Università degli Studi di Cagliari

DOTTORATO DI RICERCA

Ingegneria Industriale

Ciclo XXIV

TITOLO TESI

Observation and Control of PDE with disturbances

Settore/i scientifico disciplinari di afferenza

ING-INF/04 AUTOMATICA

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Esame finale anno accademico 2010 - 2011

Observation and Control of PDE with disturbances

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XXIV Ciclo

Contents

Abstract

In this Thesis, the problem of controlling and Observing some classes of distributed parameter systems is addressed. The particularity of this work is to consider partial differential equations (PDE) under the effect of external unknown disturbances. We consider generalized forms of two popular parabolic and hyperbolic infinite dimensional dynamics, the *heat* and *wave* equations. Sliding-mode control is used to achieve the control goals,exploiting the robustness properties of this robust control technique against persistent disturbances and parameter uncertainties.

Chapter 1

Introduction

This introductory Chapter presents the motivations which lead us to the development of this Thesis first,then in the second section, a summary of the Thesis is given, showing how it is structured in different Chapters. Finally, a list of papers and other activities derived from the present work is presented.

1.1 Motivations

There are many kind of systems whose dynamical behaviors are described by Partial Differential Equations (PDE), Curtain and Zwart (1995), Schiesser and Griffiths (2009). In the last decades, this field has broadened considerably as more realistic models have been introduced and investigated in different areas such as thermodynamics, elastic structures, fluid dynamics and biological systems, to name a few Imanuilov *et al.* (2005), Mondaini and Pardalos (2008). PDE control theory, consists of a wealth of mathematically impressive results that solve stabilization and optimal control problems. Two of the main driving principles in this development have been generality and the aim of extending the existing finite dimensional results. The latter objective has led to extending(at least) two of the basic control theoretic results to PDEs: pole placement and optimal/robust control. While these efforts have been successful, by following the extremely general finite-dimensional path ($\dot{x} = Ax + Bu$ where A and B can be any matrices), they have diverted the attention from structure-specific opportunities that exist in PDEs. Such opportunities have recently started to be capitalized on in the elegant work on distributed control of spatially invariant systems by Bamieh *et al.* (2002). In spite of the fact that optimization and control of systems governed by PDE is a very active field of research, no much have been developed for observer design. A main result in this field is described in Krstic and Smyshlyaev (2008), Krstic and Smyshlyaev (2005). Krstic and Smyshlyaev (2004) introduces backstepping as a structure-specific paradigm for parabolic PDEs (at least within the class considered) and demonstrates its capability to incorporate optimality in addition to stabilization.

1.2 Summary

The first three chapters of the Thesis recall some fundamentals which are exploited in the PDE contest in the sequel. Then in the second part is described the design of an Unknown Input Observer for diffusion PDE and an estimator of the unknown input based on second order sliding-mode. The third part of the Thesis addresses the Lyapunov-based design of second order sliding mode controllers (2-SMC) in the domain of distributed-parameters systems with uncertainties and unknown disturbances.

A summary of each Chapter is reported in the following list:

- **Chapter 2.** An introduction to Partial Derivative Equations is illustrated. In particular are described here the heat and the wave equation, and two ways to solve the them.
- **Chapter 3.** A brief survey on Variable Structure Control Systems with Sliding Modes.
- **Chapter 4.** The concept of strong observability for multi-variable linear systems and the design of an Unknown Input Observer are explained. At least an estimator of the unknown input, based on second order sliding-mode algorithm ”Super-Twisting” is proposed.
- **Chapter 5.** The methodologies described in the previous Chapter are applied to diffusion processes, where the unknown input to estimate is a non measurable disturbance or a fault in the process.
- **Chapter 6.** Lyapunov-based design of second order sliding mode controllers for PDE is described here, both diffusion and wave equations are used.
- **Chapter 7.** The Boundary control for the uncertain diffusion equation with unknown disturbance on the actuation is illustrated.
- **Chapter 8.** Conclusions and recommendations for future work are given.

1.3 Author's Publications

The topics of the preset Thesis were presented in international conferences, and a IMA Journal paper was accepted for publication in 2012.

Chapters 4 and 5 contents were published in the conference paper:

- ACD 2010: Pisano A., Scodina S., Usai E Unknown input observer with sliding mode disturbance estimator

Some of Chapter 6 contents were published in the conference paper:

- CDC 2011: Orlov Y., Pisano A., Scodina S., Usai E Lyapunov-based second-order sliding mode control for a class of uncertain reaction-diffusion processes

were accepted for publication in the *IMA Journal of Mathematical Control and Information*:

- Orlov Y., Pisano A., Scodina S., Usai E On the Lyapunov-based second order SMC design for some classes of distributed-parameter systems

Chapters 7 contents were published in the conference paper:

- VSS 2011: Orlov Y., Pisano A., Scodina S., Usai E Sliding-mode Boundary Control of Uncertain Reaction-Diffusion Processes with Spatially Varying Parameters

During the PhD the author has also made an experimental work at the UNAM University(Mexico City) about the sliding mode control of a three dimensional Crane which lead to a conference paper:

- VSS 2010: Pisano A., Scodina S., Usai E Load swing suppression in the 3-dimensional overhead crane via second-order sliding-modes

Chapter 2

Partial Derivative Equations (PDE)

In this Chapter an introduction to Partial Derivative Equations is illustrated. After a general explanation we focus on the equations that are analyzed in the Thesis, the wave and the diffusion equation. At the end we describe two kind of solutions applied for the Thesis, the modal expansion and the numerical solution by finite-differences.

2.1 Introduction

Partial differential equations (PDEs) are often used to construct models describing many basic phenomena in physics and engineering. Solving ordinary differential equations involves finding a function (or a set of functions) of one independent variable but partial differential equations involve functions of two or more variables.

Here are typical examples of the most common types of linear homogeneous PDEs, for the simplest case of just two independent variables. The wave, heat and Laplace equations are the typical representatives of three fundamental genres of partial differential equations.

- The wave equation: $u_{tt} - c^2 u_{xx} = 0$, *hyperbolic*,
- The heat equation: $u_t - \gamma^2 u_{xx} = 0$, *parabolic*,
- Laplace's equation: $u_{xx} - u_{yy} = 0$, *elliptic*.

The last column indicates the equation's type, each genre has distinctive analytical features, physical manifestations, and even numerical solution schemes. Equations governing vibrations, such as the wave equation, are typical hyperbolic. Equations modeling diffusion, such as the heat equation, are parabolic. Hyperbolic and parabolic equations both typically represent dynamical processes, and so one of the independent variables is identified as time. On the other hand, equations modeling equilibrium phenomena, including the Laplace and Poisson equations, are usually elliptic, and only involves spatial variables. Elliptic partial differential equations are associated with boundary value problems, whereas parabolic and hyperbolic equations involve initial and initial-boundary value problems.

While this tripartite classification into hyperbolic, parabolic, and elliptic equations initially appears in the bivariate context, the terminology, underlying properties, and associated physical models carry over to second order partial differential equations in higher dimensions. Most of the partial differential equations arising in applications fall into one of these three categories, and it is fair to say that the field of partial differential equation splits into three distinct subfields. Or, rather four subfields, the last containing all the equations, including higher order equations, that do not fit into the preceding categorization.

The full classification of real, linear, second order partial differential equations for scalar-valued function $u(x, y)$ depending on two variables proceeds as follows. The most general such equation has the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \tag{2.1.1}$$

where the coefficients A, B, C, D, E, F are all allowed to be functions of (x,y), as is the inhomogeneity or forcing function $G(x, y)$. The equation is *homogeneous* if and only if $G \equiv 0$. We assume that at least one of the leading coefficients A, B, C is not identically zero, as otherwise the equation degenerates to a first order equation. The key quantity that determines the type of such a partial differential equation is its *discriminant*

$$\Delta = B^2 - 4AC. \tag{2.1.2}$$

This should remind the reader of the discriminant of the quadratic equation

$$Q(\xi, \eta) = A\xi^2 + B\xi\eta + C\eta^2 + D\xi + E\eta + F = 0. \tag{2.1.3}$$

Its solution trace out a plane curve in a conic section. In the non-degenerate cases, the discriminant (2.1.2) fixes its geometrical type:

- a hyperbola when $\Delta > 0$,
- a parabola when $\Delta = 0$, or
- an ellipse when $\Delta < 0$.

This classification provides the underlying rationale for the choice of terminology used to classify second order partial differential equations.

DEFINITION 2.1.1 At a point (x,y), the linear, second order partial differential equation (2.1.1) is called

- *hyperbolic* $\Delta(x, y) > 0$,
- *parabolic* if and only if $\Delta(x, y) = 0$, but $A^2 + B^2 + C^2 \neq 0$,
- *elliptic* $\Delta(x, y) < 0$,
- *singular* $A = B = C = 0$.

In particular

- The wave equation $u_{xx} - u_{yy} = 0$ has discriminant $\Delta = 4$, and is hyperbolic.
- The heat equation $u_{xx} - u_y = 0$ has discriminant $\Delta = 0$, and is parabolic.
- The Poisson equation $u_{xx} + u_{yy} = -f$ has discriminant $\Delta = -4$, and is elliptic.

2.2 The Diffusion equation

Let us start with a physical derivation of the heat equation from first principles. Consider a bar, a thin heat-conducting body. "Thin" means that we can regard the bar as a one-dimensional continuum with no significant transverse temperature variation. We will assume that the bar is fully insulated along its length, and so heat can only enter (or leave) through its un-insulated endpoints. We use *t* to represent time, and $a \leq x \leq b$ to denote spatial position along the bar, which occupies the interval [a,b]. Our goal is to find the temperature $u(t, x)$ of the bar at position *x* and time *t*. The dynamical equations governing the temperature are based on three fundamental physical principles. First is the Law of Conservation of Heat Energy. This conservation law takes the form

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial w}{\partial x} = 0, \tag{2.2.1}$$

in which $\varepsilon(t, x)$ represent the thermal *energy density* at time *t* and position *x*, while $w(t, x)$ denotes the heat flux, i.e., the rate of flow of thermal energy along the bar. Our sign convention is that $w(t, x) > 0$ at points where heat energy flows in the direction of increasing *x*. The integrated form of the conservation law, namely

$$\frac{d}{dt} \int_a^b \varepsilon(t, x)\, dx = w(t, a) - w(t, b), \tag{2.2.2}$$

states that the rate of change in thermal energy within the bar is equal to the total heat flux passing through its un-insulated ends. The signs of the boundary terms confirm that heat flux into the bar results in an increase in temperature. The second ingredient is a constitutive assumption concerning the bar's material properties. It has been observed that, under reasonable conditions, thermal energy is proportional to temperature

$$\varepsilon(t,x) = \sigma(x)u(t,x). \tag{2.2.3}$$

The factor

$$\sigma(t,x) = \rho(x)\Xi(x).$$

is the product of the density ρ of the material and its specific heat Ξ, which is the amount of heat energy required to raise the temperature of a unit mass of the material by one degree. Note that we are assuming the medium is not changing in time, and so physical quantities such as density and specific heat depend only on position *x*. We also assume, perhaps with less physical justification, that its material properties do not depend upon the temperature; otherwise, we would be forced to deal with a much thornier nonlinear diffusion equation. The third physical principle relates heat flux and temperature. Physical experiments show that the heat energy moves from hot to cold at a rate that is in direct proportion to the temperature gradient which, in the one dimension case, means its derivative $\partial u \backslash \partial x$. The resulting relation

$$w(t,x) = -k(x)\frac{\partial u}{\partial x} \tag{2.2.4}$$

is known as *Fourier's Law of Cooling*. The proportionality factor $k(x) > 0$ is the *thermal conductivity* of the bar at position *x*, and the minus sign reflects the everyday observation that heat energy moves from hot to cold. A good heat conductor, e.g., silver, will have high conductivity, while a poor conductor, e.g., glass, will have low conductivity. Combining the three laws (**??**,2.2.3,2.2.4) produces the *linear diffusion equation*

$$\frac{\partial}{\partial t}\left(\sigma(x)u\right) = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right), \qquad a < x < b, \tag{2.2.5}$$

governing the thermodynamics of a one-dimensional medium. It is also used to model a wide variety of diffusive processes, including chemical diffusion, diffusion of contaminants in liquids and gases, population dispersion, and the spread of infectious diseases. If there is an external heat source along the length of the bar, then the diffusion equation acquires an additional inhomogeneous term:

$$\frac{\partial}{\partial t}\left(\sigma(x)u\right) = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) + h(t,x), \qquad a < x < b, \tag{2.2.6}$$

2.2.1 Initial and boundary conditions

In order to uniquely prescribe the solution $u(t, x)$, we need to specify an initial temperature distribution

$$u(t_0, x) = f(x), \qquad a \leq x \leq b. \tag{2.2.7}$$

In addiction, we must impose a suitable boundary condition at each end of the bar. There are three common types.

The first is a *Dirichlet boundary condition*, where the end is held at prescribed temperature. For example,

$$u(t, a) = \alpha(t) \tag{2.2.8}$$

fixes the temperature (possibly time-varying) at the left end.

Alternatively, the *Neumann boundary condition*

$$\frac{\partial u}{\partial x}(t, a) = \mu(t) \tag{2.2.9}$$

prescribes the heat flux $w(t, a) = -k(a)u_x(t, a)$ there. In particular, a homogeneous Neumann condition, $u_x(t, a) = 0$, models an insulated end that prevents heat energy flowing in or out. The *Robin boundary condition*

$$\frac{\partial u}{\partial x}(t, a) + ku(t, a) = \tau(t), \tag{2.2.10}$$

with $k > 0$, models the heat exchange resulting from the end of the bar being placed in a reservoir at temperature $\tau(t)$.

Each end of the bar is required to satisfy one of these boundary conditions. For example, a bar with both ends having prescribed temperature is governed by the pair of Dirichlet boundary conditions

$$u(t, a) = \alpha(t), \qquad u(t, b) = \beta(t), \tag{2.2.11}$$

whereas a bar with two insulated ends requires two homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial x}(t, a) = 0, \qquad \frac{\partial u}{\partial x}(t, b) = 0. \tag{2.2.12}$$

Mixed boundary conditions, with one end at a fixed temperature and the other insulated, are similarly formulated, e.g.,

$$u(t, a) = \alpha(t), \qquad \frac{\partial u}{\partial x}(t, b) = 0. \tag{2.2.13}$$

Finally, the periodic boundary conditions

$$u(t, a) = u(t, b), \qquad \frac{\partial u}{\partial x}(t, a) = \frac{\partial u}{\partial x}(t, b), \tag{2.2.14}$$

correspond to a circular ring obtained by joining the two ends of the bar. As before, we are assuming the heat is only allowed to flow around the ring where insulation prevents the radiation of heat from one side of the ring affecting the other side.

2.3 The Wave Equation

Newtons Second Law, states that force equals mass times acceleration, is the bedrock underlying the derivation of mathematical models describing all of classical dynamics. When applied to a one-dimensional medium, such as the transverse displacements of a guitar string, or the longitudinal motions of an elastic bar, the resulting model governing small vibrations is the second order partial differential equation

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right). \tag{2.3.1}$$

Here $u(t, x)$ represents the displacement of the string or bar at time t and position x, while $\rho(x) > 0$ denotes its density and $k(x) > 0$ its stiffness or tension, both of which are assumed to not vary with t. The right hand side of the equation represents the restoring force due to a (small) displacement of the medium from its equilibrium, whereas the left hand side is the product of mass per unit length times acceleration. We will simplify the general model by assuming that the underlying medium is homogeneous, and so both its density and stiffness are constant. Then (2.3.1) reduces to the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad where \qquad c = \sqrt{\frac{k}{\rho}} > 0 \tag{2.3.2}$$

is known as the *wave speed*.

In general, to uniquely specify the solution to any dynamical system arising from Newtons Second Law, including the wave equation (2.3.2), and the more general vibration equation (2.3.1), one must fix both its initial position and initial velocity. Thus, the initial conditions take the form

$$u(0, x) = f(x), \qquad \frac{\partial u}{\partial t}(0, x) = g(x), \tag{2.3.3}$$

where, for simplicity, we set the initial time $t_0 = 0$. The initial value problem seeks the corresponding C^2 function $u(t, x)$ that solves the wave equation (2.3.2) and has the required initial values (2.3.3). The boundary conditions are the same as for the diffusion equation.

2.4 Modal Expansion

A general method for attempting to solve PDEs is to suppose that the solution function $u(t,x)$ is a product of functions, each one depending on one only of the independent variables. This converts the PDE into two (or more) ODEs which may be solvable. For simplicity in this section we consider the heat equation with the bar composed of a uniform material, and so its density ρ, conductivity k, and specific heat Ξ are all positive constants. We also considering that the bar remains insulated along its entire length. Under this assumptions, the general diffusion equation (2.2.5) reduces to the homogeneous *heat equation*

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2} \tag{2.4.1}$$

for the temperature $u(t,x)$ at time *t* and position *x*. The constant

$$\gamma = \frac{k}{\rho \Xi} \tag{2.4.2}$$

is called the *thermal diffusivity*, and incorporates all of the bar's relevant physical properties. The solution $u(t,x)$ will be uniquely prescribed once we specify initial conditions (2.2.7) and a suitable boundary condition at both of its endpoints.

Is well known Curtain and Zwart (1995) that the separable solutions to the heat equation are based on the exponential form

$$u(t,x) = e^{-\lambda t} v(x), \tag{2.4.3}$$

where $v(x)$ depends only on the spatial variable. Functions of this form, which "separate" into a product of a function of *t* times a function of *x*, are known as *separable solutions*. Substituting (2.4.3) into (5.1.1) and erasing the common exponential factors, we find that $v(x)$ must solve the second order linear ordinary differential equation

$$-\gamma \frac{d^2 v}{dx^2} = \lambda v. \tag{2.4.4}$$

Each nontrivial solution $v(x) \neq 0$ is an *eigenfunction*, with *eigenvalue* λ, for the linear differential operator $L[v] = -\gamma v''(x)$. With the separable eigen-solutions (2.4.3) in hand, we will then be able to reconstruct the desired solution $u(t,x)$ as a linear combination, or, rather, infinite series thereof.

Consider the simple case of a uniform, insulated bar of length l that is held at zero temperature at both ends. We specify its initial temperature $f(x)$ at time $t_0 = 0$, and so the relevant initial and boundary conditions are

$$\begin{aligned} &u(t,0) = 0, \quad u(t,l) = 0, \quad t \geq 0, \\ &u(0,x) = f(x), \qquad\quad 0 \leq x \leq l. \end{aligned} \tag{2.4.5}$$

The eigensolution (2.4.3) are found by solving the Dirichlet boundary value problem

$$-\gamma\frac{d^2v}{dx^2}-\lambda v=0,\quad v(0)=0,\quad v(l)=0. \tag{2.4.6}$$

We find that if λ is either complex, or real and ≤ 0, then the only solution to the boundary value problem (2.4.6) is the trivial solution $v(x)\equiv 0$. Hence, all the eigenvalues must necessarily be real and positive. In fact, the reality and positivity of the eigenvalues does need not be explicitly checked, but, rather, follows from very general properties of positive definite boundary value problem, of which (2.4.6) is a particular case.

When $\lambda > 0$, the general solution to the differential equation is a trigonometric function

$$v(x)=a\cos wx+b\sin wx,\quad where\quad w=\sqrt{\lambda\backslash\gamma}, \tag{2.4.7}$$

and a and b are arbitrary constants. The first boundary condition requires $v(0)=a=0$. Using this to eliminate the cosine term, the second boundary condition requires

$$v(l)=b\sin wl=0. \tag{2.4.8}$$

Therefore, since $b\neq 0$ the solution is trivial and does not qualify as an eigenfunction and wl must be an integer multiple of π, and so

$$w=\frac{\pi}{l},\quad \frac{2\pi}{l},\quad \frac{3\pi}{l},\quad \dots\,. \tag{2.4.9}$$

We conclude that the eigenvalues and eigenfunction of the boundary value problem (2.4.6) are

$$\lambda_n=-\gamma\left(\frac{n\pi}{l}\right)^2,\quad v_n(x)=\sin\frac{n\pi x}{l},\quad n=1,2,3,\dots\,. \tag{2.4.10}$$

The corresponding eigensolutions (2.4.3) are

$$u_n(t,x)=\exp\left(-\frac{\gamma n^2\pi^2 t}{l^2}\right)\sin\frac{n\pi x}{l},\quad n=1,2,3,\dots\,. \tag{2.4.11}$$

Each represents a trigonometrically oscillating temperature profile that maintains its form while decaying to zero at an exponentially fast rate.

To solve the general initial value problem, we assemble the eigensolutions into an infinite series,

$$u(t,x)=\sum_{n=1}^{\infty}b_n u_n(t,x)=\sum_{n=1}^{\infty}b_n\exp\left(-\frac{\gamma n^2\pi^2 t}{l^2}\right)\sin\frac{n\pi x}{l}, \tag{2.4.12}$$

whose coefficients b_n are to be fixed by the initial conditions. Indeed, assuming that the series converges, the initial temperature profile is

$$u(0,x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = f(x). \tag{2.4.13}$$

This has the form of a Fourier sine series on the interval $[0, l]$. Thus, the coefficients are so determined by the Fourier formulae

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \qquad n = 1, 2, 3, \dots . \tag{2.4.14}$$

The resulting formula (2.4.12) describes the Fourier sine series for the temperature $u(t,x)$ of the bar at each later time $t \geq 0$. However, because we are unable to sum the series in closed form, this solution is much less satisfying than a direct, explicit formula. Nevertheless, there are important qualitative and quantitative features of the solution that can be easily gleaned from such series expansions.

If the initial data $f(x)$ is integrable (e.g., piecewise continuous), then its Fourier coefficients are uniformly bounded; indeed, for any $n \geq 1$,

$$|b_n| \leq \frac{2}{l}\int_0^l \left| f(x)\sin\frac{n\pi x}{l}\right| \,dx \leq \frac{2}{l}\int_0^l |f(x)|\,dx \equiv M. \tag{2.4.15}$$

This property holds even for quite irregular data. Under these conditions, each term in the series solution (2.4.12) is bounded by an exponentially decaying function

$$\left| b_n \exp\left(-\frac{\gamma n^2\pi^2}{l^2}t\right)\sin\frac{n\pi x}{l}\right| \leq M\exp\left(-\frac{\gamma n^2\pi^2}{l^2}t\right). \tag{2.4.16}$$

This means that, as soon as $t > 0$, most of the high frequency terms, $n \gg 0$, will be extremely small. Only the first few terms will be at all noticeable, and so the solution essentially degenerates into a finite sum over the first few Fourier modes. As time increases, more and more of the Fourier modes will become negligible, and the sum further degenerates into progressively fewer significant terms. Eventually, as $t \to \infty$, all of the Fourier modes will decay to zero. Therefore, the solution will converge exponentially fast to a zero temperature profile: $u(t,x) \to 0$ as $t \to \infty$, representing the bar in its final uniform thermal equilibrium. The fact that its equilibrium temperature is zero is the result of holding both ends of the bar fixed at zero temperature, and any initial heat energy will eventually be dissipated away through the ends. The last term to disappear is the one with the slowest decay, namely

$$u(t,x) \approx b_1\exp\left(-\frac{\gamma\pi^2}{l^2}t\right)\sin\frac{\pi x}{l}, \quad \textit{where} \quad b_1 = \frac{1}{\pi}\int_0^\pi f(x)\sin x\,dx. \tag{2.4.17}$$

Generically, $b_1 \neq 0$, and the solution approaches thermal equilibrium exponentially fast with rate equal to the smallest eigenvalue, $\lambda_1 = \gamma\pi^2\backslash l^2$, which is proportional to the thermal diffusivity divided by the square of the length of the bar. The longer the bar, or the smaller the diffusivity, the longer it takes for the effect of holding the ends at zero temperature to propagate along the entire bar. In exceptional situations, namely when $b_1 = 0$, the solution decays even faster, at a rate equal to the eigenvalue $\lambda_k = \gamma k^2\pi^2\backslash l^2$ corresponding to the first nonzero term, $b_k \neq 0$, in the series; its asymptotic shape now oscillates k times over the interval.

The heat equations smoothing effect on irregular initial data by fast damping of the high frequency modes underlies its effectiveness for smoothing out and denoising signals. We take the initial data $u(0, x) = f(x)$ to be a noisy signal, and then evolve the heat equation forward to a prescribed time $t^* > 0$. The resulting function $g(x) = u(t^*, x)$ will be a smoothed version of the original signal $f(x)$ in which most of the high frequency noise has been eliminated. Of course, if we run the heat flow for too long, all of the low frequency features will be also smoothed out and the result will be a uniform, constant signal. Thus, the choice of stopping time t^* is crucial to the success of this method. Another, closely related observation is that, for any fixed time $t > 0$ after the initial moment, the coefficients in the Fourier series (2.4.12) decay exponentially fast as $n \to \infty$. the solution $u(t, x)$ is a very smooth, infinitely differentiable function of x at each positive time t, no matter how un-smooth the initial temperature profile. We have discovered the basic smoothing property of heat flow.

THEOREM 2.4.1 If $u(t, x)$ is a solution to the heat equation with piecewise continuous initial data $f(x) = u(0, x)$, or, more generally, initial data satisfying (2.4.15), then, for any $t > 0$, the solution $u(t, x)$ is an infinitely differentiable function of x.

After even a very short amount of time, the heat equation smoothes out most, and, eventually, all of the fluctuations in the initial temperature profile. As a consequence, it becomes impossible to reconstruct the initial temperature $u(0, x) = f(x)$ by measuring the temperature distribution $h(x) = u(t, x)$ at a later time $t > 0$. Diffusion is irreversible and we cannot run the heat equation backwards in time. Indeed, if the initial data $u(0, x) = f(x)$ is not smooth, there is no function $u(t, x)$ for $t < 0$ that could possibly yield such an initial distribution because all corners and singularities are smoothed out by the diffusion process as t goes forward. Or, to put it another way, the Fourier coefficients (2.4.14) of any purported solution will be exponentially growing when $t < 0$, and so high frequency noise will completely overwhelm the solution. For this reason, the backwards heat equation is said to be *ill-posed*.

REMARK 2.4.1 The irreversibility of the heat equation points out a crucial distinction between partial differential equations and ordinary differential equations. Ordinary differential equations are always reversible and unlike the heat equation, existence, uniqueness and continuous dependence properties of solutions are all equally valid in reverse time (although the detailed qualitative and quantitative properties of solutions can very well depend up on whether time is running forwards or backwards). The irreversibility of partial differential equations modeling the diffusive processes in our universe may well be why, in our experience, Times Arrow points exclusively to the future.

2.5 Numerical Method of Finite Differences

Most differential equations are too much complicated to be solved analytically. Thus, to obtain quantitative results, one is forced to construct a sufficiently accurate numerical approximation to the solution. Even in cases, such as the heat and wave equations, where explicit solution formulas (either closed form or infinite series) exist, numerical methods still can be profitably employed. Moreover, justification of a numerical algorithm is facilitated by the possibility of comparing it with an exact solution. Moreover, the lessons learned in the design of numerical algorithms for already analytically solved problems prove to be of immense value when one is confronted with more complicated problems for which solution formulas no longer exist.

In general, to approximate the derivative of a function at a point, say $f'(x)$ or $f''(x)$, one constructs a suitable combination of sampled function values at nearby points. The underlying formalism used to construct these approximation formulae is known as the *calculus of finite differences*. Its development has a long and influent history, dating back to Newton. The resulting finite difference numerical methods for solving differential equations have extremely broad applicability, and can, with proper care, be adapted to most problems that arise in mathematics and its many applications.

The simplest finite differences approximation is the ordinary difference quotient

$$\frac{u(x+h)-u(x)}{h} \approx u'(x), \tag{2.5.1}$$

used to approximate the first derivative of the function $u(x)$. Indeed, if u is differentiable at x, then $u'(x)$ is, by definition, the limit, as $h \to 0$ of the finite difference quotients. Geometrically, the difference quotient equals the slope of the secant line through the two points $(x, u(x))$ and $(x+h, u(x+h))$ on the graph of the function. For small h, this should be a reasonably good approximation to the slope of the tangent line $u'(x)$. To see how close is an approximation to the difference quotient we assume that $u(x)$ is at least twice continuously differentiable, and examine the first order Taylor expansion

$$u(x+h) = u(x) + u'(x)h + \frac{1}{2}u''(\xi)h^2, \tag{2.5.2}$$

where ξ represents some point lying between x and $x+h$. The error or difference between the finite difference formula and the derivative being approximated is given by

$$\frac{u(x+h) - u(x)}{h} - u'(x) = \frac{1}{2}u''(\xi)h. \tag{2.5.3}$$

Since the error is proportional to h, we say that the finite difference quotient (2.5.1) is a first order approximation. When the precise formula for the error is not so important, we will write

$$u'(x) = \frac{u(x+h) - u(x)}{h} + O(h). \tag{2.5.4}$$

The $O(h)$ refers to a term that is proportional to h, or, more rigorously, bounded by a constant multiple of h as $h \to 0$.

To approximate higher order derivatives, we need to evaluate the function at more than two points. In general, an approximation to the n^th order derivative $u^(n)(x)$ requires at least $n+1$ distinct sample points. For example, let us try to approximate $u''(x)$ by sampling u at the particular points x, $x+h$ and $x-h$. Which combination of the function values $u(x-h)$, $u(x)$, $u(x+h)$ should be used? The answer to such a question can be found by consideration of the relevant Taylor expansions

$$\begin{aligned} u(x+h) &= u(x) + u'(x)h + u''(x)\frac{h^2}{2} + u'''(x)\frac{h^3}{6} + O(h^4), \\ u(x-h) &= u(x) - u'(x)h + u''(x)\frac{h^2}{2} - u'''(x)\frac{h^3}{6} + O(h^4), \end{aligned} \tag{2.5.5}$$

where the error terms are proportional to h^4. Adding the two formulae together gives

$$u(x+h) - u(x-h) = 2u(x) + u''(x)h^2 + O(h^4). \tag{2.5.6}$$

Rearranging terms, we conclude that

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2). \tag{2.5.7}$$

The result is known as the *centered finite difference approximation* to the second derivative of a function. Since the error is proportional to h^2, this is a second order approximation. A way to improve the order of accuracy of finite difference

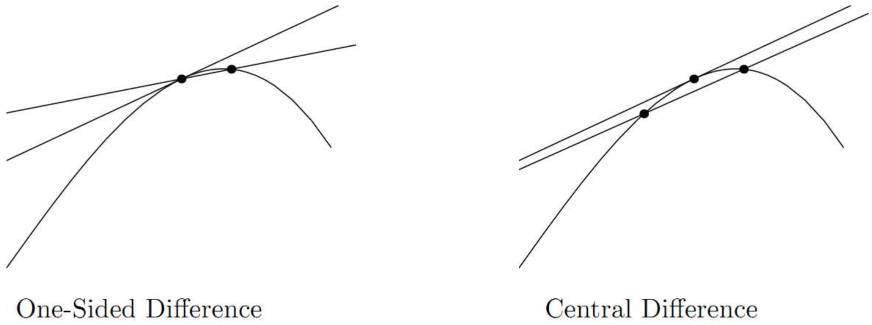

Figure 2.1: Finite Difference Approximations.

approximations is to employ more sample points. For instance, if the first order approximation (2.5.4) to the first derivative based on the two points x and $x+h$ is not sufficiently accurate, one can try combining the function values at three points x, $x+h$ and $x-h$. To find the appropriate combination of $u(x-h)$, $u(x)$, $u(x+h)$, we return to the Taylor expansions (2.5.5). To solve for $u'(x)$, we subtract the two formulae, and so

$$u(x+h) - u(x-h) = 2u'(x)h + \frac{h^3}{3} + O(h^4). \tag{2.5.8}$$

Rearranging the terms, we are led to the well-known *centered difference formula*

$$u'(x) = \frac{u(x+h) - u(x-h)}{2h} + O(h^2), \tag{2.5.9}$$

which is a second order approximation to the first derivative. Geometrically, the centered difference quotient represents the slope of the secant line through the two points $(x-h, u(x-h))$ and $x+h, u(x+h)$ on the graph of u centered symmetrically about the point x. Figure 2.1 illustrates the two approximations; the advantages in accuracy in the centered version are graphically evident. Higher order approximations can be found by evaluating the function at yet more sample points.

Many additional finite difference approximations can be constructed by similar manipulations of Taylor expansions, but these few very basic ones will suffice for our subsequent purposes. Now we apply the finite difference formulae to develop numerical solution schemes for the heat equation.

heat equation

For simplicity we will describe the case where σ and k are constant and $k\backslash\sigma = 1$.We will assume that the function $u(x,t)$ is defined for $0 \le x \le 1$ and $t \ge 0$, which solve the simple heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{2.5.10}$$

subject to the boundary conditions $u(0,t) = u(1,t) = 0$ and the initial condition $u(x,0) = h(x)$, where $h(x)$ is a given function, representing the initial temperature.

Here we discretize only the spatial domain obtaining a finite-dimensional Euclidean space R^{n-1}, **in practice we reduce the partial differential equation to a system of ordinary differential equations**. This correspond to utilizing a discrete model for heat flow rather than a continuous one.

For $0 \le i \le n$, let $x_i = i/n$ and

$$u_i(t) = u(x_i, t) = \textit{temperature in } x_i \textit{ at time } t.$$

Sice $u_0(t) = 0 = u_n(t)$ by the boundary conditions, the temperature at time t is specified by

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ u_{n-1}(t) \end{pmatrix} \tag{2.5.11}$$

a vector-valued function of one variable. The initial condition becomes

$$\mathbf{u}(0) = \mathbf{h}, \qquad \textit{where} \qquad \mathbf{h}(t) = \begin{pmatrix} h(x_1) \\ h(x_2) \\ \cdot \\ h(x_{n-1}) \end{pmatrix}. \tag{2.5.12}$$

We can approximate the first-order partial derivative by a difference quotient:

$$\frac{\partial u}{\partial x}\left(\frac{x_i + x_{i+1}}{2}, t\right) \doteq \frac{u_{i+1}(t) - u_i(t)}{x_{i+1} - x_i} = \frac{[u_{i+1}(t) - u_i(t)]}{1/n} = n\left[u_{i+1}(t) - u_i(t)\right]. \tag{2.5.13}$$

Similarly, we can approximate the second-order partial derivative:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(x_i, t) &\doteq \frac{\frac{\partial u}{\partial x}\left(\frac{x_i + x_{i+1}}{2}, t\right) - \frac{\partial u}{\partial x}\left(\frac{x_i + x_{i-1}}{2}, t\right)}{1/n} \\ &\doteq n^2\left[u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)\right]. \end{aligned} \tag{2.5.14}$$

Thus the (2.5.10) can be approximated by a system of ordinary differential equations

$$\frac{d\mathbf{u}_i}{dt} = n^2 A\mathbf{u}, \quad \textit{where} \quad A = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & \cdots & -2 \end{pmatrix} \tag{2.5.15}$$

Chapter 3

Sliding Mode Control

This Chapter presents a brief survey on Variable Structure Control Systems with Sliding Modes. Starting from a general case of sliding modes in dynamical systems with discontinuous right-hand side, classic approaches to sliding mode control systems are considered. The Chapter is based on Pisano and Usai (2011c) where the proofs of the presented theorems are shown.

