Substructurality and Residuation in Logic and Algebra

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大輔へ
待てば海路の日和あり

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Contents

Introduction

A very and natural way of introducing a logic is by using a sequent calculus, or Gentzen system. These systems are determined by specifying a set of axioms and a set of rules. Axioms are then starting points from which we can derive new consequences by using the rules. Hilbert systems consist also on a set of axioms and a set of rules that are used to deduce consequences. The main difference is that, whereas the axioms in Hilbert systems are formulas, and the rules allow to deduce certain formulas from other sets of formulas, in the case of Gentzen systems the axioms are *sequents* and the rules indicate which sequents can be inferred from other sets of sequents. By a *sequent* we understand a pair $\langle \Gamma, \Sigma \rangle$, where Γ and Σ are finite sequences of formulas. We denote the sequent $\langle \Gamma, \Sigma \rangle$ by $\Gamma \rhd \Sigma$.[1] The sequent $\Gamma \rhd \Sigma$ intends to formalize – at least in its origin – the concept "the conjunction of all the formulas of Γ implies the disjunction of all the formulas of Σ."

The notion of a sequent calculus was invented by G. Gentzen in order to give axiomatizations for Classical and Intuitionistic Propositional Logics. And the rules he gave in both cases can be grouped in different categories: because of its character, the Cut rule deserves a special category for itself; then we have the rules of introduction and elimination of each one of the connectives, both on the left and on the right – of the symbol $\rhd$ –; and finally a set of rules that do not involve any particular connective. These are known as the *structural rules*, and here we have their left versions:

$$\frac{\Gamma, \Delta \rhd \Sigma}{\Gamma, \alpha, \Delta \rhd \Sigma}\ (\textit{Weakening}) \qquad\qquad \frac{\Gamma, \alpha, \alpha, \Delta \rhd \Sigma}{\Gamma, \alpha, \Delta \rhd \Sigma}\ (\textit{Contraction})$$

$$\frac{\Gamma, \alpha, \beta, \Delta \rhd \Sigma}{\Gamma, \beta, \alpha, \Delta \rhd \Sigma}\ (\textit{Exchange})$$

These rules are necessary in Classical and Intuitionistic logics because in these logics

[1] Traditional notations for sequents are $\Gamma \Rightarrow \Sigma$ and $\Gamma \vdash \Sigma$, but since both the symbols $\Rightarrow$ and $\vdash$ have many other meanings, we prefer to denote sequents by using the less overloaded symbol $\rhd$, which can also be found in literature with this use.

the order in which we are given the premises, or if we have them repeated, is irrelevant, and we do not loose consequences if we extend the set of hypotheses. But there are other logics that do not satisfy all these rules: for instance, relevance logics and linear logic. At first, these logics were studied separately, and different theories were developed for their investigation. But later on, researches arrived to the conclusion that all of them share a common feature, which became more apparent after the work of W. Blok and D. Pigozzi. It was discovered that (pointed) residuated lattices – or FL algebras – are the algebraic counterpart of substructural logics.

In the XIX century, Boole noticed a close connection between "the laws of thought," as he put it, and algebra. After him, other mathematicians put together all the pieces and described a sort of algebras, named *Boole algebras* after him, and shed light on the connection anticipated by Boole: Boole algebras are the "natural" semantics for Classical Propositional Logic. More connections were discovered between other logics and other sorts of algebras: for instance, Heyting algebras are the "natural" semantics for Intuitionistic Propositional Logic, and MV algebras for Łukasievicz Multivalued Logic. But it was not until 1989, when Blok and Pigozzi published their book *Algebraizable Logics*, that for the first time the connections between these logics and classes of algebras were finally described with absolute precision. According to their definitions, these classes of algebras are the *equivalent algebraic sematics* of the corresponding logics. That is, these classes of algebras are the *algebraic counterparts* of the corresponding logics. Their ideas paved the way to a new branch of mathematics called *Abstract Algebraic Logic*, which investigates the connections between logics and classes of algebras, and the so-called *bridge theorems*: that is, theorems that establish bridges between some property of one realm (logic or algebra) with another property of the other realm.

The core of the connection between substructural logics and residuated lattices is that in all these logics, some theorem of the following form could always be proven:

$$\alpha, \beta \rhd \gamma \text{ is provable} \quad \Leftrightarrow \quad \beta \rhd \alpha \to \gamma \text{ is provable.}$$

Thus, we could think that the metalogical symbol ',' is acting as a real connective. More precisely, we could introduce a new connective $\cdot$, called *fusion*, and impose the following rule:

$$\alpha \cdot \beta \rhd \gamma \text{ is provable} \quad \Leftrightarrow \quad \beta \rhd \alpha \to \gamma \text{ is provable.} \tag{1}$$

Given an algebraic model with a lattice reduct, it is usually the case that the meet and join operations serve as the interpretations of the conjunction and disjunction connectives. What should be then the interpretation of the fusion? Usually, the elements of the lattice are thought as different degrees of truth, and "$\alpha \rhd \beta$ is provable" is interpreted as "for every assignment, the degree of truth of α is less than that of β." Under this natural interpretation, the condition (1) becomes:

$$a \cdot b \leqslant c \quad \Leftrightarrow \quad b \leqslant a \to c.$$

That is, the fusion is interpreted as a residuated operation on the lattice.

Being the algebraic semantics of substructural logics and containing many interesting subvarieties such as Heyting algebras, MV algebras, and lattice-ordered groups, to name a few, the variety of residuated lattices is of utmost importance to the studies of Logic and Algebra, hence our interest. In this dissertation we carry out some investigations on different problems concerning residuated lattices.

In what follows we give a brief description of the contents and organization of this dissertation. Every chapter – except for the first one, which is devoted to setting the preliminaries – starts with an introduction in which the reader will find a lengthier explanation of the subject of the chapter, the way the material is organized, and references. Thus, in order to avoid unnecessary repetitions, we will keep this introduction short.

We start by compiling in Chapter 1 all the essential well-known results about residuated lattices that we will need in the subsequent chapters. We present here the definitions of those concepts that are not specific to some particular chapter, but general. We define the variety of residuated lattices, and some of its more significant subvarieties. We also introduce nuclei, and nucleus retracts. As it is widely known, the lattice of normal convex subalgebras of a residuated lattice is isomorphic to its congruence lattice, and hence its importance. But it turns out that also the lattice of convex (not necessarily normal) subalgebras is of great significance, specially in the case of e-cyclic residuated lattices. Many of its properties depend on the fact that it is a pseudo-complemented lattice. Actually, it is a Heyting algebra. For instance, polars are special sets usually defined in terms of a certain notion of orthogonality; in the case of e-cyclic residuated lattices, polars are the pseudo-complements of the convex subalgebras. We end the chapter by briefly explaining the notions of semilinearity and projectability for residuated lattices.

In the 1960's, P. F. Conrad and other authors set in motion a general program for the investigation of lattice-ordered groups, aimed at elucidating some order-theoretic properties of these algebras by inquiring into the structure of their lattices of convex ℓ-subgroups. This approach can be naturally extended to residuated lattices and their convex subalgebras. We devote Chapters 2 and 3 to two different problems that can be framed within Conrad's program for residuated lattices. More specifically, in Chapter 2 we revisit the Galatos-Tsinakis categorical equivalence between integral GMV algebras and negative cones of ℓ-groups with a nucleus, showing that it restricts to an equivalence of the full subcategories whose objects are the projectable members of these classes. Afterwards, we introduce the notion of Gödel GMV algebras, which are expansions of projectable integral GMV algebras by a binary term that realizes a positive Gödel implication in every such algebra. We see that Gödel GMV algebras and projectable integral GMV algebras are essentially the same thing. Analogously, Gödel negative cones are those Gödel GMV algebras whose residuated lattice reducts are negative cones of ℓ-groups. Thus, we turn projectable integral GMV algebras and negative cones of projectable ℓ-groups into varieties by including this implication in their signature. We prove that there is an adjunction between the categories whose objects are the members of these varieties and whose morphisms are required to preserve implications.

We devote Chapter 3 to the study of certain kinds of completions of semilinear residuated lattices. We can find in the literature different notions of completions for residuated lattices, like for example Dedekind-McNeil completions, regular completions, complete ideal completions, ... Very often it happens that for a certain algebra in a variety of residuated lattices, those completions exists but do not belong to the same variety. That is, varieties are not closed, in general, under the operations of taking these kinds of completions. But there are other notions of completions that might have better properties in this regard. Conrad and other authors proved the existence of lateral completions, projectable completions, and orthocompletions for representable ℓ-groups, and moreover, that the varieties of representable ℓ-groups are closed under these completions. Our goal in this chapter is to prove the existence of lateral completions, (strongly) projectable completions, and orthocompletions for semilinear e-cyclic residuated lattices, as they are a natural generalization of representable ℓ-groups. We introduce all these concepts along the chapter, and prove first that every semilinear e-cyclic residuated lattice can be densely embedded into another residuated lattice which is latterly complete and strongly projectable. We obtain this lattice as a direct limit of a certain family of algebras obtained from the original lattice by taking quotients and products, so the direct limit stays in the same variety where the original algebra lives. Finally, we prove that for semilinear GMV algebras, we can find minimal dense extensions satisfying all the required properties.

