Università degli studi di Cagliari

Dottorato di Ricerca

Economia e Gestione Aziendale

Ciclo XXIII

A quantitative model for the asset liability management of a Pension Fund

Settori scientifico disciplinari di afferenza

SECS-S/06

Tutor
Prof. Marco Micocci

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Presentata da
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"Education's purpose is to replace an empty mind with an open one"

Malcom Forbes
US art collector, author, publisher (1919 - 1990)

Dedicated to my family

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Abstract

The key objective of pension plans is the delivery of retirement benefits, typically payable for life or a set period of time, to the specified group of recipients. The management of such funds entails therefore a constant monitoring of the risks exposure and a regular rebalancing of assets.
This thesis is directly related to these topics and proposes a quantitative method (mainly based on stochastic optimal control theory) to determine the optimal investment policy of a pension fund's wealth, under financial and actuarial risks.
The thesis unfolds as follows: Chapter 1 includes a basic introduction to pension systems. The topics addressed here are: how to classify pension systems, the main characteristics of each kind of system, examples of major systems and the important reforms that have been implemented in Italy; the chapter ends with a description of Italian professional order pension funds. Chapter 2 describes asset liability management techniques in pension schemes; it contains a review on major literature on asset liability management and a discussion on interested parties in this topic and on policies and instruments which can be adopted. Chapter 3 contains an original model to determine the optimal financial investment policy in a pension fund, considering both financial and actuarial risk. Moreover, the model takes care of the pension plan's sustainability, i.e. of the balance between the active and retired members. Chapter 4 is a numerical application of the model described in the previous chapter to a real Italian pension fund. Finally, in Chapter 5 conclusions are drawn related to the question asked.

Riassunto

Il principale obiettivo di un qualsiasi piano pensionistico è garantire al lavoratore il mantenimento del tenore di vita nel periodo successivo alla cessazione dell'attività lavorativa. Questo obiettivo viene raggiunto tramite l'erogazione di rendite pensionistiche, che tipicamente hanno una durata aleatoria coincidente con la vita residua del pensionato. La gestione di questo tipo di fondi, ed in particolare la gestione finanziaria, richiede un continuo monitoraggio dell'esposizione del patrimonio gestito a una serie di rischi finanziari e attuariali, nonchè un regolare ribilanciamento degli asset in cui si investe.
Il presente lavoro è direttamente collegato a queste problematiche e propone un metodo quantitativo (principalmente basato sulla teoria del controllo ottimo stocastico come strumento per determinare l'asset allocation dinamica ottimale) per stabilire la strategia di investimento migliore, tenedo conto dei rischi finanziari e attuariali in cui incorre il fondo pensione.
La tesi è stata strutturata nel seguente modo. Il Capitolo 1 inquadra l'oggetto della tesi nel più ampio spazio del sistema pensionistico; in particolare si discute della classificazione delle varie forme previdenziali e delle loro peculiari caratteristiche e ci si sofferma in modo specifico sul sistema italiano e sul suo iter di riforme. Il capitolo 2 descrive le tecniche di asset liability management nei fondi pensione; è inclusa una disamina della più importante letteratura a riguardo e un'analisi dei soggetti interessati alle scelte di asset liability management nonchè degli strumenti coi quali tale tecnica di risk management può essere resa effettiva. Il capitolo 3 presenta il modello originale proposto per determinare la strategia di investimento ottimale in un fondo pensione, tenendo conto sia del rischio finanziario che di quello attuariale. Inoltre, il modello tiene conto anche della necessità di mantenere la sostenibilità del fondo, cioè del bilanciamento costante tra partecipanti al fondo attivi (lavoratori in attività che pagano i contributi) e passivi (pensionati che ricevono la rendita previdenziale). Nel capitolo 4 è presentata un'applicazioen pratica del modello alla gestione degli investimenti di una Cassa di previdenza

di un ordine professionale italiana, mentre nel capitolo 5 vengono tratte le conclusioni scaturite dalle analisi condotte nei capitoli precedenti.

Chapter 1

The social security retirement program

I advise you to go on living solely to enrage those who are paying your annuities. It is the only pleasure I have left.

Voltaire

The social security retirement program, also called pension system, is a specific scheme to provide for retirement income. It is a part of the social security system which, as its name implies, is a scheme established to enhance the well-being of an individual against any misfortune caused by accidents, the inability to work or in the case of dependants, even death.
The aim of a pension system is to provide funds for pensioners to live without significant differences with the working period. Let us note that one could divide human life into three main phases: youth, productive years, and retirement. There are many perspectives on this division, but in this work we are interested in the economic one. Youth is therefore the period when a person, through education, acquires human capital, i.e., the ability to earn income throughout one's productive life. Acquisition of education, i.e., investment in human capital, is among the most important financial issues in a person's life. Another such issue is, of course, the accumulation of financial capital for retirement, the subject of this work. Human capital acquired mostly during one's youth is gradually used up during one's productive life. One may argue that a person should re-acquire education later in life, as the human capital accumulated early in life does become obsolete eventually, as the technology of production changes. Ironically, the blessing of longer life enjoyed by most

of the world's population in the twentieth century has made such a threat of a need for re-acquisition of human capital more likely. Nevertheless, eventually one's physical and mental ability to work is depleted, and somewhere before or at that moment, one needs to replace human capital with financial assets allowing for comparable standard of living.

The importance of the provision of a proper pension system can never be over emphasized. By 2007, more than 170 countries has adopted some form of public pension systems. Public pension expenditure are often the largest fiscal program in many countries. For example, in 2000's, in Europe, it's about 12,1 percent of total GDP (15,38% in Italy in 2008); in North America, it is about 7,1 percent; it is relatively low in Asia, about 3 percent.

1.1 Pillars of a pension system

To avoid misunderstanding, it is useful at the onset to define terms. Traditionally, specialists have divided pensions into three pillars (see Figure 1.1)

- Public pensions
- Occupational pensions
- Personal pensions

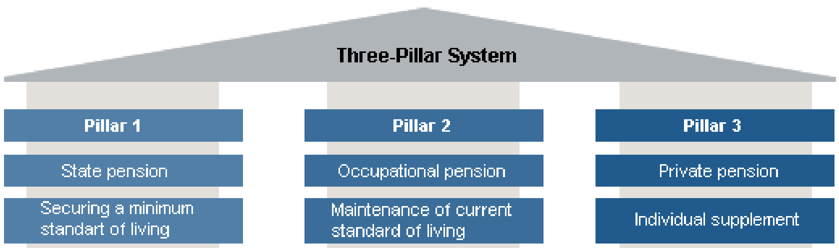

Fig. 1.1: Three Pillars Pension System

Within each pillar there are many types of pensions, sometimes referred to as "tiers", but the three categories exhaust all possibilities with respect to *providers* of pensions. There are only three sources of pensions: government schemes, schemes set up by a trade union or employers, and individual annuities. For some purposes, this is a useful way to look at pension systems, especially if the aim is to compare pension systems in different countries.

There exists an alternative framework developed by the World Bank Bank [1994] in a now famous Report titled Averting the Old Age Crisis. The authors of the Report analyse the problem of income maintenance in old age not from the perspective of pension providers but rather from the perspective of those who *participate* in retirement income schemes. Somewhat confusingly, the Report retains the terminology of "three pillars", and refers often to Pillar 1 as synonymous with "public pillar", even though the second pillar in this scheme typically is, and the third pillar could conceivably be, publicly managed. The Report defines its three pillars in this unique and useful way:

- Non-contributory (basic pension)
- Contributory (forced savings)
- Contributory (voluntary savings)

The first pillar is an anti-poverty pillar that is non-contributory and guarantees a minimum income in old age. The second is a forced savings pillar that provides benefits only to contributors, and, in general, provides the most benefits to those who contribute most. The two mandatory pillars differ only in whether benefits are flat, or related in some way to contributions. The Report is prescriptive rather than descriptive when it ≪ recommends separating the saving function from the redistribution function and placing them under different financing and managerial arrangements in two different mandatory pillars, one publicly managed and tax-financed, the other privately managed and fully funded≫. Pillar 3 is a voluntary savings pillar, available to anyone who cares to supplement the retirement income provided by the first two pillars. The first pillar protects the elderly from absolute poverty (consumption below a minimum subsistence level), whereas the second two pillars protect them from relative poverty (a fall in consumption following retirement).
The first pillar is invariably public, financed by government on a pay-as-you-go basis. It is possible to imagine private employers or trade unions providing each covered worker with a pension unrelated to earnings or contributions, but in practice this never happens.
Pillar 2 almost everywhere has traditionally been public and pay- as-you-go as well; increasingly it is private and funded, in part or in whole. The World Bank encourages governments to fund Pillar 2 and to shift management from the public to the private sector.
The third pillar is identical to the second, except that it is always funded and is typically private because participation is voluntary. It is important to note that contributions to pillars two and three need not result in pensions. Benefits can be (and often are) drawn as a lump sum or as a series of withdrawals beginning at a specified age.

All retirement income falls by definition into one of these three World Bank pillars. Wherever the state (or, conceivably, an employer, a trade union, a charity or an extended family) provides benefits to the elderly that are not related to earnings or contributions, this is Pillar 1. Pillar 2 consists of entitlements derived from mandatory contributions to a pension or retirement savings scheme. Pillar 3, we have seen, is voluntary, thus it encompasses all other retirement income.

1.2 Classification of pension plans

In this section we will present a short review of the most important types of pension plans, which may be, for the purpose of our classification, part of any of the three legs of the retirement security stool.

1.2.1 Financing mechanism in pension schemes

The first criterion for the classification of pension plans is the financial base for the liabilities for benefits promised to plan participants. There exists the two kind financial system. The first one is the **Fully Founded** system (FF). In this case the pension plan asks workers a contribute that the plan itself provides to invest and give back at the end of working life in an amount equal to how much the worker gave during his working life capitalized at the market rate. So the FF is a way to transfer agent personal richness in the time. It's a sort of saving account.
The second system is the so-called **Pay-as-you-go system** (PAYG). Here the pension plan collects workers contributions and uses them to pay pensions to pensioners. So we can understand how the PAYG is a system based on a transference in the space between current contributors to current pensioners. This can be a problem, because, as stated in Samuelson [1958] and in Aaron [1966], it is sustainable only if we have a rate of growth of total real earnings that exceeds the interest rate indefinitely and this happens when there is technological progress and/or steady population growth and excessive capital accumulation. Problems arise at this point. The first one is that in this system people often receive a pension that is a relatively high percentage (in Italy between 75% and 80%, 100% up to few years ago) of their last salary, while the pensioner contributed in a proportional way during his working life, also when he was earning less money (it provides, also, a distortion in labour market pushing at working hard old worker and inhibiting young ones). The second one depend on the fact that pensions are often object to discussions under general elections, so there is an incentive for government to use them

Pay-as-you-go	Fully Funded
Intergenerational transference between current contributors to current pensioners	Contributors' personal richness transference in the time
No Capitalization	Capitalization

Table 1.1: Pay-as-you-go vs Fully Funded financial mechanism

in order to "capture" more votes. The latter aspect is largely less probable in the Fully Founded system because the pension scheme is already determined by the contribution the pensioner gives (even though the interest rate can be changed). Table (1.1) summarizes the main features in pay-as-you-go and fully funded systems.

1.2.2 Calculation methods for benefits

There are two extremely different ways to determine benefits. On the one hand, we find **defined benefit** plans (hereafter DB), where benefits are fixed in advance by the sponsor and contributions are initially set and subsequently adjusted in order to maintain the fund in balance. DB plans are definitely preferred by workers. In fact, in the case of DB plans, the associated financial risks are supported by the plan sponsor rather than by the individual member of the plan. In fact, as we can see in Figure (1.2), the value of the pension do not depend on the value of financial assets deriving from the capitalization of workers' contributions. Actually, DB pension plans can be split into risk sharing plans and no risk sharing plans (see Table 1.2 below). Risk sharing plans contain features that allow the pension fund to mitigate the risk by sharing part of it with the sponsor and/or with the plan members. In pension plans without risk sharing, the risks stay within the pension fund. In a first step, generic pension plans that do not allow for risk sharing with the plan sponsor and/or the plan members are defined. These are the standard final pay and the standard career average plan. The main features of both plans are:

- *Final pay systems*
  We distinguish two variants of the system based on final salaries.
  - Actual final pay system: in this system, every wage increase not

Defined Benefit Pensions	
No risk sharing	Risk sharing
Final Pay • Actual • Moderate Average earned salary • Actual • Moderate	Career average with risk sharing features • Additional contribution • Conditional indexation • benefit cuts

Table 1.2: Defined Benefit Pensions

only affects the rights which will be built-up in the remaining years of service, but also in the previous built-up rights.

  – Moderate final pay system: this system only differs from the system described above, in the sense that wage increases in the last years of service do not result in a higher pension. This prevents that (extreme) wage increases in the last years of service result in a very high pension.

- *Systems based on the average earned salaries*
  Also for systems based on the average earned salaries, two variants are distinguished.

  – A system based on the actual average earned wage: in this system, every wage increase influences the pension that will be built-up in the remaining years of service. The pension over previous years of service remains unaltered.

  – An indexed system based on the earned salaries: this system is characterized by the fact that the pension based on past years of service are corrected for increases in prices or wages.

The generic pension plans that do allow for risk sharing are *career average* (CA) plans with a risk sharing feature. The risk sharing features can be additional contributions, conditional indexation and benefit cuts. An important risk sharing aspect of pension plans is the fact that the plan sponsor may be required to make additional contributions in case of underfunding. In a similar way, future contributions of plan members may be increased in an underfunding situation. These risk-sharing mechanisms do not occur in the life insurance sector and therefore deserve special attention. For further details see Peek et al. [2008].

Historically, fund managers have mainly proposed DB plans,but nowadays most of the proposed second pillar pension plans are based on a **defined contribution** scheme involving a considerable transfer of risks to workers. In defined-contribution plans (hereafter DC) contributions are fixed and benefits depend on the returns on fund's portfolio (see Figure 1.3). In particular, DC plans allow contributors to know, at each time, the value of their retirement accounts. DC pension funds provide contributors with a service of saving management, even if they do not guarantee any minimum performance. As we have already highlighted, only contributions are fixed in advance, while the final retirement account fundamentally depends on the administrative and financial skill of the fund managers. Therefore, an efficient financial management is essential to gain contributors' trust. Often the employer

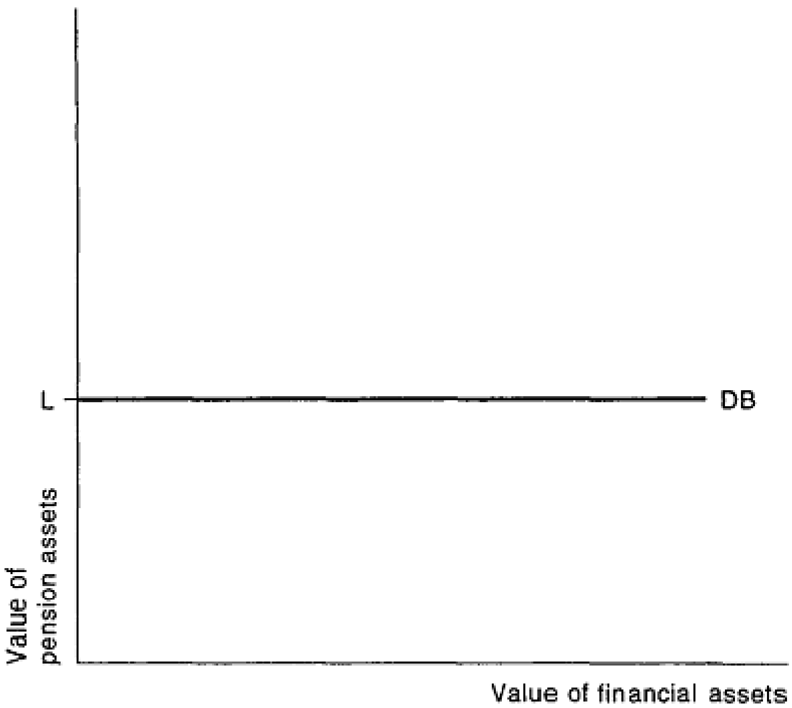

Fig. 1.2: Defined Benefit Pension Scheme

yearly transfers money (usually a percentage of the pensionable salary) to purchase a part of the employees' pension. The level of the pension depends on the number of years the pension contributions have been paid, the realized return in the years the pension has been built up, and the interest rate at the moment of retirement.

1.2.3 Types of pensions

Every type of pension provides the participant with an income after some event has happened. In this section, we discuss the most important types of pensions.

- Retirement pension.
  This is a pension for the financial care of a person, payable after pensionable age is reached. Generally, this payment is made lifelong.

- Widow's pension.
  This is a form of surviving relatives pension, that is paid to the widow(er) of a participant of the pension regulation. Generally, this payment is also made lifelong.

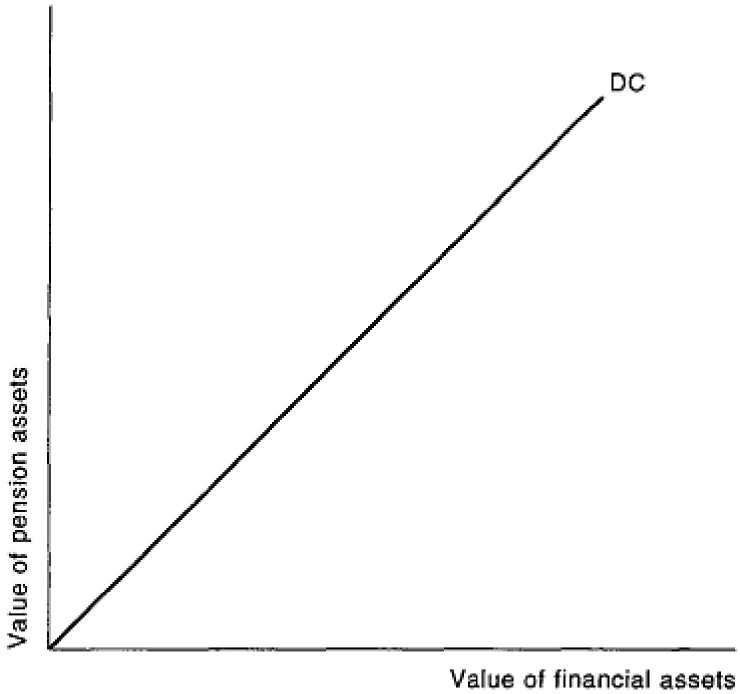

Fig. 1.3: Defined Contribution Pension Scheme

- Pension in case of disability
  This type of pension is made after the participant of the fund has become incapacitated for work.

