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Temporal Becoming and The Algebra of Time

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To my wife.

Time past and time future
What might have been and what has been
Point to one end, which is always present.
T.S. Elliott

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1

Introduction

This work is dedicated to defend one simple intuitive yet fiercely challenged claim: namely, that *time flows*. Or, better, to show that *there is a well-defined, algebraically rooted sense in which it makes sense to say that time possesses objective dynamical features.*

The idea that time flows or passes is undoubtedly part and parcel of our subjective experience:

> What we experience in one moment, glides, in the next moment, into the past. There it remains forever, irretrievable, exempt from further change [...] and yet enshrined in our memory as something that once filled out experience as an immediate present (Reichenbach, 1956, p. 1).

The key philosophical problem concerning the flow of time is thus whether this perception of transiency is just a mere shadow of our mind, or else a constitutive feature of the physical world. This problem is surely one of the most ancient and controversial philosophical puzzles, originating with the clash between the fixity of Parmenidean ontology and the fluidity of Eraclitean world. Since the second half of the twentieth century, analytic philosophy brought this problem to new life, favored by the collapse of the classical world-view brought about by the special and the general theories of relativity. Strong arguments have been raised against the logical consistency, the compatibility with scientific theories, or simply the usefulness of providing physical time with a proper motion, so that those defending the objective passage of time, or *dynamists*, are now by far a minority.

One of the main purposes of this work is to give new tenability to the dynamist view, picturing a non-standard but still adequate way of understanding the passage of time. Before tracing the guiding lines of this project, it will be therefore necessary to point out the essential features any purported model of time's passage should satisfy in order to be satisfactory.

1.1

The Ingredients of Temporal Becoming

The passage of time, or *temporal becoming*, is commonly understood as a continuous and inescapable shift of the present moment along the time axis – a conception which often goes under the name of the *moving* or *transient-now*; as such, becoming is essentially a twofold concept: on the one hand, it requires a distinction among past, present and future tenses while, on the other hand, it suggests a continuous change in the moment counting as present. To provide temporal becoming with full objective status, we are thus in need of two distinct objective ingredients:

> One is a mind-independent distinction between past and future. The other is a mind-independent continuous change of the instant of separation, the present. [...] In a word, the first ingredient, which is still static, requires a mind-independent "being present". [...] To get the idea of passage, typical of becoming, we need the second ingredient, some sort of mind-independent change (Dorato, 1995, pp. 10-11)'.

The conceptual core of temporal becoming is thus composed of both a *static* and a *dynamic* components, which any minimal account of objective temporal becoming should accordingly make sense of. In what relations do they stand as of each other?

On the one hand, it seems that the latter component is by hypothesis in need of the former one: in fact, at each time, it calls for a unique determination of the present moment, whose continuous change the dynamics of time should consist of. In the course of our discussion, we'll have the chance to see that this is not necessarily the case, since time can be provided with intrinsic dynamical features *before* any objective characterization of tenses is given. Nonetheless, we shall provisionally stick to common understanding, and take it for granted that the motion of time consists precisely in the motion of the present moment, so that the objectivity of the dynamic ingredient of temporal becoming is sufficient reason so that its static component is objective too.

Whether the converse implication is also true is more complex matter. In fact, the existence of a unique, global yet unchangeable partition of time into past, present and future would hardly seem to make any sense: if no moment of time was ever allowed to pass from futurity to pastness, then how could we really speak of past and future times at all?

However, there might exist non-dynamical ways as to how the change in the distribution of tenses could take place. Zeilicovici (1989) offered an example of how tenses might be given objective meaning without *ipso facto* demanding a moving-now conception of time. His starting point was the epistemic disparity between past and future events, on which basis he inferred the existence of a metaphysical disparity between past and future times: since time is the ordered set of events as of before and after, 'future time can be no less synthetic and no less a priori than future events; in both case we may predict existence, but may not take it for granted (Zeilicovici, 1989, p. 509)'. On this basis, he defined the present moment as the upper boundary of directly acknowledged time, whose concrete existence he thought to be certain. By contrast, he gave to all times later than the present a purely abstract or epistemic status, so that at each time the present moment should be understood as the separating element between *existing time* and merely *conjectured time*. The result was that, at each moment, the list of existing moments was substituted by *another* list, including all moments entering the previous list plus the present one; and since: 'movement is always upon the same [...] time series, while the different moments to which nowness applies belong to different [...] time series (Zeilicovici, 1989, p. 519)', this may provide us with an objective, metaphysically rooted distinction between tenses which did not require any shift or flow of the transient-now.

Indeed, there may be doubts about the efficacy of this proposal (Oaklander, 1991, 1992; Faye, 1993); but in any case, it has the merit of showing that we can make sense of the objective asymmetry of tenses *without* being committed to the dynamical conception of time. So, even granting that the dynamical ingredient of becoming is conceptually in need of an exact determination of past, present and future times, the converse implication seems not to be true.

In addition to the static and the dynamical ingredients, the moving-now interpretation of temporal becoming also provides it with a directional component: the present moment is typically thought to move irreversibly *from* past *to* future, and it is precisely the unique direction of this motion which objectively distinguishes the future from the past.

Price (2010) listed this directional ingredient, namely 'that it is an objective matter which of two non-simultaneous events is the *earlier* and which the *later*' among the constitutive features of temporal becoming. However, this hypothesis is controversial, for it seems to be no necessary condition for making sense of the motion of time to establish whether or not that motion is unidirectional. Furthermore, no direct logically connection is there between the directionality of time and the two further components of temporal becoming.

On the one hand, requiring the present moment to be moving is not trivially as much as requiring its motion to be unidirectional, i.e. both linear and irreversible: first, one may be willing to provide time with non-linear topologies (Newton-Smith, 1980), forcing the present moment to move along different trajectories; and second, even though such possibility was thought to be too exotic, there may still be the case that time moved along a closed circle, to which the relations of before and after would not apply. Conversely, the directional component of time may consist in a mere structural or topological asymmetry (*anisotropy*), independent of the internal dynamics of time (Mehlberg, 1961; Grünbaum, 1973). On the other hand, Dorato (2000a) proved that the existence of a structural asymmetry of that kind is also independent of the existence of any objective distinction of tenses, at least as long as such a distinction is made on a metaphysical basis. For this reason, we shall provisionally regard the directionality of time as a desirable but not indispensable feature any model of objective temporal becoming should satisfy.

1.2

Outline of the Work

Despite its perceptual evidence and its philosophical authority, becoming has undergone a fierce attack both in its static and in its dynamical senses. On the one front, critiques to the static component of becoming were raised at the boundary between philosophy and physics. Following the groundbreaking intuitions of Weyl (1949) and Gödel (1949a,b), philosophers of physics offered several arguments aiming to prove an objectively rooted distinction among tenses to be incompatible with relativity of simultaneity imposed by the special and the general theories of relativity. On the other front, analytic philosophers challenged the dynamical component of becoming on logical grounds: Smart (1949), Williams (1951) and Grünbaum (1967a) laid the foundations of a still living debate concerning the kinematic implications of expressions such as "time flows" or "time goes one way".

The first part of this work is dedicated to discuss arguments of both kinds. In Chapter 2, we shall face the most purely philosophical side of the dispute, namely the ontological one. The problem of objective temporal becoming is traditionally formulated as the problem whether there exists any ontological asymmetry among tenses. Our discussion will show that, contrary to some recent critiques, this problem is a metaphysically sensible one but, at the same time, it is of no use to settle the problem of objective temporal becoming. In Chapter 3, we shall face the objections coming from the special theory of relativity. On the one hand, confirming the results obtained in Chapter 2, we'll show that the representation of space-time held by the special theory of relativity does not *per se* calls for the metaphysical equivalence of tenses. On the other hand we shall see that, contrary to what is usually taken for granted, the structure of special-relativistic or temporally-oriented Minkowski space-time can support an objective, though weakened, account of tenses. Finally, Chapter 4 will face some of the major arguments which have been moved against the logical consistency of objective temporal becoming. In particular, we shall see those arguments to be effective only provided that the motion of time was understood ingenuously, in a way similar to that of a moving solid body.

In the second part of this work, we shall outline an algebraically-rooted model for objective temporal becoming. The basis for constructing our model will be offered by a generalized version of the general theory of dynamical systems, which we shall introduce in Chapter 5. Chapter 6 will be dedicated to study the different ways how a dynamical system might reverse its evolution: in particular, that study will be useful to make sense of the directionality of time. Finally, in Chapter 7 we shall provide time with a dynamical system on its own, offering a tenable model of the dynamical component of temporal becoming and, under some minor constraints, also of its static component. In particular, we shall see that such dynamical systems represent the motion of time in an entirely satisfactory way just in case their motion is completely irreversible. This way, the problem of the objectivity of temporal becoming will be reduced to that of the objectivity of the direction of time.

Chapter 8 will finally compare the dynamical interpretation of the directionality of time offered by our model with the structural interpretation given by the received view. The result of our examination will be a rebuttal of the standard approach, which we'll show to be intimately circular, in favor of an entire reversal of the problem of the direction of time: rather than wondering how the standard mathematical representation of time can be directional, we should wonder whether a less expensive, and obviously directional mathematical model of time could support our representation of physical phenomena.

2

The Ontology of Becoming

Becoming is usually understood as the *coming into being* of objects, facts and events: according to this view, what pushes the present moment towards the future is a continuous receding of the existent from futurity to pastness or, equivalently, a continuous realization of future moments, which progressively acquire existence while becoming present, while present moments, turning past, fade away into nothingness:

> What is becoming? [...] The present is the only reality. While it sleeps away, we enter into a new present, thus again remaining in the eternal Now. (Reichenbach, 1956, p. 2).
>
> Our commonplace use of tenses codifies our experience that any particular present is superseded by another whose event-content thereby "comes into being". It is this occurring of *now* or coming into being of previously future events and their subsequent belonging to the past which is called "becoming" (Grünbaum, 1967a, p. 7).

In this view, the motion of time is a product of the *ontic voltage* (Callender, 2010a) obtaining between past and future, which accounts for both the static and the dynamical components of becoming: existence is precisely what objectively distinguishes the present moment, as well as the unique motor of its change. For this reason, the problem of objective temporal becoming got intertwined with the old-dated debate between *presentism* and *eternalism*, understood as metaphysical theses respectively asserting and denying the ontological disparity of tenses.

The aim of this chapter is twofold. On the one hand, we shall defend the soundness of the presentist-eternalist debate from some recent sceptical objections, which might indirectly affect the meaningfulness of objective temporal becoming. On the other hand, we shall examine the bearings such a debate may have on the problem of finding an objective, ontological basis for the present moment.

2.1

Presentism and Eternalism

At the beginning of the twentieth century, J. E. McTaggart tried to show that time is not real (McTaggart, 1908). His refutation rested on a decomposition of time into what he referred to as the *A-series* – namely, the tripartition of events and instants into past, present and future tenses (A-determinations) – and the *B-series* – i.e. the linear order of precedence, simultaneity and succession (B-determinations). McTaggart regarded the A-series as the very essence of temporality, for time essentially involves change, and change would not be possible without tenses: at any moment of time, the death of queen Anne will always be the same death of an English queen, springing from identical causes and leading to identical effects; the sole possible properties an event could lose or acquire, and therefore the sole properties that would account for a change, are its A-determinations. However, he argued that there seems to be no sensible way in which A-determinations could be predicated of events. For, whether they were monadic properties or binary relations, they would be incompatible as of each other, and thus they could not be predicated of the same events unless they were at different times, which would obviously require temporality and hence A-determinations, circularly. Nor this circularity could be avoided by asserting that any present event is present, while it was future and it will be past, because in this way we would be constructing an A-series of A-determinations, which in its turn would be in need of its own A-series and so on, indefinitely. But if A-determinations are essential to time, and there is no way of predicating A-determinations of real events, then time itself cannot be anything real[1].

One century later, time has regained its reality, but the legacy of McTaggart's refutation still lives in the philosophical contention between presentism and eternalism (Dyke, 2002). Both theories are tangled clusters of logic, metaphysics and philosophy of language; nonetheless, we shall restrict our attention to the sole metaphysical and more fundamental side of their dispute.

Instead of denying the existence of time, eternalists (or B-theorists) moved from McTaggart's critique of pastness, presentness and futurity for reappraising the role of B-determinations. In their view, time is nothing but the linear or partial ordering of moments or events according to the relation of before and after and, therefore, there is no metaphysical disparity among what is actually past, present or future: 'There is nothing special about the present; things at other times are just as real; no time is metaphysically distinguished (Hinchliff, 1996, p 122)'. Real or concrete events are given all at once, as in a block-universe, though they distribute differently along the time axis and they present themselves successively to the human mind.

On the opposite hand, presentists (or A-theorists) follow McTaggart in making tenses an indispensable ingredient of temporality, though they provide them with ontological significance. Presentism is the claim that 'the way things are is the way things presently are', so that 'only

[1] See also Broad (1923) and Mellor (1981).

the present exists (Hinchliff, 1996, p. 123)': at each time, existence (or reality, actuality, or determination) is solely predicable of what is present, so that events progressively be-come or come into existence while shifting from futurity to presentness, and cease of existing while receding from presentness to pastness. For this reason, presentism has also been marked as an ontology of becoming, while eternalism has been labeled as an ontology of being – in which events do not *come* into existence: they simply are, since they owe the same ontological status independently on whether they are past, present or future[2].

If presentism was right, it could indeed provide us with a suitable objective criterion to distinguish the present moment from the past and the future ones, namely existence. In recent years, both presentist and eternalist metaphysics have nonetheless lost part of their initial appeal. Some philosophers casted doubts on the very existence of a real disagreement between them, arguing that presentism and eternalism merely consist in different pragmatic attitudes towards temporality. Now it seems that, if that critique was sound and if objective temporal becoming could only be grounded on a presentist metaphysics, then the question whether time flows would be trivialized in its turn. The first part of this chapter is thus dedicated to defend presentism and eternalism against this charge of metaphysical equivalence, discussing two arguments respectively put forward by Dorato (2006, 2008a) and Savitt (2006).

2.1.1 Triviality and Contradiction

There are principally two ways to talk about existence. One way is generally called *tensed*, the other *tenseless* or *detensed*. In the former sense existence is a property objects, facts or events may lose or acquire[3], and whose possession accordingly varies with time: objects, facts and events exist tensedly just in case they exist *now*, namely just in case they are co-present with the time at which they are claimed to be existing. In the latter sense, existence is a property objects, facts and events possess independently of their location in time, and on which tenses bear no effect: objects, facts and events exist tenselessly or detensedly just in case they existed in the past, they exist now or they will exist in the future, namely just in case they exist at some time and place.

Following Dorato (2006, 2008a), let us consider the presentist claim that all which exists is present, and let us compare it with the tensed-detensed dichotomy. In what sense should we understand "existence" in this case? If the presentist understood existence in the tensed sense, then it seems that she would be holding the plain triviality that all which exists now is present.

[2]Philosophers sometimes distinguish between presentism and eternalism on the one hand, understood as ontological thesis concerning the existence of past, present and future tenses, and *three-dimensionalism* and *four-dimensionalism* (or *perdurantism*) on the other, understood as analogous theses concerning the existence of things, facts or events (Sattig, 2006). The relations presentism and eternalism bear to three and four-dimensionalism, however, is controversial. For the sake of simplicity, I shall therefore treat presentism and eternalism as theses concerning both the existential status of tenses and that of the things, facts and events they host. This usage, on the other hand, is consonant with that of the arguments I shall discuss.

[3]Following Dorato (2000b) we shall consider objects, facts and events as ontologically equivalent though linguistically alternative descriptions of the very same building-blocks of physical reality; accordingly, we shall take it for granted that existence may be equivalently predicated of all.

But conversely, if she understood existence in the tenseless or detensed sense, then it seems that she would be claiming that all which ever existed (whether in the past, in the present or in the future) is just what is present, with the absurd consequence that no past or future events ever did or will take place. Symmetrically, if we took the eternalist claim that past and future events exist as being asserting that they exist tensedly, then we would reduce it to the contradictory claim that both past and future events are taking place now; and if we took it as being claiming that they exist detensedly, we would equate it to the plain truism that past and future events are either past or present or future. So, it seems that if both presentism and eternalism are confined between contradiction and triviality, then they can possess no metaphysical significance, and their purported disagreement dissolves[4].

Indeed, this argument seems to be striking a very hard blow against the metaphysical side of the presentist-eternalist contention. Nonetheless, it has a weak point. Presentism and eternalism do not only concern time. They are ontological theories, for they bear positive content *about existence*: they say what facts, objects or events existence should be predicated of. In the former case, existence is bound to presentness; in the latter case, to temporality. When the presentist claims that only those events which are present exist, she is actually offering a definition of existence which is equivalent to presentness:

> Presentism is the doctrine that [...] everything is present; more generally, [...] necessarily, it is always true that everything is (then) present (Sider, 1999, pp. 325, 326)'.
>
> Presentism is the view that only present objects exist. According to presentism, if we were to make an accurate list of all the things that exist – i.e. a list of all the things that our most unrestricted quantifiers range over – there would be not a single non-present object on the list (Markosian, 2004, p. 47).

Similarly, when the eternalist claims that past and future events exist as well, she is offering an *alternative* definition of existence which is equivalent to taking place in time:

> Presentism is the temporal analogue of the modal doctrine of actualism, according to which everything is actual. The opposite view in the philosophy of modality is possibilism, according to which nonactual things exist; its temporal analogue is eternalism, according to which *there are* such things as merely past and merely future entities (Sider, 1999, p. 326, my epmhasis).
>
> [E]ternalism [is] the view that our most inclusive domain of quantification includes past, present, and future entities (Crisp, 2004, p. 19).

In either case, existence is just what presentists and eternalists claim it to be; no additional qualification is needed. But if so, then to recognize presentness a proper (tensed) mode of

[4] See also Ludlow (2004).

existence is equivalent to join the presentist's side, while admitting that all tenses are on the same (detensed) ontological level is simply to choose for the eternalist. In other words, tensed existence is nothing but a condensed formula for the presentist thesis that existence is bound to presentness, while tenseless existence is nothing but a place-holder for the eternalist claim that existence is common to all tenses.

To make this point clearer, let us focus on Dorato's own definition of tensed existence. In his words, 'event e "exists" in the tensed sense of existence if *and only if* it exists now (Dorato, 2008a, p. 256)'. I emphasized the *only if*-clause to point out that presentness is a necessary condition to satisfy tensed existence: that is, nothing which is not present could ever exist in the tensed sense. But allowing for a way of existing which is proper to the sole present objects, facts or events, is just to discard the eternalist thesis that all tenses are ontologically on a par.

Conversely, Dorato defines tenseless (or detensed) existence as follows: 'for all present moments, event e "exists" in a tenseless sense of existence *if* and only if it has existed, exists in the present or it will exist (Dorato, 2008a, p. 256)'. This time, I emphasized the *if*-clause to underline that it is sufficient for event e to be either present, or past, or future in order to be endowed with the tenseless or detensed acceptation of existence: in other words, one may think of a way of "being there" which all past, present and future events share, independently of their location in time. Indeed, one may think that making room for a detensed acceptation of existence would not *ipso facto* rule out presentism, since providing all tenses with a common existential status does not seem to exclude that a metaphysical disparity may nevertheless hold between them, over and above the metaphysical level they shared. However, no presentist could ever coherently subscribe this claim, since for her existence should be an all-or-nothing property: there can be no different *degrees of existence* (Smith, 2002), simply because there can be no different degrees of presentness. One may admit that past, present and future events shared the same metaphysical status only if she had already given up presentism; speaking of a tenseless or detensed acceptation of existence could therefore make sense from, and only from, an eternalist point of view[5].

In sum, wondering in what sense presentists or eternalists should understand "existence" is tantamount to questioning presentism or eternalism itself, for each one respectively *consists* in understanding existence in either the tensed or detensed acceptation. In this light, putting tensed existence into presentism, or detensed existence into eternalism, amounts to equating two identical claims, while putting detensed existence into presentism, or tensed existence into eternalism, amounts to equating two contradictory statements. If there is any triviality or contradiction in here, it does not lie on presentism and eternalism themselves.

[5]One may object that some presentists actually speak of detensed existence. However, that use is purely rhetorical: when speaking of something existing in the past, or in the present or in the future they are tacitly referring to the sole present things, since according to presentism there can be nothing concrete which past and future existence can denote.

2.1.2 Metaphysical Equivalence, Different Perspectives

Savitt (2006) complained about the lack of clarity in the standard characterizations of presentism and eternalism, putting the reason of this deficiency down to 'the fact that those engaged in the debate have typically left out of consideration one term in a relational notion', since 'one has to state what eternalism and presentism are *relative to some background spacetime theory* (Savitt, 2006, p. 123)'. In accordance to the common-sense view, he restricted his attention to classical or Galilean space-time, accordingly assuming the following set of hypotheses:

CP1 Space-time is a set of events G having the structure of Galilean space-time.

CP2 In particular, Galilean space-time can be foliated uniquely into hyperplanes of simultaneity, which are equivalence classes of simultaneous events.

CP3 The present for an event e is the hyperplane of simultaneity that contains e.

CP4 Hyperplanes of simultaneity occur successively.

On this basis, he equated presentism and eternalism respectively to:

CP5 The existence of an event is its occurrence.

CE5 An event e exists if and only if $e \in G$.

Savitt saw no substantial difference between CP5 and CE5, coming to the conclusion that presentism and eternalism stand for alternative perspectives toward time, rather than for incompatible theories about existence:

> If the distinction between (classical) presentism and eternalism comes to the difference between CP5 and CE5, then the two views are compatible. One should not hastily conclude, however, that alleged difference between these venerable positions has been shown to be *merely* verbal. The difference between CP5 and CE5 reflects a difference in perspective as well as a difference in language. Presentists adopt a point of view that is close to temporal experience, confronting the actually occurring, as opposed to merely past or future, events. Eternalists consider the totality of actual, as opposed to merely possible or otherwise non-historical, events. The latter perspective seems necessary for physics, for the determination of the geometric structure of space-time. The former perspective is, as it were, that of those living inside the structure contemplated by the latter from "outside" (Savitt, 2006, p. 124).

Before moving to an evaluation of Savitt's proposal, some refinements seem to be in order. Quite surprisingly, of premises CP1-CP4, none seems to be essential to Savitt's characterization of presentism, even in its classical sense. CP5 simply equates an event's existence with its occurrence; it doesn't make any reference to global space-like foliations or absolute hyperplanes of simultaneity, nor to any linear ordering of succession to hold among them – indeed, it makes no reference to presentess at all! Similarly, Savitt's definition of eternalism seems to be independent of tenses, for CE5 does not mention any of premises CP2-CP4. To be true, there is a very simple explanation for this: Savitt took it for granted that, while reading CP5, "existence" should be interpreted tensedly and that, while reading CE5, it should be interpreted in the detensed sense. Since tensed existence is just existence at present time, while detensed existence is existence at past, present or future times, this may be thought to be sufficient to put back presentness into presentism, and tenses into eternalism. CP5 and CE5 would then read as follows:

CP5' The existence of an event at present is its occurrence.

CE5' An event e exists in the past, in the present or in the future if and only if $e \in \mathbf{G}$.

However, we cannot accept this solution: as we saw, presentism is meant to bind existence to presentness, not to characterize present existence; and, similarly, the aim of eternalism is to equate existence with temporality, not to depict what past, present or future existence should amount to. Therefore, if presentness had to enter the *existence* = *occurrence* equation demanded by CP5, it would have to appear in the occurrence's side: for the presentist understands presentness as the distinguishing feature of the *scope* of existence (Crisp, 2004), while Savitt made it an attribute of existence itself.

Restating CP5 so that it could be a faithful formulation of presentism would accordingly require to qualify occurrence as occurrence *at present time*:

CP5* The existence of an event is its occurrence at present.

In its turn, CP5* calls for a reformulation of CP3. In fact, CP3 defines presentness only relative to each event e. However, if presentness had to characterize existence, as the presentist wished, it had to be defined in a non-relational way, for otherwise existence would be a relational property in its turn. For this reason, in addition to CP3, presentists should also assume the following:

CP3* At each time, exactly one hyperplane of simultaneity qualifies as (absolutely) present.

Once again, CP5* demands that "existence" should be interpreted neither tensedly nor detensedly, for otherwise presentism would be artificially trivialized or led into a contradiction. Presentness is therefore restricted to qualify occurrence; CP1-CP4 and CP3* then play their part, determining how presentness itself should be understood in the classical context.

Symmetrically in statement CE5 past, present and future should not be predicated of existence, but of that portion of space-time to which each existing event is supposed to belong. Fortunately CP1-CP4 and CP3* guarantee that (at each time) the whole of Galilean space-time is uniquely partitioned into past, present and future times, so that we don't have to restate CE5 to make it compatible with eternalism: in this case, we only have to keep ourselves from interpreting "existence" either tensedly or detensedly, and to understand CE5 itself as a way of characterizing the detensed mode of existence.

Let us now turn to Savitt's conclusion. His main contention was that presentism and eternalism are nothing but different ways to look at the same state of affairs, for CP5 and CE5 carry no different metaphysical content. Unfortunately, he didn't offer any detailed explanation for the last clause; presumably, and reasonably, he regarded an event's occurrence and its belonging to (Galilean) space-time as being equivalent. So far, so good. But does his conclusion stand even in the face of our critique of CP5 and CE5?

If I am right, then CE5 should be confronted with CP5*-CP3*; and hence, being part of (Galilean) space-time should be confronted with lying on that unique space-like hyperplane of simultaneity which is meant to denote the present moment (Sattig, 2006, p. 62). At a first glance, once may still be willing to subscribe Savitt's conclusion: CP1-CP4 demand that the whole of space-time is invariantly foliated into linearly ordered hyperplanes of simultaneity, so that looking at space-time in its entirety and looking at one of such hypersurfaces at a time would just amount to assuming alternative but compatible perspectives on the very same thing.

However, things are not as simple as they seem. If presentism and eternalism really were equivalent theses, they had to fail in exactly the same cases. That would certainly be possible if the presentist and eternalist positions were supposed to be respectively expressed by CP5 and CE5: for there seems to be no sensible way how an event could fail to occur without failing to take place in space-time, or vice versa. However, this is no longer true while replacing CP5 with CP5*-CP3*. In fact, events may well fail to be present by occurring at a hyperplane of simultaneity which is not the present one, albeit still belonging to space-time; and therefore, as long CE5 and CP5*-CP3* disagree as whether events on past or future hyperplanes of simultaneity do exist, they bring metaphysically different content.

2.2

Incommensurability

Nonetheless, *there is* something problematic in the way presentism and eternalism are usually contrasted. Let us make one step back, and let us assume that, at any time, there exists an objective partition of time into past, present and future. To keep this statement neutral with respect to the presentist and the eternalists metaphysics, let us suppose that partition to be obtained by means of a non-metaphysically rooted criterion: for example, by holding hypotheses CP1-CP4 in Savitt's argumentation which, as we saw, are independent of the classical statements of presentism and eternalism encoded in statements CP5*-CP3* and CE5. On this basis, let us wonder what the presentist and eternalist theses would respectively add to our assumption.

On the one hand, presentists would say that all which exists lies in the present; on the other hand, eternalists would say that all which exists indifferently lies in the past, or in the present or in the future. Presentists would assign a distinct metaphysical role to tenses; eternalists would not. Presentists would agree that tenses are both objectively distinct *and* metaphysically unhomogeneous; eternalists would agree that tenses are objectively distinct, insofar as they are supported by CP1-CP4, but *not* metaphysically unhomogeneous. That's all fine: we already knew that. But, let us go on, and wonder: how could they disprove their opposers?

Let us imagine that a presentist wanted to prove that past and future did not exist: how could she do that, without assuming them to be there, in such a way that they could be predicated of non-existence? Of course, she would have the choice open to claim that past and future events are simply abstract entities: this way she could claim that past and future did not exist, just like we are capable to say that Santa Claus does not exist, without ever being in need of assuming its non-abstract existence. To say it with Markosian (2004), Socrates would be in the same boat as Santa Claus. But in that way, the presentist thesis about what concretely exists would be turned into a thesis about what is abstract, and the same point would apply as well: how could the presentist defend her claim that past and future entities are abstract, without circularly assuming that they are so because they are simply "not there"?

Let us look at the same problem from the eternalist side. The presentist is claiming that past and future events, objects or facts are merely abstract entities, whose metaphysical status is the same shared by Santa Claus. In the face of this claim, the eternalist could easily reply that, 'indicating the proper contrast class[6] will provide us with enough boats to allow them to sail separately (Savitt, 2006, p. 118)'. That is to say: if a mode of existence other than the tensed and the abstract ones was available, then past and future could be put back again among concretely existent things. But certainly, if it was there, and if its role was that of separating abstract things from concrete ones (which is precisely the function Dorato (2006, 2008a) assigned to tenseless existence), then it would also have to include the present. So, how to prove that such a mode of existence is available, without falling back again into the eternalist thesis that there is a way of being which is common to past, present and future?

There seems to be no way here to escape circularity. In sum: presentism and eternalism are not *per se* trivial, but any attempt to support them against each other would be; and this is the case, simply because they are *incommensurable*. In fact, *as ontological theses* they provide us with different ontologies and, as such, *they are speaking of different things*: presentism and eternalism are not as much different perspectives on the same "thing", as they are different choices of what should count as such (Sider, 1999). Our metaphysical discourses are bound to lie inside either one of these two choices (or whatever possible intermediate ontological thesis[7]), and they just cease to be significant outside them. For this reason, there is no way to decide in favor of any of them on purely ontological grounds, for on that level they simply cannot be compared.

[6]Following Austin (1962), both Savitt and Dorato maintained that the word "existence" has a definite meaning only to the extend that it is used in contrast to a specified class of events, objects or facts.

[7]See § 3.2.

Now it seems that, as long as a clear-cut distinction between tenses is granted, no objective basis to make that choice can be found. However, once that condition is relaxed, the distinction between presentist and eternalist ontologies becomes clearer. In fact, insofar as it denies the ontological disparity of tenses, eternalism is perfectly compatible with the denial of *any* objective distinction among them. The opposite, of course, is true for presentism. In defining existence *via* the concept of presentness, presentism clearly demands the latter to be already given clear-cut objective meaning, or else it would degenerate into a kind of idealism (Gödel, 1949b). But if so, then presentist metaphysics will never *per se* provide us with any objective account of tenses, and this just because being itself in need for such an account: rather than offering a metaphysical criterion to discriminate among past, present and future, it offers a temporal criterion to discriminate among different ontological levels.

This way we are naturally led to abandon the classical version of the presentism-eternalism debate, along with its Galilean spatio-temporal background, where the existence of an objective partition of tenses is given *ab initio* by conditions CP1-CP4. Rather, we shall investigate whether the existence of an objectively distinguished present moment is allowed by non-classical, relativistic space-times.

3

Temporal Becoming and Relativity

Newtonian or classical world consists of a unique temporal succession of three-dimensional instantaneous spaces of chronologically simultaneous events, whose relative distance is independent of any coordinate system or reference frame (Friedman, 1983, pp. 72-73); accordingly, Newtonian '[a]bsolute, true, and mathematical time' is commonly understood as a one-dimensional differentiable manifold, topologically diffeomorphic to the real line, which 'in and of itself and of its own nature, without reference to anything external, flows uniformly and by another name is called duration (Newton, 2004, p. 64)'.

Since the raise of special relativity theory (Einstein, 1952), temporal duration lost its absoluteness, and physical world ceased to be partitionable into a unique, absolute succession of instantaneous three-spaces of chronologically simultaneous events. For this reason, most scientists and philosophers of science regarded special relativity theory as a definitive refutation of the existence of a unique, objective present moment whose motion constituted the passage of time.

This chapter is dedicated to discuss some of the most debated objections which have been moved against temporal becoming on the basis of special relativity theory. In the first part, we shall discuss the relativistic counterpart of the classical ontological debate between presentism and eternalism, examining whether moving to relativistic space-time might change the result of their contention. In the second part, we shall instead examine what geometrical features of relativistic space-time may possibly play the part of the moving objective present moment demanded by objective temporal becoming.

3.1

Elements of Space-Time Theories

Relativistic space-time $\langle \mathbf{M}, \eta, \bigtriangledown \rangle$ is a four-dimensional connected manifold $\mathbf{M}$ endowed with a Lorentzian metric tensor η and affine connection $\bigtriangledown$ (Friedman, 1983). Less hermetically, we may think of it as a mathematical structure with no "holes", which locally resembles the more familiar three-dimensional Euclidean space and one-dimensional time, and in which points or events are individuated by means of quadruples of real numbers. Each physical process taking place in this manifold traces a four-dimensional curve, or *world-line*, whose instantaneous evolution is described at any point x by a tangent vector X; the collection of all tangent vectors at a given point is called the *tangent space* at that point, which we shall denote by T_x. The metric tensor η is a bilinear, symmetric and non-degenerate function from pairs of tangent vectors to real numbers. Finally, $\bigtriangledown$ is a derivative operator, intuitively determining the relative direction of any two tangent spaces at nearby points; however, since $\bigtriangledown$ is uniquely determined by η, it is often omitted.

For any point x in $\mathbf{M}$, η determines a partition of the tangent space T_x at x into *time-like* vectors, *null* vectors and *space-like* vectors, for which the quantity $\eta(X, X)$ is respectively equal, greater than and less than zero; correspondingly, a curve σ is called null, time-like or space-like depending on whether its tangent vector field is everywhere null, time-like or space-like, while any two points x and y are light-like, time-like or space-like separated if they are connected by a null, time-like or space-like curve, respectively. The set of all null vectors at x forms a double cone with vertex in x, called the *null cone* or *light-cone* of x. Intuitively speaking, the light-cone at a point x is the locus of all points of $\mathbf{M}$ which could be reached by a forward or backward light signal sent from x. Moreover, given the limiting character the speed of light has in special relativity theory, light-cones also have the role of delimiting the region of space-time containing all events which can be causally or physically connected to x. To mark the conceptual difference between light-cones and this region, we shall call the latter *causal cone*.

For any vector $X \in T_x$ tangent to a given curve σ passing through x, the *hyperplane of simultaneity* of x relative to σ is a three-dimensional cross-section of $\mathbf{M}$, coinciding with the locus of all vectors $Y \in T_x$ orthogonal to X, i.e. such that $\eta(X, Y) = 0$. In special relativity theory, the light-cone of any point x is an invariant feature of x while, due to the invariance of the speed of light, any trajectory σ through x is associated with a distinct hyperplane of simultaneity. Consequently, for any point x in special-relativistic space-time, it is possible to make a non-relative temporal orientation for all time-like curves through x, while observers moving at different speed would make different judgments as of what events space-like separated from x are simultaneous with, earlier than, or later than x: in other words, while the temporal order of time-like separated events is absolute, or invariant, that holding between space-like separated events is relative to the observer's state of motion. In general relativity theory, instead, since the shape of general-relativistic space-time is locally determined by the distribution of matter, tangent spaces at nearby points do not generally look the same: intuitively speaking, light cones of nearby points may vary in shape, size and relative orientation.

Relativistic space-time is called *temporally orientable* just in case there exists an everywhere continuous non-vanishing time-like vector field, i.e. if and only if, for any point x of $\mathbf{M}$, it is possible to select one of the two halves of the light-cone of x as pointing toward x's absolute or causal future (equivalently, its past), in such a way that this selection is invariant under any continuous translation in space-time keeping time-like vectors time-like.

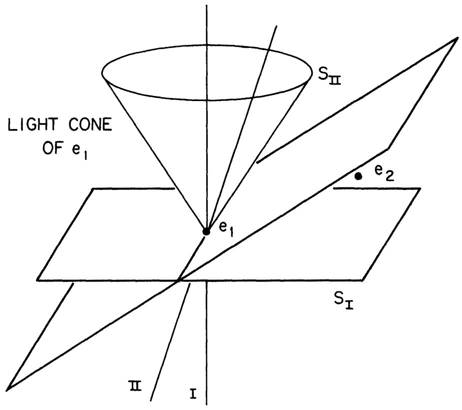

Figure 3.1: Half light-cone and hyperplanes of simultaneity in relativistic space-time (Friedman, 1983, p. 161)

In what follows, we shall concentrate on sole temporally orientable space-times. First of all, this choice is consonant with common usage. Second, assuming space-time to be temporally orientable seems to be at least a necessary condition[1] to speak of tenses in a consistent way (Earman, 1974; Clifton and Hogarth, 1995). Third, for any space-time which is not temporally orientable there exists a covering space-time which is, so that admitting non-temporally orientable space-time might appear a worthlessly expensive choice (Earman, 1986, p. 171).

However, temporal orientability may not be enough. In fact, general relativity theory allows for temporally orientable space-time models which are nonetheless everywhere filled with closed future-directed time-like curves (Gödel, 1949a): in those cases, it would be even harder to speak of past and future times, since whatever future-directed non-inertial observer would always be capable to cross her own past[2]. For this reason, we shall further restrict our attention to the space-time of special relativity theory, namely temporally oriented Minkowski space-time (Minkowski, 1952).

[1] Whether temporal orientability should also be considered sufficient for endowing (special-)relativistic space-time with objective temporal becoming is a currently debated question; among those embracing this view we may recall Dieks (2006a) and Maudlin (2007).

[2] Gödel regarded these models as a definitive refutation of objective temporal becoming. In his view, '[t]he existence of an objective lapse of time [...] means (or, at least, is equivalent to the fact) that reality consists of an infinity of layers of "now" [so that] to assume an objective lapse of time would lose every justification in these worlds (Gödel, 1949b, pp. 558, 561)', at least if co-presentness is equated with chronological simultaneity (See § 3.3). The greatest difficulty Gödel's argument faces is that of concerning surely possible, but rather improbable, cosmological models: the universe we live in is, with all evidence, not of a Gödelian kind; so what bearings do Gödel's argumentation have on the existence of temporal becoming *in our world*? Philosophers debated at length on this topic, reaching opposite results (Stein, 1970; Savitt, 1994; Dorato, 2002b).

3.1.1 Intersubjectivity, Testability, Invariance

There are at least three different ways how one might understand the objectivity of tenses, each of which entailing the previous one. The first, weakest sense, is that of being *intersubjective*, or *subject-independent*: in this sense, anything is objective just in case it can be shared by different individuals. The second, intermediate sense, is that of being *mind-independent*, namely of being independent of the perceptual and cognitive features of human subjects: in this sense, anything is objective just in case it can be observed or tested by means of artificial experimental devices. In the third and strongest sense, objectivity should be understood as being *perspective-independent*: in this sense, anything is objective just in case it is a structural or inherent feature of things, namely just in case it is independent of the way things are observed or measured.

If tenses were objective in the sole subject-independent sense, temporal becoming would be just a projection of our physiological and psychological constitution (Weyl, 1949; Grünbaum, 1967a, 1973; Savitt, 2009; Callender, 2010b), or a secondary quality just like colors, tastes and smells (Menzies and Price, 1993; Price, 1996). Those philosophers who *deny* the objectivity of becoming typically provide it with such purely inter-subjective status.

If tenses were mind-independent, but not perspective-independent, then temporal becoming would be objective both in the sense of being a distinguishing feature of human experiences and in the sense of being experimentally confirmed, albeit only under given observational condition. Incidentally, this is the type of objectivity hyperplanes of simultaneity display in special relativity theory: in fact, *all* observers moving along the same world-line should agree as of the set of events counting as simultaneous to a given time-place, whether they are human beings or purely physical clocks.

Still, this is not the kind of objectivity philosophers generally require tenses to satisfy in order to speak of objective temporal becoming. Rather, they typically require tenses to be perspective-independent, for this would make becoming not only experimentally observable, but it would also make it a structural or inherent property of the space-time manifold:

> [...] trying to establish whether the distinction between past, present and future has an objective, physical counterpart – and is therefore *mind-independent* – means trying to establish whether such a distinction is *definable* in terms of *invariant* structure of [...] space-time [...]. The requirement of definability in terms of invariant relations ensures that the candidate becoming relation be invariant for all possible observers. In our context, such invariance suffices to ensure the intersubjective validity of the becoming relation, something that should be required by any theory of objective becoming (Dorato, 2002a, p. 338).

In what follows, we shall follow this trend, so that while speaking of objective tenses or objective temporal becoming, we shall mean that there exists an invariant structure of temporally oriented Minkowski space-time to which such notions can be reduced.

3.2 Hybrid Theories, Relational Becoming

Entering the arena of space-time theories we are often asked to assume that the world, understood as the totality of the existent, consists of a four-dimensional manifold of time-places. In doing this we are forced to discard presentism as we previously defined it, as the thesis that only those events lying on the present hyperplane of simultaneity currently exist[3], since in this case it would not be possible to speak of an event not being there without *ipso facto* ruling it out of the given manifold (Callender, 2000). This is not a mandatory choice, for one may still try to preserve presentism by regarding four-dimensional spatio-temporal manifolds as purely abstract models (Sattig, 2006), but it is by far the most popular. So, in what follows, we shall concentrate on the metaphysical clash between what Callender (2000) called *hybrid theories* and the so-called *full view* of time (Dorato, 1995).

Properly speaking, both hybrid theories and the full-view subscribe the eternalist ontology, since they both agree that all physical events display the same kind of existence consisting in being part of the four-dimensional manifold representing physical reality. However, they disagree about whether all metaphysical properties, over and above mere physical existence, should be equally distributed among all physical events entering relativistic space-times: while this position is endorsed by full view theorists, hybrid theories are closer to presentism insofar as they argue that there exists some distinguishing metaphysical feature, such as reality, determinateness, determination or fixity, which at least some future events cannot display.

The main trouble with hybrid theories is that they are exposed to paradoxes of a McTaggartian type, so that they are forced to understand determinateness, determination etc. in a relational sense (Callender, 2000, p. S590-S591). In its turn, this choice bears three major consequence on their understanding of temporal becoming.

The first consequence is that hybrid theories are more flexible than presentism as of their metaphysical characterization of tenses. For this reason, they distribute into three different classes, which Dorato (1995) respectively labeled the *instant view*, the *empty view of the future* and the *half-full view of the future*: according to the first, present events possess some distinguishing metaphysical feature that neither past nor future possess; according to the second, past and present events share some relevant metaphysical feature which future events do not; finally, according to the latter, past and present events share their metaphysical properties with only that part of the future they are capable to fix or determine according to physical laws. Despite their differences, all these views nonetheless agree on considering the present moment as the temporal upper bound beyond which events cease to possess full metaphysical status. For these reasons, we may regard them as equivalent as of their contribution to the issue of objective temporal becoming.

[3] See § 2.1.2.

The second consequence is that hybrid theories, by reducing temporal becoming to the existence of a relational metaphysical property which is not equally shared by all tenses, are forced to identify temporal becoming with a binary relation holding between pairs of time-places (Dorato, 2008a).

The third and most important consequence is that the whole debate between full-view theorists and hybrid theorists reduces to the question whether any two arbitrary space-like separated events can be assigned the same metaphysical status as of each other (Dorato, 1995, pp. 147-152). For this reason it seems that, contrary to the presentism-eternalism debate, that between hybrid theories and the full view of time can be settled *before* any objective characterization of tenses is given; and for the very same reason it seems that, contrary to presentism, hybrid theories could really offer a metaphysical basis to define relativistic objective presentness: if a metaphysical asymmetry was proved to hold between any two different regions of the space-time manifold, then the present moment would be that part of space-time separating metaphysically asymmetric events.

3.2.1 The RPM Thesis

During the last decades, the philosophical debate on the relativistic counterpart of objective temporal becoming has been centered on the arguments Rietdijk (1966, 1976), Putnam (1967) and Maxwell (1985, 1988) gave in support of the full view of time. Though having been motivated by different metaphysical interests and having being supported by means of slightly different proofs, their results may nonetheless be collected under what we shall call the RPM thesis: once any two space-like separated events are assigned the same ontological features, then all time-places in Minkowski space-time must display the same features too. Rietdjik declined that claim with respect to determination, Putnam with respect to reality and Maxwell with respect to ontological fixity or definiteness. In spite of the different metaphysical content each of these notions may possibly bear, the importance of the RPM thesis for the problem of becoming stems from the fact that it denies that *any* ontological asymmetry may ever hold between different time-places, with the purported consequence of disproving the existence of any metaphysical separation of tenses.

3.2.1.1 Rietdijk: Determination and Preemption

Rietdijk's declared aim was that of proving special relativity to be committed with a rigorous determinism. Unfortunately, he never laid his metaphysical cards on the table, leaving us with the onus of extracting them from his own argumentation:

> Consider an inertial system (X_1, O_1, T_1) with the observer W_1 in O_1, as well as the inertial system (X_2, O_2, T_2), observer W_2 being in O_2. Consider both W_1 and W_2 at the time $T_1 = 0$. (X_2, O_2, T_2) moves at a constant velocity in the direction of

> O_1. X_2 intersects T_1 in P. Then we find that for W_1, observer W_2 in O_2 [$= B$] is "now". For W_2 in B, however, events B and P occur simultaneously: for W_2 event P is "present" at the same moment $T_1 = 0$ of W_1 that W_2 is "present" for W_1. [...] This "now" P for W_2 – in the "absolute future" of W_1 – is as real as, e.g., W_2 is for W_1 . Conclusion: P is completely determined at the moment $T_1 = 0$ of observer W_1. [...] With each event P_1 in the future of W_1, we can now think of an observer W_2 for whom, at the moment of observer W_1 that W_2 is "now" for W_1, P_1 is already in the past. Thus each event is determined: it is already "past" for some one in our "now" (Rietdijk, 1966, p. 341).

The key inference of Reitdjik's argumentation is that leading from 'P for W_2 [...] is as real as, e.g., W_2 is for W_1' to 'P is completely determined at the moment $T_1 = 0$ of observer W_1'. The antecedent of such inference is presumably motivated by the very general hypothesis that events standing in the same spatio-temporal relation as of their observers should also stand in the same ontological situation with respect to them. So, what auxiliary metaphysical assumptions are needed to support Rietdjik's conclusion? Rietdjik's key assumption, the one which he presumably wanted to support that inference, was the following:

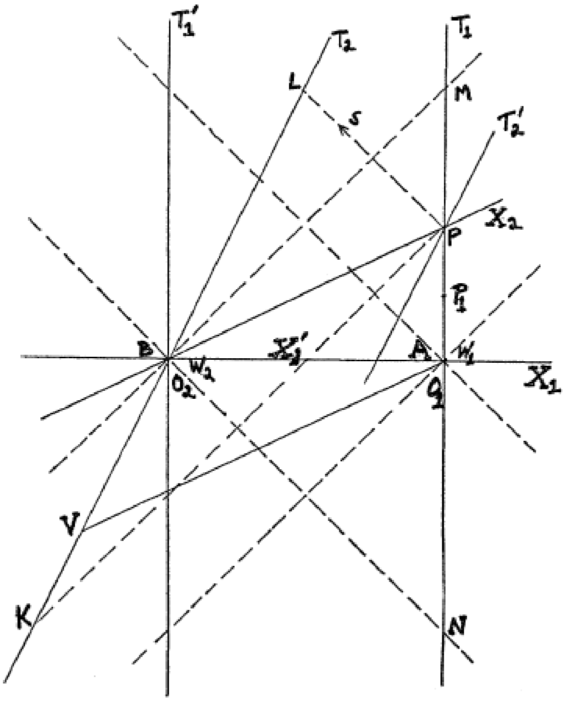

Figure 3.2: Space-time diagram of inertial observers in Rietdijk's argumentation (Rietdijk, 1966, p. 341).

> We say that an event P is (pre-) determined if, for any possible observer W_1 (that is, for all possible observers, and even for all other, e.g., physical, instances), who has P in his absolute future (that is, that the future part of W_1's course through the four dimensional continuum may eventually pass through P), we can think of a possible observer W_2 (or: there may exist an observer W_2) who can prove, at a certain moment T_p, that W_1 could not possibly have influenced event P in an arbitrary way (e.g., have prevented P) at any moment when P still was future, or was present, for W_1 , supposed that W_1 did desire to do so (Rietdijk, 1966, p. 342).

So what we must go in search for, in the end, is the way how (and the time when) W_2 could have shown that W_1 was not capable of affecting P at O_1.

The structure of Rietdjik's argumentation leads us naturally to suppose that, if W_2 was ever capable of offering such a proof, that would have happened at the sole point where his trajectory intersected $T_1 = 0$: namely, at O_2. So, what special properties does W_2 display at that point? Presumably that O_2 is the sole point at which one of W_1's hyperplanes of simultaneity intersected W_2's unique hyperplane of simultaneity containing P: in other words, O_2 is the sole point such that W_2 is isochronous to W_1 according to W_1's clock and P is isochronous to O_2 according to W_2's clock. Given this, we should wonder: is this sufficient for W_2 to prove that W_1 was not capable of affecting P at O_1? Here the weakness of Rietdjik's argumentation starts to come out. In fact *before any metaphysical significance could even be attached to that situation*, in order to make that kind of judgment, W_2 should have been capable of collecting information about the physical state of both W_1 at O_1 and P. But how could that be possible? By hypothesis, W_2 is space-like separated from both, and hence uncapable of receiving any of the signals they possibly emitted (Stein, 1968, p. 16).

