(P)$ are:

$$Det(J) = J_{22}(J_{11}J_{33} - J_{13}J_{31})$$

$$Tr(J) = (J_{11} + J_{22} + J_{33})$$

$$B(J) = (J_{11}J_{22} + J_{22}J_{33} + J_{11}J_{33} - J_{31}J_{13})$$

We determine the qualitative dynamics of the model around each of the steady states P_i^*(with $i = 1, 2, 3, 4$).

We consider the following two subsets of parameters:

$$\Theta_1 = \left\{\theta \in \Theta : \rho \in R_{++};\ 0 < \sigma < 1;\ \delta \in\ R_{++};\ \gamma < -\frac{(1-\beta)}{2\sigma\delta},\ \gamma \in R_-\right\}$$

$$\Theta_2 = \left\{\theta \in \Theta : \rho \in R_{++};\ 0 < \sigma < 1; \delta \in\ R_{++}; \gamma > -\frac{(1-\beta)}{2\sigma\delta}\right\}$$

If $\theta \in \Theta_1$, then $\gamma < -\frac{(1-\beta)}{2\sigma\delta}$; we have different equilibria; it depends on the steady state we consider:

i. $P^*1 = (0, 0, 0)$
   in this case we have always three real negative eigenvalues. Then, this is a stable equilibrium;
ii. $P^*2 = \left(0, 0, \frac{2\delta\gamma}{1-\beta^2}\right)$
   we have two real negative eigenvalues and one positive real eigenvalue. We have a continuum of equilibrium trajectories approaching the steady state P^*2 (indeterminacy);
iii. $P^*3 = \left(\frac{\rho}{\sigma}, 0, 0\right)$
   we have two real positive eigenvalues and one negative real eigenvalue.; we have a unique equilibrium trajectory approaching the steady state P^*3;
iv. $P^*4 = \left[\frac{1}{\sigma}\left(\rho - \frac{(1-\sigma)(1-\beta+2\sigma\delta\gamma)}{(1-\beta)(1-\sigma\beta^2)}\right), 0, \frac{1-\beta+2\sigma\delta\gamma}{(1-\beta)(1-\sigma\beta^2)}\right]$

If $\gamma < -\frac{(1-\beta)}{2\sigma\delta}$, then, in an easy way, we can see that: $a = J_{11} > 0; c = J_{13} > 0; g = J_{31} < 0; e = J_{22} < 0; h = J_{33} < 0$

It follows that:

$ah < 0;\ cg < 0 \qquad (a-h)^2 + 4cg < 0;$

$Det(J) = (ah - cg)e > 0$
$B(J) = (Tr(J))\,e - e^2 + \frac{DetJ}{e} > 0 \Longrightarrow (Tr(J))\,e > e^2 - \frac{DetJ}{e}$
$Tr(J) < 0 \Longrightarrow a < -(h+e)$

From a dynamic point of view, the case iv is interesting because the invariants of the Jacobian matrix change sign in the set θ_1 in a way that allows a topological structural change. By using an application of the general theorem due to Routh (see Benhabib - Perli, 1994), the number of roots of the characteristic polynomial with real positive part is given by the number of sign variations of the coefficients and the number of negative roots is given by the number of sign permanences, as shown in the following chart:

-1	$Tr(J)$	$-B(J)+\frac{Det(J)}{Tr(J)}$	$Det(J)$
$-$	$-$	$-$	$+$

There are 2 permanences of sign and 1 variation: then, the eigenvalue λ_1 has real positive part and eigenvalues $\lambda_{2/3}$ have real negative part;

$$Det(J) = (ah - cg)e > 0$$
$$B(J) = (Tr(J))\, e - e^2 + \frac{DetJ}{e} > 0 \Longrightarrow (Tr(J))\, e > e^2 - \frac{DetJ}{e} \qquad B(J) > 0$$
$$Tr(J) > 0 \Longrightarrow a > -(h+e)$$

-1	$Tr(J)$	$-B(J)+\frac{Det(J)}{Tr(J)}$	$Det(J)$
$-$	$+$	$-$	$+$

3 variations λ_1 = positive $\lambda_{2/3}$ = real positive part
we must have : $(a-h)^2 < -4cg$

We have two subsets Θ_1^A and Θ_1^B where we can have either one real positive root and two roots with negative real part (when the trace is negative) or one real positive root and two roots with positive real part, (when the trace is positive).

