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On two problems related to the Laplace operator

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Abstract

We investigate maximization of the functional $\Omega \mapsto \mathcal{E}(\Omega)$ where Ω runs in the set of compact domains of fixed volume v in any Riemannian manifold (M, g) and where $\mathcal{E}(\Omega)$ is the *mean exit time from* Ω *of the Brownian motion.* Concerning this functional, we study its critical points and prove that they are harmonic domains. We analyze the special case of the Coarea formula when we take a Morse function. We investigate minimization and maximization of the principal eigenvalue of the Laplacian under mixed boundary conditions in case the weight has indefinite sign and varies in the class of rearrangements of a fixed function g_0 defined on a smooth and bounded domain Ω in $\mathbb{R}^n$. We prove existence and uniqueness results, and in special cases, we prove results of symmetry and results of symmetry breaking for the minimizer.

Contents

Introduction

This PhD thesis is divided into two parts: the main subject of the first one is the mean exit time of the Brownian motion from a compact and connected domain Ω of a Riemannian manifold M; while, in the second one we deal with the optimization of the principal eigenvalue of the Laplacian under mixed boundary conditions.

The thesis is organized as follows: in the second chapter we develop, following the guidelines of the work by L. Cadeddu S. Gallot and A. Loi [12], an alternative analytical proof of a known fact about harmonic domains and a new remark about the Coarea Formula

$$\int_M \varphi(x)\ dv_g(x) = \int_{\inf f}^{\sup f} \left(\int_{f^{-1}(\{t\})} \frac{\varphi(x)}{\|\nabla f(x)\|}\ da_t(x) \right) dt$$

which is not valid when applied to C^∞ functions.
In the first chapter we describe in detail the original results as in [12].

Let (M, g) be an $n-$dimensional Riemannian manifold, and let Ω be a compact and connected domain in M with smooth boundary. Denoting by Δ the Laplace Beltrami operator on M associated to the Riemannian metric g, we study the following problem:

$$\begin{cases} \Delta f = 1 & \text{on } \Omega \\ f = 0 & \text{on } \partial\Omega\ . \end{cases} \tag{1}$$

It is known that it has a unique solution $f_\Omega(x) > 0$ in $H^2_{1,c}(\Omega)$, i. e. the completion of the space of C^∞ functions with compact support in the interior of Ω. In $H^2_{1,c}(\Omega)$ we define the functional E_Ω by

$$E_\Omega(f) = \frac{1}{Vol(\Omega)} \left(2 \int_\Omega f\, dv_g - \int_\Omega |\nabla f|^2\, dv_g \right).$$

Since f is a critical point of E_Ω if and only if f is a solution of (1), and since the functional E_Ω has a unique critical point (i.e. f_Ω the solution of (1)), we define the *mean exit time of the Brownian motion from* Ω as

$$\mathcal{E}(\Omega) = E_\Omega(f_\Omega) = \max_{f \in H^2_{1,c}(\Omega)} E_\Omega(f).$$

We say that a domain $\Omega \subset M$ is *harmonic* if the function $x \mapsto \|\nabla f_\Omega(x)\| = \frac{\partial f_\Omega}{\partial N}(x)$ (where $\frac{\partial}{\partial N}$ is the derivative with respect to the inner unit normal) is constant on the boundary $\partial\Omega$. A known result about harmonic domains states the following

Theorem 2.2.2. *The harmonic domains of volume v in (M,g) are exactly the critical points of the functional $\Omega \mapsto \mathcal{E}(\Omega)$, defined on the set of domains $\Omega \subset M$ with smooth boundary and with fixed volume v.*

The classical proof in [41] is based on Brownian motion techniques, but we give an alternative analytical proof using the definition of *regular smooth paths* (also called variations) and the *first variation formula* that claims that for any function k defined on a (small) neighborhood of the closure $\bar{\Omega}$ of Ω

$$\frac{d}{dt}_{|t=0}\left(\int_{\Omega_t} k \; dv_g\right) = -\int_{\partial\Omega} k\, u \; dv_g \quad , \tag{2}$$

where u is a particular function closely related to the variation of the domain Ω. The other tools are the Green formula and the introduction of an appropriate harmonic function.
As to the Coarea Formula, we observe that if (M,g) is any Riemannian manifold of dimension n and if $f : M \to \mathbb{R}$ is a C^n *Morse* function, by denoting by $\mathcal{C}(f)$ the set of critical points of f, that is the set of points $x \in M$ such that $\nabla f(x) = 0$, and by denoting by $\mathcal{S}(f)$ the image of $\mathcal{C}(f)$ under f, the Coarea Formula is valid in the following form

For any non negative continuous function φ on a Riemannian manifold (M,g),

$$\int_M \varphi(x)\; dv_g(x) = \int_{\inf f}^{\sup f}\left(\int_{f^{-1}(\{t\})} \frac{\varphi(x)}{\|\nabla f(x)\|}\; da_t(x)\right) dt \quad , \tag{3}$$

where, by $\int_{\inf f}^{\sup f}$, we intend the integral (with respect to the Lebesgue measure) on $]\inf f, \sup f[\setminus \mathcal{S}(f)$; this integral has sense, because by Sard's theorem $\mathcal{S}(f)$ has measure zero. Moreover, as we only integrate with respect to regular values t of f, $\{f = t\}$ is a submanifold of codimension 1 in M and da_t is well defined as the $(n-1)-$dimensional Riemannian measure on $\{f = t\}$ (viewed as a Riemannian submanifold of (M,g)).

In fact, if we consider a Morse function, not only $\mathcal{S}(f)$ has measure zero by Sard's theorem, but also $\mathcal{C}(f)$ has measure zero.
In the remaining part of this section we deal with the maximization of $\mathcal{E}(\Omega)$ that the authors studied in [12]. The problem of the maximization of $\mathcal{E}(\Omega)$ is really interesting: it can be seen as a generalization of a similar problem for a domain in the Euclidean plane (we report it in Appendix A of this thesis), but also when we take a Riemannian manifold (M,g) that is isoperimetric at some of its points x_0; in fact, in this case even if every geodesic ball centered at x_0 is an harmonic domain, the converse it is not true. So, since the critical points of $\mathcal{E}(\Omega)$ are harmonic domains, the previous one is a motivation to study its maxima.

In the second part we will show new results concerning the minimization and the maximization of the principal eigenvalue λ_g of the Laplacian under mixed boundary conditions in the case where the weight has indefinite sign and varies in a class of rearrangements. We study the cases where $\Omega = (0, L)$ and where Ω is a α-sector. In the last section we consider a case of symmetry breaking for the minimization problem. In the 2-dimensional case, we take $\Omega = B_{a,a+2}$, the annulus of radii a, $a + 2$ and we show that, despite the symmetry of the data, tha solution may not be symmetric. All results have been published in the article [11], a joint work with L. Cadeddu and G. Porru.

Let $\Omega \subset \mathbb{R}^2$ be a smooth and bounded domain that represents a region occupied by a population that diffuses at rate D and grows or declines locally at the rate $g(x)$. We suppose that the boundary $\partial\Omega$ is divided in two parts, Γ e $\partial\Omega\backslash\Gamma$. We also suppose that there exists a hostile population outside Γ and that there is no flux of individuals across $\partial\Omega\backslash\Gamma$. If $\phi(x, t)$ is the population density, his behavior is described by the logistic equation

$$\frac{\partial\phi}{\partial t} = D\Delta\phi + g(x)\phi - \kappa\phi^2 \quad \text{in } \Omega \times \mathbb{R}^+,$$
$$\phi = 0 \quad \text{su } \Gamma \times \mathbb{R}^+, \quad \frac{\partial\phi}{\partial\nu} = 0 \quad \text{su } (\partial\Omega\backslash\Gamma) \times \mathbb{R}^+,$$

where $\Delta\phi$ is the spatial Laplacian of $\phi(x, t)$, κ is the carrying capacity and ν is the exterior normal to $\partial\Omega$.
We consider the corresponding eigenvalue problem

$$\Delta u + \lambda g(x)u = 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{su } \Gamma, \quad \frac{\partial u}{\partial\nu} = 0 \quad \text{su } \partial\Omega\backslash\Gamma. \tag{4}$$

Many results and applications related to such eigenvalue problems are discussed in [17, 2, 37, 40].
If λ_g is the principal eigenvalue of (4), from [15] and [16] we know that we have population persistence if and only if $\lambda_g < \frac{1}{D}$.
We study the problems of minimization and maximization of λ_g where $g(x)$ varies in the set of rearrangements of a given function $g_0(x)$, $g_0(x)$ being a bounded measurable function in Ω that takes positive values in a set of positive measure.
The corresponding problem with Dirichlet boundary conditions has been investigated by many authors, see [18, 19, 20, 23] and references therein. For the case of the p-Laplacian see [22, 44]. For the case of Neumann boundary conditions see [32]. Eigenvalue problems for nonlinear elliptic equation with unilateral constrains are discussed in [26]. Finally, the problem of competition of two or more species has been discussed in [14, 38].
More generally, we study the case where Ω is a smooth bounded domain in $\mathbb{R}^N$. Let $\mathcal{G}$ be the class of rearrangements generated by g_0 and let

$$W_\Gamma^+ = \Big\{ w \in H^1(\Omega) : \ w = 0 \ \text{ on } \ \Gamma, \ \int_\Omega g\, w^2 dx > 0 \Big\}.$$

Then we have

$$\lambda_g = \inf_{w \in W_\Gamma^+} \frac{\int_\Omega |\nabla w|^2 dx}{\int_\Omega g\ w^2 dx} = \frac{\int_\Omega |\nabla u_g|^2 dx}{\int_\Omega g\ u_g^2 dx}, \tag{5}$$

Our results when Ω is a domain in $\mathbb{R}^n$ are the following

Theorem 3.2.7. *Let λ_g be defined as in* (5)*.*
(i) The problem

$$\min_{g\in\mathcal{G}} \lambda_g$$

has (at least) one solution.
(ii) If $\hat{g}$ is a minimizer then $\hat{g} = \phi\big(u_{\hat{g}}\big)$ for some increasing function $\phi(t)$.

In the proof of the previous theorem, we apply the following lemma (it follows from Lemma 2.4 of [9])

Lemma 3.2.6. *Let $\mathcal{G}$ be the set of rearrangements of a fixed function $g_0 \in L^p(\Omega)$, $p \geq 1$, and let $w \in L^q(\Omega)$, $q = p/(p-1)$. If there is an increasing function ϕ such that $\phi(w) \in \mathcal{G}$ then*

$$\int_\Omega g\, w\, dx \leq \int_\Omega \phi(w)\, w\, dx \qquad \forall g \in \overline{\mathcal{G}},$$

and the function $\phi(w)$ is the unique maximizer relative to $\overline{\mathcal{G}}$. Furthermore, if there is a decreasing function ψ such that $\psi(w) \in \mathcal{G}$ then

$$\int_\Omega g\, w\, dx \geq \int_\Omega \psi(w)\, w\, dx \qquad \forall g \in \overline{\mathcal{G}},$$

and the function $\psi(w)$ is the unique minimizer relative to $\overline{\mathcal{G}}$.

The result concerning the maximization problem is

Theorem 3.2.10. *Let λ_g be defined as in* (5)*. The problem*

$$\max_{g\in\overline{\mathcal{G}}} \lambda_g$$

has a solution; if $\int_\Omega g_0(x)dx \geq 0$, the maximizer $\check{g}$ is unique; if $\int_\Omega g_0(x)dx > 0$, we have $\check{g} = \psi\big(u_{\check{g}}\big)$ for some decreasing function $\psi(t)$; finally, if $g_0(x) \geq 0$ then the maximizer $\check{g}$ belongs to $\mathcal{G}$.

To prove the previous theorem we apply

Proposition 3.2.9. *Let λ_g be defined as in* (5)*, and let $J(g) = \frac{1}{\lambda_g}$. The map $g \mapsto J(g)$ is Gateaux differentiable with derivative*

$$J'(g;h) = \frac{\int_\Omega h\, u_g^2 dx}{\int_\Omega |\nabla u_g|^2 dx}.$$

Furthermore, if g satisfies $\int_\Omega g(x)dx \geq 0$, the map $g \mapsto \lambda_g$ is strictly concave.

In particular, we study the cases $N = 1$, when Ω is an α-sector; and finally, the case where $\Omega = B_{a,a+2}$ is an annulus. In the last case we have symmetry breaking for the minimization problem.
Firstly, concerning the one dimensional case, our results are

Theorem 3.3.2. *Let λ_g be defined as in* (5) *with $\Omega = (0, L)$ and $u(L) = 0$. Then, for all $g \in \mathcal{G}$ we have $\lambda_g \geq \lambda_{g^*}$, where g^* is the decreasing rearrangement of g.*

For proving this result we apply the definition (5) and two well known inequalities on rearrangements.
On the other hand, for solving the maximization problem, supposing that $\int_0^L g(x)dx > 0$, we construct an appropriate function $\breve{g}$, such that $\lambda_{\breve{g}}$ is the maximizer. The result is

Theorem 3.3.4. *If $\int_0^L g(x)dx > 0$, let ρ such that $\int_0^\rho g_*(x)dx = 0$. Define $\breve{g} = 0$ for $0 < x < \rho$, and $\breve{g} = g_*$ for $\rho < x < L$. If λ_g is defined as in (5) with $\Omega = (0, L)$ and $u(L) = 0$, for all $g \in \mathcal{G}$ we have $\lambda_g \leq \lambda_{\breve{g}}$.*

Secondly, concerning the α-sector, we make the assumption $0 < \alpha \leq \pi$.
Similarly to the one dimensional case we have

Theorem 3.3.8. *Let λ_g be defined as in (5), where $\Omega = D$ is the α-sector defined in (3.26) and Γ is the portion of ∂D with $r = R$. Then $\lambda_g \geq \lambda_{g^*}$, where g^* is the decreasing rearrangement of g.*

In addition, for the maximization problem we find

Theorem 3.3.10. *If $\int_D g(x)dx > 0$, let $D_\rho \subset D$ be the α-sector such that $\int_{D_\rho} g_*(x)dx = 0$. Define $\breve{g} = 0$ for $x \in D_\rho$, and $\breve{g} = g_*$ for $D \setminus D_\rho$. Let λ_g be defined as in (5), where $\Omega = D$ is the α-sector defined in polar coordinates (r, θ) as*

$$D = \{0 \leq r < R, \quad 0 < \theta < \alpha\}, \tag{6}$$

and Γ is the portion of ∂D with $r = R$. Then $\lambda_g \leq \lambda_{\breve{g}}$.

Finally, we study the case of an annulus.

Theorem 3.3.10. *Let $N = 2$ and $\Omega = B_{a,a+2}$, the annulus of radii a, $a + 2$. Suppose $g_0 = \chi_E$, where E is a measurable set contained in Ω and such that $|E| = \pi\rho^2$, $0 < \rho < 1$. Let $\mathcal{G}$ be the family of rearrangements of g_0. Consider the eigenvalue problem (3.1) in Ω with Γ being the circle with radius $a + 2$. If a is large enough then a minimizer of λ_g in $\mathcal{G}$ cannot be radially symmetric with respect to the center of $B_{a,a+2}$.*

To prove the previous theorem we take an appropriate function g that is not radially symmmetric and we find an upper bound for λ_g that is independent of a. The next step is to prove that if g is radially symmetric with respect to the center of $B_{a,a+2}$ and if Λ is a minimizer for λ_g, then $\Lambda \to \infty$ if $a \to \infty$.

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Part I

Maximization of the mean exit time of the Brownian motion in a domain

Chapter 1

Maximization of the mean exit time of the Brownian motion in a domain

1.1 Introduction

In this chapter, we start considering the following Dirichlet problem on a Riemannian manifold.
Let (M, g) be an $n-$dimensional Riemannian manifold (compact or not), d the associated Riemanian distance and dv_g the associated Riemaniann measure. We suppose that Ω is any compact connected domain in M, with *smooth boundary* $\partial\Omega$ (by this, in the case where M is compact, we also intend that the interior of $M \setminus \Omega$ is a non empty open set). If we denote by Δ the Laplace-Beltrami operator [1] on M, associated to the Riemannian metric g, we have the following

$$\begin{cases} \Delta f = 1 & \text{on } \Omega, \\ f = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

We denote by f_Ω a solution of (1.1).
Let $C_c^\infty(\Omega)$ be the space of C^∞ functions with compact support in the interior of Ω; and let $H_{1,c}^2(\Omega)$ be its completion with respect to the Sobolev norm $\|f\|_{H_1^2(\Omega)} = (\|f\|_{L^2(\Omega)}^2 + \|\nabla f\|_{L^2(\Omega)}^2)^{\frac{1}{2}}$.
The solution f_Ω belongs to $H_{1,c}^2(\Omega)$ [2]. Moreover, $f_\Omega(x) > 0$ for any $x \in \mathring{\Omega}$.

[1] By "Laplacian Δ", we mean the Laplacian with the sign convention of the geometers (which is the opposite of the sign convention of the analysts) and whose eigenvalues are non negative; thus $\Delta f = -\mathrm{Trace}(\nabla df)$ and the Euclidean Laplacian writes $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial^2 x_i}$.

[2] As the solution f_Ω of the equation (1.1) is regular, it is bounded and thus $f_\Omega \in L^2(\Omega)$; moreover, by Green's formula

$$\int_\Omega |\nabla f_\Omega|^2 \, dv_g = \int_\Omega f_\Omega \, \Delta f_\Omega \, dv_g = \int_\Omega f_\Omega \, dv_g < +\infty$$

and thus $f_\Omega \in H_1^2(\Omega)$; as f_Ω vanishes on $\partial\Omega$, then $f_\Omega \in H_{1,c}^2(\Omega)$.

We define the functional E_Ω on the space $H^2_{1,c}(\Omega)$ as follows

$$E_\Omega(f) = \frac{1}{\mathrm{Vol}(\Omega)} \left(2\int_\Omega f\, dv_g - \int_\Omega |\nabla f|^2\, dv_g \right). \tag{1.2}$$

As the solution of (1.1) is unique (see Paragraph 1.2), we have

Definition 1.1.1. *Let $\Omega \subset M$ be as above. The* mean exit time from Ω of the Brownian motion *(or briefly the mean exit-time from Ω) is the value*

$$\mathcal{E}(\Omega) = E_\Omega(f_\Omega) = \max_{f \in H^2_{1,c}(\Omega)} \left(E_\Omega(f) \right).$$

The solution f_Ω and the value $\mathcal{E}(\Omega)$ have physical meaning. The function $f_\Omega(x)$ is the first exit time from Ω of the Brownian motion starting from $x \in \Omega$; while, $\mathcal{E}(\Omega)$, called the "mean exit time from Ω of the Brownian motion", is the mean value of $f_\Omega(x)$ with respect to all initial points $x \in \Omega$ (see [41] for more details).

In the next paragraphs we deal with the maximization of the mean exit time $\mathcal{E}(\Omega)$. In particular, we maximize $\mathcal{E}(\Omega)$ when (M, g) is an isoperimetric manifold, and more generally we study the comparison of the mean exit time from Ω and its symmetrized Ω^*, where $\Omega \subset M$ and Ω^* is a particular domain in (M^*, g^*, x^*) a PIMS (Pointed model space) of (M, g). More precisely, we consider the two following set of manifolds which have a universal PIMS. Firstly, we consider the class of non compact manifolds (M, g) which have dimension $n \leq 4$. Secondly, we study the set of compact manifolds whose elements have Ricci curvature bounded from below by the Ricci curvature of the canonical sphere. Finally, we analyze the class of compact manifolds which have Cheeger's isoperimetric constant bounded from below by a positive constant H. In this case, the authors find an upper bound for $\mathcal{E}(\Omega)$ where Ω is any compact domain in (M, g) contained in the previous class.
All previous results about maximization are contained in [12].
In Appendix A we illustrate the simpler case of a Dirichlet problem in $\mathbb{R}^2$ where instead of the mean exit time $\mathcal{E}(\Omega)$ we maximize the energy integral $I(F)$.

1.2 The functional $\mathcal{E}(\Omega)$

Firstly, considering the functional E_Ω defined on $H^2_{1,c}(\Omega)$

$$E_\Omega(f) = \frac{1}{\mathrm{Vol}(\Omega)} \left(2\int_\Omega f\, dv_g - \int_\Omega |\nabla f|^2\, dv_g \right),$$

we prove that the solution f_Ω of (1.1) is unique.

Lemma 1.2.1. *The functional $f \mapsto E_\Omega(f)$ (defined on $H^2_{1,c}(\Omega)$) admits a unique critical point, which is the function f_Ω; thus f_Ω is the unique absolute maximum of E_Ω.*
As a consequence, the equation (1.1) admits f_Ω as a unique solution.

Proof. Applying Green's formula we find, for any $h \in H^2_{1,c}(\Omega)$

$$\frac{d}{dt}_{|t=0} \left(E_\Omega(f + t.h) \right) = \frac{1}{\mathrm{Vol}(\Omega)} \left(2\int_\Omega h\ dv_g - 2\int_\Omega h\ \Delta f\ dv_g \right) \quad .$$

Then

$$(\, f \text{ is a critical point of } E_\Omega\,) \iff (\, f \text{ is a solution of (1.1)}).$$

Since there exists (at least) one solution f_Ω of (1.1), the functional E_Ω admits at least one critical point.
Moreover, the functional E_Ω is strictly concave. In fact, for every $f \in H^2_{1,c}(\Omega)$ we can define

$$e(t) = E_\Omega((1-t)f_\Omega + tf) \quad t \in [0,1],$$

and if $f \neq f_\Omega$ we find that e is strictly concave and $e'(0) = 0$. We deduce that $e'(t) < 0$ for every $t \in (0,1]$ and therefore, as $e'(1) \neq 0$, f cannot be a critical point of E_Ω. And so f_Ω is the unique critical point.
Moreover, $e(1) < e(0)$ and thus $E_\Omega(f) < E_\Omega(f_\Omega)$, and so f_Ω is the (unique) absolute maximum. Consequently, the problem (1.1) admits a unique solution. □

Remark 1.2.2. There are two possible definitions of mean exit-time that can be found in the classical literature: the one considered in [12], i.e.

$$\mathcal{E}(\Omega) = \frac{1}{\mathrm{Vol}(\Omega)}\left(2\int_\Omega f_\Omega dv_g - \int_\Omega |\nabla f_\Omega|^2 dv_g\right) = \frac{1}{\mathrm{Vol}(\Omega)}\int_\Omega f_\Omega dv_g$$

and, more frequently, the functional

$$\widetilde{\mathcal{E}}(\Omega) = 2\int_\Omega f_\Omega \; dv_g - \int_\Omega |\nabla f_\Omega|^2 \; dv_g = \int_\Omega f_\Omega \; dv_g.$$

When Ω is a domain of the Euclidean plane, $\widetilde{\mathcal{E}}(\Omega) = \mathrm{Vol}(\Omega)\,.\,\mathcal{E}(\Omega)$ is the torsional rigidity of a beam whose cross-section is Ω. While, when M is Riemannian manifold of any dimension, $\mathcal{E}(\Omega)$ still has the physical meaning of "mean exit-time of the Brownian motion from Ω" . This is one reason for preferring the functional $\mathcal{E}(\Omega)$ to $\widetilde{\mathcal{E}}(\Omega)$.
Obviously, looking to the critical points of $\mathcal{E}(\Omega)$ or of $\widetilde{\mathcal{E}}(\Omega)$ (among all domains of prescribed volume v) there are two equivalent problems; in particular, looking for the maximum of $\mathcal{E}(\Omega)$ or $\widetilde{\mathcal{E}}(\Omega)$ among all domains of prescribed volume v are two equivalent problems.
However, another reason to prefer the functional $\mathcal{E}(\Omega)$ to $\widetilde{\mathcal{E}}(\Omega)$ is the following. If on the same domain Ω we change the Riemannian metric g in the homothetic metric $\lambda^2\, g$,

$$\mathcal{E}(\Omega, \lambda^2\, g) = \lambda^2\, \mathcal{E}(\Omega, g) \qquad \text{and} \qquad \widetilde{\mathcal{E}}(\Omega, \lambda^2\, g) = \lambda^{n+2}\, \widetilde{\mathcal{E}}(\Omega, g), \tag{1.3}$$

and hence $\mathcal{E}(\Omega, g)$ has the same homogeneity as the Riemannian metric g.
In fact, comparing the Laplacians of the two metrics, a direct computation gives $\Delta_{\lambda^2\, g} = \lambda^{-2}\, \Delta_g$ and thus, if f_Ω is the solution of problem (1.1) on the domain Ω endowed with the metric g, then the solution of problem (1.1) on the same domain Ω endowed with the metric $\lambda^2\, g$ is $\lambda^2\, f_\Omega$. We get

$$\widetilde{\mathcal{E}}(\Omega, \lambda^2\, g) = \int_\Omega \left(\lambda^2\, f_\Omega\right)\; dv_{\lambda^2\, g} = \lambda^{n+2}\, \widetilde{\mathcal{E}}(\Omega, g).$$

Since $\mathrm{Vol}(\Omega, \lambda^2 g) = \lambda^n \mathrm{Vol}(\Omega, g)$, we obtain

$$\mathcal{E}(\Omega, \lambda^2 g) = \frac{\widetilde{\mathcal{E}}(\Omega, \lambda^2 g)}{\mathrm{Vol}(\Omega, \lambda^2 g)} = \frac{\lambda^{n+2}\, \widetilde{\mathcal{E}}(\Omega, g)}{\lambda^n \mathrm{Vol}(\Omega, g)} = \lambda^2\, \mathcal{E}(\Omega, g). \tag{1.4}$$

But the main reason to prefer $\mathcal{E}(\Omega)$ to $\widetilde{\mathcal{E}}(\Omega)$ is Theorem 1.5.1, where there is a comparison between the mean exit-times of two domains in two different Riemannian manifolds.

1.3 Symmetrization of domains and functions

Let (M, g) and (M^*, g^*) be two Riemannian manifolds such that $\mathrm{Vol}(M, g)$ and $\mathrm{Vol}(M^*, g^*)$ are both infinite or both finite. The constant $\alpha(M, M^*)$ is defined by

$$\alpha(M, M^*) = \begin{cases} 1 & \text{if } \mathrm{Vol}(M, g) \text{ and } \mathrm{Vol}(M^*, g^*) \text{ are both infinite,} \\ \dfrac{\mathrm{Vol}(M, g)}{\mathrm{Vol}(M^*, g^*)} & \text{if } \mathrm{Vol}(M, g) \text{ and } \mathrm{Vol}(M^*, g^*) \text{ are both finite.} \end{cases}$$

Let x^* be a fixed point in M^*.

Definition 1.3.1 (Symmetrized domain)**.** *For any compact domain $\Omega \subset M$ with smooth boundary (let us recall that this also implies that the closure of Ω is a strict subset of M), its "symmetrized domain"* [3] *is defined as the geodesic ball Ω^* of (M^*, g^*), centered at x^*, such that* $\mathrm{Vol}(\Omega^*) = \alpha(M, M^*)^{-1} \mathrm{Vol}(\Omega)$.

Definition 1.3.2 (Pointed model space)**.** *(M^*, g^*, x^*) is said to be a "pointed model-space for (M, g)" if, for any compact domain $\Omega \subset M$, with smooth boundary (let us recall that this also implies that the closure of Ω is a strict subset of M), the symmetrized domain Ω^* satisfies the* **isoperimetric inequality** $\mathrm{Vol}_{n-1}(\partial\Omega) \geq \alpha(M, M^*) \mathrm{Vol}_{n-1}(\partial\Omega^*)$ *; the same manifold is said to be a "strict model-space" if, moreover, in this inequality the equality occurs iff Ω is isometric to Ω^*.*

Remark 1.3.3. When the two manifolds have different finite volumes (i. e. when $\alpha(M, M^*) \neq 1$), instead of the usual assumption $\mathrm{Vol}(\Omega^*) = \mathrm{Vol}(\Omega)$, we should make the assumption $\mathrm{Vol}(\Omega^*) = \alpha(M, M^*)^{-1} \mathrm{Vol}(\Omega)$ (which, in this case means that the relative volumes $\dfrac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(M, g)}$ and $\dfrac{\mathrm{Vol}(\Omega^*)}{\mathrm{Vol}(M^*, g^*)}$ are equal) . In fact, let us call Ω^{**} the geodesic ball of (M^*, g^*), centered at x^*, such that $\mathrm{Vol}(\Omega^{**}) = \mathrm{Vol}(\Omega)$. Then we cannot have some lower bound of $\mathrm{Vol}_{n-1}(\partial\Omega)$ in terms of $\mathrm{Vol}_{n-1}(\partial\Omega^{**})$, because, if $\mathrm{Vol}(M, g) > \mathrm{Vol}(M^*, g^*)$, then Ω^{**} does not exist when Ω is such that $\mathrm{Vol}(M^*, g^*) < \mathrm{Vol}(\Omega) < \mathrm{Vol}(M, g)$. On the other hand, if $\mathrm{Vol}(M, g) < \mathrm{Vol}(M^*, g^*)$ and if $\Omega = M \setminus B(x, \varepsilon)$, where ε is arbitrarily small and $B(x, \varepsilon)$ is any geodesic ball of radius ε in (M, g), we get that $\mathrm{Vol}_{n-1}(\partial\Omega) \leq C.\varepsilon^{n-1} << \mathrm{Vol}_{n-1}(\partial\Omega^*)$.

[3] Such a domain is written as $B(x^*, R)$, where R is the solution of the equation $\mathrm{Vol}\,(B(x^*, r)) = \alpha(M, M^*)^{-1} \mathrm{Vol}\,(\Omega)$, which exists and is unique because $\alpha(M, M^*)^{-1} \mathrm{Vol}\,(\Omega) \in \,]\,0, \mathrm{Vol}(M^*, g^*)\,[$ and $r \mapsto \mathrm{Vol}\,(B(x^*, r))$ is a continuous strictly increasing function whose image is $]\,0, \mathrm{Vol}(M^*, g^*)\,[$.