Rietdjik himself was probably aware of this flaw, and it is possibly for this reason that he added the following clause:

> It is even possible that W_2, on experiencing event B, will reduce his velocity until he is at rest with respect to W_1 during a very short time t, so that during this time, his coordinate system is inertial system (X_1', O_2', T_1') (supposing he knows his velocity towards W_1; however, this is not of fundamental interest). Thereafter, W_2 can resume his former velocity. *When, after some time, W_2 receives the light signal [coming from P], he will know that already –during the short time t – when he was at rest with respect to W_1, experiencing the same "present" as did W_1 in virtually the same inertial system, W_1 could do nothing at all to prevent event P in his absolute future* (Rietdijk, 1966, p. 342).

The aim of this remark is presumably that of making W_2's judgment retroactive, postponing it at the time-place at which he was finally able to receive the information coming from O_1 and P. However, this makes Rietdjik's argumentation no more effective. In the best case, W_2's hyperplane of simultaneity at a small neighborhood of O_2 would first include P and, soon after, W_1; however, this would not make the least change in the fact that W_2 would know first about W_1 and then about P. To know that P fell in his hyperplane of simultaneity at O_2 before W_1, he should have known when, and how long, he was actually sharing the same hyperplane of simultaneity as W_1. But if so, contrary to what Rietdijk claimed, it *is* of fundamental importance that W_2 knew his velocity relative to W_1; and, once again, this is made impossible by space-like separation.

Finally, even all of these difficulties could be sidestepped, Rietdjik's argumentation would still suffer of a serious flaw: namely, his definition of determination is not capable to distinguish a determinate event from a purely stochastic one, since all possible observers would agree that nobody could ever control it or change it at any time (Stein, 1968, p. 13, note 8).

3.2.1.2 Putnam: The Transitivity of Reality

Putnam's polemic target was the 'man on the street's view on the nature of time', which he encoded in the statement that '[a]ll (and only) things that exist *now* are real (Putnam, 1967, p. 240)'. His argumentation was based on the following set of assumptions:

i I-now am real.

ii At least one other observer is real, and it is possible for this other observer to be in motion relative to me.

iii If it is the case that all and only the things that stand on a certain relation R to me-now are real, and you-now are also real, then it is also the case that all and only the things that stand in the relation R to you-now are also real.

In order to play the part they are assigned by Putnam, assumptions (i) and (ii) should be understood to range solely over inertial observers. They are respectively meant to make reality (or whatever binary relation R could count as a criterion to determine what is real) a reflexive and non-trivial relation. The third assumption is referred to by Putnam as 'the principle that There are no Privileged Observers': in brief, it requires that "being real for" is a transitive relation (Stein, 1968).

Putnam's argumentation then follows in a manner very similar to that of Rietdijk's. Let us assume that I-now and you-now are at the same place, but moving with different velocities; hence, if R was identified with the relation of being in one's hyperplane of simultaneity then, by premise (iii), what is in your present should count as real for me-now, since you are in my present and therefore real for me. However, there are events in your hyperplane of simultaneity which are space-like separated from me and which would count as future with respect to my present coordinate system; by (iii), these events should count as real for me-now, but according to R they should not. The sole way to avoid this contradiction is, according to Putnam, to make relative simultaneity with respect to an arbitrary inertial observer a sufficient but not necessary condition for being real for that observer; however, this would have the side-effect of making all events in my (and anyone's) relative future real as well. Putnam's enthusiastic conclusion was that 'the problem of the reality and the determinateness of the future events is now solved', and that 'there are no any longer any *philosophical* problems about time (Putnam, 1967, p. 247)'.

Putnam's scenario is quite simpler than the one depicted by Rietdijk: in this case, we are faced with two inertial observers whose trajectories coincide at a space-time point and who therefore agree to be simultaneous to each other at that point, as well as of what events should then count as absolutely present and absolutely future. The major difference between Rietdijk's and Putnam's argumentations is that, in the second case, according to the man on the street's view, none of the events in the absolute future of the first observer would count as real for the second; and therefore, the sole future events Putnam could prove to be real are at best those which lie in the relative future of the former and which are space-like separated from him. So, in any case, his main conclusion would hold at best partially. This is certainly a minor shortcoming of Putnam's argument, for the latter could be easily reshaped so as to fit a scenario which is in all similar to that of Rietdijk (see, for example, Dorato (2008b)). Nonetheless, there is one further argument which may undermine Putnam's conclusions.

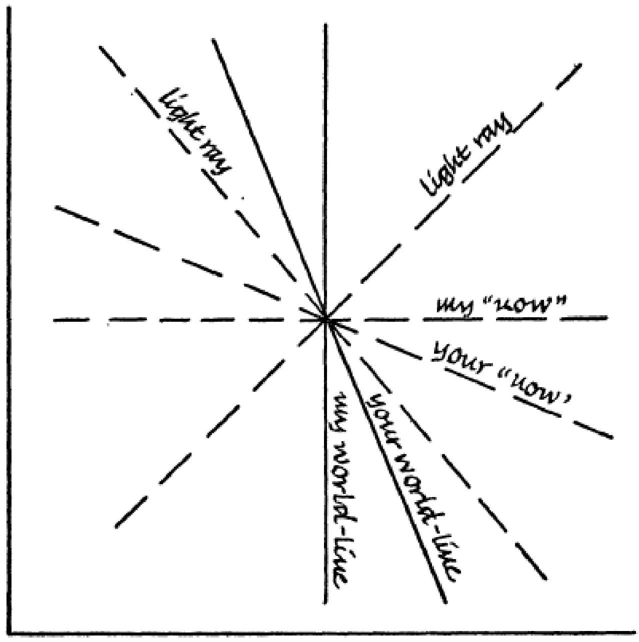

Figure 3.3: Space-time diagram of inertial observers in Putnam's argumentation (Putnam, 1967, p. 241).

The principle of no privileged observers plays as crucial a role for Putnam's argumentation as Rietdijk's definition of determination played in the previous case: in fact, it is the connecting link making it possible for different inertial observers to agree as of what events should count as real. However, in the end such an assumption is unsupported. On the one hand, Putnam never motivated it. On the other hand, the sole plausible reason one may think of to justify it is that reality should not be a subjective matter – in other words, anyone should agree as of what counts as real: is something is real for you then it must also be real for me, granted the very minimal condition that you are not just fictional. However, if that is the sole reason to support (iii), then the latter becomes superfluous. In fact, in commenting upon it, Putnam carefully demanded that 'R must be restricted to physical relations that are supposed to be independent of the choice of a coordinate system' and that 'it must not depend on anything *accidental* (Putnam, 1967, p. 241)'; but, as it was independently complained by (Stein, 1968, pp. 18-20) and (Sklar, 1985, pp. 296-297), this would suffice to make R objective in the special-relativistic context. Dismissing (iii) in favor of the sole requirement that R should be invariant would indeed support the first half of Putnam's argumentation – namely that, in Minkowski space-time, relative chronological simultaneity is no objective criterion for determining what is real – but it would not support his main conclusion.

Truly, there seem to be strong reasons in favor of the transitivity of reality. Dorato, for example, contends that 'denying transitivity would imply that what exists at a distance depends on a state of motion' and that 'by denying transitivity, two observers zooming past each other would share the same present without sharing what is real at a distance, and by simply changing reference frame (getting off a bus or jumping on an airplane), we would change what counts as real for us at a distance (Dorato, 2008b, pp. 58, 60)'. However, none of these objections affect the core of Stein's and Sklar's contention, for all they seem to require is that reality can be shared by different inertial observers, and this is guaranteed by simply demanding it to be an invariant. What invariance cannot guarantee is that "being real for", over and above being a transitive relation, held between space-like separated events. Premise (iii) should be preferred to invariance if and only if reality had to satisfy both these properties; but it is not at all clear why, in a relativistic context, one should be committed to the latter[4].

3.2.1.3 Maxwell: Determinism and Chronological Simultaneity

The broad lines of Maxwell's argumentation (Maxwell, 1985, 1988) are also shared by Rietdijk (1976), who reshaped his original argument in response to Landsberg (1972); for this reason, and despite their opposite theoretical goals, we shall follow them in parallel.

In Maxwell's view, *ontological probabilism* is the metaphysical doctrine according to which 'at any instant, there is only one past but many possible alternative futures – the fundamental laws of the universe being probabilistic and not deterministic'. In its turn, this thesis demands that 'there is a physically real difference between past and future events – the future alone containing physically, ontologically real alternative possibilities' and therefore, 'that, at any instant, there be a universal, absolute, unambiguous distinction between one past and many possible futures (Maxwell, 1985, p. 23)'. Similarly, Rietdijk maintained that 'if the indeterminism thesis makes any sense at all, there is an *objective* difference between "being determined" and "being not determined"', and hence 'from the standpoint of the adherence of the existence of indeterminism it is necessary to assume the existence, in the four-dimensional continuum, of a (straight or curved)' hyperplane H consisting of "now" events' which 'cannot depend on the velocity of any observer (Rietdijk, 1976, p. 601)'. On this basis both Maxwell and Rietdjik concluded, in a very straightforward way, that indeterminism (or ontological probabilism) is incompatible with the relativization of simultaneity demanded by the special relativity theory; Maxwell argued in favor of the former, Rietdijk in favor of the latter.

The major premise of both Maxwell and Rietdijk's argumentation is, therefore, that indeterminism is in need of an objective global foliation of space-time into non-intersecting hyperplanes of simultaneity. However, this contention is made on the basis of a highly questionable understanding of indeterminism. In fact, if we agree that indeterminism is the opposite thesis than determinism, then it is indeed possible to make it logically independent of the existence of objective global hyperplanes of simultaneity.

Perhaps the best account of determinism currently on the market is that offered by Earman (1986). He defined Laplacian or classical determinism as the property of any possible world $W_i \in \mathcal{W}$ ($\mathcal{W}$ being the collection of all physically possible worlds) such that, for any $W_j \in \mathcal{W}$, if W_i and W_j agree at a time, then they agree at all times. Indeterminism would accordingly be the

[4] See § 3.4

property of a physically possible world W_i such that, for some physically possible world W_j, W_i and W_j agree at some time, but not at all times. In this view, 'by assumption, the world-at-a-give-time is an invariantly meaningful notion (Earman, 1986, p. 13)'. That is say: if we wanted that Laplacian determinism held within a given space-time theory, then we should require that theory to be compatible with an objective partition of space-time into non-intersecting hyper-planes of simultaneity. So, special relativity theory is as much incompatible with indeterminism as it is with Laplacian determinism, and therefore Maxwell and Rietdijk's basic dilemma cannot stand. Rather, we should at best choose between Laplacian determinism or indeterminism on the one side, and special relativity theory on the other: this would still support Maxwell's option, but it would rule out Rietdijk's.

Furthermore, classical determinism is just one of a family of related conceptions of determinism, not all of which require an objective partition of tenses. For example, a possible world $W_i \in \mathcal{W}$ can be claimed to be (R_1, R_2)-deterministic exactly in case R_1 and R_2 are two arbitrary collections of similar connected regions of space-time, and for any possible world $W_j \in \mathcal{W}$, if W_i and W_j agree on space-time regions of type R_1 they also agree on space-time regions of type R_2. Indeterminism would accordingly require that, for such R_1, R_2 and W_i, there existed a physically possible world W_j such that W_i and W_j agreed at one region of type R_1 but not at the corresponding region of type R_2. In that case, neither determinism nor indeterminism would call for an objective partition of space-time into space-like hypersurfaces, consequently blocking Maxwell and Rietdjik's *reductio* once and for all.

3.2.2 Stein's Theorem

Following Maxwell, Stein (1991) linked metaphysical becoming to the existence of "stages" of definiteness, each of which should separate what is already become and settled and what is already to be determined. However, he understood "being definite" or "having become" as a transitive, reflexive, antisymmetric and non-universal binary relation R on a temporally oriented space-time manifold $\mathbf{M}$. He proved that, if Rab was true for at least one pair of events a and b in Minkowski space-time, such that b lied in the absolute past of a, then becoming should be taken to be coextensive with the relation of past causal connectability (understood in a weak sense, so as to include identity); and since past causal connectability is an objective property of Minkowski space-time, then also becoming should be. So it may seem that, if Stein was right, then Minkowski space-time could be endowed once and for all with an objective metaphysical acceptation of becoming.

The key statement of Stein's proof is well-known, and it is commonly referred to as Stein's theorem:

> If R is a reflexive, transitive relation on a Minkowski space (of any number of dimensions – of course at least two), invariant under automorphisms that preserve the time orientation, and if Rab holds for some pair of points (a, b) such that ab is a

> past-pointing (time-like or null) non-zero vector, then for any pair of points (x, y), Rxy holds if and only if xy is a past-pointing vector (Stein, 1991, p. 149).

On the other hand, Stein's conception of becoming and definiteness (or reality, or determination) differs from those held by Rietdjik, Putnam and Maxwell insofar as it is defined independently on any observer's state of motion: in his view, 'the fundamental entity, relative to which the distinction of the "already definite" from the "still unsettled" is to be made is the *here and now*; that is, the space-time point (Stein, 1991, p. 148)'. One may try to sidestep the result of his theorem by rejecting his account of becoming in favor of a world-line dependent one. Clifton and Hogarth (1995) ruled out this possibility, by formulating a more general version of Stein's theorem which could also apply to world-line dependent becoming.

For this purpose, they required world-line dependent becoming to satisfy the properties of *world-line becoming* and *world-line transitivity*, and to be implicitly defineable[5] by time-oriented metrical relations. World-line becoming requires that for any two space-time points a and b, if a lies both along the world-line of b and in its chronological past, then a has become for b. Clifton and Hogarth motivate this assumption by relying on the psychological sense a conscious observer would have that those events which he lived through had already become for her. World-line transitivity requires that for any three space-time points a, b and c and any two (non necessarily inertial) world-lines λ and λ', if a has become for b along world-line λ and b has become for c along world-line λ' then a has become for c along world-line λ'. Finally, defineability from time-oriented metrical relations requires world-line dependent becoming to be preserved under any automorphism of $\mathbf{M}$ preserving the distance between two space-time points or, which is the same, to be an objective or invariant property of temporally oriented Minkowski space-time. Given all this, they proved the following statement:

> Consider the collection of world-line dependent becoming relations associated with all world-lines (all possible observers) in time-oriented Minkowski space-time. Suppose this collection satisfies world-line becoming and world-line transitivity, and that each collection in the relation is (implicitly) defineable from time-oriented metrical relations and the relevant world-line for that relation. Then every becoming relation in the collection must be the relation of past chronological connectability, or they all must be the relation of past causal connectability, or they all must be the universal relation (Clifton and Hogarth, 1995, pp. 371-372).

By (future or past) chronological connectability and (future or past) causal connectability, Clifton and Hogarth meant, respectively, (future or past) time-like separation and (future-directed or past-directed) causal connection. However, Dorato (2000a) noticed that past chronological connectability – as well as world-line becoming – depends on the traces that past events leave in

[5]Let α and $\beta_1, ..., \beta n$ be non-logical symbols of the language L of some theory T such that α is not among the β_i; then 'α is *implicitly defineable* from the β_i in T if any two models of T which have the same domain and agree in what assign to the β_i also agree in to what they assign to α (Boolos et al., 2007, p. 266)'.

the observer's memories and, through them, on a future-directed causal connection. So, in the end, all the world-line dependent becoming relations which may obtain in Minkowski space-time should coincide with that of asymmetric past-directed causal connectability, confirming the result of Stein's theorem.

3.2.3 The Metaphysical Neutrality of Minkowski Space-Time

Clearly, in order to be demonstrable, Stein's theorem requires that any relation R meant to play the part of temporal becoming should also be strictly antisymmetric, for otherwise Rba may hold for at least one future-pointing non-zero vector ba. Furthermore, as Stein himself showed (Stein, 1991, p. 149), allowing R to hold symmetrically between any two space-time points would suffice to make it a universal relation[6]. In the light of this, his theorem proves to be just a corollary of the more basic claim that 'in Minkowski space-time [...] there are no intrinsic geometrical partitions into equivalence classes at all, besides the two trivial ones (Stein, 1968, p. 19)'.

Now it seems that, to play the part of temporal becoming, whatever binary relation should at the very least be reflexive and non-identical. Given these two minimal conditions, the above claim puts us in the front of the following double choice: (a) taking reflexivity for granted, we can only choose between (a1) a transitive but universal and (a2) a non-universal but intransitive invariant becoming relation; (b) assuming transitivity, the we can only choose between (b1) a symmetric but universal and (b2) a non-universal but antisymmetric one. The former dichotomy is precisely the one which emerged by our discussion of Putnam's argument; the second one is that underlying Stein's theorem[7]. But what is most important to notice is that, in any case, special relativity theory can lead us no further: whatever of the four options above one may be willing to choose, there is nothing left in the very structure of Minkowski space-time motivating that choice. So, once again, it seems that we are left with a matter of pure metaphysical taste.

Dorato (1995, 2000a, 2008a) repeatedly emphasized that Stein's theorem cannot, and does not want to, prove becoming to be there: at best, it can show that a relational, metaphysically rooted acceptation of becoming is *compatible* with the properties of Minkowski space-time. We may push this claim even further, and assert that Stein's theorem proves Minkowski space-time to be *neutral* with respect to the metaphysics of becoming[8].

The discussion we made of the presentism-eternalism debate in Chapter 2 led us to conclude that, insofar a well-defined distinction between tenses is supported by the given background

[6] See also (Callender, 2000).

[7] More precisely: option (a1) is that held by Putnam (1967), (a2) is that held by Stein (1968) in response to it; (b2) was also held by Stein (1991), while (b1) was endorsed by Callender (2000) as a purported refutation of Stein's theorem.

[8] Dorato (2006, 2008a) also argued in favor of this claim, but from a different perspective. In his view, there is no metaphysical distinction between an event's happening at a time-place and its coming into being then-and-there; accordingly, there is no real distinction between a metaphysics of becoming and a metaphysics of being and, as long as a space-time theory is in need of a non-empty ontology of events, it is also in need of such a weakened acceptation of becoming, which he calls *absolute*. In conclusion, if becoming is something demanded by the very postulates of all space-time theories, then special relativity theory can neither prove it nor disprove it. On absolute becoming see also Savitt (2002).

space-time theory, there is no way to choose between presentism and eternalism on purely metaphysical grounds. Our analysis of Stein's theorem has now shown that neither temporally orientable Minkowski space-time, where a geometrical characterization of tenses is not guaranteed *ex hypothesi*, is capable to offer any metaphysical basis to choose between the weak ontology of becoming held by hybrid theorists and the radical ontology of being endorsed by the full-view theorists of time.

This way, we are pushed back from metaphysics to the geometry of space-time: instead of going in search for a metaphysical account of tenses, we should better examine what structural properties of Minkowski space-time would possibly play the part of objective past, present and future times.

3.3

There's No Time Like *The* Present

Classical space-time is a temporally oriented four-dimensional differentiable manifold, topologically equivalent to $\mathbb{R}^4$. Since any point in that manifold is assigned a unique hyperplane of simultaneity, classical space-time is foliated into a unique set of three-dimensional cross-sections, each of which representing the whole of the universe at a time. This way, the classical four-dimensional manifold $\mathbf{M}$ decomposes into the cartesian product $\mathbb{R}^3 \times \mathbb{T}$ of three-dimensional absolute Euclidean space and one-dimensional absolute time: for any two arbitrary events, there exists a well-defined notion of the temporal distance holding between them, independently of the chosen coordinate system or reference frame (Friedman, 1983, p. 71-78).

In pre-relativistic space-time, hyperplanes of simultaneity offered an objective partition of all events according to their absolute temporal location, as well as a natural choice for defining the objective present of each space-time point. Co-presentness, i.e. the binary relation consisting in sharing the same present, was accordingly an equivalence relation on $\mathbf{M}$ while, at any time, the present moment was the separating element between past and future times.

Einstein's relativization of chronological simultaneity is commonly considered the major threat special relativity theory posed to the ordinary, common-sense understanding of tenses: since different inertial observers would make different judgments as of what events should count as chronologically co-present, then it seems that in the space-time of special relativity theory there can be no room for a unique, objective and globally extended "now". However, chronological simultaneity was not the sole distinguishing feature hyperplanes of simultaneity displayed in pre-relativistic space-times.

In classical mechanics, physical or causal influence was supposed to travel at finite but arbitrarily high speed. For this reason, any event x in classical space-time was given a degenerate causal cone, asymptotically coinciding with its unique hyperplane of simultaneity. Furthermore, since the causal cone of any such x met the corresponding hyperplane of simultaneity only *ad infinitum*, the latter also coincided with the the locus of events which, on the contrary, were *not* physically or causally connectible to x.

In sum, the unique hyperplane of simultaneity of each space-time point x coincided with (a) the locus of events isochronous to x, or *chronological present*, (b) the topological boundary of the locus of all events which are physically or causally connectible to x, or *limiting present*, and (c) the topological exterior of that locus, or *causal present*[9].

In Minkowski space-time, due to the existence of a finite upper limit to the speed of causal propagation, the factual coincidence between isochronism and causal separation is lost (Reichenbach, 1956, p. 40): for any time-place x, its chronological present is still conventionally associated to its hyperplane of simultaneity, while its causal present is extended to cover the whole *elsewhere region* (Eddington, 1920, p. 50) lying outside its double light-cone, possibly including x itself. Furthermore, the limiting character of the speed of light makes the topological boundary of an event's region of causal dependence coincident with its double light-cone, and hence exterior to its causal present (Friedman, 1983, pp. 159-165). This way, all the main properties of the classical notion of presentness are separated, in such a way that no space-time point could ever satisfy all of them at once, except for the location at which a given moment is supposed to take place.

On the other hand, in virtue of the very fact of being no longer factually coincident to chronological simultaneity, causal presentness and limiting presentness may offer alternative objective bases to define presentness in the special relativistic context.

3.3.1 Spatio-Temporal Coincidence

Probably the most intuitive yet troublesome choice would be that of identifying the present moment with the "here-now", namely the space-time point at which a given event takes place. This option was first put forward by Robb (1921) and later endorsed by Stein (1968, 1991), Dieks (1988) and Hinchliff (1996, 2000). Its major advantage is surely that of identifying the present moment with the sole region of space-time at which the chronological present, the causal present and limiting present of any given event intersect each other. Furthermore, reducing co-presentness to the relation of spatio-temporal coincidence would make it an equivalence relation; and, finally, restricting the present moment of any event x to the sole time-place of its occurrence would have the consequence of making it the separating element between those events lying in the absolute future of x and those which lie in its absolute past.

[9] For any subset A of a topological space $\mathcal{S}$, a point x is called an *interior point* of A if and only if there exists a neighborhood of x which is a proper subset of A, an *exterior point* of A if and only if there exists a neighborhood of x which is a proper subset of the complement of A, and a *boundary point* of A if and only if all neighborhoods of x intersect both A and its complement; accordingly, the *interior*, *exterior* and *boundary* of A is, respectively, the set of all interior points, exterior points and boundary points of A (Isham, 2001, pp. 12, 31).

Serious philosophical arguments have been moved against this conception of presentness. The most common one is that no ontological meaning could ever be attached to it without leading to a kind of metaphysical solipsism: since, according to it, no two distinct events would ever count as co-present, then no two distinct events could ever share the distinguishing ontological status accorded to the present time (Savitt, 2000; Saunders, 2002). On the very opposite hand, some philosophers have argued that endorsing the here-now view of the present with any metaphysical property would instead have the effect of distributing it universally among all time-places (Dorato, 1995; Price, 2010). However, all these arguments are of a metaphysical kind; and as long as our task is that of examining whether the structure of Minkowski space-time is by itself capable of supporting the idea of presentness, they seem to be ineffective. The major trouble with the here-now conception of the present is, rather, that it trivializes co-presentness *independently of whatever metaphysical interpretation of tenses*: as long as events are understood as point-like portions of the four-dimensional manifold, reducing the relation of co-presentness to spatio-temporal coincidence has the obvious consequence of making it coextensive with the identity relation. For this reason, we shall leave this option behind, and proceed toward less beaten tracks.

3.3.2 Light-Like Separation

The second possible option is that of identifying the present moment of each space-time point with its liming present, which is to say, with its double light-cone. Choosing this option would have the straightforward consequence of guaranteeing the objectivity of presentness by grounding it on a structural component of space-time. Moreover, it would reduce co-presentness to a suitable generalization of the classical relation of isochronism, namely that of having zero distance according to the Lorentzian metric η.

This option was indirectly discussed by Savitt (2000), whose target was the stronger thesis that the present moment of each event should be located on its past light-cone[10]. Savitt himself recognized that such proposal would have the double advantage of (a) associating co-presentness with '*bona fide* geometric structures in Minkowski space time' and (b) that 'in the limit as the speed of light approaches infinity, these structures "flatten out" to approach the hyperplanes of simultaneity that are naturally the present pre-relativistically (Savitt, 2000, p. S566)'. Nonetheless, he rejected it because leading to what he thought to be unreasonable consequences.

First, he saw no apparent reason to identify co-presentness with past-directed light-like separation with the exclusion of its future counterpart. This objection, which is entirely sound, is nevertheless not effective against the hypothesis currently under discussion.

[10]This view was discussed by Hinchliff (1996, 2000), who attributed its paternity to Godfrey-Smith (1979). However, Godfrey-Smith restricted co-presentness to 'those events which are related by the signal relation so that they are *perceived together* (or would be perceived together by some suitably located observer)', i.e. 'those to which an observer stands in a direct causal relation defined by the signal relation (Godfrey-Smith, 1979, p. 236, p. 240)'. Since he imposed no constraint on the speed of such signals, his idea of relative presentness should rather be extended so as to include past time-like separation too.

Second, he complained that, according to this view, events placed in the very earlier stages of the universe such as the source of the Cosmic Background Radiation would still count as present. However, this objection fails in the face of the conceptual separation we made between the two notions of chronological presentness and limiting presentness. Once presentness is reduced to the latter, nothing unreasonable is left in demanding that we share the same present as the Big Bang: in this sense, being co-present to a given event is just as much as being the farthest event which could ever exert any causal influence upon it in a given lapse of time, or upon which it could ever exert any causal influence in its turn. This might be counter-intuitive, but it is surely consistent.

Finally, Savitt maintained that the (past) light-cone of any given event x do not satisfy what he called a requirement of *achronality*:

> if some set of events S represents the present for event E, then no events in S should be in each other's absolute past or absolute future (that is, it should not be the case that all observers at E agree that one of the events is, say, earlier than the other) (Savitt, 2000, p. S567).

His contention is evidently based on the double assumption that light-like separated events can be arranged invariantly as of their temporal order, and that they lie in each other's absolute future (conversely, past). However, these assumptions are not as much incontrovertible as they might appear. Truly, in the context of temporally orientable space-times we commonly speak of the future and the past light cones of an arbitrarily chosen time-place x, and this may rise the impression of an absolute partition of the points lying on x's double light-cone between those which are objectively earlier than x and those which are objectively later. However, the above way of speaking may be misleading; in fact, what counts as the absolute future or past of a given event is, properly speaking, *the interior* of its future or past light-cones:

> If $\langle \mathbf{M}, \eta, \bigtriangledown \rangle$ is a temporally orientable space-time we can define in a globally consistent manner an equivalence relation $S(\ ,\ )$ on the set of time-like tangent vectors which holds between two such vectors U and V just in case they have the same time sense. The quotient of the set of time-like vectors by $S(\ ,\ )$ has two elements O_1 and O_2. The choice of one element as containing the future pointing time-like vectors is the choice of a temporal orientation or direction of time for $\langle \mathbf{M}, \eta, \bigtriangledown \rangle$ (Earman, 1974, p 18).

The equivalence relation $S(\ ,\ )$ is defined on the set of sole time-like vectors: the choice of a global temporal orientation for space-time, and hence the existence of a univocal distinction between past and future light-cones, is therefore independent of the temporal orientation of light-like vectors and on the temporal ordering of light-like separated events. In other words, it is possible to speak coherently of the absolute past and the absolute future of a given event *without* requiring that the events lying on its double light-cone are themselves past or future.

Indeed, one may well choose to redefine $S(\, , \,)$ so as to include light-like vectors too, but that choice is certainly not mandatory. In sum, there is nothing in the ordinary way of building temporally oriented Minkowski space-time which prevents light-like separated events from being achronal.

This discussion has led us to the third and final advantage of choosing light-like separation as the relativistic counterpart of co-presentness. Insofar as the equivalence relation $S(\, , \,)$ is restricted to the set of sole time-like vectors, the absolute past and future of any given event are themselves restricted to the sole interior of its past and future light-cones. Leaving the elsewhere region aside, then it is neither physically nor conceptually possible to move from the absolute past to the absolute future of any given event without crossing its limiting present – that is to say, for whatever event x, the limiting present is precisely the region of space-time separating absolute past from absolute future. Surely, this will not produce a foliation of the whole space-time manifold, but it is enough to guarantee a consistent, exhaustive and invariant decomposition of any arbitrary causal cone into past, present and future.

3.3.3 Space-Like Separation

The third possible option is to identify the present moment with the causal present, namely with the elsewhere region. Since the latter is just the topological exterior of a light-cone, this choice would provide the present moment with precisely the same kind of invariance as that of the limiting present. In addition, it would have the advantage of extending the present moment of any given event so as to span the union of all its possible hyperplanes of simultaneity, offering a plausible generalization of the classical conception of presentness as the locus of all events isochronous to a given one.

This feature of the causal present was already noticed by Weingard (1972), who endorsed this view on the basis of the critique Reichenbach (1958) and Grünbaum (1973) made of the standard definition of chronological simultaneity. In introducing the notions of chronological presentness we incidentally claimed that, in special relativity theory, the locus of events isochronous to a given time-place is *conventionally* associated to its hyperplane of simultaneity. Reichenbach and Grünbaum emphasized the conventionality of this choice, which they took to be based on the assumption of the continuity of light signals and on the constancy of the speed of light in all directions. Einstein's operationalist definition of simultaneity by means of light clocks was based on the hypothesis that light took the same time to cover equal spatial distances in opposite directions; however, given the limiting character of the speed of light, there is no physical mean to ascertain the truth of that assumption. In the end, any other definition of simultaneity between space-like separated events would work as well, though the standard choice is motivated by reasons of computational simplicity. On this basis Weingard argued that, since presentness cannot be taken to be a matter of pure convention[11], then the relativistic conventionality of chronological simultaneity makes it an unsuitable candidate for playing the

[11] See also Sklar (1985).

part of co-presentness in the special-relativistic context. Weingard did not acknowledge limiting presentness, and therefore his choice fell naturally on causal presentness.

The first objection one may possibly move against Weingard's argumentation is that whether chronological simultaneity is a conventional matter has become questionable after Malament's proof that, for each inertial world-line, standard chronological simultaneity defines the sole non-universal equivalence relation holding between space-like separated events (Malament, 1977; Dieks, 2006a). However, the key core of Weingard's account can easily be made independent on that assumption for, as he pointed out:

> [...] in terms of actual physical or experimental facts, it is the class of events that can be considered simultaneous to an event at P [as of whatever definition of simultaneity], and not the class of events absolutely [i.e. chronologically] simultaneous to the events at P, that plays the role in special relativity that the class of events simultaneous to P plays in Newtonian space-time. In each they are the class of events *that are not causally connectable* with P. And while the class of events simultaneous to an event at P, with respect to some frame of reference, is not a relativistic invariant, the class of events that can be considered simultaneous to events at P is such an invariant. It is just the class of events outside of P's light-cone (Weingard, 1972, p. 120, my emphasis).

In other words, one may simply prefer causal presentness to chronological presentness in virtue of its more straightforward physical significance.

The second possible objection is that raised by Savitt who, once again, relied on his achronality requirement: for any time-place x, there exist two distinct events y and z in x's elsewhere region such that y and z are time-like separated; accordingly, elsewhere regions are not achronal. However, Savitt's requirement seems here to be out of place. Truly, if compared to chronological simultaneity, achronality is a perfectly reasonable demand: in fact, no one would be willing to assign the same time coordinate to a pair of events which, *from any possible perspective*, appeared to be separated by a non-zero temporal gap. However, the causal present of event x was defined as the (four-dimensional) surface on which x could not exert (or could not be exerted by) any causal or physical influence. Demanding such region to be achronal would be as much as demanding that no two events which are physically or causally independent of a given one should also be physically or causally independent *of each other*. This was certainly true in the classical case, where the absence of any limiting speed to causal action produced a flattening of the causal present: in that case, the relation of causal or physical independence was given the transitive property *as a matter of fact*, as a result of having removed any physical constraint from the way causal and physical action propagated. However, there is nothing in the very notion of causal independence which would call for such transitivity. So, in the face of Weingard's proposal, Savitt's demand for achronality appears to be at least questionable.

Just like limiting presentness, causal presentness is also capable of playing the part of the separating element between past and future. Contrary to the former, however, it is compatible with a global decomposition of space-time into past, present and future, with the sole proviso of extending the absolute future (past) of any event so as to include its future (past) light-cone: as we saw, this would demand redefining temporal orientation so as to include both time-like and light-like vectors; but, as we saw, this would also be consistent with common speech.

3.3.4 Recollecting the Pieces

Just like the here-now conception of presentness was obtained as the result of intersecting chronological present, causal present and limiting present, one may similarly wonder whether their union would give rise to a suitable candidate for the present moment. In that case, the present of any event x would coincide with the four-dimensional region of space-time including both its elsewhere region and its light-cone. What properties would such unorthodox conception of the present satisfy?

First, it would include all three types of presentness, simply by definition. Second, being the union of two invariant structures of Minkowski space-time, it would be invariant in its turn. Third, it would have classical hyperplanes of simultaneity as a limiting case. Fourth, keeping the ordinary equivalence relation $S(\ , \ )$ on the set of time-like vectors, it would support a global decomposition of space-time into absolute past, present and future. Fifth, it would have a very straightforward interpretation: in fact, it would coincide precisely with that portion of space-time falling neither in the absolute past nor in the absolute future of a given event.

3.4 Towards a Dynamical Interpretation of Tenses

All the notions of presentness we discussed so far are grounded on invariant, reflexive and symmetric binary relations: being at the same time-place as, being light-like separated from, being space-like separated from, and being light-like or space-like separated from. However, none of them satisfies transitivity, except for the sole relation of spatio-temporal coincidence. Contrary to the classical case, therefore, none of the notions of co-presentness we proposed is an equivalence relation.

One may consider this failure of transitivity a very serious deficiency, since it is a primary intuitive feature of the ordinary concept of presentness that any two events which are co-present to a given one should be also co-present to each other: those who regard transitivity a necessary ingredient of co-presentness would therefore regard all the above acceptations of presentness highly unsatisfactory. Nonetheless, there are strong reasons to resist this judgment.

First, as we saw, there are no invariant partitions of the relativistic space-time manifold other that the two trivial ones; accordingly, no equivalence relation on **M** could ever play the part of co-presentness in Minkowski space-time without either relaxing invariance or falling into triviality. Under this light, transitivity seems to be the lowest price to pay. Giving up invariance would make the present moment an observer-dependent, non-structural feature of Minkowski space-time; so it seems that any objective account of co-presentness should retain invariance to the detriment of reflexivity, symmetry or transitivity. On the other hand, both reflexivity and symmetry seem to be too basic requirements to be given up: indeed, it would hardly make sense to speak of a time-place which is not present to itself, nor of two events only one of which is co-present to the other.

Second, as we already mentioned, transitivity is part and parcel of neither causal independence nor, by symmetry, of causal dependence. Hence, the sole two sources the classical notion of co-presentness inherited its transitivity from were (a) absolute chronological simultaneity and (b) the merely factual coincidence of the causal present and the limiting present with the chronological present. However, neither of these two sources persist in the relativistic case. On the one hand, chronological simultaneity lost its classical invariance, with the consequence of no longer being a reliable basis to define co-presentness. On the other hand, causal present and limiting present are separated from hyperplanes of simultaneity, so that they are no longer required to be grounded on transitive binary relations.

Should the failure of transitivity prevent us from providing the above acceptations of presentness with a satisfactory metaphysical meaning? Maybe. Nonetheless, they have the merit of having shown temporally oriented Minkowski space-time to possess enough structure to offer a purely geometric, yet objective, account of tenses. If there was anything over and above existence, determination or any other metaphysical property to set the present moment in motion, then such account would still suffice to make sense of objective temporal becoming. It is my contention that such non-metaphysical motor exists, and it lies in the very algebraic properties of the mathematical structures we normally employ to model physical time. Before turning to that issue, however, we must face the second family of objections which have been moved against objective temporal becoming, namely those concerning its dynamical component.

4

Logical Threats

While arguments coming from science typically focus on the static ingredient of becoming, those which have been moved against its dynamical component are mostly of a logical kind. In general, they contend that there is something metaphorical, if not inherently incoherent, about expressions such as "time flows", "time has a direction" or "time goes one way".

Except for McTaggart's notable antecedent, which was still based on ontological assumptions, the first two examples of purely logical arguments of this kind were first offered by Smart (1949, 1954), and they were later reproposed by Williams (1951), Black (1959) and Price (1996), just to cite few. Smart's critique consisted of two main contentions, namely that there's apparently no sensible way of accounting for the speed of time and, similarly, that there is no meaningful way to speak of the temporal framework of time's putative passage.

The third logical argument we shall discuss is due to Grünbaum (1967a), who contended that the dynamical component of becoming possesses no factual or physical meaning. Contrary to Smart's ones, Grünbaum's argument never reached philosophical fame and it was never at the center of a vast theoretical polemic, though a later version of it was discussed by Price (1996) and Maudlin (2002, 2007).

Rejecting arguments of this kind is a mandatory task for anyone who would like to sustain objective temporal becoming for, as we shall see, they have never been definitely defeated by their opposers.

4.1

The Rate and Reference of Time's Passage

Smart's objections to the passage of time have the form of a double *reductio ad absurdum*. The first may be referred to as the *no-rate* argument; in its currently debated formulation, due to

Huw Price, it states that:

> [...] if it made sense to say that time flows then it would make sense to ask how fast it flows, which does not seem to be a sensible question. Some people reply that time flows at one second per second, but even if we could live with the lack of other possibilities, this aspect misses the more basic aspect of the objection. A rate of seconds per second is not a rate at all in physical terms. It is a dimensionless quantity, rather than a rate of any sort (Price, 1996, p. 13).

Four premises evidently lie at the basis of this argument:

(1) For any x_i and any x_j, x_i passes or flows with respect to x_j only if there exists a definite rate of change of x_i with respect to x_j.

(2) For any x_i, the rate of change of x_i with respect to x_j is given in units of x_i over units of x_j.

(3) For any x_i and any x_j, the rate of change of x_i with respect to x_j is not a dimensionless quantity.

(4) For any x_i and any x_j, if $x_i = x_j$ then the ratio between units of x_i and units of x_j is a dimensionless quantity.

However, such premises are merely sufficient to establish the following conclusion:

(5) For any x_i and any x_j, if $x_i = x_j$ then x_i cannot pass with respect to x_j

and conversely, if x_i passes with respect to x_j then $x_i \neq x_j$. As a consequence, two hidden premises must be added in order to make the argument effective, namely:

(6) For any x_i, if x_i passes or flows then there must exist x_j with respect to which the passage or flow of x_i takes place.

(7) For any x_i and any x_j, if x_i flows or passes with respect to x_j, then x_j is time.

Premises (6)-(7) guarantee that

(8) If time flows or passes, then it flows or passes with respect to itself.

Premises (1)-(4) then do their own job, leading to the conclusion that

(9) Time does not flow or pass.

Smart's second argument relies instead on an infinite regression. Let us suppose that time did actually move, or flow. Since movement is a form of change with respect to time, then there would exist a second, derivative, time dimension with respect to which time could be claimed to change. However, in order to properly play the part of time, this additional time dimension would have to flow in its turn, so that a third time dimension would be needed in order to account for its passage – and so on, endlessly. If we want to avoid the regression, we must then admit that time (the first, original, time dimension) does not flow or pass. The logical structure of this argument may be reconstructed as follows:

(10) Time flows.

(6) For any x_i, if x_i passes or flows then there must exist x_j with respect to which the passage or flow of x_i takes place.

(11) For any x_i and x_j, if x_i passes with respect to x_j then x_j flows or passes.

(5) For any x_i and x_j, if x_i passes with respect to x_j then $x_j \neq x_i$.

(12) For any ordered sequence $(x_i)_{i=1}^{n}$, if for any $i = 1, ..., n-1$ it is the case that x_i passes with respect to x_{i+1} then x_n does not pass with respect to any of the x_is.

Premise (11) expresses the contention that, if time is supposed to flow or pass, then anything which is asked to play the part of time must flow or pass in its turn. (5) and (12) rule out the possibility that any time dimension could move with respect to a time dimension of less or equal degree. (11), (5) and (12) therefore jointly guarantee the infinite regression. Given these premises, and taking for granted that

(13) There cannot exist any infinite ordered sequence $(x_i)_{i=1}^{+\infty}$ such that, for any $i = 1, ..., +\infty$, x_i passes with respect to x_{i+1}

then at least one of the above premises must be false. Those who deny the passage of time will therefore choose to abandon (10), thus reaching conclusion (9).

Though apparently unrelated, Smart's arguments are nevertheless multiply logically connected. On the one hand, they both share premises (5) and (6). On the other hand, (6), (7) and (11) jointly rule out (5): for if anything could pass only with respect to time, and anything with respect to which something passes should pass in its turn, then time would be forced to pass with respect to itself. Since both arguments share hypotheses (6) and (5), then (7) and (11) cannot stand together, and hence the two arguments behave as the two prongs of a unique *reductio*: given that anything which passes must pass with respect to something, if time passed, then it should have passed either with respect to itself or with respect to something else; the no-rate argument is meant to rule out the first possibility, the argument by infinite regression the second one. This point is of a great but underrated importance, because it shows that any attempt to reject Smart's arguments by denying either the sole (7) or the sole (11) would result in merely shifting from one horn of his argumentation to the other.

4.1.1 Objections to the No-Rate Argument

The most recent responses to the no-rate argument have been inspired by Maudlin (2002, 2007), and they may be found in Phillips (2009), Raven (2010) and Skow (2010). All reject premise (4), sharing the same contention that the physical magnitudes in the nominator and denominator of a ratio do not reduce or "cancel out" the way numbers do.

4.1.1.1 Numerical Equality and Identity

Maudlin owes the merit of having raised the debate on time's passage to a new life. His polemic is especially directed against Price's version of the no-rate argument, which Price used to make room for his own solution to the problem of the direction of time. The following objection, as well as those which it inspired, is therefore to be placed in a different context than that in which Smart's logical arguments were originally formulated – and incidentally, this might have been the source of a common misunderstanding of their real target[1].

> What's Price's 'more basic' objection? This, I fear, is just a confusion. A rate of one second per second is no more a dimensionless number that an exchange rate of one dollar per dollar is free of a specified currency. Price seems to suggest that the units in a rate 'cancel out', like reducing a fraction to simplest terms. Any rate demands that one specifies the quantities put in the ratio: without the same quantities, one no longer has the same ratio. [...] Similarly, π is defined as a ratio of a length (of the circumference of an Euclidean circle) to a length (of the diameter). The ratio is length to length: length does not 'cancel out'. There is, of course, also a real number (similarly called π, but don't get confused) that stands in the same ratio to unity as the circumference of an Euclidean circle stands to its diameter. That real number is dimensionless, but it plays no role in the definition of π. [...] And the rate of passage of time at one second per second is still a rate: it, unlike π, is a measure of how much something changes *per unit time* (Maudlin, 2007, pp. 113-14).

Maudlin is evidently subscribing premise (3), admitting that no number is a rate of anything as long as it consists of the ratio of two dimensionless quantities, so that specifying which magnitudes enter the nominator and denominator of a ratio is at least a necessary condition for that ratio to be a rate: 'any rate demands that one specifies the quantities put in the ratio'. His critique to Price's version of the no-rate argument is therefore that of having confused *numerical equality* and *identity*, thus identifying dimensionless ratios between real numbers and dimensioned ratios between real-valued magnitudes. The example he proposes to clarify the alleged distinction between the two cases is that of the number π: one may alternatively conceive π as the dimensioned ratio between the circumference and the diameter of an Euclidean circle,

[1] See § 4.1.3.

both expressed in units of lengths, or as the dimensionless real number which, *as a matter of fact*, is numerically equal to that ratio. In his view 'the real number is dimensionless, but it plays no role in the definition of π', understood as a geometrical property. Perhaps, for the sake of clarity, he would better have claimed that the real number π is *equal* to the rate of the circumference c to the diameter d of any Euclidean circle. Hence, even if we express the *value* of the rate of circumference to diameter by means of a dimensionless number – namely π – the rate itself – that is c/d – is a dimensioned quantity. And so, in the very same way, one second per second is a perfectly meaningful rate even if one, thought of as the value of that rate, is not.

This line of argumentation has two main drawbacks. In the first place, it would make it difficult to make a comparison between intervals of time, or durations. For, to be effective, Maudlin's response to the no-rate argument should not only prove that seconds/second is a meaningful physical magnitude, but also that its meaning is precisely the one of a rate of passage – a point which, strangely enough, seems to have been underestimated by the supporters of Maudlin's thesis. But if so, and if magnitudes did not cancel out, how could we say, for example, that the time interval taken by an inertial runner to cover one meter is half the time he would take to cover two meters? If that runner moved at a constant rate of two meters per second, then he would take one half second to cover one meter and one second to cover two meters; hence the ratio between the two time intervals would be that of one half seconds/second. How should we read this, according to Maudlin? If the physical meaning of seconds/second was that of a speed, then what speed would one half seconds/second stand for? Certainly not that of the runner, which we know to be two meters/second. Perhaps, that of the runner's proper time? That would sound odd, for turning the above ratio upside down would suffice to make the speed of the runner's proper time increase. So, it seems that Maudlin's thesis would hold water only insofar as he could offer an independent criterion to distinguish the meaning seconds/second would have in the case of time's passage from that it would have in this case. However, there seems to be no such criterion currently on the market.

In the second place, the assumption that units of measurement in a rate do not cancel out may lead to paradoxical consequences. For example, length in space may both be measured by means of rigid rods – whose unit of measurement is meter – or as the product of speed and duration in the ideal case of a uniform motion – in which case, the unit of measurement is (meters/seconds)$\times$seconds. If seconds did not cancel out, one would be faced with two conceptually different, though numerically equivalent, metrications of space. In order to put them back together, she would then have to establish a principle of equivalence between rigid-rods lengths and speed-duration lengths, much of the sort of that holding between inertial mass and gravitational mass in general relativity. But the same would hold *mutatis mutandis* for time, and so for all fundamental physical quantities, and so for any derivative physical quantity whose definition depend on them, and so on. As a consequence, physical theory would then suffer either of an indefinite growth of magnitudes or of an indefinite growth of correspondence principles. In conclusion, there is nothing above the very fact that physical magnitudes *do* actually cancel out, which guarantees the equivalence of different operational methods of describing the same physical reality.

4.1.1.2 Numbers and Relations

One may contend that our last objection is not effective against Maudlin's argumentation, for it involves non homologous magnitudes such as meters/seconds and seconds, instead of a pair of homologous ones. However, reducing seconds in (meters/seconds)×seconds requires the algebraic operations on magnitudes to be associative, so that (meters/seconds)×seconds may equal meters×(seconds/seconds); and so, what actually cancel out in our example are two homologous quantities. Fine; but – so she would go on – this would just amount to an algebraic manipulation with no physical meaning.

This seems to be exactly the point of Phillips's reply to Olson's version of the no-rate argument (Olson, 2009):

> A rate is a *ratio of two quantities*, a relation one quantity bears *to* another. One second divided by one second is one, and one is not a rate of change, just as Olson says. But one second *per* second is not one second *divided* by one second, and it is not equal to one. One second *per* second is a ratio of time to unit time, a relation between two amounts of time, whereas neither one second divided by one second nor one is a ratio or relation of quantities. It is easy to be misled here. Fractions or quotients can sometimes be used to express ratios (and rates), at least as long as we know what it is we are expressing. But ratios are not fractions. A fraction is simply one number divided by another. Thus, $n/n = 1$, where $n \neq 0$. In contrast, a ratio, $n : m$, is the *relation* one quantity bears *to* another. It does not equal one even if $n = m$ (Phillips, 2009, p. 503).