In Chapter 4 we study the failure of the Amalgamation Property on several varieties of residuated lattices. The Amalgamation Property is of particular interest in the study of residuated lattices due to its relation with various syntactic interpolation properties of substructural logics. There are no examples to date of non-commutative varieties of residuated lattices that satisfy the Amalgamation Property. The variety of semilinear residuated lattices is a natural candidate for enjoying this property, since most varieties that have a manageable representation theory and satisfy the Amalgamation Property are semilinear. However, we prove that this is not the case, and in the process we establish that the same happens for the variety of semilinear cancellative residuated lattices, that is, it also lacks the Amalgamation Property. In addition, we prove that the variety whose members have a distributive lattice reduct and satisfy the identity $x(y \wedge z)w \approx xyw \wedge xzw$ also fails the Amalgamation Property.

In Chapter 5 we show how some well-known results of the theory of automata, in particular those related to regular languages, can be viewed within a wider framework. In order to do so, we introduce the concept of module over a residuated lattice, and show that modules over a fixed residuated lattice – that is, partially ordered sets acted upon by a residuated lattice – provide a suitable algebraic framework for extending the concept of a recognizable language as defined by Kleene. More specifically, we introduce the notion of a recognizable element of a residuated lattice by a finite module and provide a characterization of such an element in the spirit of Myhill's characterization of recognizable languages. Further, we investigate the structure of the set of recognizagle elements of a residuated lattice, and also provide sufficient conditions for a recognizable element to be recognized by a Boolean module.

We summarize in Chapter 6 the main results of this dissertation and propose some of the problems that still remain open. We end this dissertation with an appendix on directoids. These structures were introduced independently three times, and their aim is to study directed ordered sets from an algebraic perspective. The structures that we have studied in this dissertations have an underlying order, but moreover they have a lattice reduct. That is not always the case for directed ordered sets. Hence the importance of the study of directoids. We prove some properties of directoids and their expansions by additional and complemented directoids. Among other results, we provide a shorter proof of the direct decomposition theorem for bounded involute directoids. We present a description of central elements of complemented directoids. And finally we show that the variety of directoids, as well as its expansions mentioned above, all have the strong amalgamation property.

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Chapter 1

Preliminaries

In this first chapter, we introduce the basic concepts that will be used and studied in this dissertation, as well as some well-known facts about them. We have collected here only those notions that will appear in more than one chapter, in the interests of briefness. More advanced or specific concepts will be introduced in time, as we need them and our study develops. We do not give proofs for the results presented here, but we supply appropriate references for those that are not immediate. We do not intend in this chapter to provide a thorough study of the subjects that we introduce, but to keep in one place the material – which otherwise will be scattered over many articles and books – and unify notation and terminology. Unless the possibility of confusion forbids it, we will use juxtaposition to denote composition of maps, as well as other sorts of "multiplications," but most of the time the reader will discern the meaning by the context.

While we start by giving the rudiments of orders and lattices, we swiftly move forward to bring in the star concept of our research, namely residuated lattices, and related notions. We will make extensive use of Universal Algebra throughout this dissertation. We refer the reader to [18, 24, 46], where the basics and techniques of Universal Algebra are exposed. As for residuated lattices, the reader will find lucid expositions in [10, 40, 53, 72] and the references therein contained.

1.1 Partial Ordered Sets, Lattices, and Residuation

Most of the structures that we are going to consider in this dissertation are sets endowed with a certain order of its elements. Here, we introduce the very basic concepts related to partial ordered sets and lattices, and refer the reader to [32] for a comprehensive exposition of the subject.

A *partial ordered set*, or *poset*, is a pair $\mathbf{A} = (A, \leqslant)$ such that $\leqslant$ is a binary relation satisfying the following three properties:

1. *Reflexivity*: for all $a \in A$, $a \leqslant a$.
2. *Antisymmetry*: for all $a, b \in A$, if $a \leqslant b$ and $b \leqslant a$, then $a = b$.
3. *Transitivity*: for all $a, b, c \in A$, if $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$.

We say that $\leqslant$ is an *order* on A, and if $a \leqslant b$ we say that a is *lesser* or *smaller* than b and b is *greater* or *bigger* than a.

Typical examples of partial ordered sets are the natural, integer, and real numbers with their respective natural orders: $(\mathbb{N}, \leqslant)$, $(\mathbb{Z}, \leqslant)$, $(\mathbb{R}, \leqslant)$. It is well known that these posets satisfy an additional property, namely, any two elements are *comparable*, i.e. either $a \leqslant b$ or $b \leqslant a$. If this is the case, we say that the order is *total* and the partial ordered set is a *chain* or *linearly ordered*. Thus, the aforementioned sets of numbers with their natural orders are chains.

Obviously, a poset need not be a chain. For instance, the *divisibility relation*, $a \mid b$ if and only if a divides b, is an order on the set of natural numbers, but neither $3 \mid 5$ nor $5 \mid 3$, for example. Thus $(\mathbb{N}, \mid)$ is a poset but not a chain.

Given a poset $\mathbf{A} = (A, \leqslant)$, and a set $X \subseteq A$, an *upper* (*lower*) *bound* of X is an element $a \in A$ such that $b \leqslant a$ ($a \leqslant b$, respectively), for all $b \in X$. The *supremum* of X, if it exists, is the smallest of its upper bounds. Analogously, the *infimum* of X, if it exists, is the biggest of its lower bounds.

A poset is called a *lattice* if for every $a, b \in A$, the set $\{a, b\}$ has a supremum and an infimum. There is an algebraic way of presenting a lattice using precisely the existence of the supremum and infimum of every pair of elements: A *lattice* is an algebra $\mathbf{A} = (A, \wedge, \vee)$ of type (2,2), such that the operations $\wedge$ and $\vee$, called *meet* and *join*, respectively, are associative, commutative, and satisfy the *absorption laws* ($a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$). As it is well known, every algebra with such characteristics induces an order $\leqslant$ on its universe as follows: $a \leqslant b$ if and only if $a \wedge b = a$ (or equivalently $a \vee b = b$). With such definitions, the meet of a and b, $a \wedge b$, is the infimum of $\{a, b\}$ and analogously, their join is the supremum. Thus, lattices can be presented both as purely relational and as algebraic structures.

A *filter* of a poset $\mathbf{A} = (A, \leqslant)$ is a nonempty set $F \subseteq A$ that is *closed upwards*, that is, for every $a \in F$ and $b \in A$, if $a \leqslant b$ then $b \in F$. Given a lattice $\mathbf{A} = (A, \wedge, \vee)$, a *lattice-filter* (or just a *filter*) of a $\mathbf{A}$ is a filter F of the associated poset that moreover satisfies that for every $a, b \in F$, the meet $a \wedge b \in F$.

We say that a poset is *bounded above* or *bounded below* if there is a greatest element (called *top*), least element (called *bottom*), respectively, and it is *bounded* if it contains both. If the lattice is presented as an algebra, we usually add two extra constants to the language, $\perp$ and $\top$, to represent the bottom and the top, respectively. It is not difficult to see that $(\mathbb{N}, |)$ is actually a lattice, which is bounded, whose bottom is 1 and top is 0. The corresponding algebraic presentation would be $(\mathbb{N}, \mathrm{gcd}, \mathrm{lcm}, 1, 0)$, where $\mathrm{gcd}(a, b)$ and $\mathrm{lcm}(a, b)$ are the greatest common divisor and least common multiple of a and b, respectively.

Given a poset $\mathbf{A} = (A, \leqslant)$, it is easy to see that every non-empty $X \subseteq A$, equipped with the restriction of $\leqslant$ to X, is again a poset $(X, \leqslant)$. (We will not make any notational distinction between the order on A and its restriction to X, unless this leads to confusion.) We say that X *inherits* the order from $\mathbf{A}$.

Given a poset $\mathbf{A} = (A, \leqslant)$, a subset $D \subseteq A$ is said to be *directed* if for every $a, b \in D$, there is a $c \in D$ such that $a \leqslant c$ and $b \leqslant c$. That is, if every two elements of D have a common upper bound. An element a of a poset $\mathbf{A} = (A, \leqslant)$ is *compact* if any directed set $D \subseteq A$ such that the supremum of D exists and is bigger than a, contains an element $d \in D$ such that $a \leqslant d$. We denote by $\mathcal{K}(\mathbf{A})$ the set of compact elements of $\mathbf{A}$. Consider for instance the *power set* $\mathcal{P}(X)$ of a set X, that is, the set of all subsets of X. We can see that the inclusion, $\subseteq$, is an order on $\mathcal{P}(X)$. Thus, $(\mathcal{P}(X), \subseteq)$ is an ordered set and its compact elements are the finite subsets of X.