1.3 The Italian Pension System and its reforms

Italy is facing a demographic challenge substantially greater than the average for all the countries of the European Union (see 1.4 below).

Moreover, we can observe the current Italian population pyramid (Figure 1.5) and its predicted shape for 2050 (Figure 1.6). A Population Pyramid is a graph that shows a country's population by age and gender. The shape of the pyramid tells us about a country's growth rate and dependency ratio. Population pyramids have the youngest age cohort at the bottom and females on one side and males on the other. Each age-gender group is represented as a percentage of the total population or in thousands of people. For furthers details see Cackley et al. [2006], Baldini et al. [2002], Brugiavini and Peracchi

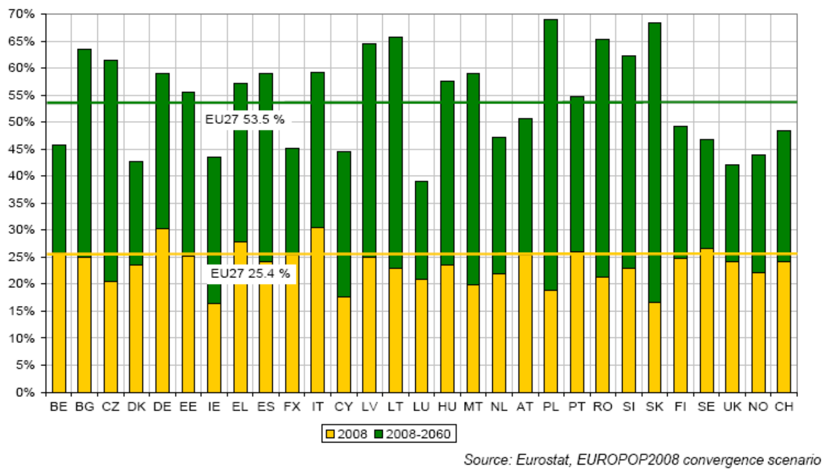

Fig. 1.4: Projected Economic Old Age Dependency Ratios for the EU member States, 2008/2060

[2004].

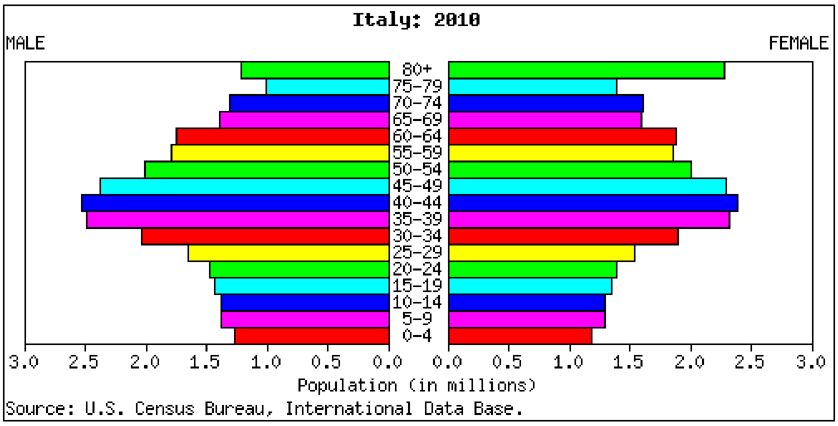

Fig. 1.5: Italian Population Pyramid in 2010 (data in millions)

Italian population pyramid warn us about the future predominance of older generations compared to the younger ones.

1.3.1 Three pillars Italian pension system

Currently, the Italian pension system consists of three pillars (see Figure 1.7):

1. Mandatory old age insurance (MOA), which also provides survivors insurance and disability benefits.

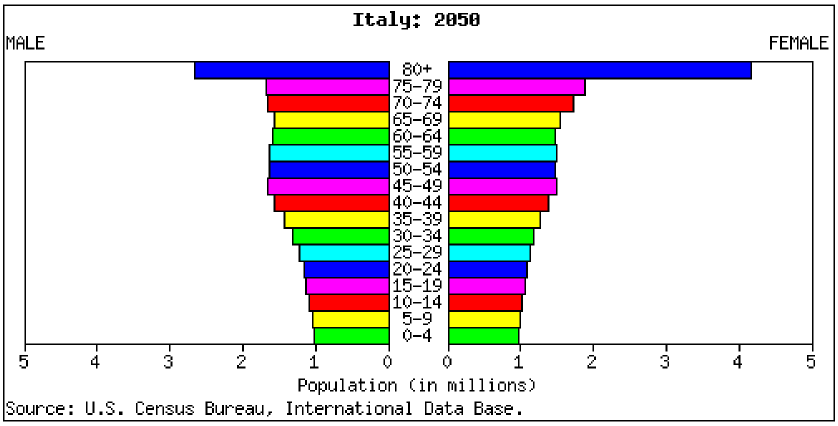

Fig. 1.6: Predicted Italian Population Pyramid for the year 2050 (data in millions)

2. Supplementary pension systems, including closed-end funds formed by employers and/or employees, and collective pension funds, which are mostly funded and voluntary.

3. Private insurance annuities or individual accounts. Supplementary pensions and private insurance substitute for the MOA in only a few cases. They are generally not mandated and they are funded.

In addition to the public pension program, Italian workers receive a severance from the *Trattmento Fine Rapporto* (TFR), which is equal to the cumulative total of 7.41 percent of earnings in each year for workers in the private sector. This is paid by employers and retained in a fund that the employers manage directly. It is paid out as a lump sum at retirement, and in the meantime serves as a source of capital for the employer. This structure is the result of a series of reforms. However, debate continues about the need for additional reforms, as additional funding or further reductions in benefits are needed to keep the program solvent. For further details see Amato and M. [2001].
The following section is devoted to a review of the reforms concerning the pension system in Italy.

Institutions of the Italian first pillar pension scheme

The current national insurance scheme consists of two principal institutions: the **Istituto Nazionale di Previdenza Sociale** (INPS) and a social security institution for public administration employees **Istituto Nazionale di**

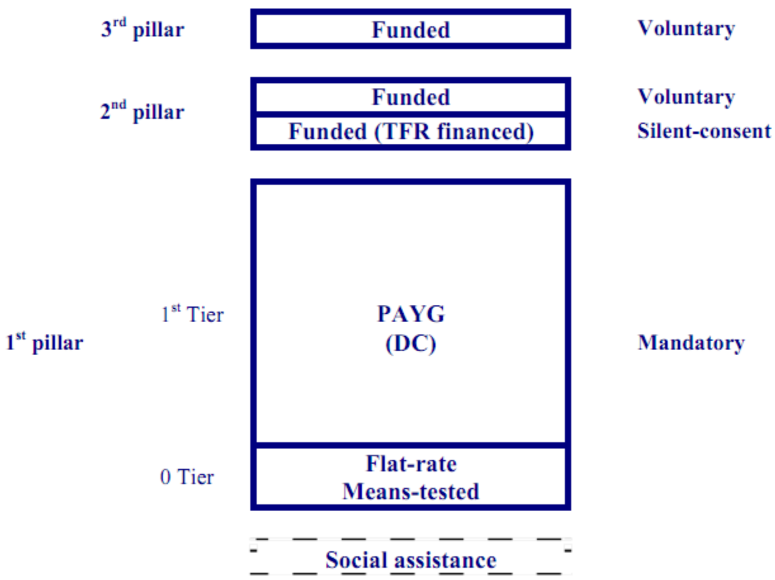

Fig. 1.7: The pillars in the Italian Pension System

Previdenza dell'Amministrazione Pubblica (INPDAP).
INPS administers the "Assicurazione Generale Obbligatoria" (AGO) (Compulsory General Insurance Scheme), of which most private sector employees are members, as well as the employees of certain semi-public bodies. All employees over 14 years of age in Italy are insured under AGO. AGO pays invalidity, old-age and survivors' benefits. Currently 10,2 million employees get one single pension by INPS, of which 6,1 millions belonging to private sector. The self-employed have their own compulsory general insurance scheme, which is broken down into four funds, administered by INPS, for the crafts sector (1,117,986 members), the trading sector (854,867 members), farming (942,042 members) and "para-subordinate" workers (717,035 members), totalling 3,631,930 members. For further details see INPS [2009].
INPDAP currently insures 3.289.700 public sector employees, mainly through the Special Fund for Public Administration Employee Pension Schemes and the Pension Fund for Local Authority Employees.
The social security institutions of self-employed members of the professions comprise 20 independent and privatised pension funds,totalling 249,443 members. The process of planned transition of the 1995 reform envisages the provisional co-existence, until 2015, of the previous earnings- related system, on the way to extinction, and the present contribution-based system.

Institutions of the Italian second pillar pension scheme

Complementary pension schemes were introduced in 1995. This additional voluntary, supplementary occupational system (**Pension Funds**), consists of both open funds and closed collectively agreed funds. The closed funds can be funded by both employers and employees as well as from the TFR. The open funds provide an annuity based on contributions. The current TFR contribution rate is 6.91%. The number of workers enrolled in a private pension fund is still low. For this reason, the Finance Act for 2007 has anticipated (with some changes) the pension reform passed some years ago (Law 243/2004 and legislative decree 252/2005) which introduced further measures in order to speed up the development of the second pillar: a) higher fiscal incentives and b) silence-as-assent for the transfer of the private severance pay (TFR).The latter means that the current severance pay accumulation will be transferred to a private pension fund, unless the worker communicates his or her refusal.

Institutions of the Italian third pillar pension scheme

The first and second pillar retirement pensions are not always as high as an individual would like it to be. The third pillar (**individual private pensions**) is supplied from a two sources:

- personal fund (FIP - forme pensionistiche individuali)
  It is an individual instrument that fulfill the purpose in supplying retirement provisions in addition to the collective ones. The funds accumulated are invested and the ensuing profit is added to the pension.

- life insurance contracts (Polizze vita)
  They are insurance contracts to provide investors supplemental retirement income via a life insurance policy featuring market appreciation, tax-deferred accumulation, and a life insurance death benefit.

1.3.2 Italian Pension System reforms: from the origins to 1992 reform

During the first half of the XX century, the Italian pension system was a public funded system, to which employers, workers, and the state contributed with not so much different contributions. Dramatically hit by the very high inflation rate during and immediately after World War II, it was changed into an unfunded public pay-as-you-go system (paygo), which was fully constructed during the years between 1957 and 1968. They were years of high rate of economic and demographic growth, which brought about a very

generous pension system (see Angrisani et al. [2001], Cackley et al. [2006], Castellino and Fornero [2001], Castellino and Fornero), Cazzola [2004].
The main features of the system built during those years are the following:

- Pensions were determined by the earnings related formula $P = c \cdot T \cdot w$, where c is the so-called internal return coefficient, T is the number of years of contribution (T could not exceed 40), and w is the reference wage. For employees of the private sector, c was 0.02, and w the average wage of the last five years of work, expressed in terms of the final year prices. The age for the old age pension was 55 for women, and 60 for men.

- "Seniority pensions" (*pensione di anzianità*) were introduced, allowing early retirement for private employees, once they had reached 35 years of contribution to the pension system, independently of the age. The benefit was computed exactly in the same way as indicated above, without any consideration of the difference in the life expectancy at the different ages of the early retirees.

- Public employees had a privileged position. For several categories of civil servants, the coefficient c was higher than 0.02. For all public employees, the reference wage was the wage of the final year of work. Every civil servant had the right to retire when he reached 20 years of seniority; in the case of a woman with two children the minimum seniority to early retirement was 14 years, six months, and one day. Again, early retirement pensions were not actuarially fair.

- In between 1957 and 1965 the public unfunded pension system was extended to all the self-employed. Artisans, shop keepers, farmers, professionals had to contribute to their new pay-go funds, but at the beginning they were allowed to get the benefit as soon as they reached the age of 65, under the condition that they had contributed at least for a year. The original pension formula related the benefit to the amount of contribution. In 1990, a new rule was enacted for self-employed, which granted pension benefits proportional to the average earnings over the last 10 years of work with an accrual rate of 2%. Having not modified the level of contributions accordingly (12% of gross income for self-employees instead of 27,4% of the gross salary for the employees, at that time), this modification caused a huge growth of future pension net liabilities. Currently, a large share of retired self-employees still receive a benefit which is integrated by the state in order to reach the national minimum level of the benefit. One should not forget that self-employment represents 30% of overall employment, in Italy.

- From 1971 to 1992, pensions were indexed both to price increases, and to the average real wage growth.
- The system allowed a uniform substitution rate, but an internal return on contribution variable in the opposite direction with the degree of seniority, with the rate of growth of wage during the working life, and with the age of retirement.

The system built in those years produced a rapid increase of pension expenditure till the first half of the nineties. Such an increase of expenditure was also supported by a large diffusion of inability pensions, as substitutes of the lacking unemployment insurance for workers of a mature age. The higher line in Figure 1.8 below shows how expenditure would have grown, in the 50 years later 1992, without any of the reforms enacted during the last decade.

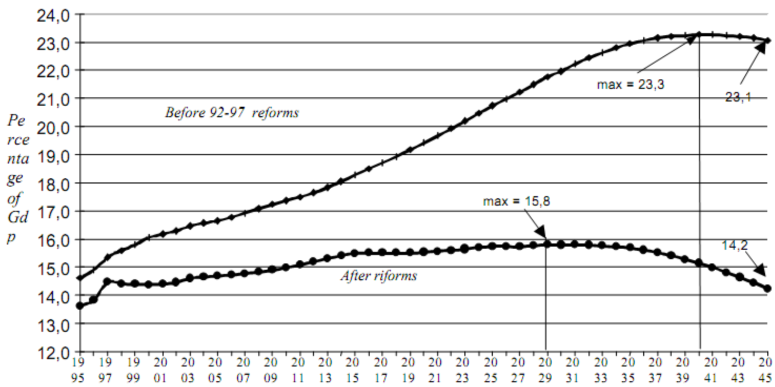

Fig. 1.8: Pension expenditure in terms of GDP

1.3.3 The first step of the reform: 1992

In 1984 the rules for inability pensions were restricted to the physical inability to work instead of the more general inability to produce income. The long process of reduction of the number of inability pensions is still underway, and it is a component of the slowdown of the growth of pension expenditure. However, it was only in September 1992 that, under the pressure of a deep currency and financial crisis, an effective process of reform started. The first Amato government introduced some important new principles in the system. The determination of the benefits should stick more to insurance principles through a stronger relationship with the amount of contributions paid during the working life. The reference wage was not to be any longer the wage of the last five years of work, but the average of the last 10 years, to be extended, in the future, to the whole working life. The formula for the computation of the first pension benefit for an individual who entered the labor market after 1992 can be expressed as follows:

$$P = 0.02 \cdot L \cdot W_0 \left[\sum_{i=1}^{L} \frac{(1+w)^{i-1} \cdot (1 + 0.01\,(L))}{L} \right]$$

where W_0 is the entry wage, w is the average growth of workers' wages and L is the seniority at retirement. Current and future generations cannot any longer afford sharing technical progress with retirees; hence the benefits were not any longer indexed to real wages. Moreover, the portion of pension higher than three times the official minimum benefit level is not any longer fully adjusted to consumer price changes.

The privileges of the civil servants started to be dismantled: a very slow process of convergence toward the system for private employees was enacted. The seniority required to a civil servant to retire early was increased.

The age for old age pension had to be progressively increased to reach 60 years for women, and 65 for men (this target has been reached in 2000). Moreover, the minimum number of years of contribution required in order to receive the benefit was increased from 15 to 20 years.

Occupational funds and individual funds should complement the paygo system, in order to face demographic transition. The legislation required was enacted, but the fiscal incentives were not enough to let those funds start.

The strongest impact of these measures was to be exerted by the abolition of the indexation of benefits to real wage growth. As a whole, all these measures helped workers to realize that the generosity on which they had built their life cycle wealth expectations was not any longer sustainable. It has been shown that households, whose breadwinner was a civil servant, felt the strongest effect.