Phillips's terminology is very unfortunate, so that some lexical clarification will be needed. For the sake of clarity, and departing from Phillips's terminology, we shall refer to a "ratio" as the numerical comparison between two quantities, whether it is dimensioned or dimensionless; in the former case, we shall speak of a "rate" while, in the latter, of a "fraction" or a "quotient". Hence, by claiming that 'a ratio, $n : m$ [...] does not equal one even if $n = m$', Phillips is claiming that rates could be numerically equal – they can be "expressed by" – but not identical, to a given quotient between dimensionless quantities. So far, he's therefore adding nothing new to what Maudlin already said. However, he would seem to be grounding the theoretical distinction between rates and quotients not on just whether they hold between dimensioned or dimensionless quantities, but rather on the difference holding between *relations* and *numbers*, the latter being understood as the results of algebraic operations. Rates are dimensioned ratios; as such, they must hold between dimensioned quantities. Fractions, instead, are dimensionless ratios; and, as such, they hold between pairs of dimensionless numbers as well as between pairs of dimensioned quantities.

To make this point clearer, let us analyze the passage we quoted above in a more detailed way. Phillips argues that '[a] fraction is simply one number *divided by* another', which may indeed raise the suspicion that fractions only obtain as ratios between dimensionless quantities, or numbers. But at the same time, he also claims that 'one second *divided by* one second *is* one': independently of the fact that one second is a dimensioned quantity, once the ratio between one second and one second is understood as a division, then it becomes *identical* to a dimensionless number, and hence to a fraction. Conversely, 'one second *per* second is not one second *divided by* one second', for '[o]ne second per second is [...] a *relation* between two amounts of time, whereas neither one second *divided by* one second nor one is a [...] *relation* of quantities'. The distinguishing feature between fractions and rates is therefore not whether the quantities entering their nominators and denominators are dimensionless or not, but whether we understand those ratios as algebraic operations or, rather, as relations.

However, there seems to be no such clear-cut distinction between relations and fractions or numbers, as the one Phillips would be in need of. Fractions or quotients are just place-holders for rational numbers, and rational numbers owe their name precisely from being relations: "rational" is what can be expressed by a "ratio", which is the Latin word for "relation". Set theory is the best testimony of the relational character of rationals: in fact, they are axiomatically produced out of integer numbers by Cartesian multiplication; but in set-theoretical terms, relations are nothing but subsets of Cartesian products.

So, if fractions or real numbers *are* relations, how to distinguish between rates and fractions? Or how to distinguish between those cases in which ratios between homologous quantities are just the product of algebraic manipulations, and those in which they are genuine physical quantities? Why should we take second/second to be a meaningful magnitude in Maudlin's case, but not in ours? Once again, a supporter of Maudlin's thesis would be in need of an independent criterion to distinguish the meaning seconds/second would have in the two cases. Unless she was able to provide such a criterion, his objection to the no-rate argument would be unsupported.

4.1.1.3 Dates and Durations

One further counterexample to the no-rate argument, similar to Maudlin's but apparently stronger, is given by Skow (2010). Let us suppose that a team of sociologists monitored how the most common birth year in a given population – let us call it MCB – changed in time; we may suppose that, due to an increase of mortality, while the MCB in 2000 was 1950, in 2001 it became 1952. Sociologists would reasonably conclude that the rate at which the MCB moved into the future was of two years per year; but if quantities canceled out, this claim would be meaningless. The apparent cogency of this example stems from its capability of easily dodging the arguments we offered so far, by picturing a case in which it would be hard to claim that quantities, or magnitudes, actually canceled out. But a deeper analysis would make it clear that the reason for which they don't is that they are simply *not the same*. For those years appearing the denominator of the MBC/year rate are measures of time intervals, or durations, while those years entering the nominator are at best gross-grained place-holders for time locations, i.e. co-ordinates, or dates: determining the most common birth year at a time is not to determine, so to say, *how long* most currently living people born, but *when* they did. So Skow's example falls short of offering a genuine case of two identical magnitudes not canceling out.

4.1.1.4 Introducing Magnitudes

Finally, Raven (2010) argues that, if homologous rates canceled out, then they could as well be introduced at pleasure. So the rate of time's passage would not solely be equal to one second per second but also, say, to one second×meter per second×meter, or even to one second×foot of floor tiles per second×foot of liquorice sticks, which would be absurd.

Indeed, it is hard to consider this objection a serious threat to the no-rate argument. If units *did* cancel out, then one second per second would not solely be equal, but also identical, to one. This would be so for all of the above putative rates, and thus there would be no absurdity at all in equating all of them. The absurdity would rise only insofar as one would like to maintain that time flows, even if magnitudes were allowed to reduce: if magnitudes canceled out, then Smart's premises would do their job, and time would not pass – at least, as of itself; so, why should we go in search of its speed? In that case, the absurdity would be that of providing time with *any* rate of passage, was it one second×foot of floor tiles per second×foot of liquorice sticks or, more simply, one second per second.

4.1.2 Common Objections

As we saw, the infinite multiplication of time-like dimensions in the argument by regression is made possible by conditions (11), (5) and (12). For this reason, standard responses to that argument concentrate on (11) or (12), albeit retaining (5). In both cases, however, condition (7) is tacitly rejected so that, if sound, the same objections would affect the no-rate argument too.

4.1.2.1 Static Hypertime

One solution for escaping both the infinite multiplication of time dimensions and the no-rate argument is, for example, that of rejecting (7) and (11), and accepting that time could move with respect to a derivative time dimension – sometimes referred to as *supertime* or *hypertime* – which is not itself in motion. This is the path traced by Schlesinger, and more recently discussed by Skow:

> In order to exhibit the sense in which the "now" moves in [our temporal dimension] TU_1, we do not require a full-blown [derivative time-dimension] TU_2. A poorer temporal universe, in which the only ordering relationship events have to one another is before and after, can provide the required container for TU_1. Suppose that

> $s_0, s_1, ..., s_n$ are successive instantaneous states of TU_1, and that $S_0, S_1, ..., S_n$ are those in TU_2. [...] Then although TU_2 has no moving "now" of its own, the movement of the "now" in TU_1 shows up in it. [...] The rate of this movement at its various stages is obtained by dividing the magnitude of the interval $s_i - s_j$ by the corresponding magnitude $S_i - S_j$ (Schlesinger, 1969, p. 6).

Two, perhaps obvious, objections may be raised against this proposal. If the motion of TU_1 could be exhaustively described by means of a static time dimension TU_2, why should us require TU_1 to move in its turn? If motion *in* time is conceivable without motion *of* time, then there's no reason why should we retain such a troublesome assertion as that time flows. And on the other hand, even if we could not dismiss this claim and had been forced to accept that TU_1 moved, then TU_1 would simply become superfluous. For if it was possible to observe its movement, and even to compute its velocity with respect to TU_2, then TU_2 would be epistemically accessible. So what would prevent us to coordinatize physical processes with respect to the latter, and to discharge TU_1 as being theoretically more expensive?

4.1.2.2 Relative Motion

An alternative family of objections rests on a relational conception of motion and speed (Webb, 1960; Schlesinger, 1969, 1985; Markosian, 1993). Objections of this kind typically reject (7) and (12), claiming (a) that a measure of the speed of physical processes with respect to time can symmetrically be turned into a measure of time with respect to those processes and hence (b) that the movement or passage of time can be meaningfully related to the latter, with no further need of invoking the existence of additional derivative time dimensions:

> [...] if one clock is chosen as a standard, and another faster clock is chosen as an alternative standard, then we can establish the ratio of the number of units ticked off by one clock over the number of units ticked off by the other, counting from an arbitrarily chosen simultaneous starting to an arbitrarily chosen simultaneous stopping, and that this ratio would represent the rate of change of the time of one clock relative to the time of the other. Thus, the time of one clock could be said to flow, and indeed, to flow with a finite measurable rate of change. [...] If it possible to say that a person's heartbeat is fast or slow in terms of seconds, it must equally be possible to say that seconds are fast or slow in terms of heartbeats (Webb, 1960, pp. 360-361).

If one's heartbeat is one beat per second, then we can indeed count seconds by counting heartbeats. This way we can evaluate *how much* time passed: physical processes may undoubtedly be used to measure durations, or intervals of time. But intervals are lengths, not rates. So what is for a physical process to measure the speed of time? In the end, objections of this kind seem to be based on a misguided conception of speed, for they typically assume that

> There is no literal or factual meaning to the expression "rate of flow" or "rate of change" except that which derives from the ratio comprised by the *number* of repeated units of one measurable "thing" over the *number* of repeated units of another measurable "thing" (Webb, 1960, p.359).

But even granting that speed could conceptually be reduced to a dimensioned ratio, not any dimensioned ratio could conversely be turned into a speed: one beat per second is *not* a measure of speed, but of frequency – and the inverse of frequency is not speed but, again, duration.

The same misunderstanding lies also at the basis of the following argument:

> [...] whenever one gives the rate of some normal change in what is admittedly the standard way, i.e. in terms of the pure passage of time, then one has likewise given the rate of the pure passage of time in terms of the first change. If I tell you that Bikila is running at the rate of twelve miles per hour in terms of the pure passage of time, for example, then I have also told you that the pure passage of time is flowing at the rate of one hour for every twelve miles run by Bikila (Markosian, 1993, p. 842).

Things are a bit more complicated in this case, for it is hard to say what physical magnitude one hour per mile would exactly amount to. But even now, what is clear is that one hour per mile is surely *not* a measure of speed (Macbeath, 1986). Velocity is a derivative magnitude and, as such, it can only be defined for differentiable trajectories. No matter what system is in motion – a massive point, a field, a distribution of purely mathematical variables – what is needed for the system to have a definite velocity at a point is that its motion is differentiable at that point. This means that the states or positions of that system should be *at least* a function of time – that is, the minimum requirement for the canonical concept of speed is that at each time the moving system should occupy exactly one position in physical or configuration space. By the same token, if the speed of time was to be computed in hours per mile, then the position of the moving present *in time* would have to be a function of space, taken as an independent variable. But it seems hard to make any sense out of this. How could space vary independently, and how could time change its position with respect to itself? Even trying to neglect this difficulty, by simply requiring that time *itself* had to be a function of space, and measuring intervals of space through the subsequent positions occupied by a given physical clock, like in Bikila's case, we would move no further: anytime the physical clock retraced its own steps, or anytime it was at rest, then different instants of time would be associated to the same location. For time to have a definite speed, the motion of the physical clock would thus not only have to be uniform, but also irreversible – in the sense of describing a non-intersecting trajectory – and interminable. Time could not have a definite rate with respect to any periodic process, including standard clocks, but it would have an infinite speed with respect to all objects at rest.

In sum, if we take relative motion to be necessarily connected with relative speed, as required by (1), then the above difficulties in determining the speed of time as of any physical process finally result in the impossibility of defining its motion in a relational way.

4.1.3 Flow without Kinematic Motion

Of the purported refutations of Smart's arguments we discussed so far, all seem to have ended up in smoke. So let us make a little experiment, trying to challenge Smart with his own weapons, and let us concede that time does *not* pass. If immobility was the sole alternative to passage, then we would be saying that time stands still; but in doing so, we would be claiming that time had zero speed – that it was moving at zero seconds per second – or that there was a derivative time dimension with respect to which time could be observed not to be in motion. In the former case, all arguments against the physical significance of a rate of seconds per second would apply as well. In the latter case, we would possibly not be able to get an infinite regression – since hypertime might symmetrically be supposed to be at rest with respect to time – but we could always rely on a principle of theoretical economy for casting doubts on the actual utility of duplicating the original time dimension. This way, the same arguments Smart directed against passage would work as well against the very opposite hypothesis of rest.

So, is there anything basically wrong in Smart's argumentation? I believe not. Rather, what is wrong is our having contrasted flow with immobility: what our experiment wanted to show is that what is at stake of the contention is *not* whether time actually passes, but whether it does so *in a kinematic way*. Since both (9) and (10) fall equally into Smart's logical trap, it is clear that time's passage cannot be its main target. Rather, his *reductio* should be directed primarily against his auxiliary hypotheses, whose aim is that of sketching the main features of time's flow, and which treat time by the same standards as a moving solid body. So, even if Smart was right in claiming that time did not pass in the usual sense, we would not be entitled to conclude that it stands still: what his twofold *reductio* actually proves is that, if time passed, then its passage could not be endowed with all those features one would require kinematic motion to possess.

If I am right, those who so far have tried to argue in favor of time's passage by rejecting Smart's argumentation were doomed to fail precisely because they did not recognize its real significance: in all cases, they faced it as if it was aimed to refuse passage *tout court*; and for this very reason, they almost invariably attempted to challenge them by providing time's passage with purely kinematic features, tacitly holding it that there could be no other form of objective passage than kinematic motion[2]. So, even if they succeeded in sidestepping Smart's critique, they would only prove that, *if* time moved, it would do so kinematically; at most, they would defuse a potential threat to becoming, but they would not make any step forward in arguing that time *does* actually pass.

[2]Indeed, Maudlin (2007) would presumably not subscribe this claim, for in his view objective temporal becoming is just the existence of a mind-dependent distinction between the past and the future directions of time. Nonetheless, as we pointed out in § 4.1.1.1, his objection to the no-rate argument works just in case one second per second is proved to be not only a physical magnitude, but also an appropriate measure *of speed*.

Markosian (1993) offers a very interesting example in this respect. He attempted to dodge the no-rate argument by denying (1), arguing that, in the case of time, talking about rates of passage simply makes no sense:

> [W]hat is essential about rate talk is that it involves a comparison between some normal change and the pure passage of time. According to this view, it does not make sense to ask about the rate of the passage of time, for to do so is to make a category mistake: the answer would have to involve a comparison between the pure passage of time and the pure passage of time, but such an answer would not make sense because the pure passage of time has a unique status among changes –it is the one to which other, normal changes are to be compared. It is the paradigm, and, as such, it alone among changes cannot be measured. If I take this line then [...] I will still be able to maintain that time literally passes (Markosian, 1993).

However, the above discussion has made it clear that, far from raising any difficulty for Smart's argumentation, this argument would point in the very same direction. Smart's primary interest was that of warning us against the hypostatization of events as something *becoming* in time – rather than simply *happening* in (or being part of) time – and on the consequent ambiguity in our use of locutions such as "the passage of time", "the flow of time" or "the river of time":

> Substances exist in space; they are related to one another in a three-dimensional order. Events *are* in time; they are related to one another in an order of earlier and later. Now if we think of events as *changing*, namely in respect of pastness, presentness and futurity, we think of them as substances changing in a certain way. But if we substantialise events, we must, to preserve some resemblance of consistency, spatialise time. "Earlier than" becomes "lower down the stream". It is easy to see how there arises the illusion of time as a river down which events float. [However] trouble arises at the boundary between our shifted system and the old one, for example, when we use "event", with its syntax shifted so as to behave like "substance" in combination with "time" with its syntax *not* shifted to behave like "space". We then get nonsense, such as "how much time_1 does it take for events *to float* a given distance (time_2) in the river?" (Smart, 1949, p. 493).

Wondering how much time would a point of time need to get to another point in time is tantamount to wondering how fast (a point of) time flows. Only at this point, the question about the speed of time rises. The proper object of contention is thus the semantical shift from time conceived as the locus of change (and hence of motion) to time as the subject of motion itself. Charging the no-rate argument of a category mistake is to grossly miss the point, for that mistake is precisely the true target of Smart's *reductio*; and, after all, it is hard to see how Markosian would be able to say that time passes *literally*, just after having denied that it could do so – or, what literal meaning of passage is there, other than the kinematic one?

4.2

The Direction of Time's Passage

If time passed, then one would reasonably be entitled to wonder toward what direction. Those who defend time's flow typically hold it that time simply moves toward the future, constantly receding from the past while continuously crossing a new present. Tenses differ objectively precisely because the future is irreversibly selected by time as the inexhaustible destination of its flow (Schuster, 1986). Block theorists reject this view as a whole, charging it of misusing a kinematic concept, exactly as they did for time's rate and reference.

4.2.1 The Theoretical Significance of Becoming

Just like Smart, Grünbaum (1967a) aimed to show the metaphorical character of transiency. Contrary to the former, however, he didn't charge it of incoherency, but of triviality; what he needed was in fact to spoil time's passage of any empirical content, so that he could reshape the problem of time's arrow in purely topological terms.

> [T]he claim that the present or now shifts in the direction of the future does invoke the transient now to single out one of the two time senses and – as we are about to see – is a mere truism like 'All bachelors are males'. For the terms "shift" or "flow" are used in their literal kinematic senses in such a way that the *spatial* direction of a shift is specified by where the shifting object is at *later* times. Hence when we speak metaphorically of the now as "shifting" temporally in a particular temporal direction, it is then simply a matter of definition that the now shifts or advances in the direction of the future. For this declaration tells us no more than that the nows corresponding to later times are later than those corresponding to earlier ones, which is just as uninformative as the truism that the earlier nows precede the later ones. [...] Being only a tautology, the kinematic metaphor of time flowing in the direction of the future does not itself render any empirical fact about the time of our experience (Grünbaum, 1967a, pp. 13-14).

Grünbaum basic premises are that (i) anything which is in motion has a direction, i.e. it occupies different positions of a given space at increasing instants of a given time, that (ii) the time and place of time's motion consist of time itself, so that becoming should be understood as present moment's being located *at* different times *at* increasing times and (iii) that time never changes its direction. This way, the claim that time moves from past to future, i.e. in the unique direction of increasing time, is translated into the claim that the present moment is located at increasing times at increasing times. Grünbaum took this claim to be trivial, and rejected it as devoid of any physical import. However, his conclusion is a *non sequitur*.

In the first place, time's passage cannot be trivially equated to the claim that 'the nows corresponding to later times are later than those corresponding to earlier ones'. This would be true only insofar as the time and locus of time's passage were necessarily located, by definition, in time itself; however, decades of philosophical debate on Smart's logical arguments have proved this hypothesis to be questionable. As we saw, defenders of time's passage would easily contend that time might instead globally move with respect to a static hypertime or, alternatively, that it might locally possess different velocities, and hence being in motion, with respect to different physical clocks; in the former case, time would move by occupying different positions in global time at subsequent hyper-times while, in the latter, it would occupy different positions in local time as a different local time was allowed to vary. Independently on their actual efficacy, these objections undeniably make it clear that whether the time and locus of time's motion coincide is *not* a matter of pure definition – for 'the present or now shifts in the direction of the future' may claim, for example, that the present would cross subsequent moments of time at subsequent hyper-times. Hence, the claim that time moves from past to future cannot be logically equated to the alleged tautology that 'the nows corresponding to later times are later than those corresponding to earlier ones', so that the logical truth of the latter cannot be shared by the former, and Grünbaum's conclusion does not follow.

In the second place, even granting that time could only pass with respect to itself and that time's passage would conceptually amount to the present moment's being at increasing times at increasing times, Grünbaum would be wrong in discarding it as a mere truism without *any* theoretical or factual import: in fact, this would have relevant consequences for Grünbaum's own philosophy of time.

4.2.1.1 Becoming and The Anisotropy of Time

One of Grünbaum's leading theses is that time's arrow should be understood topologically, rather than dynamically. In a few words, he aimed to root the linear ordering of earlier and later in time's geometrical configuration, rather than in the alleged irreversibility of its motion. Its logical argument played a crucial part in this respect (which is also testified by the number of appearances it made, sometimes word-by-word, in his philosophical production: (Grünbaum, 1967b, 1971, 1973)): by ruling out time's passage as a plain tautology with no physical meaning, it made it possible for him to recast such an ordering in purely topological terms.

So, what would be for time to be topologically asymmetric or, as he says, *anisotropic*? Since time is essentially one-dimensional, this should be understood as a lack of mirror-symmetry around any of its points or, in other words, as an objective or structural difference between the direction we call past and the one we call future. Grünbaum identified the source of this asymmetry in the 'existence of irreversible kinds of processes (Grünbaum, 1973, p 209)' where, by an irreversible process, he meant 'a process such that no counterprocess is capable of restoring the original *kind* of state of the system at another time (Grünbaum, 1967a, p. 11)': in fact, he held that 'the structure of time is not something which is apart from the particular kind of processes obtaining in the universe (Grünbaum, 1973, pp. 209, 214)', so that any factual or nomological asymmetry in the temporal evolution of physical systems should reflect in a constitutive asymmetry of time itself. On the other hand, he argued that irreversible processes are only capable to confer time an asymmetric order, not a direction. That is to say: they establish an objective distinction between the two possible orientations of the time axis, but they are not capable to single out one of these orientations as the preferred direction along which physical phenomena would be doomed to take place. For this reason, he could affirm that time may be asymmetric without being in motion, the difference between past, present and future being a purely subjective epiphenomenon:

> Although the serial relation "later than" itself does have *a* "direction" in the obvious sense of being asymmetric, the set of states ordered by it does not have *a* direction but rather exhibits a special *difference* or anisotropy as between the *two opposite directions*. Thus, when we speak of the anisotropy of time, this must not be construed as equivalent to making assertions about "the" direction of time. [Any] assertion about "the" direction of time rest on [the] incorrect supposition that there is a physical basis for becoming in the sense of the shifting of a physically defined "now" along one of the two physically distinguished directions of time. By contrast, our characterization of physical time as anisotropic involves no reference whatever to a transient division of time into past and future by a "now" whose purported unidirectional "advance" would define "the" direction of time (Grünbaum, 1973, p. 217).

One question Grünbaum left unanswered is whether, *if* time passed, this would really be enough to make it anisotropic. In general, the irreversible motion of a particle in physical or configuration space suffices to make its trajectory asymmetric with respect to the time axis in this case, however, we should determine whether time's motion would make the time axis itself asymmetric. Here is where the contested statement that earlier nows are earlier than later ones starts to play its part. Our analysis showed that Grünbaum managed to equate that statement with the claim that time is in motion only by admitting that time coincided with the place and time of its own passage. Given this premise, we may understand time as a degenerate process whose trajectory takes place in itself, and whose motion, being irreversible, would consequently produce an asymmetry in its own structure. More pictorially, the same conclusion could be reached by noticing that the first premise in Grünbaum's argumentation guarantees that, if time irreversibly flowed toward the future, then no physical process taking place in time could ever be capable of "going up time's stream", for its direction would by definition consist of the states or positions it would be in at subsequent stages of time's passage; all non-periodic[3] processes would consequently be *de facto* irreversible, which in Grünbaum's view would suffice to make time anisotropic. This way, we reached the first two significant theoretical consequences of what Grünbaum thought to be a plain triviality: given his own premises, (a) the objective passage of time would establish an objective asymmetry between past and future and (b) Grünbaum would

[3] This qualification is needed because, albeit being forced to evolve necessarily in the direction of time's flow, even in this case periodic processes would, as such, enter the same kind of state more than once.

never had been capable of trivializing time's passage without *ipso facto* trivializing its anisotropy, for any logical consequence of a tautological statement must necessarily be tautological in its turn[4].

4.2.2 Time Goes Just Where It Goes

One slightly different version of of Grünbaum's logical argument is reported by Price (1996), and has been recently discussed by Maudlin (2002, 2007):

> If time flowed, then – as with any flow – it would only make sense to assign that flow a direction with respect to a choice as to what is to count as a positive *direction* of time. In saying that the sun moves from east to west or that the hands of a clock move clockwise, we take for granted that the positive time axis lies toward what we call the future. But in the absence of some objective grounding for this convention, there isn't an objective fact as to which way the sun or the hands of the clock are 'really' moving. Of course, proponents of the view that there is an objective flow of time might see it as an advantage of their view that it does provide such an objective basis for the usual choice of temporal coordinate. The problem is that until we have such an objective basis we don't have an objective sense in which time is flowing one way rather than another (Price, 1996, p. 13).

Price's argument shares its basic premises with Grünbaum's: (i) passage is essentially related to having a definite direction, where the latter stays for being located at different places at increasing moment of a given time dimension, and (ii) time's passage is unidirectional. But while Grünbaum modeled his argument by focusing on what time's being *at different places* could mean, Price concentrated on the meaning of time's having different locations *at subsequent moments of time*, leaving the question of the locus of its motion aside. This way, instead of the alleged triviality that earlier nows are earlier than later ones, he obtained a vicious circularity.

He proceeded by confronting his premises with the dynamist claim that the constant direction of increasing time is the one of its flow. Let us suppose with him that time moved, and that its motion was unidirectional; then, by premise (i), the direction of its motion would be determined by its subsequent positions *as time increased*. Following Price, let us put aside for this moment the question concerning what those "positions" should consist of, and let us wonder: how to establish what the direction of increasing time is? Dynamists would answer that it is just the direction of time's motion. But, if so, then the direction of time's motion would be determined by its positions at increasing times, which in their turn would be determined by time's motion,

[4] In addition to the theoretical drawbacks this result would produce for Grünbaum philosophy of time, it is worth pointing out that, given Grünbaum's premises, there exists a logical connection between the dynamical and the topological features of time: if time is supposed to pass with respect to itself, and if it goes in a unique direction, then there is an objective topological distinction between past and future. On this topic, see also §§ 7.2, 7.1.3

and then by its positions at increasing times, ..., without ever reaching an ultimate basis for establishing whether 'time is flowing one way rather than another'. To avoid this circularity, one should either directly renounce time's passage or discard the hypothesis that future and past are objectively distinguished by time's motion.

Maudlin tried to avoid this conclusion by turning Price's argument upside down:

> [...] flows only have a direction because the asymmetry inherent in the passage of time provides temporal direction: from past to future. The natural thing is now to turn Price's Modus Tollens into a Modus Ponens: since there obviously is a fact about how the Mississippi flows (north to south) or how the hands of standard clocks turn (clockwise) there is equally a real distinction between the future direction in time and the past direction (Maudlin, 2007, p. 114).

However, what Maudlin actually did was simply to reaffirm the dynamist position, balancing the logical cogency of Price's argument with the intuitive force of time's asymmetry. This way, he failed in three different respects. In the first place, he misunderstood the logical structure of Price's argumentation. Price showed the dynamist defense of passage to be logically unsupported; his denial of passage stemmed from this logical pitfall in the way of a *reductio ad absurdum*. Therefore, there is no conclusion in Price's argument whose truth value could be changed in order to restore its premises, and turning Price's modus tollens into a modus ponens would demand learning to coexist with a logical circularity. In the second place, Maudlin's response is not itself free from circularity. Invoking the experienced irreversibility of physical processes as a decisive evidence of time's passage would require having ruled out any alternative source of that asymmetry at the outset. However, this would mean having assumed the very dynamist view which such an evidence were meant to support. Finally, Maudlin's objection fails exactly because being directed against his opposer. One of Price's leading theses is that the asymmetries of time and causation are secondary qualities just like colors or smells: according to this view, the experienced asymmetry of time would not reflect any property of time itself, rather than the product of its own structure together with our perspective attitude as agents participating of the local entropic gradient (Menzies and Price, 1993; Price, 1996). For this reason, Price could easily reply to Maudlin that there is no objective fact as whether the Missisippi river flows north to south, for in his view this would only be a symptom of our looking to that flow from our own perspective; and, if so, there would be no need for an objective passage of time to single out any 'real distinction' between future and past.

Nonetheless, there is a flaw in Price's argumentation: as we saw, his logical argument was constructed without imposing any condition on what the locus of time's passage should be; but, until such a question is answered, Price's conclusion holds vacuously. If there was no physical or mathematical space in which time could be placed, then time would not be capable of motion, and hence *any* statement concerning it, including those referring to its direction, would be trivially true. Not so bad – Price may say – for if there was no sensible way to speak of time's position then there would neither be any sensible way to speak of its motion, exactly as he wished. However, there *is* a sense in which time may be given a position, though in a non-conventional sense, and precisely one which could sidestep his logical snare.

Price's adherence to the no-rate argument suggests that, if he had to find a place for time's motion, he would have chosen time: what he objected to time's passage was that one second per second is not an acceptable rate because it is a dimensionless quantity; but it is clear that, to hold this position, he should have taken for granted that, if time was to have a rate, that would be a measure of time's movement *in time*. Given this premise, his argument would exactly match Grunbaum's, with the consequence of making the irreversibility of time's passage a sufficient condition for establishing its anisotropy, exactly as the dynamists would demand. Therefore, while implementing the dynamist claim that past and future are objectively distinguished by time's passage into his own premises, Price was implicitly confronting two equivalent claims; and hence, the circularity he got consists of nothing more than moving to and fro two identical statements, trying to establish which is the most fundamental.

4.3

Toward a Non-Kinematic Interpretation of Passage

Our analysis showed the real target of Smart's logical arguments to be a kinematically exhaustive representation of time's passage – one which would provide time with all the main features of kinematic motion, such as speed and a proper temporal frame of reference. Our subsequent discussion has finally confirmed that a weakened dynamical interpretation of becoming – one which, for example, allowed time to pass irreversibly with respect to itself – would possess a non-zero theoretical import, notably an objective distinction between past and future. So, what minimal features a non-kinematic account of transiency should satisfy, in order to be theoretically meaningful? In particular what properties, over and above its topological anisotropy, should time satisfy so that it could be claimed to flow in a philosophically interesting way? Before answering these question, we shall first leave the domain of pure philosophy, and enter the details of a general theory of motion.

5

Dynamical Systems on Monoids

Intuitively speaking, by a deterministic system we mean a device whose final states uniquely depend on the assigned states, together with its specific operation. In most cases, systems of this sort are supposed to be capable of working continuously or undergoing subsequent cycles, so that final states could play the part of initial states in their turn. For this reason, deterministic systems are exhaustively described by specifying the set of all their states, and the mechanisms or laws according to which they are eventually transformed into each other.

Arnold (1973) modeled deterministic systems on n-dimensional differentiable state spaces and governed by ordinary differential equations, such as those of classical mechanics and classical electromagnetism, by means of *phase flows* or *continuous dynamical system*, i.e. one-parameter groups of transformations indexed by the set $\mathbb{R}$ of time intervals, satisfying an identity and a composition requirement. More generally, deterministic systems on arbitrary non-empty state spaces and with (non-negative) integer or (non-negative) real time sets can be modeled by one-parameter families of transformations, indexed by $\mathbb{Z}^+$, $\mathbb{Z}$, $\mathbb{R}$ or $\mathbb{R}^+$. Mathematical structures of this kind usually go under the name of *mathematical dynamical systems* (Giunti, 1997):

Definition 0 (Mathematical Dynamical System)
A mathematical dynamical system*, denoted by DS is an ordered pair*

$$DS = (M, (g^t)_{t \in T}),$$

where

1. *M is a non-empty set,*
2. *T is either $\mathbb{Z}, \mathbb{Z}^+, \mathbb{R}, \mathbb{R}^+$,*
3. *$(g^t)_{t \in T}$ is a family of functions on M indexed by T,*

4. for any $x \in M$ and any $t, v \in T$,

$$g^0(x) = x \tag{5.1}$$
$$g^{t+v}(x) = g^t(g^v(x)). \tag{5.2}$$

M is called the *phase space* or *state space* of the system, including all its possible states, or *points*. T is called the *time set* of the system, and it models the time dimension through which the system is supposed to evolve. Finally, any function g^t is called a *state transition* of duration $t \in T$, or *t-advance*: to any state $x \in M$, g^t associates the state $g^t(x)$ the system displays t time after having been in x.

Examples of dynamical systems with discrete time and discrete state space are Turing machines and cellular automata; with discrete time and continuous state space: systems specified by difference equations (e.g. iterated mappings on $\mathbb{R}$); with continuous time and continuous state space: systems specified by ordinary differential equations.

Example 1 (Mathematical Dynamical System)
Let $a_1, a_2, ..., a_n, ...$ be a geometric progression with ratio r. Then, for any $n \geq 1$, the n-th element of the progression is obtained by

$$a_n = a_1 r^{n-1} \tag{5.3}$$

For any $a_i \in \{a_i\}_{i \in \mathbb{Z}^+ - \{0\}}$ and for any $n \in \mathbb{Z}^+$, let

$$g^n(a_i) = a_i r^n. \tag{5.4}$$

Then:

1. $M = \{a_i\}_{i \in \mathbb{Z}^+ - \{0\}}$ is a non-empty state space;
2. $\mathbb{Z}^+$ is a time set;
3. $(g^n)_{n \in \mathbb{Z}^+}$ is a family of functions from M to M, indexed by $\mathbb{Z}^+$:
   for any $a_i \in M$ and any $n \in \mathbb{N}$:

$$g^n(a_i) = a_i r^n = a_1 r^{i-1} r^n = a_1 r^{i-1+n} = a_{i+n} \in M; \tag{5.5}$$

4. for any $a_i \in M$ and any $n, m \in \mathbb{Z}^+$:

$$g^0(a_i) = a_i r^0 = a_i \tag{5.6}$$
$$g^{n+m}(a_i) = a_i r^{n+m} = a_i r^n r^m = g^n(g^m(a_i)). \tag{5.7}$$

Hence, $a_1, a_2, ..., a_n, ...$ is modeled by a mathematical dynamical system.

Definition 0 captures the intuitive notion of a deterministic system in the following sense. In the first place, condition (3) should be interpreted as telling us the state of the system after an evolution of an arbitrary duration $t \in T$, provided that the state of the system at the present time $t_0 \in T$ is known; in other words, if at instant t_0 the system is in state $x \in M$, then at instant $t + t_0$ the system is in state $g^t(x)$. In addition, condition (4) tells us that (a) whatever state the system is in, the evolution of duration 0 does not modify that state and (b) any evolution of duration $v + t$ can always be decomposed in two successive evolutions, the first one of duration t, and the second one of duration v[1]. Still, this definition does not make clear what mathematical structure time should *at least* possess, so that the evolution of a deterministic system can be exhaustively described. Giunti and Mazzola (2010) identify that structure with a monoid, i.e. a non-empty set along with a binary associative operation and identity element, accordingly proposing the following and more general definition of a dynamical system:

DEFINITION 1 (Dynamical System on a Monoid)
A dynamical system on a monoid L*, denoted by* DS_L*, is an ordered pair*

$$DS_L = (M, (g^t)_{t \in T}),$$

where

1. M *is a non-empty set,*
2. $L = (T, +)$ *is a monoid with identity* 0*,*
3. $(g^t)_{t \in T}$ *is a family of functions on* M *indexed by* T*,*
4. *for any* $x \in M$ *and any* $t, v \in T$*,*

$$g^0(x) = x, \tag{5.8}$$

$$g^{t+v}(x) = g^t(g^v(x)). \tag{5.9}$$

$$\tag{5.10}$$

Definition 1 is exactly analogous to Definition 5, except for substituting the time sets $\mathbb{Z}^+$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$ with the more abstract concept of a *time model* L.

Mathematical dynamical systems form a proper subclass of dynamical systems on monoids; for this reason, all basic notions employed by the theory of mathematical dynamical systems – such as those of motion, orbit, period, etc. – should be reshaped in order to fit the lighter mathematical structure of dynamical systems on monoids. In the course of this process, some of the properties of mathematical dynamical systems will inevitably get lost. This chapter is dedicated to sketch some of the fundamental features of dynamical systems on monoids, focusing in particular on how moving from mathematical dynamical systems to dynamical systems on monoids broadens the range of possible transformations a deterministic system may undergo. The next chapter will instead be dedicated to show how moving from mathematical dynamical systems, as they are ordinarily defined, to dynamical systems on monoids may change the ways we can represent reversible deterministic behavior.

[1]This condition amounts to requiring that the evolution of dynamical systems is insensitive to translations in time. This makes the time models of dynamical systems *homogeneous* (Lucas, 1973, 1984; Tung, 1985), i.e. causally irrelevant for the dynamical behavior of the associated systems (Weyl, 1922; Capek, 1961; Augustynek, 1968). Margenau (1950), Poincaré (1963) and van Fraassen (1989) made the homogeneity of time, expressed by the time-translational invariance of the laws governing their evolution, a distinguishing property of deterministic systems, this way confirming the adequacy of Definition 0.

5.1

The Algebra of Time

The notion of a dynamical system on a monoid was shaped assuming that the minimal algebraic structure on the time set that underpins a materially adequate definition of a dynamical system is that of an arbitrary monoid. To substantiate this tenet we shall focus on the directed graph that any dynamical system induces on its state space, and on a revealing link between this graph and category theory. In particular, we shall prove that such a graph can be made into a category if, and only if, the algebraic structure of the time set is that of a monoid.

5.1.1 General Dynamical Systems Theory

One may deconstruct the time model of dynamical systems on monoids until reducing it to a non-empty set along with a binary operation, or *magma*; this way, we get the notion of a *possible dynamical system*:

Definition 2 (Possible Dynamical System)
A possible dynamical system *on a magma* $L = (T, +)$*, denoted by* $\ddot{DS}_L$*, is an ordered pair*

$$\ddot{DS}_L = (M,\ (g^t)_{t\in T}),$$

such that

1. *M is a non-empty set;*
2. *$(g^t)_{t\in T}$ is a family of functions from M to M, indexed by T.*

Definition 2 is the starting point[2] from which a general theory of deterministic systems and motion could be developed. In what follows, we shall see how defining an equivalence relation on any arbitrary set of possible dynamical systems will make it possible to give a formal statement to the intuitive notion of a dynamical property and, with it, of an abstract dynamical system on a monoid. The key concepts involved in this task are those of *isomorphic dynamical systems* and *ρ-isomorphism*:

Definition 3 (ρ-Isomorphism)
Let $\ddot{DS}_{L_1} = (M_1, (g^{t_1})_{t_1\in T_1})$ *be a possible dynamical system on a magma* $L_1 = (T_1, +)$ *and let* $\ddot{DS}_{L_2} = (M_2, (g^{t_2})_{t_2\in T_2})$ *be a possible dynamical system on a magma* $L_2 = (T_2, \oplus)$*; a function f is a* ρ-isomorphism *of* $\ddot{DS}_{L_1}$ *in* $\ddot{DS}_{L_2}$ *if and only if*

1. *$\rho : T_1 \to T_2$ is an isomorphism of L_1 in L_2, and*

[2] Headers ¨ and ˙ will be used to stress the increasing complexity of the mathematical structures we shall encounter hereafter: the higher their complexity, the lower the number of points used to denote them.

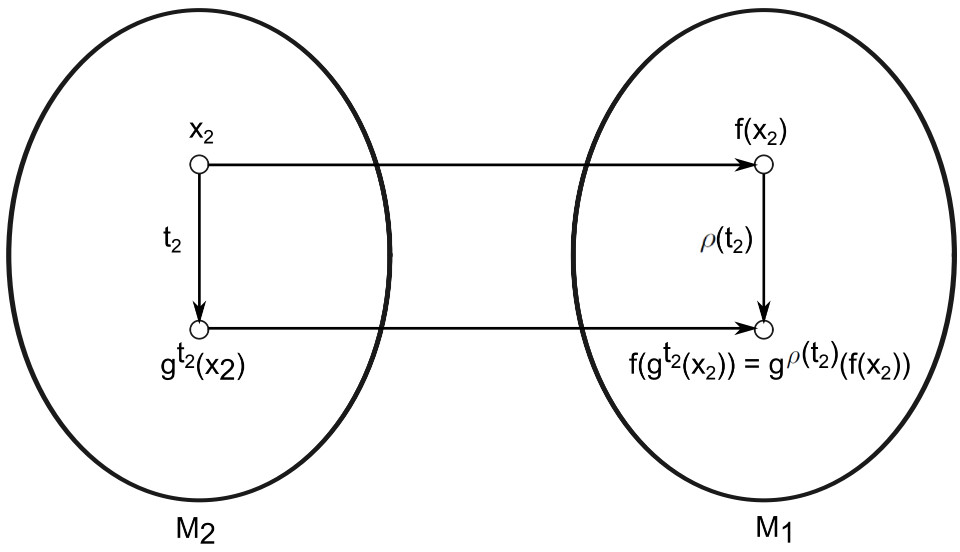

Figure 5.1: ρ-isomorphism between possible dynamical systems.

2. $f : M_1 \to M_2$ is a bijection from M_1 to M_2 such that, for any $x_1 \in M_1$ and any $t_1 \in T_1$

$$f(g^{t_1}(x_1)) = g^{\rho(t_1)}(f(x_1)). \tag{5.11}$$

DEFINITION 4 (Isomorphic Possible Dynamical Systems)
Let $\ddot{DS}_{L_1}$ be a possible dynamical system on a magma L_1 and let $\ddot{DS}_{L_2}$ be a possible dynamical system on a magma L_2. $\ddot{DS}_{L_2}$ is isomorphic *to $\ddot{DS}_{L_1}$ if and only if there exist f and ρ such that f is a ρ-isomorphism of $\ddot{DS}_{L_2}$ in $\ddot{DS}_{L_1}$.*

Theorem 5.1. Being isomorphic to *is an equivalence relation on any given set of possible dynamical systems.*

Proof
Let

- $\ddot{DS}_{L_1} = (M_1, (g^{t_1})_{t_1 \in T_1})$ be a possible dynamical system on $L_1 = (T_1, +)$,
- let $\ddot{DS}_{L_2} = (M_2, (g^{t_2})_{t_2 \in T_2})$ be a possible dynamical system on $L_2 = (T_2, \oplus)$, and
- let $\ddot{DS}_{L_3} = (M_3, (g^{t_3})_{t_3 \in T_3})$ be a possible dynamical system on $L_3 = (T_3, \odot)$.

For any possible dynamical system $\ddot{DS}_{L_1}$, let $\rho_{1,1} : T_1 \to T_1$ be the identity map on T_1 and let $f_{1,1} : M_1 \to M_1$ be the identity map on M_1; then $f_{1,1}$ is a $\rho_{1,1}$-isomorphism of $\ddot{DS}_{L_1}$ in $\ddot{DS}_{L_1}$:

- $\rho_{1,1}$ is a magma automorphism of L_1;
- $f_{1,1}$ is bijective;
- for any $x_1 \in M_1$ and any $t_1 \in T_1$

$$f_{1,1}(g^{t_1}(x_1)) = f_{1,1}(g^{\rho_{1,1}(t_1)}(x_1)) = g^{\rho_{1,1}(t_1)}(x_1) = g^{\rho_{1,1}(t_1)}(f_{1,1}(x_1)). \tag{5.12}$$

Hence, by Definition 4, $\ddot{DS}_{L_1}$ is isomorphic to $\ddot{DS}_{L_1}$.

If $\ddot{DS}_{L_1}$ is isomorphic to $\ddot{DS}_{L_2}$ then there exist a nagma isomorphism $\rho_{1,2} : T_1 \to T_2$ of L_1 in L_2 and $f_{1,2} : M_1 \to M_2$ such that $f_{1,2}$ is a $\rho_{1,2}$-isomorphism of $\ddot{DS}_{L_1}$ in $\ddot{DS}_{L_2}$. Moreover, by bijectivity of $\rho_{1,2}$ and $f_{1,2}$, there exist $(\rho_{1,2})^{-1} : T_2 \to T_1$ and $(f_{1,2})^{-1} : M_2 \to M_1$ such that $(f_{1,2})^{-1}$ is a $(\rho_{1,2})^{-1}$-isomorphism of $\ddot{DS}_{L_2}$ in $\ddot{DS}_{L_1}$:

- $(\rho_{1,2})^{-1}$ is an isomorphism of L_2 in L_1, by symmetry of magma isomorphisms;
- $(f_{1,2})^{-1}$ is bijective, by bijectivity of $f_{1,2}$;
- for any $x_2 \in M_2$ and any $t_2 \in T_2$

$$\begin{aligned} g^{(\rho_{1,2})^{-1}(t_2)}((f_{1,2})^{-1}(x_2)) &= (f_{1,2})^{-1}(f_{1,2}(g^{(\rho_{1,2})^{-1}(t_2)}((f_{1,2})^{-1}(x_2)))) \\ &= (f_{1,2})^{-1}(g^{\rho_{1,2}((\rho_{1,2})^{-1}(t_2))}(f_{1,2}((f_{1,2})^{-1}(x_2)))) \\ &= (f_{1,2})^{-1}(g^{t_2}(x_2)) \end{aligned} \tag{5.13}$$

Hence, by Definition 4, $\ddot{DS}_{L_2}$ is isomorphic to $\ddot{DS}_{L_1}$.

If $\ddot{DS}_{L_1}$ is isomorphic to $\ddot{DS}_{L_2}$ and $\ddot{DS}_{L_2}$ is isomorphic to $\ddot{DS}_{L_3}$, then there exist a magma isomorphism $\rho_{1,2} : T_1 \to T_2$ of L_1 in L_2, a magma isomorphism $\rho_{2,3} : T_2 \to T_3$ of L_2 in L_3, and $f_{1,2} : M_1 \to M_2$ and $f_{2,3} : M_2 \to M_3$ such that $f_{1,2}$ is a $\rho_{1,2}$-isomorphism of $\ddot{DS}_{L_1}$ in $\ddot{DS}_{L_2}$ and $f_{2,3}$ is a $\rho_{2,3}$-isomorphism of $\ddot{DS}_{L_2}$ in $\ddot{DS}_{L_3}$. In that case, there exist $(\rho_{2,3} \circ \rho_{1,2}) : T_1 \to T_3$ and $f_{2,3} \circ f_{1,2} : M_1 \to M_3$ such that $f_{2,3} \circ f_{1,2}$ is a $(\rho_{2,3} \circ \rho_{1,2})$-isomorphism of $\ddot{DS}_{L_1}$ in $\ddot{DS}_{L_3}$:

- $\rho_{2,3} \circ \rho_{1,2}$ is an isomorphism of L_1 in L_3, by transitivity of magma isomorphisms;
- $f_{2,3} \circ f_{1,2}$ is bijective, since function composition preserves bijectivity,
- for any $x_1 \in M_1$ and any $t_1 \in T_1$

$$f_{2,3}(f_{1,2}(g^{t_1}(x_1))) = f_{2,3}(g^{\rho_{1,2}(t_1)}(f_{1,2}(x_1))) = g^{\rho_{2,3}(\rho_{1,2}(t_1))}(f_{2,3}(f_{1,2}(x_1))). \tag{5.14}$$

Hence, by Definition 4, $\ddot{DS}_{L_1}$ is isomorphic to $\ddot{DS}_{L_3}$.

Accordingly, *being isomorphic to* is a reflexive, symmetric and transitive binary relation, i.e. an equivalence relation on any arbitrary set of possible dynamical systems . □

Let us say that a binary relation R on a given domain A *preserves* or *is compatible with* a n-ary relation Φ on A, where $n \geq 1$, if and only if for any $a_1, b_1, ...a_n, b_n \in A$, if $\Phi(a_1, ...a_n)$ and $R(a_1, b_1), ...R(a_n, b_n)$, then $\Phi(b_1, ...b_n)$.

Proposition 5.1. . *Let $\ddot{DS}_{L_1}$ be a possible dynamical system on a magma L_1 and let $\ddot{DS}_{L_2}$ be a possible dynamical system on a magma L_2; if $\ddot{DS}_{L_1}$ is a dynamical system on L_1 and $\ddot{DS}_{L_1}$ is isomorphic to $\ddot{DS}_{L_2}$, then $\ddot{DS}_{L_2}$ is a dynamical system on L_2.*

Proof
Let $\ddot{DS}_{L_1} = (M_1, (g^{t_1})_{t_1 \in T_1})$ and $\ddot{DS}_{L_2} = (M_2, (g^{t_2})_{t_2 \in T_2})$ be two possible dynamical systems on magmas $L_1 = (T_1, +)$ and $L_2 = (T_2, \oplus)$ respectively, and let $\ddot{DS}_{L_1}$ be isomorphic to $\ddot{DS}_{L_2}$. Then, by Definition 4, there must exist $\rho : T_1 \to T_2$ and $f : M_1 \to M_2$ such that f is a ρ-isomorphism of $\ddot{DS}_{L_1}$ in $\ddot{DS}_{L_2}$. If $\ddot{DS}_{L_1}$ is a dynamical system on L_1, then L_1 is a monoid, ρ is a monoid isomorphism, and L_2 is a monoid in its turn; in addition, if $0 \in T_1$ is the identity element of L_1 then $\rho(0)$ should be the identity element of L_2. Therefore:

1. M_2 is a non-empty set: by hypothesis;
2. $(g^t)_{t \in T}$ is a family of functions on M, indexed by T: by hypothesis;

3. for any $f(x_1) \in M_2$ and any $\rho(t_1), \rho(v_1) \in T_2$:

$$g^{\rho(0)}(f(x_1)) = f(g^0(x_1)) = f(x_1) \tag{5.15}$$

$$g^{\rho(t)\oplus\rho(v)} = g^{\rho(t+v)}(f(x_1)) = f(g^{t+v}(x_1)) = f(g^t(g^v(x_1))) = g^{\rho(t)}(f(g^v(x_1))) = g^{\rho(t)}(g^{\rho(v)}(f(x_1)); \tag{5.16}$$

on the other hand, due to the bijectivity of ρ and f, any $x_2 \in M_2$ is the image of exactly one $x_1 \in M_1$ with respect to f and any $t_2 \in T_2$ is the image of exactly one $t_1 \in T_1$ with respect to ρ, so that the above equalities hold for any $x_2 \in M_2$ and any $t_2, v_2 \in T_2$.

By Definition 1, $D\ddot{S}_{L_2}$ is therefore a dynamical system on a monoid. □

This allows us to speak of abstract dynamical systems on monoids in exactly the same sense we talk of abstract groups, fields, lattices, order structures, etc. We thus define:

Definition 5 (Abstract Dynamical System on a Monoid)
An abstract dynamical system on a monoid *is any equivalence class of isomorphic dynamical systems on monoids.*

By the same token, we can speak of *dynamical properties* as those properties which are *proper* to dynamical systems on monoids and are preserved by isomorphism.