Sliding-mode control has long been recognized as a powerful control method to counteract non-vanishing external disturbances and un-modeled dynamics Utkin (1992). This method is based on the deliberate introduction of sliding motions into the control system, and, since the motion along the sliding manifold proves to be uncorrupted by matched disturbances, the closed-loop system is guaranteed to exhibit strong properties of robustness against significant classes of disturbances and model uncertainties. Due to these advantages and simplicity of implementation, sliding-mode controllers have widely been used in various applications Young and Kwatny (1982). On the other hand, many important systems and industrial processes, such as flexible manipulators and chemical reactors, are governed by partial differential equations and are often described by models with a significant degree of uncertainties.

3.1 Sliding modes in discontinuous control systems

Consider a general nonlinear system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \tag{3.1.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{u} \in \mathbb{R}^q$ is the control input vector, t is time, and $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^+ \to \mathbb{R}^n$ is a vector field in the state space.

Assume that the state space is divided into 2^q subspaces $\mathcal{S}_k$ $(k = 1, 2, \ldots, 2^q)$ by the guard

$$\mathcal{G} = \{\mathbf{x} : \sigma(\mathbf{x}) = \mathbf{0}\}, \tag{3.1.2}$$

where $\sigma : \mathbb{R}^n \to \mathbb{R}^q$ is a sufficiently smooth vector function. Its ε–vicinity is defined as follows

$$\mathcal{V}_\varepsilon = \{\mathbf{x} \in \mathbb{R}^n : \|\sigma(\mathbf{x})\| \leqslant \varepsilon; \varepsilon > 0\}. \tag{3.1.3}$$

Define the control vector by a state feedback law such that

$$\mathbf{u}(t) = \ \mathbf{u}_k(\mathbf{x}) \ if \ \mathbf{x} \in \mathcal{S}_k, k \in \{1, 2, \ldots, 2^q\}, \tag{3.1.4}$$

then the following theorem holds.

THEOREM 3.1.1 Consider the nonlinear dynamics (3.1.1); if a proper ε defining (3.1.3) exists such that the control vector (3.1.4) satisfies the conditions

$$\text{sign}\left(\mathbf{J}^\sigma_\mathbf{x}(\mathbf{x}(t)) \cdot \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)\right) = -\text{sign}(\sigma), \forall \mathbf{x} \in \mathcal{V}_\varepsilon, \tag{3.1.5}$$

then the guard $\mathcal{G}$ is an invariant set in the state space and a sliding mode occurs on it.

Proof. Consider the q-dimensional vector

$$\mathbf{s} = \sigma(\mathbf{x}), \tag{3.1.6}$$

usually named *sliding variables vector*, and define the positive definite function

$$V(\mathbf{s}) = \frac{1}{2}\mathbf{s}^T\mathbf{s}. \tag{3.1.7}$$

The total time derivative of V is

$$\dot{V}(\mathbf{s}) \ = \mathbf{s}^T\dot{\mathbf{s}} = \mathbf{s}^T \text{diag}\{\text{sign}(\dot{s}_i)\}|\dot{\mathbf{s}}|. \tag{3.1.8}$$

Taking into account the implicit function theorem, (3.1.1), (3.1.4) and (3.1.5) then (3.1.8) results into

$$\dot{V}(\mathbf{s}) = -\mathbf{s}^T \text{diag}\{\text{sign}(s_i)\}|\dot{\mathbf{s}}| = -|\mathbf{s}|^T|\dot{\mathbf{s}}| < \mathbf{0}. \tag{3.1.9}$$

Therefore $V(\mathbf{s})$ is a Lyapunov function and the origin of the q-dimensional space of variables $\mathbf{s}$ is an asymptotically stable equilibrium point. □

From a geometrical point of view, condition (3.1.5) implies that within the neighborhood $\mathcal{V}_\varepsilon$ of $\mathcal{G}$ the vector field defining the state dynamics (3.1.1) is always directed towards $\mathcal{G}$ itself. Furthermore, if the magnitude of the control vector $\mathbf{u}$ components is sufficiently large so that $|\dot{s}_i| \geqslant \eta$ $(i = 1, \ldots, q)$, condition (3.1.9) satisfies the classical well-known *reaching condition* (that also make $\mathbf{s} = \mathbf{0}$ an invariant set) $\frac{1}{2}\frac{d}{dt}\|\mathbf{s}\|^2 \leq -\eta\|\mathbf{s}\|$ Utkin (1992). Therefore, the invariant set $\mathcal{G}$ is reached in a finite time $T_r \leq t_0 + \frac{\|\mathbf{s}(t_0)\|}{\eta}$ Utkin (1992), Slotine and Li (1991). $\mathbf{s}(t_0)$ ($\|\mathbf{s}(t_0)\| \leq \xi(\varepsilon) < \varepsilon$) is the sliding variables vector at initial time t_0.

From the definition of control $\mathbf{u}$ in (3.1.4), and taking into account condition (3.1.5), it is apparent that the vector field $\mathbf{f}$ defining the system dynamics (3.1.1) is discontinuous across the boundaries of the guard $\mathcal{G}$. Therefore, function $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$ has to be Lebesgue integrable on time and solution of (3.1.1) exists in the Filippov sense, Filippov (1988). Control $\mathbf{u}$ switches at infinite frequency when the system performs a sliding mode on $\mathcal{G}$, which is usually named *sliding surface* Utkin (1992).

The number of sets $\mathcal{S}_k$ partitioning the state space can be less than $2q$ if a $(q+1)$–dimensional control vector $\mathbf{u} = [u_1, \ldots, u_{q+1}]^T$, $u_i > 0$, is available. In fact the sliding variable space can be partitioned into $(q+1)$ sets $\mathcal{S}_k$, $\cap\mathcal{S}_k = \emptyset$, defining a simplex.

It is interesting to analyze the state trajectory when system (3.1.1) is constrained on $\mathcal{G}$. A simple approach to the problem is to consider the variable $\mathbf{s}$ defined by (3.1.6) as the output of the dynamical system (3.1.1), in which function $\sigma : \mathbb{R}^n \to \mathbb{R}^q$ represents the so-called output transformation. In classic sliding mode control usually condition (3.1.5) is assured by a proper choice of the control variables $\mathbf{u}$ so that matrix $\frac{\partial \dot{\mathbf{s}}}{\partial \mathbf{u}}$ has full rank in $\mathcal{V}_\varepsilon$. Then the overall system dynamics can be split into the input-output dynamics

$$\dot{\mathbf{s}}(t) = \mathbf{J}^\sigma_{\mathbf{x}}(\mathbf{x}(t)) \cdot \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) = \varphi(\mathbf{x}(t), \mathbf{u}(t), t), \tag{3.1.10}$$

and the internal dynamics

$$\dot{\mathbf{w}}(t) = \psi(\mathbf{w}(t), \mathbf{s}(t), t), \tag{3.1.11}$$

where $\mathbf{w} \in \mathbb{R}^{n-q}$ is named internal state and $\psi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^{n-q}$ is a sufficiently smooth vector function.

The relationship between the vector state $\mathbf{x}$ and the new state variables $\mathbf{s}$ and $\mathbf{w}$ is defined by a diffeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ preserving the origin and defined as follows in a vicinity of the guard $\mathcal{G}$ Isidori (1995), Slotine and Li (1991):

$$[\mathbf{s}^T, \mathbf{w}^T]^T = \Phi(\mathbf{x}) \qquad \mathbf{0} = \Phi(\mathbf{0})\ \forall \mathbf{x} \in \mathcal{V}_\varepsilon. \tag{3.1.12}$$

THEOREM 3.1.2 Assume that the diffeomorphic transformation (3.1.12) holds in the vicinity $\mathcal{V}_\varepsilon$ of the sliding manifold. Then system (3.1.1), (3.1.6) is stabilizable if a unique control $\mathbf{u}$ exists such that conditions of Theorem 1 are satisfied, the internal dynamics (3.1.11) is Bounded–Input Bounded–State (BIBS) stable and the zero dynamics

$$\dot{\mathbf{w}}(t) = \psi(\mathbf{w}(t), \mathbf{0}, t) \tag{3.1.13}$$

is stable in the Lyapunov sense.

Proof. The proof straightforwardly derives from results of Theorem 1 and the stability of the internal dynamics when the system is constrained onto $\mathcal{G}$. □

When the state is constrained onto the sliding surface $\mathcal{G}$, the system behavior is completely defined by the zero dynamics (3.1.13) Isidori (1995), taking into account the invertible relationship (3.1.12). That is, only a reduced order dynamics has to be considered during the sliding motion Utkin (1992). This "order reduction" property is a peculiar phenomenon in variable structure systems with sliding modes.

3.1.1 First order sliding mode control

Finding a feedback control (3.1.4) such that Theorem 1 holds is quite hard in the general case. Therefore, Sliding Mode Control (SMC) of uncertain systems usually refers to systems whose dynamics is affine with respect to control Utkin (1992), Edwards and purgeon (1998) i.e.,

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}(t), t) + \mathbf{B}(\mathbf{x}(t), t)\mathbf{u}(t), \tag{3.1.14}$$

where $\mathbf{A} : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is the a vector field in the state space, possibly uncertain, and $\mathbf{B}$ is a $(n \times q)$ matrix of functions $b_{ij}(\mathbf{x}(t), t) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$.

When the SMC approach is implemented, the first step of the design procedure is to define a proper system output $\mathbf{s}$, as in (3.1.6), such that the resulting internal dynamics is BIBS stable and, possibly, its zero dynamics is asymptotically stable. Then the control $\mathbf{u}$ is designed such that $\|\mathbf{s}\| \to \mathbf{0}$ in a finite time in spite of possible uncertainties.

THEOREM 3.1.3 Consider system (3.1.14), (3.1.6). Assume that the corresponding internal dynamics is BIBS stable, that the norm of its uncertain drift term $\mathbf{A}(\mathbf{x}(t), t)$ is upper bounded by a known function $F : \mathbb{R}^n \to \mathbb{R}^+$, i.e.,

$$\|\mathbf{A}(\mathbf{x}, t)\| \le F(\mathbf{x}), \tag{3.1.15}$$

and that the known square matrix $\mathbf{G}(\mathbf{x}, t) \equiv \mathbf{J}^{\sigma}_{\mathbf{x}}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x}, t) \in \mathbb{R}^{q \times q}$ is non singular $\forall \mathbf{x} \in \mathcal{V}_{\varepsilon}$, uniformly in time. Then, the set $\mathcal{G}$ in (3.1.2) is made finite time stable by means of the control law

$$\mathbf{u}(t) = -(F(\mathbf{x})\|\mathbf{J}^{\sigma}_{\mathbf{x}}\| + \eta)[\mathbf{G}(\mathbf{x}, t)]^{-1}\mathrm{sign}(\mathbf{s}), \qquad \eta > 0. \tag{3.1.16}$$

Proof.

The input-output dynamics of system (3.1.14), (3.1.6) is

$$\dot{\mathbf{s}}(t) = \mathbf{J}^{\sigma}_{\mathbf{x}}(\mathbf{x}) \cdot \mathbf{A}(\mathbf{x}, t) + \mathbf{G}(\mathbf{x}, t)\mathbf{u}(t), \tag{3.1.17}$$

Consider the positive definite function (3.1.7). Considering the time derivative of V along the trajectories of system (3.1.17), and taking into account (3.1.16), (3.1.8) yields

$$\begin{aligned}\dot{V}(\mathbf{s}) &= \mathbf{s}^T \cdot (\mathbf{J}^{\sigma}_{\mathbf{x}}(\mathbf{x}) \cdot \mathbf{A}(\mathbf{x},t) - (F(\mathbf{x})\|\mathbf{J}^{\sigma}_{\mathbf{x}}(\mathbf{x})\| + \eta)\mathrm{sign}(\mathbf{s})) \leq \\ &\leq -\eta\mathbf{s}^T \cdot \mathrm{sign}(\mathbf{s}) = -\eta\|\mathbf{s}\|_1 < -\eta\|\mathbf{s}\|_2 < 0.\end{aligned} \tag{3.1.18}$$

□

When the system control gain matrix $\mathbf{B}(\mathbf{x},t)$ is uncertain, a similar theorem can be proved if some condition about $\mathbf{B}$ is met.

THEOREM 3.1.4 Consider system (3.1.14), (3.1.6) satisfying (3.1.15). Assume that the corresponding internal dynamics is BIBS stable, that the uncertain gain matrix $\mathbf{B}(\mathbf{x},t)$ and the guard $\mathcal{G}$ are such that the square matrix $\mathbf{G}(\mathbf{x},t) \equiv \mathbf{J}^{\sigma}_{\mathbf{x}}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x},t)$ is positive definite and a known bound $\Lambda_m > 0$ exists such that

$$\Lambda_m \leq \min\{\lambda_i^{\mathbf{G}}(\mathbf{x},t); i = 1,\ldots,q\} \qquad \forall \mathbf{x} \in \mathcal{V}_\varepsilon,\ \forall t \tag{3.1.19}$$

where $\lambda_i^{\mathbf{G}}(\mathbf{x},t)$ $(i = 1,\ldots,q)$ are the eigenvalues of matrix $\mathbf{G}(\mathbf{x},t)$. Then, the set $\mathcal{G}$ in (3.1.2) is made finite time stable by means of the control law

$$\mathbf{u}(t) = -\frac{F(\mathbf{x})\|\mathbf{J}^{\sigma}_{\mathbf{x}}(\mathbf{x})\| + \eta}{\Lambda_m}\frac{\mathbf{s}}{\|\mathbf{s}\|_2}, \qquad \eta > 0. \tag{3.1.20}$$

Proof. The proof follows the same steps than previous Theorem 3.1.3. Define function $V(\mathbf{s})$ as in (3.1.7) and consider (3.1.17) and (3.1.20) into its time derivative (3.1.8). By (3.1.19) it results

$$\begin{aligned}\dot{V}(\mathbf{s}) &= \mathbf{s}^T(\mathbf{J}^{\sigma}_{\mathbf{x}}(\mathbf{x})\mathbf{A}(\mathbf{x},t) - \tfrac{F(\mathbf{x})\|\mathbf{J}^{\sigma}_{\mathbf{x}}(\mathbf{x})\|+\eta}{\Lambda_m}\mathbf{G}(\mathbf{x},t)\tfrac{\mathbf{s}}{\|\mathbf{s}\|_2}) \leq \\ &\leq -\tfrac{\eta}{\Lambda_m\|\mathbf{s}\|_2}\mathbf{s}^T\mathbf{G}(\mathbf{x},t)\mathbf{s} \leq -\eta\|\mathbf{s}\|_2 < 0.\end{aligned} \tag{3.1.21}$$

□

To counteract the uncertainty in the system model the magnitude of the control vector components has to be sufficiently large. The positive parameter $\eta > 0$ is a design parameter guaranteeing that the previously defined *reaching condition* is met. Several design methods can be found in the technical literature Utkin (1992), Edwards and purgeon (1998), Young and Kwatny (1982), Bartolini *et al.* (2008).

When the system exhibits a sliding-mode behavior, the discontinuous control (3.1.16), or (3.1.20), undergoes infinite-frequency switchings. The effect of the discontinuous and infinite-frequency switching control on the system dynamics is the same as that of the *continuous* control which allows the state trajectory to remain on the sliding surface Filippov (1988), Utkin (1992). Considering the non linear system (3.1.1) with a scalar input (i.e., $q = 1$), in Filippov (1988) it was shown that such a continuous dynamics is a convex combination of the two vector fields $\mathbf{f}_1 = \mathbf{f}(\mathbf{x}, u_1, t)$ and $\mathbf{f}_2 = \mathbf{f}(\mathbf{x}, u_2, t)$ defined on $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively, i.e.,

$$\begin{aligned} &\dot{\mathbf{x}}(t) = \mathbf{f}_0(\mathbf{x}, t), \\ &\mathbf{f}_0 = \alpha \mathbf{f}_1 + (1-\alpha)\mathbf{f}_2, \\ &\alpha = \tfrac{\nabla\sigma\cdot\mathbf{f}_2}{\nabla\sigma\cdot(\mathbf{f}_2-\mathbf{f}_1)}. \end{aligned} \tag{3.1.22}$$

The above approach to regularize differential equations with discontinuous right–end side is called the Filippov's *continuation method*. In the case of multi–input control (3.1.4) the continuous vector field $\mathbf{f}_0$ allowing the continuation of the state trajectory on $\mathcal{G}$ is still a convex combination of the 2^q vector fields $\mathbf{f}_k = \mathbf{f}(\mathbf{x}, u_k, t)$ $(k = 1, 2, \ldots, 2^q)$, i.e.,

$$\begin{aligned} &\dot{\mathbf{x}}(t) = \mathbf{f}_0(\mathbf{x}, t), \\ &\mathbf{f}_0 = \textstyle\sum_{k=1}^{2^q} \alpha_k \mathbf{f}_k, \\ &\textstyle\sum_{k=1}^{2^q} \alpha_k = 1. \end{aligned} \tag{3.1.23}$$

If the discontinuous right–hand–side of the differential equation defining the system dynamics satisfies some geometric conditions, the Filippov's continuation method can be unambiguously defined on $\mathcal{G}$.

Utkin (1992) introduced the concept of *equivalent control* as the continuous control input $\mathbf{u}_{eq}$ which is able to *maintain* the sliding-mode behaviour by nullifying $\dot{\mathbf{s}}$, i.e., it is the solution of

$$\begin{aligned} &\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}_{eq}, t), \\ &\dot{\mathbf{s}} = \tfrac{\partial\sigma}{\partial\mathbf{x}}(\mathbf{x}(t), \mathbf{u}_{eq}(t), t) \cdot \mathbf{f}(\mathbf{x}(t), \mathbf{u}_{eq}(t), t) = \mathbf{0}. \end{aligned} \tag{3.1.24}$$

With reference to affine systems (3.1.14), $\dot{\mathbf{s}}$ takes the simplified expression reported in (3.1.17) thus the equivalent control turns out to be defined as follows

$$\mathbf{u}_{eq} = -\left[\mathbf{G}(\mathbf{x}, t)\right]^{-1} \mathbf{J}_{\mathbf{x}}^{\sigma}(\mathbf{x}) \cdot \mathbf{A}(\mathbf{x}, t). \tag{3.1.25}$$

The Filippov's continuation method and the Utkin's equivalent control methods give the same continuous control law only for affine scalar control Utkin (1992) and for a limited class of systems nonlinear in the control.

The non singularity condition for the q–dimensional square matrix $\mathbf{G}(\mathbf{x}, t)$ implies that the control $\mathbf{u}$ appears explicitly in the first derivative of the sliding variable vector $\mathbf{s}$ fulfilling a kind of controllability condition, i.e., the vector relative degree Isidori (1995) between the sliding and the input variables is $[1, 1, \ldots, 1]^T$.

Such a condition could be not satisfied and then the input would not appear in the first total derivative of the sliding variable vector affecting, instead, its higher derivatives; in this case a higher-order sliding mode (HOSM) can appear.

3.2 Higher-order sliding mode control

A second order sliding mode appears when the differential inclusion $V(\mathbf{x}, t)$ defining the closed loop dynamics (3.1.1), (3.1.4) belongs to the tangential space of the sliding manifold $\mathcal{G}$ defined as in (3.1.2), i.e. Levant (1993), Fridman and Levant (1996),

$$\mathcal{G}_2 = \{\mathbf{x} : \dot{\sigma}(\mathbf{x}) = \sigma(\mathbf{x}) = \mathbf{0}.\}. \tag{3.2.1}$$

This definition can be extended to higher–order sliding manifolds as follows Levant (1993):

$$\mathcal{G}_r = \{\mathbf{x} : \frac{d^k}{dt^k}\sigma(\mathbf{x}) = \mathbf{0}, k = 0, 1, \ldots, r-1\}, \tag{3.2.2}$$

where $\sigma : \mathbb{R}^n \to \mathbb{R}^q$ is, again, a sufficiently smooth vector function and r ($r = 1, 2, \ldots$) represents the order of the so–called *sliding set.* A r^{th}–order sliding mode (r-SM) appears on the sliding manifold $\mathcal{G}$ when the state trajectory is confined on (3.2.2).

DEFINITION 3.2.1 Levant (1993) Let the r-sliding set (3.2.2) be non-empty and assume that it is locally an integral set in Filippov's sense (i.e. it consists of Filippov's trajectories of the discontinuous dynamic system). Then the corresponding motion satisfying (3.2.2) is called *r-sliding mode* (r-SM) with respect to the constraint function σ.

A Higher–Order Sliding Mode Control (HOSMC) system is implemented when the control $\mathbf{u}$ is able to constrain the system state onto (3.2.2) starting from any point in a ε-vicinity of $\mathcal{G}_r$.

HOSMC systems are difficult to design with respect to the general nonlinear dynamics (3.1.1) since it is not possible to trivially extend the definition of the sliding manifold (3.1.2) by using (3.2.2). In fact only q control variables are available and condition (3.1.5) cannot be guaranteed with respect to the resulting rq variables $\mathbf{s}, \dot{\mathbf{s}}, \ldots, \mathbf{s}^{(r)}$.

An affine time–independent structure for the nonlinear dynamics can be obtained by considering an augmented dynamics in which the control $\mathbf{u}$ is part of an augmented vector state and its time derivative $\mathbf{v} = \dot{\mathbf{u}}$ is the actual control to be designed:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{z} \\ \dot{\mathbf{u}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{x}} \\ 1 \\ 0 \\ \frac{\partial \mathbf{f}}{\partial x_{n+1}}(\mathbf{x}, \mathbf{u}, z) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{u}, z) \cdot \hat{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}, \mathbf{u}, z) \end{bmatrix} \cdot \mathbf{v}, \tag{3.2.3}$$

in which $\frac{\partial \mathbf{f}}{\partial \mathbf{u}}$ is a full–rank matrix Levant (1993). Therefore, when considering HOSMC, it is usual to refer to affine stationary nonlinear systems

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(t), \tag{3.2.4}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector (possibly augmented), $\mathbf{u} \in \mathbb{R}^q$ is the control vector, possibly the time derivative of the plant input, $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$ are sufficiently smooth vector fields and matrix, respectively, in the state space.

PROPOSITION 3.2.1 Given the system dynamics (3.2.4), a r–SMC on the manifold (3.1.2) can be designed if $n \geq rq$, $L_{\mathbf{g}} L_{\mathbf{f}}^k \sigma = \mathbf{0}$ $(k = 1, 2, \ldots, r-2$ and $L_{\mathbf{g}} L_{\mathbf{f}}^{r-1} \sigma$ has full rank. □

3.2.1 A Second-Order sliding mode controller: the Super Twisting

During the thesis was mainly used the so-called *Super–Twisting* algorithm. This is a second order sliding-mode control algoritm that ensures the continuity of the control, and it's used for systems with relative degree equal one.

The *Super–Twisting* algorithm is conceptually different from the other 2–SMC algorithms, for two reasons: first, it depends only on the actual value of the sliding variable, while the others have more information demand. Second, it is effective only for chattering attenuation purposes as far as relative degree one constraints are dealt with.

It is defined by the following dynamic controller Levant (1993):

$$\begin{aligned} u(t) &= v(t) - \lambda |s(t)|^{1/2} \mathrm{sign}(s(t)), \\ \dot{v}(t) &= -\alpha \mathrm{sign}(s(t)). \end{aligned} \tag{3.2.5}$$

where $u(t) \in \mathbb{R}$ is the input of system (3.2.4) with $q = 1$ and $s(t) \in \mathbb{R}$ is the sliding variable (i.e., the system output) (3.1.6) measuring the distance of the system from the sliding surface $\mathcal{G}$ in (3.1.2). The *Super-Twisting* controller can be considered as a nonlinear implementation of a classic PI controller with better robustness properties.

Recent findings about the *Super–Twisting* algorithm were obtained by Moreno and Osorio (2008). The authors obtained for the *Super–Twisting* algorithm a strong Lyapunov function for the first time. The introduction of a Lyapunov function allows not only to study more deeply the known properties of finite time convergence and robustness to strong perturbations, but also to improve the performance by adding linear correction terms to the algorithm. Consider the following perturbed version of (3.2.5)

$$\begin{aligned} \dot{x}_1 &= -k_1|x_1|^{1/2}\text{sign}(x_1) + x_2 + \varrho_1, \\ \dot{x}_2 &= -k_3\text{sign}(x_1) + \varrho_1. \end{aligned} \tag{3.2.6}$$

where x_i are the scalar state variables, k_i are gains to be designed, and ϱ_i are perturbation terms. The Lyapunov function that ensure the convergence in finite time of all trajectories of this system to zero, when the gains are adequately selected, and for some kinds of perturbations is (Moreno and Osorio (2008)):

$$V(x) = 2k3|x_1| + \frac{1}{2}x_2^2 + \frac{1}{2}\left(k_1|x_1|^{1/2}\text{sign}(x_1) - x_2\right)^2. \tag{3.2.7}$$

THEOREM 3.2.1 Suppose that the perturbation terms of the system (3.2.7) are globally bounded by

$$|\varrho_1| \leq \delta_1|x_1|^{1/2}, \qquad |\varrho_2| \leq \delta_2, \tag{3.2.8}$$

for some constants $\delta_1, \delta_2 \geq 0$. Then the origin $x = 0$ is an equilibrium point that is strongly globally asymptotically stable if the gains satisfy

$$\begin{aligned} &k_1 > 2\delta_1 \\ &k_3 > k_1\frac{5\delta_1 k_1 + 6\delta_2 + 4(\delta_1 + \delta_2/k_1)^2}{2(k_1 - 2\delta_1)} \end{aligned} \tag{3.2.9}$$

Moreover, all trajectories converge in finite time to the origin, upperbounded by $\frac{2V^{1/2}(x_0)}{\bar{\gamma}}$, where x_0 is the initial state and $\bar{\gamma}$ is a constant depending on the gains k_1, k_3 and the perturbation coefficients δ_1, δ_2.

For the proof please refer to the cited article Moreno and Osorio (2008).

Chapter 4

Unknown Input Observer(UIO) and estimation

In this chapter we show a brief survey of the concept of strong observability Hautus (1983), Molinari (1976) for multi-variable linear systems and the design of an Unknown Input Observer. In the last section an estimator of the unknown input, based on second order sliding-mode algorithm "Super-Twisting" Levant (1993), Moreno and Osorio (2008) is proposed.

4.1 Strong observability

The notion of strong observability has been introduced more than thirty years ago Molinari (1976); Hautus (1983) in the framework of the unknown-input observers theory. Recently it has been exploited to design robust observers based on the high-order sliding mode approach Bejarano *et al.* (2007). Consider the linear time invariant system Σ

$$\begin{aligned} \dot{x} &= Ax + Gu + Fw(t) \\ y &= Cx \end{aligned} \tag{4.1.1}$$

where $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$ are the state and output variables, $u(t) \in \mathbb{R}^h$ is a ***known input*** to the system, $w(t) \in \mathbb{R}^m$ is an ***unknown input*** term, and A, G, F, C are known constant matrices of appropriate dimension.

It's well known that (4.1.1) is observable if and only if for each initial state $x_0 = x(0)$, this status value can be determined on the basis of observation of the evolution for a finite time $t_f \geq 0$.

The system Σ is called **strongly observable** if this property holds when we use an arbitrary input function:

DEFINITION 4.1.1 Molinari (1976) Σ is called strongly observable if for all $x_0 \in \mathfrak{X}$ and for every input function u, the following holds: $y_u(t, x_0) = 0$ for all $t \geq 0$ implies $x_0 = 0$.

It's possible to connect strong observability of Σ with properties of its system matrix P_Σ. The system matrix of Σ is defined as the real polynomial matrix
Associated with this system, let us introduce the following system matrix

$$P_\Sigma(s) = \begin{pmatrix} sI - A & -F \\ C & 0 \end{pmatrix}. \tag{4.1.2}$$

The system Σ is degenerate if the rank of S is strictly less than the minimum of $[n + rank(F), n + rank(C)]$ for all values of $s \in \mathbb{C}$.

DEFINITION 4.1.2 *Invariant zeros* are those values of s that result in rank of S becoming less than the minimum of $[n + rank(F), n + rank(C)]$.

It has been shown in Molinari (1976) that the following property holds
The triplet (A, F, C) is strongly observable if and only if it has no invariant zeros.

4.2 system decoupling and UIO design

Let us make the following assumptions:

A1. The matrix triplet (A, F, C) is strongly observable

A2. $rank\,(CF) = rank\ F = m$.

If conditions A1 and A2 are both satisfied then it can be systematically found a state coordinates transformation together with an output coordinates change which decouple the unknown input ξ from a certain subsystem in the new coordinates. Such a transformation is outlined below.

For the generic matrix $J \in \mathbb{R}^{n_r \times n_c}$ with $rankJ = r$, we define $J^\perp \in \mathbb{R}^{n_r - r \times n_r}$ as a matrix such that $J^\perp J = 0$ and $rankJ^\perp = n_r - r$. Matrix $J^\perp$ always exists and, furthermore, it is not unique [1]. Let $\Gamma^+ = [\Gamma^T \Gamma]^{-1} \Gamma^T$ denote the left pseudo-inverse of Γ such that $\Gamma^+ \Gamma = I_{n_c}$, with I_{n_c} being the identity matrix of order n_c.

Consider the following transformation matrices T and U:

$$T = \begin{bmatrix} F^\perp \\ (CF)^+ C \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}, \quad U = \begin{bmatrix} (CF)^\perp \\ (CF)^+ \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}. \tag{4.2.1}$$

[1] A Matlab instruction for computing $M_b = M^\perp$ for a generic matrix M is `Mb = null(M')'`

and the transformed state and output vectors

$$\bar{x} = Tx = \begin{bmatrix} T_1 x \\ T_2 x \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}, \quad \bar{x}_1 \in R^{n-m} \quad \bar{x}_2 \in R^m \tag{4.2.2}$$

$$\bar{y} = Uy = \begin{bmatrix} U_1 y \\ U_2 y \end{bmatrix} = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix}, \quad \bar{y}_1 \in R^{p-m} \quad \bar{y}_2 \in R^m \tag{4.2.3}$$

The subcomponents of the transformed vectors take the form

$$\bar{x}_1 = F^{\perp} x, \qquad \bar{x}_2 = (CF)^{+} Cx \tag{4.2.4}$$

$$\bar{y}_1 = (CF)^{\perp} y \qquad \bar{y}_2 = (CF)^{+} y \tag{4.2.5}$$

After simple algebraic manipulations the **transformed dynamics** in the new coordinates take the form:

$$\begin{array}{rcl} \dot{\bar{x}}_1 &=& \bar{A}_{11}\bar{x}_1 + \bar{A}_{12}\bar{x}_2 + F^{\perp}Gu \\ \dot{\bar{x}}_2 &=& \bar{A}_{21}\bar{x}_1 + \bar{A}_{22}\bar{x}_2 + (CF)^{+} CGu + w(t) \\ \bar{y}_1 &=& \bar{C}_1\bar{x}_1 \\ \bar{y}_2 &=& \bar{x}_2 \end{array} \tag{4.2.6}$$

with the matrices $\bar{A}_{11}$, ..., $\bar{A}_{22}$, $\bar{C}_1$ such that

$$\begin{bmatrix} \bar{A}_{11} & \bar{A}_{11} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = TAT^{-1}, \quad \bar{C}_1 = (CF)^{\perp} C\tilde{T}_1. \tag{4.2.7}$$

It turns out that the triple (A, C, F) is strongly observable if, and only if, **the pair ($\bar{A}_{11}, \bar{C}_1$) is observable** Molinari (1976); Hautus (1983). In light of the Assumption A1, this property, that can be also understood in terms of a simplified algebraic test to check the strong detectability of a matrix triple, and its satisfaction opens the route to design stable observers for the state of the transformed dynamics (4.2.6).

The peculiarity of the transformed system (4.2.6) is that $\bar{x}_2$ is available for measurements since it constitutes a part of the transformed output vector $\bar{y}$. Hence, **state observation for system (4.2.6) can be accomplished by estimating $\bar{x}_1$ only**, whose dynamics is not affected by the unknown input vector.

The observability of the ($\bar{A}_{11}, \bar{C}_1$) pair permits the implementation of the following Luenberger observer for the $\bar{x}_1$ subsystem of (4.2.6):

$$\dot{\hat{\bar{x}}}_1 = \bar{A}_{11}\hat{\bar{x}}_1 + \bar{A}_{12}\bar{y}_2 + F^{\perp}Gu + L(\bar{y}_1 - \bar{C}_1\hat{\bar{x}}_1) \tag{4.2.8}$$

which gives rise to the error dynamics

$$\dot{e}_1 = (A - LC)e_1, \qquad e_1 = \hat{\bar{x}}_1 - \bar{x}_1 \tag{4.2.9}$$

whose eigenvalues can be arbitrarily located by a proper selection of the matrix L. Therefore, with properly chosen L we have that

$$\hat{\bar{x}}_1 \to \bar{x}_1 \quad as \quad t \to \infty \tag{4.2.10}$$

which implies that the overall system state can be reconstructed by the following relationships

$$\hat{x} = T^{-1} \begin{bmatrix} \hat{\bar{x}}_1 \\ \bar{y}_2 \end{bmatrix} \tag{4.2.11}$$

Note that the convergence of $\hat{\bar{x}}_1$ to $\bar{x}_1$ is exponential and can be made as fast as desired. Remarkably, the above estimation is correct in spite of the presence of unmeasurable, possibly very large, external inputs.

4.3 Reconstruction of the unknown inputs

An estimator can be designed which gives an exponentially converging estimate of the unknown input. Consider the following estimator dynamics

$$\dot{\hat{\bar{x}}}_2 = \bar{A}_{21}\hat{\bar{x}}_1 + \bar{A}_{22}\bar{y}_2 + v(t) \tag{4.3.1}$$

with the estimator injection input $v(t)$ yet to be specified. The next assumption is fulfilled:
A3. It can be found a constant w_d such that: $|\dot{w}(t)| \leq w_d$.