Let $\theta \in \Theta_2$; then $\gamma > -\frac{(1-\beta)}{2\sigma\delta}$.We have different equilibria, depending on the steady state considered:

i. $P^*1 = (0,0,0)$

ii. $P^*2 = \left(0, 0, \frac{2\delta\gamma}{1-\beta^2}\right)$

We have two real negative eigenvalues and one positive real eigenvalue. a continuum of equilibrium trajectories approaching the steady state P^*2 (indeterminacy).

iii. $P^*3 = \left(\frac{\rho}{\sigma}, 0, 0\right)$

We have two real positive eigenvalues and one negative real eigenvalue. We have a unique equilibrium trajectory approaching the steady state P^*3 (See BP 1994)

iv. $P^*4 = \left[\frac{1}{\sigma}\left(\rho - \frac{(1-\sigma)(1-\beta+2\sigma\delta\gamma)}{(1-\beta)(1-\sigma\beta^2)}\right), 0, \frac{1-\beta+2\sigma\delta\gamma}{(1-\beta)(1-\sigma\beta^2)}\right]$

We have two subsets Θ_2^A and Θ_2^B where we can have either three real negative eigenvalues (when $-\frac{(1-\beta)}{2\sigma\delta} < \gamma < 0$) or one negative and two positive eigenvalues (when $\gamma > 0$).

4. Hopf bifurcations

In order to find the indeterminate equilibrium close to the steady state, interesting from the dynamical point of view, we consider the equilibrium point (P^*4).

By using bifurcation theory we prove the existence of a Hopf cycle around the steady state P^*4.

We evalute the invariant elements of the Jacobian and we have that the expression for the eigenvalues is given by:

$$\begin{array}{l} \lambda_1 = J_{22} \\ \lambda_2 = \frac{1}{2}\left(J^*_{11} + J^*_{33}\right) + \frac{1}{2}\sqrt{(J^*_{11})^2 - 2J^*_{11}J^*_{33} + (J^*_{33})^2 + 4J^*_{13}J^*_{31}} \\ \lambda_3 = \frac{1}{2}\left(J^*_{11} + J^*_{33}\right) - \frac{1}{2}\sqrt{(J^*_{11})^2 - 2J^*_{11}J^*_{33} + (J^*_{33})^2 + 4J^*_{13}J^*_{31}} \end{array} \tag{4.1}$$

$$J = \left[\begin{array}{ccc} a = J^*_{11} & 0 & c = J^*_{13} \\ 0 & e = J^*_{22} & 0 \\ g = J^*_{31} & 0 & h = J^*_{33} \end{array}\right] \tag{4.2}$$

We consider the function:

$$G(\gamma) = -B(J_{P^*_4})Tr(J_{P^*_4}) + Det(J_{P^*_4}) \tag{4.3}$$

There is a parameter set Ω, in which there exists a value $\gamma = \gamma_c$ such that the function $G(\gamma)$ is equal to zero.

So, by Routh's thoerem, we can identify the number of the roots of the polinomial with positive real parts (see BP 1994).

So the conditions needed to obtain pure imaginary roots are:

Theorem. There exists a parameter value $\gamma = \gamma_c$ such that in the steady state P^*4 the Jacobian Matrix $J(P^*4)$ possesses two complex conjugated roots with real part equal to zero and the real root different from zero.