Definition 6 (Dynamical Property)
Φ *is a* dynamical property *if and only if, for any two possible dyamical systems* $D\ddot{S}_{L_1}$ *and* $D\ddot{S}_{L_2}$ *(on* L_1 *and* L_2 *respectively),*

1. *If* $D\ddot{S}_{L_1}$ *has* Φ *then* $D\ddot{S}_{L_1}$ *is a dynamical system on a* L_1*;*
2. *If* $D\ddot{S}_{L_1}$ *has* Φ *and* $D\ddot{S}_{L_1}$ *is isomorphic to* $D\ddot{S}_{L_2}$*, then* $D\ddot{S}_{L_2}$ *has* Φ*.*

Dynamical properties can thus be regarded as the specific *structural* properties of dynamical systems[3]. It is then easily shown:

Proposition 5.2. *Any two dynamical systems on monoids have exactly the same dynamical properties if and only if they are isomorphic.*

Proof
If two dynamical systems on monoids are isomorphic, then by Definition 6, they have exactly the same dynamical properties. Conversely, for any two non-isomorphic dynamical systems DS_{L_1} and DS_{L_2} on L_1 and L_2 respectively, there is a dynamical property they do not share; namely, the property of being isomorphic to DS_{L_1} (or, symmetrically, to DS_{L_2}). □

By *general dynamical systems theory* we mean the mathematical theory whose Suppes' style axiomatization (Suppes, 1957) is given by Definition 1. Since general dynamical systems theory is programmatically concerned with the study of dynamical properties, it regards any two isomorphic dynamical systems on monoids as identical. Under this light, any dynamical system on a monoid (from now on, simply *dynamical system*) should be regarded as a *model* (Giunti, 2007) of that theory.

[3] For a precise definition of a mathematical structure and of a structural property, see Chapter 8

5.1.2 Oriented Graphs and Dynamical Systems

Following Lambek (1969), we may intuitively think of a graph as a web of arrows, each of which establishes a relation of functional dependence between the two nodes it connects.

Definition 7 (Oriented Graph)
An oriented graph*, denoted by G, is an ordered quadruple*

$$G = (X, A, \sigma, \tau),$$

where

- *G and A are non-empty sets;*
- *σ and τ are functions from A to X.*

Any element of X is called an *object*, *node*, *point* or *vertex* of the graph, while any member of A is called an *arrow* or a *directed edge* of the graph. For any $a \in A$, $\sigma(a)$ is called the *source* of a and $\tau(a)$ is called its *target*: σ and τ jointly establish a well-defined orientation on any arrow entering a graph; for this reason, we further qualified graphs in the sense just specified as *oriented*.

There is an interesting link between general dynamical systems theory and graphs, as just defined. Let us first of all notice that the family of t-transitions of a possible dynamical system $\ddot{DS}_L$ naturally gives rise to a particular graph in the above specified sense. We call this graph the *transition graph* of the possible dynamical system:

Definition 8 (Transition Graph of a Possible Dynamical System)
Let $\ddot{DS}_L = (M, (g^t)_{t \in T})$ be a possible dynamical system on a magma $L = (T, +)$; the transition graph *of $\ddot{DS}_L$, denoted by $\ddot{G}(\ddot{DS}_L)$, is the ordered quadruple*

$$\ddot{G}(\ddot{DS}_L) = (X, A, \sigma, \tau)$$

such that

1. *$X = M$,*
2. *$A = \{a : \text{ for some } t \in T \text{ and some } x \in M, a = (x, t, g^t(x))\}$*
3. *$\sigma : A \to X$ is the function from A to M such that, for any triple $a \in A$, $\sigma(a)$ is the first element of a;*
4. *$\tau : A \to X$ is the function from A to M such that, for any triple $a \in A$, $\tau(a)$ is the last element of a.*

It is easy to see that

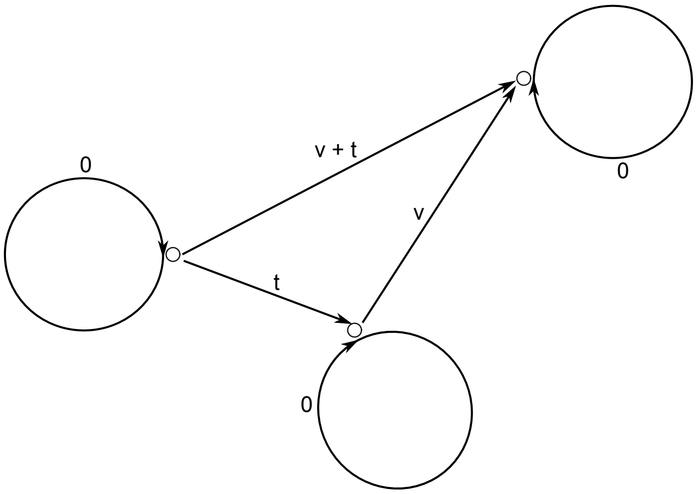

Figure 5.2: Transition graph of a possible dynamical system.

Proposition 5.3. *The transition graph of any possible dynamical system is an oriented graph.*

Proof
Let $\ddot{G}(\ddot{DS}_L) = (X, A, \sigma, \tau)$ be the transition graph of a possible dynamical system $\ddot{DS}_L = (M, (g^t)_{t \in T})$ on $L = (T, +)$. By Definition 2, M is a non-empty set and, for any $x \in M$, there exist $y \in M$ and $t \in T$ such that $y = g^t(x)$; accordingly, by Definition 8 X and A are non-empty sets. Moreover, by the same definition, σ and τ are both functions from A to X. As a consequence, by Definition 7, $\ddot{G}(\ddot{DS}_L)$ is an oriented graph. □

The notion of a possible dynamical system was shaped assuming a simple magma as a time model, and without imposing any requirement on the family of functions indexed by the latter. On the other hand, magmas can be endowed with an identity element, simply by definition. Possible dynamical systems whose time model is a magma with identity can be required to satisfy an identity and a composition condition. This way, we get the notion of a *quasi-dynamical system*:

Definition 9 (Quasi-Dynamical System)
A quasi-dynamical system *on a magma* $L = (T, +)$ *with identity* 0*, denoted by* $\dot{DS}_L$*, is an ordered pair*

$$\dot{DS}_L = (M,\ (g^t)_{t \in T}), \tag{5.17}$$

where

- *M is a non-empty set;*
- *$(g^t)_{t \in T}$ is a family of functions from M to M, indexed by T;*
- *for any $x \in M$ and any $t, v \in T$*

$$g^0(x) = x \tag{5.18}$$

$$g^{v+t}(x) = g^v(g^t(x)) \tag{5.19}$$

Quasi-dynamical systems differ from dynamical systems on monoids just for the fact that the binary operations on their time models not necessarily are associative. However, it is possible to equip the transition graph $\ddot{G}(\dot{DS}_L) = (X, A, \sigma, \tau)$ of any quasi-dynamical system $\dot{DS}_L$ on L with a family of x-identity arrows $(id_x)_{x \in X}$ and with a composition operation $\circ$, respectively corresponding to the conditions of identity and composition on the state transitions of $\dot{DS}_L$:

Definition 10 (Transition Graph (with Identity and Composition) of a Quasi-Dynamical System)
Let $\dot{DS}_L = (M, (g^t)_{t \in T})$ be a quasi-dynamical system on a magma $L = (T, +)$ with identity 0; the transition graph with identity and composition *of $\dot{DS}_L$, denoted by $\dot{G}(\dot{DS}_L)$, is the triple*

$$\dot{G}(\dot{DS}_L) = (G, (id_x)_{x \in X}, \circ) \tag{5.20}$$

such that

1. $G = (X, A, \sigma, \tau) = \ddot{G}(\dot{DS}_L)$;
2. $(id_x)_{x \in M}$ *is the family of arrows such that, for any* $x \in M$,
$$(id_x) = (x, 0, g^0(x)); \tag{5.21}$$
3. $\circ$ *is the, possibly partial, binary operation such that, for any* $a, b \in A$, *if* $a = (x, t, g^t(x))$ *and* $b = (g^t(x), v, g^v(g^t(x)))$ *for some* $t, v \in T$, *then*
$$b \circ a = (x, v + t, g^{v+t}(x))); \tag{5.22}$$
otherwise, $b \circ a$ *is undefined.*

To be sure that Definition 10 is consistent, we still need to show that, for any quasi-dynamical system $\dot{DS}_L = (M, (g^t)_{t \in T})$ on $L = (T, +)$, the set of arrows A of the corresponding transition graph $\dot{G}(\dot{DS}_L) = (X, A, \sigma, \tau)$ is closed under the composition operation $\circ$. So, let $a = (x, t, g^t(x))$ and $b = (g^t(x), v, g^v(g^t(x)))$ be any two arrows in A such that $\tau(a) = \sigma(b)$; their composition will accordingly be $b \circ a = (x, v + t, g^{v+t}(x)))$. To see that $b \circ a$ is an element of A we only have to notice that, being a magma, L is closed under its rule of composition and that, by definition of a quasi-dynamical system, g^{v+t} is a function on the state space of $\dot{DS}_L$; as a consequence, x and g^{v+t} belong to X and $v + t$ belongs to T, which is enough to guarantee that $b \circ a$ is an element of A.

Transition graphs of quasi-dynamical systems belong to a special subclass of graphs, called *deductive systems.*

Definition 11 (Deductive System)
A deductive system*, denoted by* $\mathbf{G}$*, is an ordered triple*

$$\mathbf{G} = (G, (id_x)_{x \in X}, \circ) \tag{5.23}$$

such that

1. $G = (X, A, \sigma, \tau)$ *is an oriented graph;*
2. $(id_x)_{x\in X}$ *is a family of arrows on* X *such that, for any* $x \in X$,
$$\sigma(id_x) = x \quad \text{and} \quad \tau(id_x) = x; \tag{5.24}$$
3. $\circ$ *is the possibly partial binary operation on* A, *called* arrow composition, *such that for any* $a, b \in A$, *if* $\tau(a) = \sigma(b)$, *then* $b \circ a$ *is defined and*
$$\sigma(b \circ a) = \sigma a \quad \text{and} \quad \tau(b \circ a) = \tau(b). \tag{5.25}$$

Proposition 5.4. *The transition graph (with identity and composition) of any quasi-dynamical system is a deductive system.*

Proof
Let $\dot{D}S_L = (M, (g^t)_{t\in T})$ be a quasi-dynamical system on a magma $L = (T, +)$ with identity 0 and let $\dot{G}(\dot{D}S_L) = (G, (id_x)_{x\in X}, \circ)$ be the transition graph of $\dot{D}S_L$, with $G = (X, A, \sigma, \tau)$. Then,

- by Proposition 5.3, G is an oriented graph;
- by Definition 10, $(id_x)_{x\in M}$ is a family of arrows such that, for any $x \in X = M$
$$\sigma(id_x) = \sigma(x, 0, g^0(x)) = x, \text{ and} \tag{5.26}$$
$$\tau(id_x) = \tau(x, 0, g^0(x)) = g^0(x) = x; \tag{5.27}$$
- according to Definition 10, $\circ$ is a binary operation such that for any two arrows $a = (x, t, g^t(x))$ and $b = (g^t(x), v, g^v(g^t(x)))$ in set A,
$$\sigma(b \circ a) = \sigma((x, v + t, g^{v+t}(x))) = x = \sigma(a), \tag{5.28}$$
$$\tau(b \circ a) = \tau((x, v + t, g^{v+t}(x))) = g^{v+t}(x) = g^v(g^t(x)) = \tau(b), \tag{5.29}$$

Hence, by Definition 11, $G(\dot{D}S_L)$ is a deductive system. □

Finally, *categories* are deductive systems whose operation of arrow composition is associative:

Definition 12
A deductive system $\mathbf{G} = (G, (id)_{x\in X}, \circ)$, *where* $G = (X, A, \sigma, \tau)$, *is a* category *if and only if, for any* $a, b, c \in A$, *if* $\tau(a) = \sigma(b)$ *and* $\tau(b) = \sigma(c)$, *then*
$$c \circ (b \circ a) = (c \circ b) \circ a. \tag{5.30}$$

Theorem 5.2. *The transition graph (with identity and composition)* $\dot{G}(\dot{D}S_L)$*of any quasi-dynamical system* $\dot{D}S_L$ *is a category if and only if* L *is a monoid.*

Proof
Let $\dot{D}S_L = (M, (g^t)_{t\in T})$ be a quasi-dynamical system on a magma $L = (T, +)$ with identity 0 and let $\dot{G}(\dot{D}S_L) = (G, (id)_{x\in X}, \circ)$ be the transition graph of $\dot{D}S_L$, where $G = (X, A, \sigma, \tau)$. For ease of expression let us agree that,

we can equivalently express any ordered triple of the form $a = (x, t, g^t(x))$ by means of a labeled arrow $a = x \xrightarrow{t} g^t(x)$, and let $a, b, c \in A$ be a triple of arbitrary arrows such that $a = x \xrightarrow{t} g^t(x)$, $b = g^t(x) \xrightarrow{v} g^v(g^t(x))$, $c = g^v(g^t(x)) \xrightarrow{u} g^u(g^v(g^t(x)))$. Proposition 5.4 guarantees that $\dot{G}(\dot{DS}_L)$ is a deductive system.

If L is a monoid, by the associativity of + we get:

$$\begin{aligned}(c \circ b) \circ a &= [g^v(g^t(x)) \xrightarrow{u} g^u(g^v(g^t(x))) \circ g^t(x) \xrightarrow{v} g^v(g^t(x))] \circ x \xrightarrow{t} g^t(x) \\ &= g^t(x) \xrightarrow{u+v} g^u(g^v(g^t(x))) \circ x \xrightarrow{t} g^t(x) \\ &= x \xrightarrow{(u+v)+t} g^u(g^v(g^t(x))) \\ &= x \xrightarrow{u+(v+t)} g^u(g^v(g^t(x))) \\ &= g^v(g^t(x)) \xrightarrow{u} g^u(g^v(g^t(x))) \circ x \xrightarrow{v+t} g^v(g^t(x)) \\ &= g^v(g^t(x)) \xrightarrow{u} g^u(g^v(g^t(x))) \circ [g^t(x) \xrightarrow{v} g^v(g^t(x)) \circ x \xrightarrow{t} g^t(x)] \\ &= c \circ (b \circ a); \end{aligned} \tag{5.31}$$

and therefore, by Definition 12, $\dot{G}(\dot{DS}_L)$ is a category.

On the other hand, if L is not a monoid, then there must exist $u, v, t \in T$ such that

$$u + (v + t) \neq (u + v) + t \tag{5.32}$$

and therefore, for any $x \in M$

$$x \xrightarrow{u+(v+t)} g^{u+(v+t)}(x) \neq x \xrightarrow{(u+v)+t} g^{(u+v)+t}(x); \tag{5.33}$$

as a consequence,

$$c \circ (b \circ a) \neq (c \circ b) \circ a, \tag{5.34}$$

so that, according to Definition 12, $\dot{G}(\dot{DS}_L)$ is not a category. Conversely, if $\dot{G}(\dot{DS}_L)$ is a category, then L is a monoid. □

Theorem 5.2 provides us with a justification for our claim that the minimal structure on the time set that supports a materially adequate definition of a dynamical system is at least that of a monoid. For, if it is not, the transition graph of the system cannot even be made into a category.

5.2

Basic Dynamical Concepts

General dynamical systems theory, though based on a time model as simple as a monoid, is nevertheless sufficient to define a variety of genuine dynamical concepts, as well as to prove about them significant and sometimes even surprising results. In the first part of this section, we shall give a general definition of the basic dynamical concepts of motion and orbit, together with a brief examination of their fundamental properties. In the second part, we shall define the future and past sets of an arbitrary state of a dynamical system, which will play a central role in Chapter 7, where we shall outline a dynamical interpretation of tenses.

5.2.1 Motions and Orbits

Arnold (1973) originally defined phase flows as one-parameter groups of transformations on a n-dimensional differentiable manifold, indexed by the real numbers. (Giunti, 1997) adopted a more general definition, solely requiring the state space of a mathematical dynamical system to consist of an arbitrary non-empty set and allowing for discrete time sets. In both cases, the two basic notions entering the description of the dynamical behavior of a deterministic system are those of *motion*, or *state evolution*, and *phase curve*, or *orbit*. These concepts may be generalized in a very straightforward manner in order to cover the corresponding features of dynamical systems on monoids.

By the motion of a dynamical system, we mean the function associating any lapse of time t with the state the system displays t time after having initially been set in a given state x:

Definition 13 (Motion)
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$. For any $x \in M$, the motion *(or* state evolution*) of DS_L with initial state x, denoted by g_x, is the function $g_x : T \to M$ such that, for any $t \in T$*

$$g_x = eval(g^t, x), \tag{5.35}$$

where, for any $t \in T$ and any $x \in M$,

$$eval(g^t, x) = g^t(x). \tag{5.36}$$

Definition 14 (Orbit of a Point)
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$. The orbit *of any point $x \in M$, denoted by $orb(x)$, is the set*

$$orb(x) \stackrel{def}{=} \left\{y \in M : \exists t \in T(y = g^t(x))\right\}. \tag{5.37}$$

The orbit of a point x is the image of the corresponding motion g_x. Intuitively speaking, $orb(x)$ is the set of all and sole states a dynamical system goes through after having been initially set in state x; as such, it models both its present and its future behavior. Orbits are intrinsic features of the dynamics of a deterministic system, in the sense that the property of lying in the orbit of a given point is preserved by ρ-isomorphism:

Proposition 5.5. *Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1\in T_1})$ be a dynamical system on a monoid $L_1 = (T_1, +)$, let $DS_{L_2} = (M_2, (g^{t_2})_{t_2\in T_2})$ be a dynamical system on a monoid $L_2 = (T_2, \oplus)$ and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}; then, for any $x_1, y_1 \in M_1$, $y_1 \in orb(x_1)$ if and only if $f(y_1) \in orb(f(x_1))$.*

Proof
Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1\in T_1})$ and $DS_{L_2} = (M_2, (g^{t_2})_{t_2\in T_2})$ be dynamical systems on $L_1 = (T_1, +)$ and $L_2 =$

$(T_2, \oplus)$ respectively,, let $\rho : T_1 \to T_2$ be a monoid isomorphism of L_1 in L_2 and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}. Then, for any $x_1, y_1 \in M_1$, if $y_1 \in orb(x_1)$ there exists $t_1 \in T_1$ such that

$$\begin{aligned} g^{t_1}(x_1) &= y_1 \\ f(g^{t_1}(x_1)) &= f(y_1) \\ g^{\rho(t_1)}(f(x_1)) &= f(y_1), \end{aligned} \tag{5.38}$$

and therefore, by Definition 14, $f(y_1) \in orb(f(x_1))$. Proof in the converse direction is guaranteed by the fact that $f^{-1} : M_2 \to M_1$ is a ρ^{-1}-isomorphism of DS_{L_2} in DS_{L_1}. □

Since orbits are the images of motions, ρ-isomorphisms must preserve motions in their turn. Indeed, one may think of motions and orbits as the fundamental building blocks of which the dynamics of a system is composed.

Interestingly, for any point in the state space of a dynamical system DS_L on a monoid L, it is possible to define a dynamical system DS_{x_L} on L whose state space is the orbit of x and whose state transitions are the restriction of the state transitions of DS_L to the orbit of x.

Proposition 5.6. *Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$; for any $x \in M$, $DS_{x_L} = (orb(x), (g^t|_{orb(x)})_{t\in T})$ is a dynamical system on L.*

Proof
Let $DS = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0 and let $x \in M$ be any point in its state space; then

1. $orb(x)$ is a nonempty set: including at least x,
2. T is shared with DS,
3. $(g^t|_{orb(x)})_{t\in T}$ is a family of function on $orb(x)$, indexed by T: by definition of orbit, for any $y \in orb(x)$ and any $t \in T$ there exists $v \in T$ such that

$$g^t|_{orb(x)}(y) = g^t|_{orb(x)}(g^v(x)) = g^t|_{orb(x)}(g^v|_{orb(x)}(x)) = g^{t+v}|_{orb(x)}(x) \in orb(x); \tag{5.39}$$

4. for any $y \in orb(x)$ and for any $t \in T$

$$g^0|_{orb(x)}(y) = g^0(y) = y \tag{5.40}$$

$$g^{t+v}|_{orb(x)}(y) = g^{t+v}(y) = g^t(g^v(y)) = g^t|_{orb(x)}(g^v|_{orb(x)}(y)) \tag{5.41}$$

Hence, $DS_{x_L} = (orb(x), (g^t|_{orb(x)})_{t\in T})$ is a dynamical system on L. □

This property of orbits will play an important role in Chapter 7, where we shall discuss the dynamical properties which can be attributed to time models. For the sake of the present discussion, it will be sufficient to point out that, in virtue of Proposition 5.6, any dynamical system gives rise to as many dynamical systems as the orbits it owns, and that the study of each orbit may alternatively be carried out as a study of the corresponding dynamical system. Finally, together with Proposition 5.7, Proposition 5.6 endows the set of all such dynamical systems with a partial ordering, which they naturally inherit from the relation of set inclusion holding among corresponding orbits.

Proposition 5.7. *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$; for any $x, y \in M$, $y \in orb(x)$ if and only if $orb(y) \subseteq orb(x)$.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0 and let $x, y \in M$.

- If $y \in orb(x)$, then there exists $v \in T$ such that
$$g^v(x) = y; \tag{5.42}$$
hence, for any $t \in T$ and any $g^t(y) \in orb(y)$
$$g^t(y) = g^t(g^v(x)) \in orb(x), \tag{5.43}$$
so that $orb(y) \subseteq orb(x)$.
- Conversely, if $orb(y) \subseteq orb(x)$ then, for any $t \in T$ and any $g^t(y) \in orb(y)$ there exists $v \in T$ and $g^v(x) \in orb(x)$ such that
$$g^t(y) = g^v(x); \tag{5.44}$$
therefore, assuming $t = 0$:
$$g^v(x) = g^0(y) = y, \tag{5.45}$$
so that $y \in orb(x)$.

□

One further consequence of Proposition 5.7 is that no two states x and y of a dynamical system possess crossing orbits; that is to say, whenever the orbits of any two states x and y intersect at a point, they coincide from there on:

Corollary 5.7.1. *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$; for any $x, y, z \in M$, if $z \in orb(x) \cap orb(y)$, then $orb(z) \subseteq orb(x)$ and $orb(z) \subseteq orb(y)$.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ and let $x, y, z \in M$. If $z \in orb(x) \cap orb(y)$, then $z \in orb(x)$ and $z \in orb(y)$. Hence, by Proposition 5.7, $orb(z) \subseteq orb(x)$ and $orb(z) \subseteq orb(y)$. □

Therefore, there are only three different relations orbits may bear to each other: disjunction, inclusion and merging, respectively holding in case, of any two orbits, they have no points in common, one is a subset of the other, and they meet at a point. More precisely, we say that any two orbits are merging in case they satisfy Definition 16 below:

Definition 15 (Merging Orbit)
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$; r is a merging orbit if an only if, for some $x, y \in M$, $r = orb(x)$ and

$$orb(x) \nsubseteq orb(y), \tag{5.46}$$
$$orb(y) \nsubseteq orb(x), \tag{5.47}$$
$$orb(x) \cap orb(y) \neq \emptyset. \tag{5.48}$$

Definition 16 (Orbits Merging with Each Other)
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$ and let r and s be arbitrary orbits; r is merging with *s if and only if r and s intersect, but neither one is a subset of the other.*

Disjunction, inclusion and merging are also preserved by ρ-isomorphism. In the first two cases, this is a straightforward consequence of Proposition 5.5. In the case of merging, this is shown by the following statement:

Proposition 5.8. *Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1\in T_1})$ be a dynamical system on a monoid $L_1 = (T_1, +)$, let $DS_{L_2} = (M_2, (g^{t_2})_{t_2\in T_2})$ be a dynamical system on a monoid $L_2 = (T_2, \oplus)$ and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}; for any x_1, $orb(x)$ is merging if and only if $orb(f(y_1))$ is.*

Proof
Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1\in T_1})$ and $DS_{L_2} = (M_2, (g^{t_2})_{t_2\in T_2})$ be dynamical systems on $L_1 = (T_1, +)$ and $L_2 = (T_2, \oplus)$ respectively,let $\rho : T_1 \to T_2$ be a monoid isomorphism of L_1 in L_2 and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}. Finally, let $x_1 \in M_1$. If $orb(x_1)$ is merging then, by Definition 16, for some y_1, $orb(x_1) \nsubseteq orb(y_1)$, $orb(y_1) \nsubseteq orb(x_1)$ and $orb(x_1) \cap orb(y_1) \neq \emptyset$. Hence, by Proposition 5.5,

$$orb(f(x_1)) \nsubseteq orb(f(y_1)), \tag{5.49}$$

$$orb(f(y_1)) \nsubseteq orb(f(x_1)) \tag{5.50}$$

and, for any $z_1 \in orb(x_1) \cap orb(y_1)$

$$f(z_1) \in orb(f(x_1)) \cap orb(f(y_1)); \tag{5.51}$$

accordingly, $orb(f(x_1))$ is merging. Proof in the converse direction is guaranteed by the fact that $f^{-1} : M_2 \to M_1$ is a ρ^{-1}-isomorphism of DS_{L_2} in DS_{L_1}. □

We shall see in the next chapter that possession of merging orbits is a distinguishing feature of irreversible dynamical systems[4].

5.2.1.1 Periodicity, Eventual Periodicity, Aperiodicity

In addition, orbits may be classified as *periodic*, *eventually periodic* or *aperiodic*.

Definition 17 (Periodic Point)
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0; for any $x \in M$, x is a periodic point *if and only if, for some $t \in T$*

$$t \neq 0, g^t(x) = x. \tag{5.52}$$

Definition 18 (Periodic Orbit)
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0; r is a periodic orbit *if and only if, for some $x \in M$, $r = orb(x)$ and x is a periodic point.*

[4] See Corollary 6.5.1 and Proposition 6.18.

For ease of expression, we shall occasionally talk about a period of a point x in place of a period of its orbit, which is defined below:

Definition 19 (Period)
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0; for any $t \in T$, t is a period *of $orb(x)$ if and only if*

$$t \neq 0, \tag{5.53}$$

$$g^t(x) = x. \tag{5.54}$$

Clearly, the orbit of a point x is periodic if and only if it has a period. Moreover, periods are preserved by ρ-isomorphism and, by the same token, possessing a periodic orbit must be preserved in its turn.

Proposition 5.9. *Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1\in T_1})$ be a dynamical system on a monoid $L_1 = (T_1, +)$, let $DS_{L_2} = (M_2, (g^{t_2})_{t_2\in T_2})$ be a dynamical system on a monoid $L_2 = (T_2, \oplus)$, let $\rho : T_1 \to T_2$ be a monoid isomorphism of L_1 in L_2, let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2} and let $x_1 \in M_1$. Then any $t_1 \in T_1$ is a period of $orb(x_1)$ if and only if $\rho(t_1)$ is a period of $orb(f(x_1))$.*

Proof
Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1\in T_1})$ be a dynamical system on a monoid $L_1 = (T_1, +)$ with identity 0, let $DS_{L_2} = (M_2, (g^{t_2})_{t_2\in T_2})$ be a dynamical system on a monoid $L_2 = (T_2, \oplus)$, let $\rho : T_1 \to T_2$ be a monoid isomorphism of L_1 in L_2 and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}. Finally, let $x_1 \in M_1$. For any $t_1 \in T_1$, if t_1 is a period of $orb(x_1)$ then $t_1 \neq 0$ and

$$f(x_1) = f(g^{t_1}(x_1)) = g^{\rho_{t_1}}(f(x_1)), \tag{5.55}$$

where $\rho(t_1)$ is not the identity of T_2 since, by hypothesis, ρ maps solely identity elements into idenity elements; hence, $\rho(t_1)$ is a period of $orb(f(x_1))$. Symmetrically, Proof in the converse direction is guaranteed by the fact that $f^{-1} : M_2 \to M_1$ is a ρ^{-1}-isomorphism of DS_{L_2} in DS_{L_1}. □

Points whose orbits are not periodic may nevertheless evolve into periodic points. In that case, we call their orbit *eventually periodic*:

Definition 20 (Eventually Periodic Orbit)
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$; r is an eventually periodic orbit *if and only if r is an orbit, r is not periodic and there exists $y \in r$ such that $orb(y)$ is periodic.*

Eventual periodicity is also preserved by ρ-isomorphism.

Proposition 5.10. *Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1\in T_1})$ be a dynamical system on a monoid $L_1 = (T_1, +)$, let $DS_{L_2} = (M_2, (g^{t_2})_{t_2\in T_2})$ be a dynamical system on a monoid $L_2 = (T_2, \oplus)$, let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}; and let $x_1 \in M_1$; then $orb(x_1)$ is eventually periodic if and only if $orb(f(x_1))$ is.*

Proof
Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1 \in T_1})$ be a dynamical system on a monoid $L_1 = (T_1, +)$ with identity 0, let $DS_{L_2} = (M_2, (g^{t_2})_{t_2 \in T_2})$ be a dynamical system on a monoid $L_2 = (T_2, \oplus)$, let $\rho : T_1 \to T_2$ be a monoid isomorphism of L_1 in L_2 and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}. If $orb(x_1)$ is eventually periodic, then for some $y_1 \in orb(x_1)$ and for any $t_1 \in T_1 - \{0\}$ and some $v_1 \in T_1 - \{0\}$

$$g^{t_1}(x_1) \neq x_1 \tag{5.56}$$
$$g^{v_1}(y_1) = y_1; \tag{5.57}$$

hence, by Proposition 5.5 and Proposition 5.9, $f(y_1) \in orb(f(x_1))$ and, for any $\rho(t_1) \in T_2$ and $\rho(v_1) \in T_2$,

$$g^{\rho(t_1)}(f(x_1)) \neq f(x_1) \tag{5.58}$$
$$g^{\rho(v_1)}(f(y_1)) = f(y_1), \tag{5.59}$$

where $\rho(v_1)$ is not the identity of L_2. Accordingly, $orb(f(x_1))$ is eventually periodic. Proof in the converse direction is guaranteed by the fact that $f^{-1} : M_2 \to M_1$ is a ρ^{-1}-isomorphism of DS_{L_2} in DS_{L_1}. □

Finally, if no point in the orbit of a state x is periodic, then x is called *aperiodic*.

Definition 21 (Aperiodic Orbit)
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$; r is an eventually periodic orbit *if and only if r is and orbit and it is neither periodic nor eventually periodic.*

Since both periodicity and eventual periodicity are preserved by ρ-isomorphism, so it must be aperiodicity.

Proposition 5.11. *Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1 \in T_1})$ be a dynamical system on a monoid $L_1 = (T_1, +)$, let $DS_{L_2} = (M_2, (g^{t_2})_{t_2 \in T_2})$ be a dynamical system on a monoid $L_2 = (T_2, \oplus)$, let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2} and let $x_1 \in M_1$; then $orb(x_1)$ is aperiodic if and only if $orb(f(x_1))$ is.*

Proof
Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1 \in T_1})$ be a dynamical system on a monoid $L_1 = (T_1, +)$ with identity 0, let $DS_{L_2} = (M_2, (g^{t_2})_{t_2 \in T_2})$ be a dynamical system on a monoid $L_2 = (T_2, \oplus)$, let $\rho : T_1 \to T_2$ be a monoid isomorphism of L_1 in L_2 and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}. If $orb(x_1)$ is aperiodic, then by Proposition 5.9 and Proposition 5.10, $orb(f(x_1))$ is neither periodic nor eventually periodic; hence $orb(f(x_1))$ is aperiodic. Proof in the converse direction is guaranteed by the fact that $f^{-1} : M_2 \to M_1$ is a ρ^{-1}-isomorphism of DS_{L_2} in DS_{L_1}. □

If the orbit of a point y is eventually periodic, then all points whose orbits y belongs to are certainly not aperiodic.

Proposition 5.12. *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$; for any $x, y \in M$ such that $y \in orb(x)$, if $orb(y)$ is periodic then $orb(x)$ is either periodic or eventually periodic.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0, let $x, y \in M$ such that $y \in orb(x)$ and let $orb(y)$ be periodic. So, if there exists $t \in T - \{0\}$ such that $g^t(x) = x$ then, by definition of periodic orbit, $orb(x)$ is periodic. If, on the contrary, there exists no $t \in T - \{0\}$ such that $g^t(x) = x$ then, by definition of eventually periodic orbit, $orb(x)$ is eventually periodic. □

On the other hand, if a point is aperiodic then no point along its orbit can be eventually periodic.

Proposition 5.13. *Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$; for any $x, y \in M$ such that $y \in orb(x)$, if $orb(x)$ is aperiodic then $orb(y)$ is aperiodic.*

Proof
Let $DS = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T; +)$ and let $x, y \in M$ such that $y \in orb(x)$. If $orb(y)$ is either periodic or eventually periodic, then there exists $z \in orb(y) \subseteq orb(x)$ such that $orb(z)$ is periodic; hence, by Proposition 5.12 $orb(x)$ is either periodic or eventually periodic. Conversely, if $orb(x)$ is aperiodic, then $orb(y)$ is neither periodic nor eventually periodic, i.e. $orb(y)$ is aperiodic. □

5.2.2 Future and Past

The orbit of a point x may be understood as a broad representation of both its present and its whole future history; a finer description of the evolution of a point is made possible by the following set of definitions, which will also allow for a deeper understanding of reversible dynamics.

Definition 22 (t-Future of a Point)
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0; for any state $x \in M$ and for any duration $t \in T - \{0\}$, the t-future *of x, denoted by $F^t(x)$, is defined as*

$$F^t(x) \stackrel{def}{=} \left\{y \in M : y = g^t(x)\right\} . \tag{5.60}$$

In plain words, the t-future of a point x is the image of a state transition of non-zero duration t with initial state x, i.e. a unique point in the future evolution of x or, more precisely, the singleton set of that point. On the basis of Definition 22, the whole future history of a point can be defined as the union of all its t-future histories.

Definition 23 (Future of a Point)
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0; for any state $x \in M$, the future *of x, denoted by $F(x)$, is defined as*

$$F(x) \stackrel{def}{=} \bigcup_{t\in T-\{0\}} F^t(x) = \left\{y \in M : \text{ for some } t \in T - \{0\}, y = g^t(x)\right\} . \tag{5.61}$$

The future of an arbitrary point x is openly a subset of its orbit, differing from the latter for at most x itself; as a consequence, $F(x)$ and $orb(x)$ coincide exactly in case x belongs to its own future, i.e. exactly in case x is periodic[5].

Symmetrically, the t-past history and the whole past history of a point are respectively defined as follows:

[5] See Proposition 5.15 below.

Definition 24 (t-Past of a Point)
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0; for any state $x \in M$ and for any duration $t \in T - \{0\}$, the t-past *of x, denoted by $P^t(x)$, is defined as*

$$P^t(x) \stackrel{def}{=} \left\{ y \in M : x = g^t(y) \right\}. \tag{5.62}$$

Definition 25 (Past of a Point)
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0; for any state $x \in M$, the past *of x, denoted by $P(x)$, is defined as*

$$P(x) \stackrel{def}{=} \bigcup_{t \in T - \{0\}} P^t(x) = \left\{ y \in M : \text{ for some } t \in T - \{0\}, x = g^t(y) \right\}. \tag{5.63}$$

Contrary to the case of future, the past history of a point x is generally not a subset of its orbit. In addition, for any duration t, the t-future of a state x contains exactly one state, while its t-past may very well contain several distinct states. We shall see in the following chapter that requiring the past of a point to behave exactly as its future under either of these respects leads to two distinct kinds of reversible dynamical behavior, namely reversibility[6] and logical reversibility[7].

Despite these differences, past and future are nevertheless plainly related concept: any t-future image of a point x is a point whose t-past image is x itself.

Proposition 5.14. *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$; for any $x, y \in M$ and for any $t \in T - \{0\}$, $x \in P^t(y)$ if and only if $y \in F^t(x)$*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0, let $x, y \in M$ and let $t \in T - \{0\}$.

If $x \in P^t(y)$, then there exists $t \in T - \{x\}$ such that

$$g^t(x) = y; \tag{5.64}$$

hence, by definition of t-future of a point, $y \in F^t(x)$.

If $y \in F^t(x)$, then there exists $t \in T - \{x\}$ such that

$$g^t(x) = y; \tag{5.65}$$

hence, by definition of t-past of a point, $x \in P^t(y)$. □

Corollary 5.14.1. *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$; for any $x, y \in M$, $x \in P(y)$ if and only if $y \in F(x)$*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0 and let $x, y \in M$.

[6] See § 6.2.1
[7] See § 6.1.

If $x \in P(y)$, then by Definition 25 there must exists $t \in T - \{0\}$ such that $x \in P^t(y)$; hence, by Proposition 5.14, $y \in F^t(x)$ and therefore, by Definition 23, $y \in F(x)$. If $y \in F(y)$, then by Definition 23 there must exists $t \in T - \{0\}$ such that $y \in F^t(x)$; hence, by Proposition 5.14, $x \in P^t(y)$ and therefore, by Definition 25, $x \in P(y)$. □

Concepts such as periodicity, eventual periodicity and aperiodicity of an orbit may be given further characterization by means of the notions of past and future of a point, as follows.

Proposition 5.15. *Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$; for $x \in M$, $orb(x)$ is periodic if and only if $x \in F(x)$.*

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0 and let $x \in M$.

If $orb(x)$ is periodic, then there exists $t \in T - \{0\}$ such that

$$g^t(x) = x; \tag{5.66}$$

hence, by definition of t-future of a point, $x \in F^t(x)$ and, *a fortiori*, $x \in F(x)$. If $x \in F(x)$ then

$$x \in \bigcup_{t\in T-\{0\}} F^t(x), \tag{5.67}$$

so that there must necessarily exist $t \in T - \{0\}$ such that $x \in F^t(x)$ and, by definition of t-future of a point,

$$g^t(x) = x; \tag{5.68}$$

hence, $orb(x)$ is periodic. □

Corollary 5.15.1. *Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$; for any $x \in M$, $orb(x)$ is periodic if and only if $x \in P(x)$.*

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0 and let $x \in M$. By Proposition 5.15, $orb(x)$ is periodic if and only if $x \in F(x)$; as a consequence, according to Corollary 5.14.1, $orb(x)$ is periodic if and only if $x \in P(x)$. □

Eventually periodic orbits may be accordingly be understood as the orbits of points which do not belong to their own future (past), and in whose future lies a point which, on the contrary, does. Finally, aperiodic orbits may be alternatively described as orbits none of whose points belong to their own future (past).

5.2.2.1 Special Points

The notions of future and past of a point may also be employed in studying the behavior of special points such as *fixed points* and *gardens of Eden.*

Definition 26 (Fixed Point)
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$; a point $x \in M$ is fixed *if and only if, for any $t \in T$*

$$g^t(x) = x. \tag{5.69}$$

Proposition 5.16. *Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1 \in T_1})$ be a dynamical system on a monoid $L_1 = (T_1, +)$, let $DS_{L_2} = (M_2, (g^{t_2})_{t_2 \in T_2})$ be a dynamical system on a monoid $L_2 = (T_2, \oplus)$ and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}; for any $x_1 \in M_1$, x_1 is fixed if and only if $f(x_1)$ is.*

Proof
Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1 \in T_1})$ be a dynamical system on a monoid $L_1 = (T_1, +)$ with identity 0, let $DS_{L_2} = (M_2, (g^{t_2})_{t_2 \in T_2})$ be a dynamical system on a monoid $L_2 = (T_2, \oplus)$, let $\rho : T_1 \to T_2$ be a monoid isomorphism of L_1 in L_2 and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}. For any $x_1 \in M_1$, ifx_1 is fixed, then for any $t_1 \in T_1$

$$g^{\rho(t_1)}(f(x_1)) = f(g^{t_1}(x_1)) = f(x_1), \tag{5.70}$$

so that, by surjectivity of ρ, $f(x_1)$ is fixed in its turn. Proof in the converse direction is guaranteed by the fact that $f^{-1} : M_2 \to M_1$ is a ρ^{-1}-isomorphism of DS_{L_2} in DS_{L_1}. □

Fixed points may equivalently be defined as those points whose orbits coincide with their singleton set; as such, all fixed points evidently possess periodic orbits.

Proposition 5.17. *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$; for any $x \in M$, x is fixed if and only if*

$$F(x) = \{x\} . \tag{5.71}$$

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0 and let $x \in M$.

If x is fixed, then for any $t \in T$

$$g^t(x) = x \tag{5.72}$$

and, *a fortiori*, for any $t \in T - \{0\}$

$$F^t(x) = \{y \in M : g^t(x) = y\} = \{x\} , \tag{5.73}$$

so that

$$F(x) = \bigcup_{t \in T - \{0\}} F^t(x) = \{x\} . \tag{5.74}$$

If $F(x) = \{x\}$ then, by definition

$$\{x\} = \bigcup_{t \in T - \{0\}} F^t(x) = \bigcup_{t \in T - \{0\}} \{y \in M : g^t(x) = y\} , \tag{5.75}$$

so that necessarily, for any $t \in T - \{0\}$

$$g^t(x) = x; \tag{5.76}$$

moreover, by definition of a dynamical system

$$g^0(x) = x, \tag{5.77}$$

so that for any $t \in T$

$$g^t(x) = x; \tag{5.78}$$

hence, x is fixed. □

Corollary 5.17.1. *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$; for any $x \in M$, x is fixed if and only if $orb(x) = \{x\}$.*

Proof
Let $DS = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0 and let $x \in M$.

If x is fixed, then by Proposition 5.17,

$$F(x) = \{x\}\,; \tag{5.79}$$

hence, by definition of orbit of a point and by definition of future of a point,

$$orb(x) = \bigcup_{t \in T} \{y \in M : g^t(x) = y\} = F(x) \cup \{y \in M : g^0(x) = y\} = \{x\} \cup \{x\} = \{x\}\,. \tag{5.80}$$

If $orb(x) = \{x\}$ then, by definition

$$\{x\} = \{y \in M : \text{ for some } t \in T, g^t(x) = y\}\,, \tag{5.81}$$

so that necessarily, for any $t \in T$

$$g^t(x) = x. \tag{5.82}$$

□

Equivalently, fixed points may be understood as static points, or as the common invariants of all the state transitions of a dynamical system; accordingly, the orbit of a fixed point have as many periods as the number of durations (other than the identity element) entering the given time model.

Proposition 5.18. *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0. For any $x \in M$, if x is fixed than all $t \in T - \{0\}$ are periods of $orb(x)$.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0 and let $x \in M$ be fixed. By Definition 26, for any $t \in T - \{0\}$, $g^t(x) = x$ and therefore, by definition of period of an orbit, t is a period of $orb(x)$. □

Fixed points are those at which the evolution of a deterministic system stops, since after having reached a fixed point a dynamical system can undergo no further change. Symmetrically, gardens of Eden are primitive states, points with an empty past, from which the dynamics of a deterministic system starts.

Definition 27 (Garden of Eden)
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0; a point $x \in M$ is a garden of Eden if and only if, for any $y \in M$ and any $t \in T - \{0\}$

$$g^t(y) \neq x. \tag{5.83}$$

Proposition 5.19. *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system o a monoid $L = (T, +)$; for any $x \in M$, x is a garden of Eden if and only if $P(x) = \emptyset$.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system o a monoid $L = (T, +)$ with identity 0 and let $x \in M$.

If x is a garden of Eden then, by Definition 27,

$$P(x) = \{y \in M : \text{ for some } t \in T - \{0\}\,, g^t(y) = x)\} = \emptyset. \tag{5.84}$$

Conversely, if $P(x) = \emptyset$ then, for any $t \in T - \{0\}$, for any $y \in M$

$$g^t(y) \neq x; \tag{5.85}$$

and therefore, by Definition 27 x is a garden of Eden. □

In the next chapter, we shall see that, for this reason, possession of Gardens of Eden is strictly correlated to a special type of non-reversible behavior, which we shall refer to as possession of an *incomplete past.* Furthermore, possession of a Garden of Eden is invariant under ρ-isomorphism.

Proposition 5.20. *Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1 \in T_1})$ be a dynamical system on a monoid $L_1 = (T_1, +)$, let $DS_{L_2} = (M_2, (g^{t_2})_{t_2 \in T_2})$ be a dynamical system on a monoid $L_2 = (T_2, \oplus)$ and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}; for any $x_1 \in M_1$, x_1 is a garden of Eden if and only if $f(x_1)$ is.*

Proof
Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1 \in T_1})$ be a dynamical system on a monoid $L_1 = (T_1, +)$ with identity 0, let $DS_{L_2} = (M_2, (g^{t_2})_{t_2 \in T_2})$ be a dynamical system on a monoid $L_2 = (T_2, \oplus)$, let $\rho : T_1 \to T_2$ be a monoid isomorphism of L_1 in L_2 and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}. For any $x_1 \in M_1$, if x_1 is a garden of Eden then for any $t_1 \in T_1 - \{0\}$ and any $y_1 \in M_1$

$$g^{t_1}(y_1) \neq x_1 \tag{5.86}$$

and therefore, by bijectivity of f, for any $f(y_1) \in M_2$ and any $\rho(t_1) \in T_2 - \{\rho(0)\}$

$$g^{\rho(t_1)} f(y_1) = f(g^{t_1}(y_1)) \neq f(x_1); \tag{5.87}$$

hence, $f(x_1)$ is a garden of Eden. Proof in the converse direction is guaranteed by the fact that $f^{-1} : M_2 \to M_1$ is a ρ^{-1}-isomorphism of DS_{L_2} in DS_{L_1}. □

5.3

Dynamical Systems on Commutative Monoids

Before moving to an examination of the possible types of reversible behavior a dynamical system might display, it is important to notice that dynamical systems are extremely sensitive to the algebraic structure of their time models. Hence, enriching time models results in reducing the range of possible orbits dynamical systems can display and, as we shall see, the number of ways how they can reverse their evolution. This is particularly evident in the case of commutative time models.

Intuitively, one may think that whenever two distinct states y and z both belong to the orbit of a third, distinct point x, then the orbits of z and y must intersect at a point. However, this is not guaranteed in the case of dynamical systems on non-commutative monoids. The first advantage of providing a dynamical system with a commutative time model is precisely that of ruling out this possibility, so that no state may ever possess in its future two distinct points whose orbits are disjoint.

Proposition 5.21. *Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a commutative monoid $L = (T, +)$; for any three distinct $x, y, z \in M$, if $y \in orb(x)$ and $z \in orb(x)$, then $orb(z) \cap orb(y) \neq \emptyset$.*

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a commutative monoid $L = (T, +)$ and let $x, y, z \in M$ be three distinct states. If $y \in orb(x)$ and $z \in orb(x)$, then there exist $t, v \in T$ such that

$$g^t(x) = y \text{ and} \tag{5.88}$$

$$g^v(x) = z; \tag{5.89}$$

hence, by commutativity

$$g^t(z) = g^t(g^v(x)) = g^{t+v}(x) = g^{v+t}(x) = g^v(g^t(x)) = g^v(y), \tag{5.90}$$

so that $g^t(z) = g^v(y) \in orb(z)$ and $g^t(z) = g^v(y) \in orb(y)$, and therefore $orb(z) \cap orb(y) \neq \emptyset$ □

In addition, though one may expect that all points along a periodic orbit are periodic in their turn, this is similarly not guaranteed in the case of a dynamical systems on a non-commutative monoid. However, if the time model is commutative, then any period of a point is distributed on all the points lying on its trajectory.

Proposition 5.22. *Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a commutative monoid $L = (T, +)$ with identity 0; for any $x, y \in M$ and any $t \in T - \{0\}$, if $y \in orb(x)$ and t is a period of x, then t is a period of y.*

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a commutative monoid $L = (T, +)$ with identity 0 and let $x, y \in M$. If $y \in orb(x)$ then there must exist $v \in T$ such that

$$g^v(x) = y, \tag{5.91}$$

while, if $t \in T - \{0\}$ is a period of x,

$$g^t(x) = x; \tag{5.92}$$

hence,

$$g^t(y) = g^t(g^v(x)) = g^{t+v}(x) = g^{v+t}(x) = g^v(g^t(x)) = g^v(x) = y, \tag{5.93}$$

so that t is a period of y. □

Corollary 5.22.1. *Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a commutative monoid $L = (T, +)$; for any $x, y \in M$, if $orb(x)$ is periodic and $y \in orb(x)$, then $orb(y)$ is periodic.*

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a commutative monoid $L = (T, +)$ with identity 0 and let $x, y \in M$. If $orb(x)$ is periodic, then there exists $t \in T - \{0\}$ such that t is a period of x; as a consequence, by Proposition 5.22, t is a period of y. Hence, $orb(y)$ is periodic. □

Finally, if durations commute then points are aperiodic if and only if all their future images are aperiodic; as a further consequence, it follows that if two orbits are merging, then either is aperiodic if and only if the other is too.