1.3.4 The second step: 1995-97. The notional defined contribution system

Three years later, in 1995, the Dini government enacted a radical reform of the system of computation of the benefit, based on true insurance principles, applied within a paygo system. The Prodi government in 1997 further integrated the reform. The new notional defined contribution system is a paygo system that mimics capitalzation. The amount of contribution paid to the paygo system during the whole working life is going to be capitalised at the rate of growth of nominal GDP (actually, a five years moving average of

GDP growth). Such an amount of capitalized contributions is the basis for the computation of the benefit by means of a discount rate that is a proxy of the expected long rate of growth of real GDP (1.5 per cent, in real terms because the benefit is indexed to prices).
The worker can choose the age of old age pension, once he/she is 57 years old, and is not yet 65. At 65 the retirement is compulsory. The computation of the benefit takes into account the average life expectancy (men and women) at the age the worker chooses to retire. When the new system fully works at regime, the return in terms of benefit to the contributions is the same for every worker, independently of the age of retirement, of the life time wage profile, and of the seniority at the moment of retirement. No difference between men and women age for pension will survive.
Under the new system, the benefit is approximately determined by the following simplified formula:

$$P = a_j \cdot w_0 \left[\sum_{t=0}^{T-1} (1 + y_n)^{T-t-1} (1 + g)^t \right] \gamma (y_r, \omega, s)$$

with $\frac{\partial \gamma}{\partial y_r} > 0; \frac{\partial \gamma}{\partial \omega}, \frac{\partial \gamma}{\partial s} < 0$
where a_j is the contribution rate equal to 33% and 20% respectively for employee and self employed, w_0 is the entry wage of an employee of T years of seniority, y_n is the average rate of growth of nominal GDP (the capitalisation rate) during the last T years, g is the average rate of growth of the nominal wage during the same T years, and $\gamma(.)$ is the value of a coefficient which depends on the constant discount rate $y_r = 0.015$ (the expected long run real growth rate of GDP), on the weighted average of the life expectancy of men and women (ω), and on the probability to have a survivor, combined with his/her life expectancy, s.
The notional defined contribution system introduces actuarial fairness within a paygo system; the degree of redistribution and solidarity enacted trough the pension system is reduced, though not canceled. The system reduced distortion in labor market decisions especially with regards to the choice of retirement age. Moreover, it reduces, in a sense, the perverse solidarity of the current system, where workers with less dynamic careers and lower wages finance benefits of the workers with higher and more dynamic wages. A real jungle of various privileges will be cleared: more favorable treatment both of civil servants (with special reference to military forces), and self-employees with respect to private employees will disappear. With the new system, true solidarity is going to be mainly a question of the general taxation.
If there is not any relevant long run change in income distribution between

	Before 1991		After 1992		After 1995	
Age	Employee	Self employed	Employee	Self employed	Employee	Self employed
58	71,2	67,8	61,9	61,9	54,4	33,0
60	71,2	67,8	61,9	61,9	57,8	35,0
62	71,2	67,8	61,9	61,9	61,7	37,4
65	71,2	67,8	61,9	61,9	68,7	41,6

Table 1.3: Replacement rate between pension benefit and last wage at different retirement ages for an individual who retires with seniority equals to 37 years

profits and wages, the new system tends to balance revenues and payments in a structural way. Given the anticipated real growth of GDP used as a discount rate to compute the benefit, the higher the actual rate of growth, the higher the revenues, and the benefits. Parametric reforms in 1992 and 1995 reduced the expectations for future level of pension benefits and the change in the indexation mechanism reduced the dynamic of pension benefit after retirement. The first effect can be seen by comparing the *replacement rate*, defined as the ratio between the first pension benefit and the last wage. In Table 1.3 we report the replacement rate at different age of retirement for a representative individual whose seniority at retirement is equal to 37 years. We notice that both the 1992 and the 1995 reform reduced the replacement rate. With the notional defined contribution formula the replacement rate increases with the retirement age as the formula considers explicitly the average life expectation at retirement. The parameters of the defined contribution scheme are fixed such to reach the same replacement rate that of 1992 for a 62 years old employee. The reduction of the replacement rate is stronger for self employed after 1995 because of the lower level of the contribution rate used to the computation of the pension benefit, which for this category is fixed at 20%.

To fulfill the equilibrium conditions of the system, the coefficients used to compute the benefit will be changed every ten years to take into account changes in life expectancy. If life expectancy increases, the substitution rate between wage and benefit will be reduced, but the worker might choose to work longer to compensate such a reduction.

The new system will not fully work before 25-30 years, since now; therefore the reform has dealt with the many details of the long transition process. It has created three different cohorts: the young employees, the middle seniority employees, and the mature seniority employees. The new system applies fully to the new entrants since the beginning of 1996. For the employees that at the end of 1995 had seniority, in terms of years of contribution, of not more than 17 years, the new system applies only for the remaining part of their working life. Their benefit will be the result of two parts; the first one determined according to the old computation rules, and the second one according to the new rules. Finally, for employees with seniority equal or higher than 18 years in 1995 the rules for the computation of the pension benefit remain the same as pre-1995. Early retirement benefits are completely abolished when the new system works at regime. In the mean time, for workers already at work in 1995, early retirement is progressively restricted to those who are 57 years old and have a seniority of 35 years; these conditions will apply in 2004. On that date, the pension rules for both civil servants, and private employees will be exactly the same. In other words, early retirement benefit, computed without actuarial fairness, is still allowed for employees that currently have middle, and mature seniority, when they meet the above restricted conditions.

1.3.5 International pension systems

Aging population and pension system reform is not a phenomenon only in Italy. Historically, social security systems were almost all PAYG. However, it is widely recognized that these systems generate many problems such as rising payroll tax rates, evasion and early retirement. Many countries have undertaken major or minor reforms of their pension systems in the past 30 years. Some countries went through radical reforms and changed into fully funded pension system, some chose a mixed system with a combination of both, and the rest stayed with the original PAYG pension systems and modified parameters such as contribution rate and retirement age.
The following part of this is section is dedicated to a brief description of some relevant international social security system.

Social Security System in the United States

Established in the 1930s, the US Social Security system is a mandated, public defined benefit system with very wide and compulsory participation. Few groups are permitted to opt out. The average pension benefit represents a replacement rate of approximately 50 percent of the best 35 years of salary history. Some adjustment is made to redistribute pensions to poorer participants; hence, individual replacement rates may differ from the average. To this end, the replacement rate offered to those with a poor income history is

	Pre-Reform	1992 Reform	1995 Reform
Normal retirement age	60 (men) 55 (women)	65 (men) 60 (women)	Any age, starting at 57 (men and women)
Pensionable earnings	Average of last 5 years real earnings	Career average earnings	Career contributions (capitalized at an annual rate using a five-year moving average of past GDP growth rates)
Pension benefit	2% *(pensionable earnings)*(years of tax payments ≤ 40)	2% *(pensionable earnings)*(years of taxpayments ≤ 40)	Proportional to accrued value of career contributions; proportionality factor increases with age up to age 65
Years of contributions needed for eligibility	15	20	5
Total payroll tax (employee and employer)	24.5% gross earnings	27.1% gross earnings	32,7% gross earnings

Table 1.4: Changes in Key Features of Italian Pensions During the 1990s

higher than for those with a higher income. Benefits are paid until death, include substantial indexation to in inflation (since 1972), and are extended to survivors. The system was designed along the lines of a PAYG system with current contributors largely financing pension payments. Today, the Social Security system is not the pure PAYG system, it was in the mid-1970s. With the prevailing contributions, the system was unavoidably heading toward insolvency. As a result, the Greenspan Commission in 1982-1983 recommended a sharp rise in contributions, which would permit building up a reserve, the so-called Trust Fund, to cover future shortages.

401(K) Plans and Individual Retirement Accounts in the United States

These schemes are most prevalent in the United States and are most commonly referenced when discussing funded defined contribution plans. Under the 401(K) schemes, both employers and employees contribute to these funds from pre-tax income. Participants are free to choose investment strategies from a set of chosen private providers who manage the assets for the participants. They are allowed to borrow from their account, within limits and at their discretion, but must repay under established rules. Under the Individual Retirement Accounts, individuals set up the plan directly if they meet certain eligibility criteria. Participants have sufficient choice in the structure (i) to select their asset allocation (a mix of bonds and equities; international and domestic assets) and (ii) to select preferred manager(s) from a short list of managers and mutual funds. Participants either have full discretion over asset allocation and fund selection (self-directed plans) or can delegate the responsibility to the service provider. Withdrawal of funds is permitted to finance certain activities, but if these monies are not returned before retirement, the participants incur a tax event because the Individual Retirement Accounts is no longer a tax-deferred saving. The pension is the annuity that can be purchased from service providers given the accumulation at retirement. In both systems, participants have some discretion over the level of contributions, but there are limitations on maximum contributions. These caps on contributions exist because such savings are tax deferred. However, the caps have changed over time, allowing participants to change the amount they contribute to these plans.

The Chilean Defined Contribution Model

The Chilean model, implemented in 1981, involves gradual phasing out of the PAYG plan and replacement by a mandatory DC plan. New workers can participate only in the DC plan, whereas participants in the PAYG plan can still choose between the old and new plans. Assets are managed by private companies chosen by the participant from a list approved by the government with individuals largely bearing the risk of investment performance. There are some guarantees in the Chilean system including: a low social assistance benefit to those not covered by the mandatory plan, a state-guaranteed minimum pension of approximately 25 percent of the average wage if contributions are made for at least 20 years, a minimum profitability rate guaranteed for each pension fund relative to the average for the country, and state-guaranteed annuity payments if the insurance company fails. Variations of the Chilean Scheme adopted in other parts of Latin America (in Argentina, Colombia, and Peru) offer a choice in the second pillar between a privately managed defined contribution system and a public PAYG defined benefit system. However, individuals are permitted to make voluntary contributions to their funds to allow for early retirement.

Mandatory Provident Funds

Mandatory Provident Funds are prevalent in countries formerly under the Commonwealth (such as India and Malaysia). Under these schemes, individuals contribute to the system, which then aggregates funds in a central pool. Such schemes are generally offered to private sector employees. The pooled fund is then invested in different assets, and the participants earn dividends on their contributions, which are essentially equal to the returns of the investment strategy. In some countries, dividends are smoothed over a few years of returns to reduce volatility, which leaves open the possibility that the system will be "underfunded" if a series of negative returns occurs. In Malaysia, a minimum guaranteed of 25 percent is offered to participants. Participants have individual accounts but no choice. Generally, participants are allowed to borrow against these funds to purchase house or make other investment that are deemed socially useful.

1.4 Italian Professional Orders self-administrated pension funds

The privatization of retirement funds of self-employed was implemented by the Legislative Decree (D.Lgs) 509/1994, which has enabled the transformation of 16 institutions in private associations or foundations from the first January, 1995. Those pension funds adopted a PAYG financial regime. The purpose of this law is to reorganize existing pension institutions.

The two key legislative actions immediately following are: Law 335/1995 and D.Lgs 103/1996. Thus it is possible for these institutions to acquire legal personality, making its members independent of the public welfare. The new 5 funds, built by D.Lgs 103/1996, follow a fully funded financial scheme. Table 1.5 is a list of all existing professional orders pension funds. For further details see ADePP [2005], Cackley et al. [2006], Carbone [1998], Castellino [1998], Dallocchio [2001], Inglese [2004], Trudda [2008].
The social security of professional orders is not marginal inside the Italian pension system. About 1.3 million workers, equal to 5.6% of total employment, are registered as contributors to the private pension system of Italian professional orders. The share of subscribers has increased notably for some funds (engineers and architects, lawyers and chartered accountants), others remain stationary (notaries), while other ones have experienced a reduction (accountants).

Table 1.5: Italian Professional Orders self-administrated pension funds

Cassa nazionale di previdenza e assistenza avvocati e procuratori leagli
Cassa nazionale di previdenza e assistenza tra dottori commercialisti (CNPADC)
Cassa nazionale di previdenza e assistenza geometri
Cassa nazionale di previdenza e assistenza ingegneri e architetti liberi professionisti (INARCASSA)
Cassa nazionale del notariato
Cassa nazionale di previdenza e assistenza ragionieri e periti commericiali
Ente nazionale di assistenza per gli agenti ed i rappresentanti di commercio (ENASARCO)
Ente nazionale di previdenza e assistenza consulenti del lavoro (ENPACL)
Ente nazionale di previdenza e assistenza medici (ENPAM)
Ente nazionale di previdenza e assistenza farmacisti (ENPAF)

Table 1.5: Italian Professional Orders self-administrated pension funds

Ente nazionale di previdenza e assistenza veterinari (ENPAV)
Ente nazionale di previdenza e assistenza per gli impiegati dell'agricoltura (ENPAIA)
Ente nazionale di previdenza e assistenza a favore dei biologi (ENPAB)
Ente nazionale di previdenza e assistenza dei periti industriali (EPPI)
Ente nazionale di previdenza e assistenza per gli psicologi (ENPAP)
Ente nazionale di previdenza e assistenza per gli infermieri professionali, gli assistenti sanitari e le vigilatrici di infanzia (IPASVI)
Fondo di previdenza per gli impiegati delle imprese di spedizione e agenzie marittiime
Istituto nazionale di previdenza dei giornalisti italiani (INPGI)
Opera nazionale assistenza orfani sanitari italiani (ONAOSI)
Ente previdenza e assistenza pluricategorale (attuari, chimici, dr agronomi, dr forestali, geologi)

Professional orders funds privatization resulted in self-governing management, organization and accountancy and in the adoption of a corporate governance (statute, regulations, etc) which has to obey to specific criteria with regard to:

- disclosure and transparency in relations with the fund members and in the settlement of the corporate organization to make known the roles and responsibilities of board and management to provide shareholders with a level of accountability.
- determination of the necessary requirements to perform institutional activities (in particular, honourableness and professionalism for directors and executives)

- the creation of a particular legal reserve fund.

Until 1995, Italian professional order retirement funds were administrated by the State that would step in, in case of insolvency. Since 1995, private pension plans have managed the security of a growing number of self-employed *without being sponsored by the State*. The pension funds of professional orders are now self-managed and they continue to operate according to a PAYG financing mechanism. As already highlighted in this chapter, this is an anomaly because private closed schemes are usually funded. In this particular system the financial self-sufficiency is certainly guaranteed only in the initial phase, because there are many contributors and no actual pensioners. In the long run it is necessary for the financial sustainability of the pension plan that the number of pensioners remains proportional to the number of workers. If the ratio active/retired decreases, the increase in the financial burden can entail a situation of financial disequilibrium. This is most relevant for the retirement funds of each specific professional order for which, unlike in a public system, there is indeed no intergroup compensation.
To sum up, we can say that for PAYG pension funds, in which the financial sustainability is related to a balance between the active and retired members, there is a further demographic risk source to take into account: the risk relates to future monetary cash flows necessary to ensure payments of future pensions.

Chapter 2

Asset Liability Management in a Pension Plan

There are risks and costs to a program of action. But they are far less than the long-range risks and costs of comfortable inaction.

John F. Kennedy

From the point of view of financial economics, we divide the assets into the real assets and financial assets (also known as capital assets, or securities). The real assets are used for current consumption or production, and if they are used for current production, they are a source of current income. On the other hand, the financial assets are claims to future production or consumption, i.e., claims to future income, in the form of stocks, bonds, derivative securities of various types, etc.
Pension plans, just as other financial intermediaries, such as banks, insurance companies, or mutual funds, typically hold financial assets on the asset side of their balance sheet. Sometimes, although rarely, pension plans may hold real assets, such as real estate, but even then, such assets are held for investment purposes, i.e., they are held in order to produce future income for plan participants, and consequently, they play the role of financial assets.
On the other side, a pension fund's liabilities are the benefits paid to a scheme's members and consists of a series of cash flows that the scheme must pay out in the future. The cash flows are usually calculated by an actuary (see section 2.1.4) and are based upon the aggregate forecast of all the benefits for the members. Typically, the expected cash flows are based on a snapshot of existing members and will not take account of future joiners.

Pension fund liabilities are long-dated; their calculation involves forecasting far into the future (50 years or more) to estimate what payments will be made, as well as the value that these distant payments should have today.

2.1 Asset Liability Management for Pension Plans

Asset Liability Management for pension funds is a risk management approach, which takes into account the assets, the liabilities, and also the interactions between the different policies which the board of a pension fund can apply (Berardinelli et al. [2007]). The board of a pension fund should find acceptable policies that guarantee with large probability that the **solvency** of the fund is sufficient during the planning horizon and, at the same time, all promised benefit payments will be made. The solvency is the ability of the pension fund to fulfill all promised payments in the long-run. Usually, the solvency at a certain time moment is measured as the **funding ratio**. Recall that this is the ratio of assets and liabilities.
Underfunding occurs when the funding ratio is less than one. Another way of characterizing underfunding is by saying that the surplus is negative,where the surplus is the difference between the value of the assets and the value of the liabilities. The surplus is the part of the reserves of the pension fund that is not needed for paying benefit payments. The funding ratio changes over time, mainly because of fluctuations in the liabilities and in the assets. Therefore, a pension fund rebalances its asset portfolio and adjusts for example its contribution rate regularly, in order to control changes of the funding ratio over time. In case of distress, the sponsor of the fund may have to help out with a remedial contribution.
In the ALM decision process, conflicting interests of different parties exist. In the next section, we will look in more detail at the interests of different parties. In section 2.1.2 we discuss the policies and instruments which are at the disposal of the board of pension funds. For further details see Aitken [1996], Fornero [1999], Micocci [2009], Micocci and Coppini [2001], Micocci [1999], Vigna and S. [2001].

2.1.1 Interested parties in the ALM policy of a pension plan

At least four parties are involved in the decision making process by the board of a pension fund, or are interested in its results. First of all, the *active par-*

ticipants are (or should be) interested. They are especially concerned about the level of the contribution rate. Active participants make contributions on a regular basis to the fund to build up rights concerning (some of) the different types of pensions described in section 1.2.3. If the contribution rate increases for example, the active participants have to make a larger contribution to the pension fund, which results in a lower disposable income. Older active participants are also interested in the degree of indexation of their pension: they would like to be compensated for inflation in all years. .

A second interested group consists of *retired persons* and *surviving relatives* of them. For this group, especially the indexing policy is important. Of course, they would like to receive full compensation for increases in prices or wages.

The *sponsor* of the fund is also involved. Not only does the sponsor pay a part of the regular contributions, also in case of financial distress the sponsor can play an important role. If the funding ratio drops below a certain threshold, the sponsor of the fund in some cases could contractually be forced to restore the funding ratio. On the other hand, in case of financial prosperity, the sponsor could also benefit. Note, however, that not all pension funds have a sponsor. Every pension fund related to a single company has a sponsor. Moreover, neither the government can act as a sponsor of the Italian Professional Orders funds (see section 1.4). Other funds related to companies in the same branch of industry, or funds for individuals with the same occupation, may not have a sponsor. Next to concerns about the level of the contribution rate and the level of remedial contributions and restitutions, the sponsor is also interested in the costs associated with carrying out the pension administration.

The last party discussed here is the *supervisor* of the fund. Pension funds have to justify and report their activities to the supervisor. The role of the supervisor differs from country to country.

Although all parties discussed here will be satisfied in case of financial prosperity, tensions between (some of) these groups are to be expected if the financial position of the fund is weak. Pensioners would like to receive an index-linked pension. However, this may result in even more pressure on the funding ratio, and in addition, on higher contributions by active participants or even a remedial contribution by the sponsor of the fund. On the other hand, this field of tension makes ALM problems challenging. For further details see Ambachtsheer and Don Ezra [1995].

2.1.2 Policies and instruments

The board of a pension fund has many instruments to its disposal to control the funding ratio. These are discussed in this section. The board should take into account the interests of all parties involved in the decision making process, to find the *best* policy mix. We stress here that the ALM process is considered from the perspective of the pension fund. Figure 2.1 shows the major policies and rules by which the fund can control the funding ratio.