Proof. Let $\gamma < -\frac{(1-\beta)}{2\sigma\delta}$; then it is: $a > 0 \;\; c > 0 \; g < 0 \;\; e < 0 \; h < 0$

$$\begin{array}{ll} ah < 0 \;\; cg < 0 & \\ B(J) = (Tr(J))\, e - e^2 + \frac{DetJ}{e} > 0 & \\ \qquad\quad \Longrightarrow (Tr(J))\, e > e^2 - \frac{DetJ}{e} & B(J) > 0 \\ Tr(J) < 0 \Longrightarrow a < -(h+e) & \\ Det(J) = (ah - cg)e = Det(J) > 0 & \\ (a-h)^2 + 4cg < 0 \;\; (a-h)^2 < -4cg & \end{array} \tag{4.4}$$

Considering the characteristic polynomial $-\lambda^3 + Tr(J)\lambda^2 - B(J)\lambda + Det(J) = 0$, then we have the following cases:

a) $B(J) > 0$

$$\begin{array}{cccc} -1 & Tr(J) < 0 & -B(J) + \frac{Det(J)}{Tr(J)} & Det(J) > 0 \\ - & - & - & + \end{array}$$

2 permanences,1 variation $\Rightarrow \lambda_1 =$positive $\;\lambda_{2/3} =$ real negative part

b) $Tr(J) > 0 \Longrightarrow a > -(h+e)$

-1	$Tr(J) < 0$	$-B(J) + \frac{Det(J)}{Tr(J)}$	$Det(J) > 0$
$-$	$+$	$-$	$+$

3 variations $\Rightarrow$ λ_1 =positive , $\lambda_{2/3}$ =real positive part; we must have: $(J_{11} - J_{33})^2 < -4J_{31}J_{13}$.

We determine solutions in which the function $G(\gamma)$ vanishes: $\gamma = \gamma_c^*$.

The derivative of the real part of the complex conjugate eigenvalues of $(J_{P_4^*})$ evaluated at $\gamma = \gamma_c^*$ is always different from zero.

A family of Hopf bifurcations Γ_γ emerges around the steady state P_4^* of the system (2.5), when $\gamma = \gamma_c^*$.

It follows directly from the assumptions of the Hopf bifurcation Theorem.

Thus we can infer, by the Hopf bifurcation theorem, that varying the bifurcation parameter (γ, in our case), the equilibrium point modifies its stability in correspondence of the presence of a limit cycle (attractive or repulsive).

The calculation of the stability coefficient q is done following [17].

5. Stochastic dynamics

Now we build a stochastic system that has a sunspot equilibrium as a solution; the costruction is very similar to that reported in Shigoka (1994), Benhabib - Nishimura (2006), Benhabib - Nishimura (2008).

Our system (2.5) includes one predetermined variable x_1 and two non-predetermined variables x_2 and x_3.

Let E_t the conditional expectation operator in vectorial form :

$$\left(\dot{x}_1, E_t\left(\frac{dx_2}{dt}\right), E_t\left(\frac{dx_3}{dt}\right)\right) = \phi_i(x_1, x_2, x_3) \tag{5.1}$$

with $\phi_i : U \longrightarrow R$, $U \subset R^3$, $i = 1, 2, 3$. We denote $E_t\left(\frac{dx_2}{dt}\right)$ and $E_t\left(\frac{dx_3}{dt}\right)$ as

$$E_t\left(\frac{dx_t^i}{dt} = \lim_{h \to 0^+} \frac{(x_{t+h}^i - x_t^i)}{h}\right) \ (i = 1, 2), \tag{5.2}$$

respectively.

We add a "noise" (a Wiener process) in the equations related with the control variables of the optimal choice problem. Let $s_t(\omega) = (\omega, t)$ be a random variable irrelevant to fondamental characteristic of the optimal economy: it means that it doesn't affect preferences, technology and endowment (i.e. sunspot):

$$(\dot{x}_1 dt, dx_2, dx_3) = \begin{array}{lcc} \phi_1(x_1, x_2, x_3) & + & 0 \\ \phi_2(x_1, x_2, x_3) & + & s d\omega_t \\ \phi_2(x_1, x_2, x_3) & + & d\xi_t \end{array} \tag{5.3}$$