Let (M,g) and (M^*,g^*) be two Riemannian manifolds such that (M^*,g^*,x^*) is a "pointed model-space for (M,g)" and let Ω be any compact domain with smooth boundary in M (such that $\mathrm{Vol}(\Omega) < \mathrm{Vol}(M,g)$ when (M,g) has finite volume). Let Ω^* be its "symmetrized domain", that is the geodesic ball $B(x^*,R_0)$ of (M^*,g^*) such that $\mathrm{Vol}(B(x^*,R_0)) = \alpha(M,M^*)^{-1}\,\mathrm{Vol}(\Omega)$).
Let f be any smooth non negative function on Ω which vanishes on the boundary. Let us denote by Ω_t (or by $\{f > t\}$) the set of points $x \in \Omega$ such that $f(x) > t$, and by $\{f = t\}$ the set of points $x \in \Omega$ such that $f(x) = t$. We observe that since the set of critical points of f is compact, then its image $\mathcal{S}(f)$ by f is compact and, by Sard's theorem, it has Lebesgue measure zero in $[0,\sup f]$. The elements of $[0,\sup f]\setminus\mathcal{S}(f)$ are called "regular values of f"; for any regular value t of f, $\{f = t\}$ is a smooth submanifold of codimension 1 in M, and it is equal to $\partial\Omega_t$.
For any $t \in [0,\sup f[$, let Ω_t^* be the symmetrized domain of Ω_t in the sense of Definition 1.3.1, i. e. the geodesic ball $B\left(x^*,R(t)\right)$ whose radius $R(t)$ is chosen[4] in such a way that $\mathrm{Vol}\left(B\left(x^*,R(t)\right)\right) = \alpha(M,M^*)^{-1}\,\mathrm{Vol}(\Omega_t)$. When $t = \sup f$, then $\Omega_{\sup f}$ is empty, and $\Omega^*_{\sup f}$ is the open ball of radius 0, and then $R(\sup f) = 0$.
The function $t \mapsto A(t) := \mathrm{Vol}(\Omega_t)$ is strictly decreasing. In fact, when $0 \le t < t' \le \sup f$, the set $\{x \in X \;:\; t < f(x) < t'\}$ is a nonempty open set of nonzero volume. A consequence is that the function $t \mapsto R(t)$ is also strictly decreasing[5].

Definition 1.3.4. *The function $f^* : \Omega^* \to \mathbb{R}^+$ is defined by $f^* = \bar f \circ \rho$, where $\rho(x) = d^*(x^*,x)$, where d^* is the Riemannian distance on M^* associated to g^*, and where $\bar f : [0,R_0] \to [0,\sup f]$ is defined by*

$$\begin{aligned}\bar f(r) &:= \inf\left(R^{-1}([0,r])\right) = \inf\{t \in [0,\sup f] \;:\; R(t) \le r\}\\ &= \inf\{t \;:\; A(t) \le \alpha(M,M^*)\,\mathrm{Vol}B(x^*,r)\}.\end{aligned} \tag{1.5}$$

Properties 1.3.5. *The function $\bar f$ has the following properties*

(i) $\bar f$ *is decreasing (not strictly in general),*

(ii) $\bar f(R(t)) = t$ *for every* $t \in [0,\sup f]$,

(iii) $\bar f(0) = \sup \bar f = \sup f \quad , \quad \bar f(R_0) = 0$

(iv) $\bar f(r) = \sup\{t \;:\; R(t) \ge r\} = \sup\{t \;:\; A(t) \ge \alpha(M,M^*)\,\mathrm{Vol}B(x^*,r)\}$ *for every*[6] $r \in [0,R(0)]$.

Proof. We start proving Property (i). If $r \le r'$, as $t \mapsto R(t)$ is a decreasing function, we get

$$\{t \;:\; R(t) \le r\} \subset \{t \;:\; R(t) \le r'\} \text{ and thus } \inf\{t \;:\; R(t) \le r\} \ge \inf\{t \;:\; R(t) \le r'\}$$

[4] Since $r \mapsto \mathrm{Vol}\left(B(x^*,r)\right)$ is continuous and strictly increasing, then $R(t)$ is correctly defined as the unique solution r of the equation $\mathrm{Vol}\left(B(x^*,r)\right) = \alpha(M,M^*)^{-1}\,\mathrm{Vol}(\Omega_t)$.

[5] Thus $t \mapsto R(t)$ is well defined (for every t) and injective. However, it is generally not surjective nor continuous, moreover the measure of the set $[0,R_0]\setminus\mathrm{Image}(R)$ is generally not zero.

[6] We always have $R(0) \le R_0$ and $R(0) < R_0$ iff $\mathrm{Vol}(\{f > 0\}) < \mathrm{Vol}(\Omega)$; in this last case, by construction, we have $\bar f = 0$ on the interval $[R(0),R_0]$.

and then $\bar{f}(r) \geq \bar{f}(r')$.

Property (ii) follows since by the definition of $\bar{f}$ and because of the strict monotonicity of $t \mapsto R(t)$, we have

$$\bar{f}(R(t)) = \inf\{s : R(s) \leq R(t)\} = \inf\{s : s \geq t\} = t.$$

By the construction of $\bar{f}$, we have $\sup \bar{f} \leq \sup f$. Conversely, by (ii), we have $\bar{f}(0) = \bar{f}(R(\sup f)) = \sup f$, and thus $\sup \bar{f} \geq \sup f$; we conclude that $\sup \bar{f} = \sup f$.

Moreover, by the definition of $\bar{f}$, we have

$$\bar{f}(R_0) = \inf \left(R^{-1}([0, R_0])\right) = \inf([0, \sup f]) = 0\,.$$

This proves Property (iii).

We conclude with the proof of Property (iv). We define by $E_+(r)$ (resp. $E_-(r)$) the set of values $t \in [0, \sup f]$ such that $R(t) \leq r$ (resp. $R(t) \geq r$); for every $r \in [0, R_0]$ (resp. for every $r \in [0, R(0)]$) this set is not empty.

For every $t \in E_+(r)$ and every $s \in E_-(r)$, we have $R(t) \leq r \leq R(s)$ and thus, as $R(.)$ is a strictly decreasing function, $t \geq s$, which implies that $\inf E_+(r) \geq \sup E_-(r)$ and thus $\bar{f}(r) \geq \sup E_-(r)$. For any $r \in [0, R(0)]$, if $\bar{f}(r) > 0$ we take any sequence $n \mapsto s_n$ which converges to $\bar{f}(r)$ and such that $s_n < \bar{f}(r)$ for every $n \in \mathbb{N}$, then (by (ii)) $\bar{f}\,(R(s_n)) = s_n < \bar{f}(r)$ and so (by (i)) $R(s_n) > r$, which implies that $s_n \in E_-(r)$. We conclude that $\bar{f}(r) = \sup_n(s_n) \leq \sup E_-(r)$.

If $\bar{f}(r) = 0$, as (by assumption) $r \in [0, R(0)]$, and thus $R(0) \geq r$, then the point 0 lies in $E_-(r)$ and so $\sup E_-(r) \geq 0 = \bar{f}(r)$.

We conclude that, for every $r \in [0, R(0)]$, $\bar{f}(r) \leq \sup E_-(r)$ and thus that $\bar{f}(r) = \sup E_-(r)$. □

As a consequence we have the following

Properties 1.3.6. *The properties inherited by f^* are the following:*

(i) f^* *is decreasing as a function of the distance* $d^*(x^*, \cdot)$,

(ii) $x \in \partial\Omega_t^* \implies f^*(x) = t$,

(iii) $x \in \partial\Omega^* \implies f^*(x) = 0$,

(iv) $\sup f^* = \sup f$.

Proof. We deduce Property (i) from Property 1.3.5 (i) and from the fact that, by definition, $f^* = \bar{f} \circ \rho$, which also implies that $\sup f^* = \bar{f}(0) = \sup \bar{f}$; the previous equality and Property 1.3.5 (iii) prove (iv).

As $\partial\Omega_t^*$ is the geodesic sphere of radius $R(t)$, by the definitions of ρ and f^* and by the Property 1.3.5 (ii), we get

$$x \in \partial\Omega_t^* \implies \rho(x) = R(t) \implies f^*(x) = \bar{f}(R(t)) = t;$$

this proves (ii).

Since

$$x \in \partial\Omega^* \implies \rho(x) = R_0$$

we have

$$f^*(x) = \bar{f}(R_0) = \inf \left(R^{-1}([0, R_0])\right) = \inf\left([0, \sup f]\right) = 0.$$

This proves (iii). □

1.4 The Symmetrization

The main tool which is used in comparison of the mean exit times from different domains is the following theorem.

Theorem 1.4.1 (Symmetrization). .– *Let (M, g) be a Riemannian manifold and let (M^*, g^*, x^*) be a pointed model-space for (M, g). Let f be any smooth non negative function on some domain Ω (with smooth boundary in M, let us recall that this also implies that the closure of Ω is a strict subset of M) which vanishes on the boundary. Let f^* be its symmetrized function, constructed as above on the symmetrized geodesic ball Ω^* of (M^*, g^*), centered at the point x^*. Then*

(i) f^* is Lipschitz (with Lipschitz constant $\leq \|\nabla f\|_{L^\infty}$), and thus f^* lies in $H_{1,c}(\Omega^*, g^*)$,

(ii) $\frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega f(x)^p \, dv_g(x) = \frac{1}{\mathrm{Vol}(\Omega^*)} \int_{\Omega^*} (f^*(x))^p \, dv_{g^*}(x)$ for every $p \in [1, +\infty[$,

(iii) $\frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega \|\nabla f(x)\|^2 \, dv_g(x) \geq \frac{1}{\mathrm{Vol}(\Omega^*)} \int_{\Omega^*} \|\nabla f^*(x)\|^2 \, dv_{g^*}(x)$. If, moreover, (M^*, g^*) is a strict PIMS for (M, g), then equality holds iff the set $\{f > 0\} \subset (\Omega, g)$ is isometric to the set $\{f^* > 0\} \subset (\Omega^*, g^*)$.

We start proving (ii).

Proof of Theorem 1.4.1 (ii). Let $(t_i)_{i \in I}$ be a subdivision of the interval $[0, \sup f]$, i. e. I is a finite subset of the type $\{0, 1, \ldots, N\} \subset \mathbb{N}$ and the t_i's are such that $0 = t_0 < t_1 < \ldots < t_N = \sup f$ and such that, for every $i \in \{0, 1, \ldots, N-1\}$, $t_{i+1} - t_i = \frac{\sup f}{N}$. We know that the function $t \mapsto A(t) = \mathrm{Vol}(\Omega_t)$ is strictly decreasing. We have

$$\sum_{i=1}^{N-1} (t_i)^p \;\; (A(t_i) - A(t_{i+1})) \leq \int_\Omega f^p \, dv_g \leq \sum_{i=1}^{N-1} (t_{i+1})^p \;\; (A(t_i) - A(t_{i+1}))\,,$$

Moreover, the left and the right-hand sides of this sequence of inequalities (denoted by S_N^- and S_N^+, respectively both converge to $\int_\Omega f^p \, dv_g$, because

$$\begin{aligned} S_N^+ - S_N^- &= \sum_{i=1}^{N-1} \left(t_{i+1}^p - t_i^p\right) \;\; (A(t_i) - A(t_{i+1})) \\ &\leq p \sum_{i=1}^{N-1} (t_{i+1})^{p-1} \, (t_{i+1} - t_i) \, (A(t_i) - A(t_{i+1})) \\ &\leq p \, (\sup f)^{p-1} \; \frac{\sup f}{N} \; A(0) \leq \frac{p}{N} (\sup f)^p \; \mathrm{Vol}(\Omega) \to 0 \;\text{ when } N \to \infty. \end{aligned}$$

Let us define $A^*(t) = \mathrm{Vol}(\Omega_t^*) = \mathrm{Vol}(B(x^*, R(t)))$; since $\bar{f}$ is decreasing and since $\bar{f}(R(t)) = t$, for every $t \in [0, \sup f]$ we have $\bar{f}(r) \geq t$ on $[0, R(t)]$ and $\bar{f}(r) \leq t$ on $[R(t), R_0]$, it follows that $f^* \geq t$ on $B(x^*, R(t)) = \Omega_t^*$ and that $f^* \leq t$ on its complement. Then for every $x \in B(x^*, R(t_i)) \setminus B(x^*, R(t_{i+1}))$, we have $t_i \leq f^*(x) \leq t_{i+1}$ and

$$\sum_{i=1}^{N-1} (t_i)^p \;\; (A^*(t_i) - A^*(t_{i+1})) \leq \int_{\Omega^*} (f^*)^p \, dv_{g^*} \leq \sum_{i=1}^{N-1} (t_{i+1})^p \;\; (A^*(t_i) - A^*(t_{i+1}))\,.$$

Since $A^*(t) = \alpha(M, M^*)^{-1} A(t)$ by the definition of Ω_t^* as the symmetrized domain of Ω_t , we rewrite the last sequence of inequalities as

$$\alpha(M, M^*)^{-1} \; S_N^- \leq \int_{\Omega^*} (f^*)^p \, dv_{g^*} \leq \alpha(M, M^*)^{-1} \; S_N^+;$$

taking the limits of the left and right hand-side when $N \to \infty$, we obtain

$$\alpha(M, M^*)^{-1} \int_{\Omega} (f)^p \, dv_g = \int_{\Omega^*} (f^*)^p \, dv_{g^*} ,$$

and, using the fact that $\alpha(M, M^*) = \dfrac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Omega^*)}$, we conclude the proof of (ii).

$\square$

To prove the properties (i) and (iii) of Theorem 1.4.1 we need the following results.

Regularity of $R(t)$ and $\bar{f}$

Lemma 1.4.2. *If t is a regular value of f (i. e. if $t \notin \mathcal{S}(f)$) then the function R is differentiable at the point t and*

$$R'(t) = - \frac{1}{\alpha(M, M^*) \; \mathrm{Vol}(\partial\Omega_t^*)} \int_{\{f=t\}} \frac{1}{\|\nabla f(x)\|} \; da_t(x) \neq 0 \quad . \tag{1.6}$$

Moreover, $t \mapsto R(t)$ is a diffeomorphism from $[0, \sup f] \setminus \mathcal{S}(f)$ onto its image.

Remark 1.4.3. In the right-hand side of the equality (1.6), the integral takes sense because, since t is a regular value of f , then $\nabla f(x) \neq 0$ for every $x \in \{f = t\}$, which implies that $\{f = t\}$ is a submanifold of codimension 1 of M and that da_t is well defined as the $(n-1)-$dimensional Riemannian measure on $\{f = t\}$ (viewed as a Riemannian submanifold of (M, g)).

Proof. We know that the set $\mathcal{S}(f)$ of singular values of f is compact, thus $[0, \sup f] \setminus \mathcal{S}(f)$ is an open subset of $[0, \sup f[$ on which differentiation takes sense; moreover, let t be any regular value, then there exists some $\varepsilon > 0$ such that $]t - \varepsilon, t + \varepsilon[$ only contains regular values of f , which implies that $\varphi : x \mapsto \dfrac{1}{\|\nabla f(x)\|}$ is continuous on $\{t - \varepsilon < f < t + \varepsilon\}$. For any $h \in] - \varepsilon \, , \, \varepsilon \, [$, let us denote by $I_t(h)$ the interval $] \, t + h \, , \, t \,]$ when $h < 0$ and the interval $]t, t + h]$ when $h > 0$, we may thus apply the Coarea Formula (see Theorem 2.3.1) to the function $\mathbf{1}_{f^{-1}(I_t(h))} \cdot \varphi$, which gives

$$\int_{f^{-1}(I_t(h))} \frac{1}{\|\nabla f(x)\|} \; \|\nabla f(x)\| \; dv_g(x) = \int_{I_t(h)} \left(\int_{f^{-1}(\{s\})} \frac{1}{\|\nabla f(x)\|} \; da_s(x) \right) ds.$$

Then

$$\begin{aligned} &\alpha(M, M^*) \left(\mathrm{Vol} \left[B(x^*, R(t + h)) \right] - \mathrm{Vol} \left[B(x^*, R(t)) \right] \right) \\ &= A(t + h) - A(t) = - \int_t^{t+h} \left(\int_{f^{-1}(\{s\})} \frac{1}{\|\nabla f(x)\|} \; da_s(x) \right) ds. \end{aligned} \tag{1.7}$$

It follows that $s \mapsto \mathrm{Vol}\,[B(x^*, R(s))]$ is differentiable at the point t, and hence that $s \mapsto R(s)$ is differentiable at the point t. Since $R(t+h)$ tends to $R(t)$ when h vanishes, we have

$$\mathrm{Vol}\,[B(x^*, R(t+h))] - \mathrm{Vol}\,[B(x^*, R(t))] \sim (R(t+h) - R(t))\ \mathrm{Vol}_{n-1}\,[\partial B(x^*, R(t))]$$

$$\sim h\, R'(t)\ \mathrm{Vol}_{n-1}\,[\partial B(x^*, R(t))]\,.$$

From equation (1.7) and the mean value property, we get

$$\alpha(M, M^*)\ R'(t)\ \mathrm{Vol}_{n-1}\,[\partial B(x^*, R(t))]$$
$$= - \lim_{h \to 0} \left(\frac{1}{h} \int_t^{t+h} \left(\int_{f^{-1}(\{s\})} \frac{1}{\|\nabla f(x)\|}\ da_s(x) \right) ds \right) = - \int_{f^{-1}(\{t\})} \frac{1}{\|\nabla f(x)\|}\ da_t(x).$$

This implies that $R'(t) \neq 0$ at every point $t \in [0, \sup f] \setminus \mathcal{S}(f)$, and so it is a local diffeomorphism; as $t \mapsto R(t)$ is strictly decreasing, and thus injective, it is a diffeomorphism from $[0, \sup f] \setminus \mathcal{S}(f)$ onto its image. □

Subsequently, we study the function $t \mapsto R(t)$ in a neighborhood of a singular point $t \in \mathcal{S}(f)$.

We define the functions $t \mapsto R_-(t)$ and $t \mapsto R_-(t)$ on the interval $[0, \sup f]$ by

$$R_-(t) = \begin{cases} \sup R(]t, \sup f]) & \text{when } t \in [0, \sup f[\\ 0 & \text{when } t = \sup f \end{cases}$$

$$R_+(t) = \begin{cases} R_0 & \text{when } t = 0 \\ \inf R([0, t[) & \text{when } t \in]0, \sup f] \end{cases}$$

Properties 1.4.4. *For every* $t \in [0, \sup f]$,

(i) $\forall t \in [0, \sup f[\ ,\quad R_-(t) = \lim_{s \to t\,,\ s > t} R(s)$,

(ii) $\forall t \in]0, \sup f]\ ,\quad R_+(t) = \lim_{s \to t\,,\ s < t} R(s)$,

(iii) $\forall t \in [0, \sup f]\quad R_-(t) = R(t) \leq R_+(t)$.

(iv) $\forall\, s, t \in [0, \sup f]\quad s < t \implies R_+(t) < R(s)$.

Proof. We start proving (i).For every fixed $t \in [0, \sup f[$, and for every $\varepsilon > 0$, by the definition of $R_-(t)$, there exists some $s_\varepsilon \in]\,t\ ,\ \sup f]$ such that

$$R_-(t) - \varepsilon := \sup(R(]t, \sup f])) - \varepsilon < R(s_\varepsilon) \leq \sup(R(]t, \sup f])) := R_-(t).$$

Since $R(\cdot)$ is strictly decreasing then every $s \in]t, s_\varepsilon[$ satisfies

$$R_-(t) - \varepsilon < R(s_\varepsilon) < R(s) \leq \sup(R(]t, \sup f])) = R_-(t),$$

and this proves that $R_-(t) = \lim_{s \to t\,,\ s > t} R(s)$..

Similarly, for every fixed $t \in]0, \sup f]$, and for every $\varepsilon > 0$, by the definition of $R_+(t)$, there exists some $s_\varepsilon \in\ [0\ ,\ t\,[$ such that, for every $s \in]s_\varepsilon, t\,[$,

$$R_+(t) + \varepsilon := \inf(R([0, t\,[)) + \varepsilon > R(s_\varepsilon) > R(s) \geq \inf(R([0, t\,[)) := R_+(t)\ ,$$

which proves that $R_+(t) = \lim_{s \to t\,,\ s < t} R(s)$. This proves (ii).

If $s < t$, there exists a strictly increasing sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_0 = s$ and $t_n \to t$ when $n \to +\infty$; since $t \mapsto R(t)$ is a strictly decreasing function, then the sequence $(R(t_n))_{n\in\mathbb{N}}$ is strictly decreasing and thus, applying (ii), we get

$$R(s) = R(t_0) > \lim_{n\to+\infty} (R(t_n)) = R_+(t) \quad .$$

This proves (iv).

We conclude by proving (iii). By definition, one has $R_+(0) = R_0 \geq R(0)$. Moreover, at any point $t \in]0, \sup f]$, for every $s \in [0, t[$, by monotonicity, one has $R(s) > R(t)$, and so $R_+(t) = \inf(R([0, t[)) \geq R(t)$.
Similarly, one has (by definition) $R_-(\sup f) = 0 = R(\sup f)$. Moreover, let us consider any point $t \in [0, \sup f[$ and take any decreasing sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to t$ when $n \to +\infty$ and $t_n > t$; then, by (i), $R_-(t) = \lim_{n\to+\infty} R(t_n)$. The set $\{f > t\}$ then is the increasing union of the sets $\{f > t_n\}$ (because, if $f(x) > t$ there exists n such that $f(x) > t_n > t$). Applying the property called "continuity of the measure" we get

$$A(t) = \mathrm{Vol}(\{f > t\}) = \lim_{n\to+\infty} \mathrm{Vol}(\{f > t_n\}) = \lim_{n\to+\infty} A(t_n) \quad ,$$

which implies that $\mathrm{Vol}\,[B(x^*, R(t))] = \lim_{n\to+\infty} \mathrm{Vol}\,[B(x^*, R(t_n))]$, and so

$$R(t) = \lim_{n\to+\infty} R(t_n) = R_-(t)$$

This concludes the proof of (iii). $\square$

The next lemma is about properties of $\bar{f}$.

Lemma 1.4.5. *(i) For every* $t \in [0, \sup f]$, *we have* $\bar{f}^{-1}(\{t\}) = [R_-(t), R_+(t)]$.

(ii) For every open interval $]t, t'[\subset [0, \sup f]$, *we have* $\bar{f}^{-1}(]t, t'[) =]R_+(t'), R_-(t)[$.

(iii) For every $t \in]0, \sup f]$, *we have* $\bar{f}^{-1}([0, t\,[) =]R_+(t), R_0]$.

(iv) For every $t \in [0, \sup f[$, *we have* $\bar{f}^{-1}(]t, \sup f]) = [0\,, R_-(t)[$.

(v) For every $t \in [0, \sup f]$, *we have* $\{f^* > t\} = B(x^*, R(t)) = \Omega_t^*$, *and thus the ball* $\{f^* > t\}$ *is the symmetrized of the domain* $\{f > t\}$.

Proof. We start proving (i). Since $\bar{f}$ is decreasing, in order to prove (i) it is sufficient to prove the following conditions:

$$\begin{cases} \bar{f}(R_-(t)) = \bar{f}(R_+(t)) = t & \text{for every } t \in [0, \sup f]\ , \\ \bar{f}(r) < t & \text{for every } t \in]0, \sup f] \text{ and for every } r \in]R_+(t), R_0]\ , \\ \bar{f}(r) > t & \text{for every } t \in [0, \sup f[\text{ and for every } r \in [0, R_-(t)[\ . \end{cases} \tag{1.8}$$

From Properties 1.3.5 (ii) and 1.4.4 (iii), we deduce that $\bar{f}(R_-(t)) = \bar{f}(R(t)) = t$ for every $t \in [0, \sup f]$.
When $t = 0$, by definition $R_+(0) = R_0$ and then (by Property 1.3.5 (iii)) $\bar{f}(R_+(0)) = \bar{f}(R_0) = 0$.
When $t \in]0, \sup f]$, Properties 1.4.4 (iii) and 1.3.5 (i) respectively imply that $R(t) \leq R_+(t)$ and that $\bar{f}$ is a decreasing function; then

$$\forall\, t \in [0, \sup f] \qquad \bar{f}(R_+(t)) \leq \bar{f}(R(t)) = t \quad . \tag{1.9}$$

On the other hand, for every $s \in [0,t[$, by Property 1.4.4 (iv) we get $R(0) \geq R(s) > R_+(t)$, and thus $[0,t[\subset \{s : R(s) \geq R_+(t)\}$. Applying Property 1.3.5 (iv), we get

$$t \leq \sup\{s : R(s) \geq R_+(t)\} = \bar{f}\,(R_+(t))\,.$$

From this last inequality and from (1.9) we deduce that $\bar{f}(R_+(t)) = t$ for every $t \in]0, \sup f]$. Moreover, since we have already proved that $\bar{f}(R_+(0)) = 0$, the first of the conditions (1.8) is then proved.
For every $t \in]0, \sup f]$ and for any $r > R_+(t) = \inf(R([0,t\,[))$, there exists some $s_0 \in [0,t\,[$ such that $r > R(s_0)$. Applying Properties 1.3.5 (i) and (ii) we get $\bar{f}(r) \leq \bar{f}(R(s_0)) = s_0 < t$; thus the second of the conditions (1.8) is proved.
For every $t \in [0, \sup f[$ and for any $r < R_-(t) = \sup(R(]t, \sup f]))$, there exists some $s_0 \in]t, \sup f]$ such that $r < R(s_0)$. Applying Properties 1.3.5 (i) and (ii) we get $\bar{f}(r) \geq \bar{f}(R(s_0)) = s_0 > t$; thus the third of the conditions (1.8) is proved.
This concludes the proof of (i).

We continue by proving (ii). For every $r \in \bar{f}^{-1}(]t,t'[)$, from property (i) we have

$$\bar{f}(R_-(t)) = t < \bar{f}(r) < t' = \bar{f}(R_+(t')),$$

and then, by the monotonicity of $\bar{f}$ (Property 1.3.5 (i)), we get $R_+(t') < r < R_-(t)$, which proves that $\bar{f}^{-1}(]t,t'[) \subset]\,R_+(t'), R_-(t)\,[$.
Conversely, let r be any point in the interval $]\,R_+(t'), R_-(t)\,[$, since $\bar{f}$ is monotone we get $\bar{f}(r) \in [\bar{f}(R_-(t), \bar{f}(R_+(t')] = [t,t']$, and then $r \in \bar{f}^{-1}([t,t'])$. Applying (i) we get $r \notin [R_-(t'), R_+(t')] = \bar{f}^{-1}(\{t'\})$ and $r \notin [R_-(t), R_+(t)] = \bar{f}^{-1}(\{t\})$, and so we have proved that $r \in \bar{f}^{-1}(]t,t'[)$, and then $]\,R_+(t'), R_-(t)\,[\subset \bar{f}^{-1}(]t,t'[)$. This concludes the proof of (ii).

In order to prove (iii), by (i) and by the monotonicity of $\bar{f}$ we have

$$\begin{aligned} r \in \bar{f}^{-1}([0,t[) &\implies \bar{f}(r) < t \implies \bar{f}(r) < \bar{f}(R_+(t)) \\ &\implies r > R_+(t) \implies r \in]R_+(t), R_0]; \end{aligned}$$

then $\bar{f}^{-1}([0,t\,[) \subset]R_+(t), R_0]$. Conversely, if $r > R_+(t) = \inf(R([0,t\,[))$, there exists some $s_0 \in [0,t\,[$ such that $r > R(s_0)$. Applying Properties 1.3.5 (i) and (ii) we get $\bar{f}(r) \leq \bar{f}(R(s_0)) = s_0 < t$. This proves that $]R_+(t), R_0] \subset \bar{f}^{-1}([0,t\,[)$, and thus that $\bar{f}^{-1}([0,t\,[) =]R_+(t), R_0]$.

In order to prove (iv), by (i) and by the monotonicity of $\bar{f}$, we have

$$\begin{aligned} r \in \bar{f}^{-1}(]t, \sup f]) &\implies \bar{f}(r) > t \implies \bar{f}(r) > \bar{f}(R_-(t)) \\ &\implies r < R_-(t) \implies r \in [0\,, R_-(t)[; \end{aligned}$$

then $\bar{f}^{-1}(]t, \sup f]) \subset [0\,, R_-(t)[$. Conversely, if $r < R_-(t) = \sup(R(]t, \sup f]))$, there exists some $s_1 \in]t, \sup f]$ such that $r < R(s_1)$. From Properties 1.3.5 (i) and (ii) we get $\bar{f}(r) \geq \bar{f}(R(s_1)) = s_1 > t$. This proves that $[0\,, R_-(t)[\subset \bar{f}^{-1}(]t, \sup f])$, and thus that $\bar{f}^{-1}(]t, \sup f]) = [0\,, R_-(t)[$.

We conclude proving (v). From (iv), we deduce that

$$f^* > t \iff \bar{f}\,(\rho(x)) \in]t, \sup f] \iff \rho(x) < R_-(t) \iff x \in B(x^*, R(t)) \quad ,$$

where the last equivalence is true because $R_-(t) = R(t)$ for every t by Property 1.4.4 (iii). We conclude by observing that $B(x^*, R(t))$ is, by definition, the symmetrized domain of the set $\{f > t\}$. $\square$

Let us now define the set of jumps

$$\mathcal{S}_1(f) := \{t : R_-(t) \neq R_+(t)\} = \{t : R_-(t) < R_+(t)\} \ ,$$

we have the following

Lemma 1.4.6. *$\mathcal{S}_1(f)$ is an (at most) countable subset of the set $\mathcal{S}(f)$ of singular values. Moreover, for every $t \in [0, \sup f] \setminus \mathcal{S}(f)$, one has $\bar{f}^{-1}(\{t\}) = \{R(t)\}$.*

Proof. As, by Lemma 1.4.2, $R(\cdot)$ is differentiable at any point t of the open set $[0, \sup f] \setminus \mathcal{S}(f)$ and as (by Properties 1.4.4 (i) and (ii)) $R_-(t)$ (resp. $R_+(t)$) is the limit of $R(s)$ as s tends to t from the right (resp. from the left), it follows that $R_-(t) = R(t) = R_+(t)$ for every $t \in [0, \sup f] \setminus \mathcal{S}(f)$. Consequently, $\mathcal{S}_1(f) \subset \mathcal{S}(f)$, and by Lemma 1.4.5 (i) $\bar{f}^{-1}(\{t\}) = [R_-(t), R_+(t)] = \{R(t)\}$.