Define

$$\sigma(t) = \hat{\bar{x}}_2 - \bar{x}_2 \tag{4.3.2}$$

The time derivative of $\sigma(t)$ is

$$\dot{\sigma} = \dot{\hat{\bar{x}}}_2 - \dot{\bar{x}}_2 = \bar{A}_{21}e_1(t) + v(t) - w(t) \tag{4.3.3}$$

Define the output injection $v(t)$ as follows

$$\begin{aligned} v(t) &= k_1 \left|\sigma\right|^{\frac{1}{2}} sign\sigma - k_2\sigma + \alpha(t) \\ \dot{\alpha}(t) &= -k_3 sign\sigma - k_4\sigma \end{aligned} \tag{4.3.4}$$

Considering (4.3.3) into (4.3.4) yields:

$$\dot{\sigma} = \bar{A}_{21}e_1(t) - w(t) - k_1 \left|\sigma\right|^{\frac{1}{2}} sign\sigma - k_2\sigma + \alpha(t) \tag{4.3.5}$$

To simplify the notation define

$$\Gamma(t) = \bar{A}_{21}e_1(t) + \alpha(t) - w(t) \tag{4.3.6}$$

The derivative of $\Gamma(t)$ is

$$\dot{\Gamma} = \bar{A}_{21}\dot{e}_1 + \dot{\alpha} - \dot{w} = \psi - k_3 sign\sigma - k_4\sigma \tag{4.3.7}$$

where

$$\psi = \bar{A}_{21}\dot{e}_1 - \dot{w} = \bar{A}_{21}(\bar{A}_{11} - LC_1)e_1 - \dot{w} \tag{4.3.8}$$

The error dynamics in $\sigma - \Gamma$ coordinates is:

$$\begin{aligned} \dot{\sigma} &= \Gamma - k_1 \left|\sigma\right|^{\frac{1}{2}} sign\sigma - k_2\sigma \\ \dot{\Gamma} &= \psi - k_3 sign\sigma - k_4\sigma \end{aligned} \tag{4.3.9}$$

Let the tuning parameters be chosen according to the next inequalities

$$k_1, k_3 > 0;\ k_2, k_4 \geq 0;\ min\left\{\frac{k_1}{2}, \frac{k_1k_3}{1+k_1}, k_3\right\} > M \tag{4.3.10}$$

where M is any constant such that

$$M \geq w_d + \rho^2, \quad \rho \neq 0. \tag{4.3.11}$$

Considering (4.3.9)-(4.3.11), the condition $e_1(t) \rightarrow 0$, derived from (4.2.10), guarantees that the next condition (4.3.12) holds starting from a finite time instant $\bar{T}$.

$$|\psi| \leq M, \quad t \geq \bar{T}. \tag{4.3.12}$$

The stability of (4.3.9) can be demonstrated by means of the Lyapunov function Moreno and Osorio (2008)

$$V(\sigma,\Gamma) = 2k_3|\sigma| + k_4\sigma^2 + \frac{1}{2}\Gamma^2 + \frac{1}{2}s^2(\sigma,\Gamma) \tag{4.3.13}$$

where

$$s(\sigma,\Gamma) = \Gamma - k_1|\sigma|^{\frac{1}{2}} sign\sigma \tag{4.3.14}$$

Differentiating the Lyapunov function (4.3.13) along the trajectory of the system (4.3.9) gives

$$\dot{V} \leq -|\sigma|^{-\frac{1}{2}} W(\sigma, s) \tag{4.3.15}$$

where

$$\begin{aligned} W(\sigma,s) &= [k_2 s^2|\sigma|^{\frac{1}{2}} + (\frac{k_1}{2} - M)s^2 + k_1k_2]+ \\ &+ (k_2k_3 - Mk_2)|\sigma|^{\frac{3}{2}} + (k_1k_3 - M(1-k_1))|\sigma|+ \\ &+ k_2k_4|\sigma|^{\frac{5}{2}} \geq \gamma V(\sigma,\Gamma) \end{aligned} \tag{4.3.16}$$

for some $\gamma > 0$ and for all σ, Γ, $s \in \mathbb{R}$. By taking advantage of (4.2.10) it can be easily shown that $|v(t) - w(t)| \rightarrow 0$ as $t \rightarrow \infty$. Then, under the conditions (4.3.10), the estimator (4.3.1), (4.3.9) allows one to reconstruct the unknown input $w(t)$ acting on the original system (4.1.1).

Chapter 5

Unknown input estimator for Diffusion processes

Here the methodologies described in the previous Chapter are applied to diffusion processes, where the unknown input to estimate is a non measurable disturbance or a fault in the process. The first example in section 2 considers a problem of state and disturbance estimation for a perturbed version of the diffusion PDE with collocated measurement sensors. It is assumed that the system model is corrupted by an in-domain uncertain, distributed, disturbance. The third section illustrates an approach to actuator fault detection in industrial furnaces.

5.1 Unknown disturbance estimator for the diffusion equation

Topics in this section have been published by the author in Pisano *et al.* (2010). It is assumed that the diffusion model is corrupted by an in-domain uncertain, distributed, disturbance. Related investigation were made in Demetriou and Rosen (2005) where an Unknown Input Observer (UIO) was proposed for a class of PDEs in abstract form with a concrete example developed for the perturbed diffusion-convection equation. Demetriou and Rosen (2005) basically extends to Distributed Parameter Systems (DPS) the finite dimensional results of UIO design Chen *et al.* (1996), Edwards *et al.* (2000). The key point of the design method in Demetriou and Rosen (2005) was that of selecting the sensor type and location in such a way that the resulting measurement operator fulfills certain operator equations. After deriving the finite-dimensional modal expansion of the involved PDE, here we follow a similar idea but we develop our design conditions for the approximate finite dimensional model directly.

5.1.1 Formulation of the problem

Consider a physical phenomenon represented by the space and time varying scalar field $z(x,t)$, where $0 \leq x \leq l$ is the mono-dimensional (1D) spatial variable and $t > 0$ is the time variable. Let the scalar field behavior be governed by a perturbed diffusion (PDE).

$$z_t(x,t) = \theta z_{xx}(x,t) + \psi(x,t) \tag{5.1.1}$$

where θ is the positive coefficient called *diffusivity*, $z_t(x,t)$ denotes the partial time derivative and $z_{xx}(x,t)$ denotes the second order spatial derivative. The vector field $\psi(x,t)$ represent an **uncertain** source term. We consider an uncertain source term of the type

$$\psi(x,t) = f(x)w(t) \tag{5.1.2}$$

where $f(x)$ is a **known** function, and $w(t)$ is **uncertain**. The initial conditions (ICs) are:

$$z(x,0) = z_0(x), \quad z_0(x) \in L_2[0,l] \tag{5.1.3}$$

We consider two types of boundaries conditions (BCs), namely, homogenous Neumann BCs

$$\texttt{Neumann-type} \quad z_x(0,t) = z_x(l,t) = 0 \tag{5.1.4}$$

or homogenous Dirichlet BCs

$$\texttt{Dirichlet-type} \quad z(0,t) = z(l,t) = 0. \tag{5.1.5}$$

The available measurements are the p-dimensional vector $y = [y_1\ y_2 ... y_p]$ where:

$$y_k(t) = \int_0^l s_k(x) z(x,t) dx, \quad k = 1, ..., p \tag{5.1.6}$$

$s_k(x) \in L_2(0,l)$ is a square integrable function which is determined by the location and type of the measurement sensors. In particular we consider point-wise measurements along the spatial domain hence

$$s_k = \delta(x - x_s^k) \tag{5.1.7}$$

where $\delta(\cdot)$ is the Dirac Function and x_s^k is the location of the kth measurement sensor. Then by (5.1.6) and (5.1.7)

$$y_k(t) = z(x_s^k, t), \quad k = 1, ..., p \tag{5.1.8}$$

The aim of this work is to provide the approximate reconstruction of the state $z(x,t)$ and of the unknown disturbance signal $w(t)$.

5.1.2 modal representation

Now there will be shown the modal expansion of the (5.1.1), that is generally described in the Chapter 2. In the end we achieve a common state-variable form tat is used to develop the UIO and sliding-mode estimator described in the previous Chapter.

By expanding the solution of the equation (5.1.1) in an infinite series in terms of eigenfunctions (modal expansion) it is possible to express the solution as

$$z(x,t) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x) \tag{5.1.9}$$

where $\phi_n(x)$ are the eigenfunctions corresponding to the the boundary conditions (5.1.5) or (5.1.4) and $q_n(t)$ are appropriate functions to be determined. Substituting the modal expansion for the solution $z(x,t)$ into the system we obtain an infinite-dimensional system of ODE $q_n(t)$

$$\begin{aligned} \dot{q}_n &= \lambda_n q_n + f_q^n w(t), \quad n = 1, ..., \infty. \\ y_k &= \sum_{n=1}^{\infty} q_n(t) c_k^n, \quad k = 1, ..., p. \end{aligned} \tag{5.1.10}$$

where λ_n are the eigenvalues and:

$$\begin{aligned} q_n(0) &= \frac{\int_0^l z_0(x)\phi_n(x)dx}{\int_0^l \phi_n^2(x)dx}, \quad f_q^n = \frac{\int_0^l f(x)\phi_n(x)dx}{\int_0^l \phi_n^2(x)dx} \\ c_k^n &= \int_0^l s_k(x)\phi_n(x)dx = \phi_n(x_s^k) \end{aligned} \tag{5.1.11}$$

We consider a finite number N of modes, yielding the finite dimensional approximation of (5.1.11):

$$\begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_N \end{bmatrix} = \begin{bmatrix} -\lambda_1^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & -\lambda_N^2 \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_N \end{bmatrix} + \begin{bmatrix} f_q^1 \\ \vdots \\ f_q^N \end{bmatrix} w(t) \tag{5.1.12}$$

with the output equation

$$\begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} c_1^1 & \cdots & c_1^N \\ \cdots & \cdots & \cdots \\ c_p^1 & \cdots & c_p^N \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_N \end{bmatrix} \tag{5.1.13}$$

Finally we can rewrite (5.1.12) and (5.1.13) in compact form:

$$\begin{aligned} \dot{q}\left(t\right) &= A_{md}q\left(t\right) + F_{md}w\left(t\right) \\ y\left(t\right) &= C_{md}q\left(t\right) \end{aligned} \tag{5.1.14}$$

where $q = [q_1\ q_2\ ...\ q_N]$ and

$$A_{md} = \begin{bmatrix} -\lambda_1^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & -\lambda_N^2 \end{bmatrix},\ C_{md} = \begin{bmatrix} c_1^1 & \cdots & c_1^N \\ \cdots & \cdots & \cdots \\ c_p^1 & \cdots & c_p^N \end{bmatrix},\ F_{md} = \begin{bmatrix} f_q^1 \\ \vdots \\ f_q^N \end{bmatrix} \tag{5.1.15}$$

5.1.3 Numerical simulations of the Disturbance estimator

Consider the perturbed heat equation.

$$z_t = 0.5z_{xx} + f(x)w(t) \tag{5.1.16}$$

with homogeneous Dirichlet BCs (5.1.5) where $f(x) = 2{+}6\sin(4\pi x)$ and $w(t) = 2(10{+}sin(4t))$. The initial conditions are set to $z_0(x) = sin(\pi x)$ and the location of the two sensors are: $x_c^1 = 0.1$ and $x_c^2 = 0.7$. The corresponding eigenvalues and eigenfunctions are

$$\lambda_n = (n\pi)\sqrt{0.5},\quad \phi_n(x) = sin(n\pi x),\quad n = 1, ..., \infty. \tag{5.1.17}$$

To generate the measurements and the "true" states accurately, the equation (5.1.16) has been simulated using $N = 50$ modes. Figure 5.1 shows the actual state evolution. In TEST 1 the UIO and the disturbance estimator are implemented with $N = 5$ modes. The next matrices are obtained for the original system:

$$A = \begin{bmatrix} -4.934 & 0 & 0 & 0 & 0 \\ 0 & -19.739 & 0 & 0 & 0 \\ 0 & 0 & -44.413 & 0 & 0 \\ 0 & 0 & 0 & -78.956 & 0 \\ 0 & 0 & 0 & 0 & -123.370 \end{bmatrix}$$

$$F = \begin{bmatrix} 2.546 \\ 0 \\ 0.888 \\ 6 \\ 0.509 \end{bmatrix},\ C = \begin{bmatrix} 0.338 & 0.637 & 0.860 & 0.982 & 0.987 \\ 0.827 & -0.929 & 0.218 & 0.684 & -0.987 \end{bmatrix} \tag{5.1.18}$$

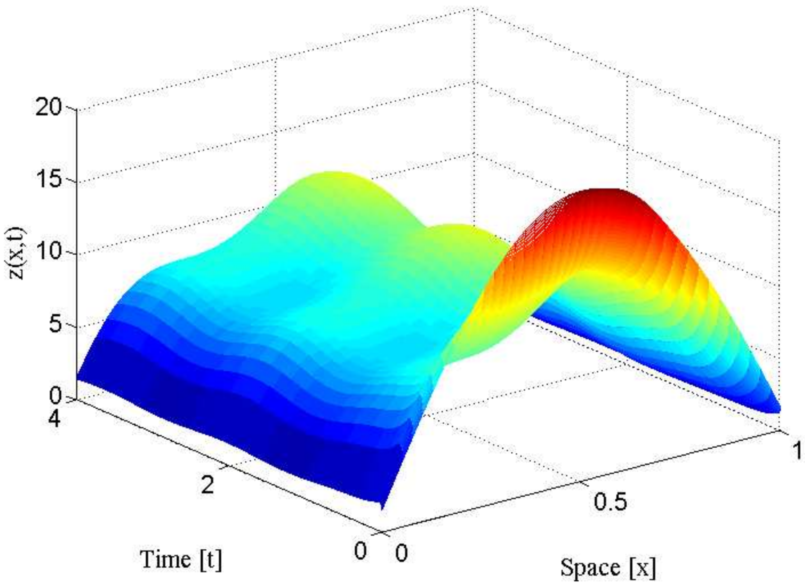

Figure 5.1: TEST 1: actual state $z(x,t)$, the BCs are not plotted.

The transformation matrices (4.2.1) described in the previous Chapter are

$$U = \begin{bmatrix} -0.737 & 1 \\ 0.081 & 0.059 \end{bmatrix}$$
$$T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -0.333 & 0 & 1 & 0 & 0 \\ -2.356 & 0 & 0 & 1 & 0 \\ -0.200 & 0 & 0 & 0 & 1 \\ 0.076 & -0.003 & 0.082 & 0.120 & 0.021 \end{bmatrix} \tag{5.1.19}$$

and the transformed system is characterized by the following matrices

$$
\begin{aligned}
\bar{A}_{11} &= \begin{bmatrix} -19.739 & 0 & 0 & 0 \\ -0.132 & -41.638 & 4.039 & 0.703 \\ -1.750 & 36.772 & -25.424 & 9.318 \\ -0.237 & 4.994 & 7.270 & -122.104 \end{bmatrix} \\
\bar{A}_{12} &= \begin{bmatrix} 0 \\ -33.510 \\ -444.132 \\ -60.318 \end{bmatrix} \\
\bar{A}_{21} &= \begin{bmatrix} -0.168 & 1.498 & -1.982 & -1.276 \end{bmatrix} \\
\bar{A}_{22} &= -62.507 \\
C_1 &= \begin{bmatrix} -1.400 & -0.417 & -0.040 & -1.716 \end{bmatrix}
\end{aligned} \tag{5.1.20}
$$

It can be checked that $\bar{A}_{11}$ and C_1 is an observable pair which confirms that the conditions A1 and A2 hold, system is strongly observable and hence we can implement the Luenberger observer (4.2.8) with the following L matrix

$$
L = \begin{bmatrix} 1834 \\ -1955 \\ -10639 \\ -836 \end{bmatrix} \tag{5.1.21}
$$

which assign all eigenvalues of the error matrix $[\bar{\Delta}_{11} - LC_1]$ the same value: -80. The variable $\hat{\bar{\mathbf{q}}}_1$ generated by the Luenberger observer is used to implement the disturbance estimator (4.3.1) and the estimator control signal $v(t)$ is obtained by setting the super-twisting gains (4.3.9) as follows: $k_1 = 44, k_2 = 0, k_3 = 10, k_4 = 0$.

Fig. 5.2 shows the spatio-temporal profile of the state estimation error $E(x,t) = z(x,t) - \hat{z}(x,t)$ where

$$
\hat{z}(x,t) = \sum_{n=1}^{5} \hat{q}_n(t)\phi_n(x)
$$

while Fig. 5.3 depicts the corresponding L_2 error norm. Fig. 5.4 show the actual and estimated profile of the disturbance $w(t)$ which confirms the good performance of the suggested estimator. In TEST 2 the observer is built considering $N = 20$ modes. The actual and estimated profiles of the disturbance are shown in Fig. 5.5. The final test (TEST 3) considers homogeneous Neumann BCs (5.1.4) instead of (5.1.5) and an observer implemented with $N = 5$ modes. Fig. 5.6, Fig. 5.7 and Fig. 5.8 show the corresponding results.

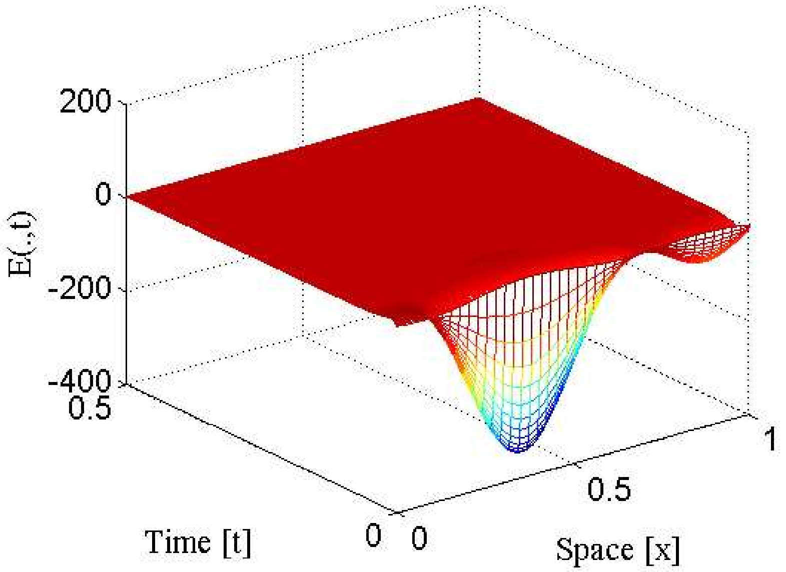

Figure 5.2: Observation error $E(x,t)$ for N=5 (TEST 1).

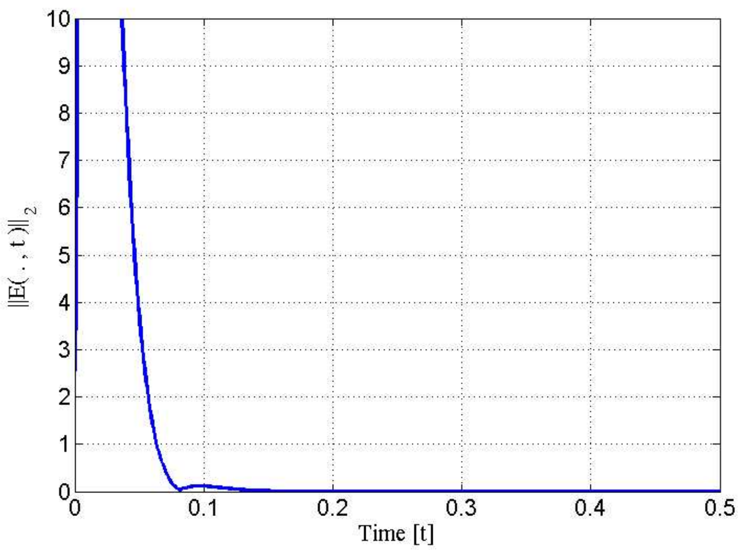

Figure 5.3: The L_2 norm of the observation error $||E(\cdot,t)||_2$ for N=5 (TEST 1).

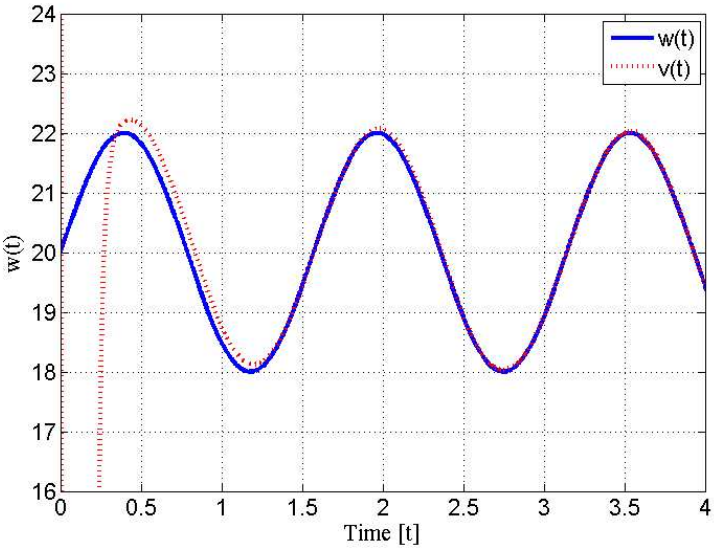

Figure 5.4: Unknown disturbance reconstruction for N=5 (TEST 1).

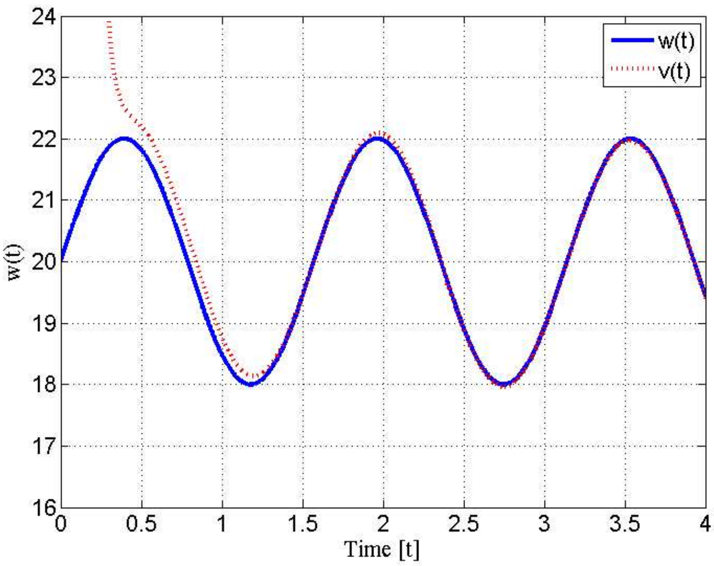

Figure 5.5: Unknown disturbance reconstruction for N=20 (TEST 2).

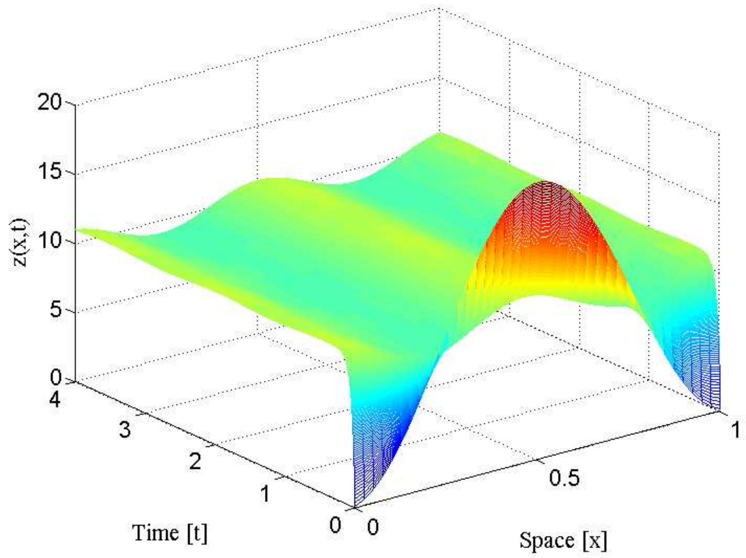

Figure 5.6: Actual state in the TEST 3 (Neumann BCs).

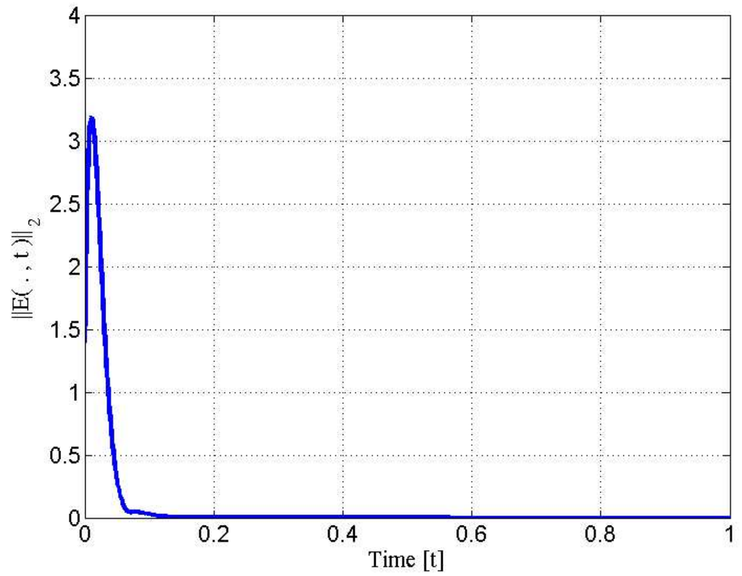

Figure 5.7: The L_2 norm of the observation error $||E(\cdot, t)||_2$ in TEST 3.

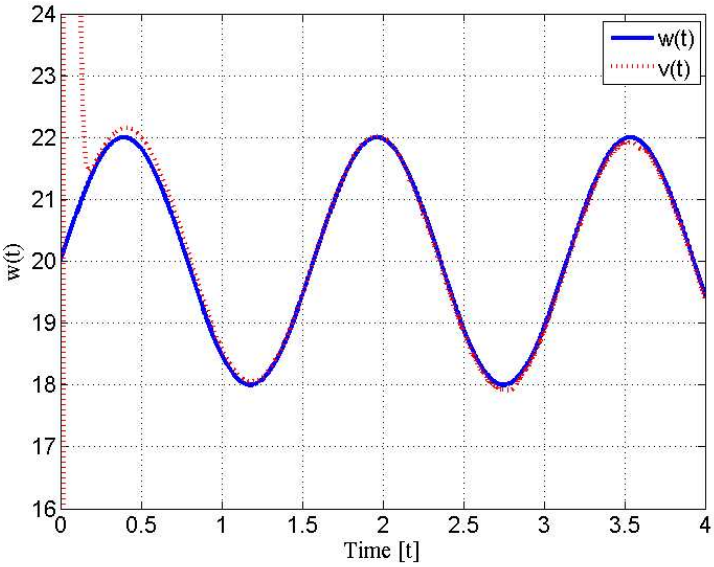

Figure 5.8: Unknown disturbance reconstruction in TEST 3.

By means of a simulation example, it is checked that both the modal approximations of the diffusion equation solution fulfills the property of strong observability when two point-wise measurements are located in the solution spatial domain. The property of Strong observability depend on the number of measurements and their location on the spatial domain, but computational problems for high order system ($N > 10$) make the Strong Observability difficult to check.

5.2 Robust actuator FDI for thermal treatment processes

This section illustrates an approach to actuator fault detection in industrial furnaces P.R.O.D.I. (2011). The method is based on unknown input observation and sliding-mode techniques described in the previous chapters.

Faults occurring in the heaters of a furnace are detected and isolated by means of a scheme that combines a linear unknown-input observer and a nonlinear, sliding mode based, disturbance estimator, along with a simple, static threshold based, residual evaluation logic. The implementation tests of the model-based FDI observer have considered a specific industrial furnace manufactured by BOSIO (”UNIOR” furnace) with near $50kW$ of nominal power, see Figure 5.9.

Figure 5.9: UNIOR furnace.

Particularly, the presented activities started from a high-fidelity furnace built by the BOSIO research group using a commercial software package ($Comsol^{TM}$) that provides user-friendly graphical model-building functionalities and accurate 1D, 2D and 3D PDE solvers. Comsol has the important feature that it can automatically export in the Matlab environment the matrices of a LTI finite dimensional approximation of the underlying PDE. Additionally, Comsol features an additional useful functionality that certain simulation parameters (e.g. the boundary conditions) can be assigned the role of ”external inputs”, which can be generated within a Simulink model and sent in real time, during the simulation run, to Comsol, that will automatically adjust the current value of the corresponding variable(s). Again, the matrices of the LTI finite dimensional (modal) approximation having such ”external inputs” as input variables and arbitrary user-selectable outputs (e.g., the temperature at given points (x, y, z) of interest) are automatically exported in Matlab-Simulink environment. This additional important functionality is exploited to develop a fully automated design procedure of the observer, with the model parameters being generated automatically by Comsol. The approach is detailed in the next Sections.

5.2.1 Comsol furnace model

The next Figure 5.10 shows a particular of the ”graphical” aspect of the Comsol simulation model. According to the specifications of the UNIOR furnace, a

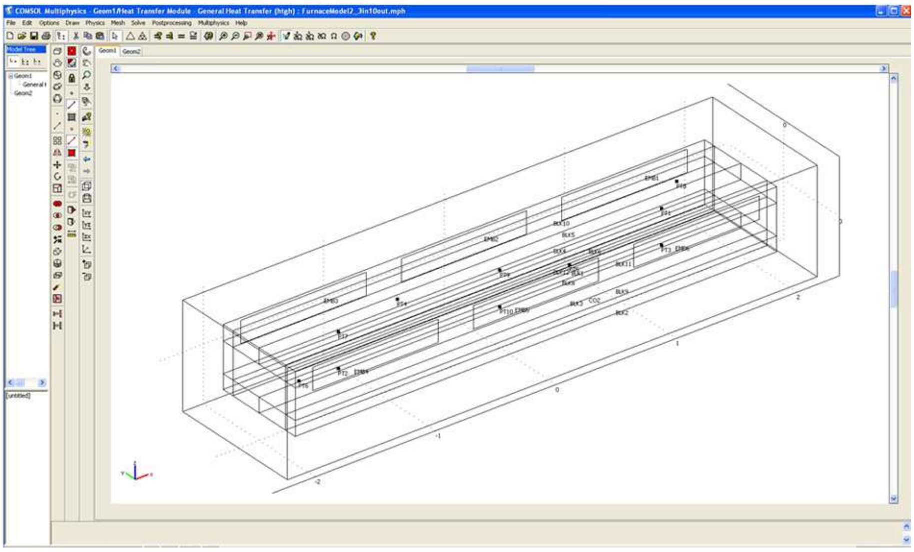

Figure 5.10: Drawing of the considered domain in Comsol.

domain with rectangular shape is considered, with an inner chamber of the same shape and six heaters (three each side). The furnace is $4m$ long. A belt is located in the interior of the furnace, by means of which the parts subject to the thermal treatment are transported inside the furnace with a belt velocity between $1m/h$ and $4m/h$. The effect of those objects, and their motion, on the overall temperature field is negligible as far as severe faults in the heating system are wanted to be detected and no accurate evaluation of the temperature field is required. The main phenomenon of thermal diffusion is modeled, with appropriate boundary conditions of Neumann type. In order to keep relatively low the order of the generated finite-dimensional models, while preserving their accuracy, more complex phenomena like convection and turbulence have been not modeled. As a matter of fact it is the task of this activity to develop a scheme for detecting faults in the heating system, which justify neglecting those complex effects that contribute only marginally to the overall ”average” temperature behavior inside the furnace. The next Figure 5.11 reports the steady state temperature profile corresponding to a constant heating power of about $8kW$ per heater.

Figure 5.11 shows a maximum temperature inside the furnace over $500K$ with such a ”constant-heating power” configuration. In real furnaces, the heating power is not constant, while it is adjusted by proper feedback temperature loops that use three thermocouples suitably located under the belt. To model this fact within Comsol, as previously said, the heating power released to the furnace by the three heater pairs is assigned to be an external input variables, that can be adjusted by

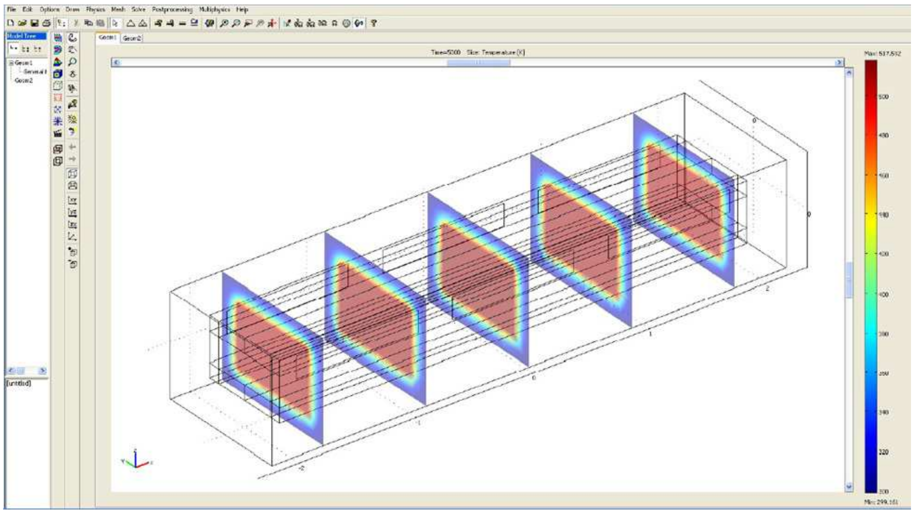

Figure 5.11: Steady-state temperature field at prescribed sections with constant heating.

feedback as time goes on. According to industrial practice the three pairs of opposite heaters (Right heaters: R1/R2; Central heaters: C1/C2; Left heaters: L1/L2, see Figure 5.10 are driven pair-wise, by three command signals, one for the right heaters R1/R2, one for the central heaters C1/C2, and one for the left heaters L1/L2. In other words, heaters R1 and R2 will be commanded by a unique reference command signal, and the same for the pairs C1/C2 and L1/L2. Overall, six temperature measurements at prescribed ”probing points” have been assigned as output variables of the Comsol model. Those six points, along with some additional ones (that are NOT used by the suggested FDI observer) have been placed in the model graphical representation (see Figure 5.10) In the location of those above-mentioned six ”probing points”, thermocouples shall be inserted in the real furnace to ”feed” the FDI algorithm during its real-time execution. Three of the selected probing points, out of the six, correspond to the locations of the thermocouples already installed in the real furnace and used for closing the temperature loops. Those sensors are thus already available in the current design of the furnace. Three additional sensors need then to be inserted in order to make the present approach applicable in the real furnaces, which are located in the upper region of the furnace. This procedure is technically easy and cheap and does not substantially impact the cost and the reliability of the furnace. On the other hand the fault detection capabilities of the scheme that is going to be illustrated will enhance the safety of operation significantly. Faults of the heating system are going to be detected and insulated.

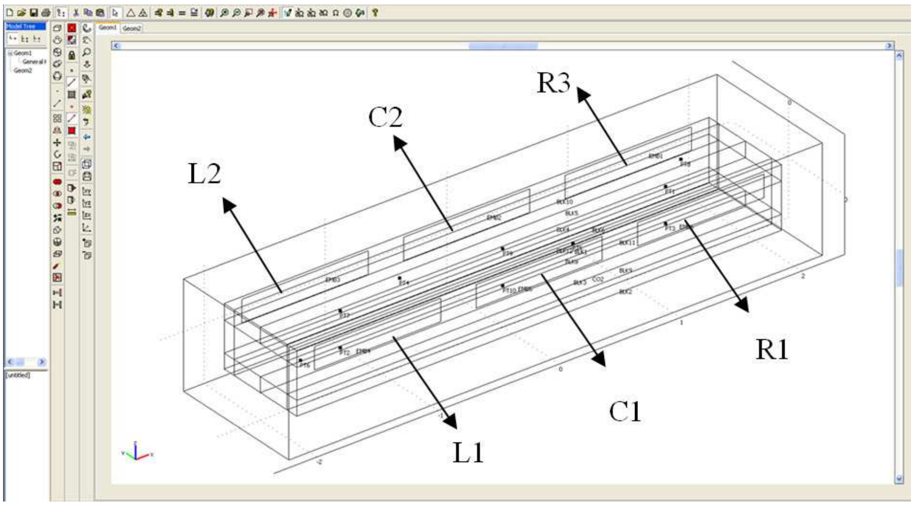

Figure 5.12: Drawing of the considered domain in Comsol.