Proposition 5.23. *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a commutative monoid $L = (T, +)$; for any $x, y \in M$ such that $y \in orb(x)$, if $orb(y)$ is aperiodic then $orb(x)$ is aperiodic.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a commutative monoid $L = (T; +)$ and let $x, y \in M$ such that $y \in orb(x)$. If $orb(y)$ is aperiodic, then it is neither periodic nor eventually periodic. According to Corollary 5.22.1, $orb(x)$ is not periodic and, therefore, $orb(x)$ is aperiodic if and only if it is not eventually periodic. So let us suppose, as a reductio, that some $z \in orb(x)$ was periodic. By Proposition 5.21, there would have existed $w \in orb(z)$ such that $w \in orb(y)$ and, by Corollary 5.22.1, $orb(w)$ was periodic. However, if that was the case, then $orb(y)$ would have been eventually periodic, contrary to the hypothesis. Hence, $orb(x)$ is aperiodic. □

Corollary 5.23.1. *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a commutative monoid $L = (T, +)$; for any $x, y \in M$ such that $y \in orb(x)$, $orb(x)$ is aperiodic if and only if $orb(y)$ is aperiodic.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a commutative monoid $L = (T; +)$ and let $x, y \in M$ such that $y \in orb(x)$. By Proposition 5.13, if $orb(x)$ is aperiodic then $orb(y)$ is aperiodic; on the other hand, by Proposition 5.23, if $orb(y)$ is aperiodic then $orb(x)$ is aperiodic. □

Corollary 5.23.2. *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a commutative monoid $L = (T, +)$; for any $x, y \in M$ whose orbits are merging, $orb(x)$ is aperiodic if and only if $orb(y)$ is aperiodic.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a commutative monoid $L = (T, +)$ and let $x, y \in M$. If $orb(x)$ and $orb(y)$ are merging, then by Definition 16 there exists $z \in M$ such that $z \in orb(x) \cap orb(y)$; that is, $z \in orb(x)$ and $z \in orb(y)$. As a consequence, by Corollary 5.23.1, $orb(x)$ is aperiodic if and only if $orb(z)$ is aperiodic, if and only if $orb(y)$ is aperiodic. □

6

Reversibile Dynamics

Intuitively speaking, a dynamical system is reversible if it is capable of recovering any of its past states; the way this goal is attained varies from case to case, depending on the logical properties of its state transitions, on the function which they consist of and, last but not least, on the algebraic properties of its time model. Therefore, moving from mathematical dynamical systems on (positive) integer or real time models to dynamical systems on monoids, we are allowed to distinguish among a cluster of different kinds of loosely speaking reversible dynamics which would otherwise get, at least partially, tangled. In the course of this chapter, we shall list six different kinds of reversible behavior – namely, *logical reversibility*, *complete past*, *complete logical reversibility*, *reversibility*, *strict reversibility* and *time invertibility.* For clearness of exposition, we shall subdivide them into two main categories, namely *proper* and *improper* types. These categories should not be thought of as being mutually exhaustive for, as we shall see, notions belonging to different categories may nevertheless be connected by the relation of logical consequence; rather, they are meant to underline the common features lying at the basis of each group of notions. Two further notions of improperly reversible dynamical behavior, namely *time symmetry* and *space invertibility*, will instead be the main focus of Chapter 8.

6.1

Improper Types of Reversibility

Improper types of reversible dynamics include logical reversibility, complete past and complete logical reversibility, which are defined according to the logical properties which the state transitions of a dynamical system might display.

Definition 28

A dynamical system $DS_L = (M, (g^t)_{t \in T})$ *on a monoid* $L = (T, +)$ *is* logically reversible *if and only if, for any* $t \in T$*,* g^t *is injective.*

Logical reversibility gives formal shape to the epistemic interpretation of reversibility: the past history of a logically reversible dynamical system is uniquely determined by the set of its state transitions, in the sense that any t-past set of whatever state is either empty or contains exactly one element; accordingly, knowledge of the whole past evolution of the system can be obtained just by knowing its operation, namely the set of all its state transitions, together with the actual state of the system.

Definition 28 is a straightforward generalization of Bennett's definition of logically reversible computational systems (Bennett, 1973) and Giunti's definition of logically reversible mathematical dynamical systems (Giunti, 1997, p. 29). Nevertheless, some properties of logically reversible mathematical dynamical systems might be lost in the course of that generalization: for example, those kind of systems typically have neither merging orbits nor eventually periodic orbits (Giunti, 1997, pp. 33-35), while this is not sure in the case of logically reversible dynamical systems whose time models are simple monoids.

Definition 29
A dynamical system $DS_L = (M, (g^t)_{t \in T})$ *on a monoid* $L = (T, +)$ *has* complete past *if and only if, for any* $t \in T$*,* g^t *is surjective.*

Complete past asserts that it is always possible to set a dynamical system so that it will reach a given state state in a chosen lapse of time; as a straightforward consequence,

Proposition 6.1. *Dynamical systems with complete past have no gardens of Eden.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system with complete past on a monoid $L = (T, +)$ with identity 0. If DS_L had a garden of Eden, then there would exist $x \in M$ such that, for any $t \in T - \{0\}$ and any $y \in M$

$$g^t(y) \neq x, \tag{6.1}$$

so that no g^t would be surjective, contrary to the hypothesis. Therefore, DS_L has no gardens of Eden. □

Symmetrically, as we already anticipated in the previous chapter, dynamical systems owning Gardens of Eden must have incomplete past history, in the sense that not all of their states have a non-empty past set[1]. However, the converse implication is not true, as it will be shown by Example 2.

Definition 30
A dynamical system $DS_L = (M, (g^t)_{t \in T})$ *on a monoid* $L = (T, +)$ *is* completely logically reversible *if and only if, for any* $t \in T$*,* g^t *is bijective.*

By definition, completely logical reversible dynamical systems are both logically reversible and with complete past; accordingly, complete logical reversibility may be understood as claiming that for any state y and duration t it is possible to choose a unique state x such that the system will move from x to y in a chosen lapse of time.

[1] See § 5.2.2.1

All improper types of reversibility considered so far are preserved by ρ-isomorphism.

Proposition 6.2. *Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1 \in T_1})$ be a dynamical system on a monoid $L_1 = (T_1, +)$, let $DS_{L_2} = (M_2, (g^{t_2})_{t_2 \in T_2})$ be a dynamical system on a monoid $L_2 = (T_2, \oplus)$ and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}. Then:*

6.2.1. *DS_{L_1} is logically reversible if and only if DS_{L_2} is;*

6.2.2. *DS_{L_1} has complete past if and only if DS_{L_2} has;*

6.2.3. *DS_{L_1} is completely logically reversible if and only if DS_{L_2} is.*

Proof
Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1 \in T_1})$ and $DS_{L_2} = (M_2, (g^{t_2})_{t_2 \in T_2})$ be dynamical systems on $L_1 = (T_1, +)$ and $L_2 = (T_2, \oplus)$ respectively, let $\rho : T_1 \to T_2$ be a monoid isomorphism of L_1 in L_2 and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}. Hence:

- Let $x_2, y_2 \in M_2$ and $t_2 \in T_2$. By bijectivity of f and ρ, we are guaranteed that, for some $x_1, y_1 \in M_1$ and some $t_1 \in T_1$, $x_2 = f(x_1)$, $y_2 = f(y_1)$ and $t_2 = \rho(t_1)$; so if DS_{L_1} is logically reversible then

$$\begin{aligned} g^{t_2}(x_2) &= g^{t_2}(y_2) \\ g^{\rho(t_1)}(f(x_1)) &= g^{\rho(t_1)}(f(y_1)) \\ f(g^{t_1}(x_1)) &= f(g^{t_1}(y_1)) \\ g^{t_1}(x_1) &= g^{t_1}(y_1) \\ x_1 &= y_1 \\ f(x_1) &= f(y_1) \\ x_2 &= y_2, \end{aligned} \tag{6.2}$$

  which makes DS_{L_2} logically reversible.

- Let $x_2 \in M_2$ and $t_2 \in T_2$. By bijectivity of f and ρ, $x_2 = f(x_1)$ for some $x_1 \in M_1$ and $t_2 = \rho(t_1)$ for some $t_1 \in T_1$. Hence, if DS_{L_1} has complete past, there exists $y_1 \in M_1$ and $y_2 = f(y_1) \in M_2$ such that

$$g^{t_1}(y_1) = x_1 \tag{6.3}$$

$$x_2 = f(x_1) = f(g^{t_1}(y_1)) = g^{\rho(t_1)}(f(y_1)) = g^{t_2}(y_2); \tag{6.4}$$

  accordingly, DS_{L_2} has complete past.

- If DS_{L_1} is completely logically reversible, then it is logically reversible and with complete past. Hence, since both logical reversibility and complete past are preserved by isomorphism, DS_{L_2} is logically reversible and with complete past in its turn, i.e. DS_{L_2} is completely logically reversible.

In all cases, proof in the converse direction is guaranteed by the fact that f^{-1} is a ρ^{-1}-isomorphism of DS_{L_2} in DS_{L_1}. □

6.2

Proper Types of Reversibility

The improper types of reversible dynamics we examined so far either express the possibility of recovering the whole past history of an arbitrary state from knowledge of the state alone, or the existence of a history that arbitrarily stretches into the past, or both. Proper types of reversible dynamical behavior, instead, refer to the capability a dynamical system has to get back to any of its states *dynamically*, namely by means of its sole state transitions.

6.2.1 Reversibility

The weakest and basic kind of properly reversible dynamics is *reversibility*:

Definition 31 (Reversibility)
A dynamical system $DS_L = (M, (g^t)_{t \in T})$ on a monoid $L = (T, +)$ is called reversible *if and only if, for any $x \in M$ and any $t \in T$, there exists $r \in T$ such that*

$$g^r(g^t(x)) = x. \tag{6.5}$$

It is easy to see that reversibility is a dynamical property, for it is both a specific property of dynamical systems and preserved by ρ-isomorphism:

Proposition 6.3. *Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1 \in T_1})$ be a dynamical system on a monoid $L_1 = (T_1, +)$, let $DS_{L_2} = (M_2, (g^{t_2})_{t_2 \in T_2})$ be a dynamical system on a monoid $L_2 = (T_2, \oplus)$ and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}. Then DS_{L_1} is reversible if and only if DS_{L_2} is.*

Proof
Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1 \in T_1})$ and $DS_{L_2} = (M_2, (g^{t_2})_{t_2 \in T_2})$ be dynamical systems on $L_1 = (T_1, +)$ and $L_2 = (T_2, \oplus)$ respectively, let $\rho : T_1 \to T_2$ be a monoid isomorphism of L_1 in L_2 and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}. If DS_{L_1} is reversible then for any $t_1 \in T_1$ and any $x_1 \in M_1$ there exists $v_1 \in T_1$ such that

$$g^{v_1}(g^{t_1}(x_1)) = g^{v_1+t_1}(x_1) = x_1. \tag{6.6}$$

Hence, for any $x_2 \in M_2$ and $t_2 \in T_2$, if

$$x_2 = f(x_1) \tag{6.7}$$
$$t_2 = \rho(t_1) \tag{6.8}$$

there exists $v_2 = \rho(v_1) \in T_2$ such that

$$\begin{aligned} f(x_1) &= f(g^{v_1+t_1}(x_1)) \\ &= g^{\rho(v_1+t_1)}(f(x_1)) \\ &= g^{\rho(v_1) \oplus \rho(t_1)}(f(x_1)) \\ &= g^{\rho(v_1)}(g^{\rho(t_1)}(f(x_1))). \end{aligned} \tag{6.9}$$

On the other hand, by bijectivity of f and ρ, for any $x_2 \in M_2$ and $t_2 \in T_2$ there always exist $x_1 \in M_1$ and $t_1 \in T_1$ such that x_2 and t_2 respectively satisfy conditions (6.7) and (6.8); accordingly, DS_{L_2} is reversible. Finally, proof in the converse direction is guaranteed by the fact that $f^{-1} : M_2 \to M_1$ is a ρ^{-1}-isomorphism of DS_{L_2} in DS_{L_1}. □

Reversibility, as just defined, is a weaker notion than that proposed by Giunti (1997), according to which reversible dynamical systems are those whose time models consist of the integer, rational or real numbers, together with arithmetical addition. In our sense, a dynamical system is reversible if and only if capable to recover its initial state after having undergone a state transition of whatever duration; equivalently, we may characterize reversible dynamical systems as those systems whose transition graphs are *invertible*, in the sense that for any arrow connecting two nodes there exists an arrow connecting them in the opposite direction, or as those systems whose past and future possible histories coincide.

Proposition 6.4. *A dynamical system $DS_L = (M, (g^t)_{t \in T})$ on a monoid $L = (T, +)$ is reversible if and only if, for any $x \in M$*

$$P(x) = F(x). \tag{6.10}$$

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ and let $x \in M$.

If DS_L is reversible, then

- For any $y \in F(x)$ there exist $t \in T - \{0\}$ and $r \in T$ such that

$$g^t(x) = y \tag{6.11}$$
$$g^r(y) = g^r(g^t(x)) = x. \tag{6.12}$$

  If $r \neq 0$, then obviously $y \in P(x)$; if $r = 0$, then $y = x$, so that $g^t(y) = x$ and $y \in P(x)$.
- For any $y \in P(x)$ there exists $t \in T - \{0\}$ and $r \in T$ such that

$$g^t(y) = x \tag{6.13}$$
$$y = g^r(g^t(y)) = g^r(x). \tag{6.14}$$

  If $r \neq 0$, then plainly $y \in F(x)$; if $r \neq 0$, then $y = x$, so that $g^t(y) = x$ and $y \in F(x)$.

As a consequence, $F(x) \subseteq P(x)$ and $P(x) \subseteq F(y)$, which is the same as

$$F(x) = P(x). \tag{6.15}$$

If $F(x) = P(x)$, then for any $t \in T - \{0\}$ there exist $y \in P(x)$ and $r \in T$ such that

$$g^t(x) = y \tag{6.16}$$
$$x = g^r(y) = g^r(g^t(x)). \tag{6.17}$$

On the other hand, for $t = 0$, condition (6.5) holds trivially. As a consequence, DS_L is reversible. □

Corollary 6.4.1. *Reversible dynamical systems have no Gardens of Eden.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ and let $x \in M$. If DS_L is reversible then, by Proposition 6.4, $P(x) = F(x)$. On the other hand, by Definition 23, $F(x)$ is necessarily non-empty, so that $P(x)$ must be non-empty in its turn; by Proposition 5.19, x is therefore not a Garden of Eden. Finally, since x was chosen arbitrarily, DS_L can have no Gardens of Eden at all. □

In the course of the preceding chapter, we had the chance to remark that a strict connection exists between the types of orbits a dynamical system may possess and the type of reversible behavior it may possibly display. The following statements confirm this claim:

Proposition 6.5. *A dynamical system $DS_L = (M, (g^t)_{t\in T})$ on a monoid $L = (T, +)$ is reversible if and only if, for any $x, y \in M$, either $orb(x) \cap orb(y) = \emptyset$ or $orb(y) = orb(x)$.*

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$.

If DS_L is reversible then, for any $y, x \in M$, if $y \in orb(x)$ there exist $t \in T$ and $r \in T - \{0\}$ such that

$$g^t(x) = y \tag{6.18}$$
$$g^r(y) = g^r(g^t(x)) = x, \tag{6.19}$$

so that $x \in orb(x)$; accordingly, by Proposition 5.7, $orb(x) \subseteq orb(y)$ and $orb(y) \subseteq orb(x)$, and therefore $orb(x) = orb(y)$. As a consequence, for any $x, y \in M$, either $orb(x) \cap orb(y) = \emptyset$ or there exists $z \in M$ such that $z \in orb(x)$, $z \in orb(y)$ and, given the above result, $orb(x) = orb(z) = orb(y)$.

If DS_L is not reversible, then there must exist $x, y \in M$ and $t \in T$ such that, for any $r \in T$

$$g^t(x) = y \tag{6.20}$$
$$g^r(y) = g^r(g^t(x)) \neq x, \tag{6.21}$$

so that $orb(x) \neq orb(y)$ and, since $y \in orb(x)$, $orb(x) \cap orb(y) \neq \emptyset$, □

Points belonging to a reversible dynamical system thus either share their orbits entirely, or they possess no common past or future image at all; as a consequence, reversible dynamical systems possess no merging orbits.

Corollary 6.5.1. *Reversible dynamical systems have no merging orbits.*

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$. If DS_L is reversible then, by Proposition 6.5, there exist no $x, y \in M$ such that $orb(x) \nsubseteq orb(y)$, $orb(y) \nsubseteq orb(x)$ and $orb(x) \cap orb(y) \neq \emptyset$; hence, DS_L has no merging orbits. □

Reversibility is logically independent of logical reversibility, as well as of complete past. Example 2 shows that reversible dynamical systems exist which are neither logically reversible nor with complete past; conversely, Example 3 will show that complete logical reversibility does not imply reversibility.

Example 2 (Reversible but Not Logically Reversible Dynamical System with Incomplete Past on a Non-commutative Monoid[2])
Let $L = (T, +)$, where $T = \{0, 1, 2, 3\}$ and the sum operation $+$ is defined by table 6.1:

In addition, let $M = \{x_i\}_{i=1}^4$ and, for any $t \in T$ and any $x_i \in M$, let $g^t(x_i)$ be defined by the following table:

Then:

- L is a non-commutative monoid: $+$ is a non-commutative, associative binary operation on T with identity 0: see Table 6.1;
- $DS_L = (M, (g^t)_{t\in T})$ is a dynamical system on the non-commutative monoid L:
    1. M is a non-emtpy set: by hypothesis,

[2]This example was suggested to me by Prof. Marco Giunti as a personal communication.

+	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	1	2	3
3	3	1	2	3

Table 6.1: The sum operation +. Read *column+row*, as reading order matters.

	g^0	g^1	g^2	g^3
x_1	x_1	x_2	x_1	x_3
x_2	x_2	x_2	x_1	x_3
x_3	x_3	x_2	x_1	x_3

Table 6.2: The t-advances family $(g^t)_{t\in T}$.

2. $(g^t)_{t\in T}$ is a family of functions on M, indexed by T: see Table 6.2,
3. for any $x_i \in M$ and any $t, v \in T$, conditions (5.8) and 5.9 hold: see Table 6.2

- DS_L is reversible: see Table 6.2;
- DS_L is not logically reversible: for all $t \in T - \{0\}$, g^t is not injective: see Table 6.2;
- DS_L has incomplete past: for all $t \in T - \{0\}$, g^t is not surjective: see Table 6.2.

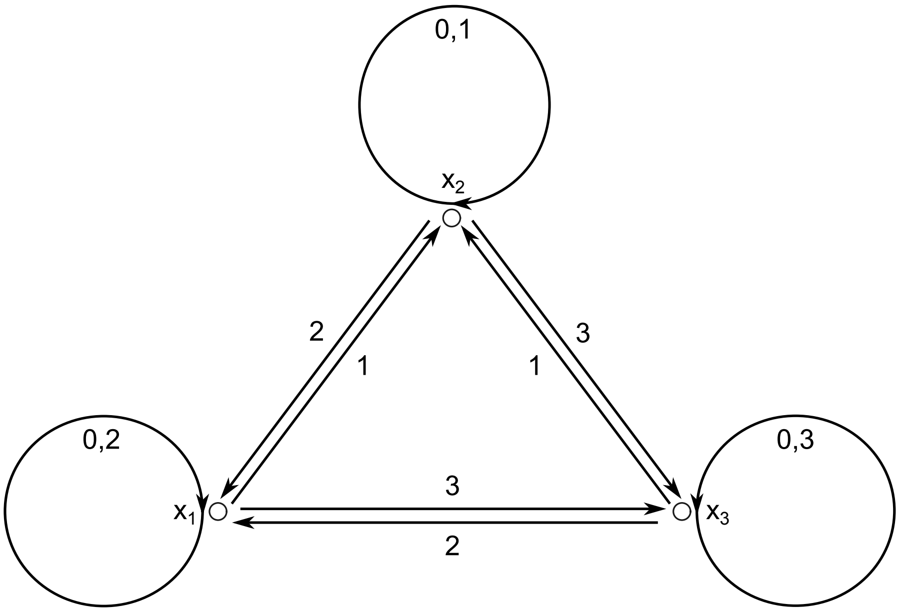

Figure 6.1: Dynamical System Described in Example 2.

The following is an example of a system that is completely logically reversible but not reversible.

Example 3 (Completely Logically Reversible and Not Reversible Dynamical System)
For any integer $x \in \mathbb{Z}$ and any non-negative integer $n \in \mathbb{Z}^+$, let $g^n(x)$ be the n-th successor of x. Then:

- $DS_L = (\mathbb{Z}, (g^n)_{n\in\mathbb{Z}^+})$ is a dynamical system on $L = (\mathbb{Z}^+, +)$:

1. $\mathbb{Z}$ is a non-empty set;
2. $(g^n)_{n\in\mathbb{Z}^+}$ is a family of functions on $\mathbb{Z}$, indexed by $\mathbb{Z}^+$;
3. for any $x \in \mathbb{Z}$ and any $n, m \in \mathbb{Z}^+$

$$g^0(x) = x + 0 = x \tag{6.22}$$
$$g^{n+m}(x) = x + (n+m) = (x+n) + m = g^m(g^n(x)). \tag{6.23}$$

- DS_L is completely logically reversible: for any $x \in \mathbb{Z}$, for any $n \in \mathbb{Z}^+$, x is the n-th successor of one and only one integer number $y \in \mathbb{Z}^+$.
- DS_L is not reversible: for any $x \in \mathbb{Z}$, for any $n, m \in \mathbb{Z}^+ - \{0\}$

$$g^m(g^n(x)) = g^{n+m}(x) = x + (n+m) \neq x. \tag{6.24}$$

6.2.2 Strict Reversibility

Reversibility only demands that, for any state x and any state transition of duration t, a reversed state transition mapping $g^t(x)$ back to x exists; however, in no way reversibility requires the reversed state transitions to be the same for all x. If that is the case, we say that the system is *strictly reversible.*

Definition 32 (Strict Reversibility)
A dynamical system $DS_L = (M, (g^t)_{t\in T})$ on a monoid $L = (T, +)$ is strictly reversible *if and only if, for any $t \in T$ there exists $r \in T$ such that, for any $x \in M$*

$$g^r(g^t(x)) = x. \tag{6.25}$$

Proposition 6.6. *Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1\in T_1})$ be a dynamical system on a monoid $L_1 = (T_1, +)$, let $DS_{L_2} = (M_2, (g^{t_2})_{t_2\in T_2})$ be a dynamical system on a monoid $L_2 = (T_2, \oplus)$ and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}. Then DS_{L_1} is strictly reversible if and only if DS_{L_2} is.*

Proof
Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1\in T_1})$ and $DS_{L_2} = (M_2, (g^{t_2})_{t_2\in T_2})$ be dynamical systems on $L_1 = (T_1, +)$ and $L_2 = (T_2, \oplus)$ respectively, let $\rho : T_1 \to T_2$ be a monoid isomorphism of L_1 in L_2 and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}. If DS_{L_1} is strictly reversible, then for any $t_1 \in T_1$ there exists $v_1 \in T_1$ such that, for any $x_1 \in M_1$

$$g^{v_1}(g^{t_1}(x_1)) = g^{v_1+t_1}(x_1) = x_1 \tag{6.26}$$

and

$$\begin{aligned} f(x_1) &= f(g^{v_1+t_1}(x_1)) \\ &= g^{\rho(v_1+t_1)}(f(x_1)) \\ &= g^{\rho(v_1)\oplus\rho(t_1)}(f(x_1)) \\ &= g^{\rho(v_1)}(g^{\rho(t_1)}(f(x_1))). \end{aligned} \tag{6.27}$$

On the other hand, by bijectivity of ρ and f, for any $t_2 \in T_2$ and any $x_2 \in M_2$ there exist $t_1 \in T_1$ and $x_1 \in M_1$ such that $t_2 = \rho(t_1)$ and $x_2 = f(x_1)$; accordingly, for any $t_2 \in T_2$ there exists $v_2 = \rho(v_1) \in T_2$ such that, for any $x_2 \in M_2$,

$$g^{v_2}(g^{t_2}(x_2)) = g^{\rho(v_1)}(g^{\rho(t_1)}(f(x_1))) = f(x_1) = x_2, \tag{6.28}$$

which makes DS_{L_2} strictly reversible. Proof in the converse direction is guaranteed by the fact that $f^{-1}: M_2 \to M_1$ is a ρ^{-1}-isomorphism of DS_{L_2} in DS_{L_1}. □

Proposition 6.6 shows that strict reversibility, as well as reversiblity, is a dynamical property. This should be no surprise, because strictly reversible dynamical systems are, by definition, reversible.

Proposition 6.7. *All strictly reversible dynamical systems are reversible.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$; if DS_L is strictly reversible, then for any $t \in T$ there exists $r \in T$ satisfying (6.25); as a consequence, by *dicto de omni*, for any $t \in T$ and any $x \in M$ there exists $r \in T$ satisfying (6.5). Hence, DS_L is reversible. □

While reversible dynamical systems may fail to be logically reversible, strict reversibility essentially demands logical reversibility. In fact, if a state transition of whatever duration t mapped different states x and z into a unique image y, then it could not be the case that the same state transition of duration r could lead y back to both x and z:

Proposition 6.8. *All strictly reversible dynamical systems are logically reversible.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a strictly reversible dynamical system o a monoid $L = (T, +)$; for any $t \in T$, if for some $x, y \in M$

$$g^t(x) = g^t(y); \tag{6.29}$$

then, by strict reversibility, there must exist $r \in T$ such that

$$x = g^r(g^t(x)) = g^r(g^t(y)) = y; \tag{6.30}$$

hence, g^t must be injective. As a consequence, DS_L is logically reversible. □

In addition, in virtue of their very logical reversibility, strictly reversible dynamical systems are also endowed with complete past and, therefore, with complete logical reversibility.

Corollary 6.8.1. *All strictly reversible dynamical systems have complete past.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a strictly reversible dynamical system o a monoid $L = (T, +)$ and let us suppose, as a reductio, that DS_L has incomplete past; then, by hypothesis, there must exist $t \in T$ and $x \in M$ such that

$$x \notin g^t(M), \tag{6.31}$$

where $g^t(M)$ is meant to denote the image set of g^t. On the other hand, by strict reversibility, there must exist $r \in T$ such that

$$g^r(g^t(M)) = M, \tag{6.32}$$

while necessarily

$$g^r(x) \in M. \tag{6.33}$$

Hence, there must exist $y \in g^t(M)$ such that

$$g^r(y) = g^r(x). \tag{6.34}$$

By Proposition 6.8, g^r must be injective, so that

$$x = y \tag{6.35}$$

and thus

$$x \in g^t(M), \tag{6.36}$$

contrary to the hypothesis. Hence, all state transitions of DS_L are surjective, and DS_L has complete past. □

Corollary 6.8.2. *All strictly reversible dynamical systems are completely logically reversible.*

Proof
Let DS_L be a strictly reversible dynamical system on a monoid L. By Proposition 6.8, DS_L is logically reversible, while by Corollary 6.8.1, it has complete past. As a consequence, DS_L is completely logically reversible. □

Thanks to complete logical reversibility, all the state transitions of a strictly reversible dynamical system possess an inverse function. This property supports the following statement:

Lemma 6.1. *Let $DS_L = (M, (g^t)_{t\in T})$ be a strictly reversible dynamical system on a monoid $L = (T, +)$ and let $t, r \in T$. If, for any $x \in M$*

$$g^r(g^t(x)) = x, \tag{6.37}$$

then

$$g^r = (g^t)^{-1}. \tag{6.38}$$

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$. If DS_L is strictly reversible then, by Corollary 6.8.2, it is completely logically reversible. Hence, for any $t \in T$, g^t has an inverse function $(g^t)^{-1}$. So let $r, t \in T$ satisfy (6.37) for any $x \in M$: if $g^r \neq (g^t)^{-1}$ then there would exist $x \in M$ such that

$$g^r(x) \neq (g^t)^{-1}(x); \tag{6.39}$$

however, since DS_L is completely logically reversible, g^t must be surjective and therefore, for some $y \in M$ such that $y = g^t(x)$,

$$g^r(g^t(y)) \neq (g^t)^{-1}(g^t(y)) = y, \tag{6.40}$$

against the hypothesis. Hence, if $r, t \in T$ satisfy (6.37) for any $x \in M$, then $g^r = (g^t)^{-1}$. □

On the other hand, any the state transition in a strictly reversible dynamical system comes equipped with an inverse state transition satisfying condition (6.37) for all possible states. For this reason, strictly reversible dynamical systems may equivalently be characterized as follows:

Proposition 6.9. *A dynamical system $DS_L = (M, (g^t)_{t\in T})$ on a monoid $L = (T, +)$ is strictly reversible if and only if for any $t \in T$ there exists $r \in T$ such that*

$$g^r = (g^t)^{-1}. \tag{6.41}$$

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$. By Definition 32 and Lemma 6.1, DS_L is strictly reversible if and only if for any $t \in T$ there exists $r \in T$ satisfying (6.37) for any $x \in M$; on the other hand, by Lemma 6.1, for any such r it must be $g^r = (g^t)^{-1}$. □

Corollary 6.9.1. *A dynamical system $DS_L = (M, (g^t)_{t\in T})$ on a monoid $L = (T, +)$ is strictly reversible if and only if for any $t \in T$ there exists $r \in T$ such that, for any $x \in M$*

$$g^t(g^r(x)) = x. \tag{6.42}$$

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$; by Proposition 6.9, DS_L is strictly reversible if and only if, for any $t \in T$, there exists $r \in T$ such that $g^r = (g^t)^{-1}$, which obtains if and only if, for any $x \in M$

$$g^t(g^r(x)) = g^t((g^t)^{-1}(x)) = x. \tag{6.43}$$

□

6.2.3 Time Invertibility

The strongest type of properly reversible dynamics we shall define is *time invertibility.*

Definition 33 (Time Invertibility)
A dynamical system $DS_L = (M, (g^t)_{t\in T})$ on a monoid $L = (T; +)$ is time-invertible *if and only if L is a group.*

Time invertibility is the clearest example of how the type of reversibility displayed by a dynamical system depends on the algebraic features of its time model, for these features explicitly enter its definition. Just like reversibility and strict reversibility, time invertibility is also a dynamical property.

Lemma 6.2. *Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1\in T_1})$ be a dynamical system on a monoid $L_1 = (T_1, +)$, let $DS_{L_2} = (M_2, (g^{t_2})_{t_2\in T_2})$ be a dynamical system on a monoid $L_2 = (T_2, \oplus)$ and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}. Then DS_{L_1} is time invertible if and only if DS_{L_2} is.*

Proof
Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1\in T_1})$ and $DS_{L_2} = (M_2, (g^{t_2})_{t_2\in T_2})$ be dynamical systems on $L_1 = (T_1, +)$ and $L_2 = (T_2, \oplus)$ respectively, let $\rho : T_1 \to T_2$ be a monoid isomorphism of L_1 in L_2 and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}. If DS_{L_1} is time-invertible, then ρ is a group isomorphism between L_1 and L_2 and therefore L_2 is a group in its turn. As a consequence, DS_{L_1} is time-invertible. Proof in the converse direction is guaranteed by the fact that $f^{-1} : M_2 \to M_1$ is a ρ^{-1}-isomorphism of DS_{L_2} in DS_{L_1}. □

In the light of Definition 33, time-invertible dynamical systems may be easily proved to be strictly reversible and hence reversible and completely logically reversible.

Proposition 6.10. *Time-invertible dynamical systems are strictly reversible.*

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be dynamical system on a monoid $L = (T, +)$ with identity 0; if DS_L is time invertible, then for any $t \in T$ there exists $-t \in T$ such that, for any $x \in M$

$$g^{-t}(g^t(x)) = g^{-t+t}(x) = g^0(x) = x; \tag{6.44}$$

hence, DS_L is strictly reversible. □

Corollary 6.10.1. *Time-invertible dynamical systems are reversible.*

Proof
By Proposition 6.10, time-invertible dynamical systems are strictly reversible; as a consequence, by Proposition 6.7, they are also reversible. □

Corollary 6.10.2. *Let DS_L be a dynamical system on a monoid L; if DS_L is time-invertible, then it is completely logically reversible.*

Proof
Let DS_L be a dynamical system on a monoid L; if DS_L is time-invertible then, by Proposition 6.10, it is strictly reversible. Hence, by Corollary 6.8.2, it is completely logically reversible. □

Strict reversibility is nevertheless not logically equivalent to time invertibility, as shown by the following example.

Example 4 (Strictly Reversible and Not Time-Invertible Dynamical System)
Let $M = \{x_i\}_{i=1}^{2}$, let $L = (\mathbb{Z}^+, +)$ be the set of non-negative integers along with arithmetical addition and, for any $n \in \mathbb{Z}^+$ and any $x_i \in M$, let us assume that

$$\text{if } n \text{ is even, then} \qquad g^n(x_i) = x_i, \tag{6.45}$$

$$\text{if } n \text{ is odd, then} \qquad g^n(x_i) = x_j, \tag{6.46}$$

taking for granted that, in all cases, $i \neq j$. Then:

- $L = (\mathbb{Z}^+, +)$ is a monoid with identity 0;
- $DS_L = (M, (g^n)_{n\in\mathbb{Z}^+})$ is a dynamical system on L: by hypothesis, M is not empty and $(g^n)_{n\in\mathbb{Z}^+}$ is a family of functions on M, indexed by $\mathbb{Z}^+$. Furthermore, for any $x_i \in M$,

$$g^0(x_i) = x_i \tag{6.47}$$

  and, for any $x_i \in M$, for any $n, m \in \mathbb{Z}^+$,

  ◦ if n is even and m is even, then $n + m$ is even and

$$g^{n+m}(x_i) = x_i = g^m(x_i) = g^n(g^m(x_i)), \tag{6.48}$$

  ◦ if n is even and m is odd, then $n + m$ is odd and

$$g^{n+m}(x_i) = x_j = g^m(x_i) = g^n(g^m(x_i)), \tag{6.49}$$

  ◦ if n is odd and m is even, then $n + m$ is odd and

$$g^{n+m}(x_i) = x_j = g^n(x_i) = g^n(g^m(x_i)), \tag{6.50}$$

  ◦ if n is odd and m is odd, then $n + m$ is even and

$$g^{n+m}(x_i) = x_i = g^m(x_j) = g^n(g^m(x_i)). \tag{6.51}$$

- DS_L is strictly reversible: for any $n \in \mathbb{Z}^+$ there exists $m \in \mathbb{Z}^+$ such that, for any $x_i \in M$

  ◦ if n is even, m is even, so that then $n + m$ is even and

$$g^{m+n}(x_i) = g^m(g^n(x_i)) = x_i \tag{6.52}$$

◦ if n is odd, m is odd, so that $n + m$ is even and

$$g^{m+n}(x_i) = g^m(g^n(x_i)) = x_i. \tag{6.53}$$

- DS_L is not time-invertible, since $\mathbb{Z}^+$ is not a group.

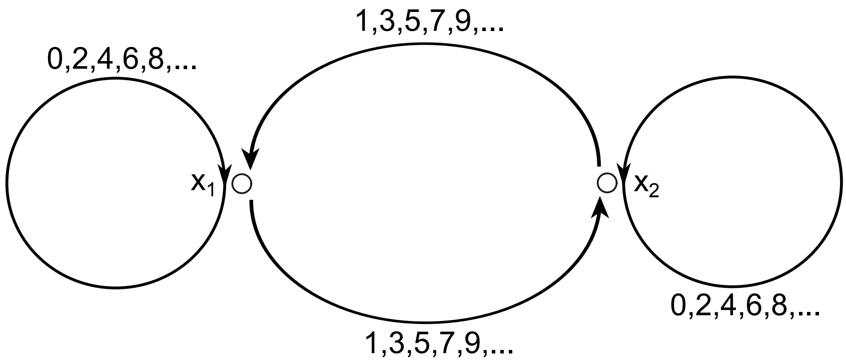

Figure 6.2: Dynamical System Described in Example 4.

Nonetheless the following, weaker implication holds:

Proposition 6.11. *Let $DS_L = (M, (g^t)_{t\in T})$ be a strictly reversible dynamical system on a monoid $L = (T, +)$; if the family $(g^t)_{t\in T}$ is injective, DS_L is time-invertible.*

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a strictly reversible dynamical system on a monoid $L = (T, +)$ with identity 0 and let $g : T \to M^M$ be the indexed family[3] $(g^t)_{t\in T}$. By Proposition 6.9 and Corollary 6.9.1, for any $t \in T$ there exists $r \in T$ such that

$$g(0) = g^0 = (g^t)^{-1} \circ g^t = g^r \circ g^t = g^{r+t} = g(r + t), \tag{6.54}$$
$$g(0) = g^0 = g^t \circ (g^t)^{-1} = g^t \circ g^r = g^{t+r} = g(t + r); \tag{6.55}$$

and therefore, if g is injective,

$$0 = g^{-1}(g(0)) = g^{-1}(g(r + t)) = r + t \tag{6.56}$$
$$0 = g^{-1}(g(0)) = g^{-1}(g(t + r)) = t + r, \tag{6.57}$$

so that r is an inverse of t. Since L is a monoid, r is unique; hence, L is a group and, by Definition 33, DS_L is time-invertible. □

On the other hand, it may be the case that the family of state transitions of a time-invertible dynamical system is not injective. For, otherwise, Proposition 6.10 and Proposition 6.11 would jointly make time invertibility collapse on strict reversibility, contrary to what Example 4 showed. Nonetheless, to be sure that the family of a time-invertible dynamical system DS_L is injective, one only needs to examine whether DS_L has at least one non-periodic orbit, or else that there

[3] Let us recall that, in general, by a family $(f^x)_{x\in X}$ of elements of a given set Y, indexed by X, we simply mean a function $f : X \to Y$.

is no period which is common to all orbits. Given Proposition 6.10, this property obtains as a direct consequence of the following:

Proposition 6.12. *Let $DS_L = (M, (g^t)_{t\in T})$ be a strictly reversible dynamical system on a monoid $L = (T, +)$; if the family $(g^t)_{t\in T}$ is not injective, then all orbits of DS_L share a common period.*

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a strictly reversible dynamical system on a monoid $L = (T, +)$ with identity 0 and let $g : T \to M^M$ be the family $(g^t)_{t\in T}$. If g is not injective, then there must exist $t, v \in T$ such that

$$g(t) = g^t = g^v = g(v) \qquad t \neq v. \tag{6.58}$$

On the other hand, by Proposition 6.9, for any $t, v \in T$ there must exist $r, u \in T$ such that

$$g^r = (g^t)^{-1} \tag{6.59}$$
$$g^u = (g^v)^{-1} \tag{6.60}$$

so that

$$\begin{aligned} g^{r+t} = g^r \circ g^t = (g^t)^{-1} \circ g^t &= g^0 \\ &= (g^t)^{-1} \circ g^v = g^r \circ g^v = g^{r+v} \\ &= (g^v)^{-1} \circ g^v = g^u \circ g^v = g^{u+v} \\ &= (g^v)^{-1} \circ g^t = g^u \circ g^t = g^{u+t} \\ = g^t \circ (g^t)^{-1} &= g^v \circ (g^t)^{-1} = g^v \circ g^r = g^{v+r} \\ &= g^v \circ (g^v)^{-1} = g^v \circ g^u = g^{v+u} \\ &= g^t \circ (g^v)^{-1} = g^t \circ g^u = g^{t+u}. \end{aligned} \tag{6.61}$$

Now let $i \in \{t, v\}$ and $j \in \{r, u\}$: if t, v, r and u where such that all possible sums of the form $i + j$ or $j + i$ were equal to 0, then by (6.58) we would get, for example,

$$t + (r + v) = t + 0 = t \neq v = 0 + v = (t + r) + v, \tag{6.62}$$

violating the associativity of $+$. Hence, at least one sum of the form $i + j$ or $j + i$ is different from 0. So let $s \in T$ be any such sum. Then, by the previous equalities (6.61), for any $x \in M$

$$g^s(x) = g^0(x) = x \qquad s \neq 0, \tag{6.63}$$

so that, by Definition 19, s is a period of all $x \in M$. □

6.3

Stronger Time Models

Except for those induced by transitivity, all the possible logical relationships which may obtain among the six different types of (proper and improper) reversible dynamics we defined so far are synthesized by Figure 6.3. Still, we have already remarked[4] that dynamical systems are very

[4] See § 5.3.

sensitive to the algebraic structure of time models, for enriching the latter results in restricting the possible types of orbits a dynamical system can display. Since there is a deep connection between the types of orbits which a dynamical system can have and its kind of loosely speaking reversibility, it is reasonable to expect that richer time models demand stronger forms of reversible dynamics. Just as before, dynamical systems on commutative monoids are a crystal clear case in this respect; the first part of this section is thus dedicated to show how reversibility may be affected by commutative time models. In the second part, we shall instead focus on the consequence the algebraic property of regularity may have on strictly reversible dynamical systems.

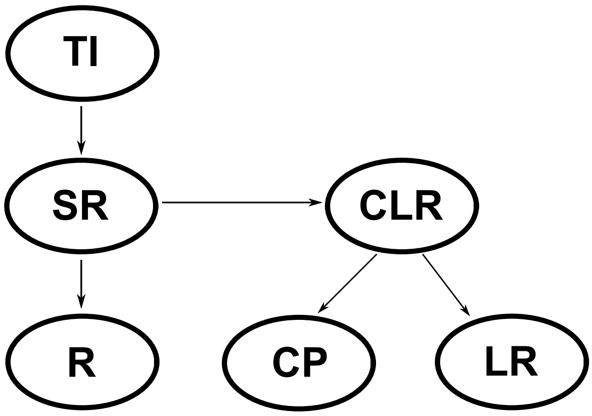

Figure 6.3: Logical relations among different types of reversible behavior.

6.3.1 Reversible Dynamics and Commutative Time Models

In introducing logical reversibility, we had the chance to notice that logically reversible dynamical systems may happen to possess eventually periodic orbits[5]; however, this possibility is ruled out in case the given time model is commutative.

Proposition 6.13. *Let $DS_L = (M, (g^t)_{t \in T})$ be dynamical system on a commutative monoid; if DS_L is logically reversible, it has no eventually periodic orbits.*

Proof
Proof of Proposition 6.13 is a straightforward generalization of that given by (Giunti, 1997, p. 33) for dynamical systems on (positive) real or (positive) integer time models. □

Similarly, reversibility *per se* implies neither logical reversibility nor complete past[6]; however, reversible dynamical systems on commutative monoids are always completely logically reversible too.

Proposition 6.14. *Let DS_L be a dynamical system on a commutative monoid $L = (T, +)$; if DS_L is reversible, then it is completely logically reversible.*

[5] See § 6.1.
[6] See Example 2

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a reversible dynamical system on the commutative monoid $L = (T, +)$.

Then, for any $t \in T$, g^t is injective: if not, for some $x, z \in M$:

$$g^t(x) = g^t(z) = y \qquad x \neq z. \tag{6.64}$$

By reversibility, there should exist $w, v \in T$ (not necessarily distinct) such that:

$$g^w(y) = x \tag{6.65}$$
$$g^v(y) = z, \tag{6.66}$$

which, by substitution, lead to

$$g^w(g^t(x)) = g^w(g^t(z)) = x \tag{6.67}$$
$$g^v(g^t(z)) = g^v(g^t(x)) = z \tag{6.68}$$

and, by a further substitution, to

$$x = g^w(g^t(z)) = g^w(g^t(g^v(g^t(x)))) \tag{6.69}$$
$$z = g^v(g^t(x)) = g^v(g^t(g^w(g^t(x)))). \tag{6.70}$$

However, by hypothesis $x \neq z$, that is:

$$g^w(g^t(g^v(g^t(x)))) \neq g^v(g^t(g^w(g^t(x)))) \tag{6.71}$$

which implies that the rule of composition between state transitions is not commutative, and with it the operation $+$ on T, contrary to the hypothesis.

Moreover, for any $t \in T$, g^t is surjective: for any $x \in M$, there should be some $y \in M$ such that

$$g^t(x) = y \tag{6.72}$$

and, by reversibility, there should be some $w \in T$ such that:

$$g^w(y) = x. \tag{6.73}$$

By substitution, this means that, for any $x \in M$ and some $w \in T$:

$$g^w(g^t(x)) = x \tag{6.74}$$

which, by commutativity, turns out to be equivalent to

$$g^t(g^w(x)) = x, \tag{6.75}$$

i.e. every $x \in M$ is the image of at least one $z = g^w(x) \in M$ with respect to g^t.

By Definition 30, DS_L is thus completely logically reversible. □

On the other hand, commutativity of the time model is not a necessary condition for a reversible dynamical system to be also completely logically reversible, as the next example shows.

Example 5 (Strictly Reversible and Completely Logically Reversible Dynamical System on a Non-Commutative Monoid)
Let T be the set of all bijective functions on $\mathbb{Z}$, let 0 be the identity function on M and $\circ$ be the standard

operation of function composition; in addition, for any $t \in T$ and any integer $x \in \mathbb{Z}$, let

$$g^t(x) = t(x). \tag{6.76}$$

Then:

- $L = (T, +)$ is a non-commutative monoid:
  - (a) T is closed with respect to $\circ$,
  - (b) $\circ$ is associative,
  - (c) 0 is the identity element with respect to $\circ$,
  - (d) $\circ$ is not commutative (e.g. the successor of the opposite of an integer is never equal to the opposite of its successor);
- $DS_L = (\mathbb{Z}, (g^t)_{t \in T})$ is a dynamical system on $L = (T, +)$:
  1. $\mathbb{Z}$ is a non-empty set,
  2. $(g^t)_{t \in T}$ is a family of functions on $\mathbb{Z}$, indexed by T,
  3. for any $x \in \mathbb{Z}$ and any $t, v \in T$:

$$g^0(x) = 0(x) = x \tag{6.77}$$
$$g^{t \circ v}(x) = (t \circ v)(x) = t(v(x)) = g^t(g^v(x)); \tag{6.78}$$

- DS_L is strictly reversible: for any $t \in T$, there exists $t^{-1} \in T$ such that, for any $x \in \mathbb{Z}$

$$g^{t^{-1}}(g^t(x)) = t^{-1}(t(x)) = x, \tag{6.79}$$

  for the inverse of a bijection is a bijection;
- DS_L is completely logically reversible: by hypothesis, for all $t \in T$, g^t is bijective. Complete logical reversibility is also guaranteed by Corollary 6.8.2.

The role of commutativity will become crucial in Chapter 8, where it will be proved to induce a special type of symmetry in the time models of dynamical systems.

6.3.2 Strict Reversibility and Regular Time Models

We saw that strictly reversible dynamical systems are always completely logically reversible, while the converse is not generally true. It is nonetheless possible to impose a sufficient condition on the time models of both logically reversible dynamical systems and dynamical systems with complete past, so that they can be strictly reversible in their turn. We say that an element t of an arbitrary monoid $L = (T, +)$ is *regular* if and only if, for some $r \in T$, $t + r + t = t$; similarly, we call a monoid regular if and only if all of its elements are (Clifford and Preston, 1961, pp. 26-27). Then:

Proposition 6.15. *Logically reversible dynamical system on regular monoids are strictly reversible.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a logically reversible dynamical system on a monoid $L = (T, +)$. Furthermore, let L be regular, so that for any $t \in T$ there exists $r \in T$ such that $t + r + t = t$; as a consequence, for any such $t, r \in T$, for any $x \in M$

$$g^t(g^r(g^t(x))) = g^{t+r+t}(x) = g^t(x). \tag{6.80}$$

Let us now suppose, as a reductio, that DS_L was not strictly reversible. In that case, there would exist $t \in T$ such that, for any $r \in T$,

$$g^r(g^t(x)) \neq x \tag{6.81}$$

for some $x \in M$. However, by logical reversibility, this would imply

$$g^t(g^r(g^t(x))) \neq g^t(x), \tag{6.82}$$

which would contradict (6.80). Hence, DS_L must be strictly reversible. □

Proposition 6.16. *Dynamical system on regular monoids having complete past are strictly reversible.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ with complete past. If L is regular, then for any $t \in T$ there exists $r \in T$ such that $t + r + t = t$; as a consequence, for any such $t, r \in T$, for any $x \in M$

$$g^t(g^r(g^t(x))) = g^{t+r+t}(x) = g^t(x). \tag{6.83}$$

By complete past, for any $t \in T$ and any $y \in M$ there are $x \in M$ and $r \in T$ such that

$$y = g^t(x); \tag{6.84}$$

thus, by (6.83) and (6.84)

$$g^t(g^r(y)) = g^t(g^r(g^t(x))) = g^{t+r+t}(x) = g^t(x) = y; \tag{6.85}$$

accordingly, by Corollary 6.9.1, DS_L is strictly reversible. □

Proposition 6.17. *Let DS_L be a dynamical system on a regular monoid L; then, the following statements are equivalent: (1) DS_L is logically reversible; (2) DS_L is strictly reversible; (3) DS_L is completely logically reversible; (4) DS_L has complete past.*

Proof
Let DS_L be a dynamical system on a regular monoid L; we shall independently prove the chain of implications (1)-(2)-(3)-(4) in the two directions. If DS_L is logically reversible then, by Proposition 6.15, it is strictly reversible; hence, by Corollary 6.8.2, it is completely logically reversible and, by Definition 30, with complete past. Conversely, if DS_L has complete past then, by Proposition 6.16, it is strictly reversible; hence, by Corollary 6.8.2, it is completely logically reversible and, by Definition 30, logically reversible. □

6.4

Non-Reversible Dynamical Systems

There are two remarkable ways how a dynamical system might fail to be reversible, which we shall refer to, respectively, as *strong irreversibility* (Giunti, 1997, p. 29) and *complete irreversibility*.