We assume that

$$E_t\left(\frac{d\omega_t}{dt}\right) = \lim_{h \to 0^+} \frac{(\omega_{t+h} - \omega_t)}{h} = E_t\left(\frac{d\xi_t}{dt}\right) = \lim_{h \to 0^+} \frac{(\xi_{t+h} - \xi_t)}{h} = 0$$

are well defined; $s \in [0, \overline{\eta})$, where $\overline{\eta}$ is a sufficiently small positive constant; dt is a Lebesgue measure; $dx_1, dx_2, d\omega_t, d\xi_t$ are Lebesgue-Stieltjes signed measures with respect to t. We assume that $d\omega_t$ is a singular signed measure of t relative to the Lebesgue measure dt.

Let $\int_t^{t+h} d\omega_t = \omega_{t+h} - \omega_t$ be and ω_t is a random variable irrelevant to the fundamentals (i.e. a sunspot variable). In the same way, we consider the random variable ξ_t such that $d\xi_t$ is also a singular signed measure of t relative to the Lebesgue measure dt and that, if $s = 0$, $\xi_t = 0$, $\forall t \geq 0$.

$\int_t^{t+h} d\xi_t = \xi_{t+h} - \xi_t$ and, and if $s = 0$, then $d\xi_t$ disappears from the system.

We specify a stochastic process $\{\varepsilon_t\}_{t\geq 0}$ generating sunspot variables in a way consistent with the formulation in the equations (2.5) and (5.1). We assume that a set of sunspot variables $\{\varepsilon_t(\omega)\}_{t\geq 0}$ is generated by a two-state continuous-time Markov process with stationary transition probabilities and that $\varepsilon_t : \Omega \to \{-1, 1\}$ for each $t \geq 0$. Let $\left[\{\varepsilon_t(\omega)\}_{t\geq 0}, (\Omega, B_\Omega, P)\right]$ be a continuous time stochastic process, where $\omega \in \Omega, B_\Omega$ is a σ-field in Ω, and P is a probability measure. We further assume that (Ω, B_Ω, P) is a complete measure space and that the stochastic process $\{\varepsilon_t(\omega)\}_{t\geq 0}$ is separable.

Let $Z = \{z(1), z(2)\}$ with $(z(1), z(2)) = (-1, 1)$, and let $\mathbf{P}(h) = [p_{ij}(h)]_{1\leq i\ ,\ j\leq 2}, h \geq 0$ denote a 2×2 stationary transition probability matrix, where $p_{ij}(h)$ is the conditional probability that $\varepsilon_t(\omega)$ moves from $\varepsilon_t(\omega) = z(i)$ to $\varepsilon_{t+h}(\omega) = z(j)$ through the length of time h under the condition $\varepsilon_t(\omega) = z(i)$.

$\sum_{j=1}^{2} p_{ij}(h) = 1$ per $i = 1, 2$ and for each $h \geq 0$. We assume that the transition probability matrix satisfies the following continuity condition:

$$\lim_{h\to 0^+} \mathbf{P}(h) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.5}$$

For a fixed ω, $\varepsilon_t = \varepsilon(t)$ considered as a function of t is called a *sample function*. A function $g(\cdot)$ will be called a *step function*, if it has only finitely many points of discontinuity in every finite closed interval, if it is identically constant in every open interval of continuity points, and if $g(t_{0^-}) \leq g(t_0) \leq g(t_{0^+})$ or $g(t_{0^+}) \leq g(t_0) \leq g(t_{0^-})$, when t_0 is a point of discontinuity. Then we have the following result:

THEOREM 3. *A sunspot equilibrium (SE) is a stochastic process* $\{x_{1t}, x_{2t}, x_{3t}\}_{t\ \geq 0}$ *with a compact support* Γ *such that it is a solution of the stochastic differential equation (5.1) (See Nishimura - Shigoka, 2006).*

It follows from Khasminskii, 1980 (Theorem 7.1.1) that our system is asymptotically stable in probability, and the coefficients of the system 5.1 satisfy an inequality

$$|b(t, X) - B| + |\sigma(t, X) - \sigma| < \gamma\, |x| \tag{5.7}$$

in a sufficiently small neighborhood of the Γ and with sufficiently small constant γ; then the solution Γ of the nonlinear system is asymptotically stable in probability.