5.2.2 Comsol state-space export

Comsol implements appropriate PDE solvers that, at the very end, lead to a finite dimensional approximating dynamics (derived by modal expansion techniques) which is a MIMO linear and time invariant system of the form

$$\begin{aligned} M\frac{dx}{dt} &= M_A x + M_B u \\ y &= Cx + Du \end{aligned} \tag{5.2.1}$$

It has been set up the Comsol model in such a way that the input vector u contain the three "command signals" for the three L/C/R heater pairs

$$u = [P_L \; P_C \; P_R] \tag{5.2.2}$$

and the output vector y contains the six temperature profiles at the probing points previously mentioned.

$$y = [T_{DL} \; T_{DC} \; T_{DR} \; T_{UL} \; T_{UC} \; T_{UR}] \tag{5.2.3}$$

with the subscript DL denotes the measurement in the Down-Left probing point, subscript UC denotes the measurement in the Upper-Central probing point, etc. As previously mentioned, the matrices of model (5.2.1) can be exported in Matlab by means of an automatic tool available in Comsol. Next Figures 6 show the two submenus of the "Export State Space Model" configuration window.

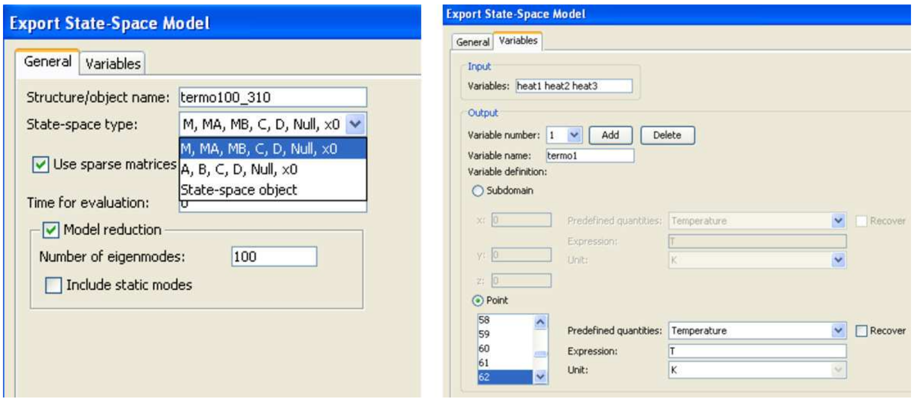

Figure 5.13: Submenus of the "Export State Space Model" configuration window and their setting.

Two models with different dimension have been generated and exported in Matlab. Note that the number of eigenmodes (that sets the order of the resulting LTI model (5.2.1)) can be selected in the corresponding "General" submenu shown in the Figure 5.13-left. The first model has been generated of order 100 (i.e., with 100 eigenmodes, see the figure 5.13-left). It will be used in Matlab as the "high accuracy model" devoted to generates the measurement signals processed by the FDI observer. Around this high accuracy model, the three temperature control loops are closed that generate the (command values for the) input vector u (the three heating powers of the left-L, central-C and right-R heating stages). Figure 5.14 report an overview of the Simulink model implementing the high-accuracy model of the furnace, with the corresponding Temperature Loops, and the FDI observer.

A closer look at the temperature loops (see Figure 5.15) shows that the same set-point, $280C$, is used for the Left (L), Central (C) and Right (R) temperature loops.

The considered controllers are simple saturated PIs.

Inside the FDI observer, the sliding-mode based UIO has been implemented by using, as the plant mathematical model, a reduced-order model with 20 eigenmodes, only. The mathematical detail of the FDI observer is illustrated in the Chapter 4.

5.2.3 Fault detection in the furnace heaters

The considered faults are:

- FAULT 1: In the left heaters, an 80% loss of the effectiveness occur at $t = 4000$

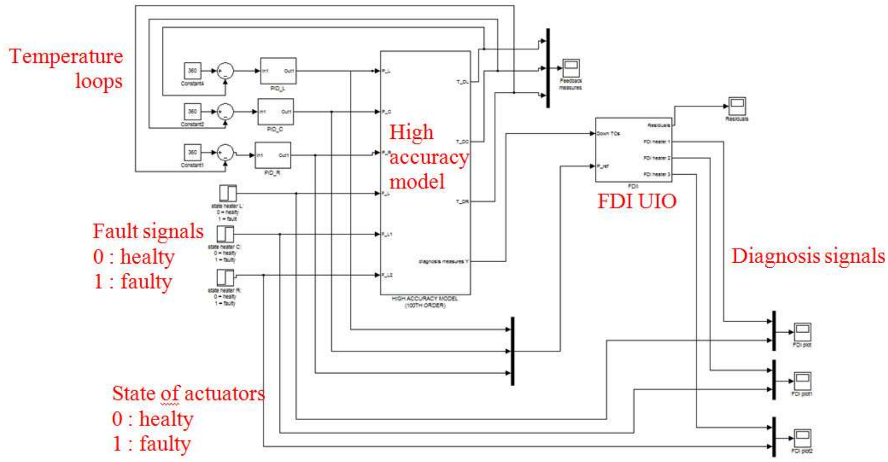

Figure 5.14: Submenus of the ”Export State Space Model” configuration window and their setting

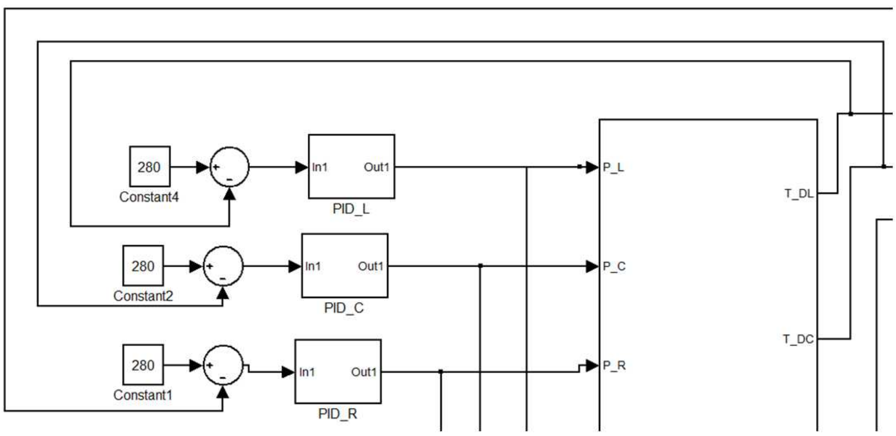

Figure 5.15: Detail of the Temperature Loops

- FAULT 2: In the central heaters, starting from the moment $t = 5000$, they begin to deliver twice the set-point power value commanded by the associate PI controller.

Basically, using the suggested FDI-UIO (4.7)-(4.11) in section 4 the actual input vector u(t) of the original plant is reconstructed, by means of the output y(t) measurement only, in accordance with (4.12). The FDI/UIO has been discretized (by fixed step Euler method) by using a sampling period of 0.01 seconds. If the reconstruction (4.12) is accurate enough, the difference between the command input vector values $u = [P_L P_C P_R]$ (output by the three temperature loops) and their actual, reconstructed, values is, then, a valid residual for FDI purposes concerning the status of the heaters. It is worth to stress that the dynamics of the reduced-order model are not the same as those of the original system, hence an error in the reconstruction of u will occur.

Some Simulation results are illustrated. The proportional and integral gain of the PIs are set to 400 and 0.02, respectively. The upper saturation of the controller output is set to $22kW$, the maximal available heating power at every heater. Lower saturation value is set to $1kW$. The three regulators have all the same parameters, and the reference temperature is $280C$ for all the loops. The next Figure 5.16 shows the temperature time history at the three points under the belt that are used to close the corresponding temperature loops. It can be seen a slight performance deterioration after that FAULT 1 occurs at $t = 4000$, and FAULT 2 then occurs at $t = 5000$. The robustness properties of the feedback controller partially compensate for the actuator faults. Correctly, FAULT 1 makes the inspected temperatures to decrease, while FAULT 2 makes them to grow.

We now analyze the residual signals associated to FAULT 1 and FAULT 2. The figure 5.17 reports the corresponding profiles along with the static threshold used for FDI. The same threshold value (9000) proved to be effective for both Faults. The residuals are sensitive to both faults, but their correct isolation is clearly feasible by using the suggested threshold value. The two Figures 5.18 show the actual and reconstructed status (0: healty/1:faulty) of the left heaters.

It is seen in Figure 5.18-down that FAULT 1 is detected after about 25 seconds from its occurrence. Figure 5.19 shows the corresponding plots for the detection of the FAULT 2. The transient is much faster than before (about 5 seconds).

Overall, the suggested method for actuator FDI has shown satisfactory performance and good fault detection capabilities.

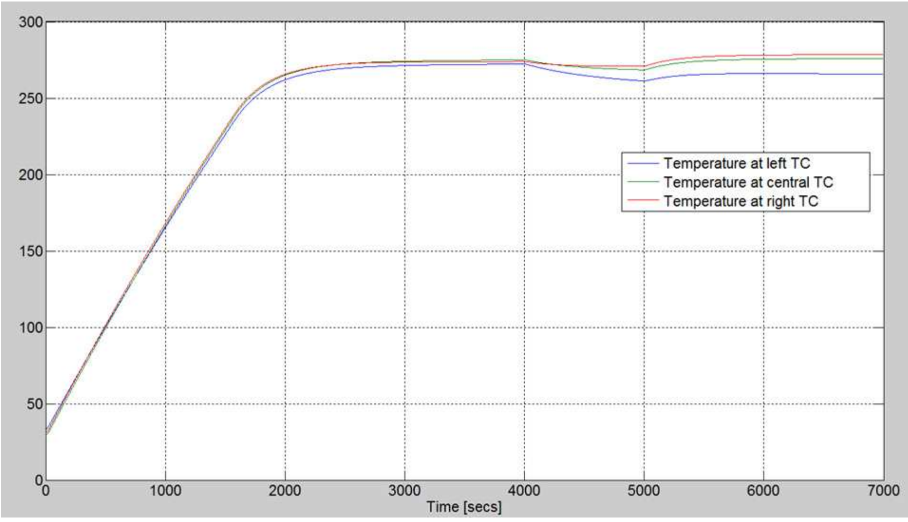

Figure 5.16: Temperature in the three thermocouples used in the feedback loops.

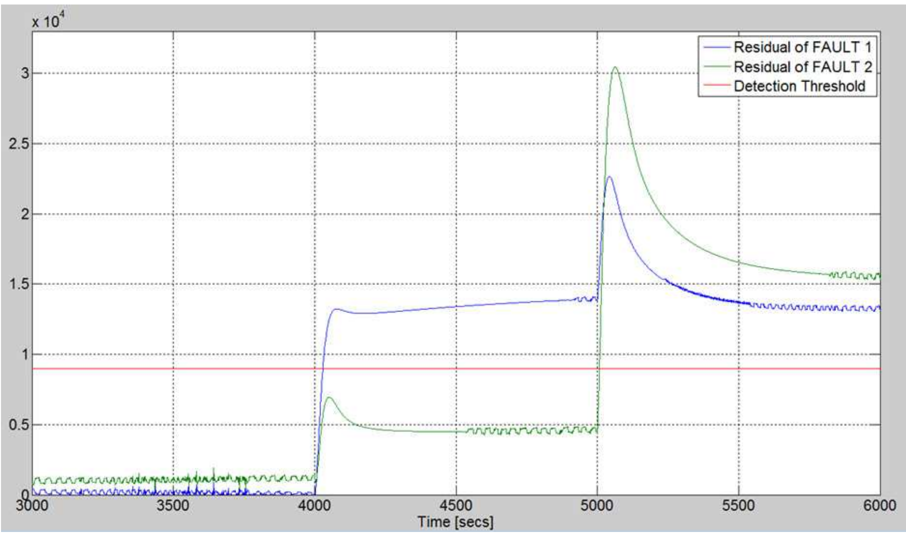

Figure 5.17: The residual signals and the detection threshold.

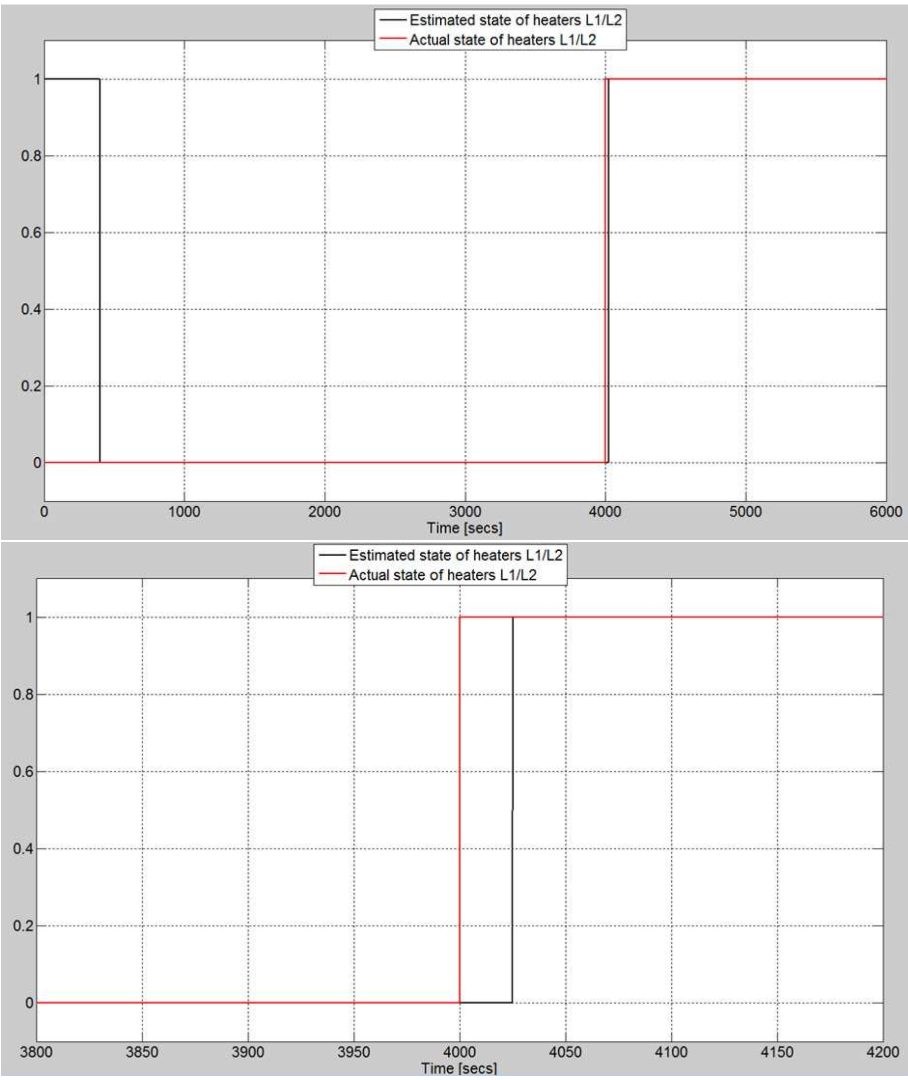

Figure 5.18: actual and estimated state of the left heaters (0: Healty 1:Faulty). Upper plot: long term behavior. Down plot: zoom across the time of FAULT 1 occurrence.

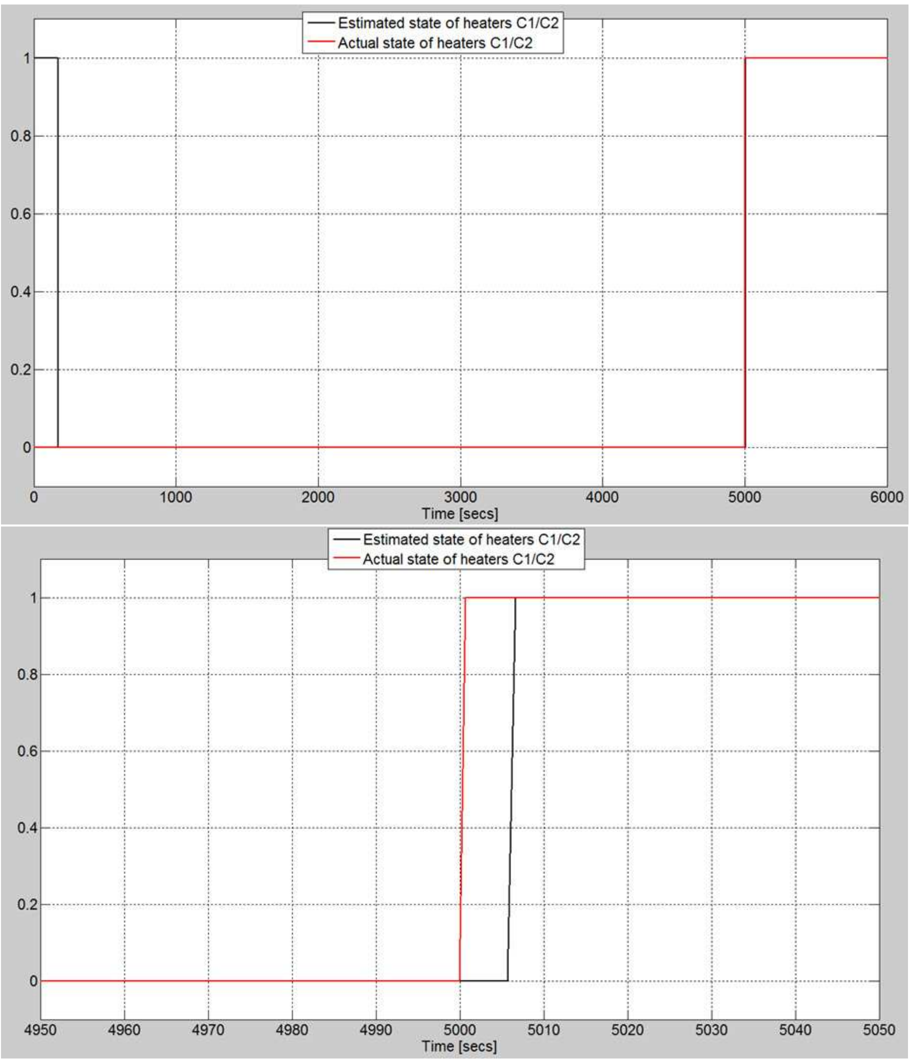

Figure 5.19: The actual and estimated state of the centrl heaters (0: Healty 1:Faulty). Upper plot: long term behavior. Down plot: zoom across the time of FAULT 1 occurrence.

Chapter 6

Distributed Sliding Mode Control design

Topics in this Chapter have been published by the author in Orlov *et al.* (2012a) and Orlov *et al.* (2011b). In this Chapter we consider generalized uncertain forms of the heat and wave equations, under the effect of an external smooth disturbance. In some recent authors publications (see Orlov *et al.* (2010); Pisano *et al.* (2011a); Orlov *et al.* (2011a)) two finite dimensional robust control algorithms, namely, the "Super-Twisting" and "Twisting" second-order sliding-mode (2-SM) controllers (see Fridman and Levant (1996); Levant (1993)) for details on these controllers) have been generalized to the infinite-dimensional setting and applied for controlling heat and wave processes, respectively. The mentioned 2-SM controllers are of special interest because in the finite dimensional setting they significantly improve the performance of sliding-mode control systems, in terms of accuracy and chattering avoidance, as compared to the standard "first-order" sliding mode control techniques (see Bartolini *et al.* (2002)).

The discontinuous control synthesis in the infinite-dimensional setting is well documented (see Levaggi (2002); Orlov and Utkin (1987); Orlov (2000); Orlov *et al.* (2004); Orlov (2009)) and it is generally shown to retain the main robustness features as those possessed by its finite-dimensional counterpart. Other robust control paradigms have been fruitfully applied in the infinite dimensional setting such as adaptive and model-reference control (see Krstic and Smyshlyaev (2008a); Demetriou *et al.* (2009)), geometric and Lyapunov-based design (see Christofides (2001)), H_∞ and LMI-based design (see Fridman and Orlov (2009)). It should be noted that the latter paradigms are capable of **attenuating** vanishing disturbances only, whereas the former discontinuous control is additionally capable of **rejecting** persistent disturbances with an *a priori* known bound on their L_2 norm.

Here we enlarge the class of controlled dynamics as compared to existing publications (cfr. Orlov *et al.* (2010); Pisano *et al.* (2011a); Orlov *et al.* (2011a)), by considering generalized forms of the heat and wave equations. More precisely, we consider the presence of some additional terms in the plant equation (dispersion and damping terms) and, furthermore, we let all the system parameters (diffusivity and dispersion coefficients, for the heat equation, and the wave velocity, the damping coefficient and the dispersion coefficient, for the wave equation) to be spatially varying and uncertain. We additionally put the constraint that the distributed control input must be a continuous (although possibly non-smooth) function of the space and time variables.

6.1 Supertwisting Synthesis of Reaction-Diffusion Processes

Consider the space- and time-varying scalar field $Q(\xi, t)$ evolving in a Hilbert space $L_2(0,1)$, where $\xi \in [0,1]$ is the monodimensional (1D) space variable and $t \geq 0$ is time. Let it be governed by the following perturbed Reaction-Diffusion Equation with spatially-varying parameters

$$Q_t(\xi,t) = [\theta_1(\xi)Q_\xi(\xi,t)]_\xi + \theta_2(\xi)Q(\xi,t) + u(\xi,t) + \psi(\xi,t), \qquad (6.1.1)$$

where $\theta_1(\cdot) \in C^1(0,1)$ is a positive-definite spatially-varying parameter called *thermal conductivity* (or, more generally, *diffusivity*), $\theta_2(\cdot) \in C(0,1)$ is another spatially-varying parameter called *dispersion* (or *reaction constant*), $u(\xi,t)$ is the modifiable source term (the distributed control input), and $\psi(\xi,t)$ represents a distributed **uncertain** disturbance source term. This uncertain term is supposed to satisfy the following conditions

$$\psi(\xi,t) \in L_2(0,1), \qquad \psi_t(\xi,t) \in W^{1,2}(0,1) \qquad (6.1.2)$$

The spatially-varying diffusivity and dispersion coefficients $\theta_1(\xi)$ and $\theta_2(\xi)$ are supposed to be uncertain, too. We consider non-homogeneous mixed boundary conditions (BCs)

$$\begin{aligned} Q(0,t) - \alpha_0 Q_\xi(0,t) &= Q_0(t) \in W^{1,2}(0,\infty), & (6.1.3)\\ Q(1,t) + \alpha_1 Q_\xi(1,t) &= Q_1(t) \in W^{1,2}(0,\infty), & (6.1.4) \end{aligned}$$

with some positive **uncertain** constants α_0, α_1. The initial conditions (ICs)

$$Q(\xi,0) = \omega_0(\xi) \in W^{2,2}(0,1) \qquad (6.1.5)$$

are assumed to meet the same BCs. Since nonhomogeneous BCs are in force, a solution of the above boundary-value problem is defined in the mild sense (see

Curtain and Zwart (1995)) as that of the corresponding integral equation, written in terms of the strongly continuous semigroup, generated by the infinitesimal plant operator.

The control task is to make the scalar field $Q(\xi,t)$ to track a given reference $Q^r(\xi,t) \in W^{2,2}(0,1)$ which should be selected in accordance with the BCs (7.2.4) and which also satisfies the following condition

$$Q^r_t \in W^{3,2}(0,1). \tag{6.1.6}$$

6.1.1 Robust Control of the Reaction-Diffusion Process

Consider the deviation variable

$$x(\xi,t) = Q(\xi,t) - Q^r(\xi,t) \tag{6.1.7}$$

whose L_2 norm will be driven to zero by the designed feedback control. The dynamics of the error variable (6.1.7) are easily derived as

$$x_t(\xi,t) = [\theta_1(\xi)x_\xi(\xi,t)]_\xi + \theta_2(\xi)x(\xi,t) + u(\xi,t) - Q^r_t(\xi,t) + \eta(\xi,t), \tag{6.1.8}$$

with the "augmented" disturbance

$$\eta(\xi,t) = [\theta_1(\xi)Q^r_\xi(\xi,t)]_\xi + \theta_2(\xi)Q^r(\xi,t) + \psi(\xi,t), \tag{6.1.9}$$

and the next ICs and homogeneous mixed BCs

$$x(\xi,0) = \omega_0(\xi) - Q^r(\xi,0) \in W^{2,2}(0,1) \tag{6.1.10}$$

$$x(0,t) - \alpha_0 x_\xi(0,t) = x(1,t) + \alpha_1 x_\xi(1,t) = 0. \tag{6.1.11}$$

Assume what follows:

ASSUMPTION 6.1.1 There exist *a priori* known constants Θ_{1m}, Θ_{1M} and Θ_{2M} such that

$$0 < \Theta_{1m} \le \theta_1(\xi) \le \Theta_{1M}, \qquad |\theta_2(\xi)| \le \Theta_{2M} \qquad for \quad all \quad \xi \in [0,1]. \tag{6.1.12}$$

ASSUMPTION 6.1.2 There exist *a priori* known constants H_0, ..., H_3, Ψ_0 and Ψ_1 such that the following inequalities hold for all $t \ge 0$

$$\|\theta_2(\cdot)Q^r_t(\cdot,t)\|_2 \le H_0, \qquad \|[\theta_2(\xi)Q^r_t(\cdot,t)]_\xi\|_2 \le H_1, \tag{6.1.13}$$

$$\|[\theta_1(\xi)Q^r_\xi(\cdot,t)]_{\xi\,t}\|_2 \le H_2, \qquad \|[\theta_1(\xi)Q^r_\xi(\cdot,t)]_{\xi\xi\,t}\|_2 \le H_3, \tag{6.1.14}$$

$$\|\psi_t(\cdot,t)\|_2 \le \Psi_0, \qquad \|\psi_{t\xi}(\cdot,t)\|_2 \le \Psi_1 \tag{6.1.15}$$

By the Assumption 6.1.2, it follows that the L_2 norm of the augmented disturbance time derivative $\eta_t(\xi,t)$, and that of its spatial derivative, fulfill the next conditions

$$\|\eta_t(\cdot,t)\|_2 \le M, \quad \|\eta_{t\xi}(\cdot,t)\|_2 \le M_\xi, \qquad \forall t \ge 0 \tag{6.1.16}$$

with

$$M = H_2 + H_0 + \Psi_0, \qquad M_\xi = H_3 + H_1 + \Psi_1 \tag{6.1.17}$$

The class of admissible "augmented" disturbances is further specified by the following additional restriction, being introduced in Pisano *et al.* (2011a):

ASSUMPTION 6.1.3 There exist *a priori* known constant M_x such that the following restriction holds uniformly beyond the origin $\|x(\cdot,t)\|_2 = 0$ in the state space $L_2(0,1)$:

$$|\eta_t(\xi,t)| \le M_x \frac{|x(\xi,t)|}{\|x(\cdot,t)\|_2}, \qquad \forall t \ge 0, \forall \xi \in [0,1] \tag{6.1.18}$$

It is worth noticing that according to the Assumption 6.1.3 an admissible disturbance has a time derivative which is not necessarily vanishing as $\|x(\cdot,t)\|_2 \to 0$ because the norm of the right-hand side of the disturbance restriction (6.1.18) remains unit according to relation $\left\| \frac{|x(\cdot,t)|}{\|x(\cdot,t)\|_2} \right\|_2 = 1$. Particularly, with $M_x \le M$ a finite-dimensional counterpart of (6.1.18) would not impose any further restrictions on admissible disturbances in addition to the first relation of (6.1.16).

It should also be noted that the assumptions on the ICs and BCs, made above, allow us to deal with strong, sufficiently smooth solutions of the uncertain error dynamics (6.1.8)-(6.1.11) in the open-loop when no control input is applied.

In order to stabilize the error dynamics it is proposed a dynamical distributed controller defined as follows

$$\begin{aligned} u(\xi,t) &= Q_t^r(\xi,t) - \lambda_1 \sqrt{|x(\xi,t)|}\ \mathtt{sign}(x(\xi,t)) - \lambda_2 x(\xi,t) + v(\xi,t) \\ v_t(\xi,t) &= -W_1 \frac{x(\xi,t)}{\|x(\cdot,t)\|_2} - W_2 x(\xi,t), \qquad v(\xi,0) = 0 \end{aligned} \tag{6.1.19}$$

which can be seen as a distributed version of the finite-dimensional "Super-Twisting" second-order sliding-mode controller (see Fridman and Levant (1996); Levant (1993)) complemented by a feed-forward term $Q_t^r(\xi,t)$ and by the two additional proportional and integral linear terms $-\lambda_2 x(\xi,t)$ and $-W_2 x(\xi,t)$. For ease of reference, the combined Distributed Super-Twisting/PI controller (6.1.19) will be abbreviated as DSTPI.

The non-smooth nature of the DSTPI controller (6.1.19), that undergoes discontinuities on the manifold $x = 0$ due to the discontinous term $\frac{x(\xi,t)}{\|x(\cdot,t)\|_2}$, requires appropriate analysis about the meaning of the corresponding solutions for the resulting discontinuous feedback system. The precise meaning of the solutions of (6.1.8), (6.1.10), (6.1.11) with the piece-wise continuously differentiable control input (6.1.19) can be defined in a generalized sense (see (Orlov (2009))) as a limiting result obtained through a certain regularization procedure, similar to that proposed for finite-dimensional systems (see Filippov (1988); Utkin (1992)). According to this procedure, the strong solutions of the boundary-value problem are only considered whenever they are beyond the discontinuity manifold $x = 0$ whereas in a vicinity of these manifolds the original system is replaced by a related system, which takes into account all possible imperfections (e.g., delay, hysteresis, saturation, etc.) in the new input function $u^\delta(x, \xi, t)$, for which there exists a strong solution $x^\delta(\xi, t)$ of the corresponding boundary-value problem with the smoothed input $u^\delta(x, \xi, t)$. In particular, a relevant approximation occurs when the discontinous term $U(x) = \frac{x(\xi,t)}{\|x(\cdot,t)\|_2}$ is substituted by the smooth approximation $U^\delta(x) = \frac{x(\xi,t)}{\delta+\|x(\cdot,t)\|_2}$. A generalized solution of the system in question is then obtained through the limiting procedure by diminishing δ to zero, thereby making the characteristics of the new system approach those of the original one. As in the finite-dimensional case, a motion along the discontinuity manifold is referred to as a "**sliding mode**".

REMARK 6.1.1 The existence of generalized solutions, thus defined, has been established within the abstract framework of Hilbert space-valued dynamic systems (cf., e.g., (Orlov , 2009, Theorem 2.4)) whereas the uniqueness and well-posedness appear to follow from the fact that in the system in question, no sliding mode occurs but in the origin $x = 0$. While being well-recognized for second order sliding mode control algorithms if confined to the finite-dimensional setting, this fact, however, remains beyond the scope of the present investigation.

The performance of the closed-loop system is analyzed in the next theorem.

THEOREM 6.1.1 Consider the perturbed diffusion/dispersion equation (6.1.1) along with the boundary conditions (6.1.4) and with the system parameters, reference trajectory and uncertain disturbance satisfying the Assumptions 6.1.1-6.1.3. Then, the distributed control strategy (6.1.19) with the parameters λ_1, λ_2, W_1 and W_2 selected according to

$$\begin{aligned} &\lambda_2 \geq \Theta_{2M}, \quad W_1 \geq \texttt{max}\left\{M + \frac{\Theta_{1M} M_\xi}{2(\lambda_2 - \Theta_{2M})}, \frac{1}{2}\frac{\Theta_{1M}}{\Theta_{1m}} M_\xi, 2M_x\right\}, \\ &\lambda_1 \geq \texttt{max}\left\{2M, \frac{2M_x}{W_1}\right\}, \quad W_2 \geq 0, \end{aligned} \tag{6.1.20}$$

guarantees that the L_2-norm $\|x(\cdot, t)\|_2$ of the tracking error tends to zero as t tends to infinity.