Definition 34 (Strong Irreversibility)
A dynamical system $DS_L = (M, (g^t)_{t \in T})$ *on a monoid* $L = (T, +)$ *is called* strongly irreversible *if and only if there exist* $x, y \in M$ *and* $t, r \in T$ *such that*

$$g^t(x) = g^r(y) \tag{6.86}$$

and, for any $v \in T$

$$g^v(x) \neq y \ \text{ and } g^v(y) \neq x. \tag{6.87}$$

In the previous chapter, we anticipated that possession of merging orbits is a distinguishing feature of irreversible dynamical system. Strong irreversibility is precisely the type of irreversible behavior which is displayed by a dynamical system with merging orbits.

Proposition 6.18. *Dynamical systems are strongly irreversible if and only if they possess merging orbits*[7].

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ and let $x, y \in M$. By Definition 16, for any $x \in M$, $orb(x)$ is merging if and only if for some $y \in M$ (i) $orb(x) \nsubseteq orb(y)$, (ii) $orb(y) \nsubseteq orb(x)$ and (iii) $orb(x) \cap orb(y) \neq \emptyset$. By Proposition 5.7 (i) obtains if and only if $x \notin orb(y)$, i.e. if and only if, for any $v \in T$, $g^v(y) \neq x$ and, similarly, (ii) obtains if and only if, for any $v \in T$, $g^v(x) \neq y$. Finally, (iii) obtains if and only if there exists $z \in M$ such that $z \in orb(x)$ and $z \in orb(y)$, which obtains if and only if there exist $t, r \in T$ such that $g^t(x) = z = g^r(y)$. By Definition 34, such conditions obtain if and only if DS_L is strongly irreversible. □

Corollary 6.18.1. *Strongly irreversible dynamical systems are not reversible.*

Proof
Let DS_L be a dynamical system on a monoid L. If DS_L is reversible, then by Corollary 6.5.1, it has no merging orbits. Hence, according to Proposition 6.18, DS_L is not strongly irreversible. Conversely, if DS_L is strongly irreversible then it is not reversible. □

Complete irreversibility is the strongest possible form of properly irreversible behavior a deterministic system might display. Intuitively, a system is completely irreversible just in case none of its states can be ever recovered dynamically, by means of the sole state transitions of the system:

Definition 35 (Complete Irreversibility)
A dynamical system $DS_L = (M, (g^t)_{t \in T})$ *on a monoid* $L = (T, +)$ *with identity* 0 *is called* completely irreversible *if and only if, for any* $t \in T$ *and any* $x \in M$, *for any* $r \in T - \{0\}$

$$g^r(g^t(x)) \neq x. \tag{6.88}$$

It is worth noticing that allowing for $r = 0$ in the above definition would have the effect of making condition (6.88) contradictory, since in that case, for $t = 0$, we would get $x \neq x$. Conversely, restricting the scope of r so that it could only range over $T - \{0\}$ has the effect of making complete reversibility logically equivalent to the following:

[7] Proposition 6.18 generalizes a similar statement included in (Giunti, 1997) to the case of dynamical systems whose time models are not necessarily commutative.

Proposition 6.19. *A dynamical system $DS_L = (M, (g^t)_{t\in T})$ on a monoid $L = (T, +)$ with identity 0 is completely irreversible if and only if, for any $x \in M$ and any $t \in T - \{0\}$, for any $r \in T - \{0\}$,*

$$g^r(g^t(x)) \neq x. \tag{6.89}$$

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0. If DS_L is completely irreversible then, by Definition 35, condition (6.89) must hold (for all $r \in T - \{0\}$) for any $x \in M$ and any $t \in T$; *a fortiori*, this must also be true (for all $r \in T - \{0\}$) for any $x \in M$ and any $t \in T - \{0\}$, proving Proposition 6.19 in one direction. To prove it in the converse direction, let us notice that, if (6.89) held (for all $r \in T - \{0\}$) for any $x \in M$ and for any $t \in T - \{0\}$, then the sole case in which (6.88) would possibly not hold would be if $t = 0$. Thus, suppose for reductio that for $t = 0$ there were $z \in M$ and $t \in T$ such that

$$g^r(g^t(x)) = g^r(g^0(x)) = g^r(x) = x. \tag{6.90}$$

In that case, by (6.90),

$$g^r(g^r(x)) = g^r(x) = x, \tag{6.91}$$

but then, (6.88) would not hold also for $t = r \in T - \{0\}$, contrary to the hypothesis that (6.89) holds for any $t \in T - \{0\}$. Therefore, the converse implication is proved. □

Complete irreversibility will play an important part in Chapter 7, where we shall study the dynamical properties of time models. In particular, we shall exploit the following properties:

Proposition 6.20. *Completely irreversible dynamical systems have no periodic orbits.*

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0. By Definition 35, if DS_L is completely irreversible then for any $x \in M$ and any $t \in T$ there exist no $r \in T - \{0\}$ such that

$$g^r(g^t(x)) = x; \tag{6.92}$$

as a consequence, if $t = 0$, for any $x \in M$ there exist no $r \in T - \{0\}$ such that

$$g^r(g^0(x)) = g^r(x) = x. \tag{6.93}$$

Therefore, by Definition 18, DS_L has no periodic orbits. □

Corollary 6.20.1. *Completely irreversible dynamical systems have no fixed points.*

Proof
By Proposition 5.18, fixed points have periodic orbits. On the other hand, by Proposition 6.20, completely irreversible dynamical system have no periodic orbits and, therefore, the can have no fixed point. □

Proposition 6.21. *A dynamical system $DS_L = (M, (g^t)_{t\in T})$ on a monoid $L = (T, +)$ is completely irreversible if and only if, for any $x \in M$,*

$$P(x) \cap F(x) = \emptyset. \tag{6.94}$$

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0. By Proposition 6.19, DS_L is not completely irreversible exactly in case there exist $x \in M$, $t \in T - \{0\}$ and $v \in T - \{0\}$ such that $g^r(g^t(x)) = x$. On the other hand, by Definition 22 and Definition 24, this happens just in case there exist

$x, y \in M$, $t \in T - \{0\}$ and $v \in T - \{0\}$ such that $y \in F^t(x)$ and $y \in P^r(x)$. By Definition 23 and Definition 25, this is equivalent to claiming that for some $x, y \in M$, $y \in F(x)$ and $y \in P(x)$, i.e. $F(x) \cap P(x) \neq \emptyset$. Conversely, for any $x \in M$, $F(x) \cap P(x) = \emptyset$ if and only if TS_L is completely irreversible. □

Corollary 6.21.1. *Completely irreversible dynamical systems are not reversible.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0 and let $x \in M$. If DS_L is completely irreversible then, by Proposition 6.21, $P(x) \cap F(x) = \emptyset$ where, by Definition 23, $F(x)$ cannot be empty; as a consequence, it must be $P(x) \neq F(x)$ and therefore, by Proposition 6.4, DS_L is not reversible. □

7

The Dynamics of Time

Having required that time should at least possess the algebraic structure of a monoid bears profound and quite unexpected consequences for our discussion concerning the objective significance of temporal becoming. This chapter is dedicated to show how time models could be provided with an internal dynamics, essentially depending on their algebraic properties and described by a special kind of dynamical systems, called *time systems*. Studying time systems, their dynamical properties and their connection with the algebraic properties of monoids will finally provide us with a rigorous and consistent characterization of the passage of time, capable to overcome most of the standard objections to becoming we met in the preceding chapters.

7.1

Time Systems

Since they are non-empty sets, time sets could play the part of state spaces as well: according to this interpretation, their elements do not model intervals, but *points* of time, or moments. Furthermore, any such set can be provided with a family of functions, indexed by the same time set, *via* a left monoid action (Clifford and Preston, 1961). This way, any time model is uniquely associated (Mazzola and Giunti, 2010) with a time system:

Definition 36 (Time System of a Monoid)
The time system *of a monoid $L = (T, +)$, denoted by TS_L, is the ordered pair*

$$TS_L = (I, (\iota^t)_{t \in T})$$

such that

$$I = T \tag{7.1}$$

and, for any $t \in T$, for any $i \in I$

$$\iota^t(i) = t + i. \tag{7.2}$$

In the specific case of time systems, elements of the state space I are called *instants* or *moments*, while state transitions on I are called *time transitions*. It is easy to see that time systems are but a special class of dynamical systems:

Proposition 7.1. *The time system TS_L of a monoid L is a dynamical system on L.*

Proof
Let $TS_L = (I, (\iota^t)_{t\in T})$ be the time system of a monoid $L = (T, +)$ with identity 0. Then

1. I is a non-empty set;
2. $(\iota^t)_{t\in T}$ is a family of functions on I, indexed by T;
3. for any $i \in I$ and any $t, v \in T$

$$\iota^0(i) = 0 + i = 0 \tag{7.3}$$
$$\iota^{t+v}(i) = (t + v) + i = t + (v + i) = \iota^t(\iota^v(i)). \tag{7.4}$$

Hence, according to Definition 1, TS_L is a dynamical system on L. □

Time systems are therefore dynamical systems whose state spaces coincide with their time sets, and whose state transitions are functional representations of their binary rules of composition. In other words, time systems are dynamical systems describing the internal dynamics of a time model $L = (T, +)$. From an intuitive point of view, for any $t \in T$ and $i \in I$, the arrow $i \xrightarrow{t} \iota^t(i)$ in the transition graph of a time system is intended to represent the flowing of time from instant i to the one from which i is separated by duration t; on this interpretation, $\iota^t(i)$ should be the instant obtained by adding (composing) duration t to instant i, just as required by (7.2).

In this view, despite their set-theoretic identity, there exists a *functional* difference between sets T and I, which depends on the different perspectives from which monoids and their time systems are observed: on the one hand, elements of T are understood algebraically as durations, or elements of a monoid; on the other hand, the same elements are understood dynamically, *qua* elements of set I, as states in a deterministic system. So, while durations can be composed according to the algebraic rule of composition of monoid L, instants themselves cannot: I is a set with no algebraic structure, or rather a set which is not required to have one. Moving from instant to instant is a dynamical operation, rather than an algebraic one: for this reason, it must be mediated by a function, notably a time transition, and precisely by one emulating the corresponding rule of composition.

7.1.1 From Algebra to Dynamics

In the course of Chapter 5, we proved isomorphic dynamical systems to be indistinguishable as of the purposes of general dynamical systems theory, for they describe exactly the same dynamics. It is then no surprise that time systems whose time models are algebraically isomorphic must be isomorphic in their turn.

Proposition 7.2. *Let L_1 be a monoid with time system TS_{L_1} and let L_2 be a monoid with time system TS_{L_2}; then any monoid isomorphism ρ of L_2 in L_1 is a ρ-isomorphism of TS_{L_2} in TS_{L_1}.*

Proof
Let $L_1 = (T_1, +)$ be a monoid with time system $TS_{L_1} = (I_1, (\iota^{t_1})_{t_1 \in T_1})$ and let $L_2 = (T_2, \oplus)$ be a monoid with time system $TS_{L_2} = (I_2, (\iota^{t_2})_{t_2 \in T_2})$. Let $\rho : T_2 \to T_1$ be a monoid isomorphism of L_2 in L_1. Hence, for any $t_2 \in T_2$ and any $i_2 \in I_2 = T_2$

$$\begin{aligned} \rho(\iota^{t_2}(i_2)) &= \rho(t_2 \oplus i_2) \\ &= \rho(t_2) + \rho(i_2) \\ &= \iota^{\rho(t_2)}(\rho(i_2)). \end{aligned} \tag{7.5}$$

Hence, according to Definition 3, ρ is a ρ-isomorphism between time systems. □

Corollary 7.2.1. *Let L_1 be a monoid with time system TS_{L_1} and let L_2 be a monoid with time system TS_{L_2}. If L_1 and L_2 are isomorphic, then TS_{L_1} and TS_{L_2} are isomorphic.*

Proof
Let L_1 be a monoid with time system TS_{L_1} and let L_2 be a monoid with time system TS_{L_2}. If L_1 and L_2 are isomorphic, then there must exist a monoid isomorphism ρ of TS_{L_2} in TS_{L_1}. By Proposition 7.2, ρ is a ρ-isomorphism of TS_{L_2} in TS_{L_1}. Therefore, by Definition 4, TS_{L_2} and TS_{L_1} are isomorphic. □

In general, the set of state transitions of whatever dynamical system DS_L can be endowed with an algebraic structure, along with the standard operation of function composition. As we already mentioned[1] , when L is a group (e.g. the real numbers), this algebraic structure is also a group, which is sometimes called the *one parameter group of transformations* of the dynamical system DS_L (Arnold, 1973; Tung, 1985). However, in the general case, we refer to this structure as *the transition algebra* of DS_L. We will see below that, when L is just a monoid, the transition algebra of DS_L is a monoid as well.

DEFINITION 37 (Transition Algebra)
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$. The transition algebra *of DS_L, denoted by TA_{DS_L}, is the ordered pair*

$$TA_{DS_L} = (H, \circ)$$

where

$$H = \{h : h = g^t \text{ for some } t \in T\} \tag{7.6}$$

and $\circ$ is the standard operation of function composition.

Proposition 7.3. *The transition algebra of a dynamical system $DS_L = (M, (g^t)_{t \in T})$ on a monoid $L = (T, +)$ with identity 0 is a monoid with identity g^0.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0 and let $TA_{DS_L} = (H, \circ)$ be the transition algebra of DS_L. Then

[1] See the introductory part of Chapter 5.

- for any $g^t, g^v \in H$

$$g^t \circ g^v = g^{t+v} \in H \tag{7.7}$$

- for any $g^t, g^v, g^w \in H$

$$g^t \circ (g^v \circ g^w) = g^t \circ g^{v+w} = g^{t+(v+w)} = g^{(t+v)+w} = g^{t+v} \circ g^w = (g^t \circ g^v) \circ g^w \tag{7.8}$$

- $g^0 \in H$ and, for any arbitrary $h = g^t \in H$,

$$g^0 \circ g^t = g^{0+t} = g^t = g^{t+0} = g^t \circ g^0. \tag{7.9}$$

Hence, TA_{DS_L} satisfies closure with respect to the composition operation $\circ$, associativity and possession of the identity element g^0. Accordingly, TA_{DS_L} is a monoid. □

In general, the time model L of a dynamical system DS_L is homomorphic, but not isomorphic, to the transition algebra TA_{DS_L}:

Proposition 7.4. *Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$; then the family $(g^t)_{t\in T}$ is a surjective monoid homomorphism from L to the transition algebra $TA_{DS_L} = (H, \circ)$ of DS_L.*

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0 and let $TA_{DS_L} = (H, \circ)$ be the transition algebra of DS_L. Finally, let $g : T \to H$ be the family $(g^t)_{t\in T}$. Then

- g maps identity element into identity element:

$$g(0) = g^0; \tag{7.10}$$

- g is structure-preserving: for any $t, v \in T$

$$g(t + v) = g^{t+v} = g^t \circ g^v = g(t) \circ g(v). \tag{7.11}$$

- g is surjective: by Definition 37, for any $h \in H$, $h = g^t$ for some t; but, by hypothesis,

$$g(t) = g^t; \tag{7.12}$$

  hence surjectivity holds.

Thus, g is a surjective monoid homomorphism of L in TA_{TS_L}. □

In addition, in the special case of time systems, the family $(\iota^t)_{t\in T}$ of state transitions is necessarily injective; accordingly,

Corollary 7.4.1. *Every monoid L is isomorphic to the transition algebra of its time system.*

Proof
Let $L = (T, +)$ be a monoid with identity 0, let $TS_L = (I, (\iota^t)_{t\in T})$ be the time system of L and let $TA_{TS_L} = (H, \circ)$ be the transition algebra of TS_L. Finally, let $\iota : T \to H$ be the family $(\iota^t)_{t\in T}$. Then:

- ι is a surjective monoid homomorphism of L in TA_{TS_L}: by Proposition 7.4;

- ι is injective: for any $t, v \in T$

$$\begin{aligned} t &\neq v \\ t + 0 &\neq v + 0 \\ \iota^t(0) &\neq \iota^v(0) \\ \iota^t &\neq \iota^v \\ \iota(t) &\neq \iota(v); \end{aligned} \tag{7.13}$$

Hence, ι is a monoid isomorphism of L in TA_{TS_L} and, accordingly, L is isomorphic to TA_{TS_L}. □

Let us also notice that, in general, a dynamical system DS_L is not isomorphic to the time system $TS_{TA_{DS_L}}$ of its transition algebra, for isomorphism between dynamical systems requires their time models to be isomorphic, while Proposition 7.4 only grants surjective homomorphism between L and TA_{DS_L}. However, Corollary 7.4.1 guarantees that the time model of a time system and the one of the time system of its transition algebra are always isomorphic. For this reason,

Proposition 7.5. *Every time system is isomorphic to the time system of its transition algebra.*

Proof
Let TS_L be the time system of a monoid L, let TA_{TS_L} be the transition algebra of TS_L and let $TS_{TA_{TS_L}}$ be the time system of TA_{TS_L}. By Corollary 7.4.1, TA_{TS_L} is isomorphic to L. Hence, by Corollary 7.2.1, $TS_{TA_{TS_L}}$ and TS_L are isomorphic. □

Proposition 7.1 and Corollary 7.4.1 jointly show that any monoid gives rise to a special dynamics, represented by its time system, that preserves its algebraic properties. Similarly, Proposition 7.3 and Proposition 7.5 show that any time system gives rise to an algebraic structure, namely its transition algebra, that preserves its dynamical properties. Time systems and monoids are thus alternative but equivalent ways of representing the same mathematical structure, whose properties are left unchanged while moving from the latter, algebraic representation, to the former, dynamical one, and back. For this very reason, the study of the algebraic properties of a monoid L can also be carried out in the form of a study of the dynamical properties of its time system TS_L, and vice versa[2].

7.1.2 The Dynamics of Identity

In the previous chapter, we introduced the study of mathematical dynamical systems as the study of their orbits and motions. Restricting the scope of such analysis to the proper subclass of time systems, the fundamental notion of orbit turns out to be identical to a well known algebraic concept.

Let $L = (T, +)$ be an arbitrary monoid; for any $t \in T$, the *principal left ideal* of t is defined as the subset of T generated by composing every element of T to the left of t (Green, 1951, p. 164).

[2] For a discussion on the relation of algebra to the study of time also see Withrow (1980).

The following statement establishes that the orbit of any instant is identical to its principal left ideal.

Proposition 7.6. *Let $TS_L = (I, (\iota^t)_{t\in T})$ be the time system of a monoid $L = (T, +)$; then for any $i \in I = T$, $orb(i)$ is the principal left ideal of i.*

Proof
Let $TS_L = (I, (\iota^t)_{t\in T})$ be the time system of a monoid $L = (T, +)$; then, by Definition 36 and the definition of an orbit, for any $i \in I = T$,

$$\begin{aligned} orb(i) &= \{j \in I : \text{ for some } t \in T, j = \iota^t(i)\} \\ &= \{j \in I : \text{ for some } t \in T, j = t + i\}, \end{aligned} \tag{7.14}$$

which is exactly the principal left ideal of i. □

One of the special features of the identity element is that of having a principal left ideal including all the elements of the monoid it belongs to. The dynamical counterpart of this property is established by the following:

Proposition 7.7. *Let $TS_L = (I, (\iota^t)_{t\in T})$ be the time system of a monoid $L = (T, +)$ with identity 0; then $orb(0) = I$.*

Proof
Let $TS_L = (I, (\iota^t)_{t\in T})$ be the time system of a monoid $L = (T, +)$ with identity 0. Then, for any $t \in T$

$$\iota^t(0) = t + 0 = t, \tag{7.15}$$

so that

$$orb(0) = \{i \in I : \text{ for some } t \in T, \iota^t(0) = i\} = T = I. \tag{7.16}$$

□

Proposition 5.6 associated the orbit of any point x in the state space of a dynamical system DS_L on a monoid L with a dynamical system DS_{x_L} on L, whose state space is equal to $orb(x)$ and whose state transitions consist of the restriction of the state transitions of DS_L to $orb(x)$; *a fortiori*, this must also be true for the orbits of instants in an arbitrary time system. Proposition 7.7 is thus asserting that the time system TS_L of any monoid L is *identical* to the dynamical system DS_{0_L} associated to the orbit of its identity element.

Corollary 7.7.1. *Let $TS_L = (I, (\iota^t)_{t\in T})$ be the time system of a monoid $L = (T, +)$ with identity 0; then $TS_L = TS_{0_L} = (orb(0), (\iota^t|_{orb(0)})_{t\in T})$.*

Proof
Let $TS_L = (I, (\iota^t)_{t\in T})$ be the time system of a monoid $L = (T, +)$ with identity 0. By Proposition 7.7, $I = orb(0)$; accordingly, for any $t \in T$, $\iota^t|_{orb(0)} = \iota^t$, so that $(\iota^t|_{orb(0)})_{t\in T} = (\iota^t)_{t\in T}$ and therefore $TS_L = TS_{0_L} = (orb(0), (\iota^t|_{orb(0)})_{t\in T})$. □

The most straightforward interpretation of Corollary 7.7.1 is that the whole dynamics of a time system is just the dynamics of the identity element of its time model. This interpretation is further supported by the following result, according to which the identity element is the unique starting point from which the whole dynamics of a time system is generated, if there is one.

Proposition 7.8. *Let $TS_L = (I, (\iota^t)_{t \in T})$ be the time system of a monoid $L = (T, +)$ with identity 0. For any $i \in I$, if i is a Garden of Eden, then $i = 0$.*

Proof
Let $TS = (I, (\iota^t)_{t \in T})$ be the time system of a monoid $L = (T, +)$ with identity 0. For any $i \in I$, if $i \neq 0$ then

$$i = i + 0 = \iota^i(0); \tag{7.17}$$

as a consequence, there exist $t = i \in T - \{0\}$ and $j = 0 \in I$ such that $\iota^t(j) = i$ and, therefore, i is not a Garden of Eden. Therefore, for any $i \in I$, if i is a Garden of Eden then $i = 0$. □

The question whether or not the identity element is a Garden of Eden, namely whether or not the identity element has a non-empty past, will reveal to be strictly related to the problem of providing the dynamics of time with a well-defined direction, as well as the one of giving a satisfactory dynamical account of tenses. In both cases, a fundamental role is played by the following property:

Proposition 7.9. *Let TS_L be the time system of a monoid $L = (T, +)$ with identity 0; then for any $i \in I$, $0 \in orb(i)$ if and only if i has a left inverse element.*

Proof
Let $TS_L = (I, (\iota^t)_{t \in T})$ be the time system of a monoid $L = (T, 0)$ with identity 0; then, for any $i \in T$,

$$\begin{aligned} 0 \in orb(i) \quad & \textit{if and only if, for some } t \in T \quad \iota^t(i) = 0 \\ & \textit{if and only if, for some } t \in T \quad t + i = 0, \end{aligned} \tag{7.18}$$

which happens exactly in case t is a left inverse of i. □

Corollary 7.9.1. *Let TS_L be the time system of a monoid $L = (T, +)$ with identity 0; TS_L has a Garden of Eden if and only if no element of L other than the identity has a left inverse element.*

Proof
Let $TS_L = (I, (\iota^t)_{t \in T})$ be the time system of a monoid with identity 0. By Proposition 7.8, TS_L has a Garden of Eden if and only if 0 is, i.e. for any $i \in I - \{0\}$, for all $t \in T$

$$\iota^t(i) \neq 0 \tag{7.19}$$

or, equivalently,

$$0 \notin orb(i) \tag{7.20}$$

which, by Proposition 7.9, obtains if and only if i has no left inverse element. □

One may wonder whether the identity could ever be periodic – at least in those cases in which the identity itself is not a Garden of Eden – so that the whole dynamics of the given time system were closed, or cyclic. The following proposition is meant to rule out this possibility.

Proposition 7.10. *Let TS_L be the time system of a monoid $L = (T, +)$ with identity 0; then 0 is not periodic.*

Proof
Let $TS_L = (I, (\iota^t)_{t \in T})$ be the time system of a monoid with identity 0; then, for any $t \in T - \{0\}$,

$$\iota^t(0) = t + 0 = t \neq 0; \tag{7.21}$$

hence, by Definition 17, 0 is not periodic. □

This property of time systems is also indirectly confirmed by the following result:

Proposition 7.11. *Let TS_L be the time system of a monoid $L = (T, +)$ with identity 0; then for any $i \in I$, i has a left inverse if and only if $orb(i) = I$.*

Proof
Let $TS_L = (I, (\iota^t)_{t \in T})$ be the time system of a monoid $L = (T, +)$ with identity 0 and let $i \in T$.

If i has a left inverse, then for some $t \in T = I$

$$\iota^t(i) = t + i = 0 \tag{7.22}$$

and thus, for any $j \in I$

$$j = \iota^j(0) = \iota^j(t + i) = j + (t + i) = (j + t) + i = \iota^{j+t}(i) \in orb(i); \tag{7.23}$$

hence, $orb(i) = I$.

If $orb(i) = I$, *a fortiori*

$$0 \in orb(i) \tag{7.24}$$

and therefore, by Proposition 7.9, i has a left inverse. □

Proposition 7.11 establishes a logical equivalence between the property of possessing a left inverse element and the one of possessing what we may call a *maximal* orbit, i.e. an orbit covering the whole state space. Clearly, identity elements own this property *by definition*, and this may be understood as the conceptual source of Proposition 7.7. Things are different in all other cases, in which this property is far from being trivial: as we shall see, providing all the elements of a monoid with a maximal orbit will make its time system reversible.

7.1.3 The Reversibility of Time

Being dynamical systems, time systems may reverse in a variety of different ways depending on the algebraic structure of their time models. In their special case, however, this dependence is made even stronger, since that structure invariably affects the logical properties of their state transitions. In particular, we shall observe a substantial strengthening of the notion of reversibility, on which most forms of reversible dynamics we discussed in the previous chapter collapse, and a sensible reduction in the range of possible combinations of reversible types of behavior a time system might display. In the end, both of these distinguishing features of time systems stem from the following result:

Proposition 7.12. *The time system TS_L of a monoid L is reversible if only if it is strictly reversible, if and only if any element of L has a left inverse.*

Proof
Let $TS = (I, (\iota^t)_{t \in T})$ be a time system on a monoid $L = (T, +)$ with identity 0.

If TS_L is reversible, then according to Definition 31, for any $i \in I$ and any $t \in T$ there must exist $r \in T$ satisfyig condition (6.5), so that

$$r + (t + i) = i. \tag{7.25}$$

In particular, fixing $i = 0$, for any $t \in T$ there must exist $r \in T$ such that

$$r + t = 0, \tag{7.26}$$

i.e. any $t \in T = I$ must possess a left inverse element.

In addition, if all elements of L have a left inverse then, for any $t \in T$, there exists $r \in T$ such that

$$r + t = 0; \tag{7.27}$$

as a consequence, for any $j \in I$

$$\iota^r(\iota^t(j)) = r + t + j = 0 + j = j. \tag{7.28}$$

and therefore, according to Definition 32, TS_L is strictly reversible.

Finally, if TS_L is strictly reversible then, by Proposition 6.7, it is also reversible, closing the circle of implications.

In sum: TS_L is reversible if and only if it is strictly reversible, if and only if any element of L has a left inverse. □

Proposition 7.12 is a clear example of how the dynamical properties of time systems essentially depend on the algebraic features of their time models: though generally not equivalent, in the special case of time systems reversibility and strict reversibility coincide because of their being equivalent to the same algebraic property: namely, that all members of a monoid possess a left inverse element. This result confirms what we already anticipated, namely that providing all the possible states of a time system with a maximal orbit is as much as making it reversible:

Corollary 7.12.1. *The time system $TS_L = (I, (\iota^t)_{t \in T})$ of a monoid $L = (T, +)$ is reversible if and only if, for any $i \in I$, $orb(i) = I$.*

Proof
Let $TS_L = (I, (\iota^t)_{t \in T})$ be the time system of a monoid $L = (T, +)$ with identity 0. By Proposition 7.12, TS_L is reversible if and only if all $i \in I = T$ have a left inverse element. On the other hand, by Proposition 7.11, this obtains if and only if, for any $i \in I$, $orb(i) = I$. □

In other terms, the orbits of all states are identical and span the entire state space.

This means that reversible time systems describe the dynamics of a unique orbit, coinciding with that of the identity element: that is to say, all the states of a reversible time system are dynamically indistinguishable.

Proposition 6.8 showed that a necessary condition for a dynamical system to be strictly reversible is that all its state transitions be injective. For, if a state transition mapped different states x and z into a unique image y, then it could not be the case that a unique state transition could lead y back to both x and z. One further consequence of Proposition 7.12 is therefore that, contrary to the general case:

Proposition 7.13. *Reversible time systems are logically reversible.*

Proof
Let $TS_L = (I, (\iota^t)_{t \in T})$ be the time system of a monoid $L = (T, +)$. If TS_L is reversible then, according to Corollary 7.12, it is also strictly reversible and therefore, by Proposition 6.8, logically reversible. □

The converse, however, is still false: a very simple counterexample is represented by the time system associated with the monoid $(\mathbb{Z}^+, +)$ consisting of the set of non-negative integers, along with arithmetical addition.

Proposition 7.12 provided a straightforward algebraic interpretation for the reversibility of time systems. Furthermore, Proposition 7.13 established a logical connection between their reversibility and their logical reversibility, though the two concepts are not logically equivalent. So, what algebraic interpretation should we give of logical reversibility?

In general, we say that a monoid $L = (T, +)$ is *left-cancellative* if and only if all $t \in T$ are *left-cancellable*, which happens whenever for any $u, v \in T$, if $t+u = t+v$ then $u = v$. Symmetrically, L is *right-cancellative* if and only all $t \in T$ are right cancellable, i.e. for any $u, v \in T$, if $u + t = v + t$ then $u = v$. Finally, a monoid is simply *cancellative* if and only if it is both left and right cancellative, i.e. if and only if all of its elements are both left- and right-cancellable (Clifford and Preston, 1961, p. 3). In the case of time systems, the dynamical property of logical reversibility corresponds to the algebraic property of left-cancellability:

Proposition 7.14. *The time system TS_L of a monoid L is logically reversible if and only if L is left-cancellative.*

Proof
Let $TS_L = (I, (\iota^t)_{t \in T})$ be the time system of a monoid $L = (T, +)$. By Definition 28, TS_L is logically reversible exactly in case, for any $t \in T$, for any $i, j \in T = I$

$$\begin{aligned} \iota^t(i) = \iota^t(j) &\quad \text{if and only if} \quad i = j \\ t + i = t + j &\quad \text{if and only if} \quad i = j, \end{aligned} \tag{7.29}$$

which obtains if and only if L is left-cancellative. □

One may think the natural counterpart of logical reversibility to be complete past, according to which all state transitions in a dynamical system are surjective. However, at least in the case of time systems, logical reversibility and complete past do not play analogous roles. Rather, complete past must be understood as the correlate of reversibility and strict reversibility:

Proposition 7.15. *The time system TS_L of a monoid L has complete past if and only if any element of L has a right inverse.*

Proof
Let $TS_L = (I, (\iota^t)_{t \in T})$ be the time system associated with a monoid $L = (T, +)$ with identity 0.

If TS_L has complete past then, according to Definition 29, for any $t \in T$ and any $i \in I$ there exists $j \in I$ such that

$$\iota^t(j) = t + j = i; \tag{7.30}$$

hence, holding $i = 0$ fixed, for any $t \in T$ there exists $j \in I = T$ such that

$$t + j = 0. \tag{7.31}$$

But if so, then any element of L has a right inverse.

Conversely, let us suppose that for any $t \in T$ there exists $i \in I = T$ such that

$$t + i = 0; \tag{7.32}$$

then for any $j \in I$,

$$\iota^t(i + j) = t + (i + j) = (t + i) + j = 0 + j = j. \tag{7.33}$$

Hence, by Definition 29, TS_L has complete past. □

In the light of Proposition 7.12, Proposition 7.15 shows that complete past plays a role analogous to that of reversibility and strict reversibility. This analogy can be pushed even further, since time systems with complete past can be proved to coincide exactly with reversible and strictly reversible ones:

Proposition 7.16. *Time systems are reversible if and only if they have complete past.*

Proof
Let $TS_L = (I, (\iota^t)_{t \in T})$ be the time system of a monoid $L = (T, +)$ with identity 0.

If TS_L is reversible, then by Proposition 7.12 for any $i \in I$ there must exist $j \in T = I$ such that

$$j + i = 0 \tag{7.34}$$

while, by the same token, there must exist $k \in T = I$ such that

$$k + j = 0. \tag{7.35}$$

Hence, by associativity:

$$k + (j + i) = (k + j) + i \tag{7.36}$$

$$k + 0 = 0 + i \tag{7.37}$$

$$k = i. \tag{7.38}$$

By (7.35) and (7.38), j is a right inverse of i. Hence, by Proposition 7.15, TS_L has complete past.

Proof in the converse direction runs similarly, *mutatis mutandis* (hint: use first Proposition 7.15 in place of Proposition 7.12 and, second, Proposition 7.12 in place of Proposition 7.15). □

In virtue of Proposition 7.16 and Proposition 7.13, all reversible time systems display both logical reversibility and complete past, and hence complete logical reversibility. In addition, contrary to the general case, reversibility is implied by complete logical reversibility in its turn.

Proposition 7.17. *Time systems are reversible if and only if they are completely logically reversible.*

Proof
By Proposition 7.13 and Proposition 7.16, reversible time systems are logically reversible and with complete past. Hence, by Definition 30, they are completely logically reversible. Conversely, completely logical reversible

time systems always possess complete past, by Definition 30; hence, by Proposition 7.16, they must be reversible too. □

In Chapter 5, we proved that any time invertible dynamical system is completely logically reversible, with complete past, strictly reversible, and reversible, but all four converse implications are false. By contrast, in the case of time systems, time invertibility is equivalent to reversibility (see Proposition 7.18 below), and thus to strict reversibility (by Proposition 7.12), complete past (by Proposition 7.16), and complete logical reversibility (by Proposition 7.17).

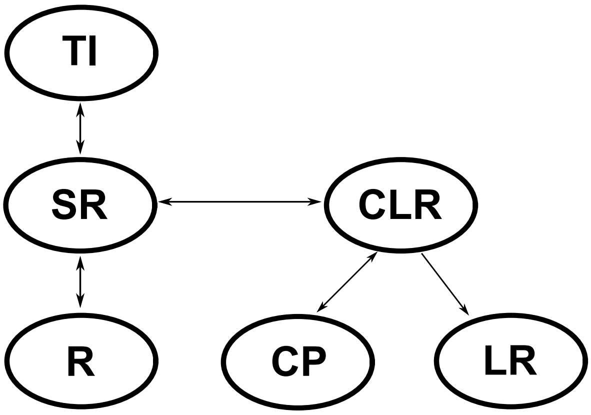

Figure 7.1: Possible types of reversible behavior displayed by time systems.

Proposition 7.18. *Time systems are reversible if and only they are time invertible.*

Proof
Let $TS_L = (I, (\iota^t)_{t\in T})$ be the time system of a monoid $L = (T, +)$ with identity 0. If TS_L is time invertible then, by Corollary 6.10.1, it is reversible. Conversely, if TS_L is reversible then, by Proposition 7.12, Proposition 7.16 and Proposition 7.15, for any $i \in T$ there exist $j, k \in T$ such that

$$j + i = 0 \tag{7.39}$$
$$i + k = 0 \tag{7.40}$$

and, by associativity

$$j = j + 0 = j + (i + k) = (j + i) + k = 0 + k = k. \tag{7.41}$$

Accordingly L is a group and, by Definition 33, TS_L is time invertible. □

The main implication of Proposition 7.18 is that, in the case of time systems, we are left with just two distinct concepts of reversible dynamics – namely, time invertibility and logical reversibility, the latter of which is entailed by the former one[3]. Symmetrically, there are just two ways time systems may be irreversible, both of which require a failure of reversibility: in the one case they might fail to be reversible, while still being logically reversible; in the other case they might fail to be logically reversible as well.

[3]See Figure 7.1.

7.2

The Dynamical Interpretation of Tenses

So far, we provided time models with intrinsic dynamical properties, which were proved to coincide exactly with those of their identity elements. What we got, however, is still not temporal becoming: to model the passage of time, we don't have just to provide the latter with proper dynamics, but we also need to make sense of its static ingredient. In other words, we still have to offer a satisfactory account of tenses, so that the dynamics we attributed to time on the basis of its algebraic property could be proved to be precisely that of the unique moving present.

7.2.1 Present States, Present Times

In general, as we already had the chance to point out, there are two distinct ways of functionally interpreting a time set. On the one hand, we can disregard its algebraic structure, and consider it as a state space, whose elements are moments or *points* in time: in this sense, the identity element is the sole element always generating the whole dynamics of a time system[4]. On the other hand, we can take the algebraic structure of time sets into account, and consider them as time models, i.e. monoids whose elements are durations, or *intervals* of time. In this sense, the identity element models the null duration, namely the one after which all states of whatever dynamical system are mapped into themselves. This interpretation of the identity, as we are going to see, is naturally associated with a dynamical conception of presentness, understood as a dynamical property predicable of different moments at different times.

Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0. In Chapter 5, we defined the future of state $x \in M$ as the set $F(x)$ of all states into which the system will evolve in a non-zero interval of time, after having been set in state x; symmetrically, we defined the past of x as the set $P(x)$ of states from which the system must have evolved in order to reach state x in a non-zero lapse of time. It is therefore intuitively straightforward to associate the present of a state x with the set including all and only those instants the system is capable of reaching a zero interval of time after having been set in state x:

Definition 38 (Present of a Point)
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0; for any state $x \in M$, the present *of x, denoted by $\Pi(x)$, is defined as*

$$\Pi(x) \stackrel{def}{=} \left\{ y \in M : y = g^0(x) \right\} . \tag{7.42}$$

[4]This is a direct consequence of Proposition 7.7, according to which the identity element always has a maximal orbit, together with the existence of time systems whose sole state having a maximal orbit is precisely the identity itself (as an example, consider the system associated with the monoid ($\mathbb{Z}^+$) consisting of the set of non-negative integers, together with arihtmetical addition).

Quite obviously, the dynamical present of any state consists precisely of that state itself:

Proposition 7.19. *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$; then, for any $x \in M$*

$$\Pi(x) = \{x\} . \tag{7.43}$$

Proof

Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0 and let $x \in M$; then, by Definition 38 and Definition 1,

$$\Pi(x) = \{y \in M : y = g^0(x)\} = \{x\} . \tag{7.44}$$

□

In general, the past, present and future of a point x in the state space of a dynamical system specify what states the system could have displayed in order to reach that point, the state displayed by the system at that point, and those it will display afterwards. In this sense pastness, presentness and futurity are thus relational or indexical features of *states*, rather than of times. However, in the case of time systems, the past, present and future of each moment i consist of the sets of moments or instants dynamically preceding, coinciding with, and following i. In this sense, pastness, presentness and futurity are relational or indexical features of *times*: for each instant i, its past, present and future sets specify what are the past, present and future times at time i. This way, once applied to time systems, the dynamical characterization of pastness, presentness and futurity given by Definition 23, Definition 25 and Definition 38 must be taken as definitions of past, present and future tenses *as such*.

How to relate this conception of presentness to the idea of a unique transient present moment, whose motion spans the whole of time? The conceptual link is offered by the following statement:

Corollary 7.19.1. *Let $TS_L = (I, (\iota^t)_{t \in T})$ be the time system of a monoid $L = (T, +)$ with identity 0; then, for any $t \in T - \{0\} = I - \{0\}$,*

$$F^t(0) = \Pi(t). \tag{7.45}$$

Proof

Let $TS_L = (I, (\iota^t)_{t \in T})$ be the time system of a monoid $L = (T, +)$ with identity 0 and let $t \in T - \{0\} = I - \{0\}$; then, by Definition 22, Definition 38 and Proposition 7.19,

$$F^t(0) = \{j \in I : j = \iota^t(0)\} = \{j \in I : j = t + 0\} = \{t\} = \Pi(t). \tag{7.46}$$

□

In plain words, Corollary 7.19.1 states that the motion of the identity element and the way the present moment is determined at each time are in fact identical: at every instant, what counts as present is the unique stage occupied by the identity element, and it is precisely the dynamical evolution of the identity which makes the unique present moment vary from time to time. Under this light, Definition 38 is just a functionally different, although dynamically equivalent interpretation of the dynamics of the identity element.

7.2.2 Separating Tenses

The discussion we made in Chapter 3 showed that any satisfactory account of tenses should at least be capable to make the present moment a constitutive or invariant property of time, coinciding with the separating element between past and future. Does the dynamical interpretation of tenses just outlined satisfy these two minimal requirements?

Intuitively, having reduced the idea of the moving present to the purely algebraic concept of the identity element is *per se* strong evidence in favor of its mind-independence. Further and even stronger evidence is nonetheless offered by the following:

Proposition 7.20. *Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1 \in T_1})$ and $DS_{L_2} = (M_2, (g^{t_2})_{t_2 \in T_2})$ be dynamical systems on $L_1 = (T_1, +)$ and $L_2 = (T_2, \oplus)$ respectively, and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}; then, for any $x_1, y_1 \in M_1$,*

7.20.1. *$y_1 \in F(x_1)$ if and only if $f(y_1) \in F(f(x_1))$;*

7.20.2. *$y_1 \in P(x_1)$ if and only if $f(y_1) \in P(f(x_1))$;*

7.20.3. *$y_1 \in \Pi(x_1)$ if and only if $f(y_1) \in \Pi(f(x_1))$.*

Proof
Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1 \in T_1})$ be a dynamical system on a monoid $L_1 = (T_1, +)$ with identity 0, let $DS_{L_2} = (M_2, (g^{t_2})_{t_2 \in T_2})$ be a dynamical system on a monoid $L_2 = (T_2, \oplus)$, let $\rho : T_1 \to T_2$ be a monoid isomorphism of L_1 in L_2 and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}. Then:

- If $y_1 \in F(x_1)$, then by Definition 23 there exist $t_1 \in T_1 - \{0\}$ such that
$$g^{t_1}(x_1) = y_1, \tag{7.47}$$
and therefore, there exist $\rho(t_1) \in T_2$ such that
$$\rho(t_1) \neq \rho(0) \text{ and} \tag{7.48}$$
$$f(y_1) = f(g^{t_1}(x)) = g^{\rho(t_1)}(f(x_1)); \tag{7.49}$$
as a consequence, by Definition 23, $f(y_1) \in F(f(x_1))$.
- If $y_1 \in P(x_1)$ then, by Corollary 5.14.1, $x_1 \in F(y_1)$; then, according to what we have just proved, $f(x_1) \in F(f(y_1))$ and again, by Corollary 5.14.1, $f(y_1) \in P(f(x_1))$.
- if $y_1 \in \Pi(x_1)$ then, by Definition 38, $x_1 = y_1$ and therefore $f(x_1) = f(y_1)$, so that $f(y_1) \in P(f(x_1))$.

In all cases, proof in the converse direction is guaranteed by the fact that f^{-1} is a ρ^{-1}-isomorphism of DS_{L_2} in DS_{L_1}. □

In the special case of time systems, Proposition 7.20 guarantees that the past, present and future of each moment are independent of the chosen time models, modulo their algebraic or dynamical equivalence. In other words, tenses are invariant dynamical properties of time systems. But are they also well-behaved? That is to say: is the dynamical present, as we defined it, the separating element between past and future?

In a sense, it is: in fact, Definition 38 was put forward as the sole possible dynamical definition of presentness which was compatible with those of future and past respectively given by Definition 23 and Definition 25. However, there is a stronger sense in which this is not so: that is to say, Definition 38, Definition 23 and Definition 25 are not sufficient to exclude that, at some time, past, present and future partially overlap each other.

The first condition which we therefore have to impose on time systems so that they can provide us with a consistent dynamical characterization of tenses is that, at each time, past and future must possess no non-empty intersection; and we already know that, according to Proposition 6.21, this condition is satisfied just in case the given time system is completely irreversible.

The second condition we must impose is that, for any possible state of a time system, its present set is entirely disjoint from both its past and future. In general, this condition is satisfied whenever a dynamical system has no periodic orbits.

Proposition 7.21. *Let* $DS_L = (M, (g^t)_{t\in T})$ *be a dynamical system on a monoid* $L = (T, +)$*. For any* $x \in M$*,*

$$\Pi(x) \cap F(x) = \emptyset \tag{7.50}$$

if and only if

$$P(x) \cap \Pi(x) = \emptyset, \tag{7.51}$$

if and only if $orb(x)$ *is not periodic.*

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0 and let $x \in M$ be an arbitrary instant. By Proposition 5.15, $orb(x)$ is periodic if and only if $x \in F(x)$. By Definition 38, this obtains if and only if

$$\Pi(x) \cap F(x) \neq \emptyset. \tag{7.52}$$

Symmetrically, $orb(x)$ is not periodic if and only if $\Pi(x) \cap F(x) = \emptyset$. Similarly, by Corollary 5.15.1, $orb(x)$ is periodic if and only if $x \in P(x)$. By Definition 38, this obtains if and only if

$$P(x) \cap \Pi(x) \neq \emptyset \tag{7.53}$$

so that, symmetrically, $orb(x)$ is not periodic if and only if $P(x) \cap \Pi(x) = \emptyset$. □

Having already required that past and future are entirely disjoint as of each other, on the other hand, is *per se* sufficient to exclude that any present moment could ever intersect its own past or future. In fact, as we already know by Proposition 6.20, completely irreversible dynamical systems possess no periodic points.

Proposition 7.22. *Let* $TS_L = (I, (\iota^t)_{t\in T})$ *be the time system of a monoid* $L = (T, +)$*; then all* $i \in I$ *satisfy conditions*

$$\Pi(i) \cap F(i) = \emptyset, \tag{7.54}$$

$$\Pi(i) \cap P(i) = \emptyset, \tag{7.55}$$

$$P(i) \cap F(i) = \emptyset \tag{7.56}$$

if and only if TS_L is completely irreversible.

Proof
Let $TS_L = (I, (\iota^t)_{t\in T})$ be the time system of a monoid $L = (T, +)$. If all $i \in I$ satisfy condition (7.54)-(7.56) then, by Proposition 6.21, TS_L is completely irreversible. Conversely, if TS_L is completely irreversible then, by Proposition 6.21, all $i \in I$ satisfy condition (7.56). Furthermore, by Proposition 6.20, TS_L must have no periodic orbits which, by Proposition 7.21, means that all $i \in I$ must also satisfy conditions (7.54) and (7.55). □

Proposition 7.22 establishes a direct connection between the existence of an anywhere clear-cut separation of time into past, present and future and the dynamical behavior of the given time model. In a sense, it provides the theoretical link between the philosophical issue of objective temporal becoming and that of the objective direction of time: speaking of a well-behaved, ever changing objective present moment is as much as speaking of the existence of a unique, irreversible direction in time's motion.