From what said above, there is the chance of "rescuing" of our model econony from a development trap, through sunspot-driven self-fulfilling expectations. For some parameter value, due to pessimistic self-fulfilling expectations, sunspot equilibria exist in some neighbourhood of the steady-state P_4^*. If the periodic orbit emerging from the steady state P_4^* is super-critical, there is no way out (Slobodyan, 2006). If the periodic solution is repelling , then there is a possibility of a way out of the orbit (in fact the growth rate of economy μ becames positive for low level of the externality γ) and the optimal path can reach another steady state P_i^*.Such situation can be understood as a poverty or development trap.

6. Simulations

Now we consider again the Jacobian matrix evaluated in the Steady State P^*4 :
$J(P^*4) =$

$$= \begin{bmatrix} \frac{\rho}{\sigma} - \\ -\frac{(1-\sigma)(1-\beta+2\sigma\delta\gamma)}{\sigma(1-\beta)(1-\sigma\beta^2)} & 0 & \frac{1-\sigma}{\sigma} \cdot \\ & & \cdot \left(\frac{\rho}{\sigma} - \frac{(1-\sigma)(1-\beta+2\sigma\delta\gamma)}{\sigma(1-\beta)(1-\sigma\beta^2)}\right) \\ 0 & -\frac{\beta}{\beta+1} \cdot \\ & \cdot \left(\frac{\rho}{\sigma} - \frac{(1-\sigma)(1-\beta+2\sigma\delta\gamma)}{\sigma(1-\beta)(1-\sigma\beta^2)}\right) + & 0 \\ & +\frac{\delta\gamma}{\beta+1} \\ \frac{(1-\beta+2\sigma\delta\gamma)}{(1+\beta)(1-\sigma\beta^2)} & 0 & \frac{1-\beta}{\beta+1}\left(\frac{\rho}{\sigma} - \frac{(1-\sigma)(1-\beta+2\sigma\delta\gamma)}{\sigma(1-\beta)(1-\sigma\beta^2)}\right) - \\ & & -2\frac{1-\beta+2\sigma\delta\gamma}{(1-\sigma\beta^2)} + \frac{2\delta\gamma}{\beta+1} \end{bmatrix}$$

In a more compact form we write:

$$J(P_4^*) = \begin{bmatrix} a & 0 & c \\ 0 & e & 0 \\ g & 0 & h \end{bmatrix}$$

Eigenvalues are solutions of the characteristic equation: $\det[J(P_4^*)] = 0$.
That is:

$$\det \begin{bmatrix} a-\lambda & 0 & c \\ 0 & e-\lambda & 0 \\ g & 0 & h-\lambda \end{bmatrix} = 0$$

So, we have:

$$(e-\lambda)\left[(a-\lambda)(h-\lambda) - cg\right] = (e-\lambda)\left[\lambda^2 - (a+h)\lambda - cg\right] = 0$$

The eigenvalues are:

$$e - \lambda = 0 \quad \Leftrightarrow \quad \lambda_1 = e$$

$$\left[\lambda^2 - (a+h)\lambda - cg\right] = 0 \quad \Leftrightarrow \quad \lambda_{2,3} = \frac{1}{2}(a+h) \pm \frac{1}{2}\sqrt{a^2 - 2ah + h^2 + 4cg}$$

So, in order to have purely imaginary eigenvalues, the following two conditions must hold:

$$1) \qquad a + h = 0 \qquad \Leftrightarrow \qquad h = -a$$

$$2) \qquad a^2 - 2ah + h^2 + 4cg < 0$$

According to the second condition, we can write:
$(a-h)^2+4cg<0 \quad \rightarrow \quad (h=-a) \rightarrow \quad 4a^2+4cg<0 \qquad a^2+cg<0$
$a^2+cg<0$