Proof of Theorem 6.1.1. Let us define the auxiliary variable

$$\delta(\xi,t) = v(\xi,t) + \eta(\xi,t) \tag{6.1.21}$$

System (6.1.8) with the control law (6.1.19) yields the following closed-loop dynamics in the new $x-\delta$ coordinates

$$\begin{aligned} x_t(\xi,t) =& [\theta_1(\xi)x_\xi(\xi,t)]_\xi - \lambda_1\sqrt{|x(\xi,t)|}\ \mathtt{sign}(x(\xi,t)) \\ &- (\lambda_2 - \theta_2(\xi))x(\xi,t) + \delta(\xi,t), \end{aligned} \tag{6.1.22}$$

$$\delta_t(\xi,t) = -W_1\frac{x(\xi,t)}{\|x(\cdot,t)\|_2} - W_2 x(\xi,t) + \eta_t(\xi,t). \tag{6.1.23}$$

In order to simplify the notation, the dependence of the system coordinates from the space and time variables (ξ,t) is omitted from this point on. Consider the following Lyapunov functional

$$V_1(t) = 2W_1\|x\|_2 + W_2\|x\|_2^2 + \frac{1}{2}\|\delta\|_2^2 + \frac{1}{2}\|s\|_2^2 \tag{6.1.24}$$

inspired from the finite-dimensional treatment (see Moreno and Osorio (2008)), where

$$s = x_t = [\theta_1(\xi)x_\xi]_\xi - \lambda_1\sqrt{|x|}\ \mathtt{sign}(x) - [\lambda_2 - \theta_2(\xi)]x + \delta. \tag{6.1.25}$$

The time derivative of $V_1(t)$ is given by

$$\begin{aligned} \dot{V}_1(t) =& \frac{2W_1}{\|x\|_2}\int_0^1 xs d\xi + 2W_2\int_0^1 xs d\xi \\ &+ \int_0^1 \delta\delta_t d\xi + \int_0^1 s s_t d\xi \end{aligned} \tag{6.1.26}$$

Let us evaluate the time derivative of the auxiliary signal s along the strong solutions of (6.1.22)-(6.1.23)

$$\begin{aligned} s_t =& [\theta_1(\xi)s_\xi]_\xi - \frac{1}{2}\lambda_1\frac{s}{\sqrt{|x|}} - [\lambda_2 - \theta_2(\xi)]s \\ &- W_1\frac{x}{\|x\|_2} - W_2 x + \eta_t \end{aligned} \tag{6.1.27}$$

Substituting (6.1.23) and (6.1.27) into (6.1.26) and rearranging it yields

$$\begin{aligned} \dot{V}_1(t) \ =& \ \frac{2W_1}{\|x\|_2}\int_0^1 xs d\xi + 2W_2\int_0^1 xs d\xi - \frac{W_1}{\|x\|_2}\int_0^1 \delta x d\xi \\ &- \ W_2\int_0^1 \delta x d\xi + \int_0^1 \delta\eta_t d\xi + \int_0^1 s[\theta_1(\xi)s_\xi]_\xi d\xi - \frac{1}{2}\lambda_1\int_0^1 \frac{s^2 d\xi}{\sqrt{|x|}} \\ &- \ \int_0^1 [\lambda_2 - \theta_2(\xi)]s^2 d\xi - \frac{W_1}{\|x\|_2}\int_0^1 xs d\xi - W_2\int_0^1 xs d\xi + \int_0^1 s\eta_t d\xi \end{aligned} \tag{6.1.28}$$

which can be manipulated as follows by virtue of Assumption 6.1.1

$$\begin{aligned}\dot{V}_1(t) \;\; &\leq \;\; -\frac{W_1}{\|x\|_2}\int_0^1 x(\delta-s)d\xi - W_2\int_0^1 x(\delta-s)d\xi + \int_0^1 s[\theta_1(\xi)s_\xi]_\xi d\xi - \\ &- \;\; \frac{1}{2}\lambda_1\int_0^1 \frac{s^2 d\xi}{\sqrt{|x|}} - [\lambda_2-\Theta_{2M}]\int_0^1 s^2 d\xi + \int_0^1 (\delta+s)\eta_t d\xi \qquad (6.1.29)\end{aligned}$$

By (6.1.25), one has

$$\begin{aligned}\delta - s &= \lambda_1\sqrt{|x|}\,\mathtt{sign}(x) + [\lambda_2-\theta_2(\xi)]x - [\theta_1(\xi)x_\xi]_\xi \qquad (6.1.30)\\ \delta + s &= 2s + \lambda_1\sqrt{|x|}\,\mathtt{sign}(x) + [\lambda_2-\theta_2(\xi)]x - [\theta_1(\xi)x_\xi]_\xi. \qquad (6.1.31)\end{aligned}$$

Due to this, and considering once more the Assumption 6.1.1, (6.1.29) can further be manipulated as

$$\begin{aligned}\dot{V}_1(t) \leq &-\frac{W_1\lambda_1}{\|x\|_2}\int_0^1 x\sqrt{|x|}\,\mathtt{sign}(x)d\xi - \frac{W_1[\lambda_2-\Theta_{2M}]}{\|x\|_2}\int_0^1 x^2 d\xi + \int_0^1 s\eta_t d\xi\\ &+ \frac{W_1}{\|x\|_2}\int_0^1 x[\theta_1(\xi)x_\xi]_\xi d\xi - W_2\lambda_1\int_0^1 x\sqrt{|x|}\,\mathtt{sign}(x)d\xi - \frac{1}{2}\lambda_1\int_0^1 \frac{s^2 d\xi}{\sqrt{|x|}}\\ &- W_2[\lambda_2-\Theta_{2M}]\int_0^1 x^2 d\xi + W_2\int_0^1 x[\theta_1(\xi)x_\xi]_\xi d\xi - \int_0^1 [\theta_1(\xi)x_\xi]_\xi \eta_t d\xi\\ &+ \int_0^1 s[\theta_1(\xi)s_\xi]_\xi d\xi + \lambda_1\int_0^1 \sqrt{|x|}\,\mathtt{sign}(x)\eta_t d\xi\\ &- [\lambda_2-\Theta_{2M}]\int_0^1 s^2 d\xi 2 - [\lambda_2-\Theta_{2M}]\int_0^1 x\eta_t d\xi.\end{aligned} \qquad (6.1.32)$$

By taking into account the BCs (6.1.11) and their time derivatives, standard integration by parts yields

$$\begin{aligned}&\int_0^1 x[\theta_1(\xi)x_\xi]_\xi d\xi =\\ &-\int_0^1 \theta_1(\xi)x_\xi^2 d\xi + \theta_1(1)x(1,t)x_\xi(1,t) - \theta_1(0)x(0,t)x_\xi(0,t) \qquad (6.1.33)\\ &\leq -\Theta_{1m}\|x_\xi\|_2^2 - \theta_1(1)\frac{x^2(1,t)}{\alpha_1} - \theta_1(0)\frac{x^2(0,t)}{\alpha_0},\\ &\int_0^1 s[\theta_1(\xi)s_\xi]_\xi d\xi =\\ &-\int_0^1 \theta_1(\xi)s_\xi^2 d\xi + \theta_1(1)s(1,t)s_\xi(1,t) - \theta_1(0)s(0,t)s_\xi(0,t) \qquad (6.1.34)\\ &\leq -\Theta_{1m}\|s_\xi\|_2^2 - \theta_1(1)\frac{s^2(1,t)}{\alpha_1} - \theta_1(0)\frac{s^2(0,t)}{\alpha_0},\end{aligned}$$

$$\begin{aligned}
&\int_0^1 [\theta_1(\xi)x_\xi]_\xi \eta_t d\xi = \\
&-\int_0^1 \theta_1(\xi)x_\xi \eta_{t\xi} d\xi + \theta_1(1)\eta_t(1,t)x_\xi(1,t) - \theta_1(0)\eta_t(0,t)x_\xi(0,t) \\
&= -\int_0^1 \theta_1(\xi)x_\xi \eta_{t\xi} d\xi - \theta_1(1)\eta_t(1,t)\frac{x(1,t)}{\alpha_1} - \theta_1(0)\eta_t(0,t)\frac{x(0,t)}{\alpha_0}.
\end{aligned} \tag{6.1.35}$$

Additional straightforward manipulations of (6.1.32) taking into account (6.1.33) and (6.1.34) yield

$$\begin{aligned}
\dot{V}_1(t) \leq &-W_1[\lambda_2 - \Theta_{2M}]\|x\|_2 - W_2[\lambda_2 - \Theta_{2M}]\|x\|_2^2 - W_1\Theta_{1m}\frac{\|x_\xi\|_2^2}{\|x\|_2} \\
&- \frac{W_1}{\|x\|_2}\theta_1(1)\frac{x^2(1,t)}{\alpha_1} - \frac{W_1}{\|x\|_2}\theta_1(0)\frac{x^2(0,t)}{\alpha_0} - W_2\Theta_{1m}\|x_\xi\|_2^2 \\
&- W_2\theta_1(1)\frac{x^2(1,t)}{\alpha_1} - W_2\theta_1(0)\frac{x^2(0,t)}{\alpha_0} - [\lambda_2 - \Theta_{2M}]\|s\|_2^2 \\
&- \Theta_{1m}\|s_\xi\|_2^2 - \theta_1(1)\frac{s^2(1,t)}{\alpha_1} - \theta_1(0)\frac{s^2(0,t)}{\alpha_0} + 2\int_0^1 s\eta_t d\xi \\
&- W_2\lambda_1\int_0^1 |x|^{3/2} d\xi - \frac{1}{2}\lambda_1\int_0^1 \frac{s^2 d\xi}{\sqrt{|x|}} - \frac{W_1\lambda_1}{\|x\|_2}\int_0^1 \sqrt{|x|}|x| d\xi \\
&+ \lambda_1\int_0^1 \sqrt{|x|}\ \mathtt{sign}(x)\eta_t d\xi + [\lambda_2 - \Theta_{2M}]\int_0^1 x\eta_t d\xi \\
&+ \int_0^1 \theta_1(\xi)x_\xi\eta_{t\xi} d\xi + \theta_1(1)\eta_t(1,t)\frac{x(1,t)}{\alpha_1} + \theta_1(0)\eta_t(0,t)\frac{x(0,t)}{\alpha_0}.
\end{aligned} \tag{6.1.36}$$

It is worth noting that by virtue of the tuning inequality $\lambda_2 > \Theta_{2M}$ in (6.1.20) all terms appearing in the right hand side of (6.1.36) are negative definite except those depending on the augmented disturbance term η_t and its spatial derivative. Some estimations involving those sign-indefinite terms are now derived by simple application of the Cauchy-Schwartz and Young's inequalities and by considering the Assumptions 6.1.1 and 6.1.2, the BCs (6.1.11) and the derived conditions (6.1.16)-(6.1.17):

$$\begin{aligned}
2\left|\int_0^1 s\eta_t d\xi\right| &\leq 2\int_0^1 |s||\eta_t| d\xi = 2\int_0^1 \frac{|s|\sqrt{|\eta_t|}\sqrt{|\eta_t|}\sqrt{|x|}}{\sqrt{|x|}} d\xi \\
&\leq \int_0^1 \frac{|\eta_t|s^2 + |\eta_t||x|}{\sqrt{|x|}} d\xi \leq M\int_0^1 \frac{s^2}{\sqrt{|x|}} d\xi + \int_0^1 \eta_t\sqrt{|x|} d\xi,
\end{aligned} \tag{6.1.37}$$

$$\left|\int_0^1 x\eta_t d\xi\right| \leq \left[\int_0^1 x^2 d\xi\right]^{1/2} \left[\int_0^1 \eta_t^2 d\xi\right]^{1/2} \leq M\|x\|_2, \tag{6.1.38}$$

$$\begin{aligned}\left|\int_0^1 \theta_1(\xi)x_\xi\eta_{t\xi}d\xi\right| &\leq \Theta_{1M}\int_0^1 |x_\xi||\eta_{t\xi}|d\xi = \Theta_{1M}\int_0^1 \frac{|x_\xi|\sqrt{|\eta_{t\xi}|}\sqrt{|\eta_{t\xi}|}\|x\|_2}{\|x\|_2}d\xi\\ &\leq \frac{1}{2}\Theta_{1M}\int_0^1 \frac{x_\xi^2|\eta_{t\xi}| + |\eta_{t\xi}|\|x\|_2^2}{\|x\|_2}d\xi\\ &\leq \frac{1}{2}\Theta_{1M}M_\xi\frac{\|x_\xi\|_2^2}{\|x\|_2} + \frac{1}{2}\Theta_{1M}M_\xi\|x\|_2.\end{aligned} \tag{6.1.39}$$

Taking into account (6.1.37)-(6.1.39), the right-hand side of (6.1.36) can be estimated as

$$\begin{aligned}\dot{V}_1(t) \leq &-(\lambda_2-\Theta_{2M})\left[W_1 - M - \frac{\Theta_{1M}M_\xi}{2(\lambda_2-\Theta_{2M})}\right]\|x\|_2 - W_2(\lambda_2-\Theta_{2M})\|x\|_2^2\\ &-\left[W_1\Theta_{1m} - \frac{1}{2}\Theta_{1M}M_\xi\right]\frac{\|x_\xi\|_2^2}{\|x\|_2} - W_2\Theta_{1m}\|x_\xi\|_2^2 - (\lambda_2-\Theta_{2M})\|s\|_2^2\\ &-\Theta_{1m}\|s_\xi\|_2^2 - W_2\lambda_1\int_0^1 |x|^{3/2}d\xi - \frac{1}{2}(\lambda_1-2M)\int_0^1\frac{s^2d\xi}{\sqrt{|x|}}\\ &-\int_0^1\sqrt{|x|}\left[\frac{W_1\lambda_1}{2\|x\|_2}|x| - \eta_t\right]d\xi - \lambda_1\int_0^1\sqrt{|x|}\left[\frac{W_1}{2\|x\|_2}|x| - \eta_t\right]d\xi\\ &-\theta_1(1)\frac{|x(1,t)|}{\alpha_1}\left[\frac{W_1}{\|x\|_2}|x(1,t)| - \eta_t(1,t)\right]\\ &-\theta_1(0)\frac{|x(0,t)|}{\alpha_0}\left[\frac{W_1}{\|x\|_2}|x(0,t)| - \eta_t(0,t)\right]\\ &-W_2\theta_1(1)\frac{x^2(1,t)}{\alpha_1} - W_2\theta_1(0)\frac{x^2(0,t)}{\alpha_0} - \theta_1(1)\frac{s^2(1,t)}{\alpha_1} - \theta_1(0)\frac{s^2(0,t)}{\alpha_0}.\end{aligned} \tag{6.1.40}$$

By virtue of Assumption 6.1.3, the next inequalities guarantee that all terms in the right hand side of (6.1.40) are negative definite

$$\lambda_2 > \Theta_{2M}, \qquad W_1 > M + \frac{\Theta_{1M}M_\xi}{2(\lambda_2-\Theta_{2M})}, \qquad W_2 > 0, \qquad W_1 > \frac{1}{2}\frac{\Theta_{1M}}{\Theta_{1m}}M_\xi, \tag{6.1.41}$$

$$\lambda_1 > 2M, \qquad W_1\lambda_1 > 2M_x, \qquad W_1 > 2M_x, \qquad W_1 > M_x \tag{6.1.42}$$

The above inequalities collected together form the tuning conditions (6.1.20). To complete the proof it remains to demonstrate that

$$\|x(\cdot,t)\|_2 \to 0 \text{ as } t\to\infty. \tag{6.1.43}$$

For this purpose, let us integrate the relation

$$\dot{\tilde{V}}(t) \leq -(\lambda_2-\Theta_{2M})\left[W_1 - M - \frac{\Theta_{1M}M_\xi}{2(\lambda_2-\Theta_{2M})}\right]\|x\|_2, \tag{6.1.44}$$

straightforwardly resulting from the negative definiteness of all terms in the right hand side of (6.1.40), to conclude that

$$\int_0^\infty \|x(\cdot,t)\|_2 dt < \infty \tag{6.1.45}$$

The inequality $\dot{V}_1(t) \leq 0$, which is readily concluded from (6.1.40) and (6.1.41)-(6.1.42) in light of the Assumption 6.1.3, guarantees that $V_1(t) \leq V_1(0)$ for any $t \geq 0$. From this, and considering (6.1.24), one can conclude that the L_2 norm of $s = x_t$ fulfills the estimation

$$\|x_t\|_2^2 \leq 2V_1(0), \qquad \forall t \geq 0 \tag{6.1.46}$$

Thus, the integrand $\omega(t) = \|x(\cdot,t)\|_2$ of (6.1.45) possesses a uniformly bounded time derivative

$$\dot{\omega}(t) = \frac{\int_0^1 x x_t d\xi}{\|x\|_2} \leq \|x_t\|_2 \leq \sqrt{2R} \tag{6.1.47}$$

on the semi-infinite time interval $t \in [0,\infty)$, where R is any positive constant such that $R \geq V_1(0)$. Convergence (6.1.43) is then verified by applying the Barbalat lemma (see Khalil (2002)). Since the Lyapunov functional (6.1.24) is radially unbounded the global asymptotic stability of the closed-loop system (6.1.8)-(6.1.11) is thus established in the L_2 space. Theorem 6.1.1 is proved. □

REMARK 6.1.2 If the spatially-varying profiles $\theta_1(\xi), \theta_2(\xi)$ of the system parameters are known, then a trivial modification of the suggested controller can be made in order to ensure the same convergence property (6.1.43) with a time dependent reference $Q_r(\xi,t) \in W^{2,2}$, too. The corresponding modified controller is

$$\begin{aligned} u(\xi,t) &= u_{ff}(\xi,t) - \lambda_1 \sqrt{|x(\xi,t)|}\ \mathtt{sign}(x(\xi,t)) - \lambda_2 x(\xi,t) + v(\xi,t), \\ v_t(\xi,t) &= -W_1 \frac{x(\xi,t)}{\|x(\cdot,t)\|_2} - W_2 x(\xi,t), \qquad v(\xi,0) = 0, \end{aligned} \tag{6.1.48}$$

with the control parameters subject to the same tuning conditions (6.1.20) and the additional feed-forward term

$$u_{ff}(\xi,t) = Q_t^r(\xi,t) - [\theta_1(\xi) Q_\xi^r(\xi)]_\xi - \theta_2(\xi) Q^r(\xi). \tag{6.1.49}$$

The proof can be easily developed by observing that the resulting external disturbance, affecting the corresponding error system, remains time-independent so that the line of reasoning used in the proof of Theorem 6.1.1 is applicable here as well. The detailed proof is thus omitted for brevity.

6.2 Twisting Synthesis of Perturbed Wave Processes

We consider a class of uncertain infinite-dimensional systems whose (y, y_t) solution is defined in the Hilbert space $L_2(0,1) \times L_2(0,1)$ and is governed by the next hyperbolic PDE with spatially varying parameters

$$y_{tt}(\xi,t) = \left[\nu^2(\xi)y_\xi\right]_\xi + \theta_1(\xi)y(\xi,t) + \theta_2(\xi)y_t(\xi,t) + u(\xi,t) + \psi(\xi,t) \quad (6.2.1)$$

where $y \in L_2(0,1)$ and $y_t \in L_2(0,1)$ are the state variables, $\xi \in [0,1]$ is the monodimensional (1D) spatial variable, and $t \geq 0$ is time. The spatially varying coefficients $\nu^2(\cdot) \in C^1(0,1)$ represents the squared value of the wave velocity, and $\theta_1(\cdot) \in C^1(0,1)$, $\theta_2(\cdot) \in C^1(0,1)$ are referred to, respectively, as the dispersion and damping coefficients, respectively. $u(\xi,t) \in L_2(0,1)$ is the modifiable source term (the distributed control input), and $\psi(\xi,t) \in L_2(0,1)$ represents a distributed **uncertain** disturbance source term. The spatially varying parameters are supposed to be **uncertain**, too, and satisfying the next Assumption:

ASSUMPTION 6.2.1 There exist *a priori* known constants Υ_m, Υ_M, Θ_1 and Θ_2 such that

$$0 < \Upsilon_m \leq \nu^2(\xi) \leq \Upsilon_M, \quad |\theta_1(\xi)| \leq \Theta_1, \quad |\theta_2(\xi)| \leq \Theta_2, \quad \forall \xi \in [0,1]. \quad (6.2.2)$$

We consider non-homogeneous mixed boundary conditions (BCs)

$$y(0,t) - \beta_0 y_\xi(0,t) = Y_0(t) \in W^{1,2}(0,\infty), \quad (6.2.3)$$
$$y(1,t) + \beta_1 y_\xi(1,t) = Y_1(t) \in W^{1,2}(0,\infty), \quad (6.2.4)$$

with some positive **uncertain** constants β_0, β_1 and functions $Y_0(t), Y_1(t) \in C^1(0,\infty)$. The initial conditions (ICs)

$$y(\xi,0) = \varphi_0(\xi) \in W^{2,2}(0,1), \quad y_t(\xi,0) = \varphi_1(\xi) \in W^{2,2}(0,1) \quad (6.2.5)$$

where $\varphi_0(\cdot), \varphi_1(\cdot) \in C^1(0,1)$ are also assumed to meet the boundary conditions (BCs) imposed on the system. As in the diffusion equation case, nonhomogeneous BCs are in general admitted, which is why a solution of the above boundary-value problem is defined in the mild sense (see Curtain and Zwart (1995)) as that of the corresponding integral equation, written in terms of the strongly continuous semigroup, generated by the infinitesimal plant operator. The control task is to make the position $y(\xi,t)$ and the velocity $y_t(\xi,t)$ to exponentially track an *a priori* given reference signal $y^r(\xi,t)$ and, respectively, its velocity $y_t^r(\xi,t)$ in the L_2 space, regardless of whichever admissible disturbance $\psi(\xi,t)$ affects the system. It is assumed throughout that the reference signal $y^r(\xi,t)$ and its time derivatives are smooth enough in the sense that

$$y^r(\cdot,t) \in W^{2,2}(0,1), \qquad y^r_t(\cdot,t) \in W^{2,2}(0,1), \qquad y^r_{tt}(\cdot,t) \in L_2(0,1), \quad \forall t \geq 0 \tag{6.2.6}$$

Apart from this, the reference signal is assumed to meet the actual BCs (6.2.3)-(6.2.4).

The **deviation variables**

$$\tilde{y}(\xi,t) = y(\xi,t) - y^r(\xi,t), \qquad \tilde{y}_t(\xi,t) = y_t(\xi,t) - y^r_t(\xi,t) \tag{6.2.7}$$

are to eventually be driven to zero in L_2 norm by the controller to be designed. By differentiating (6.2.7) and making appropriate substitutions an manipulations one derives the next PDE governing the corresponding error dynamics

$$\begin{aligned}\tilde{y}_{tt}(\xi,t) = &\left[\nu^2(\xi)\tilde{y}_\xi\right]_\xi + \theta_1(\xi)\tilde{y}(\xi,t) + \theta_2(\xi)\tilde{y}_t(\xi,t) + u(\xi,t)\\ &+ \psi(\xi,t) - y^r_{tt}(\xi,t) + \left[\nu^2(\xi)\tilde{y}^r_\xi\right]_\xi + \theta_1(\xi)y^r(\xi,t) + \theta_2(\xi)y^r_t(\xi,t)\end{aligned} \tag{6.2.8}$$

with the ICs

$$\tilde{y}(\xi,0) = \varphi_0(\xi) - \varphi^r_0(\xi) \in W^{2,2}(0,1), \qquad \tilde{y}_t(\xi,0) = \varphi_1(\xi) - \varphi^r_1(\xi) \in W^{2,2}(0,1) \tag{6.2.9}$$

and **homogeneous** BCs

$$\tilde{y}(0,t) - \beta_0\tilde{y}_\xi(0,t) = \tilde{y}(1,t) + \beta_1\tilde{y}_\xi(1,t) = 0. \tag{6.2.10}$$

The assumptions on the ICs and BCs, made above, allow us to deal with strong, sufficiently smooth solutions of the uncertain error dynamics (6.2.8)-(6.2.10) in the open-loop when no control input is applied. Just in case, the open-loop system locally possesses a unique (twice differentiable in time) strong solution $\tilde{y}(\xi,t)$ which is defined in a standard manner (see Curtain and Zwart (1995)) as an absolutely continuous function, almost everywhere satisfying the corresponding PDE rather than its integral counterpart.

The class of reference signals and admissible disturbances is specified in the next Assumptions.

ASSUMPTION 6.2.2 There exist *a priori* known constants H_0, ..., H_4 such that the reference trajectory $y^r(\xi,t)$ and its spatial and temporal derivatives meet the following inequalities for all $t \geq 0$

$$\|y^r(\cdot,t)\|_2 \leq H_0, \qquad \|y^r_t(\cdot,t)\|_2 \leq H_1, \qquad \|y^r_{tt}(\cdot,t)\|_2 \leq H_2, \tag{6.2.11}$$

$$\|[\nu^2(\xi)y^r_\xi(\cdot,t)]_\xi\|_2 \leq H_3, \qquad \|[\nu^2(\xi)y^r_{t\xi}(\cdot,t)]_\xi\|_2 \leq H_4 \tag{6.2.12}$$

ASSUMPTION 6.2.3 There exist *a priori* known constants Ψ_0, Ψ_1 such that the disturbance $\psi(\xi,t)$ and its temporal derivative meet the following inequalities for all $t \geq 0$

$$\|\psi(\cdot,t)\|_2 \leq \Psi_0 \qquad \|\psi_t(\cdot,t)\|_2 \leq \Psi_1 \qquad \forall t \geq 0 \tag{6.2.13}$$

6.2.1 Distributed Sliding Manifold Design

Define the distributed sliding variable $\sigma \in L_2(0,1)$ as follows

$$\sigma(\xi,t) = \tilde{y}_t(\xi,t) + c\tilde{y}(\xi,t), \qquad c > 0 \tag{6.2.14}$$

The motion of the system constrained on the sliding manifold $\sigma(\xi,t) = 0$ is governed by the corresponding simple first-order ordinary differential equation $\tilde{y}_t(\xi,t) + c\tilde{y}(\xi,t) = 0$ with the spatial variable ξ to be viewed as a parameter, whose solution $\tilde{y}(\xi,t)$ norm along with its time derivative exponentially tend to zero in $L_2(0,1)$. Hence the control task can be reduced to the simplified problem of steering to zero the L_2 norm of the distributed sliding variable.

In order to simplify the notation, the dependence of the system signals from the space and time variables (ξ,t) will be generally omitted from this point on. Consider the first- and second-order time derivatives of the above defined distributed sliding variable σ

$$\sigma_t = \tilde{y}_{tt} + c\tilde{y}_t, \qquad \sigma_{tt} = \tilde{y}_{ttt} + c\tilde{y}_{tt} \tag{6.2.15}$$

Differentiating the error dynamics (6.2.8) yields

$$\begin{aligned} \tilde{y}_{ttt} = {} & \left[\nu^2(\xi)\tilde{y}_{t\xi}\right]_\xi + \theta_1(\xi)\tilde{y}_t + \theta_2(\xi)\tilde{y}_{tt} \\ & + u_t + \psi_t - y^r_{ttt} + [\nu^2(\xi)y^r_{t\xi}]_\xi + \theta_1(\xi)y^r_t + \theta_2(\xi)y^r_{tt}, \end{aligned} \tag{6.2.16}$$

and now substituting (6.2.8) and (6.2.16) into the second of (6.2.15) results after simple manipulations in

$$\begin{aligned} \sigma_{tt} = {} & \left[\nu^2(\xi)(\tilde{y}_{t\xi} + c\tilde{y}_\xi)\right]_\xi + \theta_1(\xi)\left[\tilde{y}_t + c\tilde{y}\right] \\ & + \theta_2(\xi)\left[\tilde{y}_{tt} + c\tilde{y}_t\right] + u_t + cu - y^r_{ttt} - cy^r_{tt} + \overline{\psi} \end{aligned} \tag{6.2.17}$$

where

$$\overline{\psi} = \psi_t + c\psi + [\nu^2(\xi)(y^r_{t\xi} + cy^r_\xi)]_\xi + \theta_1(\xi)(y^r_t + cy^r) + \theta_2(\xi)(y^r_{tt} + cy^r_t) \tag{6.2.18}$$

is an uncertain "augmented" disturbance depending on both the disturbance ψ and the reference trajectory y^r, and their derivatives. By exploiting the Assumptions

6.2.1, 6.2.2 and 6.2.3, it follows that the next restriction on the L_2 norm of the augmented disturbance $\overline{\psi}$ holds for all $t \geq 0$

$$\|\overline{\psi}(\cdot,t)\|_2 \leq M \equiv \Psi_1 + c\Psi_0 + (H_4 + cH_3) + \Theta_1(H_1 + cH_0) + \Theta_2(H_2 + cH_1) \tag{6.2.19}$$

After simple additional manipulations one obtains that the sliding variable σ is governed by a PDE which is formally equivalent to the original wave equation (6.2.1) with a new fictitious control variable v which dynamically depends on the input u, according to

$$\sigma_{tt} = [\nu^2(\xi)\sigma_\xi]_\xi + \theta_1(\xi)\sigma + \theta_2(\xi)\sigma_t + v - y^r_{ttt} - cy^r_{tt} + \overline{\psi} \tag{6.2.20}$$

$$v = u_t + cu. \tag{6.2.21}$$

equipped with the appropriate ICs and the **homogeneous** BCs

$$\sigma(0,t) - \beta_0\sigma_\xi(0,t) = \sigma(1,t) + \beta_1\sigma_\xi(1,t) = 0. \tag{6.2.22}$$

6.2.2 Combined PD/Sliding-Mode Control of the Wave Process

In order to stabilize the uncertain dynamics (6.2.20), (6.2.22) the following distributed controller

$$v = y^r_{ttt} + cy^r_{tt} - W_1\sigma - W_2\sigma_t - \lambda_1\frac{\sigma}{\|\sigma(\cdot,t)\|_2} - \lambda_2\frac{\sigma_t}{\|\sigma_t(\cdot,t)\|_2} \tag{6.2.23}$$

is proposed for generating the fictitious control v. Controller (6.2.23) can be viewed as a mixed linear/sliding mode control algorithm, with a feed-forward term, a linear PD-type feedback term, and with the discontinuous feedback term being a distributed version of the finite-dimensional "Twisting" controller, which belongs to the class of so-called "second-order sliding-mode" controllers (2-SMCs) (see Levant (1993)).

It is worth to discuss how the actual control input $u(\xi,t)$ should be recovered from $v(\xi,t)$ once the latter has been computed according to (6.2.23). In relation (6.2.21) the spatial variable ξ can be viewed as a fixed parameter. By virtue of this fact, (6.2.21) can be interpreted as a continuum of first-order ODEs whose parameterized solutions give rise to the actual control input to be applied to the wave equation. The transfer function block $\frac{1}{s+c}$ (with s being the Laplace variable and c being the positive constant in (6.2.14)) can effectively represent the relation between signals $v(\xi,t)$ (considered as the block input) and $u(\xi,t)$ (considered as the block output). The plant control $u(\xi,t)$, obtained at the output of a dynamical filter driven by the discontinuous control $v(\xi,t)$, will be therefore a **continuous** signal with a discontinuus time derivative $u_t(\xi,t) = v(\xi,t) - cu(\xi,t)$.

The solution concept of the wave process (6.2.20)-(6.2.22) subject to the control strategy (6.2.23) is defined in the same manner as that of the diffusion process dealt with in the Section 2.

The exponential stability of the generalized wave equation subject to the control strategy (6.2.23), (7.2.10), (6.2.14) is demonstrated in Theorem 6.2.1, given below.

THEOREM 6.2.1 Consider the generalized wave equation (6.2.1) along with the initial and boundary conditions (6.2.5) and (6.2.3), and whose parameters, reference trajectory and external disturbance satisfy the Assumptions 6.2.1, 6.2.2 and 6.2.3. Consider the associated error variable (6.2.7) and the sliding variable (6.2.14). Then, the distributed control strategy (6.2.23) with the parameters W_1, W_2, λ_1 and λ_2 such that

$$W_1 > \Theta_1, \qquad W_2 > \Theta_2, \qquad \lambda_2 > M, \qquad \lambda_1 > \lambda_2 + M, \quad (6.2.24)$$

guarantees the exponential decay of the L_2 norms $\|\tilde{y}(\cdot,t)\|_2$ and $\|\tilde{y}_t(\cdot,t)\|_2$ of the solutions of 6.2.20 - 6.2.22 .

Proof of Theorem 6.2.1

Let us refer to the sliding variable dynamics (6.2.20) along with the boundary conditions (6.2.22). The closed-loop sliding variable dynamics is easily obtained by substituting (6.2.23) into (6.2.20), which yields

$$\begin{aligned}\sigma_{tt} = &[\nu^2(\xi)\sigma_\xi]_\xi - (W_1-\theta_1(\xi))\sigma - (W_2-\theta_2(\xi))\sigma_t - \\ &\lambda_1\frac{\sigma}{\|\sigma(\cdot,t)\|_2} - \lambda_2\frac{\sigma_t}{\|\sigma_t(\cdot,t)\|_2} + \overline{\psi}\end{aligned} \quad (6.2.25)$$

By the first and the second tuning inequality 6.2.24, conditions $W_1 - \theta_1(\xi) > 0$ and $W_2 - \theta_2(\xi) > 0$ hold for any admissible value of $\theta_1(\xi)$ and θ_2 for $\xi \in [0,1]$(in accordance with the Assumption 6.2.2). Consider the following Lyapunov functional $\tilde{V}(t)$

$$\begin{aligned}\tilde{V}(t) = &\frac{1}{2}\|\sqrt{W_1-\theta_1(\xi)}\,\sigma\|_2^2 + \lambda_1\|\sigma\|_2 + \frac{1}{2}\|\sigma_t\|_2^2 + \\ &\frac{1}{2}\|\nu(\xi)\sigma_\xi\|_2^2 + \frac{1}{2}\frac{\nu^2(0)}{\beta_0}\sigma^2(0,t) + \frac{1}{2}\frac{\nu^2(1)}{\beta_1}\sigma^2(1,t)\end{aligned} \quad (6.2.26)$$

The time derivative of $\tilde{V}(t)$ is given by

$$\begin{aligned}\dot{\tilde{V}}(t) = &\int_0^1 (W_1-\theta_1(\xi))\,\sigma\sigma_t d\xi + \frac{\lambda_1}{\|\sigma\|_2}\int_0^1 \sigma\sigma_t d\xi + \int_0^1 \sigma_t\sigma_{tt} d\xi \\ &+ \int_0^1 \nu^2(\xi)\sigma_\xi\sigma_{\xi t} d\xi + \frac{\nu^2(0)}{\beta_0}\sigma(0,t)\sigma_t(0,t) + \frac{\nu^2(1)}{\beta_1}\sigma(1,t)\sigma_t(1,t)\end{aligned} \quad (6.2.27)$$

By evaluating (6.2.27) along the solutions of (6.2.25), it turns out that

$$\begin{aligned}\dot{\tilde{V}}(t) = &-\int_0^1 (W_2 - \theta_2(\xi))\sigma\sigma_t d\xi - \lambda_2\|\sigma_t\|_2 + \int_0^1 \sigma_t\overline{\psi}d\xi \\ &+ \frac{\nu^2(0)}{\beta_0}\sigma(0,t)\sigma_t(0,t) + \frac{\nu^2(1)}{\beta_1}\sigma(1,t)\sigma_t(1,t) \\ &+ \int_0^1 [\nu^2(\xi)\sigma_\xi]_\xi\sigma_t d\xi + \int_0^1 \nu^2(\xi)\sigma_\xi\sigma_{\xi t}d\xi.\end{aligned} \tag{6.2.28}$$

The last term in the right hand side of (6.2.28) can be integrated by parts, and taking into account the homogeneous boundary conditions (6.2.22) it yields

$$\begin{aligned}&\int_0^1 \nu^2(\xi)\sigma_\xi\sigma_{\xi t}d\xi = \\ &[\nu^2(1)\sigma_\xi(1,t)\sigma_t(1,t) - \nu^2(0)\sigma_\xi(0,t)\sigma_t(0,t)] - \int_0^1 [\nu^2(\xi)\sigma_\xi]_\xi\sigma_t d\xi = \\ &-\frac{\nu^2(1)}{\beta_1}\sigma(1,t)\sigma_t(1,t) - \frac{\nu^2(0)}{\beta_0}\sigma(0,t)\sigma_t(0,t) - \int_0^1 [\nu^2(\xi)\sigma_\xi]_\xi\sigma_t d\xi\end{aligned} \tag{6.2.29}$$

which leads to the simplified form of $\dot{\tilde{V}}(t)$:

$$\dot{\tilde{V}}(t) \;=\; -\int_0^1 (W_2 - \theta_2(\xi))\sigma\sigma_t d\xi - \lambda_2\|\sigma_t\|_2 + \int_0^1 \sigma_t\overline{\psi}d\xi \tag{6.2.30}$$

By employing the Cauchy-Schwartz inequality (see Krstic and Smyshlyaev (2008)) and taking into account (6.2.19), one derives that

$$\left|\int_0^1 \sigma_t\overline{\psi}d\xi\right| \le \int_0^1 |\sigma_t\overline{\psi}|d\xi \le \|\sigma_t\|_2\|\overline{\psi}\|_2 \le M\|\sigma_t\|_2 \tag{6.2.31}$$

Then by (6.2.30) and (6.2.31) it follows that

$$\dot{\tilde{V}}(t) \;\le\; -(W_2 - \Theta_2)\|\sigma_t\|_2^2 - (\lambda_2 - M)\|\sigma_t\|_2 \tag{6.2.32}$$

which implies, considering 6.2.24, that the Lyapunov functional $V(t)$ is a non-increasing function of time, i.e.

$$\tilde{V}(t_2) \;\le\; \tilde{V}(t_1) \quad \forall t_2 \ge t_1 \ge 0, \tag{6.2.33}$$

Denote

$$\mathcal{D}_R = \left\{(\sigma,\sigma_t) \in L_2(0,1) \times L_2(0,1) \quad : \quad \tilde{V}(\sigma,\sigma_t) \le R\right\} \tag{6.2.34}$$

Clearly, by virtue of (6.2.33), taking any $R \geq V(0)$ the resulting domain $\mathcal{D}_R$ will be **invariant** for the error system trajectories. Our subsequent analysis will take into account that the states (σ, σ_t) belong to the domain $\mathcal{D}_R$ starting from the initial time $t = 0$ on. Note that the knowledge of the constant R is not required (see also the Remark 2).

We now demonstrate a simple Lemma that will be used along the proof.

Lemma 1. If the states (σ, σ_t) belong to the domain $\mathcal{D}_R$ (6.2.34) then the following estimates hold:

$$\|\sigma_t\|_2^2 \leq \sqrt{2R}\|\sigma_t\|_2 \tag{6.2.35}$$

$$\int_0^1 \sigma\sigma_t \, d\xi \geq -\frac{1}{2}\left[\frac{R}{\lambda_1}\|\sigma\|_2 + \|\sigma_t\|_2^2\right] \tag{6.2.36}$$

Proof of Lemma 1.