For ease of expression, we may also say that any given moment $i \in I$ satisfying conditions (7.54)-(7.56) generates a *local* partition of its time set $T = I$, in the sense that its past, present and future sets are equivalence classes on a (proper) subset of T. To get a fully satisfactory account of tenses, one may also be willing to demand any such partition to be *global*: that is to say, that the past, present and future of any moment jointly span the whole time set T. This requirement is trivially met by the identity element, since we know its orbit to range over the entire state space. However, it is not so in the general case, for which the following condition holds:

Proposition 7.23. *Let $TS_L = (I, (\iota^t)_{t\in T})$ be the time system of a monoid $L = (T, +)$; then, for any $i \in I$,*

$$P(i) \cup \Pi(i) \cup F(i) = I \tag{7.57}$$

if and only if, for all $j \in I$,

$$orb(j) \subseteq orb(i) \quad or \quad orb(i) \subseteq orb(j). \tag{7.58}$$

Proof
Let $TS_L = (I, (\iota^t)_{t\in T})$ be the time system of a monoid $L = (T, +)$ and let $i \in I$. If i satisfies condition (7.57), then for any $j \in I$, $j \in P(i)$ or $j \in \Pi(i)$ or $j \in F(i)$. Given Proposition 5.7: in the first case, by Definition 25, $orb(i) \subseteq orb(j)$; in the second case, by Definition 38, $orb(i) = orb(j)$, i.e. $orb(i) \subseteq orb(j)$ and $orb(j) \subseteq orb(i)$; in the third case, by Definition 23, $orb(j) \subseteq orb(i)$. Conversely if, for any $j \in I$, $orb(i) \subseteq orb(j)$ or $orb(j) \subseteq orb(i)$, then by Proposition 5.7 $i \in orb(j)$ or $j \in orb(i)$. By Definition 14, Definition 23 and Definition 38, in the first case it would be $i \in \Pi(j)$ or $i \in F(j)$ while, in the second case, $j \in \Pi(i)$ or $j \in F(i)$. In sum, $j \in P(i)$ or $j \in \Pi(i)$ or $j \in F(i)$ which, being j arbitrary, entails condition (7.57). □

The most straightforward consequence of Proposition 7.23 is that, to allow for a global partition of time into past, present and future, one has to rule out time models whose time systems are strongly irreversible.

Proposition 7.24. *Let $TS_L = (I, (\iota^t)_{t\in T})$ be the time system of a monoid $L = (T, +)$; if $i \in I$ satisfy condition (7.57), then TS_L has no merging orbits.*

Proof
Let TS_L be the time system of a monoid L. If all $i \in I$ satisfy condition (7.57) then, by Proposition 7.23 there can be no $i, j \in I$ such that $orb(i) \nsubseteq orb(j)$ and $orb(j) \nsubseteq orb(j)$ and therefore, by Definition 15, there can be no $i \in I$ whose orbits is merging. □

Corollary 7.24.1. *Let $TS_L = (I, (\iota^t)_{t \in T})$ be the time system of a monoid $L = (T, +)$; if all $i \in I$ satisfy condition (7.57), then TS_L is not strongly irreversible.*

Proof
Let $TS_L = (I, (\iota^t)_{t \in T})$ be the time system of a monoid $L = (T, +)$; if $i \in I$ satisfy condition (7.57) then, by Proposition 7.24, it can have no merging orbits. In that case, by Proposition 6.18, TS_L is not strongly irreversible. □

Proposition 7.22, Proposition 6.20 and Proposition 7.23 entail that a globally consistent partition of time into past present and future tenses is solely possible for a completely reversible, and thus with no periodic orbits, time system, whose set of orbits is linearly ordered by set inclusion.

7.3

An Unexpected Threat

Proposition 7.22 established a strict connection between the dynamical behavior of a time system and the existence of a well-behaved, although possibly local, partition of its time model into past, present and future tenses. Together with Corollary 6.21.1 and Proposition 7.18, it has the further consequence of making the dynamical interpretation of tenses consistent, namely anywhere well-defined, only in case the assumed time model is not a group, for only in that case the dynamics of the corresponding time system is completely irreversible.

However, physical time is ordinarily conceived as a one-dimensional differentiable manifold (or as one of the four dimensions of a four-dimensional differentiable manifold) which is diffeomorphic to the real line and, as such, owns the algebraic structure of a group. For this reason, it might seem that time systems, rather than providing us with a rigorous representation of temporal becoming, offer an indirect and unexpected refutation of the objectivity of its static ingredient. The sole possible way out of this objection is to reject its basic premise, namely to deny that, all in all, physical time must necessarily be modeled by a group. Under this light, according to Proposition 7.22 and Proposition 7.12, the existence of objective temporal becoming is essentially related to the possibility of deconstructing the mathematical representation of physical time so as to reduce its algebraic structure to that of a monoid none of whose elements has a left inverse: whether it might be possible to do that, and how, will be discussed in the next chapter.

8

Symmetry and Becoming

The discussion we made in Chapter 7 revealed the existence of a logical thread binding the algebraic structure of a time model, the dynamical properties of a time system, and the objectivity of tenses. This way, the problem of objective temporal becoming was reduced to that of the unidirectionality of time's motion which, in our approach, is deeply intertwined with that of the algebraic structure one is willing to provide time with.

This entire picture is radically different from the received view concerning the problem of the direction or the arrow of time, according to which the latter should be interpreted non-dynamically, as a structural or inherent asymmetry of the time manifold, typically consisting of its topological mirror-asymmetry, or anisotropy. In this chapter, we shall examine what relation is there between the received view and the dynamical understanding of the unidirectionality of time. In doing this, we'll have the chance to discuss the efficacy of the standard approach, as well as to face the problem whether the mathematical structures deterministic physical theories are in need of to model physical time should necessarily give rise to non-directional time systems. Our discussion will lead us to conclude that the received view can be no basis to decide whether or nor time is unidirectional, for it tacitly presupposes that time has a symmetric algebraic structure. So, rather than concentrating on whether or not standard time models are structurally symmetric, in the second part of this chapter we shall address the converse question whether or not non-symmetric time model might depict physical time in a satisfactory way.

8.1

Symmetry, Structure and Dynamics

In a sentence, the received view maintains that the direction of time should be interpreted non-dynamically, as an intrinsic asymmetry of the time manifold, consisting in a failure of the time-reversal invariance of physical laws. Our first task will be that of making sense of this claim, and to translate it into the language of general dynamical systems theory, so that it will be possible to compare its content with the results of our inquiry.

The conceptual basis of the standard view was a renewed relational account of time, inspired by the rise of the special and the general theories of relativity and definitively established by the reductionist approaches of Reichenbach (1956, 1958), Mehlberg (1961, 1980), Gold (1962), Penrose and Percival (1962), van Fraassen (1970) and Grünbaum (1973). Schematically, we may synthesize the content of this view by means of the following three tenets:

(a) Time has no dynamical properties.

(b) The direction of time, if there is one, should necessarily be understood as a mirror-asymmetry of the temporal manifold.

(c) Time has no proper structure.

(d) Speaking of the asymmetry of time, what we are actually referring to is the asymmetry physical processes display with respect to the two possible orientations of the temporal manifold they are hosted in.

(e) Physical processes are symmetric with respect to the time manifold just in case the laws governing their evolution are invariant under a reversal of the time order.

Statements (b) and (d) are corollaries of (a) and (c) respectively, while (e) is logically independent of both. General dynamical systems theory rejected the first four statements as a whole (Mazzola, 2010): as of (c) and (d) we saw that, in order to describe the evolution of deterministic systems, time models must be endowed with minimal structural properties of an algebraic kind[1]; as of (a) and (b), on those very structural properties we succeeded in constructing the internal dynamics of time systems. In what follows, however, we shall provisionally assume (b), (d) and (e) as working hypotheses: the idea is that of examining under what conditions the asymmetry in the temporal evolution of a deterministic system would affect the structure of its time model, and then to verify whether such a structural change in the time model would in its turn produce a change in the dynamics of its time system. This will make it possible to compare the results of the classical view with those of general dynamical systems theory.

8.1.1 Time-Reversal Invariance

In Weyl's classical definition, symmetry is commonly understood as 'invariance of a configuration of elements under a group of automorphic transformations (Weyl, 1952, p. i)', which is to say that 'a thing is symmetrical if there is something that you can do to it so that, after you have finished doing it, it looks the same as it did before (Feynman, 1965, p. 84)'.

[1] In this sense, our approach might possibly be ascribed to the "heretic" view initiated by Earman (1972, 1974) and later endorsed by Maudlin (2007), according to which time is in possession of both structural and directional properties on its own, over and above those of physical processes.

To get a more precise definition, we must look at the mathematical theory of models, in whose language symmetries are commonly defined. By a *model*, we mean an ordered pair $\mathbf{S} = (D, (R_i)_{i \in I})$, where D is a non-empty set of *objects*, called the *domain* of $\mathbf{S}$, and, for any $i \in \{1, ..., m, ...\}$ $(m \geq 1)$, there is n such that R_i is a n-ary relation $(n \geq 0)$ among objects of $\mathbf{S}$; for this reason, it is called the *structure* of the model $\mathbf{S}$. We say that the indexed family R_i *characterizes* the objects in the domain of $\mathbf{S}$. We say that any two models $\mathbf{S} = (D, (R_i)_{i \in I})$ and $\mathbf{S}' = (D', (R'_i)_{i \in I})$, are *homologous* just in case their structures have the same number of relations and, for any $i \in I$, relations R_i and R'_i have the same arity. For any two such models, we say that a function $\Phi : D \to D'$ is a *homomorphism* of $\mathbf{S}$ in $\mathbf{S}'$ if and only if, for any n-tuple $(x_1, ..., x_n)$ of objects in the domain of $\mathbf{S}$ and any n-ary relation R_i, if $R_i(x_1, ..., x_n)$ then $R'_i(\Phi(x_1), ..., \Phi(x_n))$. For this reason, we say that homomorphisms *preserve* the structures they are applied on, *fixing* their relations. Bijective homomorphisms whose inverse functions are themselves homomorphisms are called *isomorphisms*; an *automorphism* is any isomorphism of a model in itself. The automorphisms of any model form a group. In this context, symmetries of a model are in fact identified with its automorphisms, and any relation that is preserved by automorphisms is called *invariant* under that group (Rickles, 2008, pp. 11-12).

The concept of symmetry has gained more and more importance in contemporary philosophy of science, to the point of threatening the very idea of law (van Fraassen, 1989). One of the reasons for this success is that studying the symmetries of a model is a way to identify its essential theoretical components: intuitively, if a relation on the domain of a model is fixed by automorphism, then all structures on that model differing only as of that relation bear the same theoretical content (Belot, 2003).

Thinking of the direction of physical time as an asymmetry of the time manifold is thus to consider the temporal order of earlier and later as something which the theoretical representation of physical time cannot dispense with, in the sense that the theoretical content of our physical theories would change if the order of earlier and later was reversed. But what should the structural properties of time consist of, and how to operate a reversal of the time order?

Following the standard, *relational* approach, time has no intrinsic or structural features, all of its properties being reducible to those of the physical processes taking place inside it; accordingly, while speaking of the asymmetry of time one is actually referring to those of physical phenomena. Reversing the order of time is thus to reverse the order in which phenomena are ordinarily observed: if after having performed such a reversal things don't look the same as before, in the sense that different laws would be needed to describe them, then the structure of physical processes is affected by the chosen temporal order, and so must be the emergent properties of time itself[2].

> If the laws are not invariant under time reversal, then we could not state them without presupposing a temporal orientation on the space-time manifold – an objective

[2]For a critique of this approach, see Earman (1967), Sklar (1974) and Tegtmeier (1997).

> distinction between the two temporal directions, indicating in which one things are allowed to evolve and in which they are not. The laws themselves make reference to the distinction. If these are the fundamental laws, then we have reason to infer that the world has the structure needed to support the distinction. If the laws are symmetric under time reversal, then they do not presuppose a temporal orientation[3].
>
> They say the same thing regardless of the direction things evolve in. If these are the fundamental laws, then we would not infer a temporal orientation (North, 2008, p. 203).

The required reversal in the temporal orientation of physical processes is ordinarily performed by means of a so-called *time-reversal operator*. For this reason, whenever the laws of a theory show to be symmetric with respect to a change in the temporal orientation of physical processesthey are labeled *time-reversal invariant*. Making sense of this expression will be our primary task.

The canonical characterization of time-reversal invariance, almost ubiquitous[4] in the philosophical literature, is the following:

> [...] a theory is time reversal invariant just in case, for any sequence of instantaneous states ..., $S(t_0), S(t_1), S(t_2), ...$ allowed by the theory, the reverse sequence of *time-reversed* states ..., $S^{\mathbb{T}}(t_2), S^{\mathbb{T}}(t_1), S^{\mathbb{T}}(t_0), ...$ is also allowed, where $\mathbb{T}$ is the appropriate time reversal operator (time runs left to right) (North, 2008, pp. 206-207).

This statement, however, is quite incomplete. For what does it mean that a sequence of states is 'allowed' by a theory? Or, what should we mean by an 'appropriate' time-reversal operator? Even though not explicitly stated by North, an appropriate time reversal operator is usually required to be involutory, that is to say, if applied twice to any given state, it should return the state itself. Earman (1974), for example, explicitly identifies time reversal operators with involutory maps.

[3]North is apparently assuming the invariance of physical laws under a reversal of time to be both a sufficient and a necessary condition so that time is isotropic. However, this is a quite simplified representation of the standard view. While it is general agreed that the existence of non-invariant laws would be a certain proof of the structural asymmetry of time (and that, conversely, symmetric time would entail the existence of sole invariant laws), whether or not the converse implication is true is more controversial matter. North's account would surely be subscribed by Mehlberg, according to which 'temporal isotropy in scientific contexts is [...] *tantamount* to the covariance of the laws of nature under time reversal (Mehlberg, 1961, pp. 107-108, my emphasis)'. On the contrary, Horwich argues that, even though 'time-asymmetric laws of nature are *sufficient* condition for time to be anisotropic', 'there is no reason to regard this condition as *necessary* for anisotropy (Horwich, 1987, p. 42), since time might exhibit some purely contingent directional features which, as such, are not encoded in the laws of physics. The same contention is also supported by Reichenbach (1956), Bunge (1972) and Grünbaum (1973). For the sake of simplicity, nonetheless, we shall stick to North's reconstruction; this choice will bear no substantial implications for our discussion, leaving the major results of this chapter untouched. For a more complete survey on this topic, see Savitt (1996) and Faye (1997)

[4]Unhortodox interpretations of time-reversal invariance were given by Horwich (1987) and Albert (2000), whose principal difference with the standard view, however, concerned the specific form of the time-reversal operator. Malament (2004) and North (2008) proposed that the structural (a)symmetry of time should better be understood as the (non-)invariance of space-time theories under a reversal of the temporal orientation of the four-dimensional manifold. Similar approaches had also been proposed by Weingard (1977) and discussed by Sklar (1985) and Earman (2002).

How to apply this definition to dynamical systems? In adherence to the language of general dynamical systems theory, we shall say that a series of states is allowed by a theory exactly in case (i) DS_L is an abstract dynamical system formally encoding that theory, (ii) all members of that series are points in the state space of DS_L and (iii) each one of those states is mapped into the subsequent one by means a state transition of DS_L.

For the sake of simplicity, but without any loss of generality, we shall concentrate on series of states which are pairwise separated by equal durations. Series of these kind form what we shall call *t-sequences.*

DEFINITION 39 (t-Sequence)
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0; for any $t \in T$, a t-sequence is any ordered n-tuple of states $x_1, ..., x_n$, with $n > 1$, such that, for any x_i

$$x_{i+1} = g^t(x_i), \tag{8.1}$$

where $i = 1, ..., n - 1$.

Given Definition 39, we can say that a theory is time-reversal invariant just in case there exists an involutory time-reversal operator $\mathbb{T}$ on the state space of the abstract dynamical system DS_L modeling that theory, such that any ordered n-tuple $(x_1, ..., x_n)$ of states of DS_L is a t-sequence only if $(\mathbb{T}(x_n), ..., \mathbb{T}(x_1))$ is. In what follows, we shall outline an alternative but equivalent characterization of time-reversal invariant dynamical systems[5], based on the better-known and more flexible concept of time-symmetry.

8.1.1.1 TIME SYMMETRY

In the course of Chapter 6 we listed, among the possible types of reversible behavior a dynamical system might possibly display, logical reversibility, complete past and complete logical reversibility. These forms of reversible dynamics were called *improper*, because they do not depend on the factual capability of a system to recover its own states by means of its sole state transitions. For the same reason, we shall extend the class of improper types of reversibility so as to include the two concepts of *time symmetry* and *space invertibility.* The distinguishing feature of both concepts is that they essentially rest on the existence of a transformation on the state space of a dynamical system, which in most cases is not a state transition. For this reason, we shall group them together, forming a proper subclass of the improper types of reversible behavior which we shall refer to as *dynamical symmetries.*

DEFINITION 40 (Time Symmetry)
A dynamical system $DS_L = (M, (g^t)_{t \in T})$ on a monoid $L = (T, +)$ is time-symmetric *if and only if DS_L is completely logically reversible and there exists a function $\sim: M \to M$, called* dynamical

[5]See Corollary 8.6.1.

inversion, *such that for any* $x \in M$ *and for any* $t \in T$

$$\sim (g^t(\sim (x))) = (g^t)^{-1}(x). \tag{8.2}$$

Time symmetry is a generalization of the homonymous concept examined by Giunti (1997), and it amounts to a formal translation in the language of general dynamical system theory of what is also called *time-reversal symmetry* (Lamb and Roberts, 1998) or *time-reversibility* (Hoover, 2001).

Proposition 8.1. *Let* $DS_{L_1} = (M_1, (g^{t_1})_{t_1 \in T_1})$ *and* $DS_{L_2} = (M_2, (g^{t_2})_{t_2 \in T_2})$ *be dynamical systems on* $L_1 = (T_1, +)$ *and* $L_2 = (T_2, \oplus)$ *respectively and let* $f : M_1 \to M_2$ *be a* ρ*-isomorphism of* DS_{L_1} *in* DS_{L_2}*; then* DS_{L_1} *is time-symmetric if and only if* DS_{L_2} *is.*

Proof
Let $DS_{L_1} = (M_1, (g^{t_1})_{t_1 \in T_1})$ and $DS_{L_2} = (M_2, (g^{t_2})_{t_2 \in T_2})$ be dynamical systems on $L_1 = (T_1, +)$ and $L_2 = (T_2, \oplus)$ respectively, let $\rho : T_1 \to T_2$ be a monoid isomorphism of L_1 in L_2 and let $f : M_1 \to M_2$ be a ρ-isomorphism of DS_{L_1} in DS_{L_2}. Finally, let DS_{L_1} be time-symmetric. Since DS_{L_1} is completely logically reversible then, by Proposition 6.2.3, so is DS_{L_2}. So, let $\sim : M_1 \to M_1$ be a dynamical inversion function on M_1; then, by bijectivity of f, it is possible to define a function $\neg : M_2 \to M_2$ such that, for all $x_2 \in M_2$

$$\neg(f^{-1}(x_2)) = f(\sim (f^{-1}(x_2))). \tag{8.3}$$

So, let $t_2 \in T_2$ and $x_2 \in M_2$ such that, for some $t_1 \in T_1$ and for some $x_1 \in M_1$,

$$t_2 = \rho(t_1), \tag{8.4}$$
$$x_2 = f(x_1); \tag{8.5}$$

then:

$$\begin{aligned}
g^{t_2}(\neg(g^{t_2}(\neg(x_2)))) &= g^{\rho(t_1)}(\neg(g^{\rho(t_1)}(\neg(f(x_1))))) \\
&= g^{\rho(t_1)}(\neg(g^{\rho(t_1)}(f(\sim (x_1))))) \\
&= g^{\rho(t_1)}(\neg(f(g^{t_1}(\sim (x_1))))) \\
&= g^{\rho(t_1)}(f(\sim (g^{t_1}(\sim (x_1))))) \\
&= f(g^{t_1}(\sim (g^{t_1}(\sim (x_1))))) \\
&= f(x_1) \\
\neg(g^{t_2}(\neg(x_2))) &= \neg(g^{\rho(t_1)}(\neg(f(x_1)))) \\
&= (g^{\rho(t_1)})^{-1}((f(x_1))).
\end{aligned} \tag{8.6}$$

On the other hand, by bijectivity of ρ and f, for all $t_2 \in T_2$ and all $x_2 \in M_2$ there exist $t_1 \in T_1$ and $x_1 \in M_1$ satisfying conditions (8.4)-(8.5). This proves $\neg$ to be a dynamical inversion function on M_2 and, by Definition 40, DS_{L_2} to be time-symmetric. Proof in the converse direction is guaranteed, once again, by bijectivity of ρ and f. □

Time symmetry demands that, for any state x entering the state space M of a dynamical system, a *dynamically inverse* state $\sim (x)$ exists in M whose behavior looks exactly like that of x, if looked backwards. It is easy to verify that any function satisfying condition (8.2) appearing in Definition 40 is an involution on the state space.

Proposition 8.2. *Let $DS_L = (M, (g^t)_{t \in T})$ be a time-symmetric dynamical system on a monoid $L = (T, +)$ with identity 0; then any dynamical inversion function $\sim : M \to M$ is an involution.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a time-symmetric dynamical system on a monoid $L = (T, +)$ and let $\sim : M \to M$ be a dynamical inversion function on M; then, by condition 8.2, for any $x \in M$,

$$\sim(\sim(x)) = \sim(g^0(\sim(x))) = (g^0)^{-1}(x) = x. \tag{8.7}$$

□

One may wonder whether it is possible to define a condition analogous to that of time symmetry, which can nevertheless dispense with the hypothesis of complete logical reversibility. The notion of space invertibility is expressly shaped for this purpose:

Definition 41
A dynamical system $DS_L = (M, (g^t)_{t \in T})$ on a monoid $L = (T, +)$ is space invertible *if and only if there exists a function $\dot{\sim} : M \to M$, called* space inversion, *such that, for any $x \in M$ and any $t \in T$*

$$g^t(\dot{\sim}(g^t(\dot{\sim}(x)))) = x. \tag{8.8}$$

Clearly, time symmetry entails space invertibility, for condition (8.2) is easily transformed into (8.8) by applying g^t on both sides. In other words, the existence of a dynamical inversion function on the state space of a dynamical system makes it *ipso facto* space invertible. On the other hand, Definition 41 does not explicitly require that all state transitions of DS_L are bijective. Is it therefore possible for a dynamical system to be both space invertible and not completely logically reversible? The answer is no.

Proposition 8.3. *Space invertible dynamical systems are completely logically reversible*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ and let $x \in M$ and $t \in T$; if DS_L is space invertible then, by Definition 41, there exists a space inversion function $\dot{\sim} : M \to M$ such that, for any $x \in M$,

$$g^t(\dot{\sim}(g^t(\dot{\sim}(x)))) = x. \tag{8.9}$$

Consequently, by applying $\dot{\sim}$ on both sides,

$$\dot{\sim}(g^t(\dot{\sim}(g^t(\dot{\sim}(x))))) = \dot{\sim}(x) \tag{8.10}$$

and by substituting $\dot{\sim}(x)$ for x in (8.10):

$$\dot{\sim}(g^t(\dot{\sim}(g^t(\dot{\sim}(\dot{\sim}(x)))))) = \dot{\sim}(\dot{\sim}(x)). \tag{8.11}$$

On the other hand, by setting $t = 0$ in (8.9),

$$\dot{\sim}(\dot{\sim}(x)) = x \tag{8.12}$$

so that, by (8.12), $\dot{\sim}$ is an involution on M and, by (8.11) and (8.12), for any $t \in T$, the composed function $\dot{\sim} \circ g^t$ is also an involution.

Hence, for any $t \in T$, g^t is accordingly injective: as $\dot{\sim} \circ g^t$ is injective, for any $x, y \in M$ such that $x \neq y$

$$\begin{aligned} \dot{\sim}(g^t(x)) &\neq \dot{\sim}(g^t(y)) \\ \dot{\sim}(\dot{\sim}(g^t(x))) &\neq \dot{\sim}(\dot{\sim}(g^t(y))) \\ g^t(x) &\neq g^t(y); \end{aligned} \tag{8.13}$$

Moreover, for any $t \in T$, g^t is surjective: as $\dot{\sim} \circ g^t$ is surjective, for any $x \in M$ there is some $y \in M$ such that

$$\begin{aligned} \dot{\sim}(g^t(y)) &= x \\ \dot{\sim}(\dot{\sim}(g^t(y))) &= \dot{\sim}(x) \\ g^t(y) &= \dot{\sim}(x). \end{aligned} \tag{8.14}$$

But on the other hand, thanks to the surjectivity of $\dot{\sim}$, for any $z \in M$ there exists $x* \in M$ such that $\dot{\sim}(x*) = z$; and therefore, by substituting $x*$ for x in (8.14), for any $z \in M$ there exists $y \in M$ such that:

$$g^t(y) = z. \tag{8.15}$$

□

Proposition 8.3 bears two major consequences: on the one hand, it makes the two notions of time symmetry and space invertibility logically equivalent; on the other hand, and consequently, it makes space invertibility dependent on complete logical reversibility as well.

Corollary 8.3.1. *Dynamical systems are space invertible if and only if they are time-symmetric.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$.

If DS_L is time-symmetric, then there exists $\sim: M \to M$ such that, for any $t \in T$ and any $x \in M$

$$\begin{aligned} \sim(g^t(\sim(x))) &= (g^t)^{-1}(x) \\ g^t(\sim(g^t(\sim(x)))) &= g^t((g^t)^{-1}(x)) = x; \end{aligned} \tag{8.16}$$

hence, by Definition 41, DS_L is space invertible.

If DS_L is space invertible, then there exists $\sim: M \to M$ such that, for any $t \in T$ and any $x \in M$

$$g^t(\sim(g^t(\sim(x)))) = x. \tag{8.17}$$

On the other hand, according to Proposition 8.3, DS_L is also completely logically reversible, so that for any state transition g^t an inverse function $(g^t)^{-1}$ can be meaningfully defined. As a consequence, for any $t \in T$ and any $x \in M$

$$\sim(g^t(\sim(x))) = (g^t)^{-1}(g^t(\sim(g^t(\sim(x))))) = (g^t)^{-1}(x). \tag{8.18}$$

Hence, according to Definition 40, DS_L is time-symmetric. □

Corollary 8.3.2. *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$; then, any dynamical inversion function on M is a space-inversion function, and vice-versa.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$. If $\sim: M \to M$ is a dynamical inversion

function on M then, by Definition 40, for any $x \in M$ and any $t \in T$,

$$\begin{aligned} \sim(g^t(\sim(x))) &= (g^t)^{-1}(x) \\ g^t(\sim(g^t(\sim(x)))) &= g^t((g^t)^{-1}(x)) \\ &= x; \end{aligned} \tag{8.19}$$

accordingly, by Definition 41, $\sim$ is a space inversion functionon M.

On the other hand, if $\dot{\sim} : M \to M$ is a space inversion on M then, by Definition 41,

$$x = g^t(\dot{\sim}(g^t(\dot{\sim}(x)))). \tag{8.20}$$

On the other hand, by Definition 41 and Proposition 8.3, DS_L is completely logically reversible, which means that for any $t \in T$, g^t is bijective; accordingly, from (8.20) we get:

$$\begin{aligned} (g^t)^{-1}(x) &= (g^t)^{-1}(g^t(\dot{\sim}(g^t(\dot{\sim}(x))))) \\ . &= \dot{\sim}(g^t(\dot{\sim}(x))). \end{aligned} \tag{8.21}$$

Therefore, by Definition 40, $\dot{\sim}$ is a dynamical inversion function on M. □

Neither logical reversibility nor complete logical reversibility are sufficient for a dynamical system to be time-symmetric: in fact, time symmetry essentially depends not only on the logical properties of the given state transitions, but also on the intrinsic features of the state spaces. Example 6 below shows that, for this reason, not even the stronger condition of time-invertibility guarantees the time symmetry of a dynamical system.

Example 6 (Time-Invertible and not Time-Symmetric Dynamical System)
Let $L = (T, \circ)$ be the set of all bijective functions on $\mathbb{Z}$ together with the standard operation of function composition and let ι be the identity function on $\mathbb{Z}$. In addition, for any $x \in \mathbb{Z}$, let

$$g^t(x) = t(x). \tag{8.22}$$

Then

- $L = (T, \circ)$ is a non-commutative group:
  1. T is closed with respect to $\circ$,
  2. $\circ$ is associative,
  3. ι is the identity element with respect to $\circ$,
  4. for any $t \in T$, there exists $t^{-1} \in T$ such that

  $$t \circ t^{-1} = \iota = t^{-1} \circ t; \tag{8.23}$$

  5. $\circ$ is not commutative;
- $DS_L = (\mathbb{Z}, (g^t)_{t \in T})$ is a dynamical system on $L = (T, \circ)$:
  1. $\mathbb{Z}$ is a nonempty set
  2. $(g^t)_{t \in T}$ is a family of functions on $\mathbb{Z}$, indexed by T,
  3. for any $x \in \mathbb{Z}$ and any $t, w \in T$:

  $$g^{\iota}(x) = \iota(x) = x \tag{8.24}$$

  $$g^{t \circ w}(x) = (t \circ w)(x) = t(w(x)) = g^t(g^w(x)); \tag{8.25}$$

- DS_L is time-invertible: by hypotesis, T is a group;
- DS_L is not time-symmetric: let us suppose as a reductio that, for any $x \in \mathbb{Z}$ and any $t \in T$ a function $\sim: \mathbb{Z} \to \mathbb{Z}$ satisfying condition (8.2) existed; then, by Proposition 8.2, $\sim$ is an involution and hence a bijection; accordingly, there should exist $w \in T$ such that

$$\sim = w \tag{8.26}$$

and consequently, for any $x \in M$ and any $t \in T$:

$$\begin{aligned}
g^w(g^t(g^w(x))) &= (g^t)^{-1}(x) && \text{by (8.2)} \\
g^w(g^w(g^t(g^w(x)))) &= g^w((g^t)^{-1}(x)) && \text{applying } g^w \text{ on both sides} \\
g^t(g^w(x)) &= g^w((g^t)^{-1}(x)) && \text{being } g^w \text{ an involution} \\
g^t(g^w(x)) &= (g^w)^{-1}((g^t)^{-1}(x)) && \text{being } g^w \text{ an involution} \\
g^t(g^w(x)) &= (g^t \circ g^w)^{-1}(x) && \text{for any } t, w \in T\text{: } w^{-1} \circ t^{-1} = (t \circ w)^{-1} \\
g^t(g^w(x)) &= g^w(g^t(g^w(g^w(x)))) && \text{by (8.2)} \\
g^t(g^w(x)) &= g^w((g^t(x))) && \text{being } g^w \text{ an involution} \\
g^w(g^t(g^w(x))) &= g^w(g^w(g^t(x))) && \text{applying } g^w \text{ on both sides} \\
(g^t)^{-1}(x) &= g^w(g^w(g^t(x))) && \text{by (8.2)} \\
(g^t)^{-1}(x) &= g^t(x) && \text{being } g^w \text{ an involution}
\end{aligned} \tag{8.27}$$

  which is plainly false, not all bijections on $\mathbb{Z}$ being involutions (e.g. the successor function).

By Corollary 6.10.2, all time-invertible dynamical systems are completely logically reversible as well. Example 6 has therefore the further consequence of showing that complete logical reversibility is not a sufficient condition to make a dynamical system time-symmetric, as we said above. Finally, the following example shows a space-invertible dynamical system which is nonetheless not reversible.

Example 7 (Completely Irreversible Space-Invertible Dynamical System)
Let $L = (\mathbb{Z}^+, +)$ be the set of non-negative integers, along with arithmetical addition. Furthermore, for any $n \in \mathbb{Z}^+$, let $g^n : \mathbb{Z} \to \mathbb{Z}$ be the function such that, for all $x \in \mathbb{Z}$,

$$g^n(x) = n + x \tag{8.28}$$

and let $\dot{\sim} : \mathbb{Z} \to \mathbb{Z}$ be the function such that, for any $x \in \mathbb{Z}$,

$$\dot{\sim}(x) = -x. \tag{8.29}$$

Then,

- $DS_L = (\mathbb{Z}, (g^n)_{n \in Z^+})$ is a dynamical system on $L = (\mathbb{Z}^+, +)$:
  1. $\mathbb{Z}$ is a non-empty set,
  2. by hypothesis, $(g^n)_{n \in Z^+}$ is a family of functions on $\mathbb{Z}$ indexed by $\mathbb{Z}^+$,
  3. for any $n, m \in \mathbb{Z}^+$ and any $x \in \mathbb{Z}$,

$$g^0(x) = 0 + x = x, \tag{8.30}$$
$$g^{n+m}(x) = (n + m) + x = n + (m + x) = g^n(g^m(x)); \tag{8.31}$$

- by Definition 41, DS_L is space-invertible: for any $x \in M$ and any $n \in \mathbb{Z}^+$,

$$\dot{\sim}(g^n(\dot{\sim}(g^n(x)))) = -(g^n(-(g^n(x)))) = -(n + (-(n+x))) = -(-x) = x; \tag{8.32}$$

- by Definition 35, DS_L is completely irreversible: for any $x \in \mathbb{Z}$ and any $n \in \mathbb{Z}^+$, for any $m \in \mathbb{Z}^+ - \{0\}$,

$$g^m(g^n(x)) = m + (n + x) = (m + n) + x \neq x. \tag{8.33}$$

The complete list of the possible types of reversible behavior we studied is given in Table 8.1, while the logical relations holding among them are shown by Figure 8.1.

Properly reversible dynamics	Improperly reversible dynamics	
	Logical types	Dynamical symmetries
Time invertibility	Complete logical reversibility	Time symmetry
Strict reversibility	Logical reversibility	Space invertibility
Reversibility	Complete past	

Table 8.1: Types of reversible behavior.

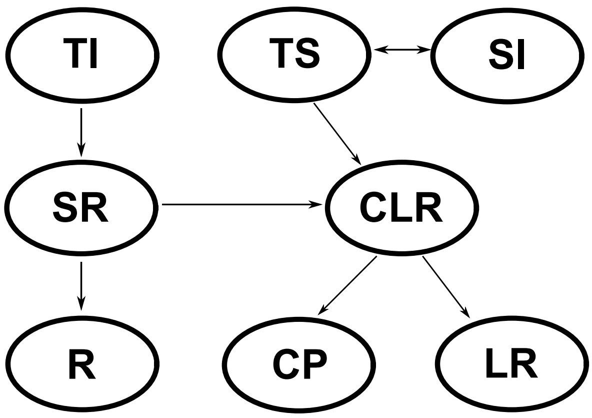

Figure 8.1: Logical relations among all possible types of reversible behavior.

8.1.1.2 Time-Reversal, Dynamical Inversion

The time-reversal operator entering the standard definition of time-reversal invariance is, all in all, just a dynamical inversion function. To prove this, we first need to notice that any two-place t-sequence between points in the state space of a time-symmetric dynamical system generates a unique reversed t-sequence between their dynamical inverses.

Proposition 8.4. *Let $DS_L = (M, (g^t)_{t \in T})$ be a time-symmetric dynamical system on a monoid $L = (T, +)$ with identity 0 and let $\sim: M \to M$ be a dynamical inversion function on M. For any $x, y \in M$ and any $t \in T$,*

$$y = g^t(x) \quad \textit{if and only if} \quad \sim(x) = g^t(\sim(y)). \tag{8.34}$$

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a time-symmetric dynamical system on a monoid $L = (T, +)$ with identity 0, let $\sim: M \to M$ be a dynamical inversion function on M, let $t \in T$ and let $x, y \in M$. By Proposition 8.2 and by definition of dynamical inversion:

$$\begin{aligned} y = g^t(x) \quad & \text{if and only if} \quad \sim (g^t(x)) = \sim (y) \\ & \text{if and only if} \quad \sim (g^t(\sim(\sim(x)))) = \sim (y) \\ & \text{if and only if} \quad (g^t)^{-1}(\sim (x)) = \sim (y) \\ & \text{if and only if} \quad \sim (x) = g^t(\sim (y)) \end{aligned} \tag{8.35}$$

□

To generalize this result to t-sequences of arbitrary length, we need a tool for building t-sequences out of each others. This tool is provided by the following statement.

Lemma 8.1. *Let $DS_L = (M, (g^t)_{t \in T})$ be a time-symmetric dynamical system on a monoid $L = (T, +)$ with identity 0, let $t \in T$ and let $x_i \in M$, with $i = 1, ..., j, ..., n$; then $(x_1, ..., x_j)$ and $(x_j, ..., x_n)$ are t-sequences if and only if $(x_1, ..., x_j, ..., x_n)$ is a t-sequence.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a time-symmetric dynamical system on a monoid $L = (T, +)$ with identity 0; let $t \in T$ and let $x_i \in M$, where $i = 1, ..., j, ..., n$. If $(x_1, ..., x_j)$ and $(x_j, ..., x_n)$ are t-sequences then, by Definition 39, condition (8.1) holds for any $i = 1, ..., j-1$ and for any $k = j, ..., n-1$ and, therefore, it also does for any $l = 1, ..., j-1, j, ..., n-1$; accordingly, by Definition 39, $(x_1, ..., x_{n+1})$ is a t-sequence. Conversely, if $(x_1, ..., x_{n+1})$ is a t-sequence, then condition (8.1) holds for any $i = 1, ..., n-1$ and, *a fortiori*, for any $i = 1, ..., j-1$ and for any $k = j, ..., n-1$; accordingly, by Definition 39, $(x_1, ..., x_j)$ and $(x_j, ..., x_n)$ are a t-sequences. □

Thanks to Proposition 8.4 and Lemma 8.1, we are in a position to build the reversed t-sequences of the dynamical inverses of whatever given t-sequence of states:

Proposition 8.5. *Let $DS_L = (M, (g^t)_{t \in T})$ be a time-symmetric dynamical system on a monoid $L = (T, +)$ with identity 0 and let $\sim: M \to M$ be a dynamical inversion function on M; then for any $x_i \in M$, with $i = 1, ..., n$, and any $t \in T$, $(x_1, ..., x_n)$ is a t-sequence if and only if $(\sim (x_n), ..., \sim (x_1))$ is a t-sequence.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a time-symmetric dynamical system on a monoid $L = (T, +)$ with identity 0, let $\sim: M \to M$ be a dynamical inversion function on M, let $t \in T$ and let $x_i \in M$, where $i = 1, ..., n$. Proof of Proposition 8.5 will proceed by induction on n.

$n = 2$ By Proposition 8.4 and Definition 39, (x_1, x_2) is a t-sequence if and only if $x_2 = g^t(x_1)$, if and only if $\sim (x_1) = g^t(\sim (x_2))$, if and only if $(\sim (x_2), \sim (x_1))$ is a t-sequence.

$n = m$ By inductive hypothesis: $(x_1, ..., x_m)$ is a t-sequence if and only if $(\sim (x_m), ..., \sim (x_1))$ is a t-sequence.

$n = m + 1$ Let $(x_1, ..., x_{m+1})$ be a t-sequence. By Lemma 8.1, $(x_1, ..., x_m)$ and (x_m, x_{m+1}) must be t-sequences too; accordingly $(\sim (x_{m+1}), \sim (x_m))$ is a t-sequence by Proposition 8.4, while $(\sim (x_m), ..., \sim (x_1))$ is a t-sequence by inductive hypothesis. Finally, by Lemma 8.1, $(\sim (x_{m+1}), ..., \sim (x_1))$ is a t-sequence too. Proof in the converse direction goes similarly, *mutatis mutandis*.

□

In accordance to Proposition 8.5, dynamical inversion functions behave like time-reversal operators on the state spaces of time-symmetric dynamical systems. Conversely, Proposition 8.6 below shows that all time-reversal operators acting on the state spaces of dynamical systems behave like dynamical inversion functions on those sets. Before laying it down, let us emphasize that any time-reversal operator, in order to be "appropriate", should be an involutory function.

Proposition 8.6. *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ and let $\sim : M \to M$ be an involutory function such that, for any $x_i \in M$, with $i = 1, ..., n$, and any $t \in T$, if $(x_1, ..., x_n)$ is a t-sequence then $(\sim (x_n), ..., \sim (x_1))$ is a t-sequence; then $\sim : M \to M$ is a dynamical inversion function on M.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0 and let $\sim : M \to M$ be an involutory function such that, for any $x_i \in M$, where $i = 1, ..., n$, and any $t \in T$, if $(x_1, ..., x_2)$ is a t-sequence then $(\sim (x_n), ..., \sim (x_1))$ is a t-sequence. Let $x, y \in M$ such that (x, y) is an a t-sequence; then, by Definition 39, keeping $n = 2$ fixed, for any $x, y \in M$ and any $t \in T$,

$$y = g^t(x). \tag{8.36}$$

By hypothesis and by (8.36)

$$\sim (x) = g^t(\sim (y)), \tag{8.37}$$

which, by substitution, leads to the following:

$$\sim (x) = g^t(\sim (g^t(x))). \tag{8.38}$$

By substituting x for $\sim (x)$ in (8.38):

$$x = \sim (\sim (x)) = g^t(\sim (g^t(\sim (x)))), \tag{8.39}$$

which, by Definition 41, proves that $\sim$ is a space inversion function on M and hence, by Corollary 8.3.2, a dynamical inversion function on that set. □

Finally, Proposition 8.5 and Proposition 8.6 jointly support the expected conclusion:

Corollary 8.6.1. *A dynamical system $DS_L = (M, (g^t)_{t \in T})$ on a monoid $L = (T, +)$ is time-symmetric if and only if there exists an involutory function $\sim : M \to M$ such that, for any $x_i \in M$, with $i = 1, ..., n$, and any $t \in T$, $(x_1, ..., x_n)$ is a t-sequence if and only if $(\sim (x_n), ..., \sim (x_1))$ is a t-sequence.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (DS_L)$; by Corollary 8.3.1 DS_L is time symmetric if and only if there exists a dynamical inversion function on M. On the other hand, by Proposition 8.2, Proposition 8.5 and Proposition 8.6, a function $\sim : M \to M$ is a dynamical inversion on M if and only if it is an involution and, for any $x_i \in M$, with $i = 1, ..., n$, and any $t \in T$, the ordered n-tuple $(x_1, ..., x_n)$ is a t-sequence if and only if $(\sim (x_n), ..., \sim (x_1))$ is a t-sequence. □

The major implication of Corollary 8.6.1 is that demanding a theory to be time-reversal invariant is just tantamount to requiring the dynamical systems modeling that theory to be time-symmetric. By itself, this result might raise no surprise. Nonetheless, it bears deep and unexpected consequences on the usual understanding of time-reversal invariance.

8.1.2 Symmetric Dynamics

Models, as we defined them, are simply abstract models of a theory whose axiomatization is given in set-theoretical terms (Suppes, 1957, p. 253). Under this light, dynamical systems may be understood as models on their own, whose domains consist of their state spaces and whose structures consist of their families of state transition functions. In general dynamical systems theory, the ordinary concept of isomorphism is naturally generalized to that of ρ-isomorphism, according to which any structure-preserving map between state spaces is parameterized to a given structure-preserving map between time models; accordingly, we can put forward the following definition:

Definition 42 (Automorphism of a Dynamical System)
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$; a function $f : M \to M$ is an automorphism *of DS_L if and only if there exists a monoid automorphism $\rho : T \to T$ such that f is a ρ-isomorphism of DS_L in DS_L.*

Definition 42 will make it possible for us to extend the usual concepts of symmetry and invariance to general dynamical systems theory. To give further plausibility to this claim, let us notice that automorphisms of dynamical systems form a group under the standard operation of function composition.

Proposition 8.7. *Let DS_L be a dynamical system on a monoid $L = (T, +)$; then the set of all automorphisms of DS_L, along with the standard operation of function composition, is a group.*

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0, let $\Gamma(DS_L)$ be the set of all automorphisms of DS_L and let $\circ$ be a suitable restriction of the standard operation of function composition with domain $\Gamma(DS_L)$; we shall prove that $(\Gamma(DS_L), \circ)$ satisfies closure, associativity, possession of the identity element and inclusion of the algebraic inverses of all given automorphisms.

Let $f_1, f_2 \in \Gamma(DS_L)$ be two automorphisms of DS_L. By Definition 42, there must exist a monoid automorphism $\rho_1 : T \to T$ and a monoid automorphism $\rho_2 : T \to T$ such that f_1 is a ρ_1-isomorphism of DS_L in DS_L and f_2 is a ρ_2-isomorphism of DS_L in DS_L. Hence,

- $\rho_1 \circ \rho_2 : T \to T$ is a monoid automorphism of L: by the properties of monoid automorphisms;
- for any $x \in M$ and any $t \in T$

$$\begin{aligned} f_1 \circ f_2(g^t(x)) &= f_1(f_2(g^t(x))) \\ &= f_1(g^{\rho_2(t)}(f_2(x))) \\ &= g^{\rho_1(\rho_2(t))}(f_1(f_2(x))) \\ &= g^{\rho_1\circ\rho_2(t)}(f_1 \circ f_2(x)), \end{aligned} \tag{8.40}$$

  so that $f_1 \circ f_2$ is a $\rho_1 \circ \rho_2$-isomorphism of DS_L in DS_L.

Since the existence of $\rho_1 \circ \rho_2$ is guaranteed by those of ρ_1 and ρ_2 then, by Definition 42, $f_1 \circ f_2$ is an automorphism of DS_L; accordingly, $f_1 \circ f_2 \in \Gamma(DS_L)$, and $(\Gamma(DS_L), \circ)$ satisfies closure.

The algebraic rule of composition $\circ$ is associative, according to the standard properties of function composition.

Let ϕ be the identity function on M and let ε be the identity function on T. Then

- ε is a monoid automorphism, whose existence is guaranteed trivially, and
- ϕ is a ε-isomorphism of DS_L in DS_L: for any $x \in M$ and any $t \in T$

$$\phi(g^t(x)) = g^t(x) = g^{\varepsilon(t)}(\phi(x)); \tag{8.41}$$

accordingly, by Definition 42, ϕ is an automorphism of DS_L, and thus $\phi \in \Gamma(DS_L)$. In addition, for any automorphism f of DS_L, for any $x \in M$ and any $t \in T$

$$f \circ \phi(g^t(x)) = f(\phi(g^t(x))) = f(g^t(x)) = \phi(f(g^t(x))) = \phi \circ f(g^t(x)), \tag{8.42}$$

proving that $\Gamma(DS_L)$ is in possession of the identity element.

Finally, for any automorphism $f \in \Gamma(DS_L))$, let rho be such that f is a ρ-isomorphism of DS_L in DS_L;

thanks to the bijectivity of ρ-isomorphisms and monoid isomorphisms, the inverse functions $f^{-1} : M \to M$ and $\rho^{-1} : T \to T$ exist, so that

- f^{-1} is a ρ^{-1}-isomorphism of DS_L in DS_L for any $x \in M$ and any $t \in T$

$$\begin{aligned} f^{-1}(g^t(x)) &= f^{-1}(g^{\rho(\rho^{-1}(t))}(f(f^{-1}(x)))) \\ &= f^{-1}(f(g^{\rho^{-1}(t)}(f^{-1}(x)))) \\ &= g^{\rho^{-1}}(f^{-1}(x)) \end{aligned} \tag{8.43}$$

- for any $x \in M$ and any $t \in T$

$$\begin{aligned} f^{-1} \circ f(g^t(x)) = f^{-1}(g^{\rho(t)}(f(x))) = g^{\rho^{-1}(\rho(t))}(f^{-1}(f(x))) = \\ = e(g^t(x)) = \\ = g^{\rho(\rho^{-1}(t))}(f(f^{-1}(x))) = f(g^{\rho^{-1}(t)}(f^{-1}(x))) = f \circ f^{-1}(g^t(x)); \end{aligned} \tag{8.44}$$

Hence, f^{-1} is an automorphism of DS_L. Together with the above results, this proves that $(\Gamma(DS_L), \circ)$ is a group. □

Proposition 8.7 confirms the adequacy of the adequacy of Definition 42. The study of the internal symmetries of dynamical systems will therefore be brought about as the study of their automorphisms.

8.1.2.1 Time-Reversal Operators Are Not Symmetries

Following the received view, the existence of a time-reversal operator on the domain of a given theory induces a mirror symmetry in the laws regulating the evolution of the systems described by that theory and, as a consequence, should be taken as positive evidence in favor of the existence of a similar symmetry in its time model. So let us wonder: under what condition does the existence of a dynamical inversion function establish a symmetry in the dynamics of a system? Or, equivalently, what further requirements should a time-symmetric dynamical system satisfy, so that a dynamical inversion function on its state space is an automorphism?

Proposition 8.8. *Let $DS_L = (M, (g^t)_{t\in T})$ be a time-symmetric dynamical system on a monoid $L = (T, +)$ and let $g : T \to M^M$ be the indexed family $(g^t)_{t\in T}$; then any dynamical inversion function $\sim: M \to M$ is an automorphism of DS_L if and only if there exists a monoid automorphism $\rho : T \to T$ such that, for any $t \in T$,*

$$g(\rho(t)) = (g(t))^{-1}. \tag{8.45}$$

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$, let $g : T \to M^M$ be the indexed family $(g^t)_{t\in T}$ and let $\sim: M \to M$ be a dynamical inversion function on M.

By Definition 40, for any $x \in M$ and any $t \in T$,

$$\sim (g^t(x)) =\sim (g^t(\sim (\sim (x)))) = (g^t)^{-1}(\sim (x)). \tag{8.46}$$

Suppose $\sim$ is an automorphism of DS_L; then, by Definition 42 and (8.46) there is a monoid automorphism $\rho : T \to T$ such that, for any $t \in T$ and any $x \in M$,

$$\sim (g^t(x)) = g^{\rho(t)}(\sim (x)) = (g^t)^{-1}(\sim (x)). \tag{8.47}$$

Therefore, since $\sim$ is an involution, by substituting $\sim (x)$ for x in (8.47),

$$g^{\rho(t)}(x) = (g^t)^{-1}(x), \tag{8.48}$$

which is just another form for (8.45).