A poset $\mathbf{A}$ is *complete* if every subset $X \subseteq A$ has a supremum (and therefore, every subset X has also an infimum). A poset $\mathbf{A}$ is *algebraic* if every element is the supremum of the set of compact elements bellow it. A complete poset need not be algebraic, nor an algebraic poset needs to be complete. $(\mathcal{P}(X), \subseteq)$ is an example of a complete poset, which is also algebraic.

Given two posets $\mathbf{A} = (A, \leqslant)$ and $\mathbf{B} = (B, \leqslant')$, a map $f : A \to B$ is *monotone* if it respects the order, i.e. if for every $a, b \in A$, if $a \leqslant b$ then $f(a) \leqslant' f(b)$. We also say that f is *order-preserving*. The map f is an *isomorphism* if it is bijective and for every $a, b \in A$, $a \leqslant b$ if and only if $f(a) \leqslant' f(b)$.

A map $f : A \to B$ is said to be *residuated*, with respect to $\mathbf{A} = (A, \leqslant)$ and $\mathbf{B} = (B, \leqslant')$, if there is another map $f^+ : B \to A$ such that for every $a \in A$ and $b \in B$,

$$f(a) \leqslant' b \quad \Leftrightarrow \quad a \leqslant f^+(b).$$

The map f^+, which is called the *residual* of f, can be shown to be uniquely determined by f. We also say that the pair (f, f^+) is a *residuated pair*. For instance, given any map $f : X \to Y$, the *direct image* map $\overline{f} : \mathcal{P}(X) \to \mathcal{P}(Y)$ is residuated, with residual f^{-1} given by: $f^{-1}(Z) = \{x \in X : f(x) \in Z\}$, for every $Z \subseteq Y$.

Residuated maps are of utmost importance in Mathematics, intimately related to Galois connections and to adjoint functors. They have many interesting properties, of which we give a brief selection in the lemma below. We refer the reader to [12] for a detailed proof of these facts.

Lemma 1.1. *If $f : \mathbf{A} \to \mathbf{B}$ is a residuated map with residual f^+, then:*

(i) *Both f and f^+ are order-preserving.*

(ii) *f preserves arbitrary existing joins and f^+ preserves arbitrary existing meets.*

(iii) *$\gamma = f^+ \circ f$ is a closure operator[1] on $\mathbf{A}$. Its associated closure system is $A_\gamma = \{f^+(b) : b \in B\}$, which inherits a partial order from $\mathbf{A}$.*

(iv) *$\delta = f \circ f^+$ is an interior operator[2] on $\mathbf{B}$. Its associated interior system is $B_\delta = \{f(a) : a \in A\}$, which inherits a partial order from $\mathbf{B}$.*

(v) *$f \circ f^+ \circ f = f$ and $f^+ \circ f \circ f^+ = f^+$.*

(vi) *The corresponding restrictions of f and f^+ determine an order-isomorphism and its inverse between $\mathbf{A}_\gamma$ and $\mathbf{B}_\delta$.*

1.2 Residuated Lattices

In this section we briefly recall basic facts about the class of (pointed) residuated lattices and some of its most important subclasses. We refer the reader to [10, 40, 53, 72] for basic results in the theory of residuated lattices. Here, we only review the background material needed in this dissertation.

A binary operation $\cdot$ on a partially ordered set $\mathbf{A} = (A, \leqslant)$ is said to be *residuated*[3] provided there exist binary operations $\backslash$ and $/$ on A such that for all $a, b, c \in A$,

$$a \cdot b \leqslant c \quad \text{if and only if} \quad a \leqslant c/b \quad \text{if and only if} \quad b \leqslant a\backslash c. \tag{Res}$$

We refer to the operations $\backslash$ and $/$ as the *left residual* and *right residual* of $\cdot$, respectively. Usually, we refer to the operation $\cdot$ as *multiplication*, specially if it is represented by the symbol '$\cdot$' or some other of the kind. We will encounter situations in which the meet operation of a lattice, $\wedge$, is residuated. Tradition prevents us from calling the meet operation a "multiplication" in these cases, also because it often collides with another operation, $\cdot$, present in the same structure, to which the name "multiplication" better

[1]See Section 1.3.

[2]See Section 1.3.

[3]Notice that this means that the operation $\cdot$ is residuated in "both coordinates," i.e. for every $a, b \in A$, the maps $a \cdot \_ : A \to A$ and $\_ \cdot b : A \to A$ are residuated, in the sense of Section 1.1, with residuals $a\backslash\_$ and $\_/b$, respectively. For this reason, sometimes $\cdot$ is said to be *biresiduated*.

fits. As usual, we write xy for $x \cdot y$, x^2 for xx and adopt the convention that, in the absence of parentheses, $\cdot$ is performed first, followed by $\backslash$ and $/$, and finally by $\vee$ and $\wedge$, if present. Sometimes we will use $\prod_{j\leqslant n} x_j$ as a shorthand notation for $x_1 \cdots x_n$.

The residuals may be viewed as generalized division operations. We tend to favor $\backslash$ in calculations, but any statement about residuated structures has a "mirror image" obtained by reading terms backwards (i.e., replacing $x \cdot y$ by $y \cdot x$ and interchanging x/y with $y\backslash x$).

We are primarily interested in the situation where $\cdot$ is a monoid operation with unit element e and the partial order $\leqslant$ is a lattice order. In this case, we add the monoid unit and the monoid product symbols to the similarity type and refer to the resulting structure $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, e)$ as a *residuated lattice*. The class of residuated lattices forms a variety (see e.g. [72, Proposition 4.5]) that we denote throughout this dissertation by $\mathcal{RL}$. We adopt the convention that when a class is denoted by a string of calligraphic letters, then the members of that class will be referred to by the corresponding string of Roman letters. Thus, for example, an RL is a residuated lattice.

The existence of residuals has the following basic consequences, which will be used throughout this dissertation without explicit reference.

Lemma 1.2. *Let* $\mathbf{L}$ *be a residuated lattice.*

(1) The multiplication preserves all existing joins in each argument; i.e., if $\bigvee X$ and $\bigvee Y$ exist for $X, Y \subseteq L$, then $\bigvee_{\substack{x\in X\\ y\in Y}} xy$ exists and

$$\left(\bigvee X\right)\left(\bigvee Y\right) = \bigvee_{\substack{x\in X\\ y\in Y}} xy.$$

(2) The residuals preserve all existing meets in the numerator and convert existing joins to meets in the denominator, i.e., if $\bigvee X$ and $\bigwedge Y$ exist for $X, Y \subseteq L$, then for any $z \in L$, $\bigwedge_{x\in X} x\backslash z$ and $\bigwedge_{y\in Y} z\backslash y$ exist and

$$\left(\bigvee X\right)\backslash z = \bigwedge_{x\in X} x\backslash z \quad \text{and} \quad z\backslash\left(\bigwedge Y\right) = \bigwedge_{y\in Y} z\backslash y,$$

and the same for $/$.

(3) The following identities[4] (and their mirror images) hold in $\mathbf{L}$*:*

(a) $y(y\backslash x) \leqslant x$;	*(c)* $x\backslash y \leqslant zx\backslash zy$;	*(e)* $xy\backslash z = y\backslash(x\backslash z)$;
(b) $(x\backslash y)z \leqslant x\backslash yz$;	*(d)* $(x\backslash y)(y\backslash z) \leqslant x\backslash z$;	*(f)* $x\backslash(y/z) = (x\backslash y)/z$;

[4]Some of them are expressed as inequalities, but are clearly equivalent to identities.

(g) $(y/(x\backslash y))\backslash y = x\backslash y$; (i) $e \leqslant x\backslash x$; (k) $(x\backslash x)^2 = x\backslash x$.

(h) $e\backslash x = x$; (j) $x(x\backslash x) = x$;

We will have the occasion to consider pointed residuated lattices. A *pointed residuated lattice* is an algebra $\mathbf{L} = (L, \cdot, \backslash, /, \vee, \wedge, e, 0)$ of signature $(2, 2, 2, 2, 2, 0, 0)$ such that $(L, \cdot, \backslash, /, \vee, \wedge, e)$ is a residuated lattice. In other words, a pointed residuated lattice is simply a residuated lattice with an extra constant 0. Pointed residuated lattices are also referred to in the literature as *FL-algebras*, as they provide algebraic semantics for the Full Lambek calculus, and its subvarieties correspond to substructural logics. We also define here a *bounded residuated lattice* to be a pointed residuated lattice with bottom element 0 (and therefore also, top element $0\backslash 0$), emphasizing that "bounded" implies that the constant 0 representing the bottom element is in the signature. Residuated lattices may be identified with pointed residuated lattices satisfying the identity $e \approx 0$.

