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Balanced metrics on complex vector bundles and the diastatic exponential of a symmetric space

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Submitted for Ph.D. Examination

07/08/2010

Abstract

This thesis deals with two different subjects: *balanced metrics on complex vector bundles* and *the diastatic exponential of a symmetric space* . Correspondingly we have two main results. In the first one we prove that if a holomorphic vector bundle E over a compact Kähler manifold (M, ω) admits a ω-balanced metric then this metric is unique. In the second one, after defining the *diastatic exponential* of a real analytic Kähler manifold, we prove that for every point p of an Hermitian symmetric space of noncompact type there exists a globally defined diastatic exponential centered in p which is a diffeomorphism and it is uniquely determined by its restriction to polydisks.

Declaration

I declare that to the best of my knowledge the contents of this thesis are original and my work except where indicated otherwise.

Acknowledgements

My thanks, sincere and owed, goes to all the people who have in someway or other allowed and encouraged my studies and the realization of my thesis.

A special word of thanks goes to my supervisor Andrea Loi who has inspired me in this discipline, for being open to dialogue, for his confidence in me, for is suggestions and corrections and for his appreciation of what I was trying to accomplish.

A big thanks goes to my colleagues who contributed, with their love for mathematics and their knowledge, to the achievement of this goal which is so important to me: Fabio Zuddas, Gilberto Spanu, Michela Zedda and Roberto Deleo.

A special thanks goes to my life-time friends: Alessandro Angioni, Andrea Casula, Giampiero Boi, Marco Piludu, Massimiliano Piras, Matteo Carboni, with whom I shared amusements, thoughts, ideas and laughter.

I would like to express a heartfelt thanks to my parents who, although they didn't fully agree with my choice, have trusted in me, allowed freedom of choice, economic support and have allowed me to gain this experience.

A special thanks to my sister Stefania, to my godfather and godmother Franco e Rossana, to my uncle Bobo, who always believed in me, unconditionally supporting my aspirations. A further thanks to my sister-in-law Valeria who has the gift of always make me smile, even in difficult moments.

My girlfriends, Elisa, has also had a special role in achieving my goal and it is her I especially dedicate my thesis to. She, with her enthusiasm for life, has always showed me the positive side of things and made the difficult moments seem much less difficult.

Contents

Introduction

This thesis deals with the following two different subjects,

1. balanced metrics on complex vector bundles,
2. the diastastic exponential of a symmetric space.

The study of these two issues has led to the writing of two articles [32] and [31], on which this thesis is based.

1. Balanced metrics on complex vector bundles.

Let $E \to M$ be a very ample holomorphic vector bundle of rank r over a compact Kähler manifold (M, ω) of complex dimension n and let h be an Hermitian metric of E. We can define a natural scalar product $\langle \cdot, \cdot \rangle_{h,\omega}$ over $H^0(M, E)$ by

$$\langle \cdot, \cdot \rangle_{h,\omega} = \frac{1}{V_\omega} \int_M h(\cdot, \cdot) \frac{\omega^n}{n!}, \tag{1}$$

where $\omega^n = \omega \wedge \cdots \wedge \omega$ and $V_\omega = \int_M \frac{\omega^n}{n!}$.

Consider the flat metric h_0 on the tautological bundle $\mathcal{T} \to G(r, N)$ and its dual metric $h_{Gr} = h_0^*$ on the quotient bundle $\mathcal{Q} = \mathcal{T}^*$, where $G(r, N)$ denotes the Grassmannian of r-dimensional complex vector subspaces of $\mathbb{C}^N$. Fix a holomorphic basis $\underline{s} = \{s_1, \ldots, s_N\}$ of $H^0(M, E)$ and consider the Kodaira map

$$i_{\underline{s}} : M \to G(r, N).$$

Consider the pull-back Hermitian metric

$$h_{\underline{s}} = i_{\underline{s}}^* h_{Gr} \tag{2}$$

on $E = i_{\underline{s}}^* Q$ and the pull-back Kähler form

$$\omega_{\underline{s}} = i_{\underline{s}}^* \omega_{Gr}.$$

on M, where ω_{Gr} is the canonical homogeneous Kähler form on $G(r, N)$.

A basis $\underline{s} = \{s_1, \ldots, s_N\}$ of $H^0(M, E)$ is called *balanced* if there exists a positive constant C such that

$$\langle s_j, s_k \rangle_{h_{\underline{s}}, \omega_{\underline{s}}} = C \delta_{jk}, \quad j, k = 1, \ldots, N. \tag{3}$$

An Hermitian metric h over E is *balanced* if $h = h_{\underline{s}}$ for a balanced basis $\underline{s}$.

Let $\omega \in c_1(E)$ be a Kähler form of M. We say that a basis $\underline{s} = \{s_1, \ldots, s_N\}$ of $H^0(M, E)$ is *ω-balanced* if

$$\langle s_j, s_k \rangle_{h_{\underline{s}}, \omega} = \frac{r}{N} \delta_{jk}, \quad j, k = 1, \ldots, N. \tag{4}$$

An Hermitian metric h over E is *ω-balanced* if $h = h_{\underline{s}}$ for a ω-balanced basis $\underline{s}$.

The concept of balanced and ω-balanced metrics on complex vector bundles was introduced by X. Wang [38] (see also [39]) following S. Donaldson's ideas [15]. It can be also defined in the non-compact case and the study of balanced metrics is a very fruitful area of research both from mathematical and physical point of view (see, e.g., [7], [8], [11], [20], [17], [27] and [28]).

In [38] X. Wang proved that, under the assumption that the Kähler form ω is integral, E is Gieseker stable if and only if $E \otimes L^k$ admits a unique ω-balanced metric (for every k sufficiently large), where $L \to M$ is a polarization of (M, ω), i.e. L is a holomorphic line bundle over M such that $c_1(L) = [\omega]_{dR}$.

On the other hand, in Lemma 2.7 of [37], R. Seyyedali shows that if a simple bundle E (i.e. $\mathrm{Aut}(E) = \mathbb{C}^* \, \mathrm{id}_E$, where $\mathrm{Aut}(E)$ denotes the group of invertible holomorphic bundle maps from E in itself) admits a balanced metric then the metric is unique. In Theorem 8, which is one of the main result of the thesis, we prove the unicity of balanced metrics for *any* vector bundle. As an application of Theorem 8 and L. Biliotti and A. Ghigi results [4] we obtain the existence and uniqueness of ω-balanced metrics over the direct sum of homogeneous vector bundles over rational homogeneous varieties (Theorem 11). We also apply our result to show the rigidity of ω-balanced Kähler maps into Grassmannians (Section 2.3) and to the study of $2\omega_{FS}$-balanced map from $\mathbb{C}P^1$ to $G(2,4)$ (Section 2.4). The proof of Theorem 8 is based on X. Wang's work on balanced metrics and on moment map techniques developed by C. Arezzo and A. Loi in [1], where it is proved the uniqueness of *balanced metrics* for holomorphic line bundles.

2. The diastastic exponential of a symmetric space.

Let (M, g) be a real analytic Kähler manifold. We say that a smooth map $\mathrm{Exp}_p : W \to M$ from a neighbourhood W of the origin of T_pM into M is a *diastatic exponential* at p if it satisfies

$$\left(d\,\mathrm{Exp}_p\right)_0 = \mathrm{id}_{T_pM},$$

$$D_p\left(\mathrm{Exp}_p(v)\right) = g_p(v,v)\,,\ \forall v \in W,$$

where D_p is Calabi's diastasis function at p (the usual exponential $\exp_p$ obviously satisfied these equations when D_p is replaced by the square of the geodesics distance from p). In this thesis we prove (Theorem 13) that for every point p of an Hermitian symmetric space of noncompact type M there exists a globally defined diastatic exponential centered in p which is a diffeomorphism and it is uniquely determined by its restriction to polydisks. An analogous result holds true in an open dense neighbourhood of every point of M^*, the compact dual of M (Theorem 14). We also provide (Theorem 16) a geometric interpretation of the symplectic duality map (recently introduced in [13]) in terms of diastatic exponentials. As a byproduct of our analysis we show (Theorem 17) that the symplectic duality map pulls back the reproducing kernel of M^* to the reproducing kernel of M.

The thesis is divided into three chapters and one appendix. The organization is as follows. In the first two sections (Section 1.1 and Section 1.2) of Chapter 1 we give the definition of balanced and ω-balanced metrics on a complex vector bundle and we describe their link with the Bergman kernel. In Section 1.3 (resp. Section 1.4) we describe the known results in the rank one case (resp. general case) on the existence and uniqueness of balanced and ω-balanced metrics. In particular in the rank one case we also describe the balanced metrics on holomorphic line bundles over homogeneous Kähler manifolds. In Section 2.1 and 2.2 of Chapter 2 we prove Theorem 8 and Theorem 11 respectively. In Section 3.3 and 3.4 we prove the rigidity of ω-balanced maps into Grassmannians and classify the $2\omega_{FS}$-balanced map from $\mathbb{C}P^1$ to $G(2,4)$. For the reader convenience at the end of the thesis we include an appendix where we summarize the basic material on moment maps needed in the proof of Theorem 8. Chapter 3 is entirely dedicated to the diastatic exponential. It is divided in three sections. In Section 3.1, we define the diastatic exponential in a neighborhood of a point of a real analytic Kähler manifold, we state our main results and we provide an explicit description of the diastatic exponential for the complex hyperbolic space and fpr polydisks. In Section 3.2. we recall the basic tools needed in the proof of our results, namely hermitian positive Jordan triple systems, spectral decomposition and their link with the hermitian symmetric spaces of noncompact type. Finally Section 3.3 contains the proof Theorem 13, Theorem 14, Theorem 16 and Theorem 17.

Chapter 1

Balanced and ω-balanced metrics

1.1 Main definitions

Let $E \to M$ be a holomorphic vector bundle of rank r over a compact complex manifold M. Denote by $H^0(M, E)$ the space of global holomorphic sections of E. Assume that the bundle E is *globally generated*, i.e. the evaluation map $s \in H^0(M, E) \mapsto s(x)$ is surjective, for every $x \in M$. Then the dual map $E_x^* \hookrightarrow H^0(M, E)^*$ is injective and determines an element of $G(r, H^0(M, E)^*)$, the Grassmannian of r-dimensional complex vector subspaces of $H^0(M, E)^*$. So we can associate to every $x \in M$ an element $i_E(x) \in G(r, H^0(M, E)^*)$. The map

$$i_E : M \to G(r, H^0(M, E)^*), \qquad x \mapsto i_E(x) \tag{1.1}$$

is called the *Kodaira map*. When this map is an embedding we call the vector bundle E *very ample*. A well–known theorem of Kodaira (see e.g. [22]) asserts that if $E \to M$ is any holomorphic vector bundle on a compact complex manifold M and $L \to M$ is a positive line bundle then, for k sufficiently large, the Kodaira map $i_{E(k)} = i_{E \otimes L^{\otimes k}}$ is an embedding. We recall that a positive line bundle $L \to M$ is a holomorphic line bundle whose first Chern class $c_1(L)$ can be represented by a Kähler form ω of M, i.e. $c_1(L) = [\omega]$. One also says that L is a *polarization* of the complex manifold M.

In order to write the Kodaira map more explicitly we fix a basis $\underline{s} = \{s_1, \ldots, s_N\}$

of $H^0(M,E)$, $N = \dim H^0(M,E)$. Therefore we can identify $G(r, H^0(M,E)^*)$ with $G(r,N)$, the Grassmannian of r-dimensional complex vector subspaces of $\mathbb{C}^N$, and the Kodaira map gives rise to a holomorphic map

$$i_{\underline{s}} : M \to G(r,N) \tag{1.2}$$

satisfying $i_{\underline{s}}^*(\mathcal{Q}) = E$, where $\mathcal{Q}$, called the *quotient bundle*, is the dual of the universal bundle $\mathcal{T} \to G(r,N)$, i.e. $Q = \mathcal{T}^*$. The map $i_{\underline{s}}$ will be called *the Kodaira map associated to the basis* $\underline{s}$. The expression of $i_{\underline{s}}$ in a local frame $(\sigma_1, \dots, \sigma_r) : U \to E$ is given by:

$$i_{\underline{s}}(x) = \begin{bmatrix} S_{11}(x) & \dots & S_{1r}(x) \\ \vdots & & \vdots \\ S_{N1}(x) & \dots & S_{Nr}(x) \end{bmatrix},\ x \in U, \tag{1.3}$$

where $s_j = \sum_{\alpha=1}^r S_{j\alpha}\sigma_\alpha$, $j = 1, \dots, N$. The square bracket denotes the equivalence class in $G(r,N) = M^*(r,N,\mathbb{C})/GL(r,\mathbb{C})$, where $M^*(r,N,\mathbb{C})$ is the set of $r \times N$ complex matrices of rank r.

Let $E \to M$ be a very ample holomorphic vector bundle of rank r over a Kähler manifold (M,ω) of complex dimension n and let h be an Hermitian metric of E. We can define a natural scalar product $\langle\cdot,\cdot\rangle_{h,\omega}$ over $H^0(M,E)$ by

$$\langle\cdot,\cdot\rangle_{h,\omega} = \frac{1}{V_\omega}\int_M h(\cdot,\cdot)\frac{\omega^n}{n!} \tag{1.4}$$

where $\omega^n = \omega \wedge \dots \wedge \omega$ and $V_\omega = \int_M \frac{\omega^n}{n!}$.

Consider the flat metric h_0 on the tautological bundle $\mathcal{T} \to G(r,N)$, i.e. $h_0(v,w) = w^*v$, and its dual metric $h_{Gr} = h_0^*$ on the quotient bundle $\mathcal{Q}$. Consider also the Plücker embedding $P : G(r,N) \to \mathbb{C}P^{\binom{N}{r}-1}$ and $\omega_{Gr} = P^*\omega_{FS}$ the Kähler form on $G(r,N)$ pull-back of the Fubini–Study form $\omega_{FS} = \frac{i}{2}\partial\bar{\partial}\log(|z_0|^2 + \dots + |z_{\binom{N}{r}-1}|^2)$ over $\mathbb{C}P^{\binom{N}{r}-1}$. Hence, we can endow $E = i_{\underline{s}}^*\mathcal{Q}$ with the pull-back Hermitian metric

$$h_{\underline{s}} = i_{\underline{s}}^* h_{Gr} \tag{1.5}$$

and the manifold M with the pull-back Kähler form

$$\omega_{\underline{s}} = i_{\underline{s}}^*\omega_{Gr}.$$

Definition 1. A basis $\underline{s} = \{s_1, \ldots, s_N\}$ of $H^0(M, E)$ is called *balanced* if there exists a positive constant C such that

$$\langle s_j, s_k \rangle_{h_{\underline{s}}, \omega_{\underline{s}}} = C\delta_{jk}, \quad j, k = 1, \ldots, N. \tag{1.6}$$

An Hermitian metric h over E is *balanced* if $h = h_{\underline{s}}$ for a balanced basis $\underline{s}$.