Let $t,\ s$ be any pair of points of $\mathcal{S}_1(f)$ such that $t < s$. Then the intervals $[R_-(t), R_+(t)] = \bar{f}^{-1}(\{t\})$ and $[R_-(s), R_+(s)] = \bar{f}^{-1}(\{s\})$ are disjoint (by Lemma 1.4.5 (i)).
The (Lebesgue) measure of the disjoint union of all the intervals $[R_-(t), R_+(t)]$, as t runs in $\mathcal{S}_1(f)$ is equal to $\sum_{t \in \mathcal{S}_1(f)} (R_+(t) - R_-(t))$, and this is bounded above by the total Lebesgue measure of the interval $[0, R_0]$, i. e. by R_0. Thus the above sum only contains a countable number of non vanishing terms. ☐

Lemma 1.4.7. *The function $t \mapsto R(t)$ is*

(i) continuous at every point of $[0, \sup f] \setminus \mathcal{S}_1(f)$,

(ii) continuous on the right at every point of $\mathcal{S}_1(f)$.

Proof. For every $t \in [0, \sup f] \setminus \mathcal{S}_1(f)$, from the definition of $\mathcal{S}_1(f)$ and from Property 1.4.4 (iii), we have $R_-(t) = R(t) = R_+(t)$. Properties 1.4.4 (i) and (ii)) then imply that $R_-(t)$ (resp. $R_+(t)$) is the limit of $R(s)$ as s tends to t from the right (resp. from the left), and so we have continuity.

Continuity from the right at every point t is an immediate consequence of Properties 1.4.4 (i) and (ii)) and of the fact that $R_-(t) = R(t)$ by Property 1.4.4 (iii). ☐

In the sequel, we apply the following result of measure theory (see for instance [48]):

Lemma 1.4.8. *Let $F : [0, a] \to \mathbb{R}$ be a decreasing function which is continuous from the right. Then there exists a positive Borel measure μ such that $\mu(]\, x, y]) = F(x) - F(y)$ for every $x,\ y \in [0, a]$ such that $x \le y$. This measure is the derivative of $-F$ in the sense of distributions.*

Since the function $t \mapsto R(t)$ is decreasing and continuous on the right (see Lemma 1.4.7), by the Lemma 1.4.8, there exist a positive Borel measure (denoted by μ_R) such that $\mu_R(]\, t, s]) = R(t) - R(s)$ for every $t,\ s \in [0, \sup f]$ such that $t \le s$.

Proposition 1.4.9. *The measure* μ_R *is the derivative of* $-R(\cdot)$ *in the sense of distributions* [7], *and thus it coincides with the (regular) measure* $-R'(t)\ dt$ *on the open subset* $[0, \sup f] \setminus \mathcal{S}(f)$ *.*

Proof. For every interval $]\, t, t'[\subset [0, \sup f] \setminus \mathcal{S}(f)$, the Lemma 1.4.2 claims that $t \mapsto R(t)$ is differentiable on the interval $]t, t'[$ and compute this derivative, which implies that

$$\begin{aligned}
\mu_R(]t,t'[) &= \lim_{n\to+\infty} \mu_R\left(\left] t+\frac{1}{n}, t'-\frac{1}{n}\right]\right) = \lim_{n\to+\infty}\left(R\left(t+\frac{1}{n}\right) - R\left(t'-\frac{1}{n}\right)\right) \\
&= \lim_{n\to+\infty} \int_{t+\frac{1}{n}}^{t'-\frac{1}{n}} \left(\frac{1}{\alpha(M,M^*)\ \mathrm{Vol}(\partial\Omega_s^*)} \int_{\{f=s\}} \frac{1}{\|\nabla f(x)\|}\ da_s(x)\right) ds \\
&= \int_{]t,t'[} \left(\frac{1}{\alpha(M,M^*)\ \mathrm{Vol}(\partial\Omega_s^*)} \int_{\{f=s\}} \frac{1}{\|\nabla f(x)\|}\ da_s(x)\right) ds.
\end{aligned}$$

We know that the set $\mathcal{S}(f)$ of singular values of f is compact, which implies that $[0, \sup f] \setminus \mathcal{S}(f)$ is an open subset of $[0, \sup f[$, then, for any open interval $]t, t'[\subset [0, \sup f[$, the subset $]t, t'[\cap ([0, \sup f[\setminus \mathcal{S}(f))$ (that we shall denote by $]t, t'[\setminus \mathcal{S}(f)$) is a disjoint countable union of open intervals. By countable additivity of measures, we obtain

$$\mu_R\left(]t,t'[\setminus\mathcal{S}(f)\right) = \int_{]t,t'[\setminus\mathcal{S}(f)} \left(\frac{1}{\alpha(M,M^*)\ \mathrm{Vol}(\partial\Omega_s^*)} \int_{\{f=s\}} \frac{1}{\|\nabla f(x)\|}\ da_s(x)\right) ds\,. \tag{1.10}$$

□

We are now able to prove

Lemma 1.4.10. $\bar{f}$ *is Lipschitz with a Lipschitz constant bounded above by* $\|\nabla f\|_{L^\infty}$ *.*

Proof. By the definition of μ_R , using the fact that it is a positive measure and the formula (1.10), for every $t,\ t' \in [0, \sup f]$ such that $t \leq t'$, we have

$$\begin{aligned}
R(t) - R(t') = \mu_R(]\,t,t']) &\geq \mu_R\left(]\,t,t'[\setminus\mathcal{S}(f)\right) \\
&= \int_{]\,t,t'[\setminus\mathcal{S}(f)} \left(\frac{1}{\alpha(M,M^*)\ \mathrm{Vol}(\partial\Omega_s^*)} \int_{\{f=s\}} \frac{1}{\|\nabla f(x)\|}\ da_s(x)\right) ds \\
&\geq \frac{1}{\|\nabla f\|_{L^\infty}} \int_{]\,t,t'[\setminus\mathcal{S}(f)} \left(\frac{\mathrm{Vol}(\partial\Omega_s)}{\alpha(M,M^*)\ \mathrm{Vol}(\partial\Omega_s^*)}\right) ds
\end{aligned} \tag{1.11}$$

[7] Although this description is not used in the sequel, we can describe the measure μ_R as the sum of a regular part, denoted by μ_R^{reg} , and of a singular part, denoted by μ_R^{sing} , where μ_R^{reg} is the measure $d\mu_R^{\mathrm{reg}}(t) = -\mathbf{1}_{[0,\sup f]\setminus\mathcal{S}(f)}(t)\, R'(t)\, dt$, whose density (with respect to the canonical Lebesgue measure) is the regular function $-R'$ given by the Lemma 1.4.2, and where μ_R^{sing} is the measure (with support on $\mathcal{S}_1(f)$): $\mu_R^{\mathrm{sing}} = \sum_{s\in\mathcal{S}_1(f)} (R_+(s) - R_-(s))\, \delta_s$, where δ_s is the Dirac measure at the point s . Justifications for this description of μ_R are given by the above definition of μ_R (based on Lemma 1.4.8), by Lemma 1.4.7 and Properties 1.4.4 (i) and (ii)) and by the proof of Lemma 1.4.6.

By assumption (M^*, g^*, x^*) is a "pointed model-space for (M,g)" (in the sense of Definition 1.3.2) and then, as Ω_t^* is the "symmetrized domain" (in the sense of Definition 1.3.2) of Ω_t, (i. e Ω_t^* is the geodesic ball $B(x^*, R(t))$ of (X,g) such that $\mathrm{Vol}(B(x^*, R(t))) = \alpha(M, M^*)^{-1}\,\mathrm{Vol}(\Omega_t)$), the isoperimetric inequality of Definition 1.3.2 gives, for every t,

$$\mathrm{Vol}_{n-1}(\partial\Omega_t) \geq \alpha(M, M^*)\,\mathrm{Vol}_{n-1}(\partial\Omega_t^*) \quad .$$

Plugging this into the equation (1.11), and recalling that $\mathcal{S}(f)$ has measure zero, we obtain

$$R(t) - R(t') \geq \frac{1}{\|\nabla f\|_{L^\infty}}\,(t' - t). \tag{1.12}$$

Since $\bar f$ is decreasing and $\bar f(R(0)) = 0$, we know that $\bar f(r) = 0$ for every $r \in [R(0), R_0]$, and so it is thus sufficient to prove that $\bar f$ is Lipschitz on the interval $[0, R(0)]$. Let us recall that, for every $r,\ r' \in [0, R(0)]$ such that $r < r'$, by the definition of $\bar f$ and Property 1.3.5 (iv), we have

$$\bar f(r') = \inf\{t' \,:\, R(t') \leq r'\} \quad , \quad \bar f(r) := \sup\{t \,:\, R(t) \geq r\}.$$

Then, for every $\varepsilon > 0$, there exist $t,\ t'$ such that $R(t) \geq r,\ R(t') \leq r'$ and $0 \leq \bar f(r) - \bar f(r') < t - t' + \varepsilon$; using this and formula (1.12), we obtain

$$0 \leq \bar f(r) - \bar f(r') - \varepsilon < t - t' \leq \|\nabla f\|_{L^\infty}(R(t') - R(t)) \leq \|\nabla f\|_{L^\infty}(r' - r) \quad ;$$

we conclude by making ε tend to zero. □

By the Rademacher theorem, since $\bar f$ is Lipschitz, it is differentiable everywhere, except on a subset $\mathcal{E}$ of Lebesgue measure zero in $[0, R_0]$. So, we can define the measure $\mu_{\bar f}$ as the (positive) measure $-\bar f'(r)\,dr$ on $[0, R_0]$. Since this measure is a measure with bounded density with respect to the Lebesgue measure, every subset of measure zero with respect to the Lebesgue measure has measure zero with respect to the measure $\mu_{\bar f}$, then $\mu_{\bar f}(\mathcal{E}) = 0$ and, moreover, for every $r,\ r' \in [0, R_0]$ such that $r \leq r'$, one gets $\mu_{\bar f}(\{r\}) = \mu_{\bar f}(\{r'\}) = 0$, and thus

$$\mu_{\bar f}([\,r, r']) = \mu_{\bar f}(]\,r, r'[) = \mu_{\bar f}(]\,r, r']) = \mu_{\bar f}([\,r, r'[) = -\int_r^{r'} \bar f'(s)\,ds = \bar f(r) - \bar f(r'). \tag{1.13}$$

The previous one proves that $\mu_{\bar f}$ is (up to the sign) the Radon-Nikodym derivative of $\bar f$ in the sense of the Lemma 1.4.8. Moreover, we have

Lemma 1.4.11. *$\bar f^{-1}(\mathcal{S}(f))$ and $[0, R_0] \setminus (R\,[\,[0, \sup f] \setminus \mathcal{S}(f)\,])$ coincide and both have $\mu_{\bar f}$-measure zero in $[0, R_0]$.*

Proof. In order to prove that $\bar f^{-1}(\mathcal{S}(f)) = [0, R_0] \setminus (R\,[\,[0, \sup f] \setminus \mathcal{S}(f)\,])$, it is sufficient to prove that $\bar f^{-1}([0, \sup f] \setminus \mathcal{S}(f)) = R\,(\,[0, \sup f] \setminus \mathcal{S}(f)\,)$. This is immediate when applying Lemma 1.4.6,

$$\bar f^{-1}([0, \sup f] \setminus \mathcal{S}(f)) = \cup_{t \notin \mathcal{S}(f)} \bar f^{-1}(\{t\}) = \cup_{t \notin \mathcal{S}(f)} \{R(t)\} = R\,(\,[0, \sup f] \setminus \mathcal{S}(f)\,)\,.$$

Let us denote by m the Lebesgue measure on $[0, \sup f]$. Since $m(\mathcal{S}(f)) = 0$, from Lebesgue measure theory we know that, for every $\varepsilon > 0$, there exists an open subset U_ε of $[0, \sup f]$ such that $\mathcal{S}(f) \subset U_\varepsilon$ and $m(U_\varepsilon) < \varepsilon$. Moreover, U_ε is a countable disjoint union of intervals of the type $]\, t_i, t_{i+1}\, [$ ($i \in I \subset \mathbb{N}$), of an interval of the type $]\, t', \sup f]$, and finally of an interval of the type $[0, t\, [$. Then, by Lemma 1.4.5 (ii), (iii) and (iv), $\bar{f}^{-1}(U_\varepsilon)$ is the countable disjoint union of the intervals $]\, R_+(t_{i+1}), R_-(t_i)\, [$, of the interval $[0\,, R_-(t')[$ and finally of the interval $]R_+(t), R_0]$. Moreover, by Lemma 1.4.5 (ii), by equality (1.13) and by Lemma 1.4.5 (i), for every $i \in I$, we have

$$\begin{aligned}&\mu_{\bar{f}}\left(\bar{f}^{-1}(]\, t_i, t_{i+1}\, [)\right) = \mu_{\bar{f}}\left(]\, R_+(t_{i+1}), R_-(t_i)\, [\right)\\ &= \bar{f}(R_+(t_{i+1})) - \bar{f}(R_-(t_i)) = t_{i+1} - t_i = m(]\, t_i, t_{i+1}\, [)\ ,\end{aligned}$$

by Lemma 1.4.5 (iv), by equality (1.13) and by Lemma 1.4.5 (i), we have

$$\mu_{\bar{f}}\left(\bar{f}^{-1}(]\, t', \sup f])\right) = \mu_{\bar{f}}\left([0\,, R_-(t')[)\right) = \bar{f}(0) - \bar{f}(R_-(t')) = \sup f - t' = m(]\, t', \sup f])\ ,$$

and finally, by Lemma 1.4.5 (iii), by equality (1.13) and by Lemma 1.4.5 (i), we have

$$\mu_{\bar{f}}\left(\bar{f}^{-1}([0, t\, [)\right) = \mu_{\bar{f}}\left(]R_+(t), R_0]\right) = \bar{f}(R_+(t)) - \bar{f}(R_0) = t - 0 = m([0, t\, [)\ .$$

From these three equalities and from the additivity of measures, we conclude that

$$\mu_{\bar{f}}\left(\bar{f}^{-1}(\mathcal{S}(f))\right) \le \mu_{\bar{f}}\left(\bar{f}^{-1}(U_\varepsilon)\right) = m(U_\varepsilon) < \varepsilon.$$

By making ε tend to zero, we have $\mu_{\bar{f}}\left(\bar{f}^{-1}(\mathcal{S}(f))\right) = 0$. □

End of the proof of the Theorem of Symmetrization

Finally, since $\dfrac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\Omega^*)} = \alpha(M, M^*)$, the following Lemma finishes the proof of the parts (i) and (iii) of the Theorem of Symmetrization 1.4.1.

Lemma 1.4.12. *The symmetrized function f^* is Lipschitz (with Lipschitz constant $\|\nabla f\|_{L^\infty}$). Moreover, if $V_{n-1}(r)$ denotes the $(n-1)-$dimensional volume of the geodesic sphere of radius r and centered at x^* in (M^*, g^*), we have*

$$\alpha(M, M^*)^{-1} \int_\Omega \|\nabla f(x)\|^2\, dv_g(x) \ge \int_0^{R_0} \bar{f}'(r)^2\, V_{n-1}(r)\, dr = \int_{\Omega^*} \|\nabla f^*(x)\|^2\, dv_{g^*}(x).$$

Proof. By the definition of the measure $\mu_{\bar{f}}$ and since the complement in $[0, R_0]$ of $R\left([0, \sup f] \setminus \mathcal{S}(f)\right)$ has $\mu_{\bar{f}}$-measure zero by the lemma 1.4.11, we have

$$\begin{aligned}&\int_{[0,R_0]} \bar{f}'(r)^2\, V_{n-1}(r)\, dr = \int_{[0,R_0]} \left(-\bar{f}'(r)\right)\, V_{n-1}(r)\, d\mu_{\bar{f}}(r)\\ &= \int_{R(\,[0,\sup f]\setminus\mathcal{S}(f)\,)} \left(-\bar{f}'(r)\right)\, V_{n-1}(r)\, d\mu_{\bar{f}}(r) = \int_{R(\,[0,\sup f]\setminus\mathcal{S}(f)\,)} \bar{f}'(r)^2\, V_{n-1}(r)\, dr\\ &= \int_{[0,\sup f]\setminus\mathcal{S}(f)} \bar{f}'(R(t))^2\, V_{n-1}(R(t))\, (-R'(t))dt,\end{aligned}$$

where the last equality follows by Lemma 1.4.2, which claims that $t \mapsto R(t)$ is a diffeomorphism from $[0, \sup f] \setminus \mathcal{S}(f)$ onto its image. Since the inverse of this diffeomorphism is the restriction of $\bar{f}$ to this (open) image by Property 1.3.5

(ii), we have $\bar{f}'(R(t))\,R'(t) = 1$ for every $t \in [0, \sup f] \setminus \mathcal{S}(f)$, and then, from the last equality and from the fact that $\mathrm{Vol}_{n-1}(\partial\Omega_t^*) = \mathrm{Vol}_{n-1}\left(\partial B(x^*, R(t))\right) = V_{n-1}(R(t))$, we have

$$\int_{[0,R_0]} \bar{f}'(r)^2\, V_{n-1}(r)\, dr = \int_{[0,\sup f]\setminus\mathcal{S}(f)} \left(-\frac{1}{R'(t)}\right) \mathrm{Vol}_{n-1}(\partial\Omega_t^*)\, dt. \tag{1.14}$$

Let us recall that, by definition, $f^* = \bar{f}\circ\rho$, where $\rho = d^*(x^*, \cdot)$, which implies, by the chain-rule and since $\|\nabla\rho\| \leq 1$ everywhere in the Lipschitz sense, that $\|\nabla f^*(x)\| = |\bar{f}'(\rho(x))|.\|\nabla\rho(x)\| \leq |\bar{f}'(\rho(x))|$ everywhere in the Lipschitz sense, and then that f^* is Lipschitz (with Lipschitz constant $\leq \|\nabla f\|_{L^\infty}$) by Lemma 1.4.10. This proves the part (i) of the Theorem of Symmetrization 1.4.1.

Moreover, since $\|\nabla\rho\| = 1$ almost everywhere (more precisely on $M^* \setminus \mathrm{Cut}(x^*)$), then $\|\nabla f^*(x)\| = |\bar{f}'(\rho(x))|$ almost everywhere. Plugging this into the equation (1.14), we obtain:

$$\begin{aligned}\int_{\Omega^*} \|\nabla f^*(x)\|^2\, dv_{g^*}(x) &= \int_{\Omega^*} |\bar{f}'(\rho(x))|^2\;\; dv_{g^*}(x) = \int_0^{R_0} |\bar{f}'(r)|^2\, V_{n-1}(r)\, dr \\ &= \int_{[0,\sup f]\setminus\mathcal{S}(f)} \left(-\frac{1}{R'(t)}\right) \mathrm{Vol}_{n-1}(\partial\Omega_t^*)\, dt,\end{aligned} \tag{1.15}$$

On the other hand, applying the Coarea Formula (Theorem 2.3.1), replacing in that formula the integrand $x \mapsto \varphi(x)$ by $x \mapsto \|\nabla f(x)\|$, and then using the Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned}&\int_\Omega \|\nabla f(x)\|^2\;\; dv_g(x) = \int_{[0,\sup f]\setminus\mathcal{S}(f)} \left(\int_{\{f=t\}} \|\nabla f(x)\|\;\; da_t(x)\right) dt \\ &\geq \int_{[0,\sup f]\setminus\mathcal{S}(f)} \frac{\mathrm{Vol}_{n-1}(\partial\Omega_t)^2}{\int_{\{f=t\}} \frac{1}{\|\nabla f(x)\|}\, da_t(x)}\, dt \\ &\geq \alpha(M, M^*)^2 \int_{[0,\sup f]\setminus\mathcal{S}(f)} \frac{\mathrm{Vol}_{n-1}(\partial\Omega_t^*)^2}{\int_{\{f=t\}} \frac{1}{\|\nabla f(x)\|}\, da_t(x)}\, dt,\end{aligned} \tag{1.16}$$

where the last inequality comes from the fact that (M^*, g^*, x^*) is a "pointed model-space for (M, g)" i. e., for any compact domain $\Omega' \subset M$, with smooth boundary, the symmetrized domain Ω'^* satisfies the isoperimetric inequality $\mathrm{Vol}_{n-1}(\partial\Omega') \geq \alpha(M, M^*)\,\mathrm{Vol}_{n-1}(\partial\Omega'^*)$.
Since $\int_{\{f=t\}} \frac{1}{\|\nabla f(x)\|}\, da_t(x) = -\alpha(M, M^*)\,\mathrm{Vol}_{n-1}(\partial\Omega_t^*)\, R'(t)$ for every $t \in [0, \sup f] \setminus \mathcal{S}(f)$ by the lemma 1.4.2, we deduce from the previous equality and from (1.15) that

$$\begin{aligned}\int_\Omega \|\nabla f(x)\|^2\;\; dv_g(x) &\geq \alpha(M, M^*) \int_{[0,\sup f]\setminus\mathcal{S}(f)} \frac{\mathrm{Vol}_{n-1}(\partial\Omega_t^*)}{-R'(t)}\, dt \\ &= \alpha(M, M^*) \int_{\Omega^*} \|\nabla f^*(x)\|^2\, dv_{g^*}(x).\end{aligned} \tag{1.17}$$

This concludes the proof of the inequality (iii).
If this inequality is an equality then, in (1.16), all inequalities are equalities. In particular $\mathrm{Vol}_{n-1}(\partial\Omega_t) = \alpha\,\mathrm{Vol}_{n-1}(\partial\Omega_t^*)$ for every $t \in [0, \sup f] \setminus \mathcal{S}(f)$ and,

if (M^*, g^*, x^*) is a strict PIMS for (M, g), this implies that Ω_t is isometric to Ω_t^*. Since $\{f > 0\}$ is the increasing union of the sets $\{f > t_n\}$ when $t_n \in [0, \sup f] \setminus \mathcal{S}(f)$ and $t_n \to 0_+$, we conclude that $\Omega_0 = \{f > 0\}$ is isometric to its symmetrized domain Ω_0^*. □

1.5 Comparison of the mean exit time $\mathcal{E}(\Omega)$ with the mean exit time $\mathcal{E}(\Omega^*)$ in a PIMS

In this section we compare the mean exit time $\mathcal{E}(\Omega)$ with the mean exit time $\mathcal{E}(\Omega^*)$ in a Pointed isoperimetric model space.

Theorem 1.5.1. *Let (M, g) be a Riemannian manifold and let (M^*, g^*, x^*) be a pointed model-space for (M, g) in the sense of Definition 1.3.2. Let Ω be any compact domain with smooth boundary in M (let us recall that this also implies that the closure of Ω is a strict subset of M), let Ω^* be the symmetrized domain, i. e. the geodesic ball of (M^*, g^*) , centered at the point x^* , such that*

$$\mathrm{Vol}(\Omega^*) = \begin{cases} \mathrm{Vol}(\Omega) & \text{if } \mathrm{Vol}(M, g) \text{ and } \mathrm{Vol}(M^*, g^*) \text{ are both infinite,} \\ \dfrac{\mathrm{Vol}(M^*, g^*)}{\mathrm{Vol}(M, g)} \mathrm{Vol}(\Omega) & \text{if } \mathrm{Vol}(M, g) \text{ and } \mathrm{Vol}(M^*, g^*) \text{ are both finite.} \end{cases}$$

then

$$\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^*) \quad .$$

Moreover, if (M^*, g^*, x^*) is a strict (pointed) model-space for (M, g) in the sense of Definition 1.3.2, then the equality $\mathcal{E}(\Omega) = \mathcal{E}(\Omega^*)$ is realized if and only if Ω is isometric to Ω^* .

Proof. Let f_Ω be the unique solution of the problem (1.1) on the domain Ω, let $(f_\Omega)^*$ be its symmetrized function. Applying the Theorem of Symmetrization 1.4.1 (ii) and (iii) we get

$$\mathcal{E}(\Omega) = E_\Omega(f_\Omega) = \frac{1}{\mathrm{Vol}(\Omega)} \left(2 \int_\Omega f_\Omega \; dv_g - \int_\Omega |\nabla f_\Omega|^2 \; dv_g \right)$$

$$\leq \frac{1}{\mathrm{Vol}(\Omega^*)} \left(2 \int_{\Omega^*} (f_\Omega)^* \; dv_{g^*} - \int_{\Omega^*} |\nabla (f_\Omega)^*|^2 \; dv_{g^*} \right) = E_{\Omega^*}\left((f_\Omega)^*\right).$$

We know that the *mean exit-time* from the domain Ω^* is the value $\mathcal{E}(\Omega^*) = \max_{u \in H^2_{1,c}(\Omega^*)} \left(E_\Omega\left(u\right)\right)$. Since by (i) of Theorem 1.4.1 $(f_\Omega)^* \in H^2_{1,c}(\Omega^*, g^*)$ it follows

$$\mathcal{E}(\Omega^*) \geq E_{\Omega^*}\left((f_\Omega)^*\right) \geq \mathcal{E}(\Omega).$$

If $\mathcal{E}(\Omega^*) = \mathcal{E}(\Omega)$, then all the inequalities are equalities. In particular

$$\int_\Omega |\nabla f_\Omega|^2 \; dv_g = \alpha(M, M^*) \int_{\Omega^*} |\nabla (f_\Omega)^*|^2 \; dv_{g^*}$$

and $E_{\Omega^*}\left((f_\Omega)^*\right) = \mathcal{E}(\Omega^*)$. Then, since the set $\{f_\Omega > 0\}$ coincides with the interior of Ω, from the equality case in part (iii) of Theorem 1.4.1 we conclude that Ω^* is isometric to Ω. □

1.6 Comparison of the mean exit times when (M, g) is an isoperimetric manifold

Let (M, g) be a Riemannian manifold, for any point x_0 we denote by $\mathrm{Cut}(x_0)$, the "cut-locus" of x_0, i. e. the union of the cut points (the points where the corresponding geodedesic ceases to be minimal) of x_0 along all of the geodesics that start from x_0. We recall that it is a closed subset of measure zero and that the exponential map $\exp_{x_0}$ is a diffeomorphism from some open subset U_{x_0} of the tangent space $T_{x_0}M$ onto $M \setminus \mathrm{Cut}(x_0)$. Let $\mathbb{S}_{x_0}$ be the euclidean unit sphere of the euclidean space $(T_{x_0}M, g_{x_0})$. We define the open subset $\widetilde{U}_{x_0} \subset]\,0\,,\,+\infty\,[\times \mathbb{S}_{x_0}$ as the pull-back of U_{x_0} by the map $(t, v) \mapsto t.v$ from $]\,0\,,\,+\infty\,[\times \mathbb{S}_{x_0}$ to $T_{x_0}M$; with this subset we can write a generalization of the usual "polar coordinates" by the notion of "normal coordinates"

$$\phi : \begin{cases} \widetilde{U}_{x_0} \to U_{x_0} \to M \setminus \mathrm{Cut}(x_0) \\ (t, v) \mapsto t.v \mapsto \exp_{x_0}(t.v) \end{cases}$$

In this coordinate system, the Riemannian measure at the point $\phi(t, v)$ is written as

$$\phi^* dv_g = \theta(t, v)\, dt\, dv \quad , \tag{1.18}$$

where dv is the canonical measure of the canonical sphere $\mathbb{S}_{x_0}$. This defines $\theta(t, v)$ as the density of the measure $\phi^* dv_g$ with respect to the measure $dt\, dv$. We use the following definition of a Riemannian manifold harmonic at x_0:

Definition 1.6.1. *Let (M, g) be a Riemannian manifold and $x_0 \in M$, (M, g) is said to be "harmonic at x_0" if the two following conditions are satisfied :*

- *U_{x_0} is empty or is a ball of the euclidean space $(T_{x_0}M, g_{x_0})$ (and thus there exists some $\beta \in]0, +\infty]$ such that $\widetilde{U}_{x_0} =]\,0\,,\,\beta\,[\times \mathbb{S}_{x_0}$).*
- *for every $t \in]\,0\,,\,\beta\,[$, $\theta(t, v)$ does not depend on v .*

Definition 1.6.2. *A Riemannian manifold (M, g) is said to be "harmonic" if it is harmonic about each of its points.*

Example 1.6.3. Spaces of revolution are "harmonic about their pole(s)", but they are generally not "harmonic" in the sense of Definition 1.6.2.
In fact, by definition, a (non compact) space of revolution (M, g) with only one pole x_0 is such that $(M \setminus \{x_0\}, g)$ is isometric to $]\,0\,,\,+\infty\,[\times \mathbb{S}^{n-1}$, endowed with a Riemannian metric of the type $(dt)^2 + b(t)^2\, g_{\mathbb{S}^{n-1}}$, where b is a smooth strictly positive function whose extension to $[0\,,\,+\infty\,[$ satisfies $b(0) = 0$ (and $b'(0) = 1$ if we want the metric to be regular at x_0), where $g_{\mathbb{S}^{n-1}}$ is the canonical metric of the sphere $\mathbb{S}^{n-1}$ and where $\{0\} \times \mathbb{S}^{n-1}$ is identified with the point x_0.
On the othe hand, a (compact) space of revolution (M, g) with two poles x_0 and x_1 is such that $(M \setminus \{x_0, x_1\}, g)$ is isometric to $]\,0\,,\,L\,[\times \mathbb{S}^{n-1}$, endowed with a Riemannian metric of the type $(dt)^2 + b(t)^2\, g_{\mathbb{S}^{n-1}}$, where b is a smooth strictly positive function whose extension to $[0\,,\,L]$ satisfies $b(0) = b(L) = 0$ (and $b'(0) = 1$, $b'(L) = -1$ if we want the metric to be regular at x_0) and where $\{0\} \times \mathbb{S}^{n-1}$ and $\{L\} \times \mathbb{S}^{n-1}$ are identified with the point x_0 and with the point x_1 respectively. In this case, we have $\phi :]\,0\,,\,\mathrm{Cut}(x_0)\,[\times \mathbb{S}_{x_0} \to M \setminus \mathrm{Cut}(x_0)$ and $\phi^* dv_g = \beta^{n-1}(t)\, dt\, dv$.