A subvariety of $\mathcal{RL}$ of particular interest is the variety $\mathcal{CRL}$ of commutative residuated lattices, which satisfies the equation $xy \approx yx$, and hence the equation $x\backslash y \approx y/x$. We always think of this variety as a subvariety of $\mathcal{RL}$, but we slightly abuse notation by listing only one occurrence of the operation $\backslash$ in describing their members.

Given an RL $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, e)$, an element $a \in A$ is said to be *integral* if $e/a = e = a\backslash e$, and $\mathbf{A}$ itself is said to be *integral* if every member of A is integral; this is equivalent to e being its top element. We denote by $\mathcal{IRL}$ the variety of all integral RLs. We call a residuated lattice *e-cyclic* if it satisfies the identity $e/x \approx x\backslash e$. Two important subvarieties of e-cyclics residuated lattices are the variety $\mathcal{CRL}$ of commutative residuated lattices and the variety $\mathcal{IRL}$ of integral residuated lattices.

Another very important subvariety of e-cyclic residuated lattices is the variety of ℓ-groups, which occupies a very special place among the varieties of residuated lattices. An element $a \in A$ is said to be *invertible* if $(e/a)a = e = a(a\backslash e)$. This is of course true if and only if a has a (two-sided) inverse a^{-1}, in which case $e/a = a^{-1} = a\backslash e$. The RLs in which every element is invertible are precisely the ℓ-groups. Perhaps a word of caution is appropriate here. An *ℓ-group* is usually defined in the literature as an algebra $\mathbf{G} = (G, \wedge, \vee, \cdot, {}^{-1}, e)$ such that $(G, \wedge, \vee)$ is a lattice, $(G, \cdot, {}^{-1}, e)$ is a group, and multiplication is order preserving (or, equivalently, it distributes over the lattice operations, see [5], [45]). The variety of ℓ-groups is term equivalent to the subvariety $\mathcal{LG}$ of $\mathcal{RL}$ defined by the equations $(e/x)x \approx e \approx x(x\backslash e)$; the term equivalence is given by $x^{-1} = e/x$, $x/y = xy^{-1}$, and $x\backslash y = x^{-1}y$. Throughout this dissertation, the members of this subvariety will be referred to as ℓ-groups *simpliciter*.

If F is a non-empty subset of a residuated lattice $\mathbf{L}$, we write F^- for the set of *negative* elements of F, that is, $F^- = \{x \in F : x \leqslant e\}$. The *negative cone* of **L** is the integral residuated lattice $\mathbf{L}^-$ with domain L^-, monoid and lattice operations the restrictions to L^- of the corresponding operations in **L**, and residuals $\backslash^-$ and $/^-$ defined by

$$x\backslash^- y = (x\backslash y) \wedge e \quad \text{and} \quad y/^- x = (y/x) \wedge e,$$

where $\backslash$ and $/$ denote the residuals in **L**.

An important variety of pointed residuated lattices is the variety of MV algebras. The class $\mathcal{MV}$ is the subvariety of commutative pointed residuated lattices axiomatized by $0 \wedge x \approx 0$ and $(x\backslash y)\backslash y \approx x \vee y$. They are the equivalent algebraic semantics of Łukasievic's infinite-valued logic (see [21] and [23]). It turns out that MV algebras are integral. Moreover, they have a very tied relation with Abelian (i.e., commutative) ℓ-groups, as is proven in [73]. Dropping some of the properties of MV algebras we can extend $\mathcal{MV}$ to a largest variety called $\mathcal{GMV}$, which stands in a similar tied relation with respect to all ℓ-groups. First, we drop the constant 0 (and therefore the axiom $0 \leqslant x$). Then we drop commutativity and give up integrality, what leads us to rewriting the axiom $(x\backslash y)\backslash y \approx x \vee y$ as:

$$(y/(x \vee y))\backslash y \approx x \vee y \approx y/((x \vee y)\backslash y). \tag{GMV}$$

Thus, a *generalized MV algebra*, or GMV algebra, is a residuated lattice satisfying (GMV). Obviously, the class $\mathcal{LG}$ is a subvariety of $\mathcal{GMV}$. It is essential to note that GMV algebras are e-cyclic and have distributive lattice reducts [41, Lemma 2.9]. Another class very important for our study will be the class of integral GMV algebras, $\mathcal{IGMV}$. It can be shown that this class is actually axiomatized, relative to $\mathcal{IRL}$, by the equations:

$$(y/x)\backslash y \approx x \vee y \approx y/(x\backslash y). \tag{IGMV}$$

It is proved in [41] that any GMV algebra is a direct sum of an ℓ-group and an integral GMV algebra.

To end this section, we introduce the class $\mathcal{LG}^-$ of *negative cones of ℓ-groups*, which is the subvariety of $\mathcal{IGMV}$, axiomatized relative to $\mathcal{IGMV}$ (see [7, Theorem 6.2]) by the equations:

$$x\backslash xy \approx y \approx yx/x. \tag{LG$^-$}$$

1.3 Nuclei on Residuated Lattices

A *closure operator* on a partial order set $(P, \leqslant)$ is a map $\gamma : P \to P$ that is order-preserving, extensive ($x \leqslant \gamma(x)$, for all $x \in P$), and idempotent ($\gamma \circ \gamma = \gamma$). Every closure operator γ has an associated *closure system* $P_\gamma = \{\gamma(x) : x \in P\}$, whose elements are called the γ-*closed* elements. The dual notion is that of an *interior operator*, which is an order-preserving, contractive ($\delta(x) \leqslant x$, for all $x \in P$), and idempotent map $\delta : P \to P$. The associated *interior system* is $P_\delta = \{\delta(x) : x \in P\}$.

A *nucleus* on an RL $\mathbf{A}$ is a closure operator γ on $\mathbf{A}$ satisfying the following condition: for all $a, b \in A$,

$$\gamma(a)\gamma(b) \leqslant \gamma(ab),$$

or equivalently, for all $a, b \in A$,

$$\gamma(\gamma(a)\gamma(b)) = \gamma(ab).$$

If $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, e)$ is an RL and γ is a nucleus on $\mathbf{A}$, the image A_γ of γ can be endowed with an RL structure as follows:

$$\mathbf{A}_\gamma = (A_\gamma, \wedge, \vee_\gamma, \cdot_\gamma, \backslash, /, \gamma(e)),$$

where

$$\gamma(a) \vee_\gamma \gamma(b) = \gamma(a \vee b), \quad \text{and} \quad \gamma(a) \cdot_\gamma \gamma(b) = \gamma(a \cdot b).$$

$\mathbf{A}_\gamma$ is called a *nucleus retract* of $\mathbf{A}$.

Nuclei on GMV algebras have a few special properties. In fact, if $\mathbf{A}_\gamma$ is a nucleus retract of an (integral) GMV algebra, then $\vee_\gamma = \vee$, $\gamma(e) = e$, and

$$\mathbf{A}_\gamma = (A_\gamma, \wedge, \vee, \cdot_\gamma, \backslash, /, e)$$

is again an (integral) GMV algebra in its own right. In particular, it follows on the one hand that nuclei on GMV algebras are lattice homomorphisms, and on the other, that nucleus retracts of negative cones of ℓ-groups (*qua* instances of IGMV algebras) are themselves IGMV algebras.

1.4 Convex Subalgebras of Residuated Lattices

In this section we list a few relevant properties of convex subalgebras of e-cyclic residuated lattices. Their proofs and additional discussion can be found in [15].

A subset C of a poset $\mathbf{P} = (P, \leqslant)$ is *order-convex* (or simply *convex*) in $\mathbf{P}$ if for every $a, b, c \in P$, whenever $a, c \in C$ with $a \leqslant b \leqslant c$, then $b \in C$. For a residuated lattice $\mathbf{L}$, we write $\mathcal{C}(\mathbf{L})$ for the set of all convex subalgebras of $\mathbf{L}$, partially ordered by set-inclusion. In fact, refer to the discussion below, it can be shown that $\mathcal{C}(\mathbf{L})$ is an algebraic and complete lattice (see Theorem 1.5).

For any $S \subseteq L$, we let $C[S]$ denote the smallest convex subalgebra of $\mathbf{L}$ containing S. As is customary, we call $C[S]$ the convex subalgebra *generated* by S and let $C[a] = C[\{a\}]$. We refer to $C[a]$ as the *principal* convex subalgebra of $\mathbf{L}$ generated by the element a. The principal convex subalgebras of $\mathcal{C}(\mathbf{L})$ are the compact members of $\mathcal{C}(\mathbf{L})$, since by Lemma 1.4.(3) below, every finitely generated convex subalgebra of $\mathbf{L}$ is principal.