Fixing $\beta=\frac{2}{3},\ \delta=\frac{1}{25},\ \rho=\frac{1}{500}$, we have after some computations:

$$a^2+cg<0 \quad \Leftrightarrow \quad \frac{3591}{3596}<\sigma<\frac{4491}{4496} \qquad (\sim 0.99861<\sigma<\sim 0.99888)$$

According to the condition 1) $a+h=0$, we have:

$$\frac{\rho}{\sigma}-\frac{(1-\sigma)}{\sigma(1-\beta)}\frac{(1-\beta+2\sigma\delta\gamma)}{\left(1-\sigma\beta^2\right)}+\frac{\rho}{\sigma}\frac{1-\beta}{\beta+1}-\frac{(1-\sigma)}{\sigma(1+\beta)}\frac{(1-\beta+2\sigma\delta\gamma)}{\left(1-\sigma\beta^2\right)}-2\frac{1-\beta+2\sigma\delta\gamma}{\left(1-\sigma\beta^2\right)}+\frac{2\delta\gamma}{\beta+1}=0$$

Again, after some computation, we get:

$$\gamma=\frac{(1-\beta)(\rho-1)}{\delta\sigma(1+\beta)}$$

Substituting the values to the parameters, we get:

$$\gamma=f(\sigma)=-\frac{499}{100\sigma}$$

Finally, choosing for σ a value inside the interval indicated above ($\sigma=0.99875$), we have: $\gamma=-\frac{3992}{799}$.

With these values we have:

$$J\left(P_4^*\right)=\begin{bmatrix} \frac{3764}{1538075} & 0 & \frac{1882}{1538075} \\ 0 & -\frac{185936}{1538075} & 0 \\ -\frac{684}{9625} & 0 & -\frac{3764}{1538075} \end{bmatrix}$$

and, in fact, the eigenvalues are:

$$\lambda_1=-\frac{185936}{1538075} \qquad \lambda_{2,3}=\pm\frac{2\sqrt{1197130790}}{7690375}i$$

that is, one eigenvalue with negative real part and the other two purely imaginary eigenvalues.

Now, we have the Hopf cycles as we indicated in the previous sections. With the Matlab software, we obtain this simulation of the trajectories giving the steady state P_4^* as initial condition:

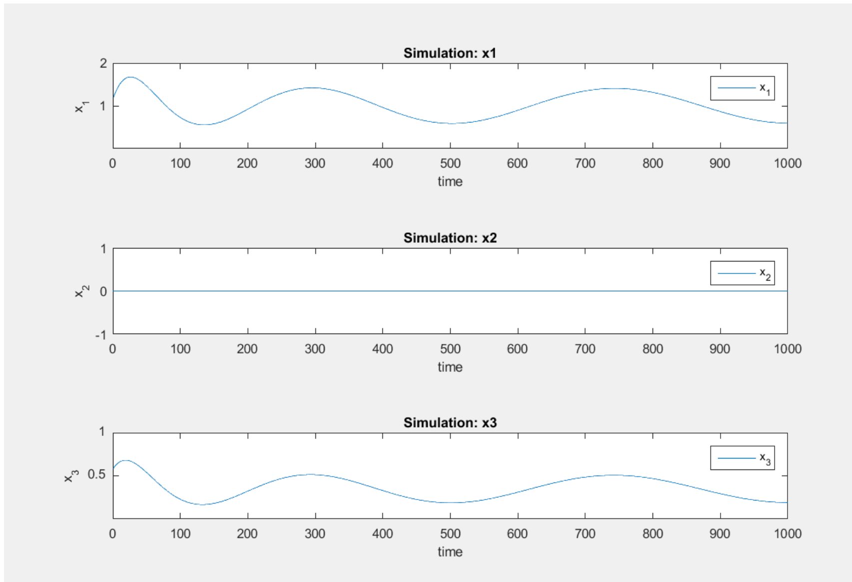

Fig. 1 - Steady state P_4 as initial condition

After an initial overshooting, after few time units we observe the Hopf cycle around the steady state.