Definition 2. Let $\omega \in c_1(E)$ be a Kähler form of M. We say that a basis $\underline{s} = \{s_1, \ldots, s_N\}$ of $H^0(M, E)$ is ω*-balanced* if

$$\langle s_j, s_k \rangle_{h_{\underline{s}}, \omega} = \frac{r}{N}\delta_{jk}, \quad j, k = 1, \ldots, N. \tag{1.7}$$

An Hermitian metric h over E is ω*-balanced* if $h = h_{\underline{s}}$ for a ω-balanced basis $\underline{s}$.

Remark 3. The choice of the constant $\frac{r}{N}$ in Definition 2 is related to the Geiseker stability of the bundle E (see Section 1.4 below for details).

1.2 The Bergman kernel

Let E be a globally generated holomorphic vector bundle of rank r over a compact Kähler manifold (M, ω). Fix an Hermitian metric h on E and let $\underline{t} = \{t_1, \ldots, t_N\}$ be an orthonormal basis of $H^0(M, E)$ with respect to the scalar product $\langle \cdot, \cdot \rangle_{h,\omega}$ given by (1.4).

The *Bergman kernel* $\mathrm{Berg}_{h,\omega} : M \to \Gamma(\mathrm{GL}(E))$ is defined by

$$\mathrm{Berg}_{h,\omega}(x) = \sum_{j=1}^{N} h(\cdot, t_j(x))t_j(x).$$

Notice that the Bergman kernel does not depend on the orthonormal basis chosen.

Let $\underline{\sigma} = (\sigma_1, \ldots, \sigma_r)$ be a local frame for E, $T = (T_{j\alpha}) \in M_{N\times r}(\mathbb{C})$ be the matrix which represents the basis $\underline{t}$ in the local frames $\underline{\sigma}$ (i.e. $t_j = \sum_\alpha T_{j\alpha}\sigma_\alpha$), and $H^h = (H^h_{\alpha\beta}) = (h(\sigma_\alpha, \sigma_\beta))$ be the matrix associated to the Hermitian metric h. We write $K^{h,\omega} = (K^{h,\omega}_{\alpha\beta}) \in M_r(\mathbb{C})$ to indicate the matrix which represents the Bergman kernel $\mathrm{Berg}_{h,\omega}$ in this local frame, namely the matrix satisfying

$$\mathrm{Berg}_{h,\omega}(x)(\sigma_\alpha(x)) = \sum_{\beta=1}^{r} K^{h,\omega}_{\alpha\beta}\sigma_\beta, \ \alpha = 1, \ldots, r.$$

In order to write the relation between $K^{h,\omega}$ and H^h set

$$\mathrm{Berg}_{h,\omega}(\sigma_\alpha) = \left(\mathrm{Berg}_{h,\omega}(\cdot)\right)(\sigma_\alpha(\cdot)).$$

Thus, for all $\alpha = 1, \dots, r$,

$$\begin{aligned}\mathrm{Berg}_{h,\omega}(\sigma_\alpha) &= \sum_{j=1}^{N} h(\sigma_\alpha, \sum_{\delta=1}^{r} T_{j\delta}\sigma_\delta) \sum_{\beta=1}^{r} T_{j\beta}\sigma_\beta \\ &= \sum_{j=1}^{N} \sum_{\beta,\delta=1}^{r} H^h_{\alpha\delta}\overline{T}_{j\delta}T_{j\beta}\sigma_\beta\end{aligned}$$

namely

$$K^{h,\omega}_{\alpha\beta} = \sum_{j=1}^{N}\sum_{\delta=1}^{r} H^h_{\alpha\delta}\overline{T}_{j\delta}T_{j\beta},\ \alpha = 1, \dots, r,$$

which in matrix notation can be written as

$$K^{h,\omega} = H^h T^* T. \tag{1.8}$$

Proposition 4. *If the Bergman kernel* $\mathrm{Berg}_{h,\omega}$ *equals* $C\,\mathrm{Id}_E$*, where* C *is a positive constant and* Id_E *is the identity bundle morphism, then*

$$C = \frac{N}{r}.$$

Proof. Observe that

$$\begin{aligned}Cr = \operatorname{tr}\mathrm{Berg}_{h,\omega} &= \sum_{j=1}^{N}\sum_{\alpha,\beta=1}^{r} H^h_{\alpha\beta}\overline{T}_{j\beta}T_{j\alpha} \\ &= \sum_{j=1}^{N} h\left(\sum_{\alpha=1}^{r} T_{j\alpha}\sigma_\alpha, \sum_{\beta=1}^{r} T_{j\beta}\sigma_\beta\right) = \sum_{j=1}^{N} h(t_j, t_j),\end{aligned}$$

so

$$Cr = \frac{1}{V(M)}\int_M \operatorname{tr}\mathrm{Berg}_{h,\omega}\frac{\omega^n}{n!} = \frac{1}{V(M)}\int_M \sum_{j=1}^{N} h(t_j, t_j)\frac{\omega^n}{n!} = N.$$

□

In the following proposition we describe the link between the Bergman kernel and the balanced condition.

Proposition 5. *Let* $\underline{t}$ *be an orthonormal basis for* $(H^0(M,E),\langle\cdot,\cdot\rangle)$*. Then we have* $\mathrm{Berg}_{h,\omega}=\frac{N}{r}\,\mathrm{Id}_E$ *(resp.*$\mathrm{Berg}_{h,\omega_{\underline{t}}}=\frac{N}{r}\,\mathrm{Id}_E$ *) if and only if* $h=\frac{N}{r}h_{\underline{t}}$ *if and only if* h *is* ω*-balanced (resp.* h *is balanced).*

Proof. In a local frame $\underline{\sigma}$ one has

$$h_{\underline{t}}(v,w)=i_{\underline{t}}^{*}h_{Gr}^{*}(v,w)=w^{*}(T^{*}T)^{-1}v.$$

Hence the matrix which represents the Hermitian product $h_{\underline{t}}$ is given by $H^{h_{\underline{t}}}=(A^{*}T^{*}TA)^{-1}$, for a certain $A\in\mathrm{GL}(E)$. By equation (1.8) we see that $\mathrm{Berg}_{h,\omega}$ ($\mathrm{Berg}_{h,\omega_{\underline{t}}}$) equals $\frac{N}{r}\,\mathrm{Id}_E$ if and only if $h=\frac{N}{r}h_{\underline{t}}$. The conclusion then follows by the fact that $\underline{s}=\sqrt{\frac{r}{N}}\underline{t}$ is a ω-balanced (resp. balanced) basis if and only if $h=\frac{N}{r}h_{\underline{t}}=h_{\underline{s}}$. □

1.3 Known results in the rank one case

In this section we assume that $E=L$ is a very ample holomorphic line bundle on M. In this case (cfr. Definition 1 and Definition 2) an Hermitian metric h on L is balanced (respectively ω-balanced) if and only if $h=h_{\underline{s}}$ where $\underline{s}=\{s_1,\dots,s_N\}$ is a basis of $H^0(M,L)$ satisfying

$$\langle\cdot,\cdot\rangle_{h_{\underline{s}},\omega_{\underline{s}}}=C\delta_{jk},\quad j,k=1,\dots,N,$$

for a positive constant C (respectively $\langle\cdot,\cdot\rangle_{h_{\underline{s}},\omega}=\frac{\delta_{jk}}{N},\quad j,k=1,\dots,N.$)

The following theorem due to Bourguignon–Li–Yau [6] shows that there is not any obstruction for the existence of ω-balanced metrics and, moreover, ω-balanced metrics are unique (we refer the reader to [2] for the application of this theorem to the estimate of the first eigenvalue of the Laplacian associated to ω).

Theorem 1. *Let* $L\to M$ *be a polarization of a compact complex manifold* M *and* ω *a Kähler form of* M*. Then the line bundle* L *admits a unique* ω*-balanced metric* h*.*

The existence and uniqueness of balanced metrics is a more difficult matter. The main results are due to S. Donaldson [15] (when $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$ is discrete) and to C. Arezzo and A. Loi [1] (for the uniqueness in the general case). Here $\frac{\mathrm{Aut}(M,L)}{\mathbb{C}^*}$ denotes the group of biholomorphisms of M which lift to holomorphic bundle maps $L \to L$ modulo the trivial automorphism group $\mathbb{C}^*$. The following theorem summarizes what is known on the existence and uniqueness of balanced metrics in the rank one case.

Theorem 2. *Let $L \to M$ be a polarization of a complex manifold M. Let ω be a Kähler form in $c_1(L)$ with constant scalar curvature. Then for m sufficiently large there exists a balanced metric h on L^m such that $[\mathrm{Ric}(h)] = [m\omega]$. Moreover, if $\tilde{h}$ is another balanced metric on L^m satisfying $[\mathrm{Ric}(\tilde{h})] = [m\omega]$ then there exists $\widehat{F} \in \mathrm{Aut}(M, L)$ such that $\widehat{F}^*\tilde{h} = h$.*

Recall that, given an Hermitian metric on L, $\mathrm{Ric}(h)$ represents the $(1,1)$-form which in a local trivialization $\sigma : U \to L \setminus \{0\}$ is given by

$$\mathrm{Ric}(h) = -\frac{i}{2}\partial\bar{\partial}\log h(\sigma(x), \sigma(x)). \tag{1.9}$$

Remark 6. The relevance of the previous theorem is that a Riemannian geometry condition as the constant scalar curvature implies a balanced condition which, by H. Luo's theorem [34], implies that the polarization L is stable in the sense of Hilbert–Mumford. It also worth mentioning that S. Zhang [46] shows that the existence of a balanced basis for $H^0(M, L)$ is equivalent to the Chow poly-stablility of the polarization L.

In the rank one case the balanced condition can be expressed in terms of a smooth function on M (see Proposition 7 below). This will allow us to describe explicit examples of balanced and ω-balanced metrics in the homogeneous case (see Theorem 3).

Given a polarization $L \to M$ of a compact Kähler manifold (M, ω), $\omega \in c_1(L)$, one can define the smooth function $\epsilon_\omega : M \to \mathbb{R}$

$$\epsilon_\omega(x) = \sum_{j=1}^{N} h(t_j(x), t_j(x)), \quad x \in M, \tag{1.10}$$

where $\{t_0, \ldots, t_N\}$ is an orthonormal basis for $(H^0(L), \langle\cdot,\cdot\rangle_{h,\omega})$ and $\mathrm{Ric}(h) = \omega$. It is easy to verify, as the notation suggests, that the function ϵ_ω depends only on the Kähler form ω and not on the Hermitian metric h with $\mathrm{Ric}(h)=\omega$ or on the orthonormal basis chosen.

Proposition 7. *Let $\omega \in c_1(L)$ be a Kähler form. Then there exists a balanced Hermitian metric h on L, such that $\mathrm{Ric}(h)=\omega$, if and only if ϵ_ω is constant.*

Proof. Let $\tilde{h}$ be an Hermitian metric such that $\mathrm{Ric}(\tilde{h})=\omega$. Pick an orthonormal basis $\underline{t}$ of $(H^0(M,L),\langle\cdot,\cdot\rangle_{\tilde{h},\omega})$ and let $i_{\underline{t}}: M \to \mathbb{C}P^{N-1}$ be the map given by (1.2). Let $\sigma: U \to L\setminus\{0\}$ be a local trivialization. Then the following equation holds:

$$\begin{aligned} i_{\underline{t}}^*\omega_{FS} &= \frac{i}{2}\partial\bar{\partial}\log\sum_{j=1}^{N}\left|\frac{t_j}{\sigma}\right|^2 \\ &= \frac{i}{2}\partial\bar{\partial}\left(\log\left(\frac{1}{\tilde{h}(\sigma,\sigma)}\right)+\log\left(\sum_{j=1}^{N}\left|\frac{t_j}{\sigma}\right|^2\tilde{h}(\sigma,\sigma)\right)\right) \\ &= \omega+\frac{i}{2}\partial\bar{\partial}\log\epsilon_\omega. \end{aligned} \tag{1.11}$$

Therefore, if h is a balanced Hermitian metric on L, such that $\mathrm{Ric}(h)=\omega$ then $h=C\tilde{h}$ for a positive constant C. Therefore $\underline{t}$ is balanced, $h=h_{\underline{t}}$ and $\omega=\omega_{\underline{t}}$, and thus by the previous equation ϵ_ω is constant. Assume now that ϵ_ω is constant. Observe that $\mathrm{Ric}(h_{\underline{t}})=i_{\underline{t}}^*\omega_{FS}$, thus by (1.11) $\mathrm{Ric}(h_{\underline{t}})=\omega$, so $h_{\underline{t}}=Ch$ for a positive constant C and $\langle t_j,t_k\rangle_{h_{\underline{t}},\omega_{\underline{t}}}=C\delta_{jk}$, $i,j=1,\dots,N$, i.e. $h_{\underline{t}}$ is a balanced metric. $\square$

The homogeneous case

Let (M,ω) be a compact simply-connected homogeneous Kähler manifold. Recall that a Kähler manifold is homogeneous if $G=\mathrm{Aut}(M)\cap\mathrm{Isom}(M)$ acts transitively on (M,ω). Assume $\omega\in c_1(L)$ for a polarization $L\to M$. We want to show (see Teorem 3 below) that there exists a homogeneous balanced metric h on L such that $\mathrm{Ric}(h)=\omega$.

We first prove that any holomorphic line bundle L over (M,ω) is homogeneous.