An important concept in the theory of ℓ-groups is the notion of an absolute value. This idea can be fruitfully generalized in the context of residuated lattices, see for example [15, 77]. Given a residuated lattice $\mathbf{L}$ and an element $x \in L$, the *absolute value* of x is the element

$$|x| = x \wedge (e/x) \wedge e.$$

If $X \subseteq L$, we set $|X| = \{|x| : x \in X\}$. We note that in the case of GMV algebras, $|x| = x \wedge (e/x)$. The proof of the following lemma is routine:

Lemma 1.3. *Let $\mathbf{L}$ be an e-cyclic residuated lattice, $x \in L$, and $a \in L^-$. The following conditions hold:*

(1) $x \leqslant e$ if and only if $|x| = x$;
(2) $|x| \leqslant x \leqslant |x|\backslash e$;
(3) $|x| = e$ if and only if $x = e$;
(4) $a \leqslant x \leqslant a\backslash e$ if and only if $a \leqslant |x|$; and
(5) if $H \in \mathcal{C}(\mathbf{L})$, then $x \in H$ if and only if $|x| \in H$.

In what follows, for a subset S of a residuated lattice $\mathbf{L}$, we write $\widehat{S}$ for the submonoid of $\mathbf{L}$ generated by S. Thus, $x \in \widehat{S}$ if and only if there exist elements $s_1, \ldots, s_n \in S$ such that $x = s_1 \cdots s_n$.

Lemma 1.4. [15] *Let $\mathbf{L}$ be an e-cyclic residuated lattice.*

1. *For $S \subseteq L$,*
$$\begin{aligned} C[S] = C[|S|] &= \left\{x \in L : h \leqslant x \leqslant h\backslash e, \text{ for some } h \in \widehat{|S|}\right\} \\ &= \left\{x \in L : h \leqslant |x|, \text{ for some } h \in \widehat{|S|}\right\}. \end{aligned}$$
2. *For $a \in L$,*
$$\begin{aligned} C[a] = C[|a|] &= \{x \in L : |a|^n \leqslant x \leqslant |a|^n\backslash e, \text{ for some } n \in \mathbb{N}\} \\ &= \{x \in L : |a|^n \leqslant |x|, \text{ for some } n \in \mathbb{N}\}. \end{aligned}$$
3. *For $a, b \in L$, $C[a] \cap C[b] = C[|a| \vee |b|]$ and $C[a] \vee C[b] = C[|a| \wedge |b|] = C[|a||b|]$.*
4. *If H is a convex subalgebra of $\mathbf{L}$, then $H = C[H^-]$.*

Lemma 1.4 yields the following results.

Theorem 1.5. [15] *If* $\mathbf{L}$ *is an e-cyclic residuated lattice, then:*

1. $\mathcal{C}(\mathbf{L})$ *is a distributive algebraic complete lattice.*
2. *The poset* $\mathcal{K}(\mathcal{C}(\mathbf{L}))$ *of compact elements of* $\mathcal{C}(\mathbf{L})$ *– consisting of the principal convex subalgebras of* $\mathbf{L}$ *– is a sublattice of* $\mathcal{C}(\mathbf{L})$.
3. *The lattice* $\mathcal{C}(\mathbf{L})$ *can be embedded (as a complete sublattice) into the congruence lattice of the lattice reduct of* $\mathbf{L}$.

1.5 Normal Convex Subalgebras of Residuated Lattices

We notice that the variety $\mathcal{RL}$ is both congruence permutable (witness the term $(z \vee (z/y)x) \wedge (x \vee (x/y)z))$ and 1-regular (witness the terms $x\backslash y \wedge e$, and $y\backslash x \wedge e$), and recall that any variety which is congruence permutable and 1-regular is, in particular, *ideal determined*: the lattice of congruence relations and the lattice of ideals (in the sense of [47]) of any algebra in the variety are isomorphic. It is proved in [10] (see also [40]) that for any RL $\mathbf{A}$, ideals of $\mathbf{A}$ coincide with convex normal subalgebras of $\mathbf{A}$, which we define in what follows.

Let $\mathbf{L}$ be a residuated lattice. Given an element $u \in L$, we define

$$\lambda_u(x) = (u\backslash xu) \wedge e \quad \text{and} \quad \rho_u(x) = (ux/u) \wedge e,$$

for all $x \in L$. We refer to λ_u and ρ_u as *left conjugation* and *right conjugation* by u. A set $X \subseteq L$ is said to be *normal* if it is closed under conjugates. An *iterated conjugation* map is a composition $\gamma = \gamma_1\gamma_2\ldots\gamma_n$, where each γ_i is a right or a left conjugation by an element $u_i \in L$. The set of all iterated conjugation maps will be denoted by Γ.

The next lemma easily follows from the definition of a normal convex subalgebra.

Lemma 1.6. [10, 53] *For a convex subalgebra* H *of a residuated lattice* $\mathbf{L}$, *the following statements are equivalent:*

(1) *H is normal.*

(2) *H is closed under all iterated conjugation maps.*

(3) *For all* $a, b \in L$, $(a\backslash b) \wedge e \in H$ *if and only if* $(b/a) \wedge e \in H$.

The set of all normal convex subalgebras of $\mathbf{L}$ will be denoted by $\mathcal{NC}(\mathbf{L})$. Given a normal convex subalgebra $\mathbf{H}$ of $\mathbf{L}$, $\Theta_{\mathbf{H}} = \{\langle x, y\rangle \in L^2 : (x\backslash y) \wedge (y\backslash x) \wedge e \in H\}$ is a congruence of $\mathbf{L}$. Conversely, given a congruence Θ, the equivalence class $[e]_\Theta$ is a normal convex subalgebra.

Lemma 1.7 ([10, 53]; see also [95] or [40]). *The lattice $\mathcal{NC}(\mathbf{L})$ of normal convex subalgebras of a residuated lattice* $\mathbf{L}$ *is isomorphic to its congruence lattice* $\mathrm{Con}(\mathbf{L})$. *The isomorphism is given by the mutually inverse maps* $\mathbf{H} \mapsto \Theta_{\mathbf{H}}$ *and* $\Theta \mapsto [e]_\Theta$.

In what follows, if $\mathbf{H}$ is a normal convex subalgebra of $\mathbf{L}$, we write $\mathbf{L}/H$ for the quotient algebra $\mathbf{L}/\Theta_{\mathbf{H}}$, and denote the equivalence class of an element $x \in L$ by $[x]_H$. We mention that $\mathcal{NC}(\mathbf{L})$ is an algebraic distributive lattice, for any residuated lattice $\mathbf{L}$. This can be verified directly, or be derived as a consequence of the fact that $\mathbf{L}$ has a lattice reduct, and hence it is congruence distributive. We also mention the trivial fact that in a commutative residuated lattice, every convex subalgebra is normal.

The following auxiliary result will be useful:

Lemma 1.8. [10, 53] *Let* $\mathbf{L}$ *be an e-cyclic residuated lattice and* $S \subseteq L$. *Consider the set* $\Gamma[|S|] = \{\gamma(a) : a \in |S|, \gamma \in \Gamma\}$. *Then:*

(1) *The normal convex subalgebra* $NC[S]$ *of* $\mathbf{L}$ *generated by* S *is*

$$NC[S] = NC[|S|] = \left\{x \in L : y \leqslant x \leqslant y \backslash e, \text{ for some } y \in \widehat{\Gamma[|S|]}\right\}$$
$$= \left\{x \in L : y \leqslant |x|, \text{ for some } y \in \widehat{\Gamma[|S|]}\right\}$$

(2) *The normal convex subalgebra* $NC[a]$ *of* $\mathbf{L}$ *generated by an element* $a \in L$ *is*

$$NC[a] = NC[|a|] = \left\{x \in L : y \leqslant x \leqslant y \backslash e, \text{ for some } y \in \widehat{\Gamma[|a|]}\right\}$$
$$= \left\{x \in L : y \leqslant |x|, \text{ for some } y \in \widehat{\Gamma[|a|]}\right\}$$

(3) $NC[|a| \vee |b|] \subseteq NC[a] \cap NC[b]$ *and* $NC[a] \vee NC[b] = NC[|a| \wedge |b|]$, *for all* $a, b \in L$.

We state the following lemma, whose proof is routine, for future reference:

Lemma 1.9. *For a convex normal subalgebra* $\mathbf{H}$ *of a residuated lattice* $\mathbf{L}$, *the following statements are equivalent:*

(i) $[a]_H = [e]_H$,

(ii) $a \in H$,

(iii) $C[a] \subseteq H$.

1.6 Pseudo-Complemented Lattices

A *pseudo-complemented lattice* is an algebra $\mathbf{L} = (L, \wedge, \vee, \neg, \top, \bot)$ of signature $(2, 2, 1, 0, 0)$ such that $(L, \wedge, \vee, \top, \bot)$ is a bounded lattice and for all $a \in L$, $\neg a = \max\{x : a \wedge x = \bot\}$.