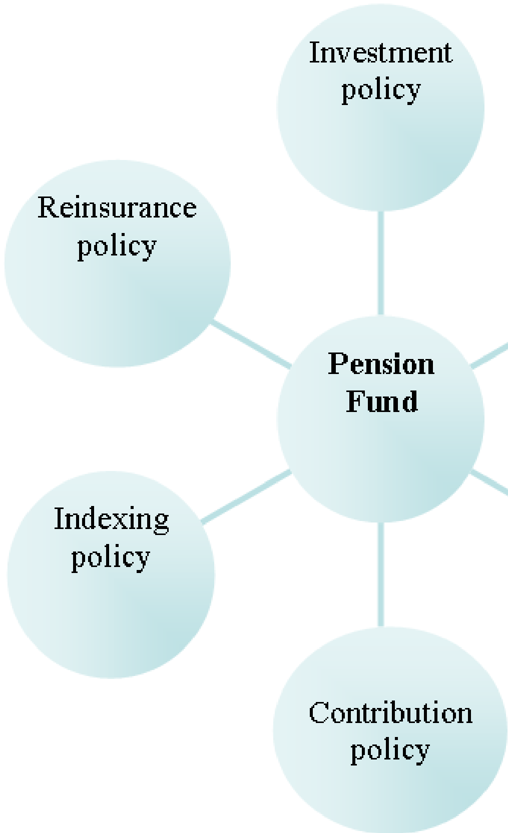

Fig. 2.1: Policies and rules of a pension plan

- **Pension policy**
  The pension policy deals with decisions with respect to the different types of pensions that the fund includes in the pension regulation. Active participants and retired people are interested in the pension policy, because they are the ones who will receive money from the pension fund.
- **Pension system**
  The rules with respect to the benefit payments are registered in the

pension rules. In these rules, the pension system is described. Especially the sponsor and the active participants are interested in the pension rules, because they have to finance the system.

- **Indexing policy**
The indexing policy is important in valuing the liabilities and (future) benefit payments. The board of a fund has to decide which base to use, for example a consumer price index, or a wage index. Moreover, generally every year again it has to be decided whether the financial position of the fund suffices to give (full) compensation. An actuary plays a key role in this decision. Retired people and active participants all would like to be compensated for increases in prices or wages. These are the parties who benefit from indexing pension rights.

- **Reinsurance policy**
Pension funds can seel out certain risks, like the risk of decease or disability, partially or entirely to an insurance company. This is called reinsurance and is part of the reinsurance policy of the pension fund. The supervisor judges the reinsurance policy of pension funds. The supervisor tries to avoid that pension funds are exposed to much risk.

- **Contribution policy**
The board of a pension fund can not only manage its liabilities, also the assets can be managed. One of the instruments to manage the assets is by means of the contribution policy. In the contribution policy, the system is chosen on which the level of the contribution rate is determined. Most pension funds use a dynamic contribution rate. In this system, the level of the contribution rate can be modified in the course of time. However, it is also possible that the different interested parties involved in the decision process agree about a fixed contribution rate. The active participants and the sponsor are the parties who are mainly interested in the level of the contribution rate, because they have to finance the system. Details about the different contribution systems that exist are beyond the scope of this thesis.

- **Investment policy**
The value of the assets is also influenced by the investment policy. In this policy, the board of the pension fund decides in which asset classes the fund invests its assets. Also the levels of the lower and upper bounds on the fraction of the total assets invested in each asset class, and rules concerning rebalancing are part of the investment policy. For example, it is possible that investments are made in indices, or that assets are

actively managed. Also investments to reduce risks, like currency hedging, are considered. The supervisor is concerned about the investment policy, because investments directly influence the risk of underfunding. Pension funds should invest their assets such that this risk is small. To do so, rules exist with respect to levels of buffers which pension funds need if they invest in certain asset classes.
This thesis is particulary concerned on investment policy and it develops in this direction.

2.1.3 Literature on Asset liability Management

The popularity of ALM in pension funds seems to have risen in recent years. Society of Actuaries (2003) provides a useful definition:

> ALM is the practice of managing a business decisions and actions taken with respect to assets and liabilities are coordinated. ALM can be defined as an ongoing process of formulating, implementing, monitoring and revising strategies related to assets and liabilities to achieve organizations financial objectives, given the organizations risk tolerances and other constraints. ALM is relevant to, and critical for, sound management of the finances of any organization that invests to meet its future cash flow needs and capital requirements

Most obviously, ALM should be the cornerstone of any pension institution's investment policy. Feinberg [2002] reports that many pension funds are now conducting more asset/liability studies mainly due to the deterioration of their funded status. She has interviewed many pension fund managers to learn the reason for the recent increase in the popularity of the asset/liability studies. The demand for these asset/liability studies has occurred due to various reasons, including: market conditions, switching from defined benefit funds to defined contribution funds, additional contributions, increased liabilities due to the baby boomers retirement and changes in the future benefits structure.
According to Chernoff [2003], a pension fund cannot just maximize its return by using traditional efficient frontier method (see further section 2.3.2). The correct way is to match pension assets against pension liabilities, and he simplifies:

> match the assets and the liabilities and go to bed

Ito [1995] argues that the aim of pension fund asset management is to provide funding for the pension liabilities, but a pension fund sponsor has also a secondary goal that is the achievement of an "earnings spread" (i.e. the positive gap between assets and liabilities), as this earnings spread can reduce the requirement for future contributions.

2.1.4 The role of the actuary in the management of pension plans

Actuary is a figure which has the expertise mainly in pension fund and life insurance areas. In both cases the actuary must rationally value products, which provide the customer with future income stream, in return for a premium stream paid to the provider. There are, however, some fundamental differences between life insurance and pension plans (see AA.VV. [1995], Aitken [1996], Anderson [1992]). First, most pension plans have fewer participants than the typical number of customers of an insurance company. Pension plans vary in their member count from as few as one or several participants to, rarely, as many as hundreds of thousands, but a typical customer base of a life insurance company is at least in tens of thousands, if not hundreds of thousands. This smaller number of participants means that random fluctuations of assets and liabilities of pension plans may have a more profound effect on plan funded status than in the case of a life insurance company. This increased uncertainty must be taken into account by the actuary when establishing pension plan liabilities. The second key difference lies in the timing of benefit payments. Pension plan participants may withdraw from the plan early, due to termination of employment. They also generally have great latitude in choosing their retirement age, within the bounds set by the early retirement age and the latest age allowed by the plan. The actual amount of the benefit will be directly influenced by the date chosen and, additionally, indirectly, the date will affect the final salary (or final salary average), again influencing the benefit amount. This makes the work of the plan actuary more challenging, especially if one wants to achieve stable normal cost, a common desire among employers.

Pension plan management requires substantial involvement of the plan actuary (see Coppini and Micocci [2002]). In the case of defined contribution plans, the actuary must assure that all applicable regulations are followed, and that existing plan assets provide appropriate level of projected benefits for plan participants. In the case of defined benefit plans the role of the plan actuary is especially pronounced: it is the actuary who values plan benefits granted.

In general, a pension plan actuary has the following responsibilities:

- to know generally the accepted pension valuation and funding methods,

- to know which methods are applicable to the plan under consideration,
- to establish appropriate assumptions for valuation,
- to estimate the effect of plan size on the stability of its funding,
- to value benefits other than retirement benefits, if granted (e.g., disability benefits);
- to model future cash flows of the plan,
- to value plan assets appropriately and
- to model sensitivity of the plan to changing parameters such as interest rates, mortality, or general economic variables.

The above list illustrates that the plan actuary must possess vast knowledge and experience in order to meet such a variety of responsibilities.

2.2 Investment Policy and Asset Allocation

In investment policy, the board of the pension fund decides about the asset classes in which the fund invests its wealth. Asset allocation is the process of choosing among possible asset classes.
A large part of financial planning consists of finding an asset allocation that is appropriate for a given investor in terms of their appetite for and ability to shoulder risk. This can depend on various factors. Asset Allocation is the product of an examination of an investor's needs and objectives. Asset allocation, done well, is a plan to invest in assets or asset classes which will best meet the needs and objectives of the investor. Investors seeking high returns and willing to expose their investments to an elevated amount of risk will allocate to equity (ownership) investments. Investors seeking stability and income will allocate to debt investments. Most investors, particularly personal investors, will find mixtures of equity and debt investments most nearly meets their needs. Asset Allocation can be practised by optimization techniques, minimizing risk for a given level of return or maximising return for a given level of risk. It also can be accomplished as goal based investing. Asset allocation techniques are based on the idea that in different years a different asset is the best-performing one. It is difficult to predict which asset will perform best in a given year. Therefore, although it is psychologically appealing to try to predict the "best" asset, proponents of asset allocation consider it risky. Experts in the field note that someone who "jumps" from

one asset to another, according to whim, may easily end up with worse results than does someone following any consistent plan.
A fundamental justification for asset allocation is the notion that different asset classes offer returns that are not perfectly correlated, hence diversification reduces the overall risk in terms of the variability of returns for a given level of expected return. Therefore, having a mixture of asset classes is more likely to meet the investor's wishes in terms of amount of risk and possible returns. In this respect, diversification has been described as "the only free lunch you will find in the investment game[1]". Academic research has painstakingly explained the importance of asset allocation and the problems of active management. This explains the steadily rising popularity of passive investment styles using index funds. Although risk is reduced as long as correlations are not perfect, it is typically forecast (wholly or in part) based on statistical relationships (like correlation and variance) that existed over some past period. Expectations for return are often derived in the same way. When such backward-looking approaches are used to forecast future returns or risks using the traditional mean-variance optimization approach to asset allocation of modern portfolio theory (MPT) (see 2.3.2), the strategy is, in fact, predicting future risks and returns based on past history. As there is no guarantee that past relationships will continue in the future, this is one of the "weak links" in traditional asset allocation strategies as derived from MPT. Other, more subtle weaknesses include the "butterfly effect", by which seemingly minor errors in forecasting lead to recommended allocations that are grossly skewed from investment mandates and/or impractical, often even violating an investment manager's "common sense" understanding of a tenable portfolio-allocation strategy.
Once asset allocation had been decided, portfolio managers try to create extra value by taking advantage of certain situations in the marketplace, or, in other words, they perform a so called tactical asset allocation. It is an active management portfolio strategy that rebalances the percentage of assets held in various categories in order to take advantage of market pricing anomalies or strong market sectors. Many papers are devoted to understanding the importance of asset allocation in explaining the total financial wealth of a fund in comparison with tactical asset allocation. In 1986, Gary P. Brinson, L. Randolph Hood, and Gilbert L. Beebower published a study about asset allocation of 91 large pension funds measured from 1974 to 1983. They replaced the pension funds' stock, bond, and cash selections with corresponding market indexes. The indexed quarterly return were found to be higher than pension plan's actual quarterly return. The two quarterly return series' linear

[1] Paul Merriman, Merriman Inc founder

correlation was measured at 96.7%, with shared variance of 93.6%. A 1991 follow-up study by Brinson et al. [1991] measured a variance of 91 5%. The conclusion of the study was that replacing active choices with simple asset classes worked just as well as, if not even better than, professional pension managers. Also, a small number of asset classes was sufficient for financial planning. Financial advisors often pointed to this study to support the idea that asset allocation is more important than all other concerns, which the Brinson [1986] study lumped together as ”market timing” (tactical asset allocation).

In 1997, William Jahnke initiated debate on this topic, attacking the Brinson [1986] study in a paper titled The Asset Allocation Hoax. The Jahnke [1997] discussion appeared in the Journal of Financial Planning as an opinion piece, not a peer reviewed article. Jahnke's main criticism, still undisputed, was that Brinson [1986]'s use of quarterly data dampens the impact of compounding slight portfolio disparities over time, relative to the benchmark. One could compound 2% and 2.15% quarterly over 20 years and see the sizable difference in cumulative return. However, the difference is still 15 basis points (hundredths of a percent) per quarter; the difference is one of perception, not fact.

In 2000, Ibbotson and Kaplan used five asset classes in their study ≪Does Asset Allocation Policy Explain 40, 90, or 100 Percent of Performance?≫. The asset classes included were large-cap US stock, small-cap US stock, non-US stock, US bonds, and cash. Ibbotson and Kaplan [2000] examined the 10 year return of 94 US balanced mutual funds versus the corresponding indexed returns. This time, after properly adjusting for the cost of running index funds, the actual returns again failed to beat index returns. The linear correlation between monthly index return series and the actual monthly actual return series was measured at 90.2%, with shared variance of 81.4%. Ibbotson and Kaplan [2000] concluded 1) that asset allocation explained 40% of the variation of returns across funds, and 2) that it explained virtually 100% of the level of fund returns. Gary Brinson has expressed his general agreement with the Ibbotson and Kaplan [2000] conclusions.

A 2000 paper by Meir Statman found that using the same parameters that explained Brinson [1986]'s 93.6% variance result, a hypothetical financial advisor with perfect foresight in tactical asset allocation performed 8.1% better per year, yet the strategic asset allocation still explained 89.4% of the variance. Statman [2000] says that strategic asset allocation is movement along the MPT efficient frontier, whereas tactical asset allocation involves movement of the efficient frontier. A more common sense explanation of the Brinson [1986] study is that asset allocation explains more than 90% of the volatility of returns of an overall portfolio, but will not explain the ending results of your portfolio over long periods of time. Hood notes in his review of the material over 20 years, however, that explaining performance over time is possible with the Brinson [1986] approach but was not the focus of the original paper.

2.2.1 Literature on optimal asset allocation in pension funds

The debate over an optimal asset allocation for a pension fund has two extreme views. One view states that bonds are the only way to match assets with liabilities, while the contradicting view recommends equity exposures. Equity and fixed income are generally the biggest investment classes in pension funds. Bodie et al. [1999] argue that a pension fund, with a financially sound sponsor corporation, should not invest in equities at all. A fully funded pension fund should only invest in fixed income assets and, thus, minimize the additional contributions. However, it is found that pension funds generally invest around 40 to 60 percent of their portfolio in equities. Bodie et al. [1999] find three reasons for these equity investments. First, a sponsor sees the defined benefit fund more like defined contribution fund: a sponsor may believe that a successful strategy may lead to extra benefits and tries to maximize benefits paid to employees. Second, sponsor believes in market timing and security selection ability. Third, a sponsor in financial distress may have an incentive to invest in riskier assets. According to Blake [2001], fixed income investments are encouraged by regulators simply because the discount rate used in pension liability calculation by actuaries and accountants is based on bond yields. This means that in order to avoid the short-term mismatch between assets and liabilities, pension fund asset allocation should be more heavily weighted toward bonds. In the U.S., pension funds have a special tax treatment and this gives them incentive to create an asset mix with a large spread between pretax and after-tax returns. Therefore, tax reasons drive pension funds to invest more in bonds than in equities (Bodie et al. [1999]). For a fully-funded healthy pension fund, Bodie [1988] recommends investments only in taxable fixed-income securities.

Black [1989] studies the role of equities in the portfolio of a pension fund. Stocks are used to achieve higher expected return, and therefore, meet the pension obligation in the future while helping to lower expected pension costs. Black [1989] acknowledges that some managers think about bonds as the only answer to hedge their pension liabilities. However, stocks also should be viewed as a hedge against a potential increase in pension liabilities.

Stocks particularly hedge against the risk of salary inflation, which causes an increase in liabilities. Black states that stock prices and the expected rate of inflation move in tandem. This is called an "economic" view of liabilities. Black [1989] divides pension liability into two categories; a narrow view and a broad view. Both of these liability types act like a security. The narrow liability is defined as a present value of all vested benefits for current employees. Hence it is only tied to past and current while not including the future. However, the narrow liability is only a snapshot of current work force, and hence, the narrow liability is changing all of the time. Hedging for the type of narrow liability is mainly performed using interest rate hedging methods and therefore, the narrow view suggests investing in bonds to hedge the liabilities. According to Black, the broad liability is the present value of all benefits to be paid, and therefore it is always greater than the narrow liability. The broad liability is the narrow liability plus salary increases, benefits to be accrued, changes in the benefits and additions to the workforce. In most cases the broad view suggests investing in stocks is superior.
Also Chun et al. [2000] argue that a growing company typically should have more equity investments, and less bonds or real-estate investments, due to the higher expected rate of return of equity. Peskin [1997] argues that pension fund's equity exposure is critical to the future contribution cost. The equity exposure varies between pension funds, and the optimal equity exposure to each fund is found by using the following factors:

- noise in liabilities: if pension fund's liabilities do not act like bonds (i.e. the relationship between bonds and liabilities is volatile) and liabilities have a lot of noise, then fund should have greater equity exposure;

- weight attached to surplus value: extraordinary equity returns generate more surpluses;

- funded status of plan: the extreme funds, both poorly funded and well-funded, should have larger equity exposures. Poorly funded funds need the upside of equity investment and well funded funds have large buffer to protect the future contributions from the downside risk.

- growth in workforce: a growing fund, with active liabilities growing faster than retired liabilities, should have more equity exposure.

Stux [1995] divides pension fund portfolio management by using two steps. First, a pension fund needs to decide which broad asset classes to invest in. Typically, the asset classes include fixed income, equities, real estate, money market instruments, venture capital and private investments.

This step is called strategic asset allocation and it is the most important part of a pension fund's asset management, as the strategic asset allocation decisions heavily affect the performance of a pension fund. The second step includes the actual implementation of the chosen strategic asset allocation by choosing internal or external fund managers, and putting in practice the particular investment strategies and security selection process. This step is also important, but has less influence on pension fund's overall performance. The liability structure of a pension fund defines how much risk the fund can take. The risk can be divided into two categories: the risk of long-term shortfall and the risk of near-term shortfall. One solution is to increase the near-term shortfall risk, and leverage existing assets to gain higher long-term return. This is called a short-term versus long-term dilemma: if the assets are growing fast enough, they will not fall below liabilities in the long-term. However, the likelihood of a shortfall increases in short-term, as the asset/liability relation is volatile. Another solution is to secure short-term needs, but that risks long-term return. A traditional view suggests that pension fund should only invest in a well-diversified fixed income portfolio, which can be duration-matched with the liability stream. The bond investment is also suggested due to the tax-advantages in some countries. Equity exposures are mainly for the higher expected rate of return on equity investments. This upside potential is especially needed in funds with younger participants, growing workforce and when salary inflation is expected.