Proposition 8. *Let $L\to M$ be a polarization of the compact complex simply-connected manifold M and let $\omega\in c_1(L)$ be a Kähler form. Then, for every $F\in G$ there exists an* *invertible holomorphic bundle map $\widehat{F} : L \to L$ such that the following diagram commutes*

$$\begin{tikzcd} L \arrow[r, "\widehat{F}"] \arrow[d] & L \arrow[d] \\ M \arrow[r, "F"] & M \end{tikzcd} \tag{1.12}$$

Proof. Consider the diagram

$$\begin{tikzcd} F^*L \arrow[r, "F^*"] \arrow[d, "\pi^*"'] & L \arrow[d, "\pi"] \\ M \arrow[r, "F"] & M \end{tikzcd}$$

where $\pi^* : F^*L \to M$ is the pull-back bundle. Since

$$c_1(F^*(L)) = F^*c_1(L) = F^*[\omega] = [F^*\omega] = [\omega] = c_1(L),$$

it follows by the fact that the first Chern class of a holomorphic line bundle $L \to M$ over a simply-connected complex manifold M is uniquely determined up the isomorphism class of L (see, e.g. p. 105 in [44]) that there exists an invertible holomorphic bundle map $\Psi : L \to F^*L$ such that

$$\begin{tikzcd} L \arrow[r, "\Psi"] \arrow[d] & F^*L \arrow[d] \\ M \arrow[r, "\mathrm{Id}"] & M. \end{tikzcd}$$

Then $\widehat{F} = \Psi \circ F^*$ is the desired bundle map. □

Theorem 3. *(cfr. [1]) Let $L \to M$ be a polarization of the compact complex manifold M and let $\omega \in c_1(L)$ be a Kähler form. Assume that (M, ω) is homogeneous. Then there exists a homogeneous balanced Hermitian metric h on L such that* $\mathrm{Ric}(h) = \omega$.

Proof. By Proposition 7 we need to show that the function $\epsilon_\omega : M \to \mathbb{R}$ defined by (1.10) is constant. Let h be an Hermitian metric of L, with $\mathrm{Ric}(h) = \omega$. Fix an orthonormal basis $\underline{s} = \{s_1, \dots, s_N\}$ for $H^0(M, L)$ with respect to $\langle \cdot, \cdot \rangle_{h,\omega}$. Given $F \in G$ let $\widehat{F}$ be its lift as in Proposition 8. Let $\sigma : U \to$ be a local trivialization for L. Then

$$\begin{aligned} \mathrm{Ric}\left(\widehat{F}^*h\right) &= \frac{i}{2}\partial\bar{\partial}\log\left((\widehat{F}^*h)(\sigma,\sigma)\right) \\ &= \frac{i}{2}\partial\bar{\partial}\log\left(h(\widehat{F}\sigma,\widehat{F}\sigma)\right) \\ &= F^*\omega = \omega. \end{aligned} \tag{1.13}$$

The basis $\left\{\widehat{F}^{-1}\left(s_1\left(F(x)\right)\right),\ldots,\widehat{F}^{-1}\left(s_N\left(F(x)\right)\right)\right\}$ of $H^0(M,L)$ is orthonormal with respect to $\langle\cdot,\cdot\rangle_{\widehat{F}^*h,\omega}$. Indeed,

$$\begin{aligned}&\langle\widehat{F}^{-1}\left(s_j\left(F(x)\right)\right),\widehat{F}^{-1}\left(s_k\left(F(x)\right)\right)\rangle_{\widehat{F}^*h,\omega}=\\&=\int_M\widehat{F}^*h\left(\widehat{F}^{-1}\left(s_j\left(F(x)\right)\right),\widehat{F}^{-1}\left(s_k\left(F(x)\right)\right)\right)\frac{\omega^n}{n!}\\&=\int_M h\left(s_j\left(F(x)\right),s_k\left(F(x)\right)\right)\frac{\omega^n}{n!}\\&=\int_M h\left(s_j\left(x\right),s_k\left(x\right)\right)\frac{(F^{-1})^*\omega^n}{n!}=\langle s_j,s_k\rangle_{h,\omega}.\end{aligned}\tag{1.14}$$

Therefore

$$\begin{aligned}\epsilon_\omega(x)&=\sum_{j=1}^N\widehat{F}^*h\left(\widehat{F}^{-1}(s_j(F(x))),\widehat{F}^{-1}(s_j(F(x)))\right)\\&=\sum_{j=1}^N h\left(s_j(F(x)),s_j(F(x))\right)=\epsilon_\omega\left(F(x)\right).\end{aligned}\tag{1.15}$$

Since G acts transitively on M ϵ_ω is forced to be constant.

Finally, given $F\in G$ and a lift $\widehat{F}$ of F, by formula (1.13) we see that there exists a constant $C>0$ such that $\widehat{F}^*h=Ch$. Therefore $\frac{1}{\sqrt{C}}\widehat{F}$ is a lift of F preserving h and so h is homogeneous. □

Example 9. Consider the line bundle $O(k)\to\mathbb{C}P^1$ where $O(k)=O(1)^{\otimes k}$ and $O(1)$ is the hyperplane bundle over $\mathbb{C}P^1$. We endow $\mathbb{C}P^1$ with the Fubini–Study Kähler form $\omega_{FS}=\frac{i}{2}\partial\bar\partial\log(|z_0|^2+|z_1|^2)\in c_1(O(1))$. A ω_{FS}-balanced basis for $H^0(\mathbb{C}P^1,O(k))$ is given by

$$\underline{s}_k=\{s_0^{(k)},\ldots,s_k^{(k)}\}=\{z_1^k,\ldots,\sqrt{\binom{k}{j}}z_0^jz_1^{k-j},\ldots,z_0^k\}.$$

Indeed, in affine coordinates $\{z_1\neq 0\}$ we have

$$\begin{aligned}\mathrm{Vol}(\mathbb{C}P^1)&=\int_{\mathbb{C}P^1}\omega_{FS}=\int_{\mathbb{C}}\frac{i}{2\left(1+|z|^2\right)^2}dzd\bar z\\&=\int_0^{2\pi}\int_0^\infty\frac{1}{(1+r)^2}drd\theta=2\pi\end{aligned},$$

integrating by part we get

$$\int_0^\infty\frac{r^j}{(1+r)^k}\frac{1}{(1+r)^2}dr=\frac{1}{\binom{k}{j}(k+1)},$$

so

$$\begin{aligned}\langle s_j^{(k)}, s_l^{(k)}\rangle_{h_{\underline{s}_k},\omega_{FS}} &= \frac{1}{\mathrm{Vol}(\mathbb{C}P^1)}\int_{\mathbb{C}P^1} h_{FS}(s_j, s_l)\omega_{FS} \\ &= \frac{1}{2\pi}\int_{\mathbb{C}}\int_{\mathbb{C}} \frac{\sqrt{\binom{k}{j}}\sqrt{\binom{k}{l}}z^j\bar{z}^l}{(1+|z|^2)^k}\frac{i}{2\left(1+|z|^2\right)^2}dzd\bar{z} \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} \frac{\sqrt{\binom{k}{j}}\sqrt{\binom{k}{l}}r^{\frac{j+l}{2}}e^{i(j-l)\theta}}{(1+r)^k}\frac{1}{(1+r)^2}drd\theta \\ &= \frac{\delta_{jl}}{k+1}\end{aligned}$$

as wished. Note that $\underline{s}_k$ is also a balanced basis. Indeed $\omega_{\underline{s}_k} = i_{\underline{s}_k}^*\omega_{FS} = \frac{i}{2}\partial\bar{\partial}\log(|z_0|^2 + |z_1|^2)^k = k\omega_{FS}$ and this implies $\langle s_j^{(k)}, s_l^{(k)}\rangle_{h_{\underline{s}_k},\omega_{\underline{s}_k}} = \langle s_j^{(k)}, s_l^{(k)}\rangle_{h_{\underline{s}_k},\omega_{FS}} = \frac{\delta_{jl}}{k+1}$.

1.4 Known results in the general case

In this section we describe the known results about existence and unicity of ω-balanced metrics on complex vector bundle of rank $r \geq 1$ (see Theorems 4, 5 and 7 below). In order to express these results we need to recall basic algebraic geometric tools also needed in the proof of our main results in the next chapter (we refer the reader to [24]).

Let L be a polarization over a complex manifold M of complex dimension n with a very ample line bundle L. Let E be an irreducible holomorphic vector bundle on M of rank r. Fix an Hermitian metric h on L and the Kähler form $\omega = \mathrm{Ric}(h)$ (see (1.9)) over M.

Definition 10. The vector bundle $E \to M$ is *Gieseker stable* (resp. *semi-stable*) if for any torsion free proper sub-sheaf $\mathcal{F} \subset E$, there exists $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$, we have

$$\frac{\chi(E(k))}{\mathrm{rank}(E)} > (resp. \geq) \frac{\chi(\mathcal{F}(k))}{\mathrm{rank}(\mathcal{F})},$$

where $E(k) = E \otimes L^k$, $\mathcal{F}(k) = \mathcal{F} \otimes L^k$ and $\chi(M, \mathcal{F}) = \sum_{k=0}^{n}(-1)^k \dim H^k(M, \mathcal{F})$ is the Euler characteristic number.

Definition 11. The Gieseker point of E

$$T_E : \bigwedge^r H^0(M, E) \to H^0(M, \det E)$$

is the map which sends $s_1 \wedge \cdots \wedge s_r \in \bigwedge^r H^0(M,E)$ to the holomorphic section of $\det E$ defined by

$$T_E(s_1 \wedge \cdots \wedge s_r) : x \mapsto s_1(x) \wedge \cdots \wedge s_r(x).$$

The group $\mathrm{GL}(H^0(M,E))$ acts on $H^0(M,E)$, therefore acts also on $\bigwedge^r H^0(M,E)$ and on $\mathrm{Hom}(\bigwedge^r H^0(M,E), H^0(M,\det E))$. The actions are given by

$$V \cdot (s_1 \wedge \cdots \wedge s_r) = V s_1 \wedge \cdots \wedge V s_r$$

and

$$(V \cdot T)(s_1 \wedge \cdots \wedge s_r) = T(V \cdot (s_1 \wedge \cdots \wedge s_r)).$$

where $T \in \mathrm{Hom}\left(\bigwedge^r H^0(M,E), H^0(M,\det E)\right)$ and $V \in \mathrm{GL}\left(H^0(M,E)\right)$.

Definition 12. Let G be a reductive group acting linearly on a vector space V. Then an element v of V is called

- *unstable* if the closure of the G orbit $\overline{Gv}$ contains 0,
- *semi–stable* if $0 \notin \overline{Gv}$,
- *stable* if Gv is closed in V and the stabilizer of v inside G is finite.

Gieseker in [19] proved that $E \to M$ is a Gieseker stable (resp. semistable) vector bundle on a polarization (M,L) if and only if there exists a constant k_0 such that for $k > k_0$ the Gieseker point $T_{E(k)}$ is stable (resp. semistable) with respect to the action of $\mathrm{SL}\left(H^0\left(M, E(k)\right)\right)$ on $\mathrm{Hom}\left(\bigwedge^r H^0\left(M, E(k)\right), H^0\left(M, \det E(k)\right)\right)$.

Theorem 4. *(X. Wang, [39]) The Gieseker point T_E is stable if and only if there exists a ω-balanced basis $\underline{s}$ of $H^0(M,E)$.*

As a consequence of this theorem and of Definition 2 we have the following:

Theorem 5. *(X. Wang, [39]) E is Gieseker stable if and only if there exists $k_0 \in \mathbb{N}$ such that for $k > k_0$ the bundle $E(k)$ admits a ω-balanced metric.*

We also recall the following result dealing with the uniqueness of ω-balanced metrics in the case E is simple, i.e. $\mathrm{Aut}(E) = \mathbb{C}^* \,\mathrm{id}_E$, where $\mathrm{Aut}(E)$ denotes the group of invertible holomorphic bundle maps from E in itself.

Theorem 6. *(R. Seyyedali, Lemma 2.7 in [37]) Let $E \to M$ be a simple complex vector bundle over a compact Kähler manifold (M, ω). If E admits a ω-balanced metric then it is unique.*

Finally, in the case of homogeneous vector bundles we have the following important result which should be compared with the analogous result in the rank one case (see Theorem 3 above) and which will be important for our applications in the next chapter.

Theorem 7. *(L. Biliotti–A. Ghigi, [4]) Let (M, ω) be a rational homogeneous variety and $E \to M$, be an irreducible homogeneous vector bundles over M. Then E admits a unique ω-balanced metric.*

Remark 13. The importance of Theorem 7 relies on the fact that in the homogeneous case in order to find a ω-balanced basis we do not need to twist the bundle E with a power of a polarization L as in Theorem 5. We refer the reader to [2] for the applications of this theorem to the estimate of the first eigenvalue of the Laplacian associated to ω (cfr. Theorem 1 above).

1.5 Concluding remarks

All results described in this chapter regarding the existence and uniqueness of balanced and ω-balanced metrics deal with vector bundles which are either irreducible or simple. This is obvious in Theorem 2 and Theorem 3 since every line bundle is irreducible and simple, in Theorem 5, since Geiseker stability implies irreducibility and in Theorem 7 (resp. Theorem 6) where the irreducibility (risp. the simpleness) is indeed an assumption. Moreover, in the case of rank $r > 1$, we only deal with ω-balanced basis and not balanced. So two problems naturally arise:

- study the existence and uniqueness of ω-balanced metrics for arbitrary vector bundles (not necessarily simple or irreducible);
- study the existence and uniqueness of balanced metrics. In particular try to find the right “stability conditions”, analogous to the Geiseker stability, which ensures the existence of balanced metrics (cfr. Remark 6 for the rank one case).

In the next chapter we treat the first problem. The balanced case is left for future research.

Chapter 2

Uniqueness of ω-balanced metrics

2.1 Statement and proof of the main result

In this section we prove the following theorem which represents one of the main results of this thesis.