We refer to $\neg a$ as the *pseudo-complement* of a. The map $\neg : L \to L$ is a self-adjoint order-reversing map, while the map sending a to its double pseudo-complement $\neg\neg a$ is a meet-preserving closure operator on $\mathbf{L}$. By a classic result due to Glivenko, the image of this closure operator is a Boolean algebra[5] $\mathbf{B_L}$ with least element $\perp$ and largest element $\top$. Any existing meets in $\mathbf{B_L}$ coincide with those in $\mathbf{L}$; the complement of a in $\mathbf{B_L}$ is precisely $\neg a$, whereas, for any pair of elements a, b of $\mathbf{B_L}$ – also referred to as *closed elements* of L –,

$$a \vee^{\mathbf{B_L}} b = \neg(\neg a \wedge \neg b).$$

A *Stonean lattice* isa pseudo-complemented lattice $\mathbf{L}$ such that for all $a \in L$, $\neg a \vee \neg\neg a = \top$. It can be easily seen that $\mathbf{L}$ is a Stonean lattice if and only if $\mathbf{B_L}$ is a sublattice of $\mathbf{L}$. Thus, in this case $\mathbf{B_L}$ coincides with the Boolean algebra of complemented elements of $\mathbf{L}$.

A *relatively pseudo-complemented lattice* is an algebra $\mathbf{A} = (A, \wedge, \vee, \to, \top)$ of signature $(2, 2, 2, 0)$ such that $(A, \wedge, \vee, \top)$ is a distributive lattice with top element $\top$ and for all $a, b, c \in A$, $a \wedge b \leqslant c$ if and only if[6] $b \leqslant a \to c$. Given $a, b \in A$, thus, $a \to b$ is the *relative pseudo-complement* of a with respect to b, namely, the greatest x such that $a \wedge x \leqslant b$. A *Heyting algebra* is an algebra $\mathbf{A} = (A, \wedge, \vee, \to, \top, \perp)$ of signature $(2, 2, 2, 0, 0)$ such that $(A, \wedge, \vee, \to, \top)$ is a relatively pseudo-complemented lattice and $\perp$ is a bottom element with respect to the lattice ordering of $\mathbf{A}$. Observe that the $(\wedge, \vee, \neg, \top, \perp)$-term reduct of a Heyting algebra, with $\neg a = a \to \perp$, is, in particular, a pseudo-complemented lattice. A *Boolean algebra* is a Heyting algebra in which the equation $\neg\neg x \approx x$ is valid.

We notice that the class of relatively pseudo-complemented lattices is a variety that is term equivalent to the subvariety $\mathcal{RPCL}$, which is axiomatized relative to $\mathcal{IRL}$ by the identity $xy \approx x \wedge y$. Clearly, $\mathcal{RPCL}$ is also a subvariety of $\mathcal{CRL}$.

A *Gödel algebra* is a Heyting algebra satisfying the equation $(x \to y) \vee (y \to x) \approx \top$ called *prelinearity*. Similarly, a *positive Gödel algebra* is a relatively pseudo-complemented lattice satisfying the same equation. It is important to recall that each interval $[b, \top]$ in a positive Gödel algebra can be made into a Stonean lattice by letting $\neg_b x = x \to b$ for all $x \in [b, \top]$.

Gödel algebras play a prominent role in Algebraic Logic because they are the equivalent variety semantics of *Gödel logic* (also known as *Dummett's logic*, or *Dummett's* **LC**), which is both an intermediate logic (i.e. an extension of intuitionistic logic) and a fuzzy logic. As an intermediate logic, it stands out for its being sound and complete with re-

[5]See the definition of Boolean algebra below.

[6]Notice that what this equivalence is saying is that the meet operation $\wedge$, which is commutative, is residuated and its residual is $\to$.

spect to linearly ordered Kripke models, and as such it received considerable attention. **LC** has been widely investigated also within the community of mathematical fuzzy logic – it was observed early on that the variety of Gödel algebras is generated by the algebra

$$([0,1], \wedge, \vee, \to, 1, 0),$$

where $\wedge$ and $\vee$ are the minimum t-norm and the maximum t-conorm respectively, while $\to$, the residual of $\wedge$, behaves as follows: for all $a, b \in [0,1]$,

$$a \to b = \begin{cases} 1 & \text{if } a \leqslant b \\ b & \text{otherwise.} \end{cases}$$

Observe that every bounded chain admits a *unique* Gödel implication, given by the above case-splitting definition. In particular, in every linearly ordered Gödel algebra $a \to b$ is $\top$ if $a \leqslant b$, and is b if $a > b$.

Every algebraic distributive lattice $\mathbf{L}$ satisfies the join-infinite distributive law: for every $a \in L$ and $\{b_i : i \in I\} \subseteq L$,

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i),$$

and hence it is relatively pseudo-complemented. That is to say, for every $a, b \in L$, the relative pseudo-complement of a with respect to b exists and is given by

$$a \to b = \bigvee \{x \in L : a \wedge x \leqslant b\}.$$

As a matter of fact, $\mathbf{L}$ has a bottom element and so it is a Heyting algebra. Moreover, we recall the following result.

Lemma 1.10. [8, Chapter IX, Theorem 8] *If* $\mathbf{L}$ *is a Heyting algebra, every interval* $[b, a]$ *in* $\mathbf{L}$*, with* $b \leqslant a$*, is pseudo-complemented and, for all* $c \in [b, a]$*, the pseudo-complement and the double pseudo-complement of* c *are respectively given by:*

$$\neg c = (c \to b) \wedge a \quad \textit{and} \quad \neg\neg c = ((c \to b) \to b) \wedge a.$$

Thus, every interval in an algebraic distributive lattice is pseudo-complemented.

1.7 Polars and Convex Subalgebras

As we have mentioned in Section 1.4, the lattice $\mathcal{C}(\mathbf{L})$ of convex subalgebras of an e-cyclic residuated lattice $\mathbf{L}$ is an algebraic distributive lattice and hence the results of the previous section apply. In particular, it is relatively pseudo-complemented. For all $X, Y \in \mathcal{C}(\mathbf{L})$, the relative pseudo-complement $X \to Y$ of X relative to Y is given by:

$$X \to Y = \max\{Z \in \mathcal{C}(\mathbf{L}) : X \cap Z \subseteq Y\}.$$

The next lemma provides an element-wise description of $X \to Y$ in terms of the absolute value, and in particular one for the pseudo-complement $X^{\perp} = X \to \{e\}$ of X.

Lemma 1.11. *If $\mathbf{L}$ is an e-cyclic residuated lattice, then $\mathcal{C}(\mathbf{L})$ is a relatively pseudo-complemented lattice. Specifically, given $X, Y \in \mathcal{C}(\mathbf{L})$,*

$$X \to Y = \{a \in L : |a| \vee |x| \in Y, \text{ for all } x \in X\}, \tag{1.1}$$

and in particular,

$$X^{\perp} = \{a \in L : |a| \vee |x| = e, \text{ for all } x \in X\}. \tag{1.2}$$

For any subset $X \subseteq L$, we define the $X^{\perp}$ as in Equation (1.2). It can be easily seen that $X^{\perp} = C[X]^{\perp}$, so $X^{\perp}$ is always a convex subalgebra. We refer to $X^{\perp}$ as the *polar* of X; in case $X = \{x\}$, we write $x^{\perp}$ instead of $\{x\}^{\perp}$ (or $C[x]^{\perp}$) and refer to it as the *principal polar* of x. Furthermore, notice that for every $X \subseteq L$, $X^{\perp} = |X|^{\perp}$, by virtue of Lemma 1.4.(1).

We state the following lemma for future reference:

Lemma 1.12. *If $\mathbf{L}$ is an e-cyclic residuated lattice, then for every $x, y \in L$,*

$$(|x| \vee |y|)^{\perp\perp} = x^{\perp\perp} \cap y^{\perp\perp}.$$

Proof. By virtue of Lemma 1.4:

$$\begin{aligned}(|x| \vee |y|)^{\perp\perp} &= C[|x| \vee |y|]^{\perp\perp} = (C[x] \cap C[y])^{\perp\perp} = C[x]^{\perp\perp} \cap C[y]^{\perp\perp} \\ &= x^{\perp\perp} \cap y^{\perp\perp}.\end{aligned}$$ □

The map ${}^{\perp} : \mathcal{C}(\mathbf{L}) \to \mathcal{C}(\mathbf{L})$ is a self-adjoint inclusion-reversing map, while the map sending $H \in \mathcal{C}(\mathbf{L})$ to its double polar $H^{\perp\perp}$ is an intersection-preserving closure operator on $\mathcal{C}(\mathbf{L})$. Therefore, a set H is a polar if and only if $H = H^{\perp\perp}$. By Glivenko's classical result, the image of this operator is a (complete) Boolean algebra[7] $\mathrm{Pol}(\mathbf{L})$ with least element $\{e\}$ and largest element L. The complement of H in $\mathrm{Pol}(\mathbf{L})$ is $H^{\perp}$ and for any family $\{H_i : i \in I\}$ in $\mathrm{Pol}(\mathbf{L})$

$$\bigvee_{i \in I}^{\mathrm{Pol}(\mathbf{L})} H_i = \Big(\bigvee_{i \in I}^{\mathcal{C}(\mathbf{L})} H_i\Big)^{\perp\perp} = \Big(\bigcup_{i \in I} H_i\Big)^{\perp\perp}.$$

We refer to $\mathrm{Pol}(\mathbf{L})$ as the *algebra of polars* of $\mathbf{L}$. Thus, $\mathrm{Pol}(\mathbf{L})$ is a complete Boolean algebra whose top and bottom elements are L and $\{e\}$, respectively.