We recall that if $\exp_{x_0}$ is is a diffeomorphism of a neighborhood V of the origin in $T_{x_0}M$, $exp_{x_0}V = U$ is called a normal neighborhood of x_0. If $B_\epsilon(0)$ is such that $B_\epsilon(0) \subset V$, we call $exp_{x_0}(B_\epsilon(0)) = B_\epsilon(x_0)$ the geodesic ball with center x_0 and radius ϵ.
Harmonic manifolds have the following properties:

Proposition 1.6.4. *If a Riemannian manifold (M, g) is harmonic (in the sense of Definition 1.6.2) then all its geodesic balls are harmonic domains (in the sense of Definition 2.2.1).*

This Proposition is an immediate consequence of the following

Proposition 1.6.5. *Let (M, g) be a Riemannian manifold and x_0 a point in M. If the Riemannian manifold (M, g) is harmonic at x_0 (in the sense of Definition 1.6.1) then every geodesic ball centered at x_0 is a harmonic domain (in the sense of Definition 2.2.1).*

Proof. We take a geodesic ball Ω of radius R centered at x_0. If $\theta(t, v)$ is the density of the measure $\phi^* dv_g$ with respect to the product measure $dt\, dv$ of $]\,0\,,\,+\infty\,[\times \mathbb{S}_{x_0}$ (as in (1.18)), then from Definition 1.6.1 we know that $\theta(t, v)$ does not depend on v and so we write it as $\theta(t)$.
We define $f : \Omega \to \mathbb{R}$ by $f(x) = u\,(d(x_0, x))$, where

$$u(r) = \int_r^R \left(\frac{\int_0^t \theta(s)\, ds}{\theta(t)} \right) dt \quad . \tag{1.19}$$

Since, in this case, $\Delta f(x) = -u''\,(d(x_0, x)) - \dfrac{\theta'\,(d(x_0, x))}{\theta\,(d(x_0, x))}\, u'\,(d(x_0, x))$ and since

$$u'(r) = -\frac{\int_0^r \theta(s)\, ds}{\theta(r)}$$

and

$$-\frac{1}{\theta(r)}\,(\theta(r)u'(r))' = 1$$

we deduce that $\Delta f = 1$. Moreover, if $x \in \partial\Omega$ then $u(R) = 0$ and so $f(x) = 0$.
Then, we deduce that $f = f_\Omega$.
For $x \in \partial\Omega$

$$\frac{\partial f}{\partial N}(x) = u'(R)\frac{\partial d}{\partial N}(x_0, x)$$

and

$$||\nabla f(x)|| = -u'(R)||\nabla d(x_0, x)|| = -u'(R),$$

so, we can conclude that Ω is harmonic.

□

Remark 1.6.6. From the last Proposition we know that, in a harmonic manifold at x_0, every geodesic ball centered at x_0 is a harmonic domain. It is also known that the converse is not true. A counterexample is given by tubular

neighborhoods, in S^3, of some geodesic circle S^1. They are examples of harmonic domains of $(\mathbb{S}^3, can.)$ which are not geodesic balls.
We take the following parametrization of $\mathbb{S}^3$

$$F : (r,t,s) \mapsto \cos r \begin{pmatrix} \cos t \\ \sin t \\ 0 \\ 0 \end{pmatrix} + \sin r \begin{pmatrix} 0 \\ 0 \\ \cos s \\ \sin s \end{pmatrix}.$$

In this parametrization the canonical metric on the sphere is

$$dr^2 + \cos r^2 dt^2 + \sin r^2 ds^2.$$

We choose the geodesic circle $C = \mathbb{S}^1$, that is $t \mapsto (\cos t, \sin t, 0, 0)$.
Since $u \mapsto \cos u(\cos t, \sin t, 0, 0) + \sin u(0, 0, \cos s, \sin s)$ is a minimizing geodesic, then the distance from $x = F(r,t,s)$ to the geodesic circle C is $\rho(x) = \rho(F(r,t,s)) = r$. The metric above implies that

$$\Delta\rho = -\frac{1}{\tan\rho} + \tan\rho,$$

and so there exists a function φ such that $\varphi \circ \rho$ satisfies $\Delta(\varphi \circ \rho) = 1$. The function φ is such that $\varphi'(r) = \frac{1}{2\tan r}$. Then the function $f(x) = \varphi\circ\rho(x) - \varphi(r_0)$ is zero on the tubular neighborhood of radius r_0 (the set of points x such that $\rho(x) < r_0$) and satisfies $\Delta f = 1$. Moreover, the tubular neighborhood of radius r_0 is a harmonic domain since the normal inner derivative on the boundary is equal to $\varphi'(r_0)$ and hence constant.
Hence, since from Theorem 2.2.2 we know that all critical points of the functional $\Omega \mapsto \mathcal{E}(\Omega)$ are harmonic domains, we are interested to the problem of study the maxima of this functional.

Definition 1.6.7. *Let (M, g) be a Riemannian manifold and let $x_0 \in M$. The manifold (M, g) is said to be* isoperimetric at x_0 *if it is harmonic at x_0 and if, for any compact domain $\Omega \subset M$ with smooth boundary, the geodesic ball Ω^* centered at x_0 with the same volume as Ω satisfies $\mathrm{Vol}_{n-1}(\partial\Omega^*) \leq \mathrm{Vol}_{n-1}(\partial\Omega)$; the same manifold is said to be* strictly isoperimetric at x_0 *if, moreover, the equality occurs iff Ω is isometric to Ω^*.*

The Euclidean space, the Hyperbolic Space and the Sphere are strictly isoperimetric at every point (see [7] sections 10 and 8.6). These are the only known examples (up to homotheties) of Riemannian manifolds which are isoperimetric at every point. If we only require the Riemannian manifolds to be isoperimetric at (at least) one point, we get much more examples.

Example 1.6.8. *A first example of a (nonstandard) space of revolution that is isoperimetrtic at its poles is the following. We consider a 2-dimensional cylinder $[0, +\infty[\times\mathbb{S}^1$ (resp. $[0, L]\times\mathbb{S}^1$) with 1 hemisphere glued to the boundary $\{0\}\times\mathbb{S}^1$ (resp. with 2 hemispheres respectively glued to the boundaries $\{0\} \times \mathbb{S}^1$ and $\{L\} \times \mathbb{S}^1$). Other examples are given by the paraboloid of revolution $z = x^2 + y^2$ or the hyperboloid of equation $x^2 + y^2 - z^2 = -1,\ z > 0$ in $\mathbb{R}^3$ (isoperimetric at their pole). More generally, a large class of nonstandard examples is given by the following theorem*

Theorem 1.6.9. ([31], Theorem 1.2) *Consider the plane $\mathbb{R}^2$ equipped with a complete and rotationally invariant Riemannian metric g such that the Gauss curvature is positive and a strictly decreasing function of the distance from the origin. Then $(\mathbb{R}^2, g)$ is isoperimetric at the origin.*

Remark 1.6.10. It is not true that every space of revolution is isoperimetric at its pole. For instance, we have the following counterexample. Let S be the hypersurface of revolution in $\mathbb{R}^3$ of equation $x^2 + y^2 + (|z| + \cos R)^2 = 1$, whose poles are $x_0 = (0, 0, 1 - \cos R)$ and $x_1 = -x_0$. The plane $y = 0$ separates S in two symmetric domains, which have the same area as the geodesic ball $B(x_0, R)$, but their boundaries are shorter than $\partial B(x_0, R)$.

Proposition 1.6.11. *Let (M, g) be a Riemannian manifold which is isoperimetric at some point $x_0 \in M$, for every $v \in]\,0\,, \mathrm{Vol}(M, g)\,[$, the functional $\Omega \mapsto \mathcal{E}(\Omega)$ (where Ω runs in the set of all compact domains in M, with smooth boundary and prescribed volume v) attains its maximum when Ω is the geodesic ball Ω^* of volume v centered at x_0 (i. e. $\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^*)$). Moreover, if (M, g) is strictly isoperimetric at x_0 then this maximum is unique, i. e. the equality $\mathcal{E}(\Omega) = \mathcal{E}(\Omega^*)$ is realized if and only if Ω is isometric to Ω^*.*

Proof. It is a consequence of Theorem 1.5.1. In fact, from Definition 1.6.7 we have that (M, g, x_0) is a PIMS for (M, g) itself in the sense of the Definition 1.3.2 and so, it is a particular case of Theorem 1.5.1 where (M, g) and (M^*, g^*) coincide and $\alpha(M, M^*) = 1$. $\square$

1.7 Comparison of the mean exit times when (M, g) is non compact

We only consider the case where (M, g) is a Cartan-Hadamard manifold.
We recall that a Cartan–Hadamard manifold is a complete simply connected Riemannian manifold with non positive sectional curvature. For these manifolds we have the Cartan–Hadamard conjecture (or Aubin's conjecture):

Conjecture 1.7.1. *The Euclidean n-dimensional space E^n, pointed at any point $x^* \in E^n$, is a strict PIMS for every Cartan-Hadamard manifold of the same dimension.*

It is known that the conjecture is true when $n \leq 4$. For $n = 2$ it was proved for the first time by A. Weil in [50], in dimension 4 it was proved by C. B. Croke [21], and in dimension 3 there is a more recent proof by B. Kleiner [36]. In higher dimensions, the conjecture is still open.
From these results we immediately get the following corollary of Theorem 1.5.1 when $\mathcal{M}$ is the class of Cartan-Hadamard manifolds of dimension at most 4:

Corollary 1.7.2. *Let (M, g) be a Cartan–Hadamard manifold of dimension $n \leq 4$. For every compact domain $\Omega \subset M$ with smooth boundary, we have*

$$\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^*),$$

where Ω^* is the Euclidean n-ball with the same volume as Ω. Moreover, the equality $\mathcal{E}(\Omega) = \mathcal{E}(\Omega^*)$ is realized if and only if Ω is isometric to an Euclidean ball.

1.8 Comparison of the mean exit times when (M, g) is compact

In this section we consider two classes of manifolds $\mathcal{M}$ whose geometry are bounded.
Firstly, we consider a class of manifolds where the Ricci curvature is bounded from below by the Ricci curvature of the canonical sphere.
Secondly, we study a class of manifolds $\mathcal{M}$ where the Cheerger's isoperimetric constant $H(M, g)$ is bounded from below by a positive constant H.

The case where $\mathrm{Ric}_g \geq (n-1) \cdot g$

The main result of this section is the following theorem:

Theorem 1.8.1. *For every complete, connected Riemannian manifold* (M, g) *whose Ricci curvature satisfies* $\mathrm{Ric}_g \geq (n-1) \cdot g$*, for every compact domain with smooth boundary* Ω *in* M*, let* Ω^* *be a geodesic ball of the canonical sphere* $(\mathbb{S}^n, g_0)$ *such that* $\dfrac{\mathrm{Vol}(\Omega^*, g_0)}{\mathrm{Vol}(\mathbb{S}^n, g_0)} = \dfrac{\mathrm{Vol}(\Omega, g)}{\mathrm{Vol}(M, g)}$*, then* $\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^*)$ *. Morever,*

(i) if there exists some domain $\Omega \subset M$ *such that* $\mathcal{E}(\Omega) = \mathcal{E}(\Omega^*)$ *then* (M, g) *is isometric to* $(\mathbb{S}^n, g_0)$ *and* Ω *is isometric to* Ω^* *.*

(ii) If there exists some domain $\Omega \subset M$ *such that*

$$\mathcal{E}(\Omega) > (1 - \delta(n, \kappa))^{\frac{2}{n}}\, \mathcal{E}(\Omega^*) \qquad \text{with} \qquad \delta(n, \kappa) = \frac{\int_0^{\frac{\varepsilon(n,\kappa)}{2}} (\sin t)^{n-1}\, dt}{\int_0^{\frac{\pi}{2}} (\sin t)^{n-1}\, dt}$$

(where $-\kappa^2$ *is a lower bound for the sectional curvature of* (M, g) *and where* $\varepsilon(n, \kappa)$ *is the Perelman constant described in Theorem 1.8.10) then* M *is diffeomorphic to* $\mathbb{S}^n$ *.*

We start describing the tools we need in order to prove the previous theorem. The first result that we use is the following isoperimetric inequality proved by M. Gromov in [30].

Theorem 1.8.2. *For every Riemannian manifold* (M, g) *whose Ricci curvature satisfies* $\mathrm{Ric}_g \geq (n-1) \cdot g$ *, for every compact domain with smooth boundary* Ω *in* M *, let* Ω^* *be a geodesic ball of the canonical sphere* $(\mathbb{S}^n, g_0)$ *such that* $\dfrac{\mathrm{Vol}(\Omega^*, g_0)}{\mathrm{Vol}(\mathbb{S}^n, g_0)} = \dfrac{\mathrm{Vol}(\Omega, g)}{\mathrm{Vol}(M, g)}$ *, then*

$$\frac{\mathrm{Vol}_{n-1}(\partial\Omega)}{\mathrm{Vol}(M, g)} \geq \frac{\mathrm{Vol}_{n-1}(\partial\Omega^*)}{\mathrm{Vol}(\mathbb{S}^n, g_0)}.$$

Moreover, this last inequality is an equality if and only if Ω *is isometric to* Ω^**. In other words, for any* $x_0 \in \mathbb{S}^n$ *,* $(\mathbb{S}^n, g_0, x_0)$ *is a strict PIMS for all the Riemannian manifolds* (M, g) *which satisfy* $\mathrm{Ric}_g \geq (n-1) \cdot g$.

Applying this Theorem and Theorem 1.5.1 we get

Corollary 1.8.3. *For every Riemannian manifold* (M, g) *whose Ricci curvature satisfies* $\mathrm{Ric}_g \geq (n-1)\cdot g$ *, for every compact domain with smooth boundary* Ω *in* M *, let* Ω^* *be a geodesic ball of the canonical sphere* $(\mathbb{S}^n, g_0)$ *such that* $\mathrm{Vol}(\Omega^*, g_0)/\mathrm{Vol}(\mathbb{S}^n, g_0) = \mathrm{Vol}(\Omega, g)/\mathrm{Vol}(M, g)$*, then* $\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^*)$*. Moreover, the equality* $\mathcal{E}(\Omega) = \mathcal{E}(\Omega^*)$ *is realized if and only if* Ω *is isometric to* Ω^* *.*

Remark 1.8.4. Since the canonical sphere $(\mathbb{S}^n, g_0)$ satisfies $\mathrm{Ric}_{g_0} = (n-1)\cdot g_0$, we may apply Theorem 1.8.2 to the sphere, and then deduce an inequality which is an equality when Ω is a geodesic ball of $(\mathbb{S}^n, g_0)$.

Remark 1.8.5. For every $K > 0$, we can extend Corollary 1.8.3 to every Riemannian manifold (M, g) which satisfies $\mathrm{Ric}_g \geq K\,(n-1)\cdot g$. In fact, we only need to make the following changes: replace the canonical sphere by the sphere of constant sectional curvature K in the statement of Corollary 1.8.3, apply Corollary 1.8.3 to the Riemannian manifold $(M, K\cdot g)$ and then use the homogeneity formula (1.4).

Remark 1.8.6. For every fixed $\beta \in]0,1[$, let $\mathcal{W}_\beta$ be the set of all domains Ω, in all the Riemannian manifolds $(M, g) \in \mathcal{M}$, such that $\mathrm{Vol}(\Omega, g)/\mathrm{Vol}(M, g) = \beta$. Then, the geodesic ball Ω^* of the canonical sphere $(\mathbb{S}^n, g_0)$ such that $\mathrm{Vol}(\Omega^*, g_0) = \beta\,\mathrm{Vol}(\mathbb{S}^n, g_0)$ is an element of $\mathcal{W}_\beta$, and Corollary 1.8.3 proves that the functional $\Omega \mapsto \mathcal{E}(\Omega)$, when restricted to the set $\mathcal{W}_\beta$, attains its absolute maximum when $\Omega = \Omega^*$ and that this maximum is strict. Moreover, by Theorem 1.8.1, if $\mathcal{E}(\Omega)$ is not far from this maximal value, then M is diffeomorphic to $\mathbb{S}^n$.

P. Bérard, G. Besson and S. Gallot generalized Theorem 1.8.2 to the case where the Ricci curvature has any sign (see [4] Theorem (2) and [28] Theorem 6.16 for a quantitatively improved version). They proved that

Theorem 1.8.7. *For any* $K \in \mathbb{R}$ *, a PIMS for all the* n*-dimensional Riemannian manifolds* (M, g) *which satisfy* $\mathrm{Ric}_g \geq (n-1)\,K\cdot g$ *and* $\mathrm{diameter}(M, g) \leq D$ *is given by the Euclidean sphere of radius* $R(K, D)$ *(PIMS at any point) where* $R(K, D)$ *is defined by*

$$R(K,D) = \begin{cases} \frac{1}{\sqrt{K}} \left(\dfrac{\int_0^{\frac{D\sqrt{K}}{2}} (\cos t)^{n-1}\, dt}{\int_0^{\frac{\pi}{2}} (\cos t)^{n-1}\, dt} \right)^{\frac{1}{n}} & \text{if } K > 0 \\[2ex] \frac{n}{2} \left(\int_0^{\frac{\pi}{2}} (\cos t)^{n-1}\, dt \right)^{-\frac{1}{n}} D & \text{if } K = 0 \\[2ex] \frac{1}{\sqrt{|K|}} \mathrm{Max} \left(\dfrac{\int_0^{D\sqrt{|K|}} (\cosh 2t)^{\frac{n-1}{2}}\, dt}{\int_0^{\pi} (\sin t)^{n-1}\, dt} \, , \, \left(\dfrac{\int_0^{D\sqrt{|K|}} (\cosh 2t)^{\frac{n-1}{2}}\, dt}{\int_0^{\pi} (\sin t)^{n-1}\, dt} \right)^{\frac{1}{n}} \right) \\ \text{if } K < 0 \end{cases}$$

In other terms, for every compact domain with smooth boundary Ω *in* M *, if* Ω^* *is a geodesic ball on the Euclidean sphere* $\mathbb{S}^n(R(K, D))$ *of radius* $R(K, D)$ *and if* Ω^{**} *is a geodesic ball of the canonical sphere* $\mathbb{S}^n(1) = (\mathbb{S}^n, g_0)$ *such that*

$$\frac{\mathrm{Vol}(\Omega^{**})}{\mathrm{Vol}(\mathbb{S}^n, g_0)} = \frac{\mathrm{Vol}(\Omega^*)}{\mathrm{Vol}(\mathbb{S}^n(R(K,D)))} = \frac{\mathrm{Vol}(\Omega, g)}{\mathrm{Vol}(M, g)} \ ,$$

then

$$\frac{\mathrm{Vol}_{n-1}(\partial\Omega)}{\mathrm{Vol}(M,g)} \geq \frac{\mathrm{Vol}_{n-1}(\partial\Omega^*)}{\mathrm{Vol}(\mathbb{S}^n(R(K,D)))} = \frac{1}{R(K,D)}\frac{\mathrm{Vol}_{n-1}(\partial\Omega^{**})}{\mathrm{Vol}(\mathbb{S}^n,g_0)}. \tag{1.20}$$

Remark 1.8.8. Firstly, we remark that the smaller $R(K,D)$ is, the stronger is the isoperimetric inequality (1.20). When $K > 0$, the Theorem 1.8.7 is sharp, because choosing $D = \frac{\pi}{\sqrt{K}}$, the sphere $\mathbb{S}^n\left(\frac{1}{\sqrt{K}}\right)$ satisfies the assumptions and the conclusion of the Theorem 1.8.7. In this case the isoperimetric inequality (1.20) is verified when $(M,g) = \mathbb{S}^n\left(\frac{1}{\sqrt{K}}\right)$; moreover, this inequality is an equality because $R(K,D) = R\left(K, \frac{\pi}{\sqrt{K}}\right) = \frac{1}{\sqrt{K}}$.

Furthermore, under the assumptions "$\mathrm{Ric}_g \geq (n-1)\,K\cdot g$" and "$(M,g)$ not isometric to $\mathbb{S}^n\left(\frac{1}{\sqrt{K}}\right)$", from Myers' theorem (and its equality case) we get that $\mathrm{diameter}(M,g) < \frac{\pi}{\sqrt{K}}$, and then we can apply the Theorem 1.8.7 with the values $K = 1$ and $D < \frac{\pi}{\sqrt{K}}$ of the constants, which implies that, under the hypothesis, $R(K,D) < \frac{1}{\sqrt{K}}$. The isoperimetric inequality (1.20) is then strictly stronger than the one of the sphere $\mathbb{S}^n\left(\frac{1}{\sqrt{K}}\right)$.

On the other hand, when $K \leq 0$, Theorem 1.8.7 is not sharp because we always have $\mathrm{diameter}\,(\mathbb{S}^n(R(K,D))) > D$, and so the sphere $\mathbb{S}^n(R(K,D))$ does not satisfy the assumptions of Theorem 1.8.7.

Applying Theorem 1.8.7 we get

Corollary 1.8.9. *Let K be an arbitrary real number (of any sign), for any n-dimensional Riemannian manifold (M,g) which satisfies $\mathrm{Ric}_g \geq (n-1)\,K\cdot g$ and $\mathrm{diameter}(M,g) \leq D$, for every compact domain with smooth boundary Ω in M, if Ω^* is a geodesic ball on the Euclidean sphere $\mathbb{S}^n(R(K,D))$ and if Ω^{**} is a geodesic ball of the canonical sphere $\mathbb{S}^n(1) = (\mathbb{S}^n, g_0)$ such that*

$$\frac{\mathrm{Vol}(\Omega^{**})}{\mathrm{Vol}(\mathbb{S}^n,g_0)} = \frac{\mathrm{Vol}(\Omega^*)}{\mathrm{Vol}(\mathbb{S}^n(R(K,D)))} = \frac{\mathrm{Vol}(\Omega,g)}{\mathrm{Vol}(M,g)}\ ,$$

then

$$\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^*) = R(K,D)^2\ \mathcal{E}(\Omega^{**}). \tag{1.21}$$

Proof. From Theorem 1.8.7 we know that the Euclidean sphere $\mathbb{S}^n(R(K,D))$ of radius $R(K,D)$ is a PIMS for the Riemannian manifold (M,g). If Ω is a compact domain with smooth boundary in M, if Ω^* is a geodesic ball on the Euclidean sphere $\mathbb{S}^n(R(K,D))$ of radius $R(K,D)$ and if Ω^{**} is a geodesic ball of the canonical sphere $\mathbb{S}^n(1) = (\mathbb{S}^n, g_0)$ such that

$$\frac{\mathrm{Vol}(\Omega^{**})}{\mathrm{Vol}(\mathbb{S}^n,g_0)} = \frac{\mathrm{Vol}(\Omega^*)}{\mathrm{Vol}(\mathbb{S}^n(R(K,D)))} = \frac{\mathrm{Vol}(\Omega,g)}{\mathrm{Vol}(M,g)}\ ,$$

then applying Lemma 1.5.1 we get

$$\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^*) = R(K,D)^2\,\mathcal{E}(\Omega^{**}),$$

where the last equality follows from the fact that the sphere of radius $R(K,D)$ is isometric to $(\mathbb{S}^n, R(K,D)^2.g_0)$ and from formula (1.4). $\square$

The last result we need in order to prove Theorem 1.8.1 is the following inequality proved by G. Perelman [43].

Theorem 1.8.10. *Let (M, g) be an n-dimensional compact Riemannian manifold. Assume that M is* not *diffeomorphic to $\mathbb{S}^n$, that* $\mathrm{Ric}_g \geq (n-1) \cdot g$ *and that the sectional curvature of (M, g) is $\geq -\kappa^2$. Then there exists a constant* $\varepsilon(n, \kappa) > 0$ *such that* $\mathrm{diameter}(M, g) \leq \pi - \varepsilon(n, \kappa)$.

Remark 1.8.11. When $K = 1$ and $D = \pi - \varepsilon(n, \kappa)$ Theorem 1.8.7 gives

$$R(K, D) = R(1, \pi - \varepsilon(n, \kappa)) = \left(1 - \frac{\int_0^{\frac{\varepsilon(n,\kappa)}{2}} (\sin t)^{n-1}\, dt}{\int_0^{\frac{\pi}{2}} (\sin t)^{n-1}\, dt} \right)^{\frac{1}{n}}. \tag{1.22}$$

Thus, with respect to the isoperimetric inequality of the canonical sphere, the isoperimetric inequality on (M, g) induced by (1.20) is improved by some factor which is bounded from below by 1.

Proof of Theorem 1.8.1: Applying Theorem 1.8.7 when the constants are $K = 1$ and $D = \mathrm{diameter}(M, g)$, we get that the Euclidean sphere $\mathbb{S}^n(R(1, D))$ of radius $R(1, D) = R(1, \mathrm{diameter}(M, g))$ is a PIMS (at any point) for the Riemannian manifold (M, g). For every compact domain with smooth boundary Ω in M, if Ω^0 is a geodesic ball on the Euclidean sphere $\mathbb{S}^n(R(1, D))$ of radius $R(1, D)$ and if Ω^* is a geodesic ball of the canonical sphere $\mathbb{S}^n(1) = (\mathbb{S}^n, g_0)$ such that

$$\frac{\mathrm{Vol}(\Omega^*)}{\mathrm{Vol}(\mathbb{S}^n, g_0)} = \frac{\mathrm{Vol}(\Omega^0)}{\mathrm{Vol}(\mathbb{S}^n(R(1, D)))} = \frac{\mathrm{Vol}(\Omega, g)}{\mathrm{Vol}(M, g)},$$

then, from Lemma 1.5.1 we obtain

$$\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^0) = R(1, D)^2\, \mathcal{E}(\Omega^*), \tag{1.23}$$

where the last equality follows from the fact that $\mathbb{S}^n(R(1, D))$ is isometric to $(\mathbb{S}^n, R(1, D)^2.g_0)$ and from formula (1.4).

We prove the second part of the Theorem 1.8.1 by contradiction applying Theorem 1.8.10. Firstly, we suppose that (M, g) is not isometric to $(\mathbb{S}^n, g_0)$. Then Myers' theorem (and its equality case)[8] implies that $\mathrm{diameter}(M, g) < \pi$, and thus, by the definition of $R(K, D)$, $R(1, D) < 1$ (where $D = \mathrm{diameter}(M, g)$). Since $R(1, D) < 1$, from the inequality (1.23), we get that, if (M, g) is not isometric to $(\mathbb{S}^n, g_0)$, then $\mathcal{E}(\Omega) < \mathcal{E}(\Omega^*)$ for every compact domain with smooth boundary Ω in M, which concludes the part (i) of the Theorem 1.8.1.
Finally, we suppose that M is not diffeomorphic to $\mathbb{S}^n$. Then, Theorem 1.8.10 implies that the value $D = \pi - \varepsilon(n, \kappa)$ is a upper bound of the diameter of (M, g). From the inequality (1.23) and the formula (1.22), we obtain that

$$\mathcal{E}(\Omega) \leq \left(1 - \frac{\int_0^{\frac{\varepsilon(n,\kappa)}{2}} (\sin t)^{n-1} dt}{\int_0^{\frac{\pi}{2}} (\sin t)^{n-1}\, dt} \right)^{\frac{2}{n}} \mathcal{E}(\Omega^*)$$

for every compact domain with smooth boundary Ω in M. This concludes the proof of part (ii) of Theorem 1.8.1.

[8] The theorem claims that if the Ricci curvature of a complete Riemannian manifold (M, g) satisfies $\mathrm{Ric}_g \geq K(n-1) \cdot g$, then its diameter is at most $\frac{\pi}{\sqrt{K}}$. Moreover if the diameter is equal to $\frac{\pi}{\sqrt{K}}$, then the manifold is isometric to a sphere of a constant sectional curvature K.

The case where $H(M,g) \geq H$

We recall the definition of *Cheeger's isoperimetric constant* $H(M,g)$, that is defined by

$$H(M,g) = \inf_{\Omega} \left(\frac{\mathrm{Vol}_{n-1}(\partial\Omega)}{\min\left[\mathrm{Vol}(\Omega)\,, \mathrm{Vol}(M \setminus \Omega)\right)} \right) \,, \tag{1.24}$$

where Ω runs in the set of all domains with smooth boundary in M.

For any $H > 0$, we denote by $\mathcal{M}_H$ the set of all Riemannian manifolds (M,g) whose Cheeger's isoperimetric constant is bounded from below by H. In other words, it is the set of the (M,g)'s that satisfy, for every domain with smooth boundary $\Omega \subset M$, the following isoperimetric inequality:

$$\mathrm{Vol}_{n-1}(\partial\Omega) \geq H \cdot \min\left(\mathrm{Vol}(\Omega), \mathrm{Vol}(M \setminus \Omega)\right) \,. \tag{1.25}$$

The result concerning this set of manifolds is the following

Theorem 1.8.12. *Let (M,g) be any compact Riemannian manifold and let Ω be any compact domain with smooth boundary in M such that* $\mathrm{Vol}(\Omega) \leq \frac{1}{2}\,\mathrm{Vol}(M)$. *Then* $\mathcal{E}(\Omega) \leq \dfrac{1}{H(M,g)^2}$.

The idea for proving the previous theorem can be found in [3] (sections IV.B.13 and IV.B.22) and in [28] (sections 5.B and Appendix A.4), where P. Bérard and S. Gallot introduce the "*double of the hyperbolic cusp*" (M^*, g_ε^*). It is constructed by endowing the manifold $M^* := \mathbb{R} \times \mathbb{S}^{n-1}$ with the Riemannian metric g_ε^* defined, at any point $(t,v) \in \mathbb{R} \times \mathbb{S}^{n-1}$, by $g_\varepsilon^* := (dt)^2 \oplus \varepsilon^2\, e^{-2\frac{H}{n-1}|t|} \cdot g_0$, where g_0 is the canonical metric of $\mathbb{S}^{n-1}$. They remark that (M^*, g_ε^*) is some kind of generalized PIMS for all the manifolds $(M,g) \in \mathcal{M}_H$, because the symmetric domains $\Omega_r^* := [r, +\infty[\times \mathbb{S}^{n-1} \subset M^*$ (i. e. the "balls" centered at the pole at infinity) realize the equality[9] in the isoperimetric inequality (1.25). For proving Theorem 1.8.12 we need the following

Lemma 1.8.13. *Let (M,g) be a Riemannian manifold. Then, for any compact domain $\Omega \subset M$ and for any non negative continuous function f on Ω which vanishes on $\partial\Omega$, one has:*

$$\forall p \in [1, +\infty[\quad , \quad \int_\Omega f^p dv_g = p \int_0^{\sup f} t^{p-1} A(t)\, dt \,.$$

For any continuous function f, we recall that we denote by Ω_t the set of points $x \in \Omega$ such that $f(x) > t$, and by $A(t)$ the volume of Ω_t.