Theorem 8. *Let E be a very ample holomorphic vector bundle over a compact Kähler manifold (M, ω). If E admits a ω-balanced metric then this metric is unique.*

Notice that if E is a very ample holomorphic vector bundle over a compact Kähler manifold (M, ω) and if $\underline{s}$ is *any* basis of $H^0(M, E)$, $F \in \mathrm{Aut}(E)$ and $U \in U(N)$, then it is immediate to verify that $i_{UF\underline{s}} = Ui_{F\underline{s}} = Ui_{\underline{s}}$, where $UF\underline{s} = (UFs_1, \ldots, UFs_N)$, and $h_{\underline{s}} = h_{UF\underline{s}}$, where $i_{\underline{s}}$ and $h_{\underline{s}}$ are given by (1.3) and (1.5) above.

Then the proof of Theorem 8 will be a consequence of the following:

Theorem 9. *If $\underline{s}$ and $\underline{\tilde{s}}$ are two balanced bases of $H^0(M, E)$ then there exist a unitary matrix $U \in U(N)$ and $F \in \mathrm{Aut}(E)$ such that $\underline{\tilde{s}} = UF\underline{s}$.*

In order to prove Theorem 9 (and hence Theorem 8) we need some preliminaries on ω-balanced basis and moment map.

Let E be a very ample holomorphic vector bundle over a compact Kähler manifold (M, ω). Let J_0 be the complex structure of E, denote by E_c the smooth complex vector bundle underlying E and write $E = (E_c, J_0)$. Let N be the complex dimension of $H^0(M, E)$ and let $\mathcal{H}$ be the (infinite dimensional) manifold consisting of pairs $(\underline{s}, J)$ where $\underline{s} = (s_1, \dots, s_N)$ is an N-uple of complex linearly independent smooth sections of E_c, J is a complex structure of E_c and each section s_j is holomorphic with respect to the complex structure J, i.e.

$$ds_j \circ I_0 = J \circ ds_j,\ j = 1, \dots, N,$$

where I_0 denotes the (fixed) complex structure of M.

Given an Hermitian metric h on E we denote by $U_h(E_c)$ the subgroup of $GL(E_c)$ consisting of smooth invertible bundle maps $E_c \to E_c$ preserving the Hermitian metric h and by $SU(N) \subset U(N)$ the group of $N \times N$ unitary matrixes with positive determinant. These groups act in a natural way on $\mathcal{H}$ as follows:

$$\Psi \cdot (\underline{s}, J) = (\Psi \underline{s}, \Psi \cdot J),\ \Psi \in U_h(E_c)$$

$$U \cdot (\underline{s}, J) = (U \underline{s}, J),\ U \in SU(N),$$

were $\Psi \underline{s} = (\Psi s_1, \dots, \Psi s_N)$, $\Psi \cdot J = \Psi J \Psi^{-1}$ and $U \underline{s} = (U s_1, \dots, U s_N)$.

Since these actions commute they induce a well-defined action of the group $\mathcal{G}_h = U_h(E_c) \times SU(N)$ on $\mathcal{H}$. The Lie algebra of $\mathcal{G}_h$ is $GL(E_c) \oplus \mathfrak{su}(N)$ and its complexification $\mathcal{G}_h^{\mathbb{C}} = GL(E_c) \times SL(N)$ naturally acts on $\mathcal{H}$ by extending the action of $\mathcal{G}_h$.

Theorem 10 (Wang [38])**.** *The manifold $\mathcal{H}$ admits a Kähler form Ω invariant for the action of $\mathcal{G}_h$ whose moment map $\mu_h : \mathcal{H} \to GL(E_c) \oplus \mathfrak{su}(N)$ is given by:*

$$\mu_h(\underline{s}, J) = (\ \sum_{j=1}^{N} h(\cdot, s_j) s_j,\ \langle s_j, s_k \rangle_h - \frac{\sum_{j=1}^{N} |s_j|_h^2}{N} \delta_{jk}\ ), \tag{2.1}$$

where $|s_j|_h^2 = \langle s_j, s_j \rangle_h = \frac{1}{V(M)} \int_M h(\cdot, \cdot) \frac{\omega^n}{n!}$. Consequently, a basis $\underline{s} = (\underline{s}, J_0)$ of $H^0(M, E)$ is balanced if and only if $\mu_{h_{\underline{s}}}(\underline{s}, J_0) = (\mathrm{Id}_E, 0)$, where $h_{\underline{s}}$ is the metric of E given by (1.5).

A key ingredient in the proof of Theorem 9 (and hence of Theorem 8) is the following:

Lemma 14. *Let* $\underline{s} = (\underline{s}, J_0)$ *be a balanced basis of* $H^0(M, E)$ *and let* $(\hat{\underline{s}}, \hat{J}) \in \mathcal{H}$ *such that:* $\mu_{h_{\underline{s}}}(\hat{\underline{s}}, \hat{J}) = (\mathrm{Id}_E, 0)$ *and* $(\hat{\underline{s}}, \hat{J})$ *lies in the same* $\mathcal{G}^{\mathbb{C}}_{h_{\underline{s}}}$*-orbit of* $(\underline{s}, J_0)$*. Then* $(\hat{\underline{s}}, \hat{J})$ *lies in the same* $\mathcal{G}_{h_{\underline{s}}}$*-orbit of* $(\underline{s}, J_0)$*, namely there exists* $(\Psi, U) \in \mathcal{G}_{h_{\underline{s}}}$ *such that* $(\Psi, U) \cdot (\hat{\underline{s}}, \hat{J}) = (U\Psi\hat{\underline{s}}, \Psi \cdot \hat{J}) = (\underline{s}, J_0)$*.*

Proof. Since $a = (\mathrm{Id}_E, 0) \in GL(E_c) \oplus \mathfrak{su}(N)$ is (obviously) invariant by the coadjoint action of $\mathcal{G}_{h_{\underline{s}}}$ it is a standard fact in moment map's theory (cfr. Thm A.3) that

$$\mu^{-1}_{h_{\underline{s}}}(a) \cap (\mathcal{G}^{\mathbb{C}}_{h_{\underline{s}}} \cdot x) = \mathcal{G}_{h_{\underline{s}}} \cdot x, \ \forall x \in \mu^{-1}_{h_{\underline{s}}}(a).$$

Then the result follows by the assumptions and by Theorem 10. □

We are now in the position to prove Theorem 9.

Proof of Theorem 9. Let $h_{\underline{s}}$ and $h_{\tilde{\underline{s}}}$ be the metric induced by $\underline{s}$ and $\tilde{\underline{s}}$ and let $\Phi \in GL(E_c)$ be such that $\Phi^* h_{\underline{s}} = h_{\tilde{\underline{s}}}$. We claim that

$$\sum_{j=1}^{N} h_{\underline{s}}(\cdot, \Phi\tilde{s}_j)\Phi\tilde{s}_j = \mathrm{Id}_E \tag{2.2}$$

and

$$\langle \Phi\tilde{s}_j, \Phi\tilde{s}_k \rangle_{h_{\underline{s}}} = \frac{r}{N}\delta_{jk}, \ j, k = 1, \dots, N. \tag{2.3}$$

Indeed

$$\mathrm{Id}_E = \sum_{j=1}^{N} h_{\tilde{\underline{s}}}(\cdot, \tilde{s}_j)\tilde{s}_j = \sum_{j=1}^{N} (\Phi^* h_{\underline{s}})(\cdot, \tilde{s}_j)\tilde{s}_j$$

and if $\underline{\sigma} = (\sigma_1, \dots, \sigma_r) : U \to E$ is a local frame then, for all $\alpha = 1, \dots, r$, one gets:

$$\begin{aligned}
\sigma_\alpha = \Phi(\Phi^{-1}(\sigma_\alpha)) &= \Phi\left(\sum_{j=1}^{N} h_{\tilde{\underline{s}}}(\Phi^{-1}(\sigma_\alpha), \tilde{s}_j)\tilde{s}_j\right) \\
&= \Phi\left(\sum_{j=1}^{N} (\Phi^* h_{\underline{s}})(\Phi^{-1}(\sigma_\alpha), \tilde{s}_j)\tilde{s}_j\right) \\
&= \Phi\left(\sum_{j=1}^{N} h_{\underline{s}}(\sigma_\alpha, \Phi\tilde{s}_j)\tilde{s}_j\right) = \sum_{j=1}^{N} h_{\underline{s}}(\sigma_\alpha, \Phi\tilde{s}_j)\Phi\tilde{s}_j,
\end{aligned}$$

where we have used the fact that $\sum_{j=1}^{N} h_{\tilde{\underline{s}}}(\cdot, \tilde{s}_j)\tilde{s}_j = \mathrm{Id}_E$, and (2.2) follows.

Moreover,

$$\begin{aligned}\langle \Phi\tilde{s}_j, \Phi\tilde{s}_k\rangle_{h_{\underline{s}}} &= \frac{1}{V(M)}\int_M h_{\underline{s}}(\Phi\tilde{s}_j, \Phi\tilde{s}_k)\frac{\omega^n}{n!}\\ &= \frac{1}{V(M)}\int_M (\Phi^* h_{\underline{s}})(\tilde{s}_j, \tilde{s}_k)\frac{\omega^n}{n!}\\ &= \frac{1}{V(M)}\int_M h_{\underline{\tilde{s}}}(\tilde{s}_j, \tilde{s}_k)\frac{\omega^n}{n!} = \frac{r}{N}\delta_{jk}\end{aligned}$$

and also (2.3) is proved.

It follows by (2.1), (2.2) and (2.3) that $(\underline{s}, J_0)$ and $(\Phi\underline{\tilde{s}}, \Phi \cdot J_0)$ are in the same level set of the moment map $\mu_{h_{\underline{s}}}$, namely

$$\mu_{h_{\underline{s}}}(\underline{s}, J_0) = \mu_{h_{\underline{s}}}(\Phi\underline{\tilde{s}}, \Phi \cdot J_0) = (\mathrm{Id}_E, 0).$$

Moreover, since $\underline{s}$ and $\underline{\tilde{s}}$ are bases of the same vector space $H^0(M, E)$ there exist a non zero constant λ and $V \in \mathrm{SL}(N)$ such that $\lambda V\underline{\tilde{s}} = \underline{s}$. Therefore

$$(\underline{s}, J_0) = (\lambda\Phi^{-1}, V) \cdot (\Phi\underline{\tilde{s}}, \Phi \cdot J_0)$$

and hence $(\underline{s}, J_0)$ and $(\Phi\underline{\tilde{s}}, \Phi J_0)$ are elements of $\mathcal{H}$ in the same $\mathcal{G}^{\mathbb{C}}_{h_{\underline{s}}}$-orbit. By Lemma 14 there exists $(\Psi, U) \in \mathcal{G}_{h_{\underline{s}}}$ such that

$$(\underline{s}, J_0) = (\Psi, U) \cdot (\Phi\underline{\tilde{s}}, \Phi \cdot J_0) = (U\Psi\Phi\underline{\tilde{s}}, (\Psi\Phi) \cdot J_0).$$

Consequently, $F = \Psi\Phi : E_c \to E_c$ preserves the complex structure J_0, i.e. $F \in \mathrm{Aut}(E)$ and $\underline{s} = UF\underline{\tilde{s}}$. □

2.2 Homogeneous vector bundles

The aim of this section is to prove the following theorem on the existence and uniqueness of ω-balanced metrics on homogeneous Kähler manifolds.

Theorem 11. *Let (M, ω) be a rational homogeneous variety and $E_j \to M$, $j = 1, \ldots, m$, be irreducible homogeneous vector bundles over M with* $\mathrm{rank}\, E_j = r_j$ *and* $\dim H^0(M, E_j) = N_j > 0$, $j = 1, \ldots, m$. *then the homogeneous vector bundle $E = \oplus_{j=1}^m E_j \to M$ admits a unique homogeneous ω-balanced metric if and only if $\frac{r_j}{N_j} = \frac{r_k}{N_k}$ for all $j, k = 1, \ldots, m$.*

We need the following result interesting on its own sake.

Lemma 15. *Let* $E \to M$ *be a holomorphic vector bundle on a compact complex manifold* M. *Suppose that* E *is a direct sum of two holomorphic vector bundles* $E_1, E_2 \to M$ *with* $\operatorname{rank} E_j = r_j$ *and* $\dim H^0(M, E_j) = N_j > 0$, $j = 1, 2$. *If* $\frac{N_1}{r_1} \neq \frac{N_2}{r_2}$ *then the Gieseker point of* E *is unstable.*

Proof. Consider the basis $\underline{s} = \{s_1, \ldots, s_{N_1+N_2}\}$ of $H^0(M, E)$ such that $\{s_1, \ldots, s_{N_1}\}$ is a basis of $H^0(M, E_1 \oplus \{0\})$ and $\{s_{N_1+1}, \ldots, s_{N_1+N_2}\}$ is a basis of $H^0(M, \{0\} \oplus E_2)$. Suppose that $\frac{N_1}{r_1} > \frac{N_2}{r_2}$. Consider the 1-parameter subgroup of $SL(N_1 + N_2)$

$$t \mapsto g(t) = \operatorname{diag}(\underbrace{t^{-N_2}, \ldots, t^{-N_2}}_{N_1 \text{ times}}, t^{N_1}, \ldots, t^{N_1}),$$

where the action on the elements of the basis $\underline{s}$ is

$$g(t)s_j = \begin{cases} t^{-N_2} s_j & \text{if } j \leq N_1 \\ t^{N_1} s_j & \text{otherwise} \end{cases}$$

Observe that the section $x \mapsto s_{j_1}(x) \wedge \cdots \wedge s_{j_r}(x)$ (with $j_1 < j_2 < \cdots < j_r$) with $r = r_1 + r_2$ is different from zero only when $j_{r_1} \leq N_1 < j_{r_1+1}$. So the action of $g(t)$ on the Gieseker point is given by:

$$g(t)T_E(s_{j_1} \wedge \cdots \wedge s_{j_r}) =$$

$$= \begin{cases} t^{r_2 N_1 - r_1 N_2} T_E(s_{j_1} \wedge \cdots \wedge s_{j_r}) & \text{if } j_{r_1} \leq N_1 < j_{r_1+1} \\ 0 & \text{otherwise} \end{cases}$$

Since $\frac{N_1}{r_1} > \frac{N_2}{r_2}$ we have

$$\lim_{t \to 0} g(t) T_E \equiv 0.$$

□

Proof of Theorem 11 By Lemma 15 and Wang's Theorem 4 we have to prove only the sufficient condition. Since $\operatorname{rank} E = \sum_{j=1}^m r_j$ and $\dim(H^0(M, E)) = \sum_{j=1}^m N_j$, it is enough to prove the theorem for $m = 2$. In Theorem 7 it is proved that each E_j, as in the statement, is a very ample bundle and admits a unique homogeneous ω-balanced metric induced by a basis $\underline{s}^j = \{s^j_1, \dots, s^j_{N_j}\}$, $j = 1, 2$. Then, the assumption $\frac{r_1}{N_1} = \frac{r_1}{N_2}$, readily implies that the basis

$$\underline{s} = ((s^1_1, 0), \dots, (s^1_{N_1}, 0), (0, s^2_1), \dots, (0, s^2_{N_2})) \tag{2.4}$$

is a homogeneous ω-balanced basis for $E_1 \oplus E_2$. Then $h_{\underline{s}} = i^*_{\underline{s}} h_{Gr}$ is the desired homogeneous balanced metric on $E_1 \oplus E_2$ which is unique by Theorem 8. $\square$

2.3 Rigidity of ω-balanced Kähler maps into Grassmannians

Let (M, ω) be a compact Kähler manifold and let $\tilde{\omega}$ a Kähler form on M. A holomorphic map $f : M \to G(r, N)$ is said to be $\tilde{\omega}$-balanced if there exist a very ample holomorphic vector bundle $E \to M$ and a $\tilde{\omega}$-balanced basis $\underline{s}$ of $H^0(M, E)$ such that $f = i_{\underline{s}}$ (thus necessarily $f^*\mathcal{Q} = E$, $r = \operatorname{rank} E$ and $N = \dim H^0(M, E)$). A $\tilde{\omega}$-balanced map $f : (M, \omega) \to G(r, N)$ is called a *Kähler* map if $f^*\omega_{Gr} = \omega$, where ω_{Gr} is the standard Kähler form on $G(r, N)$, i.e. $\mathrm{Ric}(h_{Gr}) = \omega_{Gr}$.