[7] Following the notation of the previous section, this is the algebra $\mathbf{B}_{\mathcal{C}(\mathbf{L})}$.

1.8 Semilinearity

Some prominent varieties of residuated lattices and pointed residuated lattices – including Abelian ℓ-groups and MV-algebras – are generated by their linearly ordered members. We refer to such varieties as *semilinear*,[8] and denote the variety of all semilinear residuated lattices by $\mathcal{SemRL}$. Thus, a residuated lattice is semilinear if and only if it is a subdirect product of totally ordered residuated lattices.

It is well known (see [5]) that the class $\mathcal{RepLG}$ of representable ℓ-groups form a variety and, in fact, it can be axiomatized relative to $\mathcal{LG}$ by the equation:

$$(x^{-1}yx \vee y^{-1}) \wedge e \approx e.$$

An analogous result was shown in [10] and [53]: the class $\mathcal{SemRL}$ is a variety, and it can be axiomatized, relative to $\mathcal{RL}$, by either of the equations below:

$$\lambda_u((x \vee y)\backslash x) \vee \rho_v((x \vee y)\backslash y) \approx e, \tag{SL1}$$

$$\lambda_u(x/(x \vee y)) \vee \rho_v(y/(x \vee y)) \approx e. \tag{SL2}$$

The next theorem generalizes the well-known results on representable ℓ-groups as well as all analogous results characterizing semilinear members of some classes of residuated lattices – see [35] for pseudo-MV algebras, [62] for pseudo-BL algebras, [61] for GBL algebras (DRℓ-monoids), and [92] for integral residuated lattices. The statement of the theorem refers to the following two identities, the so called *left prelinearity law* LP and the *right prelinearity law* RP:

$$((x\backslash y) \wedge e) \vee ((y\backslash x) \wedge e) \approx e, \tag{LP}$$

$$((y/x) \wedge e) \vee ((x/y) \wedge e) \approx e. \tag{RP}$$

Theorem 1.13. [15] *For a variety $\mathcal{V}$ of residuated lattices, the following statements are equivalent:*

(1) $\mathcal{V}$ is semilinear.

(2) $\mathcal{V}$ satisfies either of the equations (SL1) *and* (SL2)*.*

(3) $\mathcal{V}$ satisfies either of the prelinearity laws and the quasi-identity

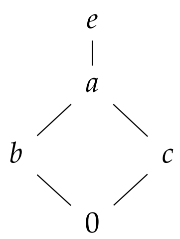
$$x \vee y \approx e \quad \Rightarrow \quad \lambda_u(x) \vee \rho_v(y) \approx e. \tag{1.3}$$

If in addition $\mathcal{V}$ is a variety of e-cyclic residuated lattices, the preceding conditions are equivalent to the condition:

[8] The more traditional, but less descriptive, name for these varieties is *representable*, specially for ℓ-groups, for which we will keep the name, subjugated by the strength of tradition.

(4) *$\mathcal{V}$ satisfies either of the prelinearity laws and for every $\mathbf{L} \in \mathcal{V}$, all (principal) polars in $\mathbf{L}$ are normal.*

It is well known, and easy to prove, that representable ℓ-groups satisfy both prelinearity laws. Thus, in view of the preceding result, a variety of ℓ-groups is semilinear if and only if all polars of every algebra in the variety are normal. Normality of polars alone is not sufficient to imply semilinearity in general. For example, the variety of Heyting algebras satisfies the normality condition on polars, since it is a variety of commutative pointed residuated lattices, but it is not semilinear. For example, the Heyting algebra below is subdirectly irreducible but not totally ordered.



1.9 Projectability

We say that an ℓ-group $\mathbf{A}$ is the *internal cardinal product* of its ℓ-subgroups $\mathbf{B}$ and $\mathbf{C}$, in symbols, $\mathbf{A} = \mathbf{B} \boxplus \mathbf{C}$, if every $a \in A$ can be written uniquely as a product bc, for some $b \in B$ and some $c \in C$, this product commutes, and moreover, given two decomposition $a_1 = b_1c_1$ and $a_2 = b_2c_2$, we have $b_1c_1 \leqslant^{\mathbf{A}} b_2c_2$ if and only if $b_1 \leqslant^{\mathbf{B}} b_2$ and $c_1 \leqslant^{\mathbf{C}} c_2$. An ℓ-group $\mathbf{A}$ is *projectable* whenever for all $a \in A$,

$$\mathbf{A} = a^{\perp} \boxplus a^{\perp\perp}, \tag{1.4}$$

where in the present context $a^{\perp} = \{b \in A : |a| \wedge |b| = e\}$ and $|a| = a \vee a^{-1}$. As proved in [88] and [89], projectable ℓ-groups coincide with ℓ-groups in which all closed intervals form a Stonean lattice, and hence they admit a Gödel implication. This result highlights that projectability is a property of ℓ-groups that is entirely determined by their order structure.

To get further insight into this, recall[9] indeed that, given an ℓ-group $\mathbf{A}$:

(1) principal polars are convex ℓ-subgroups of $\mathbf{A}$;
(2) projectability is equivalent to the property that for all $a \in A$,

$$A = a^{\perp} \vee^{\mathcal{C}(\mathbf{A})} a^{\perp\perp}. \tag{1.5}$$

[9] See [90] for a lengthier discussion of these aspects.

Thus, the projectability of an ℓ-group $\mathbf{A}$ can be described by a structural property as in (1.4) and by an order-theoretical property as in (1.5). In generalizing the notion of projectability to arbitrary e-cyclic residauted lattices, we have two options. Since we do not know whether both notions coincide always, and both of them lead to interesting results, we will give names to both of them and postpone to another time and place the discussion about which one rightfully deserves the plain name of "projectablility" – although we believe we should favor the lattice-theoretical description. Thus, we say that an e-cyclic residuated lattice $\mathbf{L}$ is *$\vee$-projectable* if every principal polar is a complemented element of $\mathcal{C}(\mathbf{L})$. That is, for all $a \in L$,

$$L = a^{\perp} \vee^{\mathcal{C}(\mathbf{L})} a^{\perp\perp}.$$

It is called *strongly $\vee$-projectable* if for every convex subalgebra $H \in \mathcal{C}(\mathbf{L})$,

$$\mathbf{L} = H^{\perp} \vee^{\mathcal{C}(\mathbf{L})} H^{\perp\perp}.$$

Equivalently, $\mathbf{L}$ is strongly $\vee$-projectable if and only if the Boolean algebra of polars $\mathrm{Pol}(\mathbf{L})$ is a sublattice of $\mathcal{C}(\mathbf{L})$, that is $\mathcal{C}(\mathbf{L})$ is a Stonean lattice.

As for the structural description of projectability, it can also be generalized as follows. A residuated lattice $\mathbf{L}$ is said to be the *internal cardinal product* of its convex subalgebras $\mathbf{B}$ and $\mathbf{C}$ – in symbols, $\mathbf{L} = \mathbf{B} \boxplus \mathbf{C}$ – if every $a \in L$ can be written uniquely as a product bc, for some $b \in B$ and some $c \in C$, this product commutes, and moreover, $a_1 = b_1c_1 \leqslant^{\mathbf{L}} b_2c_2 = a_2$ if and only if $b_1 \leqslant^{\mathbf{B}} b_2$ and $c_1 \leqslant^{\mathbf{C}} c_2$. An e-cyclic residuated lattice is *$\boxplus$-projectable* if for every $a \in L$,

$$L = a^{\perp} \boxplus a^{\perp\perp}.$$

It is called *strongly $\boxplus$-projectable* if for every convex subalgebra $H \in \mathcal{C}(\mathbf{L})$,

$$\mathbf{L} = H^{\perp} \boxplus H^{\perp\perp}.$$

Evidently, (strong) $\boxplus$-projectability implies (strong) $\vee$-projectability, since for every $H, K \in \mathcal{C}(\mathbf{L})$, $H \boxplus K \subseteq H \vee^{\mathcal{C}(\mathbf{L})} K$. It can be shown that, under certain hypothesis, both notions actually coincide, as is the case for ℓ-groups, negative cones of ℓ-groups, integral semilinear residuated lattices, and IGMV algebras (see [66]), for instance.