2.3 Optimal Asset Allocation

One of the frequent questions in finance is how to allocate a certain amount of money in different assets and at what time instant. The earliest approach to consider the optimal portfolio problem is the so-called *mean-variance approach*. It was pioneered by Markowitz [1952] and is basically a single-period model which makes an one-off decision at the beginning of the period and holds on until the end of the period.
Gradually, researchers extended this single-period model to continuous-time models (Merton [1969], Merton [1971]). By applying results from stochastic control theory (see 3.2.4) to the optimal portfolio problem, explicit solutions have been obtained for some special cases. Using stochastic optimal control theory, Merton was able to establish important financial economic principles. In the following section we recall the single-period model proposed by Markowitz [1952]. Instead, in Chapter 3 we develop a continuous-time model to get the optimal asset allocation in a Italian professional order pension fund.

2.3.1 Optimal consumption decision versus the level of interest rates

The value of financial assets is established in the capital markets, such as the stock exchanges, or the bond markets. The issue of pricing of capital assets is one of the central problems of modern financial theory. In the simplest model of a capital market, a consumer exchanges current consumption for a capital asset, which in turn is later exchanged for future consumption. The exchanges are, of course, held with the use of the monetary unit of a given national economy. In order to understand this process better, let us describe the preferences of the consumer under consideration with a utility function $U(C_{t+1}; C_t)$ where C_t is the current consumption, and C_{t+1} is the future consumption.

The consumer's problem is then to find the maximum of $U(C_{t+1}; C_t)$ given the income level of Y_t and Y_{t+1} in the future.

Let us assume that the economy offers a market for exchanging today's income for future consumption, i.e., a capital market. If a unit of such a capital asset has the price of P_t and if the consumer buys x units, then we must have:

$$Y_t - x \cdot P_t = C_t$$

where the expression C_t is nonnegative and it denotes today's consumption (we will ignore the unrealistic possibility of zero consumption). If the unit of the capital asset has the price of P_{t+1} in the future, then

$$Y_{t+1} - x \cdot P_{t+1} = C_{t+1}$$

Therefore, the consumer will maximize overall utility $U(Y_t - x \cdot P_t; Y_{t+1} - x \cdot P_{t+1})$ by choosing an optimal level of x; if the utility function U is concave and differentiable with respect to each of its variables, then by differentiating U with respect to x we obtain the following expression for the optimal level of x:

$$\frac{dU}{dx} = -\frac{\partial U}{\partial C_t} P_t + \frac{\partial U}{\partial C_{t+1}} P_{t+1} = 0 \tag{2.1}$$

Under certain assumptions this optimization problem may have a solution on the boundary of the domain of the problem function, resulting in zero consumption, but we have decided to ignore this unrealistic solution. In all other cases, 2.1 is a necessary and sufficient condition for the optimal division of income between consumption and investment.

The condition 2.1 is actually equivalent to:

$$\frac{\frac{\partial U}{\partial C_{t+1}}}{-\frac{\partial U}{\partial C_t}} = \frac{P_t}{P_{t+1}}$$

Using the economic terminology, the above equation has the following interpretation. The marginal utility of future consumption has the same relationship to today's consumption as today's price of a capital asset has to the price of the said capital asset in the future.
It should be also noted that the ratio $\frac{P_t}{P_{t+1}}$ is equal to the absolute value of the derivative $\frac{dC_{t+1}}{dC_t}$.

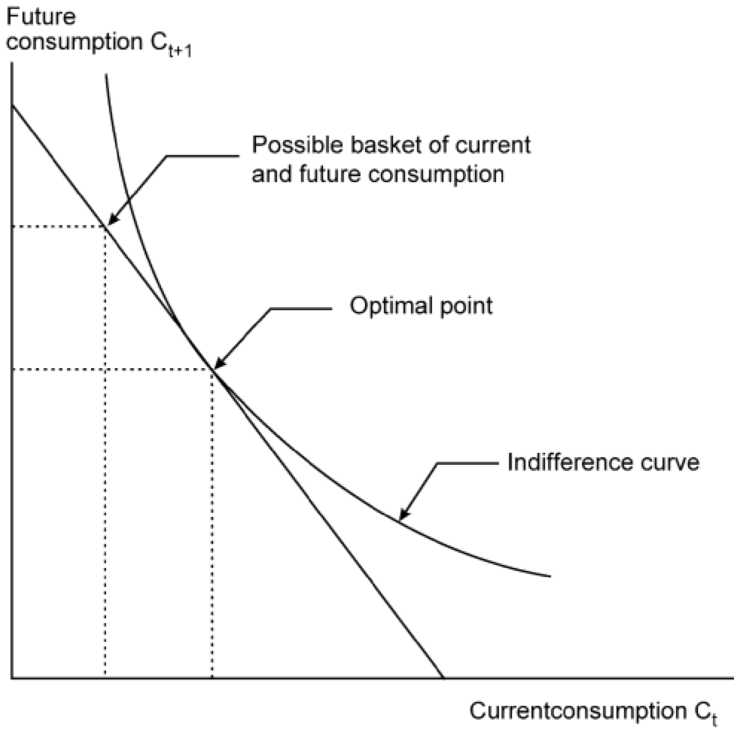

Fig. 2.2: Consumer's consumption decision

The choices available to the consumer are presented in 2.2. The optimal point is the point of tangency of an indifference curve (a curve connecting all points with the same level of $U(C_{t+1}; C_t)$ and the line given by the equation:

$$C_{t+1} = Y_{t+1} + (Y_t - C_t) \cdot \frac{P_{t+1}}{P_t} \tag{2.2}$$

obtained from the equations expressing division of income between consumption and investment at time t; and its consequences at time $t+1$; after eliminating the variable x: this is implied by the fact that the points lying on the line 2.2 represent all possible decisions of the consumer, while between two

indifference curves, the one giving higher utility lies further North-East.
The quantity $i = (\frac{P_{t+1}}{P_t}) - 1$ is the rate of return offered by the capital asset. If the price of a capital asset is deterministic, we call the rate of return the risk-free rate. In reality, however, most future prices and rates of return are quite uncertain. This uncertainty is the source of risk of capital assets. We will return to this subject later in this chapter.
Global optimization for all consumers creates the supply of savings in the economy, given by the function

$$x^*(Y_t; Y_{t+1}; i)$$

which determines the value of non-consumed income in relation to consumers' income today and in the future, and to the interest rate.
The other element creating the equilibrium in the economy is the demand for savings. That function is created as follows: for a given interest rate, the firms undertake those projects, which have a positive net present value, and reject those that have a negative net present value. The total demand is the sum of all funds needed for the projects with positive net present value. Of course, in reality this process is much more complex, as the savings plans of the consumers, as well as the projects undertaken by the firms have varying time horizons, and this causes the existence of various interest rates for various time horizons (this is called the yield curve or term structure of interest rates). Furthermore, there is a great uncertainty concerning the returns of the projects undertaken by the firms. This causes the value of capital assets, representing the right to income from the capital investment projects, to fluctuate because of the riskiness of such income, and changing risk preferences of the capital markets investors. We will attempt to address this issue now.

2.3.2 Optimization of an investment portfolio: Markowitz's model

Markowitz [1952] created the first theory of capital markets that included a consideration for the risk of capital assets. We will present the outline of his theory, commonly called the *Modern Portfolio Theory.*
In the model of Markowitz it is assumed that the rate of return of a capital asset is a random variable R. We assume that the securities market consists of N elements with random rates of return $R_1; \ldots; R_N$. The buyers and sellers of capital assets, called investors, allocate their resources to a portfolio of assets. For a given investor, the portions of this investor's portfolio invested in each of the available assets are $x_1; \ldots; x_N$. Thus the rate of return of the

portfolio, also a random variable, is

$$R_{\mathbf{x}} = \sum_{j=1}^{N} x_j R_j \tag{2.3}$$

where $\mathbf{x} = (x_1; \ldots; x_N)^\top$. Markowitz assumed that the preferences of the investors can be summarized with the expected value and the standard deviation of the rate of return of the portfolio. The standard deviation (or variance) is the measure of risk in this model. Let us note that

$$\mathbb{E}(R_{\mathbf{x}}) = \sum_{j=1}^{N} x_j \mathbb{E}(R_j) \tag{2.4}$$

and that

$$Var(R_{\mathbf{x}}) = \sum_{l=1}^{N} \sum_{j=1}^{N} x_l x_j Cov(R_l; R_j) \tag{2.5}$$

Markowitz assumed that every investor's objective is to maximize the expected value of the rate of return of the portfolio, and to minimize the standard deviation (or the variance) of that rate of return.
Only in rare situations one can find a portfolio, which indeed has both the highest expected rate of return and the lowest variance. In reality, however, most research in modern finance tends to predict that on average instruments with higher rates of return tend to have higher degree of risk, as described by the standard deviation. Thus, in practice, it takes some more work to find an optimal portfolio.
A portfolio will be called *efficient* if there is no other portfolio with a higher expected rate of return and lower variance. Thus, efficient portfolios are maximal elements in the set of all portfolios with respect to the partial order given by simultaneous increasing of the expected return and decreasing of the variance of the rate of return. In the model studied here, such maximal elements, i.e., efficient portfolios, always exist. In order to find them, one usually assumes that the investor aims to minimize the following expression:

$$Var(R_{\mathbf{x}}) - 2\tau \mathbb{E}(R_{\mathbf{x}}) \tag{2.6}$$

where $\tau > 0$ is the risk tolerance coefficient for the investor (defined as the first derivative of the utility function divided by the absolute value of the second derivative of the utility function). This minimization is performed under the constraint: $\sum_{j=1}^{N} x_j = 1$. Negative values of the x_j coefficients are allowed and refer to short position in a given security.

A short sale is a sale of a security, which is borrowed (from a broker), with that security replaced later by a purchase (covering of a short). Short sale requires a deposit (margin deposit) of a certain amount of money (typically 50% of the amount received from the sale of the security). This margin deposit may, or may not, earn interest with the broker. The money received from the sale generally does not earn interest, and is held with the broker until the short is covered. A short sale creates new supply of the security shorted, so if the shorted security pays income (e.g., dividends) that income must be produced by the short seller, in order for the holder of the newly created security to receive such income. An issue that arises in short sales is the calculation of the rate of return received by the short seller. In such a calculation, we should note that the rate of return is determined by the initial cash outlay of the investor, and the final cash flow received at the end, when the short is covered, as well as possible intermediate cash flows of dividends paid by the investor and interest received.

General formula for the effective yield earned over the period of investment is

$$i = \frac{P + I - D}{M}$$

where the symbols have the following meanings: M is the margin requirement (initial cash outlay by the short seller), D the amount of dividends paid by the short seller to the newly created security's owner, I the amount of interest earned by the short seller on the margin deposit (assuming there is no interest earned on the cash received from the initial short sale, if there is such interest, then it must be added here), and P the profit on the short sale transaction, i.e., sale proceeds short covering repurchase cost.

In order to find the desired minimum of the expression 2.6, we create the Lagrangian

$$\mathcal{L}(\mathbf{x}, \lambda) = Var(R_{\mathbf{x}}) - 2\tau\mathbb{E}(R_{\mathbf{x}}) - \lambda(\mathbf{e}^{\top}\mathbf{x} - 1)$$

where $\mathbf{e} = (1; \ldots ; 1)^{\top}$ is the unit vector in $\mathbb{R}^N$; and $\mathbf{e}^{\top}\mathbf{x}$ is the scalar product (also known as the dot product) in the same space. The parameter λ is a Lagrange multiplier. The minimization problem for 2.6 under the constraint $\sum_{j=1}^{N} x_j = 1$ may be reduced to the unconstrained minimization of $\mathcal{L}(\mathbf{x}; \lambda)$. In fact, assume that there is a $\lambda \in \mathbb{R}$ and $x_j \in \mathbb{R}^N$ such that $\mathbf{e}^{\top}\mathbf{x} = 1$ and for every $\mathbf{x} \in \mathbb{R}^N$

$$\mathcal{L}(\mathbf{x}; \lambda) \geq \mathcal{L}(\mathbf{x_l}; \lambda) = Var(R_{\mathbf{x_l}}) - 2\tau\mathbb{E}(R_{\mathbf{x_l}})$$

If $\mathbf{e}^{\top}\mathbf{x} = 1$, then from the above inequality we infer that

$$Var(R_{\mathbf{x}}) - 2\tau\mathbb{E}(R_{\mathbf{x}}) \geq Var(R_{\mathbf{x_l}}) - 2\tau\mathbb{E}(R_{\mathbf{x_l}})$$

Therefore $\mathbf{x_l}$ also minimizes 2.6 on the set of $\mathbf{x} \in \mathbb{R}^N$ such that $\mathbf{e}^\top \mathbf{x} = 1$.
Since the Lagrangian

$$\mathcal{L}(\mathbf{x};\lambda) = \sum_{l=1}^{N}\sum_{j=1}^{N} x_l x_j Cov(R_l;R_j) - 2\tau \sum_{j=1}^{N} x_j \mathbb{E}(R_j) - \lambda \left(\sum_{j=1}^{N} x_j - 1 \right)$$

,

is a convex function, it follows from the *Kuhn Tucker Theorem* that the necessary and sufficient condition for it to have a minimum at $\mathbf{x}$ is that the gradient $\nabla\mathcal{L}(.;\lambda)$ for $\mathbf{x}$ is a zero-vector and that $\mathbf{e}^\top \mathbf{x} = 1$. Since

$$\frac{\partial \mathcal{L}(\mathbf{x};\lambda)}{\partial x_k} = 2\sum_{l=1}^{N} x_l Cov(R_l;R_j) - 2\tau \mathbb{E}(R_k - \lambda$$

the necessary and sufficient condition for $\mathcal{L}$ to reach its minimum at $\mathbf{x}$ under the constraint $\mathbf{e}^\top \mathbf{x} = 1$, therefore, becomes a system of $N+1$ equations

$$\begin{cases} 2\sum_{l=1}^{N} x_l Cov(R_l;R_j) - 2\tau \mathbb{E}(R_k - \lambda = 0 \quad \text{for} k = 1,2,\ldots,N \\ \mathbf{e}^\top \mathbf{x} = 1 \end{cases} \tag{2.7}$$

in which there are also $N+1$ unknowns, $x_1;\ldots;x_N$ and λ. If we use the notation $\mathbf{\Sigma} = [Cov(R_l;R_j)]$ and $\mu = [\mathbb{E}(R_1),\mathbb{E}(R_2),\ldots,\mathbb{E}(R_N)]^\top$ then the system of equations 2.7 can be written in the following vector form:

$$\begin{cases} 2\mathbf{\Sigma x} - 2\tau\mu - \lambda\mathbf{e} = \mathbf{0} \\ \mathbf{e}^\top \mathbf{x} = 1 \end{cases} \tag{2.8}$$

where $\mathbf{0} = (0,\ldots,0)^\top$ is an N-dimensional vertical vector with all coordinates equal to zero.
Assume now that

- the matrix $\mathbf{\Sigma} = [Cov(R_l;R_j)]$ is positive definite;
- the vectors $\mathbf{e}$ and μ are linearly independent.

If we substitute $\tau = 0$ in 2.7 then we can easily get

$$\begin{cases} \mathbf{x}^{min} = \frac{\lambda}{2}\mathbf{\Sigma}^{-1}\mathbf{e} \\ \mathbf{e}^\top \mathbf{x}^{min} = 1 \end{cases} \tag{2.9}$$

where $\mathbf{x}^{min}$ is the solution of 2.7 under the condition that $\tau = 0$.
Let us multiply the left-hand side of the vector equation in 2.9 by $\mathbf{e}^\top$ and calculate that $\frac{\lambda}{2} = (\mathbf{e}^\top \mathbf{\Sigma}^{-1}\mathbf{e})^{-1}$ This eventually gives us

$$\mathbf{x}^{min} = \frac{\mathbf{\Sigma}^{-1}\mathbf{e}}{\mathbf{e}^\top \mathbf{\Sigma}^{-1}\mathbf{e}}$$

We can proceed in an analogous fashion for $\tau > 0$ and obtain a general solution $\mathbf{x}^*$ of the system of 2.7.

$$\mathbf{x}^* = \frac{\mathbf{\Sigma}^{-1}\mathbf{e}}{\mathbf{e}^\top \mathbf{\Sigma}^{-1}\mathbf{e}} + \tau \left(\mathbf{\Sigma}^{-1}\mu - \frac{\mathbf{e}^\top \mathbf{\Sigma}^{-1}\mu}{\mathbf{e}^\top \mathbf{\Sigma}^{-1}\mathbf{e}} \mathbf{\Sigma}^{-1}\mathbf{e} \right)$$

Let us use this notation:

$$\mathbf{z}^* = \mathbf{\Sigma}^{-1}\mu - \frac{\mathbf{e}^\top \mathbf{\Sigma}^{-1}\mu}{\mathbf{e}^\top \mathbf{\Sigma}^{-1}\mathbf{e}} \mathbf{\Sigma}^{-1}\mathbf{e}$$

and note that

$$\mathbf{e}^\top \mathbf{z}^* = 0$$

Therefore the solution of the problem of portfolio optimization is the set of efficient portfolios of the following form:

$$\left\{ \mathbf{x}^* = \mathbf{x}^{min} + \tau \mathbf{z}^*, \quad \tau \geq 0 \right. \tag{2.10}$$

where $\mathbf{x}^{min}$ is the solution obtained under the constraint $\tau = 0$, giving the portfolio of minimum variance, while $\mathbf{z}^*$ is a vector dependent on the expected values and covariances of returns of securities available in the market such that $\mathbf{e}^\top \mathbf{z}^* = 0$.
We say that the portfolio $\mathbf{z}^*$ is self-financing, as all long positions (i.e., securities owned in the portfolio) in it are created with funds obtained from short positions in it.