Giving other initial conditions ("random", in the sense we take a random point) we obtain other trajectories like the following:

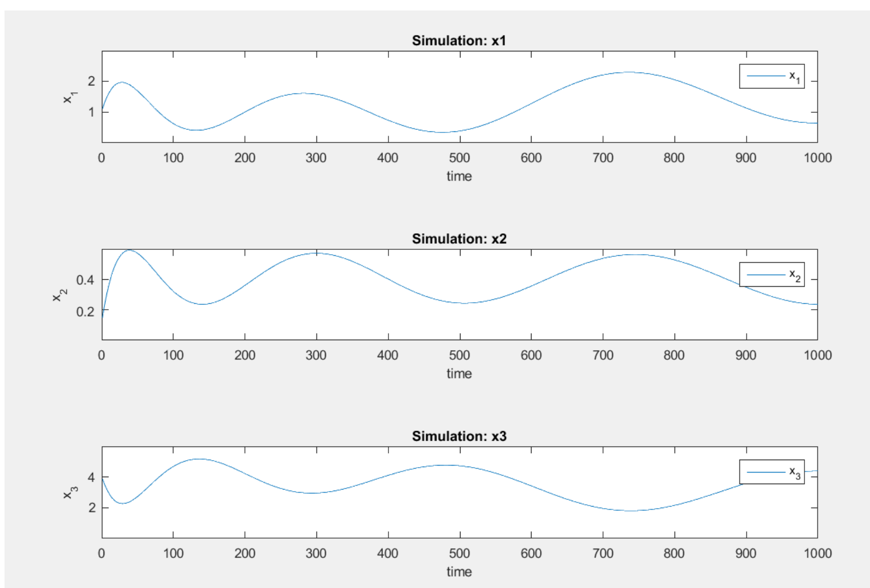

Fig. 2 - Random initial condition

These kinds of trajectories indicate the possibility of escaping the poverty (or development) trap, taking the system up towards position with higher levels of consumption, production, etc.

7. Conclusions

Sunspot equilibria are rational expectation equilibria, compatible also with some measure of market inperfections. They can be understood, from a micro-economic point of view, as the presence of macro - equilibria of underemployment. Again, they can be observed in the presence of features like externalities.

In our paper, we focused on the role of negative externalities due to over-exploitment of natural resources in determining the indeterminacy of equilibrium. As a matter of fact, economies may be both globally and locally indeterminate. Global indeterminacy refers to the balanced growth rate that is obtained in the long run and states that the initial value of consumption, crucially determines to which BGP the economy converges and, thus, the long-run balanced growth rate.

Further, that result confirms the outcome in most of the economics literature stating that the representative individual must have a relatively high intertemporal elasticity of substitution for local indeterminacy to occur if labor supply is exogenous (cf. Benhabib and Perli 1994).

Moreover, local indeterminacy around the BGP with the lower growth rate, can be observed if the parameter constellation is such that the trace of the Jacobian matrix is smaller than zero, so that both eigenvalues have negative real parts. If in that situation a certain parameter is varied, two purely imaginary eigenvalues may be observed that generate a Hopf bifurcation, which leads to stable limit cycles. In particular, we studied the possibility that the the system undergoes Hopf bifurcation, when particular values of the parameters of the model are taken into account. Further, our simulations confirm the model admits fluctuations (and stable limit cycles), in correspondence to the equilibrium and this can be sees as a way out of the poverty trap. These poverty traps (and indeterminacy) in macroeconomic models, that may be caused by externalities or increasing returns to scale, may be escaped through sunspot-driven fluctuations: authors like Slobodyan found not negligible probabilities of escaping the trap only when the initial condition is close enough to the trap boundary and the probability of escape, as expected, increases as expectations become more optimistic.

The escape happens if it choosen a random variable with bounded support as the sunspot variable. So the sunspot variable has a natural interpretation of a change in perceived present discounted wealth and it is the feature that leads the system to points of higher values of its variables.

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