Proof. Let $t_i = \frac{i}{N} \sup f$ (for every $i \in \{0, \dots N\}$). Since the function $t \mapsto A(t) = \mathrm{Vol}(\Omega_t)$ is strictly decreasing, we get

$$\sum_{i=0}^{N-1} t_i (A(t_i) - A(t_{i+1})) \leq \int_\Omega f\, dv_g \leq \sum_{i=0}^{N-1} t_{i+1}(A(t_i) - A(t_{i+1})). \tag{1.26}$$

We denote by S_N^+ and S_N^- the right and the left hand side of (1.26), respectively. These are approximations from above and below of the integral $\int_0^{\sup f} A(t)dt$.

[9] In fact, a direct computation gives: $\forall r \in \mathbb{R} \quad \dfrac{\mathrm{Vol}_{n-1}(\partial\Omega_r^*, g_\varepsilon^*)}{\min\left[\mathrm{Vol}(\Omega_r^*)\,, \mathrm{Vol}(M^* \setminus \Omega_r^*)\right]} = H\,.$

As $0 \le S_N^+ - S_N^- \le \frac{\sup f}{N} A(0)$, then when $N \to \infty$, $S_N^+ - S_N^- \to 0_+$ and so, S_N^+, S_N^- both tend to $\int_0^{\sup f} A(t)dt$. By (1.26) they also tend to $\int_\Omega f dv_g$. This proves that

$$\int_\Omega f\, dv_g = \int_0^{\sup f} A(t)\, dt. \tag{1.27}$$

Since $\mathrm{Vol}\left(\{f^p > t\}\right) = \mathrm{Vol}\left(\{f > t^{\frac{1}{p}}\}\right) = A\big(t^{\frac{1}{p}}\big)$, applying (1.27) to the function f^p we get

$$\forall p \in [1, +\infty[\qquad \int_\Omega f^p dv_g = \int_0^{\sup f^p} A\left(t^{\frac{1}{p}}\right) dt = p \int_0^{\sup f} t^{p-1} A(t) dt.$$

This concludes the proof of the lemma. □

Proof of Theorem 1.8.12 : For any compact domain $\Omega \subset M$ with smooth boundary, we denote by $\mathcal{C}(f_\Omega)$ the set of critical values of f_Ω and by $\mathcal{S}(f_\Omega) := f_\Omega\big(\mathcal{C}(f_\Omega)\big)$ the set of its singular values. Applying the definition of $\mathcal{E}(\Omega)$, Lemma 1.8.13 and since $\mathcal{S}(f_\Omega)$ has measure zero by Sard's Theorem, we get:

$$\mathrm{Vol}(\Omega)\ \mathcal{E}(\Omega) = \int_\Omega f_\Omega\, dv_g = \int_{[0,\sup f_\Omega]\setminus \mathcal{S}(f_\Omega)} A(t)\, dt, \tag{1.28}$$

For every regular value t of f_Ω we have $A(t) \le \mathrm{Vol}(\Omega) \le \mathrm{Vol}(M,g)/2$ and then, by the definition of Cheeger's isoperimetric constant, $\mathrm{Vol}_{n-1}(\partial\Omega_t) \ge H(M,g)\, A(t)$. From this inequality and from (1.28) we obtain:

$$\mathrm{Vol}(\Omega)\, \mathcal{E}(\Omega) \le \frac{1}{H(M,g)} \int_{[0,\sup f_\Omega]\setminus S(f_\Omega)} \mathrm{Vol}_{n-1}(\partial\Omega_t) dt = \frac{1}{H(M,g)} \int_\Omega |\nabla f_\Omega| dv_g,$$

where, in the last equality, we applied the Coarea Formula (see next Chapter Paragraph 2.3). Using the Cauchy–Schwarz inequality we get:

$$\mathrm{Vol}(\Omega)\ \mathcal{E}(\Omega) \le \frac{1}{H(M,g)} \left(\mathrm{Vol}(\Omega)\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla f_\Omega|^2 dv_g\right)^{\frac{1}{2}}.$$

Finally, since $\mathcal{E}(\Omega) = \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega |\nabla f_\Omega|^2 dv_g$ (as in Definition 1.1.1), we conclude that $(\mathcal{E}(\Omega))^{\frac{1}{2}} \le \frac{1}{H(M,g)}$.

Remark 1.8.14. Let us consider any Riemannian manifold $(M,g) \in \mathcal{M}_H$, from Theorem 1.8.12 we know that every domain Ω, whose relative volume β is at most $\frac{1}{2}$, satisfies $\mathcal{E}(\Omega) \le \dfrac{1}{H^2}$. We suppose that there exists a domain Ω which satisfies $\mathcal{E}(\Omega) = \dfrac{1}{H^2}$. Then, in proof of Theorem 1.8.12, all inequalities are equalities, in particular the Cauchy–Schwarz inequality is an equality and so, $|\nabla f_\Omega|^2(x)$ is constant and thus, it is equal to $\mathcal{E}(\Omega) = \dfrac{1}{H^2}$. but, this is a contradiction with the existence of a critical point of f_Ω, since Ω is supposed to be compact. In conclusion, the functional $\Omega \mapsto \mathcal{E}(\Omega)$, where Ω runs in the set of all domains, with smooth boundary and prescribed relative volume $\beta \in]0, \frac{1}{2}]$, in all the Riemannian manifolds $(M,g) \in \mathcal{M}_H$ cannot attain its maximum.

Chapter 2

On harmonic domains and on the Coarea Formula

2.1 Introduction

Following the same guidelines of the results seen in Chapter 1 we are now able to show some new results. The first one concerns with a property of harmonic domains; it is a known fact, but we give a new analytical proof. The second one is related to the famous Coarea Formula; it is a remark in the case where we take a Morse function in the formula.

2.2 On harmonic domains

Let us recall the following

Definition 2.2.1. *A domain Ω is called "harmonic" if the function $x \mapsto \|\nabla f_\Omega(x)\| = \frac{\partial f_\Omega}{\partial N}(x)$ (where f_Ω is the solution of the equation (1.1) on the domain Ω and where $\frac{\partial}{\partial N}$ is the derivative with respect to the inner unit normal) is constant on the boundary $\partial\Omega$.*

We state the following property of the mean exit time from Ω.

Theorem 2.2.2. *The harmonic domains of volume v in (M, g) are exactly the critical points of the functional $\Omega \mapsto \mathcal{E}(\Omega)$, defined on the set of domains $\Omega \subset M$ with smooth boundary and with fixed volume v.*

We give an analytic proof of this Theorem (for the classical proof based on Brownian motion see [41]).
First of all, we prove that for every x in $\partial\Omega_0 := \partial\Omega$, $f_0 := f_\Omega$, we have

$$\frac{\partial f_0}{\partial N}(x) \neq 0.$$

Then, automatically, $\frac{\partial f_0}{\partial N}(x) > 0$, since $f_0 = 0$ in $\partial\Omega$ and $f_0 > 0$ in Ω.
We denote by ∇ (resp. D) the covariant derivative on $\partial\Omega$ with the metric

induced by g (resp. of M with the metric g). By definition of the second fundamental form $\mathrm{I\!I}$ on $x \in \partial\Omega$ (Gauss-Codazzi)

$$D_X Y = \nabla_X Y + \mathrm{I\!I}(X, Y).N(x),$$

where X and Y are two tangent vector fields on $\partial\Omega$.
Taking an orthonormal frame $e_1, \ldots e_{n-1}$ at x, we have

$$\sum_{i=1}^{n-1} D_{e_i} e_i = \sum_{i=1}^{n-1} \nabla_{e_i} e_i + \sum_{i=1}^{n-1} \mathrm{I\!I}(e_i, e_i).N(x),$$

Then, if $N(x)$ is the inner unit normal,

$$\begin{aligned} 1 = \Delta f_0 &= -\sum_{i=1}^{n-1} Ddf_0(e_i, e_i) - Ddf_0(N(x), N(x)) \\ &= \sum_{i=1}^{n-1} e_i df_0(e_i) + \sum_{i=1}^{n-1} df_0(D_{e_i} e_i) - \frac{\partial^2 f_0}{\partial N^2}(x) + df_0(D_N N). \end{aligned}$$

Let c_N be the normal geodesic such that $c_N(0) = x$ and $\dot{c_N}(0) = N(x)$. Since $df_0(x) = 0$ for every vector field on $\partial\Omega$, then $e_i df_0(e_i) = 0$ and $df_0(\nabla_{e_i} e_i) = 0$. We have

$$1 = \sum_{i=1}^{n-1} \mathrm{I\!I}(e_i, e_i) df_0(N(x)) - \frac{d^2}{dt^2}_{|t=0}(f_0[c_N(t)]) + df_0(D_{\dot{c_N}(t)} \dot{c_N}(t)).$$

If $\frac{\partial f_0}{\partial N} = 0$ in x, we get

$$\frac{d^2}{dt^2}_{|t=0}(f_0[c_N(t)]) = -1$$

and from $f_0(c_N(0)) = 0$ and $df_0[\dot{c_N}(0)] = \frac{\partial f_0}{\partial N}(x) = 0$ we obtain

$$f_0(c_N(t)) = -\frac{t^2}{2} + o(t^3) < 0,$$

which is in contrast with $f_0 > 0$ in Ω.

In order to prove Theorem 2.2.2 we need the classical definition of *regular smooth paths* (called "variations") $t \mapsto \Omega_t$ in the set of domains with smooth boundary and fixed volume v, starting from $\Omega_0 = \Omega$:

Definition 2.2.3. *A (C^2) small variation Ω_t of $\Omega_0 = \Omega$ is given by a (C^2) small variation of the boundary, which is, by definition, a C^2 map $H : \partial\Omega \times]-\varepsilon, \varepsilon[\to M$ such that, for every $x \in \partial\Omega$, $H(x, 0) = x$.*

Since the variation is C^2, then for sufficiently small t's, the boundary $\partial\Omega_t = H(\partial\Omega \times \{t\})$ is a graph over $\partial\Omega$ (i. e. the orthogonal projection $\pi_t : \partial\Omega_t \to \partial\Omega$ is a diffeomorphism), Therefore $\partial\Omega_t$ still bounds a domain called Ω_t, which is a smooth variation of the domain $\Omega_0 = \Omega$.

Definition 2.2.4 (Critical point)**.** *The domain Ω is a critical point of the functional $\mathcal{E}$ (defined on the set of domains with smooth boundary and with fixed volume v in M) if $\frac{d}{dt}_{|t=0}\left(\mathcal{E}(\Omega_t)\right)$ exists and is equal to zero for every (small) variation $t \mapsto \Omega_t$ of Ω such that* $\mathrm{Vol}(\Omega_t) = v$.

As the orthogonal projection $\pi_t : \partial\Omega_t \to \partial\Omega$ is a diffeomorphism, for small t's, $\varphi_t = \pi_t^{-1}$ is a flow of diffeomorfisms whose time-trajectories are orthogonal to $\partial\Omega$, there thus exists a smooth map $w :]-\varepsilon\,,\,\varepsilon\,[\times\partial\Omega \to \mathbb{R}$ such that $\varphi_t(x) = exp_x(w(t,x).N(x))$, where $w(0,\cdot) = 0$ and N is the inner unit normal vector field. Letting $u(x) = \frac{\partial}{\partial t}_{|t=0} w(t,x)$, we then have $\frac{d}{dt}_{|t=0}\left(\varphi_t(x)\right) = u(x).N(x)$. Writing in such a way we obtain, for any function k defined on a (small) neighbourhhood of the closure $\bar{\Omega}$ of Ω, the classical "first variation formula" :

$$\frac{d}{dt}_{|t=0}\left(\int_{\Omega_t} k\; dv_g\right) = -\int_{\partial\Omega} k\, u\; dv_g \quad , \tag{2.1}$$

and thus this derivative only depends on the values of k on $\partial\Omega$.
Applying (2.1) with $k = 1$, the assumption $\forall t \quad \mathrm{Vol}\,(\Omega_t) = \mathrm{Vol}\,(\Omega) = v$ is infinitesimally equivalent to $\int_{\partial\Omega} u\; dv_g = 0$.
More precisely, we take two coordinate systems of a neighborhood U_ϵ of $\partial\Omega$ and we construct the variation $\partial\Omega_t$ of $\partial\Omega$.
Let us denote by $\tilde{f}_0$ a smooth extension of $f_0 := f_{\Omega_0} = f_\Omega$ to a (small) neighborhood of the closure $\bar{\Omega}$ of Ω, let π be the orthogonal projection on ∂M and let

$$\phi_1 \begin{cases} \partial\Omega \times (-\epsilon, \epsilon) \to U_\epsilon \\ (x,s) \mapsto exp_x(s.N(x)) \end{cases}$$

and

$$\phi_2 \begin{cases} U_\epsilon \to \partial\Omega \times (-\epsilon', \epsilon') \\ m \mapsto (\pi(m), \tilde{f}_0(m)) \end{cases}$$

be two coordinate systems (ϕ_2 is a diffeomorphism on an open set of $\partial\Omega\times(-\epsilon',\epsilon')$ since $\frac{\partial \tilde{f}_0}{\partial N}(x) \neq 0$ for every $x \in \partial\Omega$).
We define the variation $\partial\Omega_t$ on the coordinate system ϕ_2 by

$$\phi_2(\partial\Omega_t) = \{(x,s) \in \partial\Omega \times (-\epsilon', \epsilon') | s = t.v(x)\},$$

and then

$$\partial\Omega_t = \{m \in M | \tilde{f}_0(m) = t.v(\pi(m))\}.$$

In the coordinate system ϕ_1 we have

$$\phi_1^{-1}(\partial\Omega_t) = \{(x,s) | \tilde{f}_0[exp_x(s,N(x))] = t.v(x)\};$$

then we can define $w(t,x)$ from the equality

$$\tilde{f}_0[exp_x(w(t,x).N(x))] = t.v(x). \tag{2.2}$$

We get

$$\phi_1^{-1}(\partial\Omega_t) = \{(x, w(t,x)) | x \in \partial\Omega\}$$

and

$$\partial\Omega_t = \{exp_x(w(t,x).N(x)) | x \in \partial\Omega\}.$$

We obtain the following equality by deriving (2.2) with respect to t,

$$d\tilde{f}_{0_{|x}}[d(exp_x)_{|0}\frac{\partial w}{\partial t}_{|t=0}(t,x).N(x)] = v(x).$$

Since we defined $u(x) = \frac{\partial}{\partial t}_{|t=0} w(t,x)$ and $d(exp_x)_{|0} = id_{T_xM}$ we have

$$d\tilde{f}_{0_{|x}}[u(x).N(x)] = v(x)$$

and then

$$v(x) = u(x)\frac{\partial f_0}{\partial N}(x). \tag{2.3}$$

By the "first variation formula" (2.1) we get

$$\frac{d}{dt}_{|t=0}(\mathrm{Vol}\,(\Omega_t)) = -\int_{\partial\Omega} u(x) \;\; dv_g = -\int_{\partial\Omega} \frac{v(x)}{\frac{\partial f_0}{\partial N}(x)} \;\; dv_g.$$

Thus, we can choose $v(x)$ as a generic function satisfying (2.3) and also $u(x)$ is a generic function satisfying $\int_{\partial\Omega} u(x) \;\; dv_g = 0$.

We denote by V_t the harmonic function on Ω_t that is equal to $v \circ \pi(x)$ on $\partial\Omega_t$. It is known that V_t realizes the minimum of

$$f \mapsto \int_{\partial\Omega_t} |\nabla f|^2 \;\; dv_g$$

in the set of functions $f(x)$ which are equal to $v \circ \pi(x)$ on $\partial\Omega_t$, and that there exists a constant $C > 0$ such that $\int_{\partial\Omega_t} |\nabla V_t|^2 \, dv_g \leq C < +\infty$, i.e. $V_t \in H_1^2(\Omega_t)$.

Proof of Theorem 2.2.2 : Let us consider the functional $\mathcal{E}$ defined on the set of domains with smooth boundary in M and with fixed volume v. Let us denote by f_t the function f_{Ω_t}, we have

$$\mathcal{E}(\Omega_t) = E_{\Omega_t}(f_t) = \frac{1}{v}\left(2\int_{\Omega_t} f_t \; dv_g \; - \int_{\Omega_t} |\nabla f_t|^2 \; dv_g\right) = \frac{1}{v}\left(\int_{\Omega_t} f_t \; dv_g\right)$$

and thus, using the formula (2.1),

$$\begin{aligned}
&v\,(\mathcal{E}(\Omega_t) - \mathcal{E}(\Omega_0)) = \int_{\Omega_t} f_t \; dv_g \; - \int_{\Omega_0} \tilde{f}_0 \; dv_g \\
&= \int_{\Omega_t} f_t \; dv_g \; - \int_{\Omega_t} \tilde{f}_0 \; dv_g \; + \int_{\Omega_t} \tilde{f}_0 \; dv_g \; - \int_{\Omega_0} \tilde{f}_0 \; dv_g \\
&= \int_{\Omega_t} f_t \; dv_g \; - \int_{\Omega_t} \tilde{f}_0 \; dv_g \; - \; t\int_{\partial\Omega_0} u\,\tilde{f}_0 \; dv_g \; + \; o(t^2) \\
&= \int_{\Omega_t} f_t \; dv_g \; - \int_{\Omega_t} \tilde{f}_0 \; dv_g \; + \; o(t^2) \\
&= \; -\int_{\Omega_t} \tilde{f}_0 \; \Delta f_t \; dv_g \; + \int_{\Omega_t} f_t \; \Delta\tilde{f}_0 \; dv_g \; - \int_{\Omega_t} f_t \; (\Delta\tilde{f}_0 - 1) \; dv_g \; + \; o(t^2) \\
&= \; -\int_{\Omega_t} \tilde{f}_0 \; \Delta f_t \; dv_g \; + \int_{\Omega_t} f_t \; \Delta\tilde{f}_0 \; dv_g \; - \int_{\Omega_t \setminus \Omega_0} f_t \; (\Delta\tilde{f}_0 - 1) \; dv_g \; + \; o(t^2) \\
&= \; -\int_{\Omega_t} \tilde{f}_0 \; \Delta f_t \; dv_g \; + \int_{\Omega_t} f_t \; \Delta\tilde{f}_0 \; dv_g \; + \; o(t^2) \;\; ,
\end{aligned}$$

where the last equality comes from the fact that, by smoothness, $|\Delta \tilde{f}_0(x)-1| \leq C_1\, d(x,\partial\Omega) \leq C_2\, t$ and $f_t(x) \leq C_3$ when $x \in \Omega_t \setminus \Omega_0$ and that $\mathrm{Vol}\,(\Omega_t \setminus \Omega_0) \leq C_4\, t$. Using Green's formula, we get:

$$
\begin{aligned}
& v\,(\mathcal{E}(\Omega_t) - \mathcal{E}(\Omega_0)) \\
& = \; - \int_{\Omega_t} \langle \nabla \tilde{f}_0 \,,\, \nabla f_t \rangle \; dv_g \; + \; \int_{\Omega_t} \langle \nabla f_t \,,\, \nabla \tilde{f}_0 \rangle \; dv_g \\
& + \int_{\partial\Omega_t} f_t \,\frac{\partial \tilde{f}_0}{\partial \nu} \; - \int_{\partial\Omega_t} \tilde{f}_0 \,\frac{\partial f_t}{\partial \nu} \; + \; o(t^2) = \; - \int_{\partial\Omega_t} \tilde{f}_0 \,\frac{\partial f_t}{\partial \nu} \; dv_g + \; o(t^2),
\end{aligned} \tag{2.4}
$$

where ν is the normal to $\partial\Omega_t$.
We now prove that

$$
\int_{\partial\Omega_t} \tilde{f}_0 \,\frac{\partial f_t}{\partial \nu} \; dv_g = \int_{\partial\Omega_t} \tilde{f}_0 \,\frac{\partial \tilde{f}_0}{\partial \nu} \; dv_g + \; o(t^2).
$$

Since $\tilde{f}_0(x) = t\, v \circ \pi(x)$ and $\tilde{f}_0(x) = \tilde{f}_0(x) - f_t(x)$ for every $x \in \partial\Omega_t$, and V_t is harmonic in Ω_t, we have

$$
\begin{aligned}
& \int_{\partial\Omega_t} \tilde{f}_0 \left(\frac{\partial \tilde{f}_0}{\partial \nu} - \frac{\partial f_t}{\partial \nu} \right) \; dv_g = t \int_{\partial\Omega_t} v \circ \pi \left(\frac{\partial \tilde{f}_0}{\partial \nu} - \frac{\partial f_t}{\partial \nu} \right) \; dv_g \\
& = t \int_{\partial\Omega_t} (\tilde{f}_0 - f_t) \frac{\partial V_t}{\partial \nu} \; dv_g + t \int_{\Omega_t} V_t (\Delta \tilde{f}_0 - \Delta f_t) \; dv_g - t \int_{\Omega_t} \Delta V_t (\tilde{f}_0 - f_t) \; dv_g \\
& = t \int_{\partial\Omega_t} t\, v \circ \pi \frac{\partial V_t}{\partial \nu} \; dv_g + t \int_{\Omega_t \setminus \Omega_0} V_t (\Delta \tilde{f}_0 - 1) \; dv_g \\
& = t^2 \int_{\Omega_t} \Delta V_t V_t \; dv_g - t^2 \int_{\Omega_t} |\nabla V_t|^2 \; dv_g + t o(t) = o(t^2)
\end{aligned}
$$

where the last equality is follows from the fact that V_t and $\int_{\Omega_t} |\nabla V_t|^2\, dv_g$ are bounded. Thus

$$
\int_{\partial\Omega_t} \tilde{f}_0 \left(\frac{\partial \tilde{f}_0}{\partial \nu} - \frac{\partial f_t}{\partial \nu} \right) \; dv_g = o(t^2).
$$

Applying this result in (2.4) we get

$$
\begin{aligned}
& v\,(\mathcal{E}(\Omega_t) - \mathcal{E}(\Omega_0)) \; = \; - \int_{\partial\Omega_t} \tilde{f}_0 \,\frac{\partial f_t}{\partial \nu} \; dv_g + \; o(t^2) \\
& = \; - \int_{\partial\Omega_t} \tilde{f}_0 \,\frac{\partial \tilde{f}_0}{\partial \nu} \; dv_g + \; o(t^2) \; = \; - \int_{\partial\Omega_t} \tilde{f}_0 \; |\nabla \tilde{f}_0| \; dv_g + \; o(t^2) \\
& = \; - \int_{\partial\Omega_0} \tilde{f}_0(\varphi_t(x)) \; \|\nabla \tilde{f}_0\|(\varphi_t(x)) \; |\mathrm{Jac}\, \varphi_t|(x) \; dv_g(x) + \; o(t^2) \\
& = \; - \int_{\partial\Omega_0} \left(\tilde{f}_0(\varphi_t(x)) - \tilde{f}_0(x) \right) \; \|\nabla \tilde{f}_0\|(\varphi_t(x)) \; |\mathrm{Jac}\, \varphi_t|(x) \; dv_g(x) + \; o(t^2) \\
& = \; - \, t \int_{\partial\Omega_0} u(x) \; d\tilde{f}_0(N(x)) \; \|\nabla \tilde{f}_0\|(\varphi_t(x)) \; |\mathrm{Jac}\, \varphi_t|(x) \; dv_g(x) + \; o(t^2).
\end{aligned}
$$

As $(d\pi_t(v) - v)$ tends to zero (uniformly) when $t \to 0$, then $|\mathrm{Jac}\,\varphi_t|$ tends to 1 uniformly and $\|\nabla \tilde{f}_0\|(\varphi_t(x))$ tends to $\|\nabla f_0\|(x)$ when $t \to 0$. We then get

$$\mathcal{E}(\Omega_t) - \mathcal{E}(\Omega_0) \;=\; -\,t\,\frac{1}{v}\int_{\partial\Omega_0} u(x)\;\|\nabla f_0\|(x)^2\; dv_g(x) + \; o(t^2) \quad .$$

This proves that the derivative exists and satisfies the equality

$$\frac{d}{dt}_{|t=0}\left(\mathcal{E}(\Omega_t)\right) \;=\; -\frac{1}{v}\;\int_{\partial\Omega_0} u(x)\;\|\nabla f_0\|(x)^2\; dv_g(x) \quad . \tag{2.5}$$

Thus Ω_0 is a critical point of the functional $\mathcal{E}$ iff $\int_{\partial\Omega_0} u(x)\;\|\nabla f_0\|(x)^2\; dv_g(x) = 0$ for every function $u : \partial\Omega_0 \to \mathbb{R}$ such that $\int_{\partial\Omega_0} u = 0$, thus iff $x \to \|\nabla f_0\|(x)^2$ is constant on $\partial\Omega_0$; this achieves the proof of Theorem 2.2.2. $\square$

2.3 On the Coarea Formula

The famous Coarea Formula is also applied in the proof of Theorem 1.4.1. We make a remark on this formula.
Let (M, g) be any Riemannian manifold of dimension n, and let $f : M \to \mathbb{R}$ be a C^n function. Let us denote by $\mathcal{C}(f)$ the set of critical points of f, that is the set of points $x \in M$ such that $\nabla f(x) = 0$, and let $\mathcal{S}(f)$ be the image of $\mathcal{C}(f)$ by f; by Sard's theorem it has measure zero in the interval $[\inf f, \sup f]$.
In many references (see for example [3], [5] and [28]), the Coarea Formula is written as follows.
For any non negative continuous function φ on a Riemannian manifold (M, g),

$$\int_M \varphi(x)\; dv_g(x) = \int_{\inf f}^{\sup f}\left(\int_{f^{-1}(\{t\})} \frac{\varphi(x)}{\|\nabla f(x)\|}\; da_t(x)\right) dt \quad , \tag{2.6}$$

where, by $\int_{\inf f}^{\sup f}$, we intend the integral (with respect to the Lebesgue measure) on $]\inf f, \sup f[\setminus\mathcal{S}(f)$; this integral takes sense because $\mathcal{S}(f)$ has measure zero. Moreover, as we only integrate with respect to regular values t of f, $\{f = t\}$ is a submanifold of codimension 1 in M and da_t is well defined as the $(n-1)-$dimensional Riemannian measure on $\{f = t\}$ (viewed as a Riemannian submanifold of (M, g)).
By the two following counterexamples we prove that this coaera formula is not valid when applied to C^∞ functions.

A $1-$dimensional counterexample

Let $M = [-\pi, \pi]$ endowed with the usual Euclidean metric and with the Lebesgue measure. Let $f : [-\pi, \pi] \to \mathbb{R}_+$ be a smooth function such that $f(-x) = f(x)$ for every $x \in [-\pi, \pi]$ and that $f(0) = 1$, $f(\pi) = 0$, $\forall x \in [\frac{\pi}{4}, \frac{3\pi}{4}]$ $f(x) = \frac{1}{2}$, $f'(0) = f'(\pi) = 0$ and such that $\forall x \in]0, \frac{\pi}{4}[\cup]\frac{3\pi}{4}, \pi[$ $f'(x) < 0$. Let us define

$$I_1 = \int_{[-\pi,\pi]} dx \quad , \quad I_2 = \int_{[0,1]\setminus\mathcal{S}(f)}\left(\sum_{x\in f^{-1}(\{t\})} \frac{1}{|f'(x)|}\right) dt \quad ,$$

if formula (2.6) is true, then $I_1 = I_2$. Let us now compute these two integrals: it is clear that $I_1 = 2\pi$. As $\mathcal{S}(f) = \{0, \frac{1}{2}, 1\}$, for every $t \in [0,1] \setminus \mathcal{S}(f) =]0, \frac{1}{2}[\cup]\frac{1}{2}, 1[$, $f^{-1}(\{t\}$ contains exactly two symmetric points that we denote by $x(t)$ and $-x(t)$ when $t \in]0, \frac{1}{2}[$ and by $\tilde{x}(t)$ and $-\tilde{x}(t)$ when $t \in]\frac{1}{2}, 1[$, where $x(t)$, $\tilde{x}(t) < 0$; as $f(x(t)) = t$, $f(-x(t)) = t$, $f(\tilde{x}(t)) = t$, $f(-\tilde{x}(t)) = t$ the chain rule gives $x'(t) = \frac{1}{f'(x(t))} = \frac{1}{-f'(-x(t))}$ and $\tilde{x}'(t) = \frac{1}{f'(\tilde{x}(t))} = \frac{1}{-f'(-\tilde{x}(t))}$

$$
\begin{aligned}
I_2 &= \int_{]0,\frac{1}{2}[} \left(\frac{1}{f'(x(t))} + \frac{1}{-f'(-x(t))} \right) dt + \int_{]\frac{1}{2},1[} \left(\frac{1}{f'(\tilde{x}(t))} + \frac{1}{-f'(-\tilde{x}(t))} \right) dt \\
&= \int_{]0,\frac{1}{2}[} 2\, x'(t)\, dt + \int_{]\frac{1}{2},1[} 2\, \tilde{x}'(t)\, dt = 2 \left(x(\frac{1}{2}) - x(0) + \tilde{x}(1) - \tilde{x}(\frac{1}{2}) \right) \quad .
\end{aligned}
$$

As $t \mapsto x(t)$ (resp. $t \mapsto \tilde{x}(t)$) is the inverse of the map $f :] - \pi, -\frac{3\pi}{4}[\to]0, \frac{1}{2}[$ (resp. of the map $f :] - \frac{\pi}{4}, 0[\to]\frac{1}{2}, 1[$), one has $x(0) = -\pi$, $x(\frac{1}{2}) = -\frac{3\pi}{4}$, $\tilde{x}(\frac{1}{2}) = -\frac{\pi}{4}$ and $\tilde{x}(1) = 0$ and we deduce from the previous equality that $I_2 = \pi$; we conclude that $I_1 \neq I_2$ and thus the formula (2.6) is false in the present case. $\square$

A counterexample in higher dimension

The previous counterexample is not specific to the $1-$dimensional case, neither to the fact that the manifold has a non-empty boundary on which f vanishes; in fact, we are able to construct a counterexample on any compact Riemannian manifold (M, g) of any dimension:

Let f still be the function of the previous $1-$dimensional counterexample, let us fix a point $x_0 \in M$ and call i_0 the injectivity radius of (M, g) at this point. Let B_0 be the geodesic ball of radius i_0 centered at x_0. Let $\tilde{f}$ be the C^∞ extension of f to $\mathbb{R}$ such that $\tilde{f} = 0$ outside $[-\pi, \pi]$; let $u : M \to \mathbb{R}^+$ be defined by $u = \tilde{f} \circ \varrho$, where $\varrho = \frac{\pi}{i_0}\, d(x_0, \cdot)$, where $d(x_0, \cdot)$ is the Riemannian distance (on (M, g)) to the fixed point x_0. As $d(x_0, \cdot)$ is C^∞ on the geodesic ball B_0 and as $u = 0$ outside B_0, u is C^∞; moreover, by construction, u is strictly positive on B_0. Let us define

$$
J_1 = \int_{\mathbb{M}} dv_g = \mathrm{Vol}\,(M, g) \quad , \quad J_2 = \int_{[0,1]\setminus\mathcal{S}(u)} \left(\int_{u^{-1}(\{t\})} \frac{1}{\|\nabla u(x)\|}\, da_t(x) \right) dt \quad ;
$$

if formula (2.6) is true it will be $J_1 = J_2$. Let us now compute J_2.