We also have

$$\mathbb{E}(R_{\mathbf{x}^*}) = \mathbb{E}(R_{\mathbf{x}^{min}}) + \tau \mathbb{E}(R_{\mathbf{z}^*})$$

and

$$Var(R_{\mathbf{x}^*}) = Var(R_{\mathbf{x}^{min}}) + \tau^2 \mathbb{E}(R_{\mathbf{z}^*}) \tag{2.11}$$

The first of the above equalities is obtained from **??** and 2.10. It can be shown easily, by using the definitions of $\mathbf{z}^*$ and $\mathbf{x}^{min}$, that

$$(\mathbf{x}^{min})^\top \mathbf{\Sigma} \mathbf{z}^* = 0$$

and this completes the proof of the equality **??**.
The set of efficient portfolios in the plane, which has the portfolio variance on the x-axis and the expected return of a portfolio on the y-axis, is called the *efficient frontier*. This set is a parabola (see 2.3). If, instead, we place the standard deviation of the portfolio return on the x-axis, then the efficient frontier so created is a hyperbola.

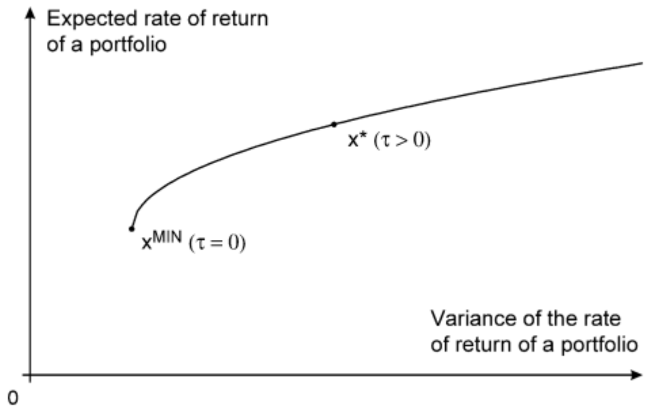

Fig. 2.3: Efficient frontier

2.4 Risks faced by a pension fund

The key objective of a pension plan is a secure delivery of pension benefits to the plan participants. The plan sponsor has a choice of a variety of funding methods and risk management strategies. All actuarial funding methods have one characteristic in common: they require that at least some of the future liabilities of the plan are financed with capital assets purchased today. A pension plan has assets in the form of financial instruments, while its liabilities are set actuarially in relation to benefits granted. While holding securities as assets is designed to assure delivery of benefits, it does become a source of additional risks for the plan.
What kind of risks does a pension plan face? The Society of Actuaries Committee on Valuation and Related Matters (1979) charged with, among others, this question, defined the following three key kinds of risks faced by a life or annuity insurance enterprise.

- C1 risk: asset default and depreciation risk, i.e., the risk of suffering losses in equities or bond losses due to credit risk (but not due to interest rate risk). This is an asset-side risk.
- C2 risk: pricing risk, i.e., the risk that the product issued by the financial intermediary has been issued at an inappropriate price (e.g., it did not provide the intermediary with proper compensation for risks that the intermediary has assumed). This is a liabilities-side risk.
- C3 risk: interest rate risk or, more generally, asset-liability management

risk, i.e., the risk that assets and liabilities of the intermediary may respond differently to changes in market prices or indexes (notably, changes in interest rates).

- C4 risk: the general business risk, caused by the management, political and regulatory environment, or other socio-economic factors.

The level of C1 risk is a function of the investment policy assumed by the intermediary. In the case of insurance companies and pension plans, C2 risk is clearly the responsibility of an actuary. For a pension plan, plan liabilities may turn out to be higher than assumed, due to factors such as mortality, salaries, withdrawal, or retirement pattern, diverging from the actuarial assumptions used in valuation. But the asset-liability management risk C3 is a function of how an intermediary manages its assets and liabilities together. Such integrated comprehensive risk management is a relatively new idea in the management of financial intermediaries.
The key issue of asset-liability management is the question of whether assets and liabilities exhibit similar behavior when market conditions change, and if their behavior is different, whether the financial intermediary is compensated properly for assuming that risk. Furthermore, an intermediary should be able to understand and control the risk assumed (see Artzner et al. [1999]). Clearly, an intermediary cannot control the level of interest rates. However, one of the main objectives of asset liability management is to create a comprehensive and integrated policy of dealing with the changes in interest rates. Interest rate risk can cause insolvency of a financial intermediary. For a pension plan we arrive at the following Principle of Asset Liability Management:

> Asset liability management, and interest rate risk management in particular, for a pension plan, should be based on the principle that either assets and liabilities should behave in the same way under changing market conditions, or if they diverge under changing market conditions, the pension plan managers should understand that risk, be able to control it, and be compensated for assuming it.

Chapter 3

A quantitative model for optimal asset allocation in a PAYGO pension fund under financial and actuarial risk

The first rule is not to lose.
The second rule is not to forget the first rule.

Warren Buffet

3.1 Research questions and research methodology and methods

3.1.1 Research questions

The purpose of this thesis is precisely to identify the optimal dynamic investment strategy (i.e. the optimal asset allocation rebalancing rule over a certain time span) as a function of the portfolio manager's targets in a PAYG pension plan (such as an Italian Professional Order pension fund). In particular, we consider that the portfolio optimization problem faced in this case presents time-varying market returns (financial risk) and most of all, it is a long-term investor problem (see Brennan et al. [1997], Blake et al. [2000], Haberman and Sung [1994], Hainaut [2006], Menoncin [2002]). In general,

the problem of long-term investments is a well-established research field introduced by Samuelson [1969] and Merton [1969], Merton [1971]. Since then, it is well understood that a short-term portfolio optimization can be very different from long-term portfolio optimization. In this chapter, continuous-time modeling along the lines of Merton [1971] is pursued.
Moreover, the proposed model entails another source of risk, typically faced by a pension fund: actuarial risk. This comes from the possibility that the assumptions that actuaries implement to manage a specific pension plan may turn out wrong or somewhat inaccurate. In particular, we deal with assumptions concerning future cash flows coming from expected contributions and benefits making them random rather then deterministic.
Furthermore, we believe that, in dealing with a PAYG pension fund, is particulary important to take expressly into account the sustainability of the fund, i.e. the balance between the active and retired members. To express this concept in a mathematical way, we ground on an empiric indicator to measure the financial sustainability of a pension funds, expressed by the ratio of the fund value to the current expenditure for pensions. The Italian legislation proposes, as indicator for the retirement funds of the professional orders, the ratio of the fund value to 5 times the current expenditure for pensions. In the model, we assume that the asset manager tries, on the one hand, to maximize the above mentioned ratio and, on the other hand, to keep it in equilibrium in order to avoid useless extreme speculative investments.
Over the past few years, a vast literature regarding demographic risks and the sustainability in public PAYG system has been developed (see Feldstein [1996] ; Feldstein [2005]) on the reform of the Social Security in the USA). Pemberton [2000] analyzes the transition from PAYG to funded pensions. Sinn [2000] and Blake [2000] illustrate the characteristics and the differences between PAYG and funded system (for further details on this topic see Cannas et al. [2010]). There are few contributions on private PAYG scheme. Ferrara [2002] analyzes the equilibrium of a PAYG pension fund with defined benefit, where new entrants are random variables depending on the employment rate. Menoncin [2005] studies the allocation problem for a pension fund, which behaves according to a PAYG rule: considering the total number of workers and pensioners as random variables. He shows how the demographic risk affects the optimal portfolio in a cyclical way. Melis and Trudda [2009] propose a model for the evolution of a PAYG pension fund, with stochastic new entrants and global asset return.

3.1.2 research methodology

The whole research carried out in this work is based on a positive-deductive methodology. Deductive reasoning works from the more general to the more specific, sometimes this is informally called a ”top-down” approach. Conclusions follows logically from premises. As we have pointed out above, this work is based on arguments founded on rules and accepted principles (e.g. to describe asset manager preferences or financial market behaviour) which are generally used for deductive reasoning. From an epistemological point of view, this research can be assessed as taking a positivist position that advocates the application of the methods of the natural sciences to the study of the social reality an beyond. For further details on business research methods the reader is referred to Bryman and Bell [2007].
Doubtless, many quantitative methods represented the real toolbox to perform this research, the most of them comes from mathematical theory.
The next section is devoted to a detailed presentation of the mathematical methods used to construct the model presented in Section 3.3. According to the random nature of the variable that come into play in this research, we have made a large use of instruments coming from probability theory (see Björk [2008], Oksendal [2003]).

3.2 Mathematical methods

3.2.1 Stochastic processes

Definition 1. *A stochastic process $W(t)$ is called Brownian motion if it satisfies the following conditions:*

1. *Independence:* $W(t + \Delta t) - W(t)$ *is independent of* $\{W(\tau)\}$ *for all* $\tau \leq t$*;*

2. *Stationarity: The distribution of* $W(t+\Delta t) - W(t)$ *does not depend on* t*;*

3. *Continuity:* $\lim_{\Delta t \to 0} \frac{P(|W(t+\Delta t) - W(t)| \geq \delta)}{\Delta t} = 0$ *for all* $\delta > 0$

Please note that the third assumption is expressed with probabilities: discontinuities in sample functions can only occur with probability zero. Hence, there is a version of the Brownian motion with all sample functions continuous.
This definition induces the distribution of the process $W(t)$.

Theorem 1 (Normally distributed increments of Brownian motion)**.** *If $W(t)$ is a Brownian motion, then $W(t) - W(0)$ is a normal random variable with mean μt and variance $\sigma^2 t$, where μ and σ are constant real numbers.*

As a result of this theorem, we have the following density function of a Brownian motion:

$$f_{W(t)}(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}} \tag{3.1}$$

An irritating property of Brownian motion is that its sample paths are not differentiable. This is easily verified in the mean-square sense:

$$\mathbb{E}\left[\left(\frac{W(t+\Delta t) - W(t)}{\Delta t}\right)^2\right] = \frac{\mathbb{E}[(W(t+\Delta t) - W(t))^2]}{\Delta t^2} = \frac{\sigma^2}{\Delta t^2} \tag{3.2}$$

This diverges for $\Delta t \to 0$ and therefore $W(\dot{})$ is not differentiable in L^2.
The Brownian motion $W(\dot{})$ (starting at $W(0) = 0$) has many more bizarre and intriguing properties. Some of them are listed below:

- Autocovariance function: $\mathbb{E}(W(t) - \mu t)(W(\tau) - \mu\tau) = \sigma^2 \min(t, \tau)$
- $Var\left\{\frac{W(t)}{t}\right\} = \frac{\sigma^2}{t}$
- $\lim_{t\to\infty} \frac{W(t)-\mu t}{t} = 0$ with probability 1
- The total variation of the Brownian motion over a finite interval $[0, T]$ is infinite.
- The "sum of squares" of a drift-free Brownian motion is deterministic: $\lim_{N\to\infty} \sum_{k=1}^{N} (W(k \cdot \frac{T}{N}) - W((k-1) \cdot \frac{T}{N}))^2 = \sigma^2 T$. Important consequence: Whenever the term dW^2 appears in a stochastic differential equation, it should be replaced by $\sigma^2 dt$.
- Zero-crossings: In a finite interval $[0, T]$, every sample of a drift-free Brownian motion has infinitely many zero-crossings. The set of zero-crossings is dense in $[0, T]$, i.e., no sample path has isolated zero-crossings.

Definition 2 (Standard Brownian motion)**.** *A Brownian motion is called standard if:*

$$W(0) = 0 \tag{3.3}$$

$$\mathbb{E}[W(t)] = 0 \quad (\mu = 0) \tag{3.4}$$

$$\mathbb{E}[W^2(t)] = t \quad (\sigma^2 = 1) \tag{3.5}$$

In the sequel, a Brownian motion is assumed to be a standard Brownian motion unless explicitly stated otherwise. In most cases, we use the differential form

$$dW(t) = \lim_{\tau \to 0} W(t + \tau) \tag{3.6}$$

with $\mathbb{E}[dW(t)] = 0$and the sum-of-squares property $\mathbb{E}[dW^2(t)] = dt$ The generalization of a Brownian motion from the scalar case to the vector case is straightforward: The scalar drift parameter μ becomes a vector; and the "volatility parameter" σ and the "intensity parameter" σ^2 become symmetric, positive-definite matrices. The notation in the vector case will be Σ instead of σ^2 and $\Sigma^{1/2}$ instead of σ. In the case of a vector-valued standard Brownian motion, it will be assumed that the component processes of the vector are mutually independent.

3.2.2 Stochastic differential equations

A non-standard Brownian motion $X(\Delta)$ satisfies the stochastic differential equation

$$dX(t) = \mu dt + \sigma dW(t) \tag{3.7}$$

$$X(0) = 0 \tag{3.8}$$

where $W(\Delta)$ is a standard Brownian motion.
The geometric Brownian motion $X(\Delta)$ is described by the differential equation

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t) \tag{3.9}$$

It is popular in financial engineering for modeling stock prices.
In the most general nonlinear case, the stochastic differential equation for a stochastic process can be written as follows:

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t) \tag{3.10}$$

3.2.3 Stochastic calculus

Due to the "sum-of-squares" property of the Brownian motion, the rules of differentiation in the stochastic case differ from those in the deterministic case.
Consider the following problem: given a stochastic differential equation for the process $X(\dot{})$

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t) \tag{3.11}$$

$$X(t_0) = X_0 \tag{3.12}$$

find the differential equation for the process $Y(t)$ which is a function of $X(t)$,

$$Y(t) = \phi(t, X(t)) \tag{3.13}$$

where the function $\phi(t, X(t))$ is continuously differentiable in t and twice continuously differentiable in X.
Let us do a Taylor series expansion of 3.13 up to second-order terms:

$$\begin{aligned} dY(t) = \phi_t(t, X)dt + \phi_X(t, X)[f(t, X(t))dt + g(t, X(t))dW(t)] + \frac{1}{2}\phi_{tt}(t, X)dt^2 + \\ \frac{1}{2}\phi_{XX}(t, X)[f(t, X(t))dt + g(t, X(t))dW(t)]^2 + \phi_{tX}(t, X)[f(t, X(t))dt + \\ g(t, X(t))dW(t)]dt + h.o.t. \end{aligned} \tag{3.14}$$

Notice that the term $dW^2(t)$ appears when the square factor of $\phi_{tt}(t, X)$ is expanded. Replacing it by dt and retaining only the terms of first order yield the following result:

$$\begin{aligned} dY(t) = [\phi_t(t, X) + \phi_X(t, X)f(t, X(t)) + \\ \frac{1}{2}\phi_{XX}(t, X)g^2(t, X(t))dt \\ + \phi_X(t, X)g(t, X(t))dW(t) \end{aligned} \tag{3.15}$$

$$Y(t_0, X_0) = \phi(t_0, X(t_0)) \tag{3.16}$$

3.2.4 Stochastic optimal control

In this section, the following stochastic optimal control problem is considered for a dynamic system with the state vector $x(t) \in \mathbb{R}^n$, the admissible control vector $u(t) \in U \subseteq \mathbb{R}^m$ (where U is a time-invariant, convex, and closed subset of $\mathbb{R}^m$, and the standard vector Brownian motion $W(t) \in \mathbb{R}^k$.

Problem: for the dynamic system described by the stochastic differential equation

$$dx(t) = f(x(t), u(t))dt + g(x(t), u(t))dW(t)dW(t) \tag{3.17}$$

with the given deterministic initial state x_0 at the fixed initial time t_0,

$$x(t_0) = x_0 \tag{3.18}$$

find a piecewise continuous control vector $u(t) \in U$ for all times t in the fixed time interval $[t_0, t_1]$, such that the objective functional

$$J = \mathbb{E}\left[K(x(t_1)) + \int_{t_0}^{t_1} L(x(t), u(t))dt\right] \tag{3.19}$$

is maximized.

Hamilton-Jacobi-Bellman Theory:

Theorem 2 (Stochastic Hamilton-Jacobi-Bellman Theorem). *If the partial differential equation*

$$-\mathcal{J}_t(x,t) = \max_{u \in U}\left[L(x(t), u(t)) + \mathcal{J}_x f(x,u) + \frac{1}{2} tr(\mathcal{J}_x x(x,t) g(x,u) g^{\top}(x,u))\right]$$

(3.20)

with the boundary condition

$$\mathcal{J}(x(t_1)) = K(x)$$

admits a unique solution, the globally optimal state feedback control law is

$$u(x,t) = arg \max_{u \in U}\left[L(x(t), u(t)) + \mathcal{J}_x f(x,u) + \frac{1}{2} tr(\mathcal{J}_x x(x,t) g(x,u) g^{\top}(x,u))\right] \tag{3.21}$$

3.3 The model

In this section we have solved the optimal asset allocation problem for a pension fund which operates in a PAYG system and periodically revises its investment strategies. We assume that the fund manager is concerned with guarantee the sustainability (stated by a certain sustainability ratio S_t) of the fund wealth and, at the same time, he has to face fund's risk exposure. The variable S_t is observable and we use the Hamilton-Jacobi-Bellman framework to find the optimal investment strategy for the pension fund.

3.3.1 The financial market

Let us consider a financial market consisting of different assets, such as stocks, bonds with different maturities, or various other kinds of financial assets; the main objective is that of deriving the dynamics of the so-called *self-financing portfolio*. In such a financial market, time is continuous and trading takes place continuously and frictionless.
At any time $t \in [0, T]$, where T is the length of the fund manager's investment horizon, we consider a financial market composed of n ($n > 1$) risky assets

$$F_{1t}, F_{2t}, \cdots, F_{it}, \cdots, F_{nt} \qquad 1 \leq i \leq n$$

driven by geometric Brownian motions

$$dF_{it} = m_i F_{it} dt + \sum_{j=1}^{n} \sigma_{ij} F_{it} dW_{it}^{F_i}$$

where

$F_{i0}, m_i, \sigma_{ij} \in \mathbb{R}_+$. $(W_{it}^{F_i})_{t\geq 0}$ is a Brownian motion defined on a probability space $(\Omega^F, \mathcal{F}^F, P^F)$ where $(\mathcal{F}^F)_{t\geq 0})$ is the filtration generated by $(W_t^F)_{t\geq 0}$. The (n, n) matrix of $\sigma_{i,j}$ is noted s and is the Choleski's decomposition of the variance-covariance matrix of the assets $\Sigma = \sigma^\top \sigma$.