Example 16. Let $M = \mathbb{C}P^1$ and $\omega_\lambda = \lambda\omega_{FS}$, where ω_{FS} is the Fubini-Study Kähler form and λ is a positive real number. Then, it is not hard to see that the holomorphic map

$$f : \mathbb{C}P^1 \to G(2, 4) : [z_0, z_1] \mapsto \begin{bmatrix} z_0 & 0 \\ 0 & z_0 \\ z_1 & 0 \\ 0 & z_1 \end{bmatrix} \tag{2.5}$$

is a ω_λ-balanced map for all λ. Moreover, f is Kähler with respect to $2\omega_{FS}$ i.e. $f^*\omega_{Gr} = 2\omega_{FS}$. (It follows by definition that if $\underline{s}$ is a ω-balanced basis of $H^0(M, E)$ then $\underline{s}$ is still $\lambda\omega$-balanced for $\lambda > 0$).

Note that in the previous example $f^*\mathcal{Q} = O(1) \oplus O(1)$, where $O(1)$ is the hyperplane bundle on $\mathbb{C}P^1$ and $f^*\omega_{Gr} = 2\omega_{FS}$. On the other hand, there exist holomorphic maps

$\tilde{f} = i_{\underline{\tilde{s}}} : \mathbb{C}P^1 \to G(2,4)$ (where $\underline{\tilde{s}}$ is a basis of $O(1) \oplus O(1)$) satisfying these two conditions but for which it cannot exist a unitary transformation U of $G(2,4)$ such that $\tilde{f} = Uf$ (cfr. [10]). An example is given by:

$$\tilde{f} : \mathbb{C}P^1 \to G(2,4), \quad [z_0, z_1] \mapsto \begin{bmatrix} {z_0}^2 & z_0\bar{z}_1\frac{1}{2}\left(\sqrt{3}-1\right) \\ -z_0z_1\frac{1}{2}\left(\sqrt{3}-1\right) & |z_0|^2 + \frac{1}{2}|z_1|^2\sqrt{3} \\ -z_0z_1\frac{1}{2}\left(\sqrt{3}+1\right) & -\frac{1}{2}|z_1|^2 \\ {z_1}^2 & \bar{z}_0z_1\frac{1}{2}\left(1-\sqrt{3}\right) \end{bmatrix}.$$

This phenomenon is due to the fact that the rigidity of Kähler maps into $G(r, N)$ with $r \geq 2$ does not in general holds true (see, e.g., [9], [10], [21], [40], [41]), in contrast with the case $r = 1$ where one has the celebrated Calabi's rigidity theorem for Kähler maps into projective spaces.

On the other hand the following theorem, which is the main result of this section, shows the rigidity of $\tilde{\omega}$-balanced Kähler embedding.

Theorem 12. *Let $E \to M$ be a very ample complex vector bundle over a compact Kähler manifold (M, ω). Assume that E admits a $\tilde{\omega}$-balanced metric h such that* $\mathrm{Ric}(h) = \omega$*, where $\tilde{\omega}$ is a Kähler form on M. Then there exists a unique (up to a unitary transformations of $G(r, N)$) $\tilde{\omega}$-balanced Kähler embedding $f : M \to G(r, N)$ such that $f^*Q = E$.*

Proof. Let $\underline{s}$ be a $\tilde{\omega}$-balanced basis of $H^0(M, E)$ and let $f = i_{\underline{s}} : M \to G(r, N)$ be the associated Kodaira's map. By Theorem 9 f is the unique (up to a unitary transformations of $G(r, N)$) $\tilde{\omega}$-balanced embedding such that $f^*Q = E$. So it remains to show that $f^*\omega_{Gr} = \omega$. Fix a local frame $(\sigma_1, \dots, \sigma_r) : U \to E$. In this local frame $f : U \to G(r, N)$ is given by (1.3). Then, the local expression of $\omega = \mathrm{Ric}(h)$ and $f^*\omega_{Gr}$ are given respectively by $-\frac{i}{2}\partial\bar{\partial}\log\det(S^*S)^{-1}$ and $\frac{i}{2}\partial\bar{\partial}\log\det(S^*S)$. $\square$

2.4 Kähler maps of $(\mathbb{C}P^1, 2\omega_{FS})$ in $G(2,4)$

In this section we prove that the only $2\omega_{FS}$-balanced Kähler maps (see Section 2.3), among the Kähler maps $f : (\mathbb{C}P^1, 2\omega_{FS}) \to G(2,4)$ are those unitarily equvalent to the map $F : (\mathbb{C}P^1, 2\omega_{FS}) \to G(2,4)$ given by

$$F([z,w]) = \begin{bmatrix} z & 0 \\ 0 & z \\ w & 0 \\ 0 & w \end{bmatrix}. \tag{2.6}$$

Q.–S. Chi and Y. Zheng [10] proved that every Kähler map $f : (\mathbb{C}P^1, 2\omega_{fS}) \to G(2,4)$ is unitarily equivalent to one element of the following family of Kähler maps

$$f_t : \mathbb{C}P^1 \to G(2,4), \quad [z,w] \mapsto \begin{bmatrix} z^2 & z\overline{w}\left(\cos(t) - \sin(t)\right) \\ -zw\left(\cos(t) - \sin(t)\right) & |z|^2 + |w|^2 \sin(2t) \\ -zw\left(\cos(t) + \sin(t)\right) & -|w|^2 \cos(2t) \\ w^2 & -\overline{z}w\left(\cos(t) - \sin(t)\right) \end{bmatrix}$$

Therefore we are reduced to investigate which Kähler maps among the family f_t are $2\omega_{FS}$-balanced. The calculation are done with the help of software of symbolic calculus.

First of all we study the family f_t when $t \neq \frac{\pi}{4}, \frac{3\pi}{4}$. The expression of f_t in local coordinates of $\mathbb{C}P^1$ and $G(2,4)$ can be written

$$[z,1] \mapsto \begin{bmatrix} -\frac{z}{\cos(t)+\sin(t)} & 0 \\ -\frac{2\sin(t)\cos(t)}{2(\cos(t))^2-1} & -\frac{z}{\cos(t)-\sin(t)} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.7}$$

Therefore $E_t = f_t^* \mathcal{T}^* = O(1) \oplus O(1)$ and the basis $\underline{s}^{(t)}$ of $H^0(\mathbb{C}P^1, E_t)$ satisfying $f_t = i_{\underline{s}^{(t)}}$ is given by

$$\begin{aligned} \underline{s}^{(t)} &= \left\{ \left(\frac{z}{\cos(t) + \sin(t)}, 0 \right), \left(-\frac{2\sin(t)\cos(t)}{2(\cos(t))^2 - 1}, -\frac{z}{\cos(t) - \sin(t)} \right), (1,0), (0,1) \right\} \\ &= \{s_1^{(t)}, \dots, s_4^{(t)}\}. \end{aligned} \tag{2.8}$$

With a direct calculus we see that the matrix $\mathcal{H}^{\underline{s}^{(t)}} = \langle s_j^{(t)}, s_k^{(t)}\rangle_{h_{s(t)}, 2\omega_{FS}} = \frac{1}{2}I$ if and only if $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. Substituting these values of t in (2.7) we deduce immediately that $f_0, f_{\frac{\pi}{2}}, f_\pi, f_{\frac{3\pi}{2}}$ are unitary equivalent to the embedding

$$[z, 1] \mapsto \begin{bmatrix} z & 0 \\ 0 & z \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

To conclude we have to prove that $f_{\frac{\pi}{4}}$ and $f_{\frac{3\pi}{4}}$ are not $2\omega_{FS}$-balanced.

We consider first the case $t = \frac{\pi}{4}$. In local coordinates $f_{\frac{\pi}{4}}$ is written as

$$[z, 1] \mapsto \begin{bmatrix} 0 & z^2 \\ 1 & 0 \\ 0 & -\sqrt{2}z \\ 0 & 1 \end{bmatrix}$$

We see that $E_{\frac{\pi}{4}} = f^*_{\frac{\pi}{4}}\mathcal{T}^* = O(2) \oplus O(0)$ and that the basis $\underline{s}^{(\frac{\pi}{4})} = \{s_1^{(\frac{\pi}{4})}, \ldots, s_4^{(\frac{\pi}{4})}\} = \{(0, z^2), (1, 0), (0, -\sqrt{2}z), (0, 1)\}$ of $H^0(E, \mathbb{C}P^1)$ is such that $i_{\underline{s}^{(\frac{\pi}{4})}} = f_{\frac{\pi}{4}}$. The map is not $2\omega_{FS}$-balanced, indeed the matrix $\mathcal{H}^{\underline{s}^{(\frac{\pi}{4})}} = \langle s_j^{(\frac{\pi}{4})}, s_k^{(\frac{\pi}{4})}\rangle_{\underline{s}^{(\frac{\pi}{4})}}$ is given by

$$\begin{bmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We study now the last case $t = \frac{3\pi}{4}$. The local expression of $f_{\frac{3\pi}{4}}$ is given by

$$[z, 1] \mapsto \begin{bmatrix} \sqrt{2}z & -z^2 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We see that the immersion is not full, so $f_{\frac{3\pi}{4}}$ can not be induced by a basis of $H^0(M, E)$ and therefore $f_{\frac{3\pi}{4}}$ is not $2\omega_{FS}$-balanced.

Final remark Theorem 12 can be restated as follows: *Let* $E \to M$ *be a very ample complex vector bundle over a compact Kähler manifold* (M, ω)*. Suppose that* $H^0(M, E)$ *admits two* $\tilde{\omega}$*-balanced basis* $\underline{s}_1 = \{s_{1_1} \dots, s_{1_N}\}$ *and* $\underline{s}_2 = \{s_{2_1} \dots, s_{2_N}\}$ *where* $\tilde{\omega}$ *is a Kähler form. Then*

$$i_{\underline{s}_1} = U i_{\underline{s}_2}$$

for a unitary transformation $U \in U(N)$*.*

On the other hand by Definition 2, the $\tilde{\omega}$-balanced condition on $\underline{s}_1$ and $\underline{s}_2$ is equivalent to $\mathcal{H}^{\underline{s}_1, \tilde{\omega}} = \mathcal{H}^{\underline{s}_2, \tilde{\omega}} = \mathrm{Id}$, where $\mathcal{H}^{\underline{s}_l, \tilde{\omega}} = \langle s_{l_j}, s_{l_k} \rangle_{h_{\underline{s}_l}, \tilde{\omega}}$ for $l = 1, 2$. Therefore, it is natural to ask if, under the (necessary) condition that $\mathcal{H}^{\underline{s}_1}$ and $\mathcal{H}^{\underline{s}_2}$ are similar matrices, there exists $U \in U(N)$ such that

$$i_{\underline{s}_2} = U i_{\underline{s}_1}.$$

The answer is no. For example if $t_2 = t_1 + \pi$ (with $t_1, t_2 \neq \frac{\pi}{4}, \frac{3\pi}{4}$) and $\underline{s}^{(t_1)}$ and $\underline{s}^{(t_2)}$ are the basis given in (2.8), then $\mathcal{H}^{\underline{s}^{(t_1)}, 2\omega_{FS}} = \mathcal{H}^{\underline{s}^{(t_2)}, 2\omega_{FS}}$, but $h_{\underline{s}^{(t_1)}} \neq h_{\underline{s}^{(t_2)}}$.

Chapter 3

The diastatic exponential of a symmetric space

3.1 Statements of the main results

Let M be a n-dimensional complex manifold endowed with a real analytic Kähler metric g. For a fixed point $p \in M$ let $D_p : U \to \mathbb{R}$ be the *Calabi diastasis function*, defined in the following way. Recall that a Kähler potential is an analytic function Φ defined in a neighborhood of a point p such that $\omega = \frac{i}{2}\partial\bar{\partial}\Phi$, where ω is the Kähler form associated to g. By duplicating the variables z and $\bar{z}$ a potential Φ can be complex analytically continued to a function $\tilde{\Phi}$ defined in a neighborhood U of the diagonal containing $(p, \bar{p}) \in M \times \bar{M}$ (here $\bar{M}$ denotes the manifold conjugated to M). The diastasis function is the Kähler potential D_p around p defined by

$$D_p(q) = \tilde{\Phi}(q, \bar{q}) + \tilde{\Phi}(p, \bar{p}) - \tilde{\Phi}(p, \bar{q}) - \tilde{\Phi}(q, \bar{p}).$$

If $d_p : \exp_p(V) \subset M \to \mathbb{R}$ denotes the geodesic distance from p then one has:

$$D_p(q) = d_p(q)^2 + O\left(d_p(q)^4\right)$$

and $D_p = d_p^2$ if and only if g is the flat metric. We refer the reader to the seminal paper of E. Calabi [9] for more details and further results on the diastasis function (see also

[30], [29] and [12]).