As $\mathcal{S}(u) = \mathcal{S}(f) = \{0, \frac{1}{2}, 1\}$, for every $t \in [0,1] \setminus \mathcal{S}(f) =]0, \frac{1}{2}[\cup]\frac{1}{2}, 1[$, $u^{-1}(\{t\})$ is the geodesic sphere $S(x_0, R(t))$ of (M, g) centered at x_0 whose radius $R(t)$ satisfies, for every $x \in S(x_0, R(t))$,

$$
\begin{aligned}
& f\left(\frac{\pi}{i_0}\, R(t) \right) = u(x) = t \\
& \|\nabla u(x)\| = |f'(\varrho(x)|.\|\nabla \varrho(x)\| = -\frac{\pi}{i_0}\, f'\left(\frac{\pi}{i_0}\, R(t) \right) = -\frac{1}{R'(t)}\, ,
\end{aligned} \tag{2.7}
$$

where the right-hand equalities come from the chain rules and from the fact that $\|\nabla d(x_0, \cdot)\| = 1$ outside the cut-locus of x_0. Let $L(r)$ be the $(n-1)-$dimensional volume of the sphere of radius r, applying the gradient estimate (2.7), we get

$$J_2 = \int_{]0,\frac{1}{2}[\cup]\frac{1}{2},1[} -R'(t)\, L(R(t))\; dt = \int_{R(1)}^{R_-(\frac{1}{2})} L(r)\, dr + \int_{R_+(\frac{1}{2})}^{R(0)} L(r) dr \ , \quad (2.8)$$

where $R_+(\frac{1}{2})$ (resp. $R_-(\frac{1}{2})$) is the limit of $R(t)$ when $t \to \frac{1}{2}$ with $t < \frac{1}{2}$ (resp. with $t > \frac{1}{2}$). As (by the first equality (2.7)) $t \mapsto \dfrac{\pi}{i_0} R(t)$ is the inverse of the (restricted) map $f :]0, \frac{\pi}{4}[\cup]\frac{3\pi}{4}, \pi[\longrightarrow]\frac{1}{2}, 1[\cup]0, \frac{1}{2}[$, one has $\dfrac{\pi}{i_0} R(0) = \pi$, $\dfrac{\pi}{i_0} R_+(\frac{1}{2}) = \frac{3\pi}{4}$, $\dfrac{\pi}{i_0} R_-(\frac{1}{2}) = \frac{\pi}{4}$ and $\dfrac{\pi}{i_0} R(1) = 0$. Plugging these estimates into the equality (2.8), we obtain:

$$\begin{aligned} J_2 &= \mathrm{Vol}\left[B\left(x_0, R_-\left(\frac{1}{2}\right)\right)\right] - \mathrm{Vol}\left[B(x_0, R(1))\right] + \mathrm{Vol}\left[B(x_0, R(0))\right] \\ &\quad - \mathrm{Vol}\left[B\left(x_0, R_+\left(\frac{1}{2}\right)\right)\right] \\ &= \mathrm{Vol}\left[B(x_0, i_0)\right] + \mathrm{Vol}\left[B\left(x_0, \frac{i_0}{4}\right)\right] - \mathrm{Vol}\left[B\left(x_0, \frac{3\, i_0}{4}\right)\right] \\ &< \mathrm{Vol}\left[B(x_0, i_0)\right] \leq \mathrm{Vol}(M, g) \ ; \end{aligned}$$

it follows that $J_1 \neq J_2$ and thus the formula (2.6) is false in the present case.

□

Theorem 2.3.1. (Coarea Formula, see for example [7], pp 104-7)*. Let (M, g) be any Riemannian manifold of dimension n, and let $f : M \to \mathbb{R}$ be a C^n function; for any measurable function φ on M,*

$$\int_M \varphi(x)\ \|\nabla f(x)\|\ dv_g(x) = \int_{\inf f}^{\sup f} \left(\int_{f^{-1}(\{t\})} \varphi(x)\ da_t(x) \right) dt \quad .$$

Concerning the first formula (2.6) we find out that

Proposition 2.3.2. *Formula* (2.6) *is valid for every Morse function f, but not for C^∞ functions f whose set $\mathcal{C}(f)$ of critical points admits interior points.*

Proof. Let us denote by M' the open set $M \setminus \mathcal{C}(f)$; when f is C^n, the function $x \mapsto \dfrac{1}{\|\nabla f(x)\|}$ is continuous on M'; we may thus apply the above Theorem 2.3.1 on the manifold M' and replace $\varphi(x)$ by $\dfrac{\varphi(x)}{\|\nabla f(x)\|}$ in the integrals; this implies that

$$\int_{M'} \frac{\varphi(x)}{\|\nabla f(x)\|}\ \|\nabla f(x)\|\, dv_g(x) = \int_{]\inf f, \sup f[\setminus \mathcal{S}(f)} \left(\int_{f^{-1}(\{t\}) \cap M'} \frac{\varphi(x)}{\|\nabla f(x)\|}\ da_t(x) \right) dt\, ;$$

from this and from the fact that $f^{-1}(\{t\}) \cap M' = f^{-1}(\{t\})$ for every $t \in]\inf f, \sup f[\setminus \mathcal{S}(f)$ [because $f^{-1}(\{t\}) \subset M \setminus f^{-1}(\mathcal{S}(f)) \subset M \setminus \mathcal{C}(f)$] we deduce that

$$\int_{M\setminus\mathcal{C}(f)} \varphi(x)\ dv_g(x) = \int_{]\inf f,\sup f[\setminus\mathcal{S}(f)} \left(\int_{f^{-1}(\{t\})} \frac{\varphi(x)}{\|\nabla f(x)\|}\ da_t(x) \right) dt\ . \tag{2.9}$$

If f is a Morse function (or, more generally, if $\mathcal{C}(f)$ has measure zero), then $\int_M \varphi(x)dv_g(x) = \int_{M\setminus\mathcal{C}(f)} \varphi(x)dv_g(x)$; this last equality and the equality (2.9) prove the formula (2.6) in this case.

When f is a C^∞ function whose set $\mathcal{C}(f)$ of critical points admits interior points, there exists continuous functions φ such that $\varphi = 0$ on $M \setminus \mathcal{C}(f)$ and that $\int_M \varphi(x)dv_g(x) \neq 0 = \int_{M\setminus\mathcal{C}(f)} \varphi(x)dv_g(x)$; this last inequality and equality (2.9) prove that the formula (2.6) is false in this case. $\square$

Remark 2.3.3. In the two previous examples, we take C^∞ functions whose sets of critical points have not measure zero. In fact the two functions f and u are C^∞ and the sets $\mathcal{C}(f)$ and $\mathcal{C}(u)$ contain respectively $[-\frac{3\pi}{4}, -\frac{\pi}{4}] \cup [\frac{\pi}{4}, \frac{3\pi}{4}]$ and $B\left(x_0, \frac{3i_0}{4}\right) \setminus B\left(x_0, \frac{i_0}{4}\right)$, and so their measures are strictly positive.

Appendices

Appendix A

Maximization of the energy integral in the plane

We analyze a Dirichlet problem in the plane that is studied in [13].
We consider a disc B in the plane, a subset $F \subset B$ with positive measure, a real number q such that $0 \leq q < 1$ and the Dirichlet problem

$$\begin{cases} -\Delta u = \chi_F u^q & \text{in } B \\ u = 0 & \text{on } \partial B. \end{cases} \tag{A.1}$$

We consider only the positive solution u in the Sobolev space $H_0^1(B)$. The corresponding energy integral is

$$I(F) = \int_B |\nabla u|^2 \, dx. \tag{A.2}$$

When $q = 0$, problem (A.1) has the following physical meaning. It models the position of equilibrium of an ideal membrane that is fixed on the boundary ∂B and such that there is a weight with unitary density in the region F.
In [13] we find the following result of symmetry preservation:

Proposition A.0.4. *Let $0 < \beta < |B|$ and let $\mathcal{F}$ be the class of all subsets F of B such that $|F| = \beta$. A maximizer of the energy integral $I(F)$ corresponding to problem (A.1) with $0 \leq q < 1$ in $\mathcal{F}$ is a disc concentric with B.*

Its proof is essentially based on the following properties on rearrangements [33]:

- if f and g are non negative measurable functions in B then

  $$\int_B f(x)g(x)\,dx \leq \int_B f^*(x)g^*(x)\,dx \quad \text{(Hardy-Littlewood inequality)}$$

  where f^* and g^* are the decreasing rearrangements of f and g, respectively;

- if $u \in H_0^1(B)$, $u \geq 0$ then $u^* \in H_0^1(B)$ and

  $$\int_B |\nabla u|^2\,dx \geq \int_B |\nabla u^*|^2\,dx \quad \text{(Polya-Szegő inequality)}$$

  where u^* is the decreasing rearrangement of u.

Proof. It is known that a positive solution u of problem (A.1) satisfies

$$\begin{aligned}\int_B |\nabla u|^2\,dx &= \frac{q+1}{1-q}\int_B \left(\frac{2}{q+1}\chi_F u^{q+1} - |\nabla u|^2\right)dx \\ &= \frac{q+1}{1-q}\sup_{w\in H_0^1(B)}\int_B \left(\frac{2}{q+1}\chi_F |w|^{q+1} - |\nabla w|^2\right)dx.\end{aligned} \tag{A.3}$$

Moreover, if w is a maximizer in the above integral, also $|w|$ is a maximizer. Thus, we can consider the superior for $w \geq 0$. In addition, since u is the unique solution of problem (A.1), then the maximizer u is unique in the class of positive functions.
Applying the previous inequalities on rearrangements we get

$$\begin{aligned}I(F) = \int_B |\nabla u|^2\,dx &= \frac{q+1}{1-q}\int_B \left(\frac{2}{q+1}\chi_F u^{q+1} - |\nabla u|^2\right)dx \\ &\leq \frac{q+1}{1-q}\int_B \left(\frac{2}{q+1}\chi_{F^*} (u^*)^{q+1} - |\nabla u^*|^2\right)dx \\ &\leq \frac{q+1}{1-q}\int_B \left(\frac{2}{q+1}\chi_{F^*} (z)^{q+1} - |\nabla z|^2\right)dx = I(F^*),\end{aligned} \tag{A.4}$$

where z is the solution of problem (A.1) when $F = F^*$.
The last inequality concludes the proof. $\square$

Part II

Optimization of the principal eigenvalue under mixed boundary conditions

Chapter 3

Optimization of the principal eigenvalue

3.1 Introduction

In this chapter we consider an eigenvalue problem which is the generalization of the following biological problem. We suppose that $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, biologically Ω is a region where a population lives diffusing at rate D and growing or declining locally at a rate $g(x)$ (more precisely $g(x) > 0$ means a local growth and $g(x) < 0$ a local decline). We suppose that the boundary $\partial\Omega$ is divided in two parts: Γ and $\partial\Omega \setminus \Gamma$, so that the $1-$Lebesgue measure of Γ is positive. If $\phi(x,t)$ is the population density, the behavior of the population is described by the logistic equation

$$\frac{\partial \phi}{\partial t} = D\Delta\phi + (g(x) - \kappa\phi)\phi \quad \text{in } \Omega \times \mathbb{R}^+.$$

We have Dirichlet boundary condition and Neumann boundary condition, respectively,

$$\phi = 0 \quad \text{on } \Gamma \times \mathbb{R}^+, \quad \frac{\partial \phi}{\partial \nu} = 0 \quad \text{on } (\partial\Omega \setminus \Gamma) \times \mathbb{R}^+.$$

We denote by $\Delta\phi$ the spatial laplacian of $\phi(x,t)$, κ is the carrying capacity and ν is the exterior normal to the boundary $\partial\Omega$.thus there is a hostile population outside across Γ and no flux (in either direction) of individuals across $\partial\Omega \setminus \Gamma$. Considering the associated eigenvalue problem

$$\Delta u + \lambda g(x) u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \setminus \Gamma$$

and its (positive) principal eigenvalue λ_g, we know (see [15, 16]) that there is population persistence if and only if $\lambda_g < 1/D$.
Given a bounded measurable function g_0 in Ω, we investigate a more general problem where Ω is a smooth bounded domain in $\mathbb{R}^N$ and analyse the problems of minimization and maximization of λ_g, when the weight $g(x)$ varies in the set of rearrangements $\mathcal{G}$ of a given function $g_0(x)$. The study of these problems is related to the problem of finding out the most convenient spatial arrangement

for favorable and unfavorable resources in the region Ω.
We study the cases where $\Omega = (0, L)$ and where Ω is an α-sector in depth.
In the last section we consider a case of symmetry breaking for the minimization problem.
Throughout this part, a decreasing function will be a non-increasing function. If $E \subset \mathbb{R}^N$ is a measurable set we denote with $|E|$ its Lebesgue measure. Moreover, we say that two measurable functions $f(x)$ and $g(x)$ have the same rearrangement in Ω if

$$|\{x \in \Omega : f(x) \geq \beta\}| = |\{x \in \Omega : g(x) \geq \beta\}| \quad \forall \beta \in \mathbb{R}.$$

3.2 Optimization of the principal eigenvalue in $\mathbb{R}^n$

Let Ω be a bounded smooth domain in $\mathbb{R}^N$, and let $g_0(x)$ be a bounded measurable function in Ω which takes positive values in a set of positive measure. Suppose Γ is a portion of $\partial\Omega$ with a positive $(N-1)$-Lebesgue measure. Let $\mathcal{G}$ be the class of rearrangements generated by g_0. For $g \in \mathcal{G}$, we consider the eigenvalue problem

$$\Delta u + \lambda g(x) u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \setminus \Gamma. \tag{3.1}$$

We are interested in the principal eigenvalue, that is, a positive eigenvalue to which corresponds a positive eigenfunction. Denoting by

$$W_\Gamma^+ = \Big\{ w \in H^1(\Omega) : \; w = 0 \; \text{ on } \; \Gamma, \; \int_\Omega g\, w^2 dx > 0 \Big\},$$

we have

$$\lambda_g = \inf_{w \in W_\Gamma^+} \frac{\int_\Omega |\nabla w|^2 dx}{\int_\Omega g\; w^2 dx} = \frac{\int_\Omega |\nabla u_g|^2 dx}{\int_\Omega g\; u_g^2 dx}. \tag{3.2}$$

Remark 3.2.1. We observe that if u_g is a minimizer then it is solution of (3.1). In fact, if u_g is a minimizer we have

$$\frac{\int_\Omega |\nabla (u_g + \epsilon\varphi)|^2 dx}{\int_\Omega g (u_g + \epsilon\varphi)^2 dx} \geq \frac{\int_\Omega |\nabla u_g|^2 dx}{\int_\Omega g u_g^2 dx},$$

for admissible $(u_g + \epsilon\varphi)$. Then,

$$\int_\Omega g u_g^2 dx \int_\Omega |\nabla (u_g + \epsilon\varphi)|^2 dx \geq \int_\Omega |\nabla u_g|^2 dx \int_\Omega g (u_g + \epsilon\varphi)^2 dx,$$

and so,

$$\int_\Omega g u_g^2 dx \int_\Omega \left(|\nabla u_g|^2 + \epsilon^2 |\nabla\varphi|^2 + 2\epsilon \nabla u_g \cdot \nabla\varphi \right) dx \geq \int_\Omega |\nabla u_g|^2 dx \int_\Omega g(u^2 + \epsilon^2\varphi^2 + 2\epsilon u_g \varphi) dx.$$

Since ϵ^2 is an infinitesimal of higher order than ϵ, we have

$$2\epsilon \int_\Omega \nabla u_g \cdot \nabla\varphi dx \geq 2\epsilon \frac{\int_\Omega |\nabla u_g|^2 dx}{\int_\Omega g u_g^2 dx} \int_\Omega g u_g \varphi dx.$$

If $\epsilon > 0$
$$\int_\Omega \nabla u_g \cdot \nabla \varphi dx \geq \frac{\int_\Omega |\nabla u_g|^2 dx}{\int_\Omega g u_g^2 dx} \int_\Omega g u_g \varphi dx,$$
and if $\epsilon < 0$
$$\int_\Omega \nabla u_g \cdot \nabla \varphi dx \leq \frac{\int_\Omega |\nabla u_g|^2 dx}{\int_\Omega g u_g^2 dx} \int_\Omega g u_g \varphi dx.$$
Then
$$\int_\Omega \nabla u_g \cdot \nabla \varphi dx = \frac{\int_\Omega |\nabla u_g|^2 dx}{\int_\Omega g u_g^2 dx} \int_\Omega g u_g \varphi dx,$$
and so u_g is solution of (3.1).
The converse is also true: if u_g is a solution of (3.1) it is also a minimizer. In fact, from (3.1) we have
$$-\int_\Omega \Delta u_g u_g dx = \lambda_g \int_\Omega g u_g^2 dx,$$
and then by the boundary conditions we conclude that
$$\lambda_g = \frac{\int_\Omega |\nabla u_g|^2 dx}{\int_\Omega g u_g^2 dx}.$$

Moreover, u_g is positive. In fact, if u_g is a minimizer, so is $|u_g|$, hence, $|u_g|$ satisfies equation (3.1). By Harnack's inequality (see, for example, [49], Theorem 1.1) we have $|u_g| > 0$ in Ω. By continuity, we have either $u_g > 0$ or $u_g < 0$. We also note that u_g is unique up to a positive constant. It is a consequence of the following theorem.

Theorem 3.2.2. *Suppose that λ is the principal eigenvalue and u is a positive eigenfunction that solve problem (3.1). If there exists another function v such that $-\Delta v = \lambda g v$, then $u = cv$ for some positive constant c.*

The proof of uniqueness can be found in [34], although in [34] the authors study the $p-$laplacian.

Proof. We consider the weak form of problem (3.1). For $u, \varphi \in H^1(\Omega)$ we have
$$-\int_\Omega \Delta u \varphi dx = \lambda \int_\Omega g u \varphi dx.$$
Integrating by parts
$$-\int_{\partial\Omega} \frac{\partial u}{\partial x_i} \nu_i \varphi dx + \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx = \lambda \int_\Omega g \varphi dx,$$
and so, using the boundary conditions
$$\int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx = \lambda \int_\Omega g \varphi dx, \qquad \forall \varphi \in H^1(\Omega).$$
The function φ is a test function, we choose it appropriately
$$\varphi = u - u\left(\frac{v}{u}\right)^2.$$

We have

$$\nabla\varphi = \nabla u + \nabla u \left(\frac{v}{u}\right)^2 - 2\left(\frac{v}{u}\right)\nabla v,$$

then

$$\int_\Omega \nabla u \cdot \left(\nabla u + \nabla u \left(\frac{v}{u}\right)^2 - 2\left(\frac{v}{u}\right)\nabla v\right) dx = \lambda \int_\Omega g(u^2 - v^2)dx,$$

and so

$$\int_\Omega |\nabla u|^2 + |\nabla u|^2 \left(\frac{v}{u}\right)^2 - 2\left(\frac{v}{u}\right)\nabla u \cdot \nabla v dx = \lambda \int_\Omega g(u^2 - v^2)dx. \tag{3.3}$$

For the other solution v we have

$$\int_\Omega \nabla v \cdot \nabla\varphi dx = \lambda \int_\Omega guv,$$

choosing $\varphi = v - v\left(\frac{u}{v}\right)^2$ we find

$$\int_\Omega |\nabla v|^2 + |\nabla v|^2 \left(\frac{u}{v}\right)^2 - 2\left(\frac{u}{v}\right)\nabla v \cdot \nabla u dx = \lambda \int_\Omega g(v^2 - u^2)dx. \tag{3.4}$$

We sum (3.3) and (3.4) and we find

$$\int_\Omega (u^2 + v^2)\left(\frac{\nabla u}{u} - \frac{\nabla v}{v}\right)^2 dx = 0,$$

and then

$$\frac{\nabla u}{u} = \frac{\nabla v}{v}. \tag{3.5}$$

Since

$$\nabla\left(\frac{u}{v}\right) = \frac{\nabla u}{v} - \frac{u}{v^2}\nabla v,$$

then

$$\frac{v}{u}\nabla\left(\frac{u}{v}\right) = \frac{\nabla u}{u} - \frac{\nabla v}{v}.$$

Using (3.5), we have

$$\nabla\left(\frac{u}{v}\right) = 0,$$

and so, on a connected set

$$\frac{u}{v} = c.$$

□

We also have the following

Theorem 3.2.3. *Suppose that there exist λ_g and u_g that solve (3.1). If there exist a positive number Λ and a positive function v such that*

$$\Delta v + \Lambda g(x)v = 0 \quad in\ \Omega, \quad v = 0 \quad on\ \Gamma, \quad \frac{\partial v}{\partial \nu} = 0 \quad on\ \partial\Omega \setminus \Gamma, \tag{3.6}$$

then $\Lambda = \lambda_g$ and $v = cu_g$ for some positive constant c.

The proof is the same of [35], Corollary 5.6, however, in [35] the authors consider the case of Dirichlet boundary conditions.

Proof. We argue by contradiction. Suppose that $\lambda_g < \Lambda$.
We take
$$u_k = \begin{cases} k & u_g \geq k \\ u_g & u_g < k \end{cases},$$
it is a test function. In the weak formulation (3.1) becomes

$$\int_\Omega \nabla u_k \cdot \nabla u_k dx = \lambda_g \int_\Omega g u_g u_k dx. \tag{3.7}$$

For (3.6), we choose $\varphi = \frac{u_k^2}{v+\epsilon}$ and we find

$$\int_\Omega \nabla v \cdot \left(2\frac{u_k \nabla u_k}{v+\epsilon} - \left(\frac{u_k}{v+\epsilon}\right)^2 \nabla v \right) dx = \Lambda \int_\Omega g v \frac{u_k^2}{v+\epsilon} dx. \tag{3.8}$$

Subtracting (3.8) from (3.7) we find

$$\int_\Omega \left(\nabla u_k - \frac{u_k}{v+\epsilon} \nabla v \right)^2 dx = \lambda_g \int_\Omega g u_g u_k dx - \Lambda \int_\Omega g v \frac{u_k^2}{v+\epsilon} dx,$$

and then
$$0 \leq \lambda_g \int_\Omega g u_g u_k dx - \Lambda \int_\Omega g v \frac{u_k^2}{v+\epsilon} dx \quad \forall \epsilon > 0.$$
If $\epsilon \to 0$ and $k \to \infty$ we have

$$0 \leq \lambda_g \int_\Omega g u_g^2 dx - \Lambda \int_\Omega g u_g^2 dx = (\lambda_g - \Lambda) \int_\Omega g u_g^2 dx$$

and then $\lambda_g \geq \Lambda$ contradicting the hypothesis. □

For studying the problems

$$\inf_{g \in \mathcal{G}} \lambda_g, \quad \sup_{g \in \mathcal{G}} \lambda_g,$$

we need the following results proved in [8] and [9]. We denote with $\overline{\mathcal{G}}$ the weak closure of $\mathcal{G}$ in $L^p(\Omega)$. It is well known that $\overline{\mathcal{G}}$ is convex and weakly sequentially compact (see for example Lemma 2.2 of [9]).

Lemma 3.2.4. *Let $\mathcal{G}$ be the set of rearrangements of a fixed function $g_0 \in L^\infty(\Omega)$, and let $u \in L^p(\Omega)$, $p \geq 1$. There exists $\hat{g} \in \mathcal{G}$ such that*

$$\int_\Omega g\, u\, dx \leq \int_\Omega \hat{g}\, u\, dx \qquad \forall g \in \overline{\mathcal{G}}.$$

Proof. It follows from Lemma 2.4 of [9]. □

Lemma 3.2.5. *Let $g : \Omega \mapsto \mathbb{R}$ and $w : \Omega \mapsto \mathbb{R}$ be measurable functions, and suppose that every level set of w has measure zero. Then there exists an increasing function ϕ such that $\phi(w)$ is a rearrangement of g. Furthermore, there exists a decreasing function ψ such that $\psi(w)$ is a rearrangement of g.*

Proof. The assertions follow from Lemma 2.9 of [9]. □

Lemma 3.2.6. *Let $\mathcal{G}$ be the set of rearrangements of a fixed function $g_0 \in L^p(\Omega)$, $p \geq 1$, and let $w \in L^q(\Omega)$, $q = p/(p-1)$. If there is an increasing function ϕ such that $\phi(w) \in \mathcal{G}$, then*

$$\int_\Omega g\, w\, dx \leq \int_\Omega \phi(w)\, w\, dx \qquad \forall g \in \overline{\mathcal{G}},$$

and the function $\phi(w)$ is the unique maximizer relative to $\overline{\mathcal{G}}$. Furthermore, if there is a decreasing function ψ such that $\psi(w) \in \mathcal{G}$, then

$$\int_\Omega g\, w\, dx \geq \int_\Omega \psi(w)\, w\, dx \qquad \forall g \in \overline{\mathcal{G}},$$

and the function $\psi(w)$ is the unique minimizer relative to $\overline{\mathcal{G}}$.

Proof. The assertions follow from Lemma 2.4 of [9]. □

We recall that the $L^q(\Omega)$ topology on $L^p(\Omega)$ is the weak topology if $1 \leq p < \infty$, and the weak* topology if $1 \leq p \leq \infty$ [6].

Firstly, we investigate the problem of minimization.
Let $\overline{\mathcal{G}}$ be the closure of $\mathcal{G}$ with respect to the weak* topology of $L^\infty(\Omega)$. Recall that $\overline{\mathcal{G}}$ is convex and weakly sequentially compact.

Theorem 3.2.7. *Let λ_g be defined as in* (3.2).
(i) The problem

$$\min_{g\in\mathcal{G}} \lambda_g$$

has (at least) a solution.
(ii) If $\hat{g}$ is a minimizer then $\hat{g} = \phi\big(u_{\hat{g}}\big)$ for some increasing function $\phi(t)$.

Proof. If g_n is a minimizing sequence for $\inf_{g\in\mathcal{G}} \lambda_g$, we have

$$I = \inf_{g\in\mathcal{G}} \lambda_g = \lim_{n\to\infty} \lambda_{g_n} = \lim_{n\to\infty} \frac{\int_\Omega |\nabla u_{g_n}|^2 dx}{\int_\Omega g_n\, u_{g_n}^2\, dx}. \tag{3.9}$$

We can suppose the sequence λ_{g_n} is decreasing, therefore,

$$\int_\Omega |\nabla u_{g_n}|^2 dx \leq C_1 \int_\Omega g_n\, u_{g_n}^2\, dx \leq C_2 \int_\Omega u_{g_n}^2 dx, \tag{3.10}$$

for suitable constants $C_1,\ C_2$. Let us normalize u_{g_n} so that

$$\int_\Omega u_{g_n}^2 dx = 1. \tag{3.11}$$

By (3.10) and (3.11) we deduce that the norm $\|u_{g_n}\|_{H^1(\Omega)}$ is bounded by a constant independent of n. Therefore (see [29]), a sub-sequence of u_{g_n} (denoted again by u_{g_n}) converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to some function $z \in H^1(\Omega)$ with $z(x) \geq 0$ in Ω, $z = 0$ on Γ and

$$\int_\Omega z^2 dx = 1.$$

Moreover, since the sequence g_n is bounded in $L^\infty(\Omega)$, there is a subsequence (denoted again by g_n) which converges to some $\eta \in \overline{\mathcal{G}}$ in the weak* topology of $L^\infty(\Omega)$. We have

$$\int_\Omega g_n u_{g_n}^2 dx - \int_\Omega \eta\, z^2 dx = \int_\Omega (g_n - \eta)\, z^2 dx + \int_\Omega g_n(u_{g_n}^2 - z^2) dx.$$

Since

$$\lim_{n\to\infty} \int_\Omega (g_n - \eta)\, z^2 dx = 0$$

and since

$$\Big| \int_\Omega g_n(u_{g_n}^2 - z^2) dx \Big| \le C_3 \|u_{g_n} + z\|_{L^2(\Omega)} \|u_{g_n} - z\|_{L^2(\Omega)},$$

we find

$$\lim_{n\to\infty} \int_\Omega g_n u_{g_n}^2 dx = \int_\Omega \eta\, z^2 dx \ge 0. \tag{3.12}$$

Furthermore, since $\left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}$ is a norm equivalent to the usual norm in $H^1(\Omega)$ with $u_\Gamma = 0$, we have

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_{g_n}|^2 dx \ge \int_\Omega |\nabla z|^2 dx. \tag{3.13}$$

We claim that

$$\int_\Omega \eta\, z^2 dx > 0.$$

Indeed, passing to the limit as $n \to \infty$ in

$$\int_\Omega \nabla u_{g_n} \cdot \nabla \psi\, dx = \lambda_{g_n} \int_\Omega g_n\, u_{g_n} \psi\, dx$$

we get

$$\int_\Omega \nabla z \cdot \nabla \psi\, dx = I \int_\Omega \eta\, z\, \psi\, dx, \quad \forall \psi \in H^1(\Omega),$$

and

$$\int_\Omega |\nabla z|^2\, dx = I \int_\Omega \eta\, z^2 dx.$$

If we had $\int_\Omega |\nabla z|^2\, dx = 0$, we would have $z = 0$, contradicting the condition $\int_\Omega z^2 dx = 1$. And so, the claim follows.