3.3.2 Contributions and pensions

We assume that the stream of contributions Γ_t and benefits B_t is stochastic and modeled by Geometric Brownian motions. We recall that contributions and benefits are respectively positive and negative cash flows. In particular, we denote with $d\Gamma_t$ e dB_t the dynamic of the payment process related respectively to contributions and benefits of all living members at a certain time t:

$$d\Gamma = \mu_\Gamma(t)\Gamma_t dt + \sigma_\Gamma(t)\Gamma_t dW_t^L \tag{3.22}$$

$$dB = \mu_B(t)B_t dt + \sigma_B(t)B_t dW_t^L \tag{3.23}$$

where $\mu_\Gamma(t)$, $\mu_B(t)$, $\sigma_\Gamma(t)$ e $\sigma_B(t)$ are time functions and $(W_t^L)_{t\geq 0}$ is a Brownian motion defined on the probability space $(\Omega^L, \mathcal{F}^L, P^L)$. Moreover, let define a reserve of the fund as

$$R_t = \alpha \cdot B_t \tag{3.24}$$

where $\alpha \in \mathbb{R}$. This stochastic variable is introduced to take into account the Italian legislation indicator for the retirement funds of the professional orders, that is the ratio of the fund value to a multiple of the current expenditure for pensions, here stated as the reserve R_t.

By applying Ito's lemma to (3.24), we obtain the dynamic of R_t:

$$dR_t = \mu_B(t) R_t dt + \sigma_B(t) R_t dW_t^L \tag{3.25}$$

3.3.3 The managed wealth and the sustainability ratio

Given the n-dimensional price process $\{F_t\}_{t\geq 0}$, a Markovian relative portfolio strategy is any n-dimensional process $\{\Pi_t\}_{t\geq 0}$ of the form $\Pi_t = \Pi(t, F_t)$ for some function $\Pi : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$.
Let the $\mathcal{F}_t^L$-adapted one-dimensional consumption process be $\{B_t - \Gamma_t\}_{t\geq 0}$.
The fund wealth is the value process of a self-financing portfolio-consumption, whose stochastic differential equation is

$$dA_t = \left(m^\top \Pi_t A_t \Gamma_t - B_t\right) dt + A_t \Pi_t^\top \sigma dW_t^F \tag{3.26}$$

where $m^\top$ is the n-vector containing expected asset returns (transpose matrix of m), σ is the $(n \times n)$ assets covariance matrix.
The proportion of A_t invested in the i^{th} risky asset at time t is denoted π_t^i.
We assume that this constraint holds:

$$\pi_t^N = 1 - \sum_{i=1}^{N-1} \pi_t^i$$

Let us define the sustainability ratio S_t:

$$S_t = \frac{A_t}{R_t} \tag{3.27}$$

We obtain S_t's dynamic in this way:

$$dS_t = d\frac{A_t}{R_t} = A_t \cdot d\left(\frac{1}{R_t}\right) + \frac{1}{R_t} \cdot dA_t \tag{3.28}$$

where

$$d\left(\frac{1}{R_t}\right) = -\frac{1}{R_t} \cdot \left[\left(\mu_B(t) - \sigma_B(t)^2\right) dt + \sigma_B(t) dW_t^L\right] \tag{3.29}$$

Substituing 3.26 and 3.29 in 3.28, we obtain:

$$dS_t = \left(\left(m^\top \Pi_t - \mu_P(t) + \sigma_P(t)^2\right) S_t + \frac{1}{\alpha}(\Psi - 1)\right) dt + \left(\Pi_t^\top \sigma dW_t^F - \sigma_P(t) dW_t^L\right) S_t \tag{3.30}$$

where

$$\Psi_t = \frac{\Gamma_t}{B_t} \tag{3.31}$$

3.3.4 The fund objective function

We define the pension fund asset manager optimization problem as: $\forall t \in [0,T]$

$$V(S_t, \Psi_t, t) = \max_{\Pi_t \in \mathbb{R}^n} \mathbb{E}\left[\int_0^T U(S_t)dt\right] \tag{3.32}$$

s.t.

$$dS_t = \left((m^\top \Pi_t + \mu_B(t) + \sigma_B(t)^2)S_t + \frac{1}{\alpha}(\Psi_t - 1) \right) dt + \sigma \Pi^\top S_t dW^F - \sigma_B(t) S_t dW^L \tag{3.33}$$

$$d\Psi_t = \Psi_t \left(\mu_\Gamma(t) - \mu_B(t) + \sigma_\Gamma(t)^2 - \sigma_\Gamma(t) \cdot \sigma_B(t) \right) dt + \left(\sigma_\Gamma(t) - \sigma_B(t)\right) \Psi_t dW_t^L \tag{3.34}$$

$$S(0) = S_0$$
$$\Psi(0) = \Psi_0$$

$$\pi_t^n = 1 - \sum_{i=1}^{n-1} \pi_t^i$$

We assume that the fund manager seeks to maximize continuously the utility arising from the sustainability ratio, via an adapted investment policy Π_t, which is the control variable of our model. The preferences of the manager are reflected by a quadratic utility function having two components. A first one, proportional to the sustainability ratio, is related to the fact that the fund manager wants to maximize the sustainability ratio. The second component, proportional to the square of the difference between the sustainability ratio and a benchmark time dependant ratio $TS(t)$, penalizes the spread between the current ratio and a target one. Utility function is:

$$U(S_t) = kS_t - (1-k)(S_t - TS(t))^2$$

where $k \in [0,1]$ and $TS(t)$ is a deterministic function. In order to establish the relation between the value function and the optimal investment policy, we try to solve Hamilton-Jacobi-Bellman's equation.
Hypothesis:

- there exists an optimal control law Π_t
- the optimal value function $V(S_t, \Psi_t, t)$ is regular in the sense that $V \in \mathcal{C}^{2,2,1}$

For all fixed vector π, the partial differential equation operator $\mathcal{L}$ is defined by:

$V(S_t, \Psi_t, t)$ satisfy the Hamilton-Jacobi-Bellman equation:

$$\begin{cases} V_t + \sup_{\pi_t} [U(S_t) + \mathcal{L}^{\pi} V] = 0 \\ V(T, s, \psi) = U(S_T) \end{cases} \tag{3.35}$$

The control is π_t:

$$\Pi_t = M_1 \pi_t + M_2 \tag{3.35}$$

3.3.5 The optimal dynamic portfolio strategy

Chapter 4

An application to an Italian Professional Order pension fund

Just as eating against one's will is injurious to health, so study without a liking for it spoils the memory, and it retains nothing it takes in.

Leonardo Da Vinci

4.1 Data

In order to give a numerical illustration of the model developed in the previous chapter, we take data from a real Italian Professional Order pension fund. Our model parameters are related both to the pension fund calculus performed by the actuary and to the financial market data. The next sections are dedicated to show the paratemers we used in this numerical example.

4.1.1 Pension Fund data

The pension fund financial situation can be summarized by the parameters written in Table 4.1. As we can see, the asset manager has to manage an huge amount of money.

Moreover, we assume that the asset manager set an investment horizon of ten years ($T = 10$), and that current benefits multiplier $\alpha = 5$. To describe an asset manager averse to risk we set the risk aversion coefficient $k = 0.1$.

Table 4.1: Data on an Italian Professional Order pension fund (year 2009, in Euros)

Pension fund's wealth (A)	1.600.000.000
Contributions (Γ)	423.898.092
Benefits (B)	365.481.348
S $(A/5\dot{B})$	0.88
Ψ (Γ/B)	1.16

Figures from 4.1 to 4.6 represent the other parameters related to the fund that we described in the previous chapter.

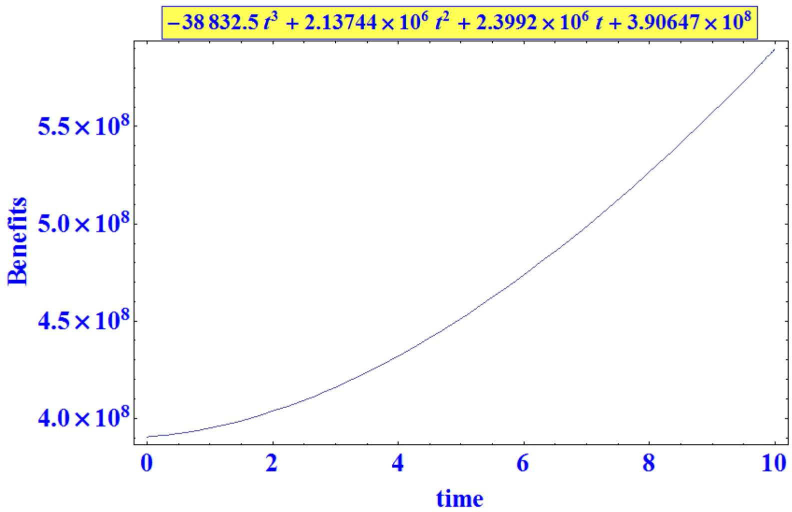

Fig. 4.1: Expected average benefits' path

4.2 Results

This section is devoted to show the optimal allocation law for each of the 12 asset classes presented in the previous section. We present surfaces graphs in which we can find the optimal number of shares for every asset class in every point of time and for every sustainability level.