Equation (6.2.35) comes from the following trivial chain of implications

$$\tilde{V}(t) \leq R \;\; \Rightarrow \frac{1}{2}\|\sigma_t\|_2^2 \leq R \;\; \Rightarrow \;\; \|\sigma_t\|_2 \leq \sqrt{2R} \Rightarrow \;\; \|\sigma_t\|_2^2 \leq \sqrt{2R}\|\sigma_t\|_2 \tag{6.2.37}$$

A similar procedure results in

$$\tilde{V}(t) \leq R \;\; \Rightarrow \;\; \lambda_1\|\sigma\|_2 \leq R \;\; \Rightarrow \;\; \|\sigma\|_2 \leq \frac{R}{\lambda_1} \tag{6.2.38}$$

By applying the well-known inequality $ab \geq -\frac{1}{2}(a^2 + b^2)$ it follows that

$$\int_0^1 \sigma\sigma_t \, d\xi \geq -\frac{1}{2}\left[\|\sigma\|_2^2 + \|\sigma_t\|_2^2\right] = -\frac{1}{2}\left[\|\sigma\|_2\|\sigma\|_2 + \|\sigma_t\|_2^2\right] \tag{6.2.39}$$

Being coupled together, relations (6.2.37)-(6.2.39) yield (6.2.36), which proves the Lemma. □

Now consider the "augmented" functional

$$\begin{aligned} V_R(t) = \tilde{V}(t) + \kappa_R \int_0^1 \sigma\sigma_t \, d\xi + \frac{1}{2}\kappa_R\|\sqrt{W_2 - \theta_2(\xi)}\,\sigma\|_2^2 = \\ = \frac{1}{2}\|\sqrt{W_1 - \theta_1(\xi)}\,\sigma\|_2^2 + \lambda_1\|\sigma\|_2 + \frac{1}{2}\|\sigma_t\|_2^2 + \frac{1}{2}\|\nu(\xi)\sigma_\xi\|_2^2 \\ + \frac{1}{2}\frac{\nu^2(0)}{\beta_0}\sigma^2(0,t) + \frac{1}{2}\frac{\nu^2(1)}{\beta_1}\sigma^2(1,t) \\ + \kappa_R \int_0^1 \sigma\sigma_t \, d\xi + \frac{1}{2}\kappa_R\|\sqrt{W_2 - \theta_2(\xi)}\,\sigma\|_2^2 \end{aligned} \tag{6.2.40}$$

where κ_R is a positive constant. In light of the Lemma 1, function $V_R(t)$ can be estimated as

$$V_R(t) \geq \frac{1}{2}\left(W_1 - \Theta_1 + \kappa_R(W_2 - \Theta_2)\right)\|\sigma\|_2^2 + \left(\lambda_1 - \frac{\kappa_R R}{2\lambda_1}\right)\|\sigma\|_2 + \frac{1}{2}(1-\kappa_R)\|\sigma_t\|_2^2 + \frac{1}{2}\|\nu(\xi)\sigma_\xi\|_2^2 + \frac{1}{2}\frac{\nu^2(0)}{\beta_0}\sigma^2(0,t) + \frac{1}{2}\frac{\nu^2(1)}{\beta_1}\sigma^2(1,t) \tag{6.2.41}$$

Since $W_1 - \Theta_1 > 0$ and $W_2 - \Theta_2 > 0$, as previously noticed, then, provided that the positive coefficient κ_R is selected sufficiently small according to

$$0 < \kappa_R \leq \mathtt{min}\ \left\{\frac{2\lambda_1^2}{R}, 1\right\}, \tag{6.2.42}$$

the augmented functional (6.2.40) is thus proved to be positive definite within the invariant domain $\mathcal{D}_R$, and it can be then used as a candidate Lyapunov functional to analyze the stability of the error dynamics. Let us compute the time derivative of $V_R(t)$ along the solutions of (6.2.25). Simple manipulations yield

$$\begin{aligned}\dot{V}_R(t) = &-\|\sqrt{W_2 - \theta_2(\xi)}\ \sigma_t\|_2^2 - \lambda_2\|\sigma_t\|_2 + \int_0^1 \sigma_t\overline{\psi}d\xi \\ &+ \kappa_R\|\sigma_t\|_2^2 + \kappa_R\int_0^1 [\nu^2(\xi)\sigma_\xi]_\xi \sigma d\xi - \kappa_R\|\sqrt{W_1 - \theta_1(\xi)}\ \sigma\|_2^2 \\ &- \kappa_R\lambda_1\|\sigma\|_2 - \frac{\kappa_R\lambda_2}{\|\sigma_t\|_2}\int_0^1 \sigma\sigma_t d\xi + \kappa_R\int_0^1 \sigma\overline{\psi}d\xi.\end{aligned} \tag{6.2.43}$$

Let us compute upperbounds to the sign-indefinite terms of (6.2.43). Equation (6.2.31) has previously been derived, which is rewritten in a similar form with the signal σ replacing σ_t:

$$\kappa_R\left|\int_0^1 \sigma\overline{\psi}d\xi\right| \leq \kappa_R\int_0^1 |\sigma\overline{\psi}|d\xi \leq \kappa_R M\|\sigma\|_2 \tag{6.2.44}$$

Apart from this, the next inequality can readily be derived by employing the Couchy-Schwartz inequality (see Krstic and Smyshlyaev (2008)):

$$\begin{aligned}\frac{\kappa_R\lambda_2}{\|\sigma_t\|_2}\left|\int_0^1 \sigma\sigma_t d\xi\right| &\leq \frac{\kappa_R\lambda_2}{\|\sigma_t\|_2}\int_0^1 |\sigma\sigma_t|\, d\xi \\ &\leq \frac{\kappa_R\lambda_2}{\|\sigma_t\|_2}\sqrt{\int_0^1 \sigma^2 d\xi}\sqrt{\int_0^1 \sigma_t^2 d\xi} = \kappa_R\lambda_2\|\sigma\|_2\end{aligned} \tag{6.2.45}$$

Then, the following estimate can be written by substituting (6.2.44) and (6.2.45) into (6.2.43), considering the Assumption 6.2.1, and noticing that the equality

$$\kappa_R \int_0^1 [\nu^2(\xi)\sigma_\xi]_\xi \, \sigma d\xi = -\frac{\nu^2(1)}{\beta_1}\sigma^2(1,t) - \frac{\nu^2(0)}{\beta_0}\sigma^2(0,t) - \|\nu^2(\xi)\sigma_\xi\|_2^2 \quad (6.2.46)$$

holds due to the BCs (6.2.22):

$$\begin{aligned}\dot{V}_R(t) \leq & -(W_2-\Theta_2)\|\sigma_t\|_2^2 - \rho\|\sigma_t\|_2 - \kappa_R(\lambda_1-\lambda_2-M)\|\sigma\|_2 \\ & - \kappa_R(W_1-\Theta_1)\|\sigma\|_2^2 - \kappa_R\Upsilon_m\|\sigma_\xi\|_2^2 - \frac{\nu^2(1)}{\beta_1}\sigma^2(1,t) - \frac{\nu^2(0)}{\beta_0}\sigma^2(0,t),\end{aligned} \quad (6.2.47)$$

$$\rho = \left(\lambda_2 - M - \kappa_R\sqrt{2R}\right), \quad (6.2.48)$$

Therefore, employing the parameter tuning conditions (6.2.24) and introducing one more restriction

$$\kappa_R \leq \mathtt{min} \quad \left\{\frac{2\lambda_1^2}{R}, 1, \frac{\lambda_2 - M}{\sqrt{2R}}\right\} \quad (6.2.49)$$

about the coefficient κ_R beyond (6.2.42), it readily follows that all terms appearing in the right-hand side of (6.2.47) are negative definite. It can be then concluded that

$$\dot{V}_R(t) \leq -c_R\left(\|\sigma\|_2 + \|\sigma\|_2^2 + \|\sigma_t\|_2 + \|\sigma_t\|_2^2 + \|\sigma_\xi\|_2^2 + \sigma^2(1,t) + \sigma^2(0,t)\right), \quad (6.2.50)$$

where

$$c_R = \mathtt{min}\left\{W_2-\Theta_2,\ \rho,\ \kappa_R(W_1-\Theta_1),\ \kappa_R(\lambda_1-\lambda_2-M),\ \kappa_R\Upsilon_m,\ \frac{\Upsilon_m}{\beta_1}, \frac{\Upsilon_m}{\beta_0}\right\} \quad (6.2.51)$$

Applying the Cauchy-Schwartz integral inequality and considering (6.2.38) yields

$$\left|\int_0^1 \sigma\sigma_t \, d\xi\right| \leq \|\sigma\sigma_t\|_1 \leq \|\sigma\|_2\|\|\sigma_t\|_2 \leq \frac{R}{\lambda_1}\|\sigma_t\|_2 \quad (6.2.52)$$

Now substituting (6.2.52) into (6.2.40) and upper-estimating further the resulting right-hand side in light of the Assumption 6.2.1 yields the next estimation

$$V_R(t) \leq w_R\left(\|\sigma\|_2^2 + \|\sigma\|_2 + \|\sigma_t\|_2^2 + \|\sigma_t\|_2 + \|\sigma_\xi\|_2^2 + \sigma^2(1,t) + \sigma^2(0,t)\right) \quad (6.2.53)$$

$$w_R = \mathtt{max}\left\{\frac{1}{2}\left(W_1-\Theta_1+\kappa_R(W_2-\Theta_2)\right),\ \lambda_1,\ \frac{1}{2},\ \kappa_R\frac{R}{\lambda_1},\ \frac{1}{2}\Upsilon_M,\ \frac{\Upsilon_M}{\beta_1},\ \frac{\Upsilon_M}{\beta_0}\right\} \quad (6.2.54)$$

Hence, the following differential inequality holds

$$\dot{V}_R(t) \leq -c_1 V_R(t), \qquad c_1 = \frac{c_R}{w_R} \tag{6.2.55}$$

thereby ensuring the exponential convergence of $\|\sigma\|_2$, $\|\sigma_t\|_2$, and $\|\sigma_\xi\|_2$ to zero as $t \to \infty$. It remains to prove that the L_2 norm of the tracking error $\tilde{y}(\xi, t)$ and that of its derivative tend exponentially to zero. Indeed, the inequality

$$\|\sigma(\cdot, t)\|_2 \leq c_2 V_R(t), \qquad c_2 = \frac{2\lambda_1}{2\lambda_1^2 - \kappa_R R}, \tag{6.2.56}$$

is straightforwardly derived from (6.2.41) whereas by (6.2.14), the spatiotemporal evolution of $\tilde{y}(\xi, t)$ is governed by

$$\tilde{y}_t(\xi, t) = -c\tilde{y}(\xi, t) + \sigma(\xi, t), \qquad c > 0. \tag{6.2.57}$$

In (6.2.57), the sliding variable $\sigma(\xi, t)$ can be viewed as an external driving input, exponentially decaying in L_2 norm according to (6.2.56). Then computing the time derivative of the Lyapunov functional $W(t) = \|\tilde{y}\|_2$ along dynamics (6.2.57) yields

$$\dot{W}(t) = -cW(t) + \frac{1}{W(t)} \int_0^1 \tilde{y}\sigma d\xi \tag{6.2.58}$$

Since

$$\left| \int_0^1 \tilde{y}\sigma d\xi \right| \leq \|\tilde{y}\|_2 \|\sigma\|_2 \leq W(t)\|\sigma\|_2 \tag{6.2.59}$$

by combining (6.2.56), (6.2.58) and (6.2.59) it follows that

$$\dot{W}(t) \leq -cW(t) + \|\sigma\|_2 \leq -cW(t) + c_2 V_R(t). \tag{6.2.60}$$

It is now clear that relations (6.2.55)-(6.2.57), and (6.2.60), coupled together, ensure the exponential decay of $\|\tilde{y}(\cdot, t)\|_2$ and $\|\tilde{y}_t(\cdot, t)\|_2$. Theorem 1 is thus proved. □

6.3 Numerical Simulations

For solving the PDEs governing the closed-loop systems, standard finite-difference approximation method is used by discretizing the spatial solution domain $\xi \in [0, 1]$ into a finite number of N uniformly spaced solution nodes $\xi_i = ih$, $h = 1/(N + 1)$, $i = 1, 2, ..., N$. The value $N = 100$ has been used in the present simulations. The resulting 100-th order discretized system is implemented in Matlab-Simulink and solved by fixed-step Euler integration method with constant step $T_s = 10^{-4}s$.

6.3.1 Reaction-Diffusion equation

Consider the perturbed Reaction-Diffusion equation (7.2.1) with the spatially varying parameters given by

$$\theta_1(\xi) = 0.1 + 0.02\sin(1.3\pi\xi), \tag{6.3.1}$$

$$\theta_2(\xi) = 1 + 0.1\sin(3.5\pi\xi), \tag{6.3.2}$$

mixed-type BCs

$$Q(0,t) - \alpha_0 Q_\xi(0,t) = Q(1,t) + \alpha_1 Q_\xi(1,t) = 20 - 5\pi, \tag{6.3.3}$$

and ICs

$$Q(\xi,0) = 20 + 10sin(6\pi\xi). \tag{6.3.4}$$

We choose the next spatially varying reference profile

$$Q^r(\xi,t) = 20 + 5sin(\pi\xi), \tag{6.3.5}$$

which meets the actual BCs. A spatially varying disturbance term is considered in the form

$$\psi(\xi) = 5\mathtt{sin}(2.5\pi\xi). \tag{6.3.6}$$

By (6.3.2), the bound Θ_{2M} in (7.2.8)can be readily overestimated by any $\Theta_{2M} > 1.1$ Then, the controller gains are set in accordance with (6.1.20) to the values

$$W_1 = 20, \quad \lambda_1 = 20, \quad W_2 = 20, \quad \lambda_2 = 20. \tag{6.3.7}$$

The left plot in Figure 6.1 depicts the solution $Q(\xi,t)$, which converges to the given reference profile as confirmed by the contractive evolution of the tracking error L_2 norm $\|x(\cdot,t)\|_2$ shown n the Figure 6.1-right. Figure 6.2 depicts the control input $u(\xi,t)$ which, as expected, appears to be a smooth function of both time and space. The attained results confirm the validity of the presented analysis.

6.3.2 Generalized wave equation

Consider the perturbed equation (6.2.1) with spatially-varying parameters:

$$\nu^2(\xi) = 0.1 + 0.02\sin(2\pi\xi), \tag{6.3.8}$$

$$\theta_1(\xi) = -(1 + \sin(1.2\pi\xi)), \tag{6.3.9}$$

$$\theta_2(\xi) = -(5 + 3\sin(3\pi\xi)), \tag{6.3.10}$$

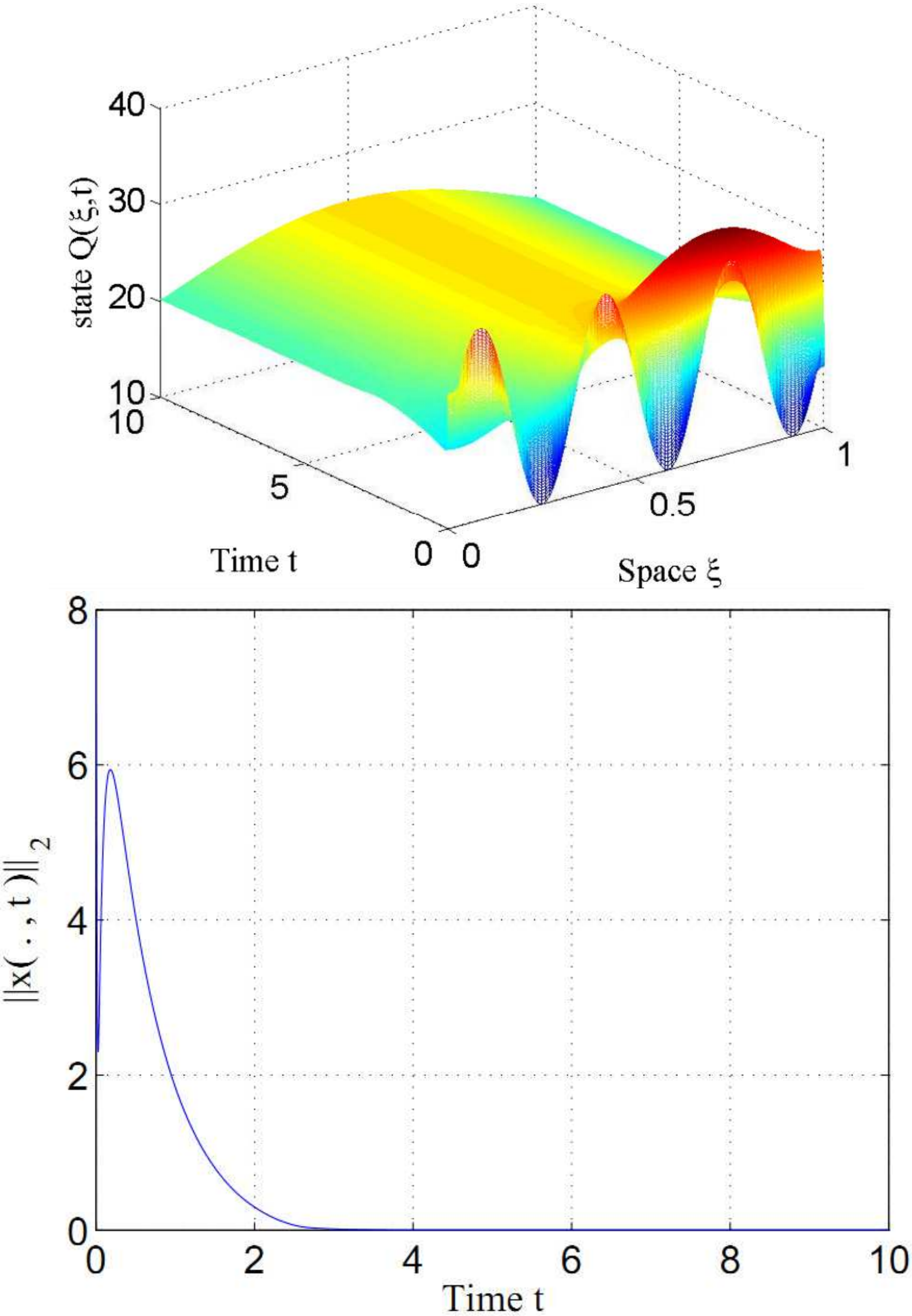

Figure 6.1: The solution $Q(\xi, t)$ (top plot) and the tracking error L_2 norm (bottom plot) for the controlled reaction-diffusion equation

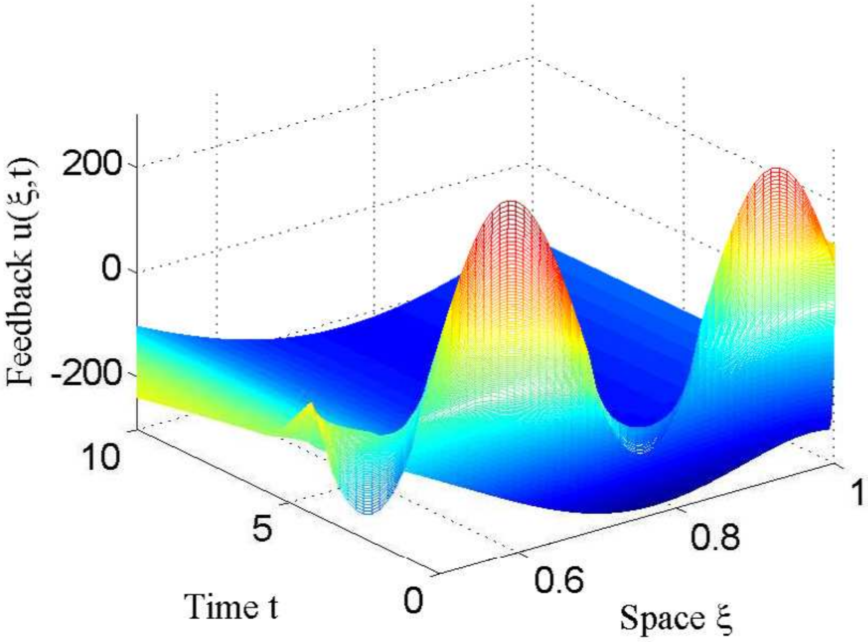

Figure 6.2: Distributed control input $u(\xi, t)$ of the controlled reaction-diffusion equation.

and mixed BC's

$$y(0,t) - y_\xi(0,t) = y(1,t) + y_\xi(1,t) = 0. \tag{6.3.11}$$

The bounds $\Theta_1 = 2,\ \Theta_2 = 8$ to the uncertain system parameters (see the Assumption 3.1) are taken into account for the controller tuning. The initial conditions in (6.2.5) are set to $\varphi_0(\xi) = 10\sin(6\pi\xi)$, $\varphi_1(\xi) = 0$. The reference profile is set to $y^r(\xi,t) = 2\sin(\pi\xi)\sin(\pi t)$. The bounds $H_0 = 2$, $H_1 = 6$, $H_2 = 20$, $H_3 = 3$, $H_4 = 96$ to the norms of its derivatives as in (6.2.11)-(6.2.12) are considered. The disturbance is set to $\psi(\xi,t) = 10\sin(5\pi\xi)\sin(2\pi t)$. The upperbounds $\Psi_0 = 10$, $\Psi_1 = 63$, are considered in the restrictions (6.2.13). The distributed sliding manifold $\sigma(\xi,t)$ has been implemented with the parameter $c = 2$. Parameter M in (6.2.19) is chosen as $M = 400$. The controller parameters are set in accordance with (6.2.24) as $W_1 = 2$,$W_2 = 10$, $\lambda_2 = 500$ and $\lambda_1 = 1000$. Figure 6.3.2 reports two different views of the solution $y(\xi,t)$. Figures 6.3.2 and 6.5 show the corresponding plots of the distributed control $u(\xi,t)$ and of the tracking error L_2 norm $\|\tilde{y}(\cdot,t)\|_2$. Good performance of the proposed control algorithm is concluded from the graphics that confirm the theoretical properties of the proposed distributed controller. The continuity of the applied distributed control input particularly follows from the inspection of Figure 6.3.2.

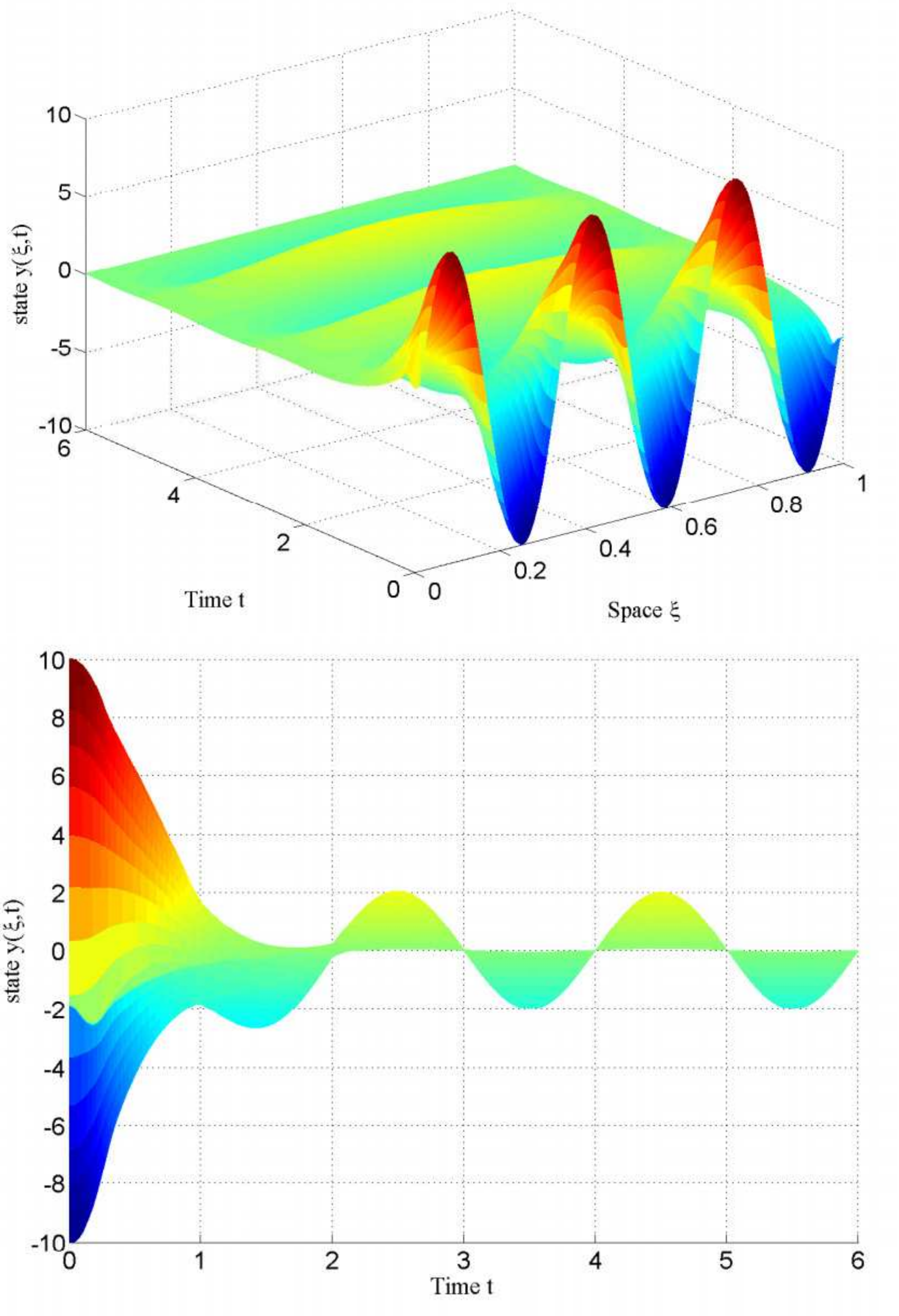

Figure 6.3: Different views of the solution $y(\xi, t)$ of the controlled generalized wave equation.

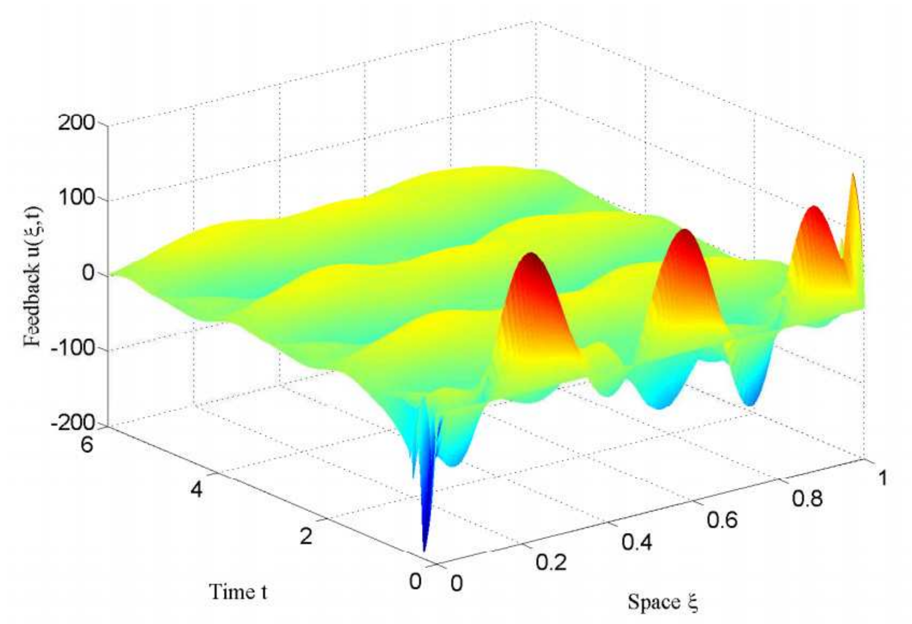

Figure 6.4: Generalized wave equation test: distributed control $u(\xi, t)$.

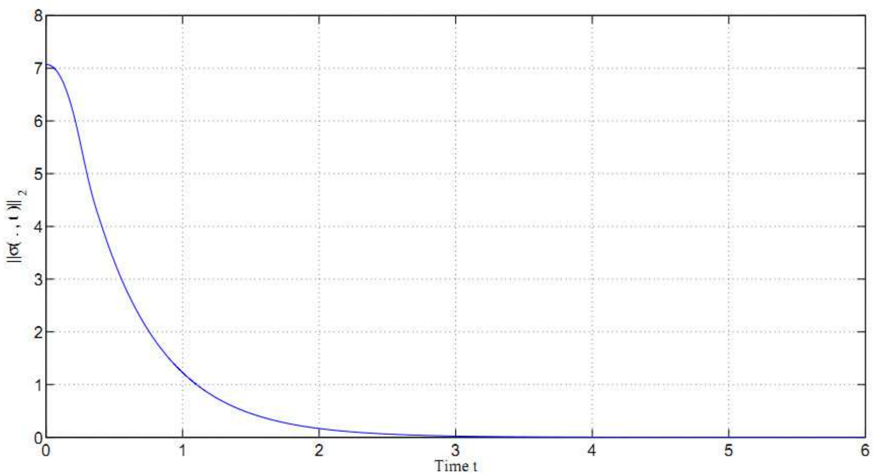

Figure 6.5: Generalized wave equation test: Tracking error L_2 norm $\|\tilde{y}(\cdot, t)\|_2$.

Chapter 7

Boundary Sliding Mode control design

In this chapter the Boundary control for the uncertain diffusion equation with unknown disturbance on the actuation is illustrated. At first section we develop the Boundary control stabilization problem of a one-dimensional uncertain reaction-diffusion process powered with a Dirichlet type actuator from one of the boundaries. The heat flux at the controlled boundary is the only measured signal, the uncertain diffusion and reaction parameters are admitted to be spatially varying, and the system is also affected by a sufficiently smooth boundary disturbance, which is not available for measurements and can be also unbounded in magnitude. The proposed robust synthesis is based on a dynamic input extension, and it is formed by the relay control algorithm and a linear term, suitably combined. A continuous stabilizing boundary control law is suggested to achieve exponential stability under some restrictions on the uncertain parameters spatial profiles characterstics.

In the second section the heat process is governed by an uncertain parabolic partial differential equation (PDE) with mixed boundary conditions. The process exhibits an unknown spatially varying diffusivity parameter, and is affected by a smooth uncertain boundary disturbance which is, possibly, unbounded in magnitude. The proposed robust synthesis is formed by the linear feedback design and by the "Twisting" second-order sliding-mode control algorithm, suitably combined and re-worked in the infinite-dimensional setting.

A non-standard Lyapunov functional is invoked to prove the global asymptotic stability of the resulting closed-loop systems in a suitable Sobolev space. The proofs are accompanied by a set of simple tuning rules for the controller parameters. The effectiveness of the developed controls scheme are always supported by simulation results.

7.1 Boundary Sliding-mode Control of Uncertain Reaction-Diffusion Processes

Topics in this section have been published by the author in Orlov *et al.* (2012b).

We address the stabilization problem for a class of reaction-diffusion processes with spatially varying uncertain coefficients and subject to an external boundary disturbance. Dirichlet-type boundary actuation is assumed, and the heat flux in the controlled boundary is the only required sensing. The proposed controller, whose effectiveness requires some restrictions on the spatial profiles of the uncertain parameters, provides for the global exponential stability of the system in the space $W^{1,2}(0,1)$. With respect to the closely related work Cheng *et al.* (2011), where a constant-parameters reaction-diffusion was studied, we allow the system parameters to be spatially varying and, furthermore, we achieve exponential stability in a larger Sobolev space involving spatial derivatives up to the first order In Cheng *et al.* (2011) only constant parameters were taken into account, and stability was only assured in the standard $L_2(0,1)$ space. It is worth to mention that in the above publication unstable reaction processes were dealt with by means of a backstepping transformation (see Krstic and Smyshlyaev (2008)) requiring the distributed measurement of the state variable over the entire spatial domain. In this paper we limit the scope of our investigations to control systems based on boundary control and sensing, only, which makes it impossible, or at least very challenging, to cover the unstable case. The main contribution of this paper is, thus, the asymptotic rejection of a possibly unbounded disturbance by means of a continuous control action rather than the boundary stabilization of an internally unstable reaction-diffusion process which was achieved in Cheng *et al.* (2011) under more restrictive conditions on the plant equations and assuming the availability of distributed sensing.

7.1.1 Problem statement

Consider the space- and time-varying scalar field $w(x,t)$ evolving in a Sobolev space $W^{1,2}(0,1)$ where $x \in [0,1]$ is the monodimensional spatial variable and $t \geq 0$ is the time variable. Let it be governed by the following reaction-diffusion equation with spatially-varying parameters

$$w_t(x,t) = [\theta(x)w_x(x,t)]_x - c(x)w(x,t) \tag{7.1.1}$$

where $\theta(\cdot) \in C^1(0,1)$ is the positive-definite spatially-varying diffusivity parameter and $c(\cdot) \in C^1(0,1)$ is the spatially-varying reaction parameter. The initial condition (IC) is

$$w(x,0) = w^0(x) \in W^{2,2}(0,1). \tag{7.1.2}$$

Throughout, we consider controlled and perturbed Dirichlet-type BC's of the form

$$w(0,t) = 0, \quad w(1,t) = u(t) + \psi(t), \tag{7.1.3}$$

where $u(t) \in \mathbb{R}$ is a modifiable source term (boundary control input) and $\psi(t) \in \mathbb{R}$ is an uncertain and sufficiently smooth disturbance.

The heat flux $w_x(1,t)$ at the controlled boundary is the only measured signal.

The class of initial functions and admissible disturbances is specified by the following assumption.

ASSUMPTION 7.1.1 *The initial function (7.1.2) is compatible to the next perturbed BC's*

$$w^0(0) = 0, \;\; w^0(1) = \psi(0) \tag{7.1.4}$$

whereas the disturbance $\psi(t)$ is twice continuously differentiable, and there exists a constant M such that

$$|\psi_t(t)| \le M, \quad t \ge 0 \tag{7.1.5}$$

The spatially varying parameters of the system are supposed to satisfy the next restrictions

ASSUMPTION 7.1.2 There exist constants Θ_m, Θ_{M1}, C_m, C_{M1}, Θ_x and C_x such that

$$\begin{aligned}
&0 < \Theta_m \le \theta(x), \quad \forall x \in [0,1] \\
&\theta(1) < \Theta_{M1} \\
&C_m \le c(x), \quad \forall x \in [0,1] \\
&|c(1)| \le C_{M1} \\
&|\theta_x(x)| \le \Theta_x \quad\; \forall x \in [0,1]. \\
&|c_x(x)| \le C_x \quad\; \forall x \in [0,1].
\end{aligned} \tag{7.1.6}$$

The task is that of guaranteeing the global asymptotic stability of the system trajectories in the space $W^{1,2}(0,1)$ despite the uncertainty in the system parameters and the effects of the boundary disturbance.

Note that the sign of constant C_m in (7.1.6) is unspecified. If it is negative, then the resulting system may be open loop unstable.

ASSUMPTION 7.1.3 The constants M, Θ_{M1}, and C_{M1} are a-priori known, and the following restrictions are further assumed on the system parameters:

$$\Theta_x \le 2\Theta_m \tag{7.1.7}$$

$$\Theta_m + 2C_m \geq C_x + \frac{\Theta_x + \Theta_{M1} + C_{M1}}{2} \tag{7.1.8}$$

The proposed dynamic controller is

$$u_t(t) = -\lambda_1 w_x(1,t) \; - \lambda_2 \operatorname{sign} w_x(1,t), \quad u(0) = 0 \tag{7.1.9}$$

where λ_1, λ_2, are constant tuning parameters, is currently under study, where, according to (7.1.4), the initial control value $u(0)$ is set to zero to verify the compatibility[1] $w^0(1) = u(0) + \psi(0)$ to the BC's (7.1.3) The time derivative of the control (7.1.9) is composed of a continuous linear part (the first member) and a discontinuous part. The actual plant control $u(t)$, calculated by integrating the discontinuous derivative (7.1.9), will be therefore a continuous function of time.