In [29] it is proven that there exists an open neighbourhood S of the zero section of the tangent bundle TM of M and a smooth embedding $\nu : S \to TM$ such that $p \circ \nu = p$, where $p : TM \to M$ is the natural projection, satisfying the following conditions: if one writes

$$\nu (p, v) = (p, \nu_p (v)) , \ (p, v) \in S$$

then the diffeomorphism

$$\nu_p : T_pM \cap S \to T_pM \cap \nu (S)$$

satisfies

$$(d\nu_p)_0 = \mathrm{id}_{T_pM}$$

$$D_p \left(\exp_p (\nu_p (v)) \right) = g_p (v, v) , \ \forall v \in T_pM \cap S,$$

where $\exp_p : V \subset T_pM \to M$ denotes the exponential map at p (V is a suitable neighbourhood of the origin of T_pM where the restriction of $\exp_p$ is a diffeomorphism). Thus, the smooth map

$$\mathrm{Exp}_p := \exp_p \circ \nu_p : T_pM \cap S \to M$$

satisfies

$$\left(d \, \mathrm{Exp}_p \right)_0 = \mathrm{id}_{T_pM} \tag{3.1}$$

$$D_p \left(\mathrm{Exp}_p (v) \right) = g_p (v, v) , \ \forall v \in W. \tag{3.2}$$

In analogy with the exponential at p (which satisfies $d_p \left(\exp_p (v) \right) = \sqrt{g_p (v, v)}$, $\forall v \in V$) any smooth map $\mathrm{Exp}_p : W \to M$ from a neighbourhood W of the origin of T_pM into M satisfying (3.1) and (3.2) will be called a *diastatic exponential* at p.

It is worth pointing out (see [5] for a proof) that $\exp_p$ is holomorphic if and only if the metric g is flat and it is not hard to see that the same assertion holds true for a diastatic exponential Exp_p.

In this thesis we study the diastatic exponentials for the Hermitian symmetric spaces of noncompact type (HSSNT) and for their compact duals. The following examples deal with the rank one case and it will be our prototypes for the general case.

Example 17. Let $\mathbb{C}H^n = \{z \in \mathbb{C}^n \mid |z|^2 = |z_1|^2 + \cdots + |z_n|^2 < 1\}$ be the complex hyperbolic space endowed with the hyperbolic metric, namely the metric g^{hyp} whose associated Kähler form is given by $\omega^{\mathrm{hyp}} = -\frac{i}{2}\partial\bar{\partial}\log\left(1 - |z|^2\right)$. Thus the diastasis function $D_0^{\mathrm{hyp}} : \mathbb{C}H^n \to \mathbb{R}$ and the exponential map $\exp_0^{\mathrm{hyp}} : T_0\mathbb{C}H^n \cong \mathbb{C}^n \to \mathbb{C}H^n$ around the origin $0 \in \mathbb{C}^n$ are given respectively by

$$D_0^{\mathrm{hyp}}(z) = -\log\left(1 - |z|^2\right)$$

and

$$\exp_0^{\mathrm{hyp}}(v) = \tanh(|v|)\frac{v}{|v|}, \quad \exp_0^{\mathrm{hyp}}(0) = 0.$$

It is then immediate to verify that the map $\mathrm{Exp}_0^{\mathrm{hyp}} : T_0\mathbb{C}H^n \to \mathbb{C}H^n$ given by:

$$\mathrm{Exp}_0^{\mathrm{hyp}}(v) = \sqrt{1 - e^{-|v|^2}}\,\frac{v}{|v|}, \quad \mathrm{Exp}_0^{\mathrm{hyp}}(0) = 0, \quad v = (v_1, \ldots v_n)$$

satisfies $\left(d\,\mathrm{Exp}_0^{\mathrm{hyp}}\right)_0 = \mathrm{id}_{T_0\mathbb{C}H^n}$ and

$$D_0^{\mathrm{hyp}}\left(\mathrm{Exp}_0^{\mathrm{hyp}}(v)\right) = g_0^{\mathrm{hyp}}(v, v) = |v|^2, \quad \forall v \in T_0\mathbb{C}H^n = \mathbb{C}^n.$$

Hence $\mathrm{Exp}_0^{\mathrm{hyp}}$ is a diastatic exponential at 0. Notice that $\mathrm{Exp}_0^{\mathrm{hyp}}$ is characterized by the fact that it is *direction preserving*. More precisely, if $F : T_0\mathbb{C}H^n \to \mathbb{C}H^n$ is a diastatic exponential satisfying $F(v) = \lambda(v)\,v$, for some smooth nonnegative function $\lambda : \mathbb{C}^n \to \mathbb{R}$, then $F = \mathrm{Exp}_0^{\mathrm{hyp}}$.

Example 18. Let $P = \left(\mathbb{C}H^1\right)^\ell$ be a polydisk. If z_k, $k = 1, \ldots, \ell$, denotes the complex coordinate in each factor of P and $v = (v_1, \ldots, v_\ell) \in T_0P \cong \mathbb{C}^\ell$. Then the diastasis $D_0^P : P \to \mathbb{R}$, the exponential map $\exp_0^P : T_0P \to P$ and a diastatic exponential

$\mathrm{Exp}_0^P : T_0P \to P$ at the origin are given respectively by:

$$D_0^P(z) = -\sum_{k=1}^{\ell} \log\left(1 - |z_k|^2\right),$$

$$\exp_0^P(v) = \left(\tanh(|v_1|)\frac{v_1}{|v_1|}, \dots, \tanh(|v_\ell|)\frac{v_\ell}{|v_\ell|}\right), \quad \exp_0^{\mathrm{hyp}}(0) = 0,$$

$$\mathrm{Exp}_0^P(v) = \left(\sqrt{1 - e^{-|v_1|^2}}\frac{v_1}{|v_1|}, \dots, \sqrt{1 - e^{-|v_\ell|^2}}\frac{v_\ell}{|v_\ell|}\right), \quad \mathrm{Exp}_0^P(0) = 0. \tag{3.3}$$

Let now M be an HSSNT which we identify with a bounded symmetric domain of $\mathbb{C}^n$ centered at the origin $0 \in \mathbb{C}^n$ equipped with the hyperbolic metric g^{hyp}, namely the Kähler metric whose associated Kähler form (in the irreducible case) is given by

$$\omega^{\mathrm{hyp}} = \frac{i}{2g}\partial\bar{\partial}\log K_M.$$

Here $K_M(z, \bar{z})$ (holomorphic in the first variable and antiholomorphic in the second one) denotes the reproducing kernel of M and g its genus. By using the rotational symmetries of M one can show that the diastasis function at the origin $D_0^{\mathrm{hyp}} : M \to \mathbb{R}$ is globally defined and reads as

$$D_0^{\mathrm{hyp}}(z) = \frac{1}{g}\log K_M(z, \bar{z}),$$

(see [30] for a proof and further results on Calabi's function for HSSNT). Notice also that, by Hadamard theorem, the exponential map $\exp_0^{\mathrm{hyp}} : T_0M \to M$ is a global diffeomorphism.

The following theorem which is the first result of this chapter, contains a description of the diastatic exponential for HSSNT.

Theorem 13. *Let $\left(M, g^{\mathrm{hyp}}\right)$ be an HSSNT. Then there exists a globally defined diastatic exponential $\mathrm{Exp}_0^{\mathrm{hyp}} : T_0M \to M$ which is a diffeomorphism and is uniquely determined by the fact that ${\mathrm{Exp}_0^{\mathrm{hyp}}}_{|T_0P} = \mathrm{Exp}_0^P$ for every polydisk $P \subset M$, $0 \in P$, where Exp_0^P is given by (3.3). In particular ${\mathrm{Exp}_0^{\mathrm{hyp}}}_{|T_0N} = \mathrm{Exp}_0^N$ for every complex and totally geodesic submanifold $N \subset M$ through 0.*

Consider now the Hermitian symmetric spaces of compact type (HSSCT). Let us consider first the compact duals of Examples 17 and 18.

Example 19. Let $\mathbb{C}P^n$ be the complex projective space endowed with the Fubini–Study metric g^{FS}, namely the metric whose associated Kähler form is given by

$$\omega^{FS} = \frac{i}{2}\partial\bar{\partial}\log\left(|Z_0|^2 + \cdots + |Z_n|^2\right)$$

for a choice of homogeneous coordinates $Z_0, \ldots, Z_n$. Let $p_0 = [1, 0, \ldots, 0]$ and consider the affine chart $U_0 = \{Z_0 \neq 0\}$. Thus we have the following inclusions

$$\mathbb{C}H^n \subset \mathbb{C}^n \cong U_0 \subset \mathbb{C}P^n, \tag{3.4}$$

where we are identifying U_0 with $\mathbb{C}^n$ via the affine coordinates

$$U_0 \to \mathbb{C}^n : [Z_0, \ldots, Z_N] \mapsto \left(z_1 = \frac{Z_1}{Z_0}, \ldots, z_n = \frac{Z_n}{Z_0}\right).$$

Under this identification we make no distinction between the point p_0 and the origin $0 \in \mathbb{C}^n$. The Calabi's diastasis function $D_0^{FS} : U_0 \to \mathbb{R}$ around $p_0 \equiv 0$ and the exponential map $\exp T_0\mathbb{C}P^n \to U_0$ are respectively given by

$$D_0^{FS}(z) = \log\left(1 + |z|^2\right)$$

and

$$\exp_0^{FS}(v) = \left(\tan(|v|)\,\frac{v}{|v|}\right), \quad \exp_0(0) = 0.$$

Observe that D_0^{FS} blows up at the points belonging to $\mathbb{C}P^n \setminus U_0$ which is the cut locus of p_0 with respect to the Fubini–Study metric. We denote this set by $\mathrm{Cut}_0(\mathbb{C}P^n)$.

It is not hard to verify that the map

$$\mathrm{Exp}_0^{FS} : T_0\mathbb{C}P^n \to \mathbb{C}P^n \setminus \mathrm{Cut}_0(\mathbb{C}P^n)$$

given by

$$\mathrm{Exp}_0^{FS}(v) = \sqrt{e^{|v|^2} - 1}\,\frac{v}{|v|}, \quad \mathrm{Exp}_0^{FS}(0) = 0,$$

is a diastatic exponential at 0, namely it satisfies $\left(d\,\mathrm{Exp}_0^{FS}\right)_0 = \mathrm{id}_{T_0\mathbb{C}P^n}$ and

$$D_0^{FS}\left(\mathrm{Exp}_0^{FS}(v)\right) = g_0^{FS}(v, v) = |v|^2, \quad \forall v \in T_0\mathbb{C}P^n.$$

Example 20. Let $P^* = \left(\mathbb{C}P^1\right)^\ell$ be a (dual) polydisk. If z_k, for $k = 1, \dots, \ell$, denotes the affine coordinate in each factor of P^* and $v = (v_1, \dots, v_\ell) \in T_0 M^* \cong \mathbb{C}^\ell$ then it is immediate to see that the diastasis $D_0^{P^*} : P^* \to \mathbb{R}$, the exponential map $\exp_0^{P^*} : T_0 P^* \to P^*$ and a diastatic exponential $\mathrm{Exp}_0^{P^*} : T_0 P^* \to P^*$ at the origin are given respectively by:

$$D_0^{P^*}(z) = \sum_{k=1}^{\ell} \log\left(1 + |z_k|^2\right),$$

$$\exp_0^{P^*}(v) = \left(\tan(|v_1|)\frac{v_1}{|v_1|}, \dots, \tan(|v_\ell|)\frac{v_\ell}{|v_\ell|}\right), \quad \exp_0(0) = 0,$$

$$\mathrm{Exp}_0^{P^*}(v) = \left(\sqrt{e^{|v_1|^2} - 1}\frac{v_1}{|v_1|}, \dots, \sqrt{e^{|v_\ell|^2} - 1}\frac{v_\ell}{|v_\ell|}\right), \quad \mathrm{Exp}_0^{P^*}(0) = 0. \tag{3.5}$$

Given an arbitrary HSSNT M of genus g let us denote by M^* its compact dual equipped with the Fubini–Study metric g^{FS}, namely the pull-back of the Fubini–Study metric of $\mathbb{C}P^N$ via the Borel–Weil embedding $M^* \to \mathbb{C}P^N$ (see [13] for details). Let $0 \in M^*$ be a fixed point and denote by $\mathrm{Cut}_0(M^*)$ the cut locus of 0 with respect to the Fubini–Study metric. In the irreducible case the Kähler form ω^{FS} associated to g^{FS} is given (in the *affine* chart $M^* \setminus \mathrm{Cut}_0(M^*)$) by

$$\omega^{FS} = \frac{i}{2g}\partial\bar{\partial}\log K_{M^*},$$

where

$$K_{M^*}(z, \bar{z}) = 1/K_M(z, -\bar{z}). \tag{3.6}$$

We call K_{M^*} the *reproducing kernel* of M^*. Notice that K_{M^*} is the weighted Bergman kernel for the (finite dimensional) complex Hilbert space consisting of holomorphic functions f on $M^* \setminus \mathrm{Cut}_0(M) \subset M^*$ such that $\int_{M^* \setminus \mathrm{Cut}_0(M)} |f|^2 \left(\omega^{FS}\right)^n < \infty$ (see [30] and also [16] for a nice characterization of symmetric spaces in terms of K_{M^*}). Notice that when $M = \mathbb{C}H^n$ then $g = n+1$, $K_M(z, \bar{z}) = \left(1 - |z|^2\right)^{-(n+1)}$, $K_{M^*}(z, \bar{z}) = \left(1 + |z|^2\right)^{n+1}$ and the Borel–Weil embedding is the identity of $\mathbb{C}P^n$.