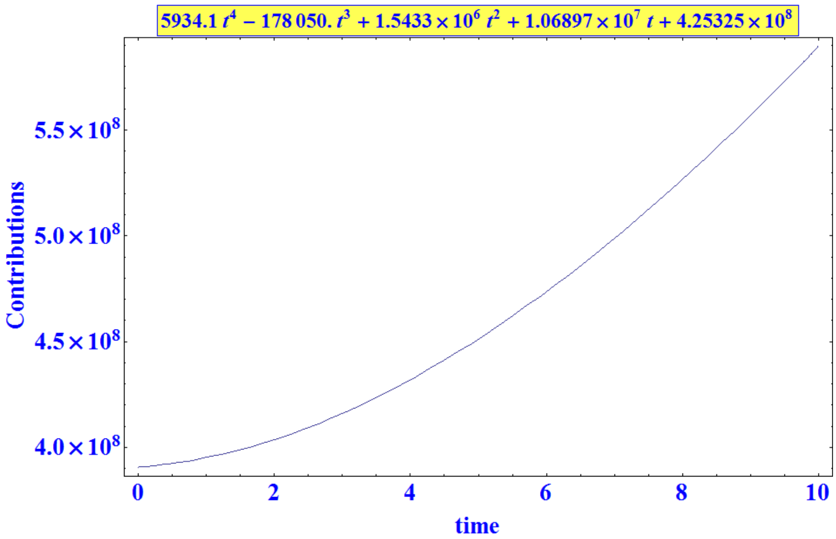

Fig. 4.2: Expected average contributions' path

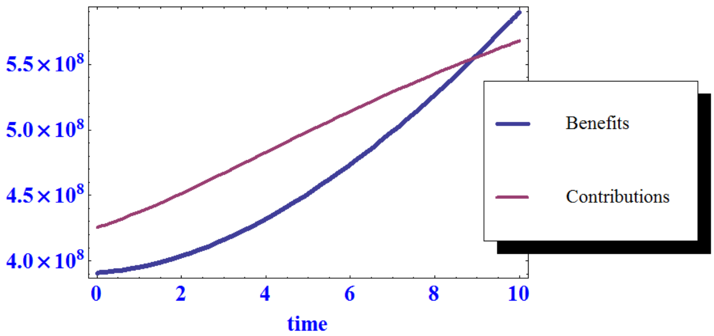

Fig. 4.3: Average contributions and benefits' paths

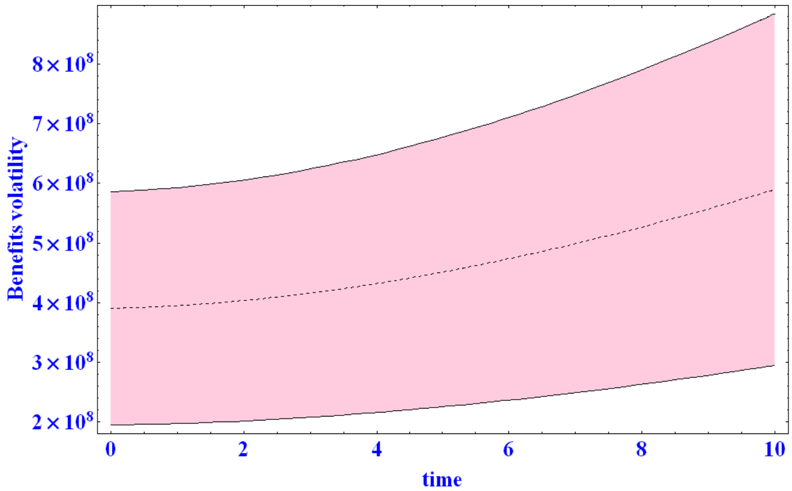

Fig. 4.4: Volatility range of benefits' path

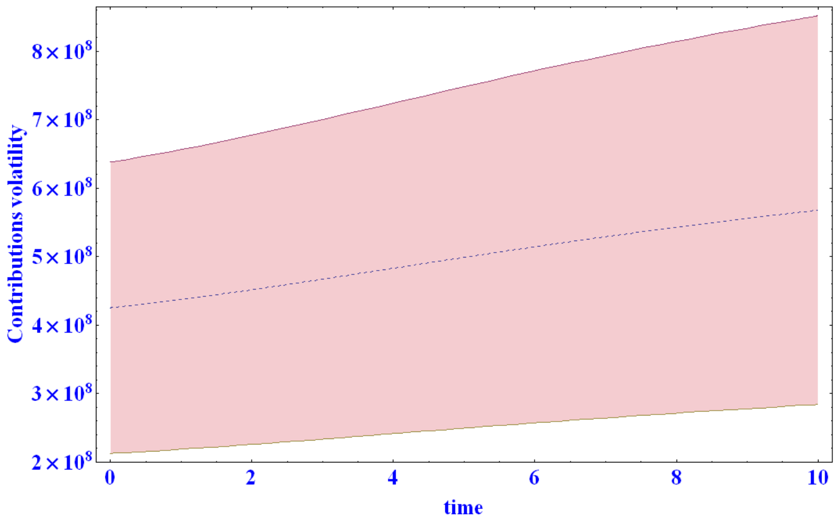

Fig. 4.5: Volatility range of contributions' path

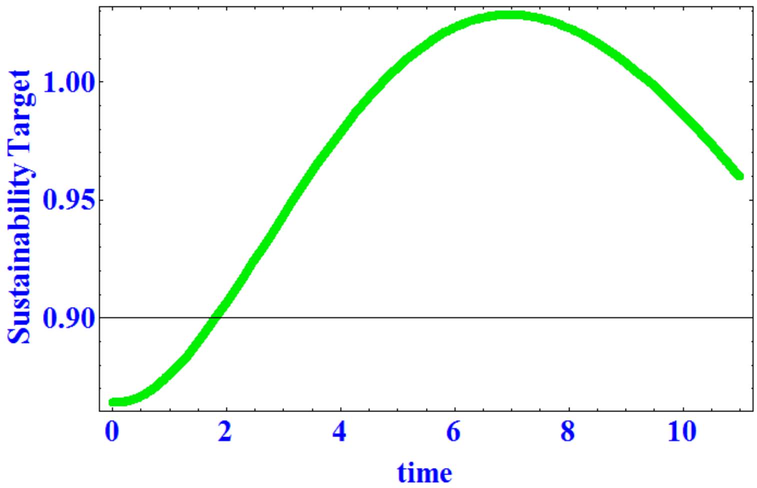

Fig. 4.6: Target sustainability ratio's path

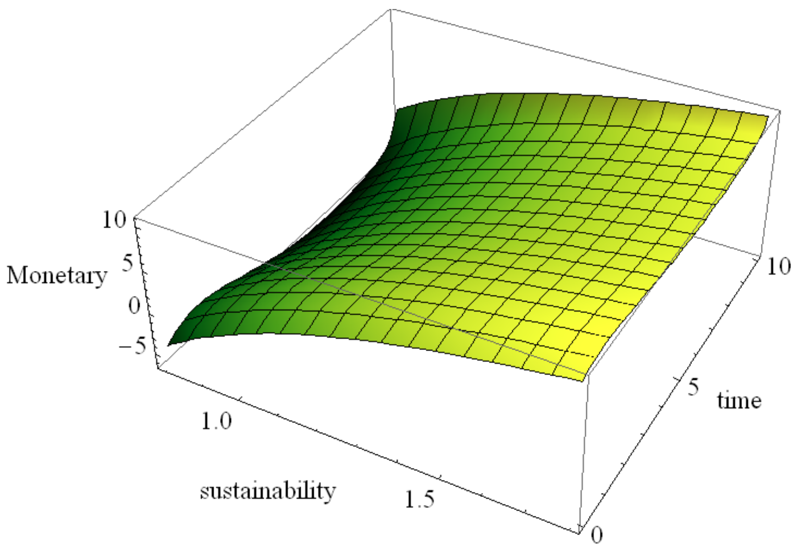

Fig. 4.7: Monetary

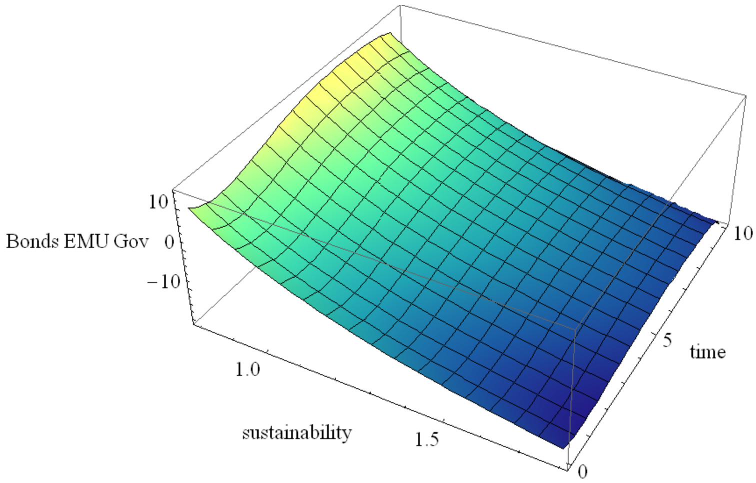

Fig. 4.8: Bonds Governments EMU

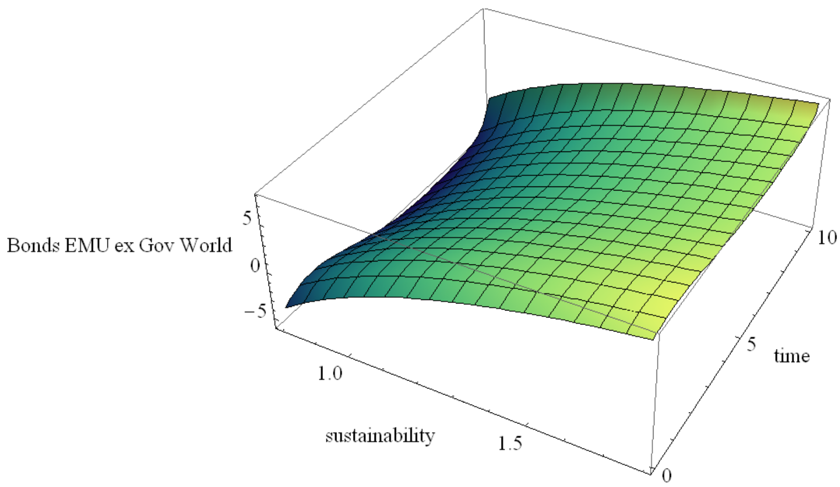

Fig. 4.9: Bond Governments World ex EMU

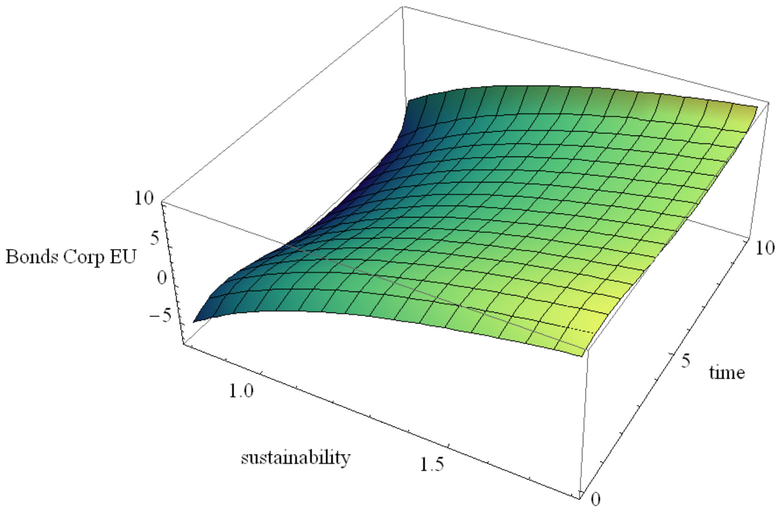

Fig. 4.10: Bonds Corporations EMU

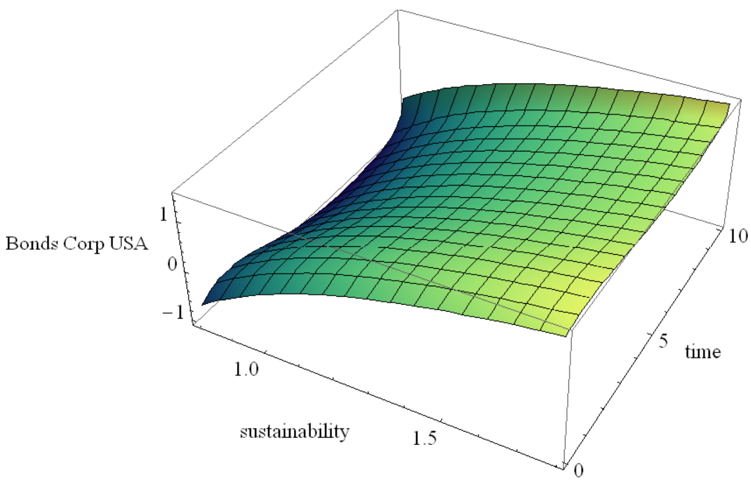

Fig. 4.11: Bonds Corporations USA

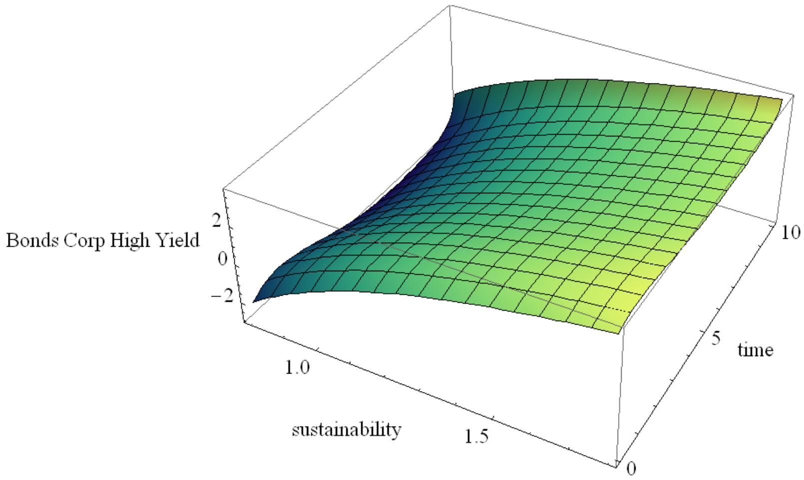

Fig. 4.12: Bond Corporation High Yield

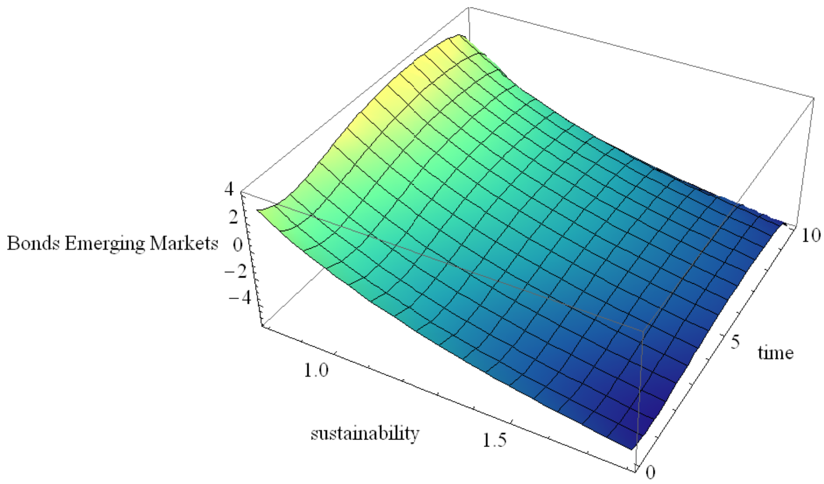

Fig. 4.13: Bond Corporation Emerging Markets

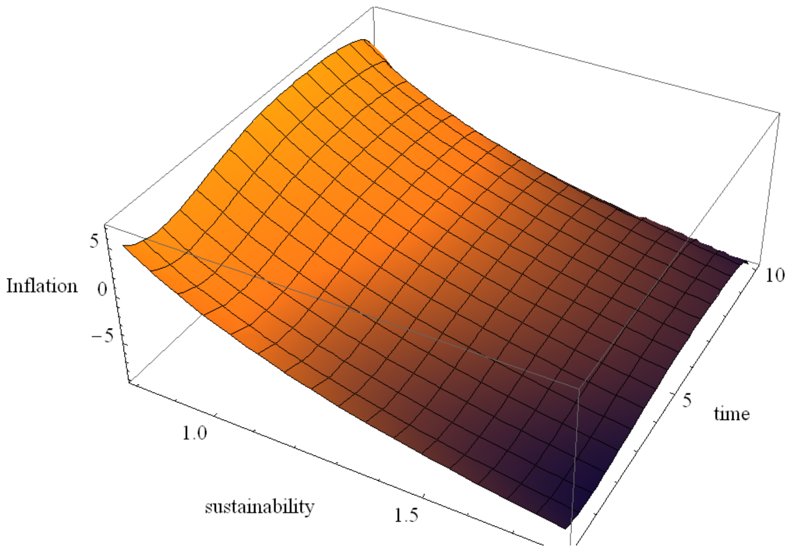

Fig. 4.14: Inflation

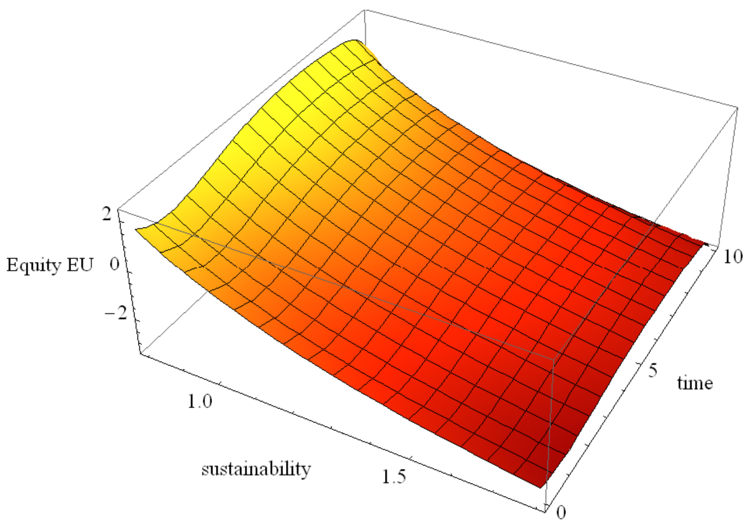

Fig. 4.15: Equity EU

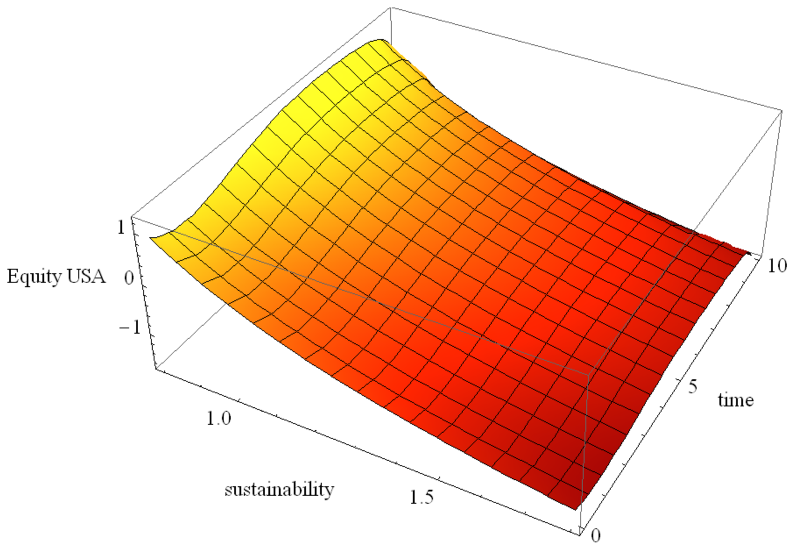

Fig. 4.16: Equity USA

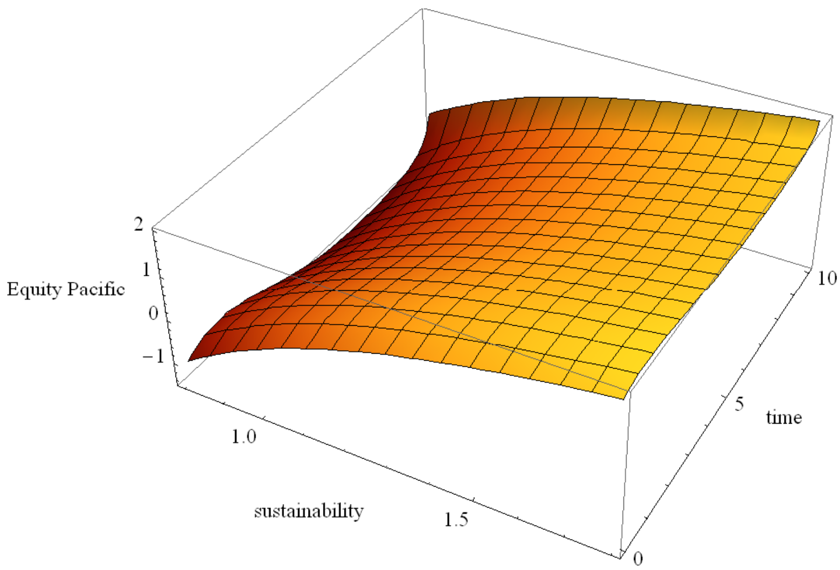

Fig. 4.17: Equity Pacific

Chapter 5

Conclusions

The aim of a pension system is to provide funds for pensioners to live without significant differences with the working period. In the past twenty years, several reforms have affected the public retirement systems prevailing in European countries, including Italy. It has been argued that the decreasing birthrate and the rising life expectancy would make unsustainable such systems where presently active workers pay for the pension of presently alive retirees. On the other hand, in a system based upon individual investment plans in financial markets, workers face a high uncertainty on their future pension wealth: their assets may suffer important drawdowns once they retire, thus drastically reducing their standards of living as retirees. There is therefore a need for a collective institution (Pension funds) able to insure present and future retirees and provide a smoothing of retirement benefits over time by sharing these risks over several generations.
Italy is facing a demographic challenge substantially greater than the average for all the countries of the European Union. Currently, the Italian pension system consists of three pillars: mandatory public old age insurance, supplementary collective pension funds and private individual insurance annuities. This thesis focuses in particular on the pension funds of professional orders. They are part of the fist pillar, but are now private self-managed companies, continuing to operate according to a PAYG financing mechanism. As often highlighted in this thesis, this is an anomaly because private closed schemes are usually funded and included in second pillar. In this particular system the financial self-sufficiency is certainly guaranteed only in the initial phase, because there are many contributors and no actual pensioners. In the long run it is necessary for the financial sustainability of the pension plan that the number of pensioners remains proportional to the number of workers. If the ratio active/retired decreases, the increase in the financial burden can entail a situation of financial disequilibrium. This is most relevant for the retire-

ment funds of each specific professional order for which, unlike in a public system, there is indeed no intergroup compensation.
In analyzing the risks of a pension plan there are of course many crucial issues. Pension funds, just as other financial intermediaries, such as banks, insurance companies, or mutual funds, typically hold financial assets on the asset side of their balance sheet. Such assets are held for investment purposes, i.e., they are held in order to produce future income for plan participants. The board of the pension fund is involved Ii investment policy, i.e. it decides about the asset classes in which the fund invests its wealth. Asset allocation is the process of choosing among possible asset classes.
A large part of financial planning consists of finding an asset allocation that is appropriate for a given investor in terms of their appetite for and ability to shoulder risk. This can depend on various factors. Asset Allocation is the product of an examination of an investor's needs and objectives. Asset allocation, done well, is a plan to invest in assets or asset classes which will best meet the needs and objectives of the investor. Investors seeking high returns and willing to expose their investments to an elevated amount of risk will allocate to equity (ownership) investments. Investors seeking stability and income will allocate to debt investments. Most investors, particularly personal investors, will find mixtures of equity and debt investments most nearly meets their needs. Asset Allocation can be practised by optimization techniques, minimizing risk for a given level of return or maximising return for a given level of risk. It also can be accomplished as goal based investing. The exposure to losses of a financial market investment can badly fall on pension fund wealth, so without a careful analysis of the potential danger, the investment could cause catastrophical consequence at the worst. With the experience of recent failure of large financial institutions, sufficient risks control measures are clearly essential and the regulators have set restrictions on limiting the exposure to market risks.
Moreover, pension funds world entails another source of risk: actuarial risk. This comes from the possibility that certain assumptions implemented to manage a specific pension plan may turn out wrong or somewhat inaccurate. In particular, we deal with assumptions concerning future cash flows coming from expected contributions and benefits.

This thesis has proposed a quantitative model to solve the optimal dynamic asset allocation problem faced by the pension fund asset manager. In our model we address both financial and actuarial sources of risk (financial market returns, contributions and benefits are random variables) and we take into account also the sustainability of the plan, which is both a key indicator to manage the fund and a tool of control used by regulators.

In a first step, we detail a quantitative method to determine the optimal investment policy performed by a fund manager who tries to maximize the pension scheme's sustainability . The optimal asset allocation is precisely determined continuously over a certain time horizon and for every sustainability lavel.
In a second step, we apply our model to an Italian pension fund in order to give a numerical illustration of the proposed model.

The methodology presented in this work can probably be applied to a wider range of problem than the one developed in this thesis. There is still interesting issues that motives further researches. For e.g. it can be interesting to restrict the optimal portfolio law in order to avoid short seeling of the asset managed. . . . of the fund to the market risk.

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Appendix A

Some basic elements of Probability Theory

The reader is referred to **?** and **?** We call *probability space* the triple $(\Omega, \mathcal{F}, P)$ where

1. $\Omega \neq \emptyset$ is a set called *sample ? space*. By a financial point of view, it represents every possible state of the market.

2. $\mathcal{F}$ is a $\sigma - algebra$, i.e. a family of subsets of Ω with the following properties:

   (a) $\emptyset \in \mathcal{F}$

   (b) if $A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F}$

   (c) $A_1, A_2, \dots \in \mathcal{F} \Rightarrow A := \bigcup_{n=1}^{\infty} A_i \in \mathcal{F}$

   The subsets F of Ω which belong to $\mathcal{F}$ are called $\mathcal{F} - measurable$ sets. Ina probability context these sets are called *events*

3. P is a *probability measure* on a *measurable space* $(\Omega, \mathcal{F})$. P is a function $P : \mathcal{F} \longrightarrow [0, 1]$ such that:

   (a) $P(\emptyset) = 0$

   (b) $P(\Omega) = 1$

   (c) if $\{A_i\}_{i=1}^{\infty} \in \mathcal{F}$ is disjoint, then $P \bigcup_{i=1}^{\infty} A_i = \sum_{i=1}^{\infty} P(A_i)$

   $P(A)$ is the probability of the event A, i.e. "'the probability that the event A occurs"'. In particular, if $P(A) = 1$ it can be said that "'A occurs with probability 1, or almost surely (a.s.)"'

If $\Omega = \mathbb{R}^n$, $\mathcal{F} = \mathcal{B}$ is a *Borel* $\sigma - algebra$, i.e.the smallest $\sigma - algebra$ containing all open sets of $\mathbb{R}^n$.
If $\Omega = \mathbb{R}^n$ and $\mathcal{F} = \mathcal{B}$, then the probability measures defined in this space is called *distribution*.