7.1.2 Main result

To achieve the control goal, the system state is augmented through a dynamic input extension by inserting an integrator at the plant input. The performance of the closed-loop system is analyzed in the next theorem.

THEOREM 7.1.1 Consider the reaction-diffusion equation (7.1.1) along with the initial and boundary conditions (7.1.2), (7.1.3), and with the system parameters and uncertain disturbance satisfying the Assumptions 7.1.1, 7.1.2, and 7.1.3. Then, the dynamical boundary control strategy (7.1.9) with the parameters λ_1, λ_2 selected according to the inequalities

$$\lambda_1 > \frac{\Theta_{M1} + C_{M1}}{2}, \quad \lambda_2 > M, \tag{7.1.10}$$

guarantees the global asymptotic stability of the closed-loop system in the space $W^{1,2}(0,1)$.

Proof of Theorem 7.1.1. We take the next Lyapunov function

$$V(t) = \frac{1}{2}||w(\cdot,t)||^2_{1,2} = \frac{1}{2}||w(\cdot,t)||^2_2 + \frac{1}{2}||w_x(\cdot,t)||^2_2 \tag{7.1.11}$$

whose time derivative is

$$\begin{aligned} \dot{V}(t) = &\int_0^1 w(\xi,t) w_t(\xi,t) \, d\xi \\ + &\int_0^1 w_x(\xi,t) w_{xt}(\xi,t) \, d\xi \end{aligned} \tag{7.1.12}$$

[1]See, e.g., Vazquez and Krstic (2007) for the need of certain compatibility conditions in the dynamic boundary control synthesis.

Let us evaluate the two integral terms in (7.1.12) along the solutions of (7.1.1)-(7.1.4) with the boundary control (7.1.9), (7.1.10). The first integral is manipulated as

$$\begin{aligned}&\int_0^1 w(\xi,t)w_t(\xi,t)\ d\xi\\ &=\int_0^1 w(\xi,t)\left[[\theta(\xi)w_x(\xi,t)]_x - c(\xi)w(\xi,t)\right]\ d\xi\\ &=\int_0^1 w(\xi,t)\ [\theta(\xi)w_x(\xi,t)]_x\ d\xi - \int_0^1 c(\xi)w^2(\xi,t)\ d\xi\end{aligned} \tag{7.1.13}$$

The first term in the right hand side of (7.1.13) can be integrated by parts, and further manipulated by taking account (7.1.6) and the BCs (7.1.3) as

$$\begin{aligned}&\int_0^1 w(\xi,t)\ [\theta(\xi)w_x(\xi,t)]_x\ d\xi = \theta(1)w(1,t)w_x(1,t)\\ &-\theta(0)w(0,t)w_x(0,t) - \int_0^1 \theta(\xi)w_x^2(\xi,t)\ d\xi\\ &\le \theta(1)w(1,t)w_x(1,t) - \Theta_m||w_x(\cdot,t)||_2^2\end{aligned} \tag{7.1.14}$$

Concerning the term $\int_0^1 c(\xi)w^2(\xi,t)\ d\xi$, by (7.1.6) it can be estimated as

$$-\int_0^1 c(\xi)w^2(\xi,t)\ d\xi \le -C_m||w(\cdot,t)||_2^2 \tag{7.1.15}$$

If C_m is negative then the estimation (7.1.15) implies a destabilizing effect as it adds a positive contribution into the right hand side of (7.1.12).

The second integral term in (7.1.12) can be integrated by parts and then evaluated along the solutions of (7.1.1)-(7.1.4), yielding

$$\begin{aligned}&\int_0^1 w_x(\xi,t)w_{\xi t}(\xi,t)\ d\xi = w_t(1,t)w_x(1,t)\\ &-w_t(0,t)w_x(0,t) - \int_0^1 w_t(\xi,t)w_{xx}(\xi,t)\ d\xi\\ &= w_t(1,t)w_x(1,t) - \int_0^1 w_{xx}(\xi,t)[\theta(\xi)w_x(\xi,t)]_x\ d\xi\\ &+\int_0^1 c(\xi)w(\xi,t)w_{xx}(\xi,t)\ d\xi\end{aligned} \tag{7.1.16}$$

Substituting the dynamic controller (7.1.9) into the first term of (7.1.16) we get

$$\begin{aligned}&w_t(1,t)w_x(1,t) = w_x(1,t)[u_t(t)+\psi_t(t)]\\ &= -\lambda_1 w_x^2(1,t) - \lambda_2 w_x(1,t)\text{sign } w_x(1) + \psi_t(t)w_x(1,t)\end{aligned} \tag{7.1.17}$$

Let us resolve and manipulate separately the two integral terms of (7.1.16). One obtains

$$
\begin{aligned}
&-\int_0^1 w_{xx}(\xi,t)[\theta(\xi)w_x(\xi,t)]_x \; d\xi \\
&= -\int_0^1 w_{xx}(\xi,t)[\theta_x(\xi)w_x(\xi,t)+\theta(\xi)w_{xx}(\xi,t)] \; d\xi \\
&= -\int_0^1 \theta_x(\xi)w_x(\xi,t)w_{xx}(\xi,t) \; d\xi - \int_0^1 \theta(\xi)w_{xx}^2(\xi,t) \; d\xi \\
&\leq -\int_0^1 \theta_x(\xi)w_x(\xi,t)w_{xx}(\xi,t) \; d\xi - \Theta_m||w_{xx}(\cdot,t)||_2^2
\end{aligned}
\tag{7.1.18}
$$

The sign-indefinite term $\int_0^1 \theta_x(\xi)w_x(\xi,t)w_{xx}(\xi,t) \; d\xi$ can be estimated by means of the triangle inequality as

$$
\begin{aligned}
&\left|\int_0^1 \theta_x(\xi)w_x(\xi,t)w_{xx}(\xi,t) \; d\xi\right| \\
&\leq \Theta_x \int_0^1 |w_x(\xi,t)w_{xx}(\xi,t)| \; d\xi \\
&\leq \frac{\Theta_x}{2}||w_x(\cdot,t)||_2^2 + \frac{\Theta_x}{2}||w_{xx}(\cdot,t)||_2^2
\end{aligned}
\tag{7.1.19}
$$

By considering (7.1.19) into (7.1.18) it yields

$$
\begin{aligned}
&-\int_0^1 w_{xx}(\xi,t)[\theta(\xi)w_x(\xi,t)]_x \; d\xi \\
&\leq -\left[\Theta_m - \frac{\Theta_x}{2}\right]||w_{xx}(\cdot,t)||_2^2 + \frac{\Theta_x}{2}||w_x(\cdot,t)||_2^2
\end{aligned}
\tag{7.1.20}
$$

Integrating by parts the last term in (7.1.16), and considering the BCs (7.1.3), one obtains

$$
\begin{aligned}
&\int_0^1 c(\xi)w(\xi,t)w_{xx}(\xi,t) \; d\xi \\
&= c(1)w(1,t)w_x(1,t) - \int_0^1 c_x(\xi)w(\xi,t)w_x(\xi,t) \; d\xi \\
&- \int_0^1 c(\xi)w_x^2(\xi,t) \; d\xi
\end{aligned}
\tag{7.1.21}
$$

Let us estimate the integral terms appearing in the right hand side of (7.1.21). Considering (7.1.6), and by applying the triangle inequality, the sign-indefinite integral $\int_0^1 c_x(\xi)w(\xi,t)w_x(\xi,t)\ d\xi$ can be estimated as

$$\begin{aligned}&\left|\int_0^1 c_x(\xi)w(\xi,t)w_x(\xi,t)\ d\xi\right| \le C_x\int_0^1 |w(\xi,t)w_x(\xi,t)|\ d\xi\\ &\le \frac{C_x}{2}||w(\cdot,t)||_2^2+\frac{C_x}{2}||w_x(\cdot,t)||_2^2\end{aligned} \tag{7.1.22}$$

Concerning the term $\int_0^1 c(\xi)w_x^2(\xi,t)\ d\xi$, by (7.1.6) it can be estimated as

$$-\int_0^1 c(\xi)w_x^2(\xi,t)\ d\xi \le -C_m\|w_x(\cdot,t)\|_2^2 \tag{7.1.23}$$

If C_m is negative then the estimation (7.1.23) implies a destabilizing effect as it adds a positive contribution into the right hand side of (7.1.12).

By collecting together the above derived relationships (7.1.13)-(7.1.23) it can be further manipulated (7.1.12) as

$$\begin{aligned}&\dot V(t) \le \theta(1)w(1,t)w_x(1,t) - \Theta_m||w_x(\cdot,t)||_2^2\\ &-C_m||w(\cdot,t)||_2^2-\lambda_1 w_x^2(1,t)-\lambda_2|w_x(1,t)|+\psi_t(t)w_x(1,t)\\ &-\left[\Theta_m-\frac{\Theta_x}{2}\right]||w_{xx}(\cdot,t)||_2^2+\frac{\Theta_x}{2}||w_x(\cdot,t)||_2^2\\ &+c(1)w(1,t)w_x(1,t)+\frac{C_x}{2}||w(\cdot,t)||_2^2+\frac{C_x}{2}||w_x(\cdot,t)||_2^2\\ &-C_m\|w_x(\cdot,t)\|_2^2 \le -\left[\Theta_m+C_m-\frac{\Theta_x}{2}-\frac{C_x}{2}\right]||w_x(\cdot,t)||_2^2\\ &-C_m||w(\cdot,t)||_2^2-\lambda_1 w_x^2(1,t)-(\lambda_2-M)|w_x(1,t)|\\ &-\left[\Theta_m-\frac{\Theta_x}{2}\right]||w_{xx}(\cdot,t)||_2^2+\frac{C_x}{2}||w(\cdot,t)||_2^2\\ &+(\theta(1)+c(1))w(1,t)w_x(1,t)\end{aligned} \tag{7.1.24}$$

By (7.1.6), and by applying Young's inequality, one can derive the next estimation

$$\begin{aligned}&|(\theta(1)+c(1))w(1,t)w_x(1,t)| \le \frac{\Theta_{M1}+C_{M1}}{2}w^2(1,t)\\ &+\frac{\Theta_{M1}+C_{M1}}{2}w_x^2(1,t)\end{aligned} \tag{7.1.25}$$

By (7.1.3), the next relation holds

$$w(1,t)=\int_0^1 w_x(\xi,t)\ d\xi, \tag{7.1.26}$$

Squaring both sides of (7.1.26), and successively applying simple estimations taking into account relation (A.0.3), it yields

$$\begin{aligned} w^2(1,t) &= \left[\int_0^1 w_x(\xi,t)\ d\xi\right]^2 \\ &\leq \left[\int_0^1 |w_x(\xi,t)|\ d\xi\right]^2 = ||w_x(\cdot,t)||_1^2 \leq ||w_x(\cdot,t)||_2^2, \end{aligned} \tag{7.1.27}$$

Thus by considering (7.1.27) into (7.1.25) we get

$$\begin{aligned} |(\theta(1)+c(1))w(1,t)w_x(1,t)| &\leq \frac{\Theta_{M1}+C_{M1}}{2}||w_x(\cdot,t)||_2^2 \\ &+ \frac{\Theta_{M1}+C_{M1}}{2}w_x^2(1,t) \end{aligned} \tag{7.1.28}$$

By considering (7.1.28) into (7.1.24) one further obtains

$$\begin{aligned} \dot{V}(t) \leq &- \left[\Theta_m + C_m - \frac{\Theta_x + C_x + \Theta_{M1} + C_{M1}}{2}\right] ||w_x(\cdot,t)||_2^2 \\ &- \left[\lambda_1 - \frac{\Theta_{M1}+C_{M1}}{2}\right] w_x^2(1,t) - (\lambda_2 - M)|w_x(1,t)| \\ &- \left[\Theta_m - \frac{\Theta_x}{2}\right] ||w_{xx}(\cdot,t)||_2^2 + \left[\frac{C_x}{2} - C_m\right] ||w(\cdot,t)||_2^2 \end{aligned} \tag{7.1.29}$$

The last term in (7.1.29) can be bounded by exploiting the relation

$$||w(\cdot,t)||_2^2 < ||w_\xi(\cdot,t)||_2^2 \tag{7.1.30}$$

that holds due to the BCs (7.1.3). It yields

$$\left[\frac{C_x}{2} - C_m\right] ||w(\cdot,t)||_2^2 \leq \left[\frac{C_x}{2} - C_m\right] ||w_x(\cdot,t)||_2^2 \tag{7.1.31}$$

Thus it can be manipulated (7.1.29) as

$$\begin{aligned} &\dot{V}(t) \leq \\ &- \left[\Theta_m + 2C_m - C_x - \frac{\Theta_x + \Theta_{M1} + C_{M1}}{2}\right] ||w_x(\cdot,t)||_2^2 \\ &- \left[\lambda_1 - \frac{\Theta_{M1}+C_{M1}}{2}\right] w_x^2(1,t) - (\lambda_2 - M)|w_x(1,t)| \\ &- \left[\Theta_m - \frac{\Theta_x}{2}\right] ||w_{xx}(\cdot,t)||_2^2 \end{aligned} \tag{7.1.32}$$

Due to the relations (7.1.8), (7.1.7), and the tuning conditions (7.1.10), all terms in the right hand side of (7.1.32) are negative definite. Let

$$\rho = \left[\Theta_m + 2C_m - C_x - \frac{\Theta_x + \Theta_{M1} + C_{M1}}{2} \right] \tag{7.1.33}$$

which is strictly positive due to (7.1.8). The next chain of inequalities can be derived by (7.1.32) and (7.1.30)

$$\begin{aligned} \dot{V}(t) &\leq -\rho||w_x(\cdot,t)||_2^2 = -\frac{1}{2}\rho||w_x(\cdot,t)||_2^2 - \frac{1}{2}\rho||w_x(\cdot,t)||_2^2 \\ &\leq -\frac{1}{2}\rho||w(\cdot,t)||_2^2 - \frac{1}{2}\rho||w_x(\cdot,t)||_2^2 = \rho V(t) \end{aligned} \tag{7.1.34}$$

relation (7.1.34) proves the global asymptotic stability of the closed-loop system in the space $W^{1,2}(0,1)$. The Theorem is proven, ■.

7.1.3 Simulation results

Consider the perturbed heat equation (7.1.1)-(7.1.3) with the next spatially varying diffusivity and reaction coefficient:

$$\theta(x) = 2 + 0.2\sin(0.5\pi x), \tag{7.1.35}$$

$$c(x) = -0.1\sin(\pi x), \tag{7.1.36}$$

The disturbance $\psi(t)$ is set to

$$\psi(t) = \cos(2\pi t) + t. \tag{7.1.37}$$

The initial conditions have been set to

$$w^0(x) = 2\sin(\pi x). \tag{7.1.38}$$

The magnitude of the disturbance time derivative ψ_t can be easily upper-estimated as $M = 7.5$. The constants in (7.1.2) are easily estimated as well according to:

$$\begin{aligned} &\Theta_m = 2, \quad \Theta_{M1} = 2, \quad \Theta_x = 0.4, \\ &C_m = -0.1, \quad C_{M1} = 0, \quad C_x = 0.4. \end{aligned} \tag{7.1.39}$$

The restrictions (7.1.7)-(7.1.8) are satisfied and the controller (7.1.9) has been implemented with the parameters $\lambda_1 = 1$, $\lambda_2 = 8$, which are selected in accordance with (7.1.10).

For solving the PDE, governing the closed-loop system behaviour, a standard finite-difference approximation method is used by discretizing the spatial solution domain $\xi \in [0,1]$ into a finite number of N uniformly spaced solution nodes $\xi_i = ih$, $h = 1/(N+1)$, $i = 1,2,...,N$. The boundary nodes $\xi_0 = 0$ and ξ_{N+1} are not included in the state vector of the discretized system. The value $N = 80$ has been used in the presented simulations. The resulting 80-th order discretized system is solved by fixed-step Euler method with step $T_s = 0.001s$. The Figure 7.1 shows the open loop solution $w(x,t)$ with no feedback control ($u(t)$=0). The unbounded growth of the state is due to the selected external dusturbance (7.2.76), which grows unbluded and is not compensated by the boundary feedback control which is set to zero in this first test. Figures 7.2 and 7.3 show the solution $w(x,t)$ in the "controlled" test, and the corresponding applied boundary control $u(t)$. The figure 7.2 confirms the satisfactory performance of the control system, in terms of state stabilization, while figure 7.3 shows that, as expected, the applied boundary control is a continuous signal.

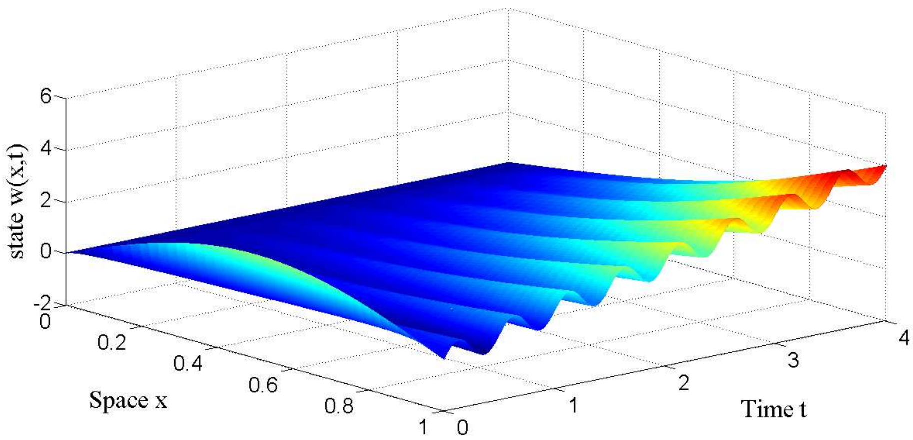

Figure 7.1: The $w(x,t)$ solution in the open-loop test ($u(t) = 0$).

Using a sliding mode control algorithm with a linear term, the problem of the boundary global asymptotic stabilization of an uncertain heat process is solved in the presence of a persistent smooth disturbance, which is generally speacking with an arbitrary shape. The proposed control law is synthesized by passing a certain discontinuous output through an integrator, it is therefore continuous, and the chattering phenomenon is thus attenuated. Along with this, the proposed infinite-dimensional treatment retains robustness features against non-vanishing matched disturbances similar to those possessed by its finite-dimensional counterpart.

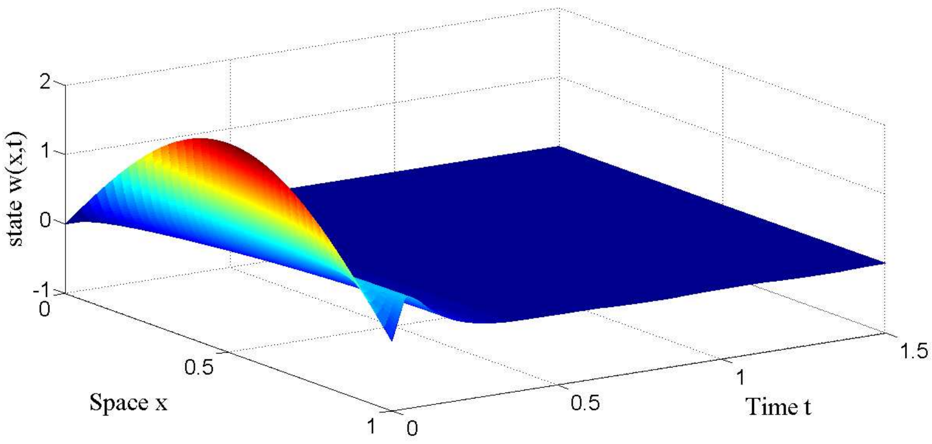

Figure 7.2: The $w(x,t)$ solution with the suggested feedback control.

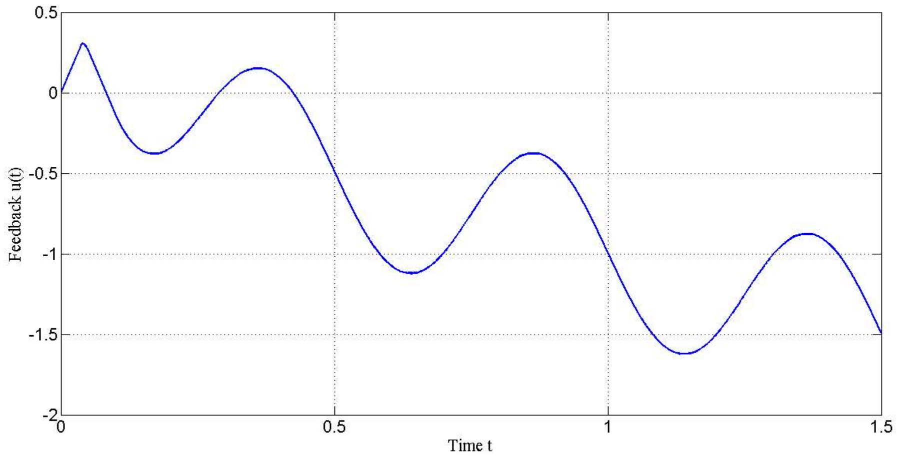

Figure 7.3: The boundary control $u(t)$.

7.2 Boundary second-order sliding-mode control

The topics of this section are based on Orlov *et al.* (2011c).

The primary concern of the section is the regulation of an uncertain heat process with collocated boundary sensing and actuation. The boundary control problem for heat processes was studied, e.g., in Boskovic *et al.* (2001); Fridman and Orlov (2009); Krstic and Smyshlyaev (2008b) under more strict assumptions on the admitted uncertainties and perturbations compared to those made in the present work. Here we address the boundary control problem for an uncertain heat process, governed by a parabolic partial differential equation (PDE) with a scalar spatial variable $\xi \in [0, 1]$ and with Robin's boundary conditions (i.e., mixed boundary conditions are admitted in contrast to that of Pisano and Orlov (2011b) where only Neumann's ones were under study). An appropriate extension of second-order sliding mode (2-SM) control techniques Fridman and Levant (1996); Orlov (2009) allows us to address the following main features:

- The diffusivity parameter is admitted to be uncertain
- Only collocated boundary sensing and actuation are assumed to be available.
- The proposed controller is simple to implement and to tune, and rejects a class of non-vanishing matched perturbations of arbitrary shape, possibly unbounded in magnitude, requiring just the knowledge of a constant upper bound to the magnitude of the disturbance time derivative.
- The plant input is continuous, whereas its first-order time derivative is discontinuous.
- The global asymptotic stability of the error system is achieved in the Sobolev space $W^{2,2}(0, 1)$.

In the closely related recent publication Chen *et al.* (1996) a similar problem has been studied by combining an integral-type first-order sliding mode controller and a backstepping transformation (see Krstic and Smyshlyaev (2008)). A similar dynamics as that considered in the present paper, with Dirichlet (instead of Robin's) BCs, has been dealt with in the above work. However, the controller tuning inequalities resulting from the presented Lyapunov analysis depend on the spatiotemporal derivatives of the solution, which are, normally, not available for feedback in practice, thereby making the result presented in Chen *et al.* (1996) of local nature. The main advances we achieve in the present work as compared to Pisano and Orlov (2011b) are listed:

- The positive diffusivity parameter is admitted to have an uncertain spatially-varying profile, whereas it was assumed constant in Pisano and Orlov (2011b)
- Mixed BCs are considered here, whereas just Neumann BCs were considered in Pisano and Orlov (2011b)
- A space varying reference is considered in the present work whereas a constant (i.e., time- and spatially-invariant) reference was taken into account in Pisano and Orlov (2011b).

In the resulting closed-loop system, the discontinuous 2-SM controller is connected to the plant input through a dynamical filter (an integrator) thereby augmenting the system state with its time derivative. While passing through the filter, the discontinuous signal is smoothed out, and the so-called chattering phenomenon, extremely undesired in practice, is thus attenuated. Due to such a dynamic input extension, the global asymptotic stabilization of the underlying uncertain heat process is achieved in a stronger norm of a Sobolev space, involving spatial state derivatives up to the second order. The stability proof is based on a non-smooth Lyapunov functional construction and it leads to a set of simple tuning rules for the controller parameters.

7.2.1 Problem formulation

Consider the space- and time-varying scalar field $Q(\xi, t)$ with the monodimensional spatial variable $\xi \in [0, 1]$ and time variable $t \geq 0$. Let it be governed by a perturbed version of the parabolic PDE which is commonly referred to as the "**Heat Equation**":

$$Q_t(\xi, t) = [\theta(\xi) Q_\xi(\xi, t)]_\xi \tag{7.2.1}$$

where Q_t and $Q_{\xi\xi}$ denote temporal and second-order spatial derivatives, respectively, and $\theta(\cdot) \in C^1(0, 1)$ is a positive-definite spatially-varying parameter called *thermal conductivity* (or, more generally, *diffusivity*). The initial condition (IC) is given by

$$Q(\xi, 0) = Q^0(\xi) \in W^{2,2}(0, 1). \tag{7.2.2}$$

Throughout, we assume controlled and perturbed Robin's (i.e., mixed) boundary conditions (BCs) of the form

$$\begin{aligned} Q_\xi(0, t) &= \alpha_0 Q(0, t) + \beta_0 &(7.2.3)\\ Q_\xi(1, t) &= -\alpha_1 Q(1, t) + \beta_1 + u(t) + \psi(t), &(7.2.4) \end{aligned}$$

where α_i and β_i ($i = 0, 1$), are proper constants such that $\alpha_0 \geq 0$ and $\alpha_1 \geq 0$, $u(t) \in \mathbb{R}$ is a modifiable source term (boundary control input) and $\psi(t) \in \mathbb{R}$ represents an **uncertain** sufficiently smooth disturbance.

We consider the time-independent and spatially varying reference $Q^r(\xi)$ which satisfies the boundary value problem

$$[\theta(\xi)Q^r_\xi(\xi)]_\xi = 0 \tag{7.2.5}$$
$$Q^r_\xi(0) = \alpha_0 Q^r(0) + \beta_0 \tag{7.2.6}$$
$$Q^r(1) = Q^r_1 \tag{7.2.7}$$

for an arbitrary, user-selectable, constant Q^r_1.

The class of admissible disturbances is specified by the following restriction on their time derivative.

ASSUMPTION 7.2.1 The disturbance $\psi(t)$ is differentiable and there exists an *a priori* known constant M such that

$$|\psi_t(t)| \leq M \tag{7.2.8}$$

for almost all $t \geq 0$.

The spatially varying diffusivity is supposed to satisfy the next restriction

ASSUMPTION 7.2.2 There exist *a priori* known constants Θ_m, Θ_M such that

$$0 < \Theta_m \leq \theta(\xi) \leq \Theta_M, \quad \forall \xi \in [0, 1]. \tag{7.2.9}$$

With the assumptions above the evolution of the considered heat process is studied in the Sobolev space $W^{2,2}(0, 1)$ and the control objective is to steer the $W^{2,2}$-norm of the deviation

$$x(\xi, t) = Q(\xi, t) - Q^r(\xi) \tag{7.2.10}$$

of the scalar field $Q(\xi, t)$ from the *a priori* given reference to zero, despite the presence of an uncertain, arbitrarily shaped, smooth boundary disturbance $\psi(t)$ fulfilling the Assumption 1. Boundary sensing at $\xi = 1$ of the deviation $x(\xi, t)$ and of its time derivative $x_t(\xi, t)$ is assumed to be the only available information on the state of the system. The deviation variable $x(\xi, t)$ is governed by the heat equation

$$x_t(\xi, t) = [\theta(\xi)x_\xi(\xi, t)]_\xi \tag{7.2.11}$$

subject to the next Robin-type BCs

$$x_\xi(0, t) - \alpha_0 x(0, t) = 0 \tag{7.2.12}$$
$$x_\xi(1, t) + \alpha_1 x(1, t) = u(t) + \psi(t) + \gamma_1, \tag{7.2.13}$$

with the constant

$$\gamma_1 = \beta_1 - Q^r_\xi(1) - \alpha_1 Q^r_1, \tag{7.2.14}$$

which can be derived by considering (7.2.10), and its spatial derivative $x_\xi(\xi,t) = Q_\xi(\xi,t) - Q^r_\xi(\xi)$, along with the conditions (7.2.4) and (7.2.7). The corresponding ICs are

$$x(\xi,0) = x^0(\xi), \qquad x^0(\xi) = Q^0(\xi) - Q^r(\xi) \tag{7.2.15}$$

It is worth noticing that the disturbance-free system (7.2.11)-(7.2.15) in open-loop is only stable, rather than asymptotically stable. Thus, the modifiable control variable $u(t)$ should be designed in order to make the zero solution $x(\xi,t) = 0$ of the closed-loop system (7.2.11)-(7.2.15) globally asymptotically stable in the $W^{2,2}$-space despite the presence of an unknown disturbance $\psi(t)$ affecting the state of the system through its boundary. Since non-homogeneous boundary conditions are in force, the meaning of the boundary-value problem (7.2.11)-(7.2.15) is subsequently viewed in the mild sense.

The mild solutions, if any, coincide with those of the following PDE in distributions

$$x_t(\xi,t) = [\theta(\xi)x_\xi(\xi,t)]_\xi + \theta(1)[u(t) + \psi(t) + \gamma_1]\delta(\xi - 1) \tag{7.2.16}$$

subject to the homogeneous Robin BCs

$$\begin{aligned} x_\xi(0,t) - \alpha_0 x(0,t) &= 0 && (7.2.17)\\ x_\xi(1,t) + \alpha_1 x(1,t) &= 0, && (7.2.18) \end{aligned}$$

and to the ICs (7.2.15). Indeed, (weak) solutions of the boundary-value problem (7.2.16)-(7.2.18) are defined by means of the corresponding Green function, yielding the same integral equation

7.2.2 Main result

To achieve the control goal, the system state is augmented through a *dynamic input extension* by inserting an integrator at the plant input. The control derivative $u_t(t)$ is then regarded as a fictitious control variable to be generated by a suitable feedback mechanism.

The following dynamic controller

$$\begin{aligned} u_t(t) &= -\lambda_1 \text{sign}\, x(1,t) - \lambda_2 \text{sign}\, x_t(1,t) - W_1 x(1,t)\\ &- W_2 x_t(1,t) \qquad u(0) = 0 \end{aligned} \tag{7.2.19}$$

is currently under study, where the initial condition $u(0)$ is set to zero for certainty. In the above relation, λ_1, λ_2, W_1 and W_2 are constant parameters subject to the inequalities

$$\lambda_2 > M, \;\; \lambda_1 > \lambda_2 + M, \;\; W_1 > \frac{1}{2}\frac{\Theta_M}{\Theta_m}, \;\; W_2 > 0, \tag{7.2.20}$$

The time derivative (7.2.19) of the control input contains a discontinuous part (the first two terms) and a continuous linear part. The discontinuous components implement the well-known "Twisting" 2-SMC algorithm Levant (1993). The combined use of the Twisting and linear feedback was suggested in Orlov (2009). The main novelty here is the application of this algorithm to regulate an infinite dimensional system from its boundary.

REMARK 7.2.1 Since the dynamic control input is governed by the ordinary differential equation (7.2.19) with discontinuous (multi-valued) right-hand side, the precise meaning of the solutions of the distributed parameter system (7.2.11)-(7.2.15), driven by the discontinuous dynamic controller (7.2.19), is then specified in the sense of Filippov (1988). Extension of the Filippov concept towards the infinite-dimensional setting may be found in Levaggi (2002); Orlov (2009). As in the finite-dimensional case, a motion along the discontinuity manifold is referred to as a sliding mode.

The proposed dynamic controller makes explicit use of $Q(1,t)$ and $Q_t(1,t)$ for feedback. Despite the state derivative is normally not permitted to use in the synthesis (as it generally induces algebraic loops) its use becomes acceptable when a dynamic input extension is performed, similar to that of the present paper where the input signal passes through an integrator. By virtue of this, the system state is augmented by Q_t being viewed as a component of the augmented state vector (Q, Q_t). We simply assume the following.

ASSUMPTION 7.2.3 The closed-loop system (7.2.16)-(7.2.19) possesses a unique mild solution $x(\cdot,t) \in W^{2,2}(0,1)$ whose time derivative $x_t(\cdot,t) \in W^{2,2}(0,1-)$[2] constitutes a (weak) solution of the distribution boundary-value problem

$$x_{tt}(\xi,t) = [\theta(\xi)x_{t\xi}(\xi,t)]_\xi + \theta(1)\{u_t[y](t) + \psi_t(t)\}\delta(\xi-1) \tag{7.2.21}$$

$$\begin{aligned} x_{t\xi}(0,t) - \alpha_0 x_t(0,t) &= 0, \\ x_{t\xi}(1,t) + \alpha_1 x_t(1,t) &= 0. \end{aligned} \tag{7.2.22}$$

with respect to $x_t(\xi,t)$, which is formally obtained by differentiating (7.2.16)-(7.2.18) in the time variable.

[2]This inclusion means (see the Notation Subsection 1.1 for the meaning of $W^{2,2}(0,1-)$) that at any time instant $x_t(\cdot,t) \in W^{2,2}(0,1-\varepsilon)$ for any $\varepsilon \in (0,1)$, i.e., x_t, along with a regular component of class $W^{2,2}(0,1)$, may contain an impulsive (Dirac) function, atomized at the boundary $\xi = 1$.

Since relation (7.2.16), coupled to Assumption 2, ensures that $x_t(\cdot,t) \in W^{l,2}(0,1-\varepsilon)$ for $l=0,1,2$ and any $\varepsilon \in (0,1)$, whereas $\|x_t(\cdot,t)\|_{W^{l,2}(0,1+\varepsilon)}$ escapes to infinity for an arbitrarily small positive ε, it becomes reasonable to define the $W^{l,2}$-norm of $x_t(\cdot,t)$ on the interval $(0,1)$ as follows

$$\|x_t\|_{l,2} = \lim_{\varepsilon\downarrow 0} \|x_t\|_{W^{l,2}(0,1-\varepsilon)},\ l=0,1,2. \tag{7.2.23}$$

Clearly, given $x_t \in W^{l,2}(0,1),\ l=0,1,2$ (that occurs if $u+\psi \equiv 0$), the above norm coincides with the standard $W^{l,2}$-norm. Behind this, relation (7.2.23) extends the $W^{l,2}$-norm concept to those distributions x_t, which are regular within the interval $(0,1)$ and singular components of which are atomized at the boundary $\xi = 1$. The above definitions of the $W^{l,2}$-norms (with $l = 0,1,2$) of the time derivative of the mild solutions in question apply throughout. The following relation

$$\|x_t\|_2 = \|[\theta(\xi)x_\xi(\xi,t)]_\xi\|_2 \tag{7.2.24}$$

being valid on the mild solutions, is particularly concluded from (7.2.16) and (7.2.23) with $l=0$.

Along with the technical lemmas of the next subsection, relation (7.2.24) will be instrumental in our further derivation. We are now in a position to state our main result.