Observe that, as in the previous examples, D_0^{FS} is globally defined in $M^* \setminus \mathrm{Cut}_0(M^*)$

(see [43] for a proof) and it blows up at the points in $\mathrm{Cut}_0(M^*)$. Moreover

$$D_0^{FS}(z) = \frac{1}{g} \log K_{M^*}(z, \bar{z}), \quad z \in M^* \setminus \mathrm{Cut}_0(M^*).$$

Furthermore (see e.g. [45]) $M^* \setminus \mathrm{Cut}_0(M^*)$ is globally biholomorphic to T_0M and if 0 denotes the origin of M one has the following inclusions (analogous of (3.4))

$$M \subset T_0M = T_0M^* \cong M^* \setminus \mathrm{Cut}_0(M^*) \subset M^*. \tag{3.7}$$

We are now in the position to state our second result which is the dual counterpart of Theorem 13.

Theorem 14. *Let (M^*, g^{FS}) be an HSSCT. Then there exists a globally defined diastatic exponential* $\mathrm{Exp}_0^{FS} : T_0M^* \to M^* \setminus \mathrm{Cut}_0(M^*)$ *which is uniquely determined by the fact that for every (dual) polydisk $P^* = (\mathbb{C}P^1)^s \subset M^*$ its restriction to T_0P^* equals the map* $\mathrm{Exp}_0^{P^*}$ *given by (3.5). In particular* $\mathrm{Exp}_0^{FS}{}_{|T_0N^*} = \mathrm{Exp}_0^{N^*}$ *for every complex and totally geodesic submanifold $N^* \subset M^*$ through* 0.

The key ingredient for the proof of Theorem 13 and Theorem 14 is the theory of Hermitian positive Jordan triple systems (HPJTS). In [13] this theory has been the main tool to study the link between the symplectic geometry of an Hermitian symmetric space $(M, \omega^{\mathrm{hyp}})$ and its dual (M^*, ω^{FS}) where ω^{hyp} (resp. ω^{FS}) is the Kähler form associated to g^{hyp} (resp. g^{FS}). The main result proved there, is the following theorem (an alternative proof of this theorem can be found in [14]).

Theorem 15. *Let M be an HSSNT and $B(z,w)$ its associated Bergman operator (see next section). Then the map*

$$\Psi_M : M \to M^* \setminus \mathrm{Cut}_0(M^*), \quad z \mapsto B(z,z)^{-\frac{1}{4}} z, \tag{3.8}$$

called the symplectic duality map, *is a real analytic diffeomorphism satisfying*

$$\Psi_M^* \omega_0 = \omega^{\mathrm{hyp}}$$

and

$$\Psi_M^* \omega^{FS} = \omega_0,$$

where ω_0 *is the flat Kähler form on* T_0M*. Moreover, for every complex and totally geodesic submanifold* $N \subset M$ *one has* $\Psi_{M|N} = \Psi_N$*.*

Here ω_0 denotes the Kähler form on M obtained by the restriction of the flat Kähler form on $T_0M = \mathbb{C}^n$.

The following theorem which represents our third result provides a geometric interpretation of the symplectic duality map in terms of diastatic exponentials.

Theorem 16. *Let* M *be a HSSNT and* M^* *be its compact dual. Then the symplectic duality map can be written as*

$$\Psi_M = \mathrm{Exp}_0^{FS} \circ \left(\mathrm{Exp}_0^{\mathrm{hyp}}\right)^{-1} : M \to M^* \setminus \mathrm{Cut}_0\left(M^*\right),$$

where $\mathrm{Exp}_0^{\mathrm{hyp}} : T_0M \to M$ *and* $\mathrm{Exp}_0^{FS} : T_0M^* \to M^* \setminus \mathrm{Cut}_0\left(M^*\right)$ *are the diastatic exponentials at* 0 *of* M *and* M^* *respectively.*

Our fourth (and last) result is the following theorem which shows that the "algebraic manipulation" (3.6) which allows us to pass from K_M to K_{M^*} can be realized via the symplectic duality map.

Theorem 17. *Let* K_M *be the reproducing kernel for an HSSNT and let* K_M^* *be its dual. Then*

$$K_{M^*} \circ \Psi_M = K_M,$$

where $\Psi_M : M \to M^* \setminus \mathrm{Cut}_0\left(M^*\right)$ *is the symplectic duality map.*

This chapter contains other two sections. In the first one we recal some standard facts about HSSNT and HPJTS. In the second one we prove our main results: Theorem 13, Theorem 14, Theorem 16 and Theorem 17.

3.2 Basic tools for the proofs of the main results

Hermitian positive Jordan triple systems

We refer the reader to [36] (see also [33]) for more details of the material on Hermitian positive Jordan triple systems.

An Hermitian Jordan triple system is a pair $(\mathcal{M}, \{,,\})$, where $\mathcal{M}$ is a complex vector space and $\{,,\}$ is a map

$$\{,,\} : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$$

$$(u, v, w) \mapsto \{u, v, w\}$$

which is $\mathbb{C}$-bilinear and symmetric in u and w, $\mathbb{C}$-antilinear in v and such that the following *Jordan identity* holds:

$$\{x, y, \{u, v, w\}\} - \{u, v, \{x, y, w\}\} = \{\{x, y, u\}, v, w\} - \{u, \{v, x, y\}, w\}.$$

For $x, y, z \in \mathcal{M}$ considered the following operator

$$T(x, y)\, z = \{x, y, z\}$$

$$Q(x, z)\, y = \{x, y, z\}$$

$$Q(x, x) = 2Q(x)$$

$$B(x, y) = \mathrm{id}_{\mathcal{M}} - T(x, y) + Q(x)\, Q(y)\,.$$

The operators $B(x, y)$ and $T(x, y)$ are $\mathbb{C}$-linear, the operator $Q(x)$ is $\mathbb{C}$-antilinear. $B(x, y)$ is called the *Bergman operator.* For $z \in V$, the *odd powers* $z^{(2p+1)}$ of z in the Jordan triple system V are defined by

$$z^{(1)} = z \qquad z^{(2p+1)} = Q(z)\, z^{(2p-1)}.$$

An Hermitian Jordan triple system is called *positive* if the Hermitian form

$$(u \mid v) = \operatorname{tr} T(u, v)$$

is positive definite. An element $c \in \mathcal{M}$ is called *tripotent* if $\{c, c, c\} = 2c$. Two tripotents c_1 and c_2 are called *(strongly) orthogonal* if $T(c_1, c_2) = 0$.

HSSNT associated to HPJTS

M. Koecher ([25], [26]) discovered that to every HPJTS $(\mathcal{M}, \{,,\})$ one can associate an Hermitian symmetric space of noncompact type, i.e. a bounded symmetric domain M centered at the origin $0 \in \mathcal{M}$. The domain M is defined as the connected component containing the origin of the set of all $u \in \mathcal{M}$ such that $B(u, u)$ is positive definite with respect to the Hermitian form $(u, v) \mapsto \operatorname{tr} T(u, v)$. *We will always consider such a domain in its (unique up to linear isomorphism) circled realization.* The reproducing kernel K_M of M is given by

$$K_M(z, \bar{z}) = \det B(z, z) \tag{3.9}$$

and so when M is irreducible

$$\omega^{\mathrm{hyp}} = -\frac{i}{2g} \partial\bar{\partial} \log \det B.$$

The HPJTS $(\mathcal{M}, \{,,\})$ can be recovered by its associated HSSNT M by defining $\mathcal{M} = T_0 M$ (the tangent space to the origin of M) and

$$\{u, v, w\} = -\frac{1}{2}\left(R_0(u, v) w + J_0 R_0(u, J_0 v) w\right), \tag{3.10}$$

where R_0 (resp. J_0) is the curvature tensor of the Bergman metric (resp. the complex structure) of M evaluated at the origin. The reader is referred to Proposition III.2.7 in [3] for the proof of (3.10). For more informations on the correspondence between $HPJTS$ and $HSSNT$ we refer also to p. 85 in Satake's book [42].

Totally geodesic submanifolds of HSSNT

In the proof of our theorems we need the following result.

Proposition 21. *Let M be a HSSNT and let $\mathcal{M}$ be its associated $HPJTS$. Then there exists a one to one correspondence between (complete) complex totally geodesic submanifolds through the origin and sub-$HPJTS$ of $\mathcal{M}$. This correspondence sends $T \subset M$ to $\mathcal{T} \subset \mathcal{M}$, where $\mathcal{T}$ denotes the HPJTS associated to T.*

Spectral decomposition and functional calculus

Let $\mathcal{M}$ be a HPJTS. Each element $z \in \mathcal{M}$ has a unique *spectral decomposition*

$$z = \lambda_1 c_1 + \cdots + \lambda_s c_s \qquad (0 < \lambda_1 < \cdots < \lambda_s),$$

where $(c_1, \ldots, c_s)$ is a sequence of pairwise orthogonal tripotents and the λ_j are real numbers called eigenvalues of z. For every $z \in \mathcal{M}$ let $\max\{z\}$ denote the largest eigenvalue of z, then $\max\{\cdot\}$ is a norm on $\mathcal{M}$ called the *spectral norm.* The HSSNT M associated to $\mathcal{M}$ is the open unit ball in $\mathcal{M}$ centered at the origin (with respect to the spectral norm M), i.e.

$$M = \{z = \sum_{j=1}^{s} \lambda_j c_j \mid \max\{z\} = \max_j\{\lambda_j\} < 1\}. \tag{3.11}$$

Using the spectral decomposition, it is possible to associate to an *odd* function $f : \mathbb{R} \to \mathbb{C}$ a map $F : \mathcal{M} \to \mathcal{M}$ as follows. Let $z \in \mathcal{M}$ and let

$$z = \lambda_1 c_1 + \cdots + \lambda_s c_s, \quad 0 < \lambda_1 < \cdots < \lambda_s$$

be the spectral decomposition of z. Define the map F by

$$F(z) = f(\lambda_1) c_1 + \cdots + f(\lambda_s) c_s. \tag{3.12}$$

If f is continuous, then F is continuous. If

$$f(t) = \sum_{k=0}^{N} a_k t^{2k+1}$$

is a polynomial, then F is the map defined by

$$F(z) = \sum_{k=0}^{N} a_k z^{(2k+1)} \qquad (z \in \mathcal{M}).$$

If f is analytic, then F is real-analytic. If f is given near 0 by

$$f(t) = \sum_{k=0}^{\infty} a_k t^{2k+1},$$

then F has the Taylor expansion near $0 \in V$:

$$F(z) = \sum_{k=0}^{\infty} a_k z^{(2k+1)}.$$

Example 22. Let $P = \left(\mathbb{C}H^1\right)^\ell \subset \left(\mathbb{C}^\ell, \{,,\}\right)$ be the polydisk embedded in its associated HPJTS $\left(\mathbb{C}^\ell, \{,,\}\right)$. Define $\tilde{c}_j = \left(0, \dots, 0, e^{i\theta_j}, 0, \dots, 0\right)$, $1 \leq j \leq \ell$. The $\tilde{c}_j$ are mutually strongly orthogonal tripotents. Given $z = \left(\rho_1 e^{i\theta_1}, \dots, \rho_\ell e^{i\theta_\ell}\right) \in \left(\mathbb{C}H^1\right)^\ell$, $z \neq 0$, then up to a permutation of the coordinates, we can assume $0 \leq \rho_1 \leq \rho_2 \leq \dots \leq \rho_\ell$. Let i_1, $1 \leq i_1 \leq \ell$, the first index such that $\rho_{i_1} \neq 0$ then we can write

$$z = \rho_{i_1}\left(\tilde{c}_{i_1} + \dots + \tilde{c}_{i_2-1}\right) + \rho_{i_2}\left(\tilde{c}_{i_2} + \dots + \tilde{c}_{i_3-1}\right) + \dots + \rho_{i_s}\left(\tilde{c}_{i_s} + \dots + \tilde{c}_{i_{s+1}-1}\right)$$

with $0 < \rho_{i_1} < \rho_{i_2} < \dots < \rho_{i_s} = \rho_\ell$ and $i_{s+1} = \ell + 1$. The c_j's, defined by $c_j = \tilde{c}_{i_j} + \dots + \tilde{c}_{i_{j+1}-1}$, are still mutually strongly orthogonal tripotents and $z = \lambda_1 c_1 + \dots + \lambda_s c_s$ with $\lambda_j = \rho_{i_j}$, is the spectral decomposition of z. So the diastatic exponential given in (3.3) can be written as

$$\mathrm{Exp}_0^P(z) = \left(\sqrt{1 - e^{-|z_1|^2}}\frac{z_1}{|z_1|}, \dots, \sqrt{1 - e^{-|z_\ell|^2}}\frac{z_\ell}{|z_\ell|}\right) = \sum_{j=1}^{s}\left(1 - e^{-\lambda_j^2}\right)^{\frac{1}{2}} c_j$$

and $\mathrm{Exp}_0^P(0) = 0$.

We are now in the position to prove our main results. In all the following proofs we can assume, without loss of generality, that M is irreducible. Indeed, in the reducible case the Bergman operator is the product of the Bergman operator of each factor and therefore the same holds true for the diastatic exponential and for the symplectic duality map.