Now, by Lemma 3.2.4, we find some $\hat{g} \in \mathcal{G}$ such that

$$\int_\Omega \eta\, z^2 dx \le \int_\Omega \hat{g}\, z^2 dx.$$

Using this estimate and recalling the variational characterization of $\lambda_{\hat{g}}$ we find

$$I = \frac{\int_\Omega |\nabla z|^2 dx}{\int_\Omega \eta\, z^2 dx} \ge \frac{\int_\Omega |\nabla z|^2 dx}{\int_\Omega \hat{g}\, z^2 dx} \ge \lambda_{\hat{g}} \ge I.$$

Therefore,

$$\inf_{g\in\mathcal{G}} \lambda_g = \lambda_{\hat{g}}.$$

Part (i) of the theorem is proved.

Let us prove that $\hat{g} = \phi(u_{\hat{g}})$ for some increasing function ϕ. By

$$\frac{\int_\Omega |\nabla w|^2 dx}{\int_\Omega g\, w^2 dx} \geq \frac{\int_\Omega |\nabla u_{\hat{g}}|^2 dx}{\int_\Omega \hat{g}\, u_{\hat{g}}^2 dx} \quad \forall g \in \mathcal{G},\ \forall w \in W_\Gamma^+,$$

with $w = u_{\hat{g}}$ we get

$$\int_\Omega g\, u_{\hat{g}}^2 dx \leq \int_\Omega \hat{g}\, u_{\hat{g}}^2 dx \quad \forall g \in \mathcal{G}. \tag{3.14}$$

The function $u_{\hat{g}}$ satisfies the equation

$$-\Delta u_{\hat{g}} = \lambda_{\hat{g}} \hat{g}\, u_{\hat{g}}. \tag{3.15}$$

Recall that $u_{\hat{g}} > 0$ in Ω. By equation (3.15), the function $u_{\hat{g}}$ cannot have flat zones neither in the set

$$F_1 = \{x \in \Omega : \hat{g}(x) < 0\}$$

nor in the set

$$F_2 = \{x \in \Omega : \hat{g}(x) > 0\}.$$

By Lemma 3.2.5, there is an increasing function $\phi_1(t)$ such that $\phi_1(u_{\hat{g}}^2)$ is a rearrangement of $\hat{g}(x)$ on $F_1 \cup F_2$. Define

$$\alpha = \inf_{x \in \Omega \setminus F_1} u_{\hat{g}}^2(x).$$

Using (3.14), one proves that $u_{\hat{g}}^2(x) \leq \alpha$ in F_1 (see Lemma 2.6 of [10] for details). Now define

$$\beta = \sup_{x \in \Omega \setminus F_2} u_{\hat{g}}^2(x).$$

Using (3.14) again one shows that $u_{\hat{g}}^2(x) \geq \beta$ in F_2.

Since

$$\sup_{F_1} \phi_1(u_{\hat{g}}^2) = \sup_{F_1} \hat{g}(x) \leq 0$$

we have $\phi_1(t) \leq 0$ for $t < \alpha$. Similarly, since

$$\inf_{F_2} \phi_1(u_{\hat{g}}^2) = \inf_{F_2} \hat{g}(x) \geq 0$$

we have $\phi_1(t) \geq 0$ for $t > \beta$. We put

$$\tilde{\phi}(t) = \begin{cases} \phi_1(t) & \text{if } 0 \leq t < \alpha \\ 0 & \text{if } \alpha \leq t \leq \beta \\ \phi_1(t) & \text{if } t > \beta. \end{cases}$$

The function $\tilde{\phi}(t)$ is increasing. Furthermore, $\tilde{\phi}(u_{\hat{g}}^2)$ is a rearrangement of $\hat{g}(x)$ in Ω (the functions $\hat{g}$ and $\tilde{\phi}(u_{\hat{g}}^2)$ have the same rearrangement on $F_1 \cup F_2$, and both vanish on $\Omega \setminus (F_1 \cup F_2)$). By (3.14) and Lemma 3.2.6 we must have $\hat{g} = \tilde{\phi}(u_{\hat{g}}^2)$. Part (ii) of the theorem follows with $\phi(t) = \tilde{\phi}(t^2)$. $\square$

The following is a continuity result.

Proposition 3.2.8. *Let λ_g be defined as in (3.2). Suppose $g_n \in \mathcal{G}$, $g \in \overline{\mathcal{G}}$ and $g_n \rightharpoonup g$ as $n \to \infty$ with respect to the weak* convergence in $L^\infty(\Omega)$.*
(i) If $g(x) > 0$ in a subset of positive measure then

$$\lim_{n\to\infty} \lambda_{g_n} = \lambda_g.$$

(ii) If $g(x) \leq 0$ in Ω then

$$\lim_{n\to\infty} \lambda_{g_n} = +\infty.$$

Proof. To prove Part (i), we use an argument similar to that used in [22] (proof of Lemma 4.2) in case of Dirichlet boundary conditions and $g(x) \geq 0$. Let u_{g_n} be the eigenfunction corresponding to g_n normalized so that

$$\int_\Omega u_{g_n}^2\, dx = 1.$$

We have

$$\lambda_{g_n} = \frac{\int_\Omega |\nabla u_{g_n}|^2 dx}{\int_\Omega g_n u_{g_n}^2\, dx} \leq \frac{\int_\Omega |\nabla u_g|^2 dx}{\int_\Omega g_n u_g^2 dx},$$

where u_g is the principal eigenfunction corresponding to g normalized so that

$$\int_\Omega u_g^2 dx = 1.$$

Since

$$\lambda_g = \frac{\int_\Omega |\nabla u_g|^2 dx}{\int_\Omega g\, u_g^2 dx},$$

we have

$$\lambda_{g_n} \leq \frac{\int_\Omega |\nabla u_g|^2 dx}{\int_\Omega g_n u_g^2 dx} = \lambda_g \frac{\int_\Omega g\, u_g^2 dx}{\int_\Omega g_n u_g^2 dx}.$$

The assumption $g_n \rightharpoonup g$ with respect to the weak* convergence in $L^\infty(\Omega)$ yields

$$\lim_{n\to\infty} \int_\Omega g_n u_g^2 dx = \int_\Omega g u_g^2 dx.$$

Therefore, for $\epsilon > 0$ we find ν_ϵ such that, for $n > \nu_\epsilon$ we have

$$\lambda_{g_n} < \lambda_g + \epsilon.$$

It follows that

$$\limsup_{n\to\infty} \lambda_{g_n} \leq \lambda_g.$$

To find the complementary inequality we use the equation

$$-u_{g_n} \Delta u_{g_n} = \lambda_{g_n} g_n u_{g_n}^2.$$

Integrating over Ω, recalling that $u_{g_n} = 0$ on Γ, that the normal derivative of u_{g_n} on $\partial\Omega \setminus \Gamma$ vanishes, and using the inequality $\lambda_{g_n} < \lambda_g + \epsilon$ (for n large), we find a constant C such that

$$\int_\Omega |\nabla u_{g_n}|^2 dx \leq (\lambda_g + \epsilon) \int_\Omega g_n u_{g_n}^2\, dx \leq C,$$

where we have used the boundedness of g_n and the normalization of u_{g_n}.Then the norm $\|u_{g_n}\|_{H^1(\Omega)}$ is bounded by a constant independent of n. A sub-sequence of u_{g_n} (denoted again by u_{g_n}) converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to some function $z \in H^1(\Omega)$ with $z \geq 0$, $z = 0$ on Γ, and

$$\int_\Omega z^2 dx = 1.$$

Consequently,

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_{g_n}|^2 dx \geq \int_\Omega |\nabla z|^2 dx.$$

Moreover, arguing as in the proof of Theorem 3.2.7 we get

$$\lim_{n\to\infty} \int_\Omega g_n u_{g_n}^2 dx = \int_\Omega g\, z^2 dx,$$

and as in the proof of Theorem 3.2.7 we cannot have

$$\int_\Omega |\nabla z|^2\, dx = \int_\Omega \eta\, z^2 dx = 0.$$

Therefore, it follows that

$$\liminf_{n\to\infty} \lambda_{g_n} = \liminf_{n\to\infty} \frac{\int_\Omega |\nabla u_{g_n}|^2 dx}{\int_\Omega g_n u_{g_n}^2 dx} \geq \frac{\int_\Omega |\nabla z|^2 dx}{\int_\Omega g\, z^2 dx} \geq \lambda_g.$$

Part (i) of the proposition is proved.

To prove Part (ii), we argue by contradiction. Suppose there is a sub-sequence of λ_{g_n}, still denoted λ_{g_n}, and a real number M such that

$$\lambda_{g_n} = \frac{\int_\Omega |\nabla u_{g_n}|^2 dx}{\int_\Omega g_n u_{g_n}^2 dx} \leq M$$

and

$$\int_\Omega u_{g_n}^2\, dx = 1.$$

It follows that

$$\int_\Omega |\nabla u_{g_n}|^2 dx \leq M \int_\Omega g_n u_{g_n}^2 dx \leq \tilde{M}.$$

Therefore, there is a sub-sequence of u_{g_n} (denoted again by u_{g_n}) which converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to some function $z \in H^1(\Omega)$, $z(x) \geq 0$ $z = 0$ on Γ, and such that

$$\int_\Omega z^2 dx = 1.$$

Moreover, up to a subsequence, we may suppose that

$$\lim_{n\to\infty} \lambda_{g_n} = \tilde{\lambda}.$$

For $n = 1, 2, \ \ldots$ we have

$$\int_\Omega \nabla u_{g_n} \cdot \nabla \psi dx = \lambda_{g_n} \int_\Omega g_n u_{g_n} \psi dx \quad \forall \psi \in H^1(\Omega).$$

Letting $n \to \infty$ we find

$$\int_\Omega \nabla z \cdot \nabla \psi dx = \tilde{\lambda} \int_\Omega g z \psi dx \quad \forall \psi \in H^1(\Omega).$$

By the latter equation we find $z \in C^1(\Omega)$. Furthermore, putting $\psi = z$ we find

$$\int_\Omega |\nabla z|^2 dx = \tilde{\lambda} \int_\Omega g z^2 dx \leq 0,$$

where the assumption $g(x) \leq 0$ has been used. It follows that $|\nabla z| = 0$ in Ω. Therefore, $z = 0$, contradicting the condition $\int_\Omega z^2 dx = 1$. □

We recall that a function f is said to be *Gateaux diffferentiable* at x if there exists a bounded linear operator $T_x \in \mathcal{B}(X, Y)$ such that for any $v \in X$,

$$\lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} = T_x v.$$

For the maximization problem we need the following result.

Proposition 3.2.9. *Let λ_g be defined as in (3.2), let $g > 0$ in a subset of positive measure, and let $J(g) = \frac{1}{\lambda_g}$. The map $g \mapsto J(g)$ is Gateaux differentiable with derivative*

$$J'(g; h) = \frac{\int_\Omega h\, u_g^2 dx}{\int_\Omega |\nabla u_g|^2 dx}.$$

Furthermore, if g satisfies $\int_\Omega g(x) dx > 0$, the map $g \mapsto \frac{1}{\lambda_g}$ is strictly convex.

Proof. In case we have Dirichlet boundary conditions, the proof of this proposition is well known (see, for example, Proposition 2.1 of [17]). The same proof also works under our boundary conditions.
We start proving that the map $g \mapsto \lambda_g$ is Gateaux differentiable.
We take a subsequence u_{g_n} as in Theorem 3.2.7 that converges in $L^2(\Omega)$ to z. We claim that z is the maximizer u_g of $J(g)$. In fact, from

$$J(g_n) = \frac{\int_\Omega g_n u_{g_n}^2 dx}{\int_\Omega |\nabla u_{g_n}|^2},$$

$$\lim_{n \to \infty} \int_\Omega g_n u_{g_n}^2 dx = \int_\Omega g z^2 dx,$$

$$\liminf_{n \to \infty} \int_\Omega |\nabla u_{g_n}|^2 dx \geq \int_\Omega |\nabla z|^2 dx$$

and from Proposition 3.2.8 we get

$$J(g) \leq \frac{\int_\Omega g z^2 dx}{\int_\Omega |\nabla z|^2 dx} \leq J(g).$$

By the uniqueness of the maximizer we must have $z = u_g$.
Let $t_n > 0$ be a sequence such that $t_n \to 0$ as $n \to \infty$. Let $f, g \in \overline{\mathcal{G}}$ and let

$g_n = g + t_n(f-g)$. Since $g(x) > 0$ in a subset of positive measure, for t_n small enough, also $g_n(x) > 0$ in a subset of positive measure. Since

$$J(g) + \frac{\int_\Omega (g_n - g)u_g^2 dx}{\int_\Omega |\nabla u_g|^2 dx} \le J(g_n) \le J(g) + \frac{\int_\Omega (g_n - g)u_{g_n}^2 dx}{\int_\Omega |\nabla u_{g_n}|^2 dx},$$

we get

$$J(g) + t_n \frac{\int_\Omega (f - g)u_g^2 dx}{\int_\Omega |\nabla u_g|^2 dx} \le J(g_n) \le J(g) + t_n \frac{\int_\Omega (f - g)u_{g_n}^2 dx}{\int_\Omega |\nabla u_{g_n}|^2 dx}.$$

Then

$$\frac{\int_\Omega (f - g)u_g^2 dx}{\int_\Omega |\nabla u_g|^2 dx} \le \frac{J(g + t_n(f-g)) - J(g)}{t_n} \le \frac{\int_\Omega (f - g)u_{g_n}^2 dx}{\int_\Omega |\nabla u_{g_n}|^2 dx}.$$

From

$$\lim_{n\to\infty} \frac{\int_\Omega (f - g)u_{g_n}^2 dx}{\int_\Omega |\nabla u_{g_n}|^2 dx} = \frac{\int_\Omega (f - g)u_g^2 dx}{\int_\Omega |\nabla u_g|^2 dx},$$

and as the sequence t_n is arbitrary, we have

$$\lim_{t\to 0} \frac{J(g + t(f-g)) - J(g)}{t} = \frac{\int_\Omega (f - g)u_g^2 dx}{\int_\Omega |\nabla u_g|^2 dx}. \tag{3.16}$$

We continue proving the convexity of $\frac{1}{\lambda_g}$. Let $t \in (0,1)$. From (3.2), denoting by u_t the maximizer $u_{tf+(1-t)g}$ we get

$$\begin{aligned}\frac{1}{\lambda tf + (1-t)g} &= \frac{\int_\Omega [tf + (1-t)g]u_t^2\,dx}{\int_\Omega |\nabla u_t|^2\,dx} = t\frac{\int_\Omega fu_t^2\,dx}{\int_\Omega |\nabla u_t|^2\,dx} + (1-t)\frac{\int_\Omega gu_t^2\,dx}{\int_\Omega |\nabla u_t|^2\,dx} \\ &\le t\frac{\int_\Omega fu_f^2\,dx}{\int_\Omega |\nabla u_f|^2\,dx} + (1-t)\frac{\int_\Omega gu_g^2\,dx}{\int_\Omega |\nabla u_g|^2\,dx} = t\frac{1}{\lambda_f} + (1-t)\frac{1}{\lambda_g}.\end{aligned}$$

Since $\int_\Omega g_0(x)\,dx > 0$, then $\int_\Omega g(x)\,dx > 0$ for all $g \in \overline{\mathcal{G}}$. For $f, g \in \overline{\mathcal{G}}$, we suppose that equality holds in the previous inequality for some $t \in (0,1)$, then

$$\begin{aligned}\frac{1}{\lambda_{tf+(1-t)g}} &= \frac{\int_\Omega [tf + (1-t)g]u_t^2\,dx}{\int_\Omega |\nabla u_t|^2\,dx} \\ &= t\frac{\int_\Omega fu_f^2\,dx}{\int_\Omega |\nabla u_f|^2\,dx} + (1-t)\frac{\int_\Omega gu_g^2\,dx}{\int_\Omega |\nabla u_g|^2\,dx} = t\lambda_f + (1-t)\lambda_g.\end{aligned}$$

Since

$$\frac{1}{\lambda_{tf+(1-t)g}} = t\frac{\int_\Omega fu_t^2\,dx}{\int_\Omega |\nabla u_t|^2\,dx} + (1-t)\frac{\int_\Omega gu_t^2\,dx}{\int_\Omega |\nabla u_t|^2\,dx},$$

it follows that

$$\frac{\int_\Omega fu_t^2\,dx}{\int_\Omega |\nabla u_t|^2\,dx} = \frac{\int_\Omega fu_f^2\,dx}{\int_\Omega |\nabla u_f|^2\,dx}$$

and

$$\frac{\int_\Omega gu_t^2\,dx}{\int_\Omega |\nabla u_t|^2\,dx} = \frac{\int_\Omega gu_g^2\,dx}{\int_\Omega |\nabla u_g|^2\,dx},$$

and so, by uniqueness of the maximizer, we must have $u_t = u_f = u_g$. Moreover, $\lambda_f = \lambda_g$. Since

$$-\Delta u_f = \lambda_f f u_f \qquad \text{a.e. in } \Omega,$$

and

$$-\Delta u_g = \lambda_g g u_g \qquad \text{a.e. in } \Omega,$$

then

$$\lambda_f f u_f = \lambda_g g u_g \qquad \text{a.e. in } \Omega,$$

and then, we get $f = g$ a.e. in Ω. □

Theorem 3.2.10. *Let λ_g be defined as in* (3.2) *and let $\int_\Omega g_0(x)\,dx > 0$. The problem*

$$\max_{g \in \overline{\mathcal{G}}} \lambda_g$$

has a unique solution; furthermore, we have $\breve{g} = \psi\big(u_{\breve{g}}\big)$ for some decreasing function $\psi(t)$; finally, if $g_0(x) \geq 0$ then the maximizer $\breve{g}$ belongs to $\mathcal{G}$.

Proof. Since the functional $g \mapsto \lambda_g$ is continuous with respect to the weak* topology of $L^\infty(\Omega)$ (by Proposition 3.2.8), and since $\overline{\mathcal{G}}$ is weakly compact, a maximizer $\breve{g}$ exists in $\overline{\mathcal{G}}$. The uniqueness of the maximizer follows from the strict convexity of $\frac{1}{\lambda_g}$ (see Proposition 3.2.9). Furthermore, since $\int_\Omega g_0(x)dx > 0$, the maximizer $\breve{g}$ is positive in a subset of positive measure, therefore, $\lambda_{\breve{g}}$ is finite and $u_{\breve{g}}(x) > 0$ a.e. in Ω. If $0 < t < 1$ and if $g_t = \breve{g} + t(g - \breve{g})$, since $J(g)$ is differentiable (see Proposition 3.2.9), we have

$$J(\breve{g}) \leq J(g_t) = J(\breve{g}) + t\frac{\int_\Omega (g - \breve{g})u_{\breve{g}}^2 dx}{\int_\Omega |\nabla u_{\breve{g}}|^2 dx} + o(t) \quad \text{as} \quad t \to 0.$$

Then

$$\int_\Omega (g - \breve{g})u_{\breve{g}}^2 dx \geq 0.$$

Equivalently, we have

$$\int_\Omega g\; u_{\breve{g}}^2 dx \geq \int_\Omega \breve{g}\; u_{\breve{g}}^2 dx \quad \forall g \in \overline{\mathcal{G}}. \tag{3.17}$$

The function $u_{\breve{g}}$ satisfies the equation

$$-\Delta u_{\breve{g}} = \lambda_{\breve{g}} \breve{g}\; u_{\breve{g}}. \tag{3.18}$$

By equation (3.18), the function $u_{\breve{g}}$ cannot have flat zones neither in the set $F_3 = \{x \in \Omega : \breve{g}(x) > 0\}$ nor in the set $F_4 = \{x \in \Omega : \breve{g}(x) < 0\}$. By Lemma 3.2.5, there is a decreasing function $\psi_1(t)$ such that $\psi_1(u_{\breve{g}}^2)$ is a rearrangement of $\breve{g}(x)$ on $F_3 \cup F_4$. Following the proof of Theorem 2.1 of [10], we introduce the class $\mathcal{W}$ of rearrangements of our maximizer $\breve{g}$. Of course, $\mathcal{W} \subset \overline{\mathcal{G}}$. Define

$$\gamma = \inf_{x \in \Omega \setminus F_3} u_{\breve{g}}^2(x).$$

Using (3.17), one proves that $u_{\breve{g}}^2(x) \leq \gamma$ in F_3. Define

$$\delta = \sup_{x \in \Omega \setminus F_4} u_{\breve{g}}^2(x).$$

Using (3.17) again one shows that $u_{\breve{g}}^2(x) \geq \delta$ in F_4. Now we put

$$\tilde{\psi}(t) = \begin{cases} \psi_1(t) & \text{if } \ 0 \leq t < \gamma \\ 0 & \text{if } \ \gamma \leq t \leq \delta \\ \psi_1(t) & \text{if } \ t > \delta. \end{cases}$$

The function $\tilde{\psi}(t)$ is decreasing and $\tilde{\psi}(u_{\breve{g}}^2)$ is a rearrangement of $\breve{g}(x)$ in Ω. Indeed, the functions $\breve{g}$ and $\tilde{\psi}(u_{\breve{g}}^2)$ have the same rearrangement on $F_3 \cup F_4$, and both vanish on $\Omega \setminus (F_3 \cup F_4)$. By (3.17) and Lemma 3.2.6 we must have $\breve{g} = \tilde{\psi}(u_{\breve{g}}^2) \in \mathcal{W}$.

We observe that, in general, the maximizer $\breve{g}$ does not belong to $\mathcal{G}$ (see next Theorem 3.11). Assuming $g_0(x) \geq 0$, we can prove that $\breve{g} \in \mathcal{G}$. Indeed, by (3.18), the function $u_{\breve{g}}$ cannot have flat zones in the set $F = \{x \in \Omega : \breve{g}(x) > 0\}$. If $|F| < |\Omega|$, since $\breve{g} \in \overline{\mathcal{G}}$, by Lemma 2.14 of [9] we have $|F| \geq |\{x \in \Omega : g_0(x) > 0\}|$. Therefore there is $g_1 \in \mathcal{G}$ such that its support is contained in F. By Lemma 3.2.6, there is a decreasing function $\psi_1(t)$ such that $\psi_1(u_{\breve{g}}^2)$ is a rearrangement of $g_1(x)$ on F. Define

$$\gamma = \inf_{x \in \Omega \setminus F} u_{\breve{g}}^2(x).$$

Using (3.17), one proves that $u_{\breve{g}}^2(x) \leq \gamma$ in F. By using equation (3.17) once more we find that $u_{\breve{g}}^2(x) < \gamma$ a.e. in F. Now define

$$\tilde{\psi}(t) = \begin{cases} \psi_1(t) & \text{if } \ 0 \leq t < \gamma \\ 0 & \text{if } \ t \geq \gamma. \end{cases}$$

The function $\tilde{\psi}(t)$ is decreasing and $\tilde{\psi}(u_{\breve{g}}^2)$ is a rearrangement of $g_1 \in \mathcal{G}$ on Ω. Indeed, the functions g_1 and $\tilde{\psi}(u_{\breve{g}}^2)$ have the same rearrangement on F, and both vanish on $\Omega \backslash F$. By (3.17) and Lemma 3.2.6 we must have $\breve{g} = \tilde{\psi}(u_{\breve{g}}^2) \in \mathcal{G}$. Hence, in case of $|F| < |\Omega|$, the conclusion follows with $\psi(t) = \tilde{\psi}(t^2)$. If $|F| = |\Omega|$, the proof is easier and we do not need the introduction of the function g_1. $\square$

Theorem 3.2.11. *Suppose $u \in H^2(\Omega) \cap C^0(\Omega)$ with $u = 0$ on Γ and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega \setminus \Gamma$. Here $\Gamma \subset \partial\Omega$ is smooth and is supposed to have a $(N-1)$-Lebesgue positive measure. Let $u(x) > 0$ in Ω and*

$$-\Delta u = \Lambda \psi(u) u \quad a.e. \ in \ \ \Omega$$

for some $\Lambda > 0$ and some decreasing bounded function ψ. Then, either $\Delta u \leq 0$ or $\Delta u \geq 0$ a.e. in Ω.

Proof. We argue by contradiction. Suppose that the essential range of Δu contains positive and negative values. Since $u > 0$ and $-\Delta u = \Lambda \psi(u) u$, $\psi(t)$ takes positive and negative values for $t > 0$. Let

$$\beta = \sup\{t : \psi(t) \geq 0\},$$

$$\Omega_\beta = \{x \in \Omega : u(x) > \beta\}.$$

By our assumptions, the open set Ω_β is not empty. On the other hand, since ψ is decreasing and $u > 0$ we have

$$-\Delta u < 0 \ \text{ in } \ \Omega_\beta, \quad u = \beta \ \text{ on } \ \partial\Omega_\beta \setminus \Gamma_\beta \ \text{ and } \ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \ \Gamma_\beta,$$

where Γ_β is a suitable subset of $\partial\Omega \setminus \Gamma$. By second Hopf's boundary Lemma, u cannot have its maximum value on Γ_β. Therefore, the maximum principle for subharmonic functions yields $u(x) \leq \beta$ in Ω_β. This contradicts the definition of Ω_β. □

3.3 Symmetry

3.3.1 The one dimensional case

We take $N = 1$ and $\Omega = (0, L)$. First of all, we recall a well known result on rearrangement that will be useful in the sequel.
Let $f : \Omega \to \mathbb{R}$ be a measurable function, we denote by f^* the decreasing rearrangement of f, and by f_* the increasing rearrangement of f.

Lemma 3.3.1. *Let $N = 1$ and $\Omega = (0, L)$.*
(i) If $f(x)$ and $g(x)$ belong to $L^\infty(\Omega)$ then

$$\int_\Omega f_*(x)g^*(x)dx \leq \int_\Omega f(x)g(x)dx \leq \int_\Omega f^*(x)g^*(x)dx. \tag{3.19}$$

(ii) If $u \in H^1(\Omega)$, $u(x) \geq 0$ and $u(L) = 0$, then $u^* \in H^1(\Omega)$, $u^*(x) \geq 0$, $u^*(L) = 0$ and

$$\int_\Omega (u')^2 dx \geq \int_\Omega ((u^*)')^2 dx. \tag{3.20}$$

Proof. See, for example, [1, 33]. □

Theorem 3.3.2. *Let λ_g be defined as in* (3.2) *with $\Omega = (0, L)$ and $u(L) = 0$. Then, for all $g \in \mathcal{G}$ we have $\lambda_g \geq \lambda_{g^*}$.*

Proof. In the one dimensional case we have

$$\lambda_g = \frac{\int_\Omega (u_g')^2 dx}{\int_\Omega g\, u_g^2 dx}, \tag{3.21}$$

where $g \in \mathcal{G}$ and u_g is a corresponding (positive) principal eigenfunction.
Since $u_g > 0$ and $(u_g^*)^2 = (u_g^2)^*$, by (3.19) we find

$$\int_\Omega g\, u_g^2\, dx \leq \int_\Omega g^*\, (u_g^*)^2 dx, \tag{3.22}$$

and moreover $u_g^*(L) = 0$. Applying (3.19), (3.20), and remembering the variational characterization of λ_{g^*} we have

$$\lambda_g = \frac{\int_\Omega (u_g')^2 dx}{\int_\Omega g\, u_g^2 dx} \geq \frac{\int_\Omega ((u_g^*)')^2 dx}{\int_\Omega g^*\, (u_g^*)^2 dx} \geq \frac{\int_\Omega (u_{g^*}')^2 dx}{\int_\Omega g^*\, u_{g^*}^2 dx} = \lambda_{g^*}.$$

□

We apply the previous theorem to the following example.

Example 3.3.3. For $0 < \alpha \leq \beta < L$, let $g(t) = 1$ on a subset E with measure α, $g(t) = -1$ on a subset F with measure $L - \beta$, and $g(t) = 0$ on $(0, L) \setminus (E \cup F)$. By Theorem 3.3.2, a minimizer is the decreasing rearrangement of g

$$g^* = \begin{cases} 1, & 0 < t < \alpha, \\ 0, & \alpha \leq t \leq \beta, \\ -1, & \beta < t < L. \end{cases}$$

If $\Lambda > 0$ is the corresponding principal eigenvalue and u is a corresponding eigenfunction, we have to study

$$-u'' = \begin{cases} \Lambda u, & 0 < t < \alpha, \\ 0, & \alpha \leq t \leq \beta, \\ -\Lambda u, & \beta < t < L, \end{cases}$$

with $u'(0) = u(L) = 0$. We can easily solve this boundary value problem. We find

$$u = \begin{cases} \cos(\sqrt{\Lambda}\, t), & 0 < t < \alpha, \\ At + B & \alpha \leq t \leq \beta, \\ K \sinh(\sqrt{\Lambda}(L - t)), & \beta < t < L. \end{cases}$$

Since the function u must be continuous and differentiable for $t = \alpha$ and for $t = \beta$, the constants Λ, A, B and K must satisfy the conditions

$$\begin{cases} \cos(\sqrt{\Lambda}\, \alpha) = A\alpha + B \\ -\sqrt{\Lambda} \sin(\sqrt{\Lambda}\, \alpha) = A, \end{cases}$$

and

$$\begin{cases} K \sinh(\sqrt{\Lambda}(L - \beta)) = A\beta + B \\ -K\sqrt{\Lambda} \cosh(\sqrt{\Lambda}(L - \beta)) = A. \end{cases}$$

And so, Λ and K must satisfy

$$\sin(\sqrt{\Lambda}\, \alpha) = K \cosh(\sqrt{\Lambda}(L - \beta))$$

and

$$\cos(\sqrt{\Lambda}\, \alpha) + \alpha\sqrt{\Lambda} \sin(\sqrt{\Lambda}\, \alpha) = K \sinh(\sqrt{\Lambda}(L - \beta)) + K\beta\sqrt{\Lambda} \cosh(\sqrt{\Lambda}(L - \beta)).$$

Therefore,

$$\cot(\sqrt{\Lambda}\, \alpha) = \tanh(\sqrt{\Lambda}(L - \beta)) + \sqrt{\Lambda}(\beta - \alpha). \tag{3.23}$$

The function $y(t) = \cot(t\, \alpha)$ satisfies, for $0 < t < \pi/(2\alpha)$,

$$y(0) = +\infty, \quad y'(t) < 0, \quad y\Big(\frac{\pi}{2\alpha}\Big) = 0.$$

Moreover, the function $z(t) = \tanh(t(L - \beta)) + t(\beta - \alpha)$ satisfies, for $0 < t$,

$$z(0) = 0, \quad z'(t) > 0, \quad z(t) < 1 + t(\beta - \alpha).$$

We conclude that equation (3.23) has a unique solution $\Lambda = \Lambda(\alpha)$ such that

$$\frac{1}{\alpha} \arctan \frac{1}{1 + \sqrt{\Lambda}(\beta - \alpha)} < \sqrt{\Lambda} < \frac{\pi}{2\alpha}.$$

Note that $\Lambda \to \infty$ as $\alpha \to 0$.