Conversely, suppose that there is a monoid automorphism ρ such that, for any $t \in T$, condition (8.45) holds. Then, for any $t \in T$ and any $x \in M$, (8.48) holds too. By substituting $\sim (x)$ for x in (8.48), and by (8.46), we get (equation: time-reversal invariance 2), and so $\sim$ is an automorphism. □

Unfortunately, there seems to be no general criterion to determine whether a monoid authomorphism ρ of the sort required exists, for its behavior depends both on the algebraic properties of a monoid and on the specific form of the family of state transitions of a dynamical system having it as a time model. In fact, the existence of such monoid automorphism is not a trivial matter:

Proposition 8.9. *Let $DS_L = (M, (g^t)_{t\in T})$ be a time-symmetric dynamical system on a monoid $L = (T, +)$ and let $g : T \to M^M$ be the family $(g^t)_{t\in T}$; if a function $\rho : T \to T$ exists such that all $t \in T$ satisfy condition (8.45), then DS_L is strictly reversible.*

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$ and let $g : T \to M^M$ be the family $(g^t)_{t\in T}$. Finally, let us suppose that a function $\rho : T \to T$ exists such that, for any $t \in T$, ρ satisfies condition (8.45). Hence, for any $t \in T$ there is $r \in T$ such that

$$r = \rho(t), \tag{8.49}$$

$$g^r = g(r) = g(\rho(t)) = (g(t))^{-1} = (g^t)^{-1}; \tag{8.50}$$

hence, by Proposition 6.9, DS_L is strictly reversible. □

In the light of the logical independence existing between time-symmetry and strict reversibility, Proposition 8.8 and Proposition 8.9 deal the first blow on the received view on time-reversal invariance, for they establish once and for all that dynamical inversion functions are not, *qua tales*, automorphisms of dynamical systems. That is to say: time-reversal invariance, commonly understood as the mere existence of a time-reversal operator, is *not* the same thing as the invariance of a given theory under such operator.

How was it possible that such a difference was never acknowledged before? First of all, let us notice that all groups are naturally equipped with a function satisfying the condition demanded by Proposition 8.8:

Lemma 8.2. *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on $L = (T, +)$ and let $g : T \to M^M$ be the indexed faimily $(g^t)_{t \in T}$; if L is a group, then the function mapping any $t \in T$ to its algebraic inverse satisfies (8.45).*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a group $L = (T, +)$, let $g : T \to M^M$ be the indexed faimily $(g^t)_{t \in T}$ and let $\rho : T \to T$ be the function on T such that, for any $t \in T$:

$$\rho(t) = -t; \tag{8.51}$$

then, for any $t \in T$,

$$g(\rho(t)) = g(-t) = g^{-t} = (g^t)^{-1} = (g(t))^{-1}. \tag{8.52}$$

□

Furthermore, being a commutative groups is both a sufficient and necessary condition so that for any such function to be a group automorphism of its domain:

Lemma 8.3. *Let $L = (T, +)$ be a group; the function $\rho : T \to T$ mapping any $t \in T$ to its algebraic inverse is an involutory monoid automorphism on L if and only if L is commutative.*

Proof
Let $L = (T, +)$ be a group with identity 0 and let $\rho : T \to T$ be the function mapping any $t \in T$ to its algebraic inverse element. Then, ρ is an involution: for any $t \in T$,

$$\rho(\rho(t)) = -(-t) = t. \tag{8.53}$$

Furthermore, ρ maps the identity element into itself and it is bijective, while it is structure-preserving if and only if, for any $t, v \in T$

$$t + v = \rho(\rho(t + v)) = \rho(\rho(t) + \rho(v)) = -(-t + (-v)) = -(-(v)) + (-(-t)) = v + t, \tag{8.54}$$

namely if and only if L is commutative; accordingly, ρ is an involutory monoid automorphism on L if and only if L is commutative.

□

The joint product of Lemma 8.2 and Lemma 8.3 is thus the following:

Proposition 8.10. *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on $L = (T, +)$; if L is a commutative group, then any dynamical inversion function $\sim: M \to M$, if it exists, is an automorphism of DS_L.*

Proof
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on $L = (T, +)$ and let $g : T \to M^M$ be the indexed faimily $(g^t)_{t \in T}$. If L is a commutative group, then by Lemma 8.2 the function $\rho : T \to T$ mapping any $t \in T$ to its algebraic inverse satisfies (8.45). In addition, by Lemma 8.3, ρ is a monoid automorphism. Thus, by Proposition 8.8, any dynamical inversion function on M, if it exists, is an automorphism of DS_L. □

Proposition 8.10 provides us with a plausible explanation for the widespread confusion between the mere existence of a time-reversal operator and the invariance of a theory under time-reversal. Philosophical reflection on time-reversal invariance typically focuses on the topological features of time models, rather than on their algebraic properties; as a consequence, time is almost invariably, albeit quite incautiously, supposed to be topologically diffeomorphic to the real line, this way inheriting at least enough algebraic structure to make it a commutative group (Torretti, 2007, pp. 736-738). Under these circumstances, Proposition 8.10 makes the existence of a monoid automorphism of the kind specified by Proposition 8.8 trivial, so that the two distinct concepts of time-reversal invariance and invariance of a theory under the time-reversal operator become, in the common picture, logically equivalent.

8.1.2.2 The Standard Model of Time

Once the conceptual distinction between the mere existence of a time-reversal operator and the invariance of a theory under a reversal of the time order has been made clear, the received view on the problem of the direction of time should better be reformulated as follows: a structural symmetry in the time model of a given theory exists if and only if there exists a time-reversal operator on the domain of the theory *and* such a theory is invariant under that operator. The exact translation of this statement in the language of general dynamical systems theory is the following: a structural symmetry in the time model $L = (T, +)$ of a dynamical system $DS_L = (M, (g^t)_{t \in T})$ exists if and only if DS_L is time-symmetric and the dynamical inversion functions on M are automorphisms of DS_L.

Unfortunately, there seems to be no general recipe to determine how the existence of such automorphisms would affect the algebraic properties of a time model, so that it is not possible to study whether the received view is generally sound. Nonetheless, it is possible to formulate some additional hypotheses, which could make it easier to examine whether that view stands or falls in a relatively wide range of cases.

One crucial assumption underlying the received view is that one should be capable to infer the properties of a time model from those of the state transitions it indexes. The cheapest way to lay down this assumption is to suppose the existence of a structure-preserving map from the one-parameter monoid of state transitions of a dynamical system, namely its transition algebra, to its time model. Since, by Proposition 7.4, any family of state transitions is a surjective monoid homomorphism from the time model of a dynamical system to the set of functions it indexes, this amounts to requiring the transition algebra of a dynamical system to be isomorphic to its time model or, equivalently, to requiring its family of state transitions to be injective.

This condition might appear to be very restrictive at a first glance, but it is not really so. In the first place, as it was shown by Proposition 8.9, dynamical systems which are invariant under time-reversal are necessarily strictly reversible and, by Proposition 6.12, strictly reversible dynamical systems whose families of state transitions are not injective all possess at least one period which is common to all points in their state spaces. Hence, restricting our attention to those systems whose families of state transitions are injective, we are only ruling out systems whose entire evolution is periodically reset. If the issue at stake is to determine whether the fundamental laws of physics call for the existence of any substantial difference between past and future, this seems to be no great loss.

We already know from Proposition 6.11 that all strictly reversible dynamical systems whose families of state transition are injective must *ipso facto* be time-invertible, i.e. their time models must possess the algebraic shape of a group. So, in the first place, what further condition should such time-symmetric, time-invertible dynamical systems satisfy, in order to be invariant under time-reversal?

Lemma 8.4. *Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$ and let the indexed family $g : T \to M^M$ be injective; if a function $\rho : T \to T$ satisfying (8.45) exists, it is unique.*

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$ and let the indexed family $g : T \to M^M$ be injective; finally, let both $\rho : T \to T$ and $\rho' : T \to T$ satisfy (8.45). Then, thanks to the injectivity of g, for any $t \in T$

$$\rho(t) = g^{-1}(g(\rho(t))) = g^{-1}((g(t))^{-1}) = g^{-1}(g(\rho'(t))) = \rho'(t), \tag{8.55}$$

so that ρ and ρ' coincide. □

Together with Lemma 8.2, Lemma 8.4 is telling us that, in case the family of state transitions of a strictly reversible dynamical system is injective, then not only the time model of that system is a group, but the sole function capable of satisfying the condition stated in Proposition 8.8 is the one mapping any element of that group to its algebraic inverse. On the other hand, we know by Lemma 8.3 that such a function is a monoid automorphism if and only if the given group is commutative. So, along with Proposition 8.10, we get the following:

Proposition 8.11. *Let $DS_L = (M, (g^t)_{t\in T})$ be a time-symmetric dynamical system on a monoid $L = (T, +)$; if the indexed family $(g^t)_{t\in T}$ is injective, then any dynamical inversion function $\sim: M \to M$ is an automorphism of DS_L if and only if L is a commutative group.*

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a time-symmetric dynamical system on a monoid $L = (T, +)$ and let the family $(g^t)_{t\in T}$ be injective. If L is a commutative group then, by Proposition 8.10, any dynamical inversion function on M is an automorphism of DS_L. Conversely, if there exists a dynamical inversion function $\sim: M \to M$ such that $\sim$ is an automorphism on DS_L then, by Proposition 8.9, DS_L is strictly reversible and, by Proposition 6.11, time-invertible. In addition, by Proposition 8.8 there must exist a monoid automorphism $\rho : T \to T$ satisfying (8.45); but on the other hand, by Lemma 8.4, ρ must be unique and therefore, by Lemma 8.2, it must be identical to the function mapping any $t \in T$ into its algebraic inverse. Since by Lemma 8.3 that function is a monoid automorphism just in case L is commutative, then L is a commutative group. □

To sum up: the received view is based on a double inference: (i) from the (non-)existence of a time-reversal operator to the (non-)invariance of a theory under such operator, and (ii) from such (non-)invariance to the structural (a)symmetry of time. Proposition 8.11 shows that, in order to allow for both those inferences, one has precisely to model time with a commutative group. This result bears remarkable consequences on the very hard-core of the received view, for it easy to prove such a relatively rich time model to be invariably endowed with a mirror symmetry.

8.1.2.3 Symmetric Time, Circular Approach

Lemma 8.3 showed that any commutative group comes naturally equipped with a symmetry, consisting of the function mapping all of its elements to the corresponding algebraic inverses; what consequences has all this on the dynamical behavior of the corresponding time system?

One straightforward consequence of Proposition 8.11 is that all time systems whose time models are commutative groups are themselves invariant under a reversal of time. Hence,

Corollary 8.11.1. *Let* $TS_L = (I, (\iota^t)_{t \in T})$ *be the time system of a commutative group* L*; then any dynamical inversion function* $\sim: I \to I$*, if it exists, is an automorphism of* L*.*

Proof
Let $TS_L = (I, (\iota^t)_{t \in T})$ be the time system of a commutative group $L = (T, +)$ and let $\sim: I \to I$ be a dynamical inversion function. By Proposition 7.5, the indexed family $(\iota^t)_{t \in T}$ is a monoid isomorphism of L in TA_{TS_L} and, as such, it is injective; as a consequence, by Proposition 8.11, $\sim$ is an automorphism of TS_L. □

But on the other hand, *all* the time systems of commutative groups are time-symmetric:

Proposition 8.12. *The time system of any commutative group is time-symmetric.*

Proof
Let $TS_L = (I, (\iota^t)_{t \in T})$ be the time system of a commutative group $L = (T, +)$ with identity 0. Let $\sim: I \to I$ be the function such that, for any $i \in I$

$$\sim (i) = -i; \tag{8.56}$$

thus, for any $i \in I$ and for any $t \in T = I$

$$\begin{aligned} (t - i) + (\sim (t - i)) &= 0 \\ -i + (\sim (t - i)) &= -t \\ \sim (t - i) &= i + (-t). \end{aligned} \tag{8.57}$$

As a consequence, by commutativity

$$\sim (\iota^t(\sim (i))) = \sim (\iota^t(-i)) = \sim (t - i) = i + (-t) = -t + i = (\iota^t)^{-1}(t). \tag{8.58}$$

In addition, by hypothesis, TS_L is completely logically reversible. Hence, by Definition 40, TS_L is time-symmetric. □

Corollary 8.11.1 and Proposition 8.12 jointly show that the internal dynamics of all commutative groups is invariant under a reversal of time. Intuitively, this means that for any such time model

$L = (T, +)$ and the related time system $TS_L = (I, (\iota^t)_{t \in T})$, one can reverse the way the family $(\iota^t)_{t \in T}$ operates, exchanging the image of any $t \in T$ with that of its algebraic inverse, without changing the dynamics of TS_L.

To sum up: once a one-to-one correspondence between the structure of physical processes and that of physical time is assumed, then the invariance of the given theory under a time-reversal operator guarantees that the algebraic structure of time is that of a commutative group, and hence to possess an internal symmetry. In our language, provided that the family of functions of a dynamical system DS_L is injective, if that system is time-symmetric and invariant under dynamical inversion then its time model is a commutative group. In that case, the time system TS_L of its time model is time-symmetric too, and its dynamical inversion function is an automorphism as well.

Conversely, if a time-reversal operator exists on the domain of the given theory, but the theory is not invariant as of it, then the given time model cannot be a commutative group. Speaking in the language of dynamical systems this means that, if DS_L is time-symmetric but its dynamical inversion functions are not automorphisms of DS_L, then L is not a commutative group. By Proposition 8.8, there are two distinct cases in which this may happen. On the one hand, a function ρ satisfying condition (8.45) may exists, although it is not a monoid automorphism of the given time model. In that case, by Lemma 8.2 and Lemma 8.4, ρ is precisely the function mapping any element of the time model to its algebraic inverse, which makes the time model a group, even though not a commutative one. On the other hand, the required function ρ may not even exist. In that case, there would exist some duration in the given time model, which indexed a state transition whose inverse function, instead, was not indexed by any duration: by Proposition, 6.9, the given system would not be strictly reversible, which would make its time model not even a group. In the former case, the dynamics of the given time model would still be reversible, but not time-symmetric. In the latter case, it would neither be reversible.

This results would seem to vindicate the received view once and for all because, together with Proposition 8.11, they support the inference from the time-reversal (non-)invariance of a theory to the dynamical (non-)equivalence of the future and the past directions of time.

However, thing are not just as straightforward as they seem. For let us suppose that we were given a time-symmetric dynamical system and let us suppose, once again, that the family of its state transitions was injective; then how could we know whether that system was invariant under time-reversal? Following Proposition 8.8, we should go in search of a certain monoid automorphism on its time model; according to Proposition 8.11 that would be as much as wondering whether or not time has the algebraic structure of a commutative group. But here troubles would start coming up.

If we had direct acquaintance with the algebraic structure of time, we would be capable *ipso facto* of determining both its dynamical and its topological properties *without* having to look at the time-reversal invariance of theories. In that case, the received view would be simply useless. But on the other hand, if we had no way of deciding whether or not time is a group, then we would simply be incapable of determining in what cases a time-symmetric dynamical system was also invariant under a reversal of the time direction. In that case, the received view would become inapplicable.

The problem with the received view is that, according to Proposition 8.11, in order to support the very inferences (i) from the existence of a time-reversal operator to the invariance of the given theory under time-reversal and (ii) from the invariance of that theory to the structural properties of its time model, it must take it for granted that such a time model has at least as much structure as it is needed to speak of a *reversal* of the direction of time, in the sense that its time system is both reversible (by Proposition 7.18) and time-symmetric (by Proposition 8.12). But, according to Corollary 8.11.1, that is precisely as much as being demanding that time has the same structure in both directions, putting the cart before the horses.

So, even conceding that the structure of time essentially depended on that of physical processes, the whole problem should be restated as follows. If the basic laws of physics were not time-reversal invariant, namely if no time-reversal operator existed, then we would be sure that time had no internal symmetry. But, on the other hand, if the fundamental laws of physics were all time-reversal invariant, as they seem to be[6], then we could only conclude that time had no proper direction just in case it was modeled by a commutative group, as it ordinarily is. So, the fundamental question becomes: does it really need to?

8.2

Too Much Time?

Symmetries are a guide to the intrinsic features of the phenomena theories describe. But symmetries may also obtain as a result of providing theories with too much mathematical structure, if compared to that which is strictly needed to model such phenomena. In those cases, a theory would offer multiple and equivalent representations for the same phenomenon, so that it would be possible in principle to discard the redundant or superfluous part of its mathematical apparatus without in any way affecting its descriptive efficacy. But how to distinguish between genuine symmetries, and those which only depend on superfluous theoretical structure?

Following Ismael and van Fraassen (2003), we may understand a theory as a theoretical ontology, together with a set of laws. Elements of the ontology should possess enough structure so that only some relations are allowed; this way, only one of the metaphysically possible worlds which could obtain from combining the elements of a given ontology is chosen. The role of laws is that of selecting a subset of points in a metaphysically possible world, corresponding to those worlds which are physically possible, i.e. which may obtain if the right conditions were satisfied. In set-theoretical terms, we may look to physically possible worlds as uninterpreted models of a theory,

[6]For a discussion on this topic see Horwich (1987); Callender (1995); Hutchison (1995a,b); Albert (2000); Earman (2002).

namely as purely mathematical objects satisfying the set-theoretical predicate defining the latter. Once mathematical structures of this kind are given, they should be linked to phenomena by means of the empirical interpretation of their ontologies. Those features which enter the empirical interpretation of a theory but not its mathematical formulation are referred to by Ismael and van Fraassen as *qualities*. In their view, automorphisms on a theory which suggest the existence of superfluous structure in its mathematical formulation are precisely those which (i) preserve satisfaction of laws, mapping physically possible worlds to physically possible worlds (and which are therefore called *symmetries of the laws*) and (ii) preserve all the qualitative features of a model. Following Healey (2009), we may qualify transformations of this kind as *theoretical symmetries*, namely non-trivial automorphisms of the set of models of a theory, connecting models which might equivalently be used to represent the same *empirical situation*.

So far we understood dynamical systems as uninterpreted models (physically possible worlds) of the general theory of dynamical systems, whose ontology is shaped in set-theoretical terms through the very definition of a dynamical system, and whose laws are given by the specific form of their state transitions. Entering the details of how dynamical systems can be interpreted empirically would go beyond the scope of our discussion; what is worth noticing in this context is that (i) whatever the specific form it may take, the very least requirement one should move to the empirical interpretation of a dynamical system is that of being capable of representing all the dynamically relevant features of a physical phenomenon, in which case we shall call it *adequate*, and (ii) in general, the empirical interpretation of a dynamical systems leaves part of its mathematical structure uninterpreted (Giunti, 2007, 2010b). We shall call that part of a dynamical system which is given empirical interpretation the *empirical substructure* of that system (van Fraassen, 1980, p. 64), while the remaining part of that system we shall call *surplus structure* (Redhead, 1975, pp. 87-88).

Surplus structures are undoubtedly redundant, in the sense of being negligible while applying a theory to a given phenomenon; nonetheless, their redundancy might be relative to a given empirical domain, and hence it might disappear as a result of extending the application domain of a theory. Still, redundant structure in the sense envisaged by Ismael and van Fraassen, if there is some, must be searched precisely among the surplus structure of a theory. More precisely, if part of a mathematical structure is redundant in Ismael and van Fraassen's sense, then (a) it is a proper substructure of the given one, (b) there exists at least one adequate interpretation of that substructure and (c) there exists at least one adequate interpretation of the given structure leaving that substructure uninterpreted. In plain words, this means that we could model exactly the same phenomena we model thanks to that substructure by means of a completely different part of the theoretical machinery we are given, in such a way that no relevant empirical phenomenon is left uninterpreted.

8.2.1 Splitting Time in Two

There are principally two different ways to decompose a dynamical system into its proper substructures. The first is to partition its state space into mutually disjoint sets of dynamically connected points (Giunti, 2010a); the second is to split their time models into distinct non-trivial submonoids. In the latter case, which is the one we shall concentrate on, we speak of the *temporal sections* of a dynamical system.

Definition 43 (Temporal Section of a Dynamical System)
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$; a temporal section *of DS_L on a proper submonoid $L' = (T', +|_{T'})$ of L', denoted, by $DS_{L|_{L'}}$, is the ordered pair*

$$DS_{L|_{L'}} = (M, (g^t)_{t\in T'})$$

.

Proposition 8.13. *The temporal sections of a dynamical system are dynamical systems.*

Proof
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0, let $L' = (T', +|_{T'})$ be a proper submonoid of L and let $DS_{L|_{L'}} = (M, (g^t)_{t\in T'})$ be the temporal section of DS_L on L; then $DS_{L|_{L'}} = (M, (g^t)_{t\in T'})$ satisfies all conditions required by Definition 1 to be a dynamical system on L', simply by hypothesis.
□

It is easy to verify that any time-symmetric dynamical system on a linearly ordered commutative group has at least two isomorphic temporal sections, whose temporal orders are opposite to each other.

Let us begin by defining the *positive part* of a linearly ordered group $L = (T, +, \leq)$ with identity 0 as the ordered triple $L^+ = (T^+, +|_{T^+}, \leq|_{T^+})$, where $T^+ = \{t \in T : 0 \leq t\}$, $+|_{T^+}$ is the restriction of $+$ to the set T^+ and, similarly, $\leq|_{T^+}$ is the restriction of $\leq$ to the same set. Symmetrically, let us define the *negative part* of $L = (T, +, \leq)$ as the triple $L^- = (T^-, +|_{T^-}, \leq|_{T^-})$, where $T^- = \{t \in T : t \leq 0\}$. It is then easy to show all linearly ordered commutative groups to be composed of two disjoint (modulo the identity element) specular submonoids.

Lemma 8.5. *The positive and negative parts of any linearly ordered group are linearly ordered submonoids of that group.*

Proof
Let $L = (T, +, \leq)$ be a linearly ordered group with identity 0 and let $L^+ = (T^+, +|_{T^+}, \leq|_{T^+})$ and $L^- = (T^-, +|_{T^-}, \leq|_{T^-})$ be its positive and negative parts. To prove Lemma 8.5, we shall only concentrate on L^+, since the analogous proof for L^- goes similarly, *mutatis mutandis*. By definition, T^+ is a proper subset of L, including the identity element. Hence, to prove that L^+ is a submonoid of L, we only have to show that T^+ is closed with respect to $+|_{T^+}$. So, as a reductio, let us suppose that for some $t, v \in T^+ - \{0\}$, it was the case that

$t +|_{T^+} v \in T^-$; then by definition of T^+ and T^-, we would get

$$\begin{aligned} t +|_{T^+} v \leq t \quad &\text{and} \quad t +|_{T^+} v \leq v, \\ t + v \leq t \quad &\text{and} \quad t + v \leq v, \\ (-t + t) + v \leq -t + t \quad &\text{and} \quad t + (v + (-v)) \leq v + (-v), \\ v \leq 0 \quad &\text{and} \quad t \leq 0, \end{aligned} \tag{8.59}$$

that is, $v \in T^-$ and $t \in T^-$, contrary to the hypothesis. Finally, if either $t = 0$ or $v = 0$ then closure holds trivially. To prove that T^+ is linearly ordered by $\leq|_{T^+}$, it is sufficient to notice that $\leq|_{T^+}$ is transitive, antisymmetric, connected and compatible with $+$ by inheritance from $\leq$. □

It should be remarked that the time systems of the positive and negative parts of a given linearly ordered group L are not temporal sections of its time system TS_L, for they do not possess its entire state space. Rather, the positive and negative parts of L split the time model of TS_L into two specular components, each of which being the time model of a temporal sections of TS_L.

Proposition 8.14. *Let $L = (T, +, \leq)$ be a linearly ordered group with identity 0, let $L^+ = (T^+, +|_{T^+}, \leq|_{T^+})$ and $L^- = (T^-, +|_{T^-}, \leq|_{T^-})$ be its positive and negative parts, let $L^-* = (T^-, +|_{T^-}, \geq|_{T^-})$ be the linearly ordered monoid obtained from L^- by reversing its linear order and let $\rho : T \to T$ be the function mapping any $t \in T$ to its algebraic inverse; then:*

(i) the restriction $\rho|_{T^+} : T^+ \to T^-$ of ρ

1. *is an isomorphism of $(T^+, +|_{T^+})$ in $(T^-, +|_{T^-})$ if and only if L is commutative;*
2. *is an isomorphism of $(T^+, \leq|_{T^+})$ in $(T^-, \geq|_{T^-})$.*

(ii) the restriction $\rho|_{T^-} : T^- \to T^+$ of ρ

1. *is an isomorphism of $(T^-, +|_{T^-})$ in $(T^+, +|_{T^+})$ if and only if L is commutative;*
2. *is an isomorphism of $(T^-, \leq|_{T^-})$ in $(T^+, \geq|_{T^+})$.*

Proof

In order to prove Proposition 8.14, we only need to prove statement (i), statement (ii) following similarly. So let $L = (T, +, \leq)$ be a linearly ordered group with identity 0, let $L^+ = (T^+, +|_{T^+}, \leq|_{T^+})$ be its positive part, let $L^- = (T^-, +|_{T^-}, \leq|_{T^-})$ be its negative part, and let $L^-* = (T^-, +|_{T^-}, \geq|_{T^-})$ be the linearly ordered monoid obtained from L^- by reversing its linear order. Furthermore, let $\rho : T \to T$ be the function mapping any $t \in T$ to its algebraic inverse, and finally let $\rho|_{T^+} : T^+ \to T$ be the restriction of ρ to T^+. By Lemma 8.5, L^+ is a linearly ordered monoid; then:

1. $\rho|_{T^+}$ is an isomorphism from $(T^+, \leq|_{T^+})$ to $(T^-, \geq|_{T^-})$:
   - it is bijective: by hypothesis, any $t \in T^+$ has a unique algebraic inverse $-t \in T^-$, while any $-t \in T^-$ is the algebraic inverse of exactly one $t \in T^+$;
   - for any $t, v \in T^+$:

$$\begin{aligned} t &\leq v \\ \rho|_{T^+}(v) + t + \rho|_{T^+}(t) &\leq \rho|_{T^+}(v) + v + \rho|_{T^+}(t) \\ \rho|_{T^+}(v) &\leq \rho|_{T^+}(t) \\ \rho|_{T^+}(t) &\geq \rho|_{T^+}(v). \end{aligned} \tag{8.60}$$

2. $\rho|_{T^+}$ is an isomorphism of $(T^+, +|_{T^+})$ to $(T^-, +|_{T^-})$ if and only if L is commutative:

- it is bijective: as before;
- it maps identity element into identity element:

$$\rho|_{T^+}(0) = -0 = 0; \tag{8.61}$$

- it is structure-preserving if and only if L is commutative:

(a) if L is commutative then, for any $t, v \in T^+$:

$$\rho(t+|_{T^+}v) = \rho(t+v) = -(t+v) = -v+(-t) = -t+(-v) = \rho(t)+\rho(v) = \rho(t)+|_{T^-}\rho(v) \tag{8.62}$$

(b) if L is not commutative then, for some $t, v \in T^+$ and thanks to the bijectivity of $\rho|_{T^+}$:

$$\begin{aligned} t+|_{T^+}v &\neq v+|_{T^+}t \\ \rho(t+|_{T^+}v) &\neq \rho(v+|_{T^+}t) \\ &\neq \rho(v+t) \\ &\neq -(v+t) \\ &\neq -t+(-v) \\ &\neq -t+|_{T^-}(-v) \\ &\neq \rho(t)+|_{T^-}\rho(v). \end{aligned} \tag{8.63}$$

□

Symmetries of this kind, producing two specular structures out of a given one, we may generally call *chiral symmetries*. However, we are not interested in chiral time models as such: what is of main interest for our purposes is that time-reversal operators are capable of transferring chiral symmetries from time models to their associated dynamical systems.

Proposition 8.15. *Let $DS_L = (M, (g^t)_{t\in T})$ be a time-symmetric dynamical system on a linearly ordered commutative group $L = (T, +, \leq)$ with positive part $L^+ = (T^+, +|_{T^+}, \leq|_{T^+})$ and negative part $L^- = (T^-, +|_{T^-}, \leq|_{T^-})$, and let $L^-* = (T^-, +|_{T^-}, \geq|_{T^-})$ be the linearly ordered monoid obtained from L^- by reversing its linear order. Furthermore, let $\rho : T \to T$ be the function mapping any $t \in T$ to its algebraic inverse and let $\rho|_{T^+} : T^+ \to T$ and $\rho|_{T^+} : T^- \to T$ be the restrictions of ρ to T^+ and T^- respectively; then*

(i) any dynamical inversion function on M is a $\rho|_{T^+}$-isomorphism of $DS_{L|_{L^+}}$ in $DS_{L|_{L^-*}}$, and

(ii) any dynamical inversion function on M is a $\rho|_{T^-}$-isomorphism of $DS_{L|_{L^-*}}$ in $DS_{L|_{L^+}}$.

Proof

Let $DS_L = (M, (g^t)_{t\in T})$ be a time-symmetric dynamical system on a linearly ordered commutative group $L = (T, +, \leq)$ with positive part $L^+ = (T^+, +|_{T^+}, \leq|_{T^+})$ and negative part $L^- = (T^-, +|_{T^-}, \leq|_{T^-})$. By Lemma 8.5, L^+ and L^- are linearly ordered submonoids of L; so, let $L^-* = (T^-, +|_{T^-}, \geq|_{T^-})$ be the linearly ordered monoid obtained from L^- by reversing its linear order. By Definition 43, $DS_{L|_{L^+}} = (M, (g^t)_{t\in T^+})$ and $DS_{L|_{L^-*}} = (M, (g^t)_{t\in T^-})$ are temporal sections of DS_L and hence, by Proposition 8.13, they are dynamical systems. In addition, let $\rho : T \to T$ be the function mapping any $t \in T$ to its algebraic inverse and let $\rho|_{T^+} : T^+ \to T$

and $\rho|_{T^+} : T^- \to T$ be the restrictions of ρ to T^+ and T^- respectively; by Proposition 8.14, they are monoid isomorphisms. Finally, let $\sim: M \to T$ be a dynamical inversion function; then for any $t \in T^+$ and any $x \in M$

$$\sim (g^t(x)) = \sim (g^t(\sim (\sim (x))) = (g^t)^{-1}(\sim (x)) = g^{-t}(\sim (x)) = g^{\rho|_{T^+}(t)}(\sim (x)), \tag{8.64}$$

while for any $t \in T^-$, for any $x \in M$

$$\sim (g^t(x)) = \sim (g^t(\sim (\sim (x))) = (g^t)^{-1}(\sim (x)) = g^{-t}(x) = g^{\rho|_{T^-}(t)}(\sim (x)); \tag{8.65}$$

accordingly, by Definition 3, (i) $\sim$ is a $\rho|_{T^+}$-isomorphism of $DS_{L|_{L+}}$ in $DS_{L|_{L-*}}$ and (ii) $\sim$ is a $\rho|_{T^-}$-isomorphism of $DS_{L|_{L-*}}$ in $DS_{L|_{L+}}$. □

Proposition 8.15 makes all time-symmetric dynamical systems on a linearly ordered commutative group decomposable into two chiral temporal sections; what consequences does this property bear on their empirical interpretation?

Let $DS_L = (M, (g^t)_{t \in T})$ be a time-symmetric dynamical system of the kind just described and let $DS_{L|_{L+}}$ and $DS_{L|_{L-*}}$ be two chiral temporal sections of DS_L. Let us start by interpreting the sole $DS_{L|_{L+}}$; since it shares the same state space as DS_L but only part of its time model, this can intuitively be done by simply applying any given empirical interpretation of DS_L, with the sole proviso of leaving the negative part of L temporarily uninterpreted.

For the sake of simplicity, let us assume the given interpretation of DS_L to be adequate. Once the empirical interpretation of $DS_{L|_{L+}}$ has been performed, let us wonder: is there still anything left in the empirical phenomenon which can be modeled by DS_L?

Certainly, no dynamically relevant magnitude other than time, for by hypothesis the state space of $DS_{L|_{L+}}$ is identical to that of DS_L, whose interpretation we supposed to be adequate. So, our question becomes: is there any interval of physical time which has not been modeled by the empirical interpretation we gave of $DS_{L|_{L+}}$?

Interpreting temporal intervals is just as much as interpreting state transitions, for we know a surjective monoid homomorphism to exist between the time model of any given dynamical system and its transition algebra, which under the current hypothesis is bijective. So, is there any physical process left in the given phenomenon which was not included in our interpretation of $DS_{L|_{L+}}$? Since $DS_{L|_{L+}}$ and $DS_{L|_{L-*}}$ are isomorphic, we have good reasons to suppose that this cannot be the case: in fact, for any transition $g^{-t} : x \to g^{-t}(x)$ taking place in the negative temporal section $DS_{L|_{L-*}}$ of DS_L there must exist a transition $g^t : \sim (x) \to (g^t(\sim (x)))$ being its exact positive duplicate in $DS_{L|_{L+}}$, and vice versa. Hence, the empirical substructure of DS_L might possibly reduce to either the sole $DS_{L|_{L+}}$ or the sole $DS_{L|_{L+*}}$.

If that was really the case, given the empirical interpretation of one of a pair of chiral temporal sections of a dynamical system, the other one would become surplus structure. In addition, being the two temporal sections isomorphic, the converse situation may obtain as well. Hence, all of the three conditions pointing to the existence of superfluous theoretical structure would be satisfied: (a) each of a pair of chiral temporal sections of a dynamical system is a proper substructure of that system, (b) each one is adequately interpreted whenever the given dynamical system is, and (c) there may exist at least one adequate interpretation of that system leaving one of its two chiral temporal sections uninterpreted.

8.2.2 When Worlds Collide

Ismael and van Fraassen lingered on chiral symmetries for a while, wondering whether they might be evidence of superfluous theoretical structure. In particular, they noticed that 'there should be a strong suspicion of superfluous structure if two distinct worlds are related by a transformation that has some world as a fixed point (Ismael and van Fraassen, 2003, p. 387)', which is precisely the type of symmetry displayed by chiral temporal sections of dynamical systems. In the meanwhile, however, they warned us against being too confident with symmetries of this type, for discarding either of the two chiral components of a given structure as a mere duplicate of the other could lead to the paradoxical consequence of denying the existence of their composition.

The same warning was also raised, in a more specific way, by Earman. In his words, the hypothesis we discussed so far would be ascribable to what he called the Reichenbach-Gold view, according to which any two models differing only as their time order 'are not descriptions of two different physically possible worlds but rather are "equivalent descriptions" of one and the same world (Earman, 1974, p. 23)'. One of the major difficulties he attributed to this view is that, 'if on the Reichenbach-Gold position, all possible worlds are not to collapse into a single one, there must be some objective distinguishing feature which separates them and which can be ascertained to hold independently of the direction of time (Earman, 1974, p. 23)'; however, he didn't see any distinguishing feature of that kind.

Is the interpretation we gave of chiral dynamical systems affected by these type of shortcomings? I submit that it is not. Ismael and van Fraassen discussed chiral symmetries as transformations of purely mathematical structures, or *symmetries of worlds*. So, while speaking of discarding either of two chiral structures as a mere redundant copy of the other, they were referring to mathematical objects of which no empirical interpretation was yet been given. In our view, instead, what should be dismissed as redundant is not either of two isomorphic chiral dynamical systems, but its empirical interpretation. Under this light, time-symmetric, time-invertible dynamical systems obtaining as a composition of two isomorphic chiral temporal sections are perfectly legitimate mathematical objects, which might nonetheless be too expensive to model empirical phenomena.

On the other hand, Earman's objection seems to be going too far. Why should it be that, following the Reichenbach-Gold view, *all* possible worlds should collapse on each other? Even granting, with him, that the time-reversal operator had no privileged status among the possible types of symmetry a theory might display, that would still be not enough to support an overall collapse of all its possible models. In general, structural equivalence is mediated by a symmetry; so at best, one may argue that all *symmetric* kinematic models of a theory, independently of the type of symmetry relating them, would be pairwise equivalent to each other. This, however, would raise no difficulty: truly, symmetric kinematic models would amount to merely different instantiations of the same abstract model, precisely as we assumed so far; but in no way this would make all abstract models of a theory reduce to one. In our case, the objective distinguishing feature keeping different abstract models separated as of each other would be precisely their dynamical non-equivalence.

Can we thus conclude that the invariance under dynamical inversion of dynamical systems on commutative groups is a certain symptom of the redundancy of their time models? Unfortunately, not. Surely, symmetries of this kind are positive evidence in favor of such a redundancy, but they are not yet a proof. To establish once and for all that our representation of the physical world could dispense with time models which are commutative groups, one should be able to prove that all differential equations physical laws consist of could lead to the same results if they were restricted to the sole domain of non-negative (non-positive) real numbers, provided they were modeled by time-symmetric dynamical systems.

Giving this proof goes beyond the aim of our discussion. However, what is of our interest is to underline that modeling physical time as a commutative group is not a mandatory choice. But if so then the possibility of providing time with a completely irreversible time-system, and hence with a well-behaved dynamically grounded representation of tenses, is still open, and the hypothesis of objective temporal becoming with it.

9

Conclusion

Chapter 2 was dedicated to examine whether the philosophical dispute between presentist and eternalist ontologies might cast some light on the problem of objective temporal becoming, either proving or disproving the existence of an ontological basis on which tenses could be objectively established. Our conclusion was that, insofar as this contention is framed inside a classical space-time background, there can be no decisive argument in favor of either position, nor they can be of any utility for the problem of objective temporal becoming, for in that case the existence of an absolute partition of space-time into subsequent layers of co-presentness is guaranteed *ab initio*.

In the subsequent chapter, we accordingly left the classical scenario in favor of that of special relativity theory. Rather than confronting eternalism with presentism, in that case we held an eternalist ontology, examining the contention between its radical, full-view interpretation and its moderate, hybrid ones. Once again, we concluded that there is nothing in the very structure of the assumed space-time framework which could tip the scales in favor of either position, giving either a definite refutation or confirmation of the existence of a metaphysically distinguished present moment. Rather, we showed that more than one objective and exhaustive definition of co-presentness other than chronological simultaneity, albeit necessarily weakened, can be supported by that structure.

Chapter 4 was instead dedicated to discuss the logical consistency and the significance of providing time with dynamical properties. The main result of our discussion was that the idea of objective temporal becoming is neither internally inconsistent nor meaninglessness, at least as long as one is willing to accept that time could move or pass in a specific, non-kinematic sense.

This way, we defused some of the major threats to objective temporal becoming, though at the cost of renouncing a straightforward metaphysical interpretation of tenses and a naive interpretation of the dynamical features of time's motion. Moving to general dynamical systems theory, we were finally capable to reconceive the movement of time as a non-metaphysical albeit mind-independent and indispensable component of our representation of deterministic systems.

Chapter 5 outlined the foundations of a general theory of deterministic motion, one of whose minimal requirements is that time should be modeled by a monoid. This result was the starting point for both the analysis of the various types of reversible dynamics we analyzed in Chapterr 6 and for the construction of time-systems we made in Chapter 7. In particular, time systems show that a different way of understanding the objective passage of time is available, consisting in nothing more than its algebraic features. Finally, we proved that the model of objective temporal becoming offered by time systems is tenable just in case physical time can be given a weaker algebraic structure than that of a group, while in Chapter 8 we gave positive, though not conclusive, evidence in favor of this thesis.

9.1 Discussion

Time systems thus offer a very general model for objective temporal becoming, rooting all basic ingredients of time's passage in the sole algebraic properties of time models. In the first place, they provide any time model with a proper dynamics, which we showed to coincide with the motion of the identity element. In the second place, we showed that motion to be at the basis of a dynamical interpretation of tenses, which we proved to be both objective and, under some very general conditions, entirely consistent. The time has come to examine whether such model is also capable to overcome the philosophical objections which we saw to stand against objective temporal becoming, or whether it may possibly be susceptible to any objection on its own.

9.1.1 Setting Time in Motion

In the course of Chapter 4, we discussed two logical objections to the claim that time flows, namely Smart's charge of inconsistency, articulated along his two-pronged *reductio*, and Grünbaum's charge of triviality, later reshaped in the form of Price's charge of circularity.

The analysis we made showed that, rather than denying the passage of time as such, the aim of Smart's argumentation was to deny that time could move in a way analogous to that of solid bodies, owing definite kinematic properties such as a determinate instantaneous speed, a well-defined position in space *at different times*, etc. Time systems offer a consistent, non-kinematic model for the dynamical component of temporal becoming, which is capable to provide time with a well-defined acceptation of passage, while escaping both horns of Smart's critique at once.

To begin with, the state spaces and the time models of time systems always coincide: the distinction between time sets and sets of instants is merely *functional*, namely they are just different interpretations of the very same set. This way, time systems identify the temporal reference of becoming with time itself, this way dodging the second of Smart's *reductiones* at the very outset. What about the first one? Its major premise was that, if the motion of time was to be measured with respect to time itself, then time had to possess a well-defined instantaneous speed. General dynamical systems theory refuted this claim twice. In the general case, it made it possible to speak of the dynamical evolution of deterministic systems with discrete state spaces or on discrete time sets, for which no standard definition of speed is available. On the other hand, in the special case of time systems, it made it possible to prove that providing time models with internal dynamics is just a different way to represent their algebraic structure, so that requiring it to be mathematically modeled by a monoid is all that we need to set time in motion.

Our discussion of Grünbaum's argument, instead, led us to conclude that claiming that time moves from past to future, rather than being a mere truism, is a theoretically meaningful assertion, whose content is that there exists a well-defined distinction between the past and the future directions of time. The study we made of time systems and of the dynamical interpretation of tenses they support confirms this conclusion: to say that time moves invariably *from* past *to* future is just as much as saying that the dynamics of the given time system is completely irreversible, which Proposition 7.22 showed in its turn to be logically equivalent to the existence of an everywhere clear-cut distinction among tenses.

Incidentally, time systems also refute a minor objection which have occasionally been moved against the objectivity of temporal becoming, namely that the scientific description of phenomena simply can do away with it, nothing in the laws of physics requiring the existence of a unique moving now (Smart, 1955; Grünbaum, 1967a; Park, 1972): since our description of deterministic systems cannot dispense with assuming time to be a monoid, as we saw, for that very reason it is also forced to provide time with a proper dynamics.

9.1.2 Back to Geometry

One possible critique which can be raised against our model is that, as we had the chance to notice while discussing the here-now conception of the present[1], reducing co-presentness to a binary relation which is coextensive to the identical relation may result in trivializing the very idea of the objective present:

> It would [...] be a complete trivialization of the thesis of the mind-*in*dependence of becoming to [say] that, *by definition*, an event occurring at a certain clock time t has the unanalyzable attribute of nowness at t (Grünbaum, 1967a, p. 27).

One may charge the dynamical acceptation of presentness, given by Definition 38, of precisely this type of deficiency: in fact, it identifies the present of each moment with the corresponding image of the time transition of null duration, which we know very well to model the identity relation on the time set.

This objection is nonetheless rejected by Corollary 7.19.1, in the light of which *being present at time* $i = t$ can be reduced to the property of lying in the t-future image of the identity element.

[1] See § 3.3.1.

The dynamical acceptation of presentness is accordingly neither unanalyzable nor trivial: in fact, it brings positive content concerning the dynamics of the identity element, the algebraic structure of time models, and the way state transitions are temporally indexed; all in all, Definition 38 is telling us something positive about the way we expect deterministic systems to behave.

There is still another, more serious difficulty the dynamical interpretation of tenses must overcome. Moving from the usual space-time framework to general dynamical systems theory, we were forced to abandon the standard ontology of events or time-places in favor of an ontology of states *and* times. This ontological gap seems to be blocking any attempt to transfer our results directly into a space-time scenario. The time sets of dynamical systems were so far treated independently of their state spaces, as they were entirely distinguishable, either structurally or functionally, as of the latter ones; however, relativistic theories consider time a non-separable component of the four-dimensional spatio-temporal manifold (Earman, 1970). So, how to embed time models in relativistic space-times?

Unfortunately, there seems to be no straightforward answer to this question. The most intuitive solution would be that of interpreting time sets through the proper times of moving particles; however, that solution would fail in the case of complex systems, whose evolution depends on the behavior of several, possibly remote, components. Perhaps a more exotic solution could be that envisaged by Rietdijk (1985) and Peacock (1992), according to whom the present moment should consist, at each time, of a hypersurface of constant action, i.e. a three-dimensional layer of events on which the product of energy and duration is constant. Since the total energy of a closed deterministic system is ordinarily assumed to be fixed, each layer would provide us with a spatio-temporal representation of the instantaneous state of a system, while subsequent layers would describe the evolution of that system in time; the one-dimensional time-like cross-section of the union of all such layers would accordingly describe its time model. Unfortunately, the instantaneous distribution of energy in a deterministic system is not necessarily homogeneous, so that according to this proposal different components of a unique state might happen to belong to different nows. Furthermore, such proposal might give rise to privileged frames of reference (Clifton and Hogarth, 1995).

Nevertheless, the model we built may reveal not to be so strongly committed to a classical world view, as long as we renounce to the very idea of a global instantaneous state and we regard becoming as a purely local matter (Dieks, 2006a): after all, wasn't it the very conceptual core of Einstein's critique of chronological simultaneity? Maybe. From this point of view, the motion of time would really look like that of an ideal fluid, whose particles would all move independently as of each other, even though all downstream. Perhaps, *pace* Smart, "the river of time" might possibly be no so unfit image to describe temporal becoming.

9.1.3 What Makes Time Special

Perhaps, in providing time with a proper dynamics, we have gone a bit too far. As all monoids have an associated time system whose state-space is identical to the domain of the monoid, there is a definite sense in which all monoids pass or flow. But then, consider a dynamical system whose state space represents the possible positions of a material particle moving in one-dimensional space. Similar to time-systems, the state-space of this system is identical to its time set, namely the set of real numbers. What could then prevent us from saying that space is flowing as well?

This possible objection overlooks the functional distinction which we repeatedly observed to hold between instants and durations: what we called instants or moments have in general the role of modeling the subsequent stages of an arbitrary algebraic structure, understood as a system which is capable to undergo a deterministic evolution; durations have in general the function of modeling the temporal distance separating those stages, as well as that of identifying the functions connecting them. So, while the domain of any monoid can serve both as the state space and the time set of a time system, its elements are attached different meanings as of they are understood as moments or durations. In the first case, they are the states through which a quite special dynamics evolves (namely, the dynamics established by equation (7.2). In the second case, however, *independently of this specific dynamics*, such elements would count as durations, or intervals *of time*. Nevertheless, no monoids other than those whose associated dynamics is the one specified by (7.2) could ever be claimed to flow or pass with respect to themselves, i.e. to possess an intrinsic dynamics: for only in this case the time models and the associated dynamical systems turn out to be equivalent descriptions of essentially the same entity, as Corollary 7.4.1 and Proposition 7.5 formally show.

But why is it so? What is there, which impose us to understand time models as mathematical representations *of time*? There seem to be two, related reasons for this. On the one hand, as Smart's own argumentation pointed out, we can speak of movement or passage only by referring to a temporal dimension. In other words, taking place in time is an essential theoretical component of the very notion of dynamics, so that insofar as dynamical systems are understood to model the evolution of deterministic systems, their time models must be understood *ipso facto* as being modeling time. On the other hand, the role of time models is that of indexing the family of transformations which govern the evolution of deterministic systems; so to say, they provide the one parameter which makes the transition algebra of a dynamical system a *one-parameter* monoid of transformations. Skow (2007) and Callender (2010b) independently argued that fulfilling this role is precisely what makes time different from space: in order to keep laws as simple as possible one has to keep the number of independent variables as small as possible, and durations are precisely the unique independent parameter one needs to get the simplest possible laws: '[t]ime is the measure of change: its existence simply consists of there being functions giving the magnitudes of other quantities at different times. So time is given as the totality of possible arguments of such functions (Dummett, 2000, p. 509)'

In this view, time flows exactly because, and only insofar as, it is an independent variable. To understand this claim, let us begin with the basic intuition that anything which moves should possess different states *at different times*. Here's the crucial point: dynamics essentially requires instants to vary, but what is there to make them *change*, if not the brute fact that times flows? To make this point another way: dynamical laws require the trajectory of any moving object to be a function of time; but what should be time a function of? Surely not of space, nor of any physical process, for otherwise we would run into the very same kind of absurdities we encountered while discussing the objections to Smart's second argument[2]. It is essential to the very concept of motion (and hence to that of speed) that it should be determined with respect to a physical variable whose values are capable to change freely, i.e. without being functionally related to any other quantity entering the dynamical picture: It is this change which time's passage consists of, a change which is both a brute fact of experience and a logical requirement of physical laws: if we cannot arrest the course of events, it is exactly because there is nothing in the physical world on which time could depend.

[2]See § 4.1.2.2.

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