THEOREM 7.2.1 *Consider the perturbed heat process (7.2.1)-(7.2.4) subject to the dynamic control strategy (7.2.19), (7.2.20). Let Assumptions 1 and 2 be satisfied. Then the solutions (x,x_t) of the resulting error boundary-value problem (7.2.21)-(7.2.22) are globally asymptotically stable in the space $W^{2,2}(0,1)\times L_2(0,1-)$.*

Instrumental Lemmas

We now present several technical lemmas that will be instrumental in the subsequent proof of Theorem 7.2.1.

LEMMA 7.2.1 Let $z(\xi) \in L_2(0,1)$. Then, the following inequality holds:

$$\|z(\cdot)\|_1 \le \|z(\cdot)\|_2 \tag{7.2.25}$$

Proof of Lemma 7.2.1: Given $z(\xi), h(\xi) \in L_2(0,1)$, the Cauchy-Schwartz inequality states that

$$\int_0^1 |z(\xi)h(\xi)|d\xi \le \sqrt{\int_0^1 z^2(\xi)d\xi}\sqrt{\int_0^1 h^2(\xi)d\xi}. \tag{7.2.26}$$

Setting $h(\xi) = 1$ straightforwardly specifies (7.2.26) to relation (7.2.25). Lemma 7.2.1 is proved. □

LEMMA 7.2.2 Let $z(\xi) \in W^{1,2}(0,1)$. Then, the following inequality holds:

$$\|z(\cdot)\|_2^2 \le 2(z^2(i) + \|z_\xi(\cdot)\|_2^2), \quad i = 0, 1. \tag{7.2.27}$$

Proof of Lemma 7.2.2: Given $z(\xi) \in W^{1,2}(0,1)$, it is absolutely continuous and therefore,

$$z(\xi) = z(0) + \int_0^\xi z_\xi(\eta)d\eta, \; for\ any\ \xi \in [0,1]. \tag{7.2.28}$$

which, considering Lemma 1, can be estimated as

$$\begin{aligned} |z(\xi)| &\le |z(0)| + \int_0^\xi |z_\xi(\eta)|d\eta \le |z(0)| + \int_0^1 |z_\xi(\eta)|d\eta \\ &= |z(0)| + \|z_\xi(\cdot)\|_1 \le |z(0)| + \|z_\xi(\cdot)\|_2 \end{aligned} \tag{7.2.29}$$

Now squaring both sides of (7.2.29), applying the well-known inequality $2ab < a^2+b^2$, and integrating both sides over the spatial domain $\xi \in [0,1]$, yield (7.2.27) with $i = 0$. The proof of (7.2.27) with $i = 1$ becomes identical under the change of coordinate $\zeta = 1 - \xi$. Lemma 7.2.2 is proved. □

LEMMA 7.2.3 The functional

$$\begin{aligned} \tilde{V}(x, x_t) &= \lambda_1\theta(1)|x(1,t)| + \frac{1}{2}\theta(1)W_1x^2(1,t) \\ &+ \frac{1}{2}\|x_t(\cdot,t)\|_2^2, \end{aligned} \tag{7.2.30}$$

being computed on the mild solutions (x, x_t) of the boundary-value problem (7.2.21)-(7.2.22), upper estimates the weighted $W^{2,2}(0,1) \times L_2(0,1-)$-norm of these solutions in the sense that

$$\alpha(\|x(\cdot,t)\|_{2,2,\theta}^2 + \|x_t(\cdot,t)\|_2^2) \le \tilde{V}(x, x_t) \tag{7.2.31}$$

for at an arbitrary time instant $t \ge 0$ and some positive constant α

Proof of Lemma 7.2.3: Successively applying relation (7.2.27) with $i = 1$ to a mild solution $z = x(\xi, t)$ and then to the term $z = \theta(\xi)x_\xi(\xi,t)$ yields

$$\|x(\cdot,t)\|_2^2 \le 2(x^2(1,t) + \|x_\xi(\cdot,t)\|_2^2), \tag{7.2.32}$$

$$\|\theta(\cdot)x_\xi(\cdot,t)\|_2^2 \le 2(\theta^2(1)x_\xi^2(1,t) + \|[\theta(\cdot)x_\xi(\cdot,t)]_\xi\|_2^2). \tag{7.2.33}$$

By exploiting the next relation

$$\Theta_m^2\|x_\xi(\cdot,t)\|_2^2 \le \|\theta(\cdot)x_\xi(\cdot,t)\|_2^2 \le \Theta_M^2\|x_\xi(\cdot,t)\|_2^2, \tag{7.2.34}$$

which is a trivial consequence of (7.2.9), it can be further manipulated (7.2.33) so as to obtain

$$\begin{aligned}\|x_\xi(\cdot,t)\|_2^2 &\le \frac{2}{\Theta_m^2}(\Theta_M^2 x_\xi^2(1,t) + \|[\theta(\cdot)x_\xi(\cdot,t)]_\xi\|_2^2) = \\ &= \rho_1 x_\xi^2(1,t) + \rho_2\|[\theta(\cdot)x_\xi(\cdot,t)]_\xi\|_2^2 \end{aligned}\tag{7.2.35}$$

with the positive constants ρ_1 and ρ_2 beng implicitly defined. By taking into account the BC (7.2.18), the above relations (7.2.32) and (7.2.35) can be rewritten in the form

$$\begin{aligned}\|x(\cdot,t)\|_2^2 \le 2x^2(1,t) + 2\rho_1\alpha_1^2 x^2(1,t) \\ +2\rho_2\|[\theta(\cdot)x_\xi(\cdot,t)]_\xi\|_2^2 \end{aligned}\tag{7.2.36}$$

$$\|x_\xi(\cdot,t)\|_2^2 \le \rho_1\alpha_1^2 x^2(1,t) + \rho_2\|[\theta(\cdot)x_\xi(\cdot,t)]_\xi\|_2^2. \tag{7.2.37}$$

Employing relation (7.2.24), it follows from (7.2.36)-(7.2.37) that

$$\begin{aligned}\|x(\cdot,t)\|_{2,2,\theta}^2 &= \|x(\cdot,t)\|_2^2 + \|x_\xi(\cdot,t)\|_2^2 + \|[\theta(\cdot)x_\xi(\cdot,t)]_\xi\|_2^2 \\ &\le (2+3\rho_1\alpha_1^2)x^2(1,t) + (3\rho_2+1)\|[\theta(\cdot)x_\xi(\cdot,t)]_\xi\|_2^2 \\ &= (2+3\rho_1\alpha_1^2)x^2(1,t) + (3\rho_2+1)\|x_t(\cdot,t)\|_2^2\end{aligned}$$

and taking into account (7.2.30), the validity of (7.2.31) is thus concluded for all $t \ge 0$ and for some positive α. Lemma 7.2.3 is proved. □

LEMMA 7.2.4 Let a set

$$\begin{aligned}\mathcal{D}_R^{\tilde{V}} = \{&(z(\xi),h(\xi)) \in W^{2,2}(0,1)\times L_2(0,1-) : \\ &\tilde{V}(z,h) \le R\}\end{aligned}\tag{7.2.38}$$

be determined by means of functional (7.2.30) and be specified with some positive R. Then the following conditions

$$\int_0^1 z(1)h(\xi)\,d\xi \ge -\frac{1}{2}\left[\frac{R}{\lambda_1\Theta_m}|z(1)| + \|h\|_2^2\right] \tag{7.2.39}$$

$$\|h\|_2^2 \le 2R, \quad \|h\|_2 \le \sqrt{2R}, \quad \|h\|_2^2 \le \sqrt{2R}\|h\|_2 \tag{7.2.40}$$

hold for an arbitrary $(z(\xi),h(\xi)) \in \mathcal{D}_R^{\tilde{V}}$.

Proof of Lemma 7.2.4: The following implications hold in light of the inequalities (7.2.9):

$$\begin{aligned}\tilde{V}(z,h) &= \theta(1)\lambda_1|z(1)| + \frac{1}{2}\theta(1)W_1 z^2(1) + \frac{1}{2}\|h\|_2^2 \leq R \\ &\Rightarrow \theta(1)\lambda_1|z(1)| \leq R \\ &\Rightarrow \quad |z(1)| \leq \frac{R}{\lambda_1\theta(1)} \leq \frac{R}{\lambda_1\Theta_m} \end{aligned} \tag{7.2.41}$$

Furthermore, applying the well-known inequality $ab \geq -\frac{1}{2}(a^2+b^2)$ yields:

$$\begin{aligned}&\int_0^1 z(1)h(\xi)\, d\xi \geq -\frac{1}{2}(z^2(1) + ||h||_2^2) \\ &= -\frac{1}{2}(|z(1)||z(1)| + ||h||_2^2). \end{aligned} \tag{7.2.42}$$

Being coupled together, (7.2.41) and (7.2.42) immediately result in (7.2.39). In turn, the relations (7.2.40) follow from the trivial chain of implications (that consider the positive definiteness of $\theta(1)$):

$$\begin{aligned}\tilde{V}(z,h) &= \theta(1)\lambda_1|z(1)| + \frac{1}{2}\theta(1)W_1 z^2(1) + \frac{1}{2}\|h\|_2^2 \leq R \\ &\Rightarrow \quad \frac{1}{2}\|h\|_2^2 \leq R \;\Rightarrow\; \|h\|_2 \leq \sqrt{2R} \;\Rightarrow \\ &\Rightarrow \quad \|h\|_2^2 \leq \sqrt{2R}\|h\|_2. \end{aligned} \tag{7.2.43}$$

Lemma 7.2.4 is thus proved. □

proof of Theorem 7.2.1

By Lemma 7.2.3, functional (7.2.30) is positive definite along the mild solutions (x, x_t) of the boundary-value problem (7.2.21)-(7.2.22). The time derivative of (7.2.30) along such solutions is

$$\begin{aligned}\dot{\tilde{V}}(t) &= \lambda_1\theta(1)x_t(1,t)sign(x(1,t)) \\ &+ W_1\theta(1)x(1,t)x_t(1,t) + \int_0^1 x_t x_{tt} d\xi \\ &= \lambda_1\theta(1)x_t(1,t)sign(x(1,t)) + W_1\theta(1)x(1,t)x_t(1,t) \\ &+ \int_0^1 x_t\left([\theta(\xi)x_{t\xi}(\xi,t)]_\xi + \theta(1)[u_t(t)+\psi_t(t)]\delta(x-1)\right) d\xi \\ &= \lambda_1\theta(1)x_t(1,t)sign(x(1,t)) + W_1\theta(1)x(1,t)x_t(1,t) \\ &+ \int_0^1 x_t[\theta(\xi)x_{t\xi}(\xi,t)]_\xi d\xi + \theta(1)x_t(1,t)[u_t(t)+\psi_t(t)]. \end{aligned} \tag{7.2.44}$$

The integral term in the right hand side of (7.2.44), being integrated by parts by taking into account the homogeneous BC's (7.2.22), yields

$$\int_0^1 x_t[\theta(\xi)x_{t\xi}(\xi,t)]_\xi d\xi = \theta(1)x_t(1,t)x_{t\xi}(1,t)$$
$$-\theta(0)x_t(0,t)x_{t\xi}(0,t) - \int_0^1 \theta(\xi)x_{t\xi}^2 d\xi =$$
$$-\theta(1)\alpha_1 x_t^2(1,t) - \theta(0)\alpha_0 x_t^2(0,t) - \int_0^1 \theta(\xi)x_{t\xi}^2 d\xi \tag{7.2.45}$$

By substituting (7.2.19) into the last term of (7.2.44) one obtains

$$\begin{aligned} \theta(1)x_t(1,t)[u_t(t)+\psi_t(t)] = \theta(1)x_t(1,t)u_t(t) \\ +\theta(1)x_t(1,t)\psi_t(t) = -\theta(1)\lambda_1\, x_t(1,t)\text{sign}\, x(1,t) \\ -\theta(1)\lambda_2 x_t(1,t)\text{sign}\, x_t(1,t) - \theta(1)W_1 x_t(1,t)x(1,t) \\ -\theta(1)W_2 x_t^2(1,t) + \theta(1)x_t(1,t)\psi_t(t) \end{aligned}$$

and the next simplification

$$\begin{aligned} \dot{\tilde{V}}(t) &= -\lambda_2\theta(1)|x_t(1,t)| - \theta(1)(W_2+\alpha_1)x_t^2(1,t) \\ &- \int_0^1 \theta(\xi)x_{t\xi}^2 d\xi - \theta(0)\alpha_0 x_t^2(0,t) \\ &+ \theta(1)x_t(1,t)\psi_t(t) \end{aligned} \tag{7.2.46}$$

of the time derivative (7.2.44) of the Lyapunov functional (7.2.30) is then obtained. Due to the upper bound (7.2.8) on the time derivative of the boundary disturbance, one obtains

$$|\theta(1)x_t(1,t)\psi_t(t)| \le \theta(1)M|x_t(1,t)|, \tag{7.2.47}$$

By (7.2.47), and considering as well the inequality (7.2.9), relation (7.2.46) is further manipulated to

$$\begin{aligned} \dot{\tilde{V}}(t) &\le -\Theta_m(\lambda_2 - M)|x_t(1,t)| - \Theta_m(W_2+\alpha_1)x_t^2(1,t) \\ &- \Theta_m\alpha_0 x_t^2(0,t) - \Theta_m\|x_{t\xi}\|_2^2. \end{aligned} \tag{7.2.48}$$

Due to (7.2.20) and (7.2.48), the Lyapunov functional $\tilde{V}(t)$, being computed along the mild solutions of the closed-loop system, is a non-increasing function of time:

$$\tilde{V}(t_2) \le \tilde{V}(t_1) \quad \forall t_2 \ge t_1 \ge 0. \tag{7.2.49}$$

Clearly, by virtue of (7.2.49), the domain $\mathcal{D}_R^{\tilde{V}}$, given by (7.2.38) with an arbitrary $R \geq \tilde{V}(0)$, is **invariant** for the system trajectories. Thus, the subsequent analysis will take into account that the mild solutions (x, x_t) stay in the domain $\mathcal{D}_R^{\tilde{V}}$ forever.

Now consider the *augmented* functional

$$\begin{aligned}\tilde{V}_R(t) &= \tilde{V}(t) + \frac{1}{2}\kappa_R\theta(1)(W_2+\alpha_1)x^2(1,t) \\ &+ \kappa_R \int_0^1 x(1,t)x_t(\xi,t)\, d\xi \end{aligned} \qquad (7.2.50)$$

where κ_R is a sufficiently small positive constant to subsequently be specified. Note that the integral term in the right-hand side of (7.2.50) is sign-indefinite, and therefore, the positive-definiteness of the Lyapunov functional (7.2.50) has to be analyzed.

By Lemma 7.2.4 specified with $z = x$ and $h = x_t$, and considering the inequality (7.2.9), in the domain $\mathcal{D}_R^{\tilde{V}}$ function $\tilde{V}_R$ can be lower estimated as

$$\begin{aligned} \tilde{V}_R(x,x_t) &\geq \lambda_1\Theta_m|x(1,t)| \\ &+\frac{1}{2}\Theta_m[W_1+\kappa_R(W_2+\alpha_1)]x^2(1,t) \\ &+\frac{1}{2}\|x_t\|_2^2 - \frac{\kappa_R}{2}\left[\frac{R}{\lambda_1\Theta_m}|x(1,t)| + \|x_t\|_2^2\right] \\ &= \left(\lambda_1\Theta_m - \frac{\kappa_R R}{2\lambda_1\Theta_m}\right)|x(1,t)| + \frac{1}{2}(1-\kappa_R)\|x_t\|_2^2 \\ &+\frac{1}{2}\Theta_m(W_1+\kappa_R(W_2+\alpha_1))x^2(1,t) \end{aligned} \qquad (7.2.51)$$

Let us specify $\kappa_R > 0$ such that

$$\kappa_R < \mathtt{min} \ \left\{\frac{2\lambda_1^2\Theta_m^2}{R}, 1\right\}. \qquad (7.2.52)$$

Then, it follows from (7.2.51), (7.2.52) that the augmented functional (7.2.50) is lower estimated by functional (7.2.30) as

$$\tilde{V}_R \quad (x, x_t) \geq \mu\tilde{V}(x, x_t) \qquad (7.2.53)$$

provided that

$$\mu = \mathtt{min} \ \left\{1 - \frac{\kappa_R R}{2\lambda_1^2\Theta_m^2}, \frac{W_1+\kappa_R(W_2+\alpha_1)}{W_1}, (1-\kappa_R)\right\} \qquad (7.2.54)$$

It means that along with (7.2.30), the functional $\tilde{V}_R$ is positive definite on the mild solutions (x, x_t) of the boundary-value problem (7.2.21)-(7.2.22) within the invariant set $\mathcal{D}_R^{\tilde{V}}$.

Let us now evaluate the time derivative of $\tilde{V}_R(t)$:

$$\dot{\tilde{V}}_R = \dot{\tilde{V}} + \kappa_R\theta(1)(W_2 + \alpha_1)x(1,t)x_t(1,t)$$
$$+\kappa_R \int_0^1 x_t(1,t)x_t(\xi,t)d\xi + \kappa_R \int_0^1 x(1,t)x_{tt}(\xi,t)\xi. \qquad (7.2.55)$$

By utilizing the first inequality of (7.2.40) specified with $h = x_t$ and applying Lemma 7.2.1, the magnitude of the first integral term in the right hand side of (7.2.55) is upper-estimated by

$$\left|\kappa_R \int_0^1 x_t(1,t)x_t(\xi,t)d\xi\right| \leq \kappa_R|x_t(1,t)| \int_0^1 |x_t(\xi,t)|d\xi$$
$$\leq \kappa_R|x_t(1,t)|\|x_t\|_2 \leq \sqrt{2R}\kappa_R|x_t(1,t)|. \qquad (7.2.56)$$

By straightforward integration one finds that the last integral term in (7.2.55) can be manipulated as follows

$$\kappa_R x(1,t) \int_0^1 x_{tt}(\xi,t)d\xi$$
$$= \kappa_R x(1,t)$$
$$\times \int_0^1 \left([\theta(\xi)x_{t\xi}(\xi,t)]_\xi + \theta(1)[u_t(t) + \psi_t(t)]\delta(x-1)\right) d\xi$$
$$= \kappa_R x(1,t)[\theta(1)x_{t\xi}(1,t) - \theta(0)x_{t\xi}(0,t)]$$
$$+\kappa_R x(1,t)\theta(1)(u_t(t) + \psi_t(t)) \qquad (7.2.57)$$

Considering the BCs (7.2.22), the terms in the right hand side of (7.2.57) can be further elaborated as

$$\kappa_R x(1,t)[\theta(1)x_{t\xi}(1,t) - \theta(0)x_{t\xi}(0,t)] =$$
$$-\kappa_R\theta(1)\alpha_1 x(1,t)x_t(1,t) - \kappa_R\theta(0)\alpha_0 x(1,t)x_t(0,t) \qquad (7.2.58)$$

$$\kappa_R\theta(1)x(1,t)(u_t(t) + \psi_t(t)) = -\kappa_R\theta(1)\lambda_1|x(1,t)|$$
$$-\kappa_R\theta(1)\lambda_2 x(1,t)\text{sign } x_t(1,t) - \kappa_R\theta(1)W_1x^2(1,t)$$
$$-\kappa_R\theta(1)W_2x(1,t)x_t(1,t) + \kappa_R\theta(1)x(1,t)\psi_t(t). \qquad (7.2.59)$$

The next relation follows by applying the Young's inequality

$$|\kappa_R\theta(0)\alpha_0 x(1,t)x_t(0,t)| \leq \kappa_R\theta(0)\left(\frac{1}{2}x^2(1,t) + \frac{1}{2}\alpha_0^2 x_t^2(0,t)\right),$$

and the following estimates

$$|\kappa_R\theta(1)\lambda_2 x(1,t)\text{sign } x_t(1,t)| \le \kappa_R\theta(1)\lambda_2|x(1,t)| \tag{7.2.60}$$

$$|\kappa_R\theta(1)x(1,t)\psi_t(t)| \le \kappa_R\theta(1)M|x(1,t)| \tag{7.2.61}$$

hold for the corresponding terms in (7.2.59) by virtue of Assumption 1. Employing (7.2.46)-(7.2.48), (7.2.56)-(7.2.61), and the inequality (7.2.9), the time derivative (7.2.55) is finally manipulated to

$$\begin{aligned}\dot{V}_R(t) &\le -\Theta_m\left(\lambda_2 - M - \frac{\kappa_R\sqrt{2R}}{\Theta_m}\right)|x_t(1,t)| \\ &- \Theta_m(W_2+\alpha_1)x_t^2(1,t) \\ &- \frac{1}{2}\Theta_m\alpha_0\left(2-\kappa_R\alpha_0\right)x_t^2(0,t) \\ &- \Theta_m\|x_{t\xi}\|_2^2 - \kappa_R\Theta_m[(\lambda_1-\lambda_2)-M]|x(1,t)| \\ &- \kappa_R\left(W_1\Theta_m - \frac{1}{2}\Theta_M\right)x^2(1,t).\end{aligned} \tag{7.2.62}$$

It is clear that all the terms appearing in the right-hand side of (7.2.62) are non-positive provided that the tuning condition (7.2.20), imposed on the controller parameters, hold and, in place of (7.2.52), the next more restrictive condition on the coefficient κ_R is additionally satisfied:

$$\kappa_R < \min\left\{\frac{2\lambda_1^2\Theta_m^2}{R}, 1, \frac{\Theta_m(\lambda_2-M)}{\sqrt{2R}}, \frac{2}{\alpha_0}\right\}. \tag{7.2.63}$$

By Lemma 7.2.2, specialized with $z = x$, the mild solutions $x(\xi,t) \in W^{2,2}(0,1)$ satisfy the estimate (7.2.27). Moreover, its spatial and temporal derivatives $x_\xi(\xi,t) \in W^{1,2}(0,1)$ and $x_t(\xi,t) \in W^{1,2}(0,1)$ satisfy the next estimates

$$\|z_\xi(\cdot,t)\|_2^2 \le 2(z_\xi^2(i,t) + \|z_{\xi\xi}(\cdot,t)\|_2^2) \tag{7.2.64}$$

$$\|z_t(\cdot,t)\|_2^2 \le 2(z_t^2(i,t) + \|z_{t\xi}(\cdot,t)\|_2^2) \tag{7.2.65}$$

for $i = 0, 1$ and for almost all $t \ge 0$, which result from (7.2.27) by substituting $x_\xi(\xi,t)$ and $x_t(\xi,t)$ for $z(\xi,t)$, respectively. By (7.2.65) with $i = 1$ it yields

$$x_t^2(1,t) + \|x_{t\xi}\|_2^2 \ge \frac{1}{2}\|x_t\|_2^2 \tag{7.2.66}$$

In light of the above, the next estimate can be made

$$-\Theta_m(W_2+\alpha_1)x_t^2(1,t) - \Theta_m\|x_{t\xi}\|_2^2 \le -\Theta_m\gamma_1\|x_t\|_2^2 \tag{7.2.67}$$

$$\gamma_1 = \frac{1}{2}\min\{W_2 + \alpha_1, 1\}. \tag{7.2.68}$$

Relation (7.2.62) can further be manipulated to

$$\begin{aligned}
\dot{\tilde{V}}_R(t) &\le -\Theta_m\left(\lambda_2 - M - \frac{\kappa_R\sqrt{2R}}{\Theta_m}\right)|x_t(1,t)| \\
&- \Theta_m\gamma_1\|x_t\|_2^2 - \frac{1}{2}\Theta_m\alpha_0\left(2-\kappa_R\alpha_0\right)x_t^2(0,t) \\
&- \kappa_R\Theta_m[(\lambda_1-\lambda_2)-M]|x(1,t)| \\
&- \kappa_R\left(W_1\Theta_m - \frac{1}{2}\Theta_M\right)x^2(1,t) \\
&\le -\gamma_2(|x(1,t)| + x^2(1,t) + \|x_t\|_2^2)
\end{aligned} \tag{7.2.69}$$

$$\gamma_2 = \Theta_m \min\{\kappa_R[(\lambda_1-\lambda_2)-M], \kappa_R\left(W_1 - \frac{1}{2}\frac{\Theta_M}{\Theta_m}\right), \gamma_1\}. \tag{7.2.70}$$

On the other hand, (7.2.51) is readily estimated as

$$\tilde{V}_R(t) \ \ge\ \gamma_3(|x(1,t)| + x^2(1,t) + \|x_t\|_2^2) \tag{7.2.71}$$

with positive

$$\gamma_3 = \min\left\{\left(\lambda_1\Theta_m - \frac{\kappa_R R}{2\lambda_1\Theta_m}\right), \frac{1}{2}\Theta_m(W_1 + \kappa_R(W_2+\alpha_1)) , \frac{1}{2}(1-\kappa_R)\right\}. \tag{7.2.72}$$

Relations (7.2.69) and (7.2.71), coupled together, result in

$$\dot{\tilde{V}}_R(t) \ \le\ -\frac{\gamma_2}{\gamma_3}\tilde{V}_R(t) \tag{7.2.73}$$

that establishes the exponential convergence of $\tilde{V}_R(t)$, initialized within (7.2.38), to zero as $t \to \infty$.

To complete the proof it remains to note that due to the upper estimate (7.2.53) of the functional $\tilde{V}(t)$ by the functional $\tilde{V}_R(t)$, it follows that $\tilde{V}(t)$, being computed on the mild solutions (x, x_t) of the boundary-value problem (7.2.21)-(7.2.22), converges to zero, too:

$$\tilde{V}(t) \to 0 \quad \text{as} \quad t \to \infty,$$

and by virtue of Lemma 7.2.3, the local asymptotic stability of (7.2.21)-(7.2.22) with the augmented state (x, x_t) in the $W^{2,2}(0,1) \times L_2(0,1-)$-space is established with the initial set (7.2.38). Since the initial set (7.2.38) can be specified with an arbitrarily large $R > 0$ the **global** asymptotic stability in the $W^{2,2}(0,1) \times L_2(0,1-)$-space is then concluded. Theorem 7.2.1 is thus proved. □

7.2.3 Simulations results

Consider the perturbed heat equation (7.2.1) with constant diffusivity $\theta = 1$. The parameters of the uncontrolled Robin's BC (7.2.3) are set as $\alpha_0 = 1$ and $\beta_0 = -5$.

The boundary value problem (7.2.5)-(7.2.7) specialized for a constant diffusivity has a solution which linearly depends on the spatial variable

$$Q^r(\xi) = Q_0^r + \xi(Q_1^r - Q_0^r), \tag{7.2.74}$$

where the reference boundary value $Q^r(1) = Q_1^r$ is arbitrarily selected as $Q_1^r = 15$, and the resulting value for Q_0^r derives from the other parameters according to

$$Q_0^r = \frac{Q_1^r - \beta_0}{1 + \alpha_0} = 10 \tag{7.2.75}$$

which is obtained by imposing the BC (7.2.3) on the solution (7.2.74). β_1 is arbitrarily set to the value $\beta_1 = 1$. The disturbance $\psi(t)$ is set to

$$\psi(t) = 4\mathtt{cos}(0.5\pi t). \tag{7.2.76}$$

The magnitude of the disturbance time derivative ψ_t can be easily upper-estimated as $M = 6.5$, as required by (7.2.8). The initial conditions have been set to $Q^0(\xi) = 3 + 2\mathtt{sin}(4\pi\xi)$.

Controller (7.2.19) has been implemented with the parameters $\lambda_1 = 15$, $\lambda_2 = 7$, $W_1 = W_2 = 1$ which are selected in accordance with (7.2.20).

For solving the PDE, governing the closed-loop system behaviour, a standard finite-difference approximation method is used by discretizing the spatial solution domain $\xi \in [0, 1]$ into a finite number of N uniformly spaced solution nodes $\xi_i = ih$, $h = 1/(N+1)$, $i = 1, 2, ..., N$. The boundary nodes $\xi_0 = 0$ and ξ_{N+1} are not included in the state vector of the discretized system. The value $N = 40$ has been used in the first simulations. The resulting 40-th order discretized system is solved by fixed-step Euler method with step $T_s = 0.001s$. The Figures 7.4 and 7.5 show the solution $Q(\xi, t)$ and the applied boundary control $u(t)$. It can be seen that the solution converges to the linear reference (7.2.74)along the entire solution domain, and that the applied boundary control is a continuous function.

Using a dynamic version of a second-order sliding mode control algorithm, the problem of the boundary global asymptotic stabilization of an uncertain heat process is solved in the presence of a persistent smooth disturbance, which is generally speaking unbounded and with an arbitrary shape. The proposed control law is synthesized by passing a certain discontinuous output through an integrator, it is therefore continuous, and the chattering phenomenon is thus attenuated. Along with this, the proposed infinite-dimensional treatment retains robustness features against non-vanishing matched disturbances similar to those possessed

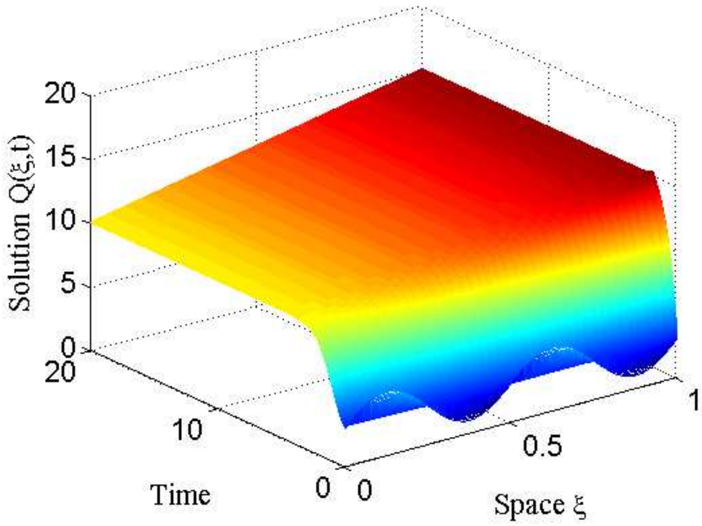

Figure 7.4: The solution $Q(\xi, t)$.

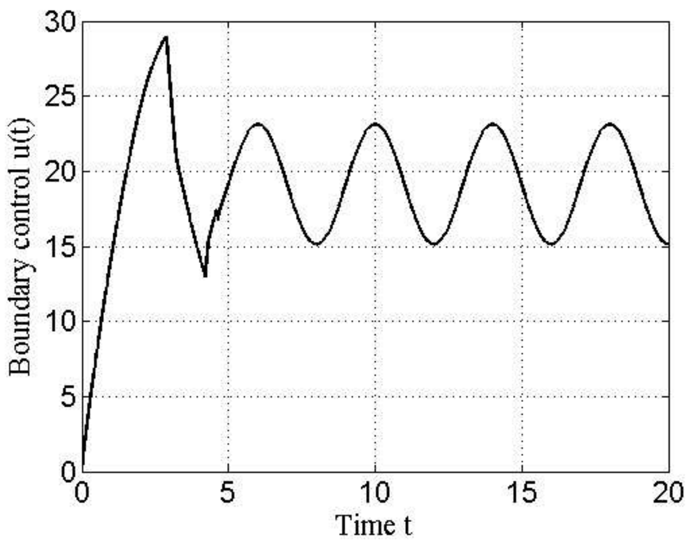

Figure 7.5: The boundary control $u(, t)$

by its finite-dimensional counterpart. Finite-time convergence of the proposed algorithm, which would be the case if confined to a finite dimensional treatment, cannot be proved using the proposed Lyapunov functional, and it remains among other actual problems to be tackled in the future within the present framework.

Chapter 8

Conclusions

In this Thesis has been addressed the problem of controlling and Observing some classes of distributed parameter (PDE) systems under the effect of external unknown disturbances. Sliding-mode control algorithms, and in particular the *Super–Twisting* algorithm, are used to achieve the control goals.

Numerical simulations show the applicability of the suggested approaches to the considered classes of PDE, Matlab and simulinc are used as the calculation tools.

As regards the Observation achievements an Unknown Input Observer and its estimator for diffusion equation are proposed. The observer/estimator design is carried out by making reference to a finite dimensional modal decomposition of the solution, and Point-wise measurements are considered to observe the system. A combined state and output transformation is applied to the resulting finite dimensional approximation, yielding a special form for the transformed system that allows the implementation of a linear observer for reconstructing the system state and a sliding mode observer for reconstructing the unknown input. The property of Strong observability depend on the number of measurements and their location on the spatial domain, but computational problems for high order system ($N > 10$) make the Strong Observability difficult to check. Future developments could find better approaches to check this property, for example using the system matrix 4.1.2. Another future objective, very challenging, is to find an Unknown input and estimator for PDEs without reducing the equation to a finite dimensional system.

As regards the Distributed control of PDE the *Super-Twisting* and the *Twisting* 2-SMC algorithms have been used in conjunction with linear PI and PD controllers. The two resulting schemes have been applied to solve the tracking control problems for heat and wave processes subject to persistent disturbances of arbitrary shapes and with spatially varying uncertain plant parameters. In the end the problem of Boundary global asymptotic stabilization is addressed, where a dynamical sliding mode control algorithm with a linear term and a dynamic version of a second-order sliding mode control algorithm the stabilization are used to achieve the stabilization goal in the presence of a persistent smooth disturbance. The proposed control laws are synthesized by passing a certain discontinuous output through an integrator, it is therefore continuous, and the chattering phenomenon is thus attenuated.

For distributed and boundary control algorithms the stability of the resulting error dynamics are proven by means of appropriate ad-hoc Lyapunov functionals, in appropriate Sobolev spaces. Along with this, the proposed infinite-dimensional treatments retain robustness features against non-vanishing matching disturbances similar to those possessed by their finite-dimensional counterparts. Finite-time convergence of the proposed algorithms, which would be the case if confined to a finite dimensional treatment, cannot be proved using the proposed Lyapunov functionals, and it remains among other actual problems to be tackled in the future within the present framework. Future development could be the integration of this algorithms with backstepping methodology Krstic and Smyshlyaev (2008) both for control and for observation problems. Future research could be deal with the application of the proposed algorithms to more complicated PDEs, such as Navier Stokes, Maxwell or Burgers equations.

Appendix A

Notation

The notation used throughout is fairly standard. $L_2(0,1)$ stands for the Hilbert space of square integrable functions $z(\zeta),\ \zeta \in (0,1)$, whose L_2-norm is given by

$$\|z(\cdot)\|_2 = \sqrt{\int_0^1 z^2(\zeta)d\zeta}. \tag{A.0.1}$$

Define

$$\|z(\cdot)\|_1 = \sqrt{\int_0^1 |z(\zeta)|d\zeta}. \tag{A.0.2}$$

and note that the next well known relation holds for all $z \in L_2(0,1)$

$$\|z(\cdot)\|_1 \leq \|z(\cdot)\|_2 \tag{A.0.3}$$

$W^{l,2}(a,b)$ denotes the Sobolev space of absolutely continuous scalar functions $z(\zeta),\ \zeta \in [a,b]$ with square integrable derivatives $z^{(i)}(\zeta)$ up to the order $l \geq 1$.

In particular $W^{0,2}(0,1)$ denotes the Hilbert space $L_2(0,1)$. $W^{1,2}(0,1)$ denotes the Sobolev space of absolutely continuous scalar functions $z(\zeta)$ on $(0,1)$ with square integrable derivative $z_\zeta(\zeta)$ and the norm

$$\|z(\cdot)\|_{1,2} = \sqrt{\|z(\cdot)\|_2^2 + \|z_\zeta(\cdot)\|_2^2} \tag{A.0.4}$$

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