For the maximization problem we find the following

Theorem 3.3.4. *If* $\int_0^L g(x)dx > 0$*, let* ρ *such that* $\int_0^\rho g_*(x)dx = 0$*. Define* $\breve{g} = 0$ *for* $0 < x < \rho$*, and* $\breve{g} = g_*$ *for* $\rho < x < L$*. If* λ_g *is defined as in* (3.2) *with* $\Omega = (0, L)$ *and* $u(L) = 0$*, for all* $g \in \mathcal{G}$ *we have* $\lambda_g \le \lambda_{\breve{g}}$*.*

Proof. Let u_g be a principal eigenfunction corresponding to $g \in \mathcal{G}$, and let $u_{\breve{g}}$ be a principal eigenfunction corresponding to $\breve{g}$. Then

$$\lambda_g = \frac{\int_\Omega (u_g')^2 dx}{\int_\Omega g\ u_g^2 dx} \le \frac{\int_\Omega \left(u_{\breve{g}}'\right)^2 dx}{\int_\Omega g\ u_{\breve{g}}^2 dx}. \tag{3.24}$$

In addition, the function $u_{\breve{g}}$ solves the problem

$$-u_{\breve{g}}'' = \lambda_{\breve{g}} \breve{g} u_{\breve{g}},\ \ u'(0) = u(L) = 0.$$

From $u_{\breve{g}}'' = 0$ on $(0, \rho)$ and $u'(0) = 0$, we deduce that the function $u_{\breve{g}}$ is a positive constant on $(0, \rho)$. Moreover, since

$$-u_{\breve{g}}'(x) = \lambda_{\breve{g}} \int_0^x \breve{g} u_{\breve{g}} dt \ge 0,$$

$u_{\breve{g}}$ is decreasing on $(0, L)$. (Recall that we write decreasing instead of non-increasing). Then

$$u_{\breve{g}} = u_{\breve{g}}^*, \quad u_{\breve{g}}^2 = (u_{\breve{g}}^2)^*.$$

Since g_* is increasing, applying (3.19) we find

$$\int_0^L g\ u_{\breve{g}}^2 dx \ge \int_0^L g_*\ u_{\breve{g}}^2 dx. \tag{3.25}$$

Moreover, since $u_{\breve{g}}$ is a constant on $(0, \rho)$ and since $\int_0^\rho g_* dx = 0$, we have

$$\int_0^L g_*\ u_{\breve{g}}^2 dx = c^2 \int_0^\rho g_* dx + \int_\rho^L g_*\ u_{\breve{g}}^2 dx = \int_0^L \breve{g}\ u_{\breve{g}}^2 dx.$$

Then, by (3.25) we have

$$\int_0^L g\ u_{\breve{g}}^2 dx \ge \int_0^L \breve{g}\ u_{\breve{g}}^2 dx.$$

From the latter inequality and (3.24) we conclude

$$\lambda_g \le \frac{\int_\Omega (u_{\breve{g}}')^2 dx}{\int_\Omega \breve{g}\ u_{\breve{g}}^2 dx} = \lambda_{\breve{g}}.$$

$\square$

The next example is an application of the previous theorem.

Example 3.3.5. For $0 < \alpha < L$, let $g(t) = 1$ on a subset E with measure $L - \alpha$, and $g(t) = 0$ on $(0, L) \setminus E$. By Theorem 3.3.4, the maximizer is the function

$$g_* = \begin{cases} 0, & 0 < t < \alpha, \\ 1, & \alpha < t < L. \end{cases}$$

If $\Lambda > 0$ is the corresponding principal eigenvalue and u is a corresponding eigenfunction, we have to study

$$-u'' = \begin{cases} 0, & 0 < t < \alpha, \\ \Lambda u, & \alpha < t < L, \end{cases}$$

with $u'(0) = u(L) = 0$. We can solve easily this boundary value problem. We find

$$u = \begin{cases} 1, & 0 < t < \alpha, \\ \sin(\sqrt{\Lambda}(L - t)), & \alpha < t < L. \end{cases}$$

Since the function u must be continuous and differentiable for $t = \alpha$, we obtain that $\Lambda > 0$ must satisfy the condition

$$\sqrt{\Lambda} = \frac{\pi}{2(L - \alpha)}.$$

Remark 3.3.6. The statements of Theorems 3.3.2 and 3.3.4 still hold for the following problem. Let $\Omega = (0, L) \times (0, \ell)$. Let

$$\begin{cases} u = 0 & \text{on } \{x_1 = L\}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_2 = \ell\}. \end{cases}$$

We suppose that the function g_0 depends on x_1 only, and the class $\mathcal{G}$ is the (restricted) family of all rearrangements of g_0 in Ω depending on x_1 only. In this particular case, the principal eigenfunctions depend on x_1 only, and the optimization of the corresponding principal eigenvalue is essentially a one dimensional problem.

3.3.2 The α-sector

In this paragraph the domain Ω in polar coordinates (r, θ) is

$$D = \{0 \le r < R, \ \ 0 < \theta < \alpha\}, \tag{3.26}$$

where we suppose $0 < \alpha \le \pi$.
For a function $f \in L^2(D)$, we denote by f^* the radial decreasing rearrangement of f and by f_* the radial increasing rearrangement of f. We recall that f^* depends only on r and it is decreasing and f_* depends only on r and it is increasing (see [1] for more details).
We will use the following

Lemma 3.3.7. *If $f, g \in L^2(D)$ we have*

$$\int_D f_* g^* dx \le \int_D fg \ dx \le \int_D f^* g^* dx. \tag{3.27}$$

If $u \in H^1(D)$, $u \ge 0$ and $u = 0$ on $r = R$, then, $u^* \in H^1(D)$, $u^* \ge 0$ and $u^* = 0$ on $r = R$. Furthermore,

$$\int_D |\nabla u|^2 dx \ge \int_D |\nabla u^*|^2 dx. \tag{3.28}$$

Proof. See [1], pages 73-75. □

For the minimization problem we have

Theorem 3.3.8. *Let λ_g be defined as in* (3.2)*, where $\Omega = D$ is the α-sector defined in* (3.26) *and Γ is the portion of ∂D with $r = R$. Then $\lambda_g \geq \lambda_{g^*}$.*

Proof. If λ_g is the corresponding principal eigenvalue, applying inequalities (3.27) and (3.28) we have

$$\lambda_g = \frac{\int_D |\nabla u_g|^2 dx}{\int_D g u_g^2 dx} \geq \frac{\int_D |\nabla u_g^*|^2 dx}{\int_D g^*(u_g^*)^2 dx} \geq \frac{\int_D |\nabla u_{g^*}|^2 dx}{\int_D g^* u_{g^*}^2 dx} = \lambda_{g^*}.$$

We observe that, since $u_g \geq 0$ in D and it vanishes on Γ, also $u_g^* \geq 0$ in D and vanishes on Γ. □

Example 3.3.9. *We take E and F as two disjoint subsets of D,and let $\mathcal{G}$ be the class generated by $g_0 = \chi_E - \chi_F$. Then we find $g^* = \chi_{\hat{E}} - \chi_{\hat{F}}$, where*

$$\hat{E} = \Big\{(r,\theta) \in D : \, r^2 \leq \frac{2|E|}{\alpha}\Big\},$$

$$\hat{F} = \Big\{(r,\theta) \in D : \, r^2 \geq R^2 - \frac{2|F|}{\alpha}\Big\}.$$

Theorem 3.3.10. *If $\int_D g(x)dx > 0$, let $D_\rho \subset D$ be the α-sector such that $\int_{D_\rho} g_*(x)dx = 0$. Define $\breve{g} = 0$ for $x \in D_\rho$, and $\breve{g} = g_*$ for $D \setminus D_\rho$. Let λ_g be defined as in* (3.2)*, where $\Omega = D$ is the α-sector defined in* (3.26) *and Γ is the portion of ∂D with $r = R$. Then $\lambda_g \leq \lambda_{\breve{g}}$.*

Proof. Let u_g be a principal eigenfunction corresponding to $g \in \mathcal{G}$, and let $u_{\breve{g}}$ be a principal eigenfunction corresponding to $\breve{g}$. Then we have

$$\lambda_g = \frac{\int_D |\nabla u_g|^2 dx}{\int_D g\, u_g^2 dx} \leq \frac{\int_D |\nabla u_{\breve{g}}|^2 dx}{\int_D g\, u_{\breve{g}}^2 dx}. \tag{3.29}$$

On the other hand, the function $u_{\breve{g}}$ solves the problem

$$-\Delta u_{\breve{g}} = \lambda_{\breve{g}} \breve{g} u_{\breve{g}} \;\; \text{in } D,$$

with $u = 0$ on Γ and $u_\theta = 0$ on the segments $\theta = 0$ and $\theta = \alpha$. Since the solution $u_{\breve{g}}$ is radial and since $\breve{g} = 0$ for $x \in D_\rho$ its derivative (with respect to r) is a constant in $(0,\rho)$.Moreover, since $u'(0) = 0$, this constant must be zero, and then the function $u_{\breve{g}}$ is a positive constant in D_ρ. Furthermore, since

$$-r u_{\breve{g}}'(r) = \lambda_{\breve{g}} \int_0^r t \breve{g} u_{\breve{g}} dt \geq 0,$$

$u_{\breve{g}}(r)$ is decreasing on $(0, R)$. Then

$$u_{\breve{g}} = u_{\breve{g}}^*, \;\; u_{\breve{g}}^2 = (u_{\breve{g}}^2)^*.$$

Hence, since g_* is increasing, by (3.27) we have

$$\int_D g\, u_{\breve{g}}^2 dx \geq \int_D g_*\, u_{\breve{g}}^2 dx. \tag{3.30}$$

Moreover, since $u_{\breve{g}}$ is a constant in D_ρ and since $\int_{D_\rho} g_* dx = 0$, we find

$$\int_D g_*\, u_{\breve{g}}^2 dx = c^2 \int_{D_\rho} g_* dx + \int_{D\setminus D_\rho} g_*\, u_{\breve{g}}^2 dx = \int_D \breve{g}\, u_{\breve{g}}^2 dx.$$

Therefore, by (3.30) we find

$$\int_D g\, u_{\breve{g}}^2 dx \geq \int_D \breve{g}\, u_{\breve{g}}^2 dx.$$

by the latter inequality and (3.29) we conclude

$$\lambda_g \leq \frac{\int_D |\nabla u_{\breve{g}}|^2 dx}{\int_D \breve{g}\, u_{\breve{g}}^2 dx} = \lambda_{\breve{g}}.$$

□

Example 3.3.11. We take two disjoint subset E and F of D such that $|E| > |F|$. If $\mathcal{G}$ is the class generated by $g_0 = \chi_E - \chi_F$, thenthe maximum of λ_g is attained in $\breve{g} = \chi_G$, where G is the set

$$G = \Big\{(r,\theta) \in D :\, r^2 \geq R^2 - \frac{2(|E| - |F|)}{\alpha}\Big\}.$$

If $|E| \leq |F|$, we have $\sup \lambda_g = +\infty$.

3.4 Symmetry breaking

For the minimization problem, we find that, despite the symmetry of the data, the solution may not be symmetric.

Theorem 3.4.1. *Let $N = 2$ and $\Omega = B_{a,a+2}$, the annulus of radii a, $a+2$. Suppose $g_0 = \chi_E$, where E is a measurable set contained in Ω and such that $|E| = \pi\rho^2$, $0 < \rho < 1$. Let $\mathcal{G}$ be the family of rearrangements of g_0. Consider the eigenvalue problem* (3.1) *in Ω with Γ being the circle with radius $a+2$. If a is large enough then a minimizer of λ_g in $\mathcal{G}$ cannot be radially symmetric with respect to the center of $B_{a,a+2}$.*

We recall that

$$\lambda_g = \inf\left\{\frac{\int_\Omega |\nabla w|^2 dx}{\int_\Omega g\, w^2 dx},\quad w \in H^1(\Omega):\quad w = 0 \;\text{ on }\; \Gamma,\quad \int_\Omega g\, w^2 dx > 0\right\}. \tag{3.31}$$

The idea for the proof is the following. Firstly we take a function g_0 that is not radially symmetric to the centre of $B_{a,a+2}$ and we prove that λ_g has an upper bound independent of a. Then, we suppose that $g = \chi_E$ where E is radially symmetric with respect to the centre of the annulus. Finally, we prove that $\lambda_g \to \infty$ as $a \to \infty$.

Proof. Let $E = B_\rho$ be a disc with radius ρ and such that its center x_0 lies on $|x| = a+1$. If $g = \chi_{B_\rho}$, the function $z = (\rho^2 - |x - x_0|^2)^+$ vanishes on Γ, hence, if $|x - x_0| = r$,

$$\lambda_g \leq \frac{\int_{B_\rho} |\nabla z|^2 dx}{\int_{B_\rho} z^2 dx} = \frac{\int_0^\rho 4r^3 dr}{\int_0^\rho r(\rho^2 - r^2)^2 dr} = \frac{6}{\rho^2}. \tag{3.32}$$

We observe that this upper bound is independent of a.

Now we suppose $g = \chi_E$, with E radially symmetric with respect to the center of $B_{a,a+2}$. With $r = |x|$, put $g(x) = h(r) = \chi_{E_1}$, E_1 being the intersections of E with a ray of B_{a+2}. The corresponding eigenfunction is radially symmetric (by uniqueness), and the inferior in (3.31) can be taken over all $v \in H^1_{rad}$ (the class of radially symmetric functions in $H^1(\Omega)$) with $v(a+2) = 0$. We have

$$\lambda_g = \inf\left\{ \frac{\int_a^{a+2} r(v')^2 dr}{\int_a^{a+2} rhv^2 dr}, \quad v \in H^1_{rad} : \ v(a+2) = 0 \right\}.$$

We find

$$\frac{\int_a^{a+2} r(v')^2 dr}{\int_a^{a+2} rhv^2 dr} \geq \frac{\int_a^{a+2} a(v')^2 dr}{\int_a^{a+2} (a+2)hv^2 dr} = \frac{a}{a+2} \frac{\int_a^{a+2} (v')^2 dr}{\int_a^{a+2} hv^2 dr}.$$

The 1-measure of E_1 depends on the location of E, however we have

$$|E_1| \leq \sqrt{a^2 + \rho^2} - a := \ell.$$

Note that $\ell \to 0$ as $a \to \infty$.

Using classical inequalities for decreasing rearrangements we find

$$\frac{\int_a^{a+2} (v')^2 dr}{\int_a^{a+2} h\, v^2 dr} \geq \frac{\int_a^{a+2} ((v^*)')^2 dr}{\int_a^{a+2} h^*\, (v^*)^2 dr} \geq \frac{\int_{-1}^1 (w')^2 dt}{\int_{-1}^1 g^*\, w^2 dt}, \tag{3.33}$$

where $w(t) = v^*(r), \ t = r - (a+1)$, and

$$g^* = \begin{cases} 1, & -1 < t < -1 + \ell, \\ 0, & -1 + \ell < t < 1. \end{cases}$$

We have

$$\lambda_g \geq \frac{a}{a+2} \inf\left\{ \frac{\int_{-1}^1 (w')^2 dt}{\int_{-1}^1 g^* w^2 dt} : \quad w \in H^1(-1,1), \quad w(1) = 0 \right\} = \frac{a}{a+2} \Lambda_{g^*}. \tag{3.34}$$

To find $\Lambda = \Lambda_{g^*}$, we look for a positive solution of the problem

$$-z'' = \begin{cases} \Lambda z, & -1 < t < -1 + \ell, \\ 0, & -1 + \ell < t < 1, \end{cases}$$

with $z'(-1) = z(1) = 0$. We find

$$z = \begin{cases} \cos(\sqrt{\Lambda}(t+1)), & -1 < t < -1 + \ell, \\ A(1-t) & -1 + \ell < t < 1, \end{cases}$$

where A, and Λ satisfy

$$\cos(\sqrt{\Lambda}\ell) = A(2-\ell), \quad \sqrt{\Lambda}\sin(\sqrt{\Lambda}\ell) = A.$$

It follows that

$$\sqrt{\Lambda}\tan(\sqrt{\Lambda}\ell) = \frac{1}{2-\ell}.$$

Since $\ell \to 0$ as $a \to \infty$, the latter equation shows that we must have $\Lambda \to \infty$ as $a \to \infty$. Then, by (3.34), we find also $\lambda \to \infty$ as $a \to \infty$. The latter result together with (3.32) allow us to conclude that a minimizer $\hat{g}$ of $g \mapsto \lambda_g$ cannot be symmetric for a large. $\square$

Remark 3.4.2. If we switch the role of Γ and Γ_1 in the Theorem 3.4.1, that is Γ is the circle of radius a and Γ_1 is the circle of radius $a+2$, the same proof works with the following change. We have

$$\lambda_g = \inf\left\{ \frac{\int_a^{a+2} r(v')^2 dr}{\int_a^{a+2} rhv^2 dr}, \quad v \in H^1_{rad} : \ v(a) = 0 \right\}.$$

In this case $\ell := a+2-\sqrt{(a+2)^2-\rho}$ and $|E_1| \geq \ell$.
Instead of 3.33, we use classical inequalities for increasing rearrangements and we find

$$\frac{\int_a^{a+2}(v')^2 dr}{\int_a^{a+2} h\, v^2 dr} \geq \frac{\int_a^{a+2}((v_*)')^2 dr}{\int_a^{a+2} h_*\,(v_*)^2 dr} \geq \frac{\int_{-1}^1 (w')^2 dt}{\int_{-1}^1 g_*\, w^2 dt},$$

where $w(t) = v^*(r)$, the change of variable is the same $t = r-(a+1)$, and

$$g_* = \begin{cases} 1, & -1 < t < 1-\ell, \\ 0, & 1-\ell < t < 1. \end{cases}$$

We have

$$\lambda_g \geq \frac{a}{a+2} \inf\left\{ \frac{\int_{-1}^1 (w')^2 dt}{\int_{-1}^1 g_* w^2 dt} : \quad w \in H^1(-1,1), \quad w(1) = 0 \right\} = \frac{a}{a+1}\Lambda_{g_*}. \tag{3.35}$$

To find $\Lambda = \Lambda_{g_*}$, we look for a positive solution of the problem

$$-z'' = \begin{cases} \Lambda z, & -1 < t < 1-\ell, \\ 0, & 1-\ell < t < 1, \end{cases}$$

with $z(-1) = z'(1) = 0$. We have

$$z = \begin{cases} A(1+t), & -1 < t < 1-\ell, \\ \cos(\sqrt{\Lambda}(1-t)) & 1-\ell < t < 1, \end{cases}$$

where A, and Λ satisfy

$$A(2-\ell) = \cos(\sqrt{\Lambda}\ell), \quad A = \sqrt{\Lambda}\sin(\sqrt{\Lambda}\ell).$$

It follows that

$$\sqrt{\Lambda}\tan(\sqrt{\Lambda}\ell) = \frac{1}{2-\ell}.$$

And so, the conclusion is the same.

For the maximizer the situation is different. In fact, since we have uniqueness of the maximizer (for a class $\mathcal{G}$ generated by $g_0 = \chi_E$), we cannot have symmetry breaking for any annulus.

In Remark 3.3.6 we considered the bi-dimensional problem (3.1) in the rectangle $(0,L)\times(0,\ell)$ with Γ being the portion of $\partial\Omega$ with $x_1 = L$, and $\mathcal{G}$ being a class of rearrangements of a function g depending on x_1 only. We observed that it is essentially a one dimensional problem. One may ask what happens if $\mathcal{G}$ is the entire family of rearrangements. We prove that for large ℓ we have a sort of symmetry breaking.

Theorem 3.4.3. *Let $N = 2$ and $\Omega = (0, L) \times (0, \ell)$, $\ell \geq L$. Suppose $g_0 = \chi_E$, where E is a measurable set contained in Ω and such that $|E| = \pi\rho^2$, $0 < \rho < \frac{L}{2}$. Let $\mathcal{G}$ be the family of rearrangements of g_0. Consider the eigenvalue problem* (3.1) *in Ω with Γ being the portion of $\partial\Omega$ with $x_1 = L$. If ℓ is large enough then a minimizer of λ_g in $\mathcal{G}$ cannot be a set of the kind $K \times (0, \ell)$.*

Proof. If $E = K \times (0, \ell)$, our problem is essentially one dimensional, which we have treated in Remark 3.3.6. The minimum of the eigenvalue (for this kind of sets E) is attained when $E = (0, \tau) \times (0, \ell)$, with $\tau = \frac{\pi\rho^2}{\ell}$. By Example 3.3.3 (with $\alpha = \tau$ and $\beta = L$) it is clear that $\lambda_g \to \infty$ as $\ell \to \infty$. On the other hand, if we take $E = B_\rho$, a ball with radius ρ, settled in Ω far from $\partial\Omega$, the same computation which leads to (3.32) shows that λ_{B_ρ} is bounded independently of ℓ. The theorem follows.

Bibliography

[1] **C. Bandle**, *Isoperimetric Inequalities and Applications*, Pitman Monographs and Studies in Mathematics, Vol. **7**, Pitman, Boston, 1980.

[2] **F. Belgacem**, *Elliptic Boundary Value Problems with Indefinine Weights: Variational Formulations of the Principal Eigenvalue and Applications*, Pitman Research Notes in Mathematics, **368**, Longman, Harlow, Essex, UK, 1997.

[3] **P. Bérard**, *Spectral Geometry: Direct and Inverse Problems*, Lecture Notes in Math. **1207**, Springer-Verlag, 1987.

[4] **P. Bérard, G. Besson, S. Gallot** *Sur une inégalité isopérimétrique qui généralise celle de Paul Lévy-Gromov*, Invent. math. **80** (1985), 295-308.

[5] **M. Berger, P. Gauduchon, E. Mazet**, *Le Spectre d'une Variété Riemannienne*, Lecture Notes in Math. **194**, Springer-Verlag, 1971.

[6] **Brezis H.**, *Analisi Funzionale*, Liguori Editore, 1986.

[7] **Burago Y.D., Zalgaller V.A.**, *Geometric Inequalities*, Grundlehren der math. Wiss. **285**, Springer-Verlag, 1988.

[8] **G.R. Burton**, *Rearrangements of functions, maximization of convex functionals and vortex rings*, Math. Ann. **276** (1987), 225-253.

[9] **G.R. Burton**, *Variational problems on classes of rearrangements and multiple configurations for steady vortices*, Ann. Inst. Henri Poincaré. **6** (4) (1989), 295–319.

[10] **G.R. Burton and J.B. McLeod**, *Maximisation and minimisation on classes of rearrangements*, Proc. Roy. Soc. Edinburgh Sect. A, **119** (3-4) (1991), 287–300.

[11] **L. Cadeddu, M. A. Farina, G. Porru**, *Optimization of the principal eigenvalue under mixed boundary conditions*, Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 154, 1–17.

[12] **L. Cadeddu, S. Gallot, A. Loi**, *Maximizing mean exit-time of the Brownian motion on Riemannian manifolds*, Springer Vienna, Monatsh. Math. **176** (4) (2015), 551–570.

[13] **L. Cadeddu, G. Porru**, *Symmetry breaking in problems involving semilinear equation*, Discrete and continuous dynamical systems, Supplement 2011, (pp. 219-228).

[14] **S. Cano-Casanova and J. López-Gómez**, *Permanence under strong aggressions is possible*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **20** no. 6 (2003),(1989), 999–1041.

[15] **R.S. Cantrell and C. Cosner**, *Diffusive logistic equations with indefinite weights: population models in disrupted environments*, Proc. Roy. Soc. Edinburgh, **112A** (1989), 293–318.

[16] **R.S. Cantrell and C. Cosner**, *The effects of spatial heterogeneity in population dynamics*, J. Math. Biol., **29** (1991), 315–338.

[17] **C. Cosner, F. Cuccu and G. Porru**, *Optimization of the first eigenvalue of equations with indefinite weights*, Advanced Nonlinear Studies, **13** (2013), 79–95.

[18] **S. Cox**, *The two phase drum with the deepest note*, Japan J. Indust. Appl. Math., **8** (1991), 345–355.

[19] **S.J. Cox and J.R. McLaughlin**, *Extremal eigenvalue problems for composite membranes, I*, Appl. Math. Optim., **22** (1990), 153–167.

[20] **S.J. Cox and J.R. McLaughlin**, *Extremal eigenvalue problems for composite membranes, II*, Appl. Math. Optim., **22** (1990), 169–187.

[21] **C. B. Croke**, *A sharp four-dimensional isoperimetric inequality*, Comment. Math. Helv. 59 no. 2, (1984) 187-192.

[22] **F. Cuccu, B. Emamizadeh and G. Porru**, *Optimization of the first eigenvalue in problems involving the p-Laplacian*, Proc. Amer. Math. Soc., **137** (2009), 1677–1687.

[23] **F. Cuccu and G. Porru**, *Optimization in eigenvalue problems*, Dyn. Contin. Discrete Impuls. Syst. Ser. A, Math. Anal., **10** (2003), 51–58.

[24] **H. Federer**, *Geometric measure theory, Die Grundlehren der mathematischen Wissenschaften*, Band 153, New York: Springer-Verlag New York Inc. (1969), cap. 2 e 3.

[25] **H. Federer**, *Curvature measures*, Trans. Amer. Math. Soc. vol.93 (1959) pp.418-491.

[26] **M.E. Filippakis, N.S. Papageorgiu and S. Vasile**, *Eigenvalue problems for nonlinear elliptic equations with unilateral constains*, Nonlinear Analysis, T.M.A., **69** (2008), 85–109.

[27] **W. H. Fleming, R. Rishel**, *An integral formula for total gradient variation*, Archiv der Mathematik 11 (1) (1960), 218-222.

[28] **S. Gallot**, *Inégalités isopérimétriques et analytiques sur les variétés riemanniennes*, Astérisque **163-164**, 1988, Soc. Math. Fr. Edit.

[29] **D. Gilbarg and N.S. Trudinger**, *Elliptic Partial Differential Equations of Second Order*, Springer Verlag, Berlin, 1977.

[30] **M. Gromov**, *Paul Lévy's isoperimetric inequality*, Prépublication IHES, 1980.

[31] **H. Howards, M. Hutchings, F. Morgan**, *The isoperimetric problem on surfaces of revolution of decreasing Gauss curvature*, Trans. AMS 352, n. 11, (2000), 4889-4909.

[32] **K. Jha and G. Porru**, *Minimization of the principal eigenvalue under Neumann boundary conditions*, Numerical Functional Analysis and Optimization, **32** (2011), 1146–1165.

[33] **B. Kawohl**, *Rearrangements and Convexity of Level Sets in PDE*, Lectures Notes in Mathematics, **1150**, Berlin, 1985.

[34] **B. Kawohl and M. Belloni**, *A direct uniqueness proof for equations involving the p-Laplace operator*, Manuscripta Math., **109** (2002), 229–231.

[35] **B. Kawohl, M. Lucia and S. Prashanth**, *Simplicity of the first eigenvalue for indefinite quasilinear problems*, Adv. Differential Equations, **12** (2007), no. 4, 407–434.

[36] **B. Kleiner**, *An isoperimetric comparison theorem*, Invent. Math. 108 (1992), no. 1, 37-47.

[37] **P. Lindqvist**, *On the equation* $\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$. Proc. Amer. Math. Soc. **109** (1990), 157–164. Addendum ibidem **116**(1992), 583–584.

[38] **J. López-Gómez**, *Permanence under strong competition*, Dynamical systems and applications, 473488, World Sci. Ser. Appl. Anal., **4**, World Sci. Publ., River Edge, NJ, 1995.

[39] **J. Maly, D. Swanson, W. Ziemer**, *The co-area formula for Sobolev mappings*, Transactions of the American Mathematical Society 355 (2) (2002), 477-492.

[40] **A. Manes and A.M. Micheletti**, *Un'estensione della teoria variazionale classica degli autovalori per operatori ellittici del secondo ordine*, Boll. Unione Mat. Ital. Ser IV, **7** (1973), 285–301.

[41] **McDonald P.**, *Isoperimetric Conditions, Poisson Problems and Diffusions in Riemannian manifolds*, Potential Anal. 16 (2002), 115-138.

[42] **J. Milnor**, *Morse Theory*, Princeton University Press, 1963.

[43] **G. Perelman**, *A diameter sphere theorem for manifolds of positive Ricci curvature*, Math. Z. 218 (1998), p. 595-596.

[44] **G. Porru**, *Optimization problems for eigenvalues of p-Laplace equations with indefinite weights*, Boll. Accademia Gioenia di Scienze Naturali - Catania, Vol. 46, n. 376 (2013), 44–56.

[45] **A. Savo**, *A Mean-value Lemma and Applications*, Bull. Soc. math. France 129 (4), 2001, pp. 505-542.

[46] **A. Savo**, *Uniform Estimates and the Whole Asymptotic Series of the Heat Content on Manifolds*, Geometriae Dedicata 73, 1998, pp. 181-214.

[47] **G. Talenti**, *Elliptic equations and rearrangements*, Ann. Scuola Norm. Sup. Pisa **3** (1976), 697-718.

[48] **M. E. Taylor**, *Measure Theory and Integration*, Graduate Study in Mathematics, Vol. 76, AMS, RI, 2006.

[49] **N.S. Trudinger**, *On Harnack type inequalities and their applications to quasilinear elliptic equations.* Comm. on Pure and Applied Math. Vol. XX (1967), 721–747.

[50] **A. Weil**, *Sur les surfaces à courbure négative*, C. R. Acad. Sci. Paris 182 (1926), 1069-1071.