3.3 Proofs of the main results

Proof of Theorem 13 Consider the odd smooth function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \left(1 - e^{-t^2}\right)^{\frac{1}{2}}\frac{t}{|t|},\ f(0) = 0$$

and the map $F : T_0M \to M \subset T_0M$ associated to f by (3.12), namely

$$F(z) = \sum_{j=1}^{s}\left(1 - e^{-\lambda_j^2}\right)^{\frac{1}{2}} c_j, \tag{3.13}$$

where $z = \lambda_1 c_1 + \dots + \lambda_s c_s$ is the spectral decomposition of $z \in T_0M$. Note that, by (3.11), $F(T_0M) \subset M$ and (3.13) is indeed the spectral decomposition of $F(z)$. We will show that $\mathrm{Exp}_0^{\mathrm{hyp}} := F$ is a diastatic exponential at the origin for M satisfying the conditions of Theorem 13. First,

$$\left(d\,\mathrm{Exp}_0^{\mathrm{hyp}}\right)_0(v) = \lim_{r\to 0^+}\frac{d}{dr}\sum_{j=1}^{s}\left(1-e^{-(r\mu_j)^2}\right)^{\frac{1}{2}} d_j = \sum_{j=1}^{s}\mu_j d_j = v,$$

where $v = \mu_1 d_1 + \cdots + \mu_s d_s$ is the spectral decomposition of $v \in T_0M$. Hence $\mathrm{Exp}_0^{\mathrm{hyp}}$ is a diastatic exponential if one shows that $D_0^{\mathrm{hyp}}\left(\mathrm{Exp}_0^{\mathrm{hyp}}(z)\right) = g_0^{\mathrm{hyp}}(z,z)$. In order to prove this equality observe that (see [36] for a proof)

$$B(z,z)\,c_j = \left(1-\lambda_j^2\right)^2 c_j, \quad j = 1,\dots,s, \tag{3.14}$$

$$\det B(z,z) = \prod_{j=1}^{s}\left(1-\lambda_j^2\right)^g,$$

$$g_0^{\mathrm{hyp}}(z,z) = \frac{1}{g}\operatorname{tr}T(z,z) = \sum_{j=1}^{s}\lambda_j^2.$$

Thus (3.9) yields,

$$D_0^{\mathrm{hyp}}(z) = -\frac{1}{g}\log\det B(z,z) = -\log\prod_{j=1}^{s}\left(1-\lambda_j^2\right) \tag{3.15}$$

and so

$$D_0^{\mathrm{hyp}}\left(\mathrm{Exp}_0^{\mathrm{hyp}}(z)\right) = -\log\prod_{j=1}^{s}\left[1-\left(1-e^{-\lambda_j^2}\right)\right] = \sum_{j=1}^{s}\lambda_j^2 = g_0^{\mathrm{hyp}}(z,z),$$

namely the desired equality. Moreover, the map $G : M \subset T_0M \to T_0M$ induced by the odd smooth function

$$g(t) = \left(-\log(1-t^2)\right)^{\frac{1}{2}}\frac{t}{|t|},\ g(0) = 0,$$

namely

$$G(z) = \sum_{j=1}^{s}\left(-\log(1-\lambda^2)\right)^{\frac{1}{2}} c_j,$$

is the inverse of $\mathrm{Exp}_0^{\mathrm{hyp}}$ and so $\mathrm{Exp}_0^{\mathrm{hyp}} : T_0M \to M$ is a diffeomorphism.

In order to prove the second part of the theorem let $P \subset M$ be a polydisk through the origin. Thus equality $\mathrm{Exp}_0^{\mathrm{hyp}}{}_{|T_0P} = \mathrm{Exp}_0^P$ follows by Proposition 21, Example 22 and formula (3.13). Moreover $\mathrm{Exp}_0^{\mathrm{hyp}}$ is determined by its restriction to polydisks since it is well-known that $\forall z \in T_0 M$ there exists a polydisk $P \subset M$ such that $0 \in P$ and $z \in T_0 P$ (see, e.g. [23] and also [18]).

Proof of Theorem 14 Let $z = \lambda_1 c_1 + \cdots + \lambda_s c_s$ be a spectral decomposition of $z \in M^* \setminus \mathrm{Cut}_0(M^*) \cong T_0 M$. In analogy with the non compact case one has

$$B(z, -z)\, c_j = \left(1 + \lambda_j^2\right)^2 c_j$$

$$\det B(z, -z) = \prod_{j=1}^{s} \left(1 + \lambda_j^2\right)^g .$$

$$g_0^{FS}(z, z) = \lambda_j^2.$$

Thus, by (3.6), Calabi's diastasis function at the origin for g^{FS} is given by:

$$\begin{aligned} D_0^{FS}(z) &= -\frac{1}{g} \log K_{M^*}(z, \bar{z}) = \frac{1}{g} \log[K_M(z, -\bar{z})] = \frac{1}{g} \log[\det B(z, -z)] \\ &= \frac{1}{g} \log \prod_{j=1}^{s} \left(1 + \lambda_j^2\right) \end{aligned} \tag{3.16}$$

Define $\mathrm{Exp}_0^{FS} : T_0 M^* \cong T_0 M \to M^* \setminus \mathrm{Cut}_0(M^*) \cong T_0 M$ as the map associated to the real function $f^*(t) = \left(e^{t^2} - 1\right)^{\frac{1}{2}} \frac{t}{|t|}$ by (3.12), namely

$$\mathrm{Exp}_0^{FS}(z) = \sum_{j=1}^{s} \left(e^{\lambda_j^2} - 1\right)^{\frac{1}{2}} c_j. \tag{3.17}$$

Thus, following the same line of the proof of Theorem 13, one can show that Exp_0^{FS} is the diastatic exponential at 0 uniquely determined by its restriction to polydisks.

Proof of Theorem 16 By (3.8) and (3.14)

$$\Psi_M(z) = B(z, \bar{z})^{-\frac{1}{4}}(z) = \frac{\lambda_j}{\left(1 - \lambda_j^2\right)^{\frac{1}{2}}} c_j \tag{3.18}$$

By the very definition of the diastatic exponential $\mathrm{Exp}_0^{\mathrm{hyp}}$ for the hyperbolic metric its inverse $\left(\mathrm{Exp}_0^{\mathrm{hyp}}\right)^{-1} : M \to T_0 M$ reads as:

$$\left(\mathrm{Exp}_0^{\mathrm{hyp}}\right)^{-1}(z) = \sum_{j=1}^{s} \left(-\log\left(1 - \lambda_j^2\right)\right)^{\frac{1}{2}} c_j,$$

Then, by (3.17) and (3.18),

$$\mathrm{Exp}_0^{FS} \circ \left(\mathrm{Exp}_0^{\mathrm{hyp}}\right)^{-1}(z) = \Psi_M(z)$$

and this concludes the proof of Theorem 16.

Proof of Theorem 17 Since $D_0^{\mathrm{hyp}} = \frac{1}{g}\log K_M$ and $D_0^{FS} = \frac{1}{g}\log K_{M^*}$, equation $K_{M^*} \circ \Psi_M = K_M$ is equivalent to $D_0^{FS} \circ \Psi_M = D_0^{\mathrm{hyp}}$ which is a straightforward consequence of (3.15), (3.16) and (3.18).

Appendix A

Moment map

In this appendix we describe the main properties of the moment map needed in this thesis. For a more detailed treatment of the subject we refer the reader to [35].

Let G be a compact Lie group which acts on a symplectic manifold (M, ω) by symplectomorphism, i.e.

$$g : M \to M$$

is a symplectomorphism for every $g \in G$ and

$$gh(x) = g(h(x)) \quad \text{for all } g, h \in G,\ x \in M$$

$$e(x) = x \quad \text{for all } x \in M$$

where e is the identity element of G. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathcal{X}(M)$ the Lie algebra of vector field on M, there is a natural Lie algebra homomorphism $\mathfrak{g} \to \mathcal{X}(M, \omega) : \xi \mapsto X_\xi$ defined by

$$X_\xi(x) = \frac{d}{dt}(\exp(t\xi)(x))_{|t=0}.$$

In particular it satisfies

$$X_{Ad_{g^{-1}}\xi} = g^* X_\xi \tag{A.1}$$

and

$$X_{[\xi,\eta]} = [X_\xi, X_\eta] \tag{A.2}$$

for $\xi, \eta \in \mathfrak{g}$ and $g \in G$. The first equality is a consequence of the following property of the exponential map

$$\exp(Ad_g \eta) = g \exp(\eta) g^{-1},$$

indeed

$$X_{Ad_{g^{-1}}\xi}(x) = \frac{d}{dt}_{|t=0} \big(\exp(tAd_{g^{-1}}\xi)(x)\big) = \frac{d}{dt}_{|t=0} g \exp(t\eta) g^{-1} = g^*(X_\xi(g_t^{-1}(x)))$$

Now we prove that (A.1) implies (A.2).

$$\begin{aligned} [X_\xi, X_\eta] &= \mathcal{L}_{X_\eta} X_\xi = \frac{d}{dt}_{|t=0} g_t^*(X_\xi(g_t^{-1}(x))) \\ &= \frac{d}{dt}_{|t=0} X_{Ad_{g_t^{-1}}\xi}(x) = X_{\frac{d}{dt}_{|t=0} Ad_{g_t}\xi} = X_{ad_\eta \xi} \\ &= X_{[\eta,\xi]} \end{aligned}$$

Since G acts symplectly, it follow that X_ξ is a symplectic vector field, i.e. the 1-form $i_{X_\xi}\omega$ is closed. This follows from the Cartan's formula for the Lie derivative

$$\mathcal{L}_{X_\xi}\omega = i_{X_\xi} d\omega + d(i_{X_\xi}\omega).$$

For any smooth function $H : M \to \mathbb{R}$ is defined the vector field X_H by the following identity

$$i_{X_H}\omega = dH,$$

so we can define the Poisson bracket $\{\cdot,\cdot\} : (F, H) \mapsto \omega(X_F, X_H)$ which induce a Lie algebra structure on $C^\infty(M)$.

If for each $\xi \in \mathfrak{g}$ the vector field X_ξ is Hamiltonian, that is the 1-form $i_{X_\xi}\omega$ is exact, then we can define a map $\xi \mapsto H_\xi \in C^\infty(M)$ such that $dH_\xi = i_{X_\xi}\omega$. Observe that the function H_ξ is determines up to a constant. If H_ξ can be choosen such that the map

$$\mathfrak{g} \to C^\infty(M) : \xi \mapsto H_\xi$$

is a Lie algebra homomorphism, then the action is called Hamiltonian.

Assume from now on that the action of G is Hamiltonian.

Definition A.1. *The map*

$$\mu : M \to \mathfrak{g}^*$$

is called **moment map** *for the action of* G *if the formula*

$$H_\xi(p) = \langle \mu(p), \xi \rangle$$

defines a Lie algebra homomorphism $\mathfrak{g} \to C^\infty(M) : \xi \mapsto H_\xi$*. As a consequence of the definition we get*

$$d\langle \mu(x), \xi \rangle = i_{X_\xi}\omega$$

Lemma A.2. *The* G *action "commutes" with the moment map in the following sense*

$$\mu(g \cdot x) = \mathrm{Ad}^*_{g^{-1}}(\mu(x)) \qquad g \in (G) \tag{A.3}$$

Proof. Let $g_t = \exp(t\eta)$ be a curve such that $g_0 = 1_{\mathcal{G}}$ and $g_1 = g$, then we have:

$$\begin{aligned} \omega(g_t^* X_\xi(g_t \cdot x), g_t^* X_\eta(g_t \cdot x))_x &= \omega(X_{\mathrm{Ad}_{g^{-1}}\xi}, X_{\mathrm{Ad}_{g^{-1}}\eta}) \\ &= \{\langle \mu(x), \mathrm{Ad}_{g^{-1}}\xi \rangle, \langle \mu(x), \mathrm{Ad}_{g^{-1}}\eta \rangle\} \\ &= \langle \mu(x), [\mathrm{Ad}_g \xi, \mathrm{Ad}_g \eta] \rangle \\ &= \langle \mu(x), \mathrm{Ad}_g[\xi, \eta] \rangle \end{aligned}$$

and

$$\begin{aligned} &\frac{d}{dt}(\langle \mu(g_t \cdot x), \xi \rangle - \langle \mathrm{Ad}^*_{g_t^{-1}}(\mu(x)), \xi \rangle) = \\ &\quad = \omega(X_\xi(g_t \cdot x), X_\eta(g_t \cdot x))_{g_t \cdot x} - \frac{d}{dt}\langle \mu(x), \mathrm{Ad}_{g_t^{-1}}(\xi) \rangle \\ &\quad = \omega(g_t^* X_\xi(g_t \cdot x), g_t^* X_\eta(g_t \cdot x))_x - \langle \mu(x), \mathrm{Ad}_{g_t^{-1}}[\xi, \eta] \rangle \\ &\quad = \langle \mu(x), \mathrm{Ad}_{g_t^{-1}}[\xi, \eta] \rangle - \langle \mu(x), \mathrm{Ad}_{g_t^{-1}}[\xi, \eta] \rangle = 0. \end{aligned}$$

The conclusion follows by observing that $\mu(g_0 \cdot x) = \mathrm{Ad}^*_{g_0^{-1}}(\mu(x)) = \mu(x)$. ☐

Theorem A.3. *Suppose that the complexification* $G^{\mathbb{C}}$ *also acts on* M*. If* a *is an element of* $\mathfrak{g}^*$ *fixed by the coadjoint action, then for every* $x \in \mu^{-1}(a)$ *we have*

$$\mu^{-1}(a) \cap (G^{\mathbb{C}} \cdot x) = G \cdot x.$$

Proof. By Lemma A.2 we deduce that $\mu^{-1}(a)$ is G invariant and the inclusion $\mu^{-1}(a) \cap (G^{\mathbb{C}} \cdot x) \supset G \cdot x$ is an immediate consequence. Chosen $y \in \mu^{-1}(a) \cap (G^{\mathbb{C}} \cdot x)$ we have

$$\begin{aligned} y &= g \cdot x & g &\in G^{\mathbb{C}} \\ &= p \cdot k \cdot x & p &\in \exp(i\mathfrak{g}) \text{ and } k \in \exp(\mathfrak{g}) \end{aligned}$$

to conclude we have to prove that $y = k \cdot x$. Let $y(t) = \exp(ti\xi)k \cdot x$ be a curve such that $y(1) = y$ and $y(0) = k \cdot x$, and $f(t) = \langle \mu(y(t)), \xi \rangle$.

$$\dot{y}(t) = \frac{d}{ds}(\exp(si\xi) \cdot y(t))_{|s=0} = X_{i\xi}(y(t)) = JX_{\xi}(y(t))$$

so by definition of moment together with the previous equation, we get

$$\dot{f}(t) = \omega(X_{\xi}(y(t)), \dot{y}(t)) = \omega(-J\dot{y}(t), \dot{y}(t)) = |\dot{y}|^2.$$

Since the action of G is ω-preserving $\dot{f}(t) = |\dot{y}|^2$ must be constant. Thus observing that $\mu(y(0)) = \mu(y(1)) = a$, we conclude that $\dot{f}(t) = |\dot{y}|^2 \equiv 0$ and therefore $y = k \cdot x$. □

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