Chapter 2

Projectable ℓ-groups and Algebras of Logic: Categorical and Algebraic Connections

2.1 Introduction

In the 1960's, P. F. Conrad launched a general program for the investigation of lattice-ordered groups ([25], [26], [27], [28]), aimed at capturing relevant information about these algebras by inquiring into the structure of their lattices of convex ℓ-subgroups (as opposed to convex *normal* ℓ-subgroups, which had traditionally received greater attention in that they bijectively correspond to congruences). The chief idea behind this program is a working hypothesis to the effect that many significant properties of ℓ-groups are, in essence, either purely lattice-theoretical, or at least such that the underlying group structure does not play a predominant role. A class of ℓ-groups that is known to be characterized purely in terms of its order structure is the class of *projectable* ℓ-groups – namely, ℓ-groups in which every principal polar is a cardinal summand.[1] C. Tsinakis, in fact, has established that an ℓ-group is projectable if and only if each one of its intervals is a Stonean lattice; as a consequence, projectability is preserved under lattice isomorphisms. Also, the negative cone of an ℓ-group is projectable if and only if its lattice reduct can be endowed with a positive Gödel implication ([88], [89]).

[1]Projectable ℓ-groups are first-class citizens in the theory of lattice-ordered groups: recall, for example, that every representable ℓ-group can be embedded into a member of this class [9, 20, 29] and that conditionally σ-complete ℓ-groups are projectable. Further examples arise in functional analysis, namely, vector lattices with the principal projection property [68].

While Conrad's program led to remarkable outcomes in its original domain of application (for a survey, see [4]), a natural continuation of such consists in extending it to *residuated lattices* ([40, 72]), generalizations of ℓ-groups that also include MV algebras, Heyting algebras, and several other classes of algebras of prime importance for mathematical logic.[2] Here, the principal objects of research become the lattices of convex subalgebras (in the integral case, the lattices of multiplicative filters). Some detailed investigations along these lines have been carried out in recent years [15]. One of the results obtained so far within this extended Conrad's program [66] is a characterization of projectability for integral and distributive residuated lattices satisfying the quasiequation:

$$x \vee y \approx e \quad \Rightarrow \quad xy \approx x \wedge y,$$

which closely matches the aforementioned description of projectable ℓ-groups. Ledda, Paoli, and Tsinakis have indeed shown that a member of this class is projectable if and only if the order dual of each interval $[a, e]$ is a Stonean lattice.

In general, for integral and distributive residuated lattices, admitting a positive Gödel implication is a stronger condition than being projectable [66, Example 15], although it is equivalent in some especially well-behaved cases. A case in point is given by *integral GMV algebras (IGMV algebras)* [41], simultaneous generalizations of MV algebras to the unbounded and noncommutative case. IGMV algebras, to within isomorphism, can be viewed as nucleus retractions of negative cones of ℓ-groups – actually, it was shown in [41] that the categories of IGMV algebras and negative cones of ℓ-groups with a nucleus are equivalent. It is then natural to conjecture that such an equivalence restricts to an equivalence of the subcategories whose objects are the projectable members of these classes of algebras, and perhaps that we can take advantage of the previously cited lattice-theoretical description of projectable IGMV algebras to establish this result. The main aim of this chapter is to investigate the extent to which this conjecture is correct.

The chapter is structured as follows. In Section 2.2, we go over some preliminary notions needed in the sequel. In Section 2.3, we show that an analogue of the Galatos-Tsinakis equivalence result can be reproduced in our setting:

Theorem A (See Theorem 2.14.). *The categories of projectable IGMV algebras and of negative cones of projectable ℓ-groups with a nucleus are equivalent.*

A crucial step in establishing Theorem A is showing that any projectable IGMV algebra can be represented as a nucleus retract of the negative cone of some projectable

[2] See Section 1.2.

ℓ-group. In the same section, we also introduce *Gödel GMV algebras* as expansions of projectable IGMV algebras by a binary term that realizes a positive Gödel implication in every such algebra; in light of the above, Gödel GMV algebras and projectable IGMV algebras amount to essentially the same thing. Similarly, *Gödel negative cones* are those Gödel GMV algebras whose RL reducts are negative cones of ℓ-groups. Including the Gödel implication in the signature enables us to view the above-mentioned classes of algebras as *varieties* in the expanded type, with all the familiar benefits that result in similar cases. In Section 2.4, we point out the exact relationship between these notions:

Theorem B (See Theorem 2.28.)**.** *There is an adjunction between the categories whose objects are, respectively, Gödel GMV algebras and Gödel negative cones with a retraction and a dense nucleus on the image of the retraction.*

2.2 Background

2.2.1 Compactly Stonean Lattices

Recall from Section 1.6 that a pseudo-complemented lattice $\mathbf{L} = (L, \wedge, \vee, \neg, \bot, \top)$ is called a *Stonean lattice* if for all $a \in L$, $\neg a \vee \neg\neg a = \top$, that $\mathbf{L}$ is a Stonean lattice if and only if $\mathbf{B_L}$ is a sublattice of $\mathbf{L}$, and that in this case $\mathbf{B_L}$ coincides with the Boolean algebra of complemented elements of $\mathbf{L}$.

Also remember that *positive Gödel algebras* and *Gödel algebras* are obtained from relatively pseudo-complemented lattices and Heyting algebras, respectively, by imposing the the equation $(x \to y) \vee (y \to x) \approx \top$, and each interval $[b, \top]$, which we will also denote by $\uparrow b$, in a (positive) Gödel algebra can be made into a Stonean lattice by letting $\neg_b x = x \to b$ for all $x \in [b, \top]$.

As we said before, every algebraic distributive lattice $\mathbf{L}$ is a Heyting algebra, in which the relative pseudo-complement of a with respect to b is given by $a \to b = \bigvee\{x \in L : a \wedge x \leqslant b\}$, and we have the following result.

Lemma 2.1. [8, Chapter IX, Theorem 8] *If $\mathbf{L}$ is a Heyting algebra, every interval $[b, a]$ in $\mathbf{L}$, with $b \leqslant a$, is pseudo-complemented and, for all $c \in [b, a]$, the pseudo-complement and the double pseudo-complement of c are respectively given by:*

$$\neg c = (c \to b) \wedge a \quad \textit{and} \quad \neg\neg c = ((c \to b) \to b) \wedge a.$$

Thus, every interval in an algebraic distributive lattice is pseudo-complemented. In particular, if $a = \top$ then $\uparrow b$ is itself an algebraic distributive lattice and the compact elements in $\uparrow b$ are exactly those of the form $b \vee c$, for c a compact element of L. Recall that we denote by $\mathcal{K}(\mathbf{L})$ the set of all compact[3] elements of the lattice $\mathbf{L}$.

An algebraic distributive lattice $\mathbf{L}$ is called *compactly Stonean* if it satisfies $\neg c \vee \neg\neg c = \top$, for all $c \in \mathcal{K}(\mathbf{L})$. Observe that a compactly Stonean lattice need not be Stonean. In view of Lemma 2.1, $\uparrow b$ is compactly Stonean if and only if, for all $c \in \mathcal{K}(\mathbf{L})$,

$$(c \to b) \vee ((c \to b) \to b) = \top.$$

It is shown in [66, Proposition 19] that:

Lemma 2.2. *Let $\mathbf{L}$ be an algebraic distributive lattice whose compact elements form a sublattice $\mathcal{K}(\mathbf{L})$ of $\mathbf{L}$. The conditions below are equivalent:*

(1) for all $b \in L$, $\uparrow b$ is compactly Stonean;
(2) for all $b \in L$ and for all $c \in \mathcal{K}(\mathbf{L})$, $(c \to b) \vee ((c \to b) \to b) = \top$;

and imply the mutually equivalent conditions:

(3) for all $c, b \in \mathcal{K}(\mathbf{L})$, $(c \to b) \vee ((c \to b) \to b) = \top$;
(4) for all $a, b \in \mathcal{K}(\mathbf{L})$, with $b \leqslant a$, $[b, a] \cap \mathcal{K}(\mathbf{L})$ is a Stonean lattice.

The next result will turn out to be useful in what follows.

Proposition 2.3. *Let $\mathbf{L}$ and $\mathbf{M}$ be isomorphic algebraic and distributive lattices such that $\mathcal{K}(\mathbf{L})$ and $\mathcal{K}(\mathbf{M})$ are subuniverses of $\mathbf{L}$ and $\mathbf{M}$, respectively. Suppose $\varphi : \mathbf{L} \to \mathbf{M}$ is an isomorphism. Then:*

1. *φ preserves pseudo-complements.*
2. *$\mathbf{L}$ is compactly Stonean if and only if $\mathbf{M}$ is such.*
3. *For $a \in L$, $\neg a$ is complemented if and only if $\varphi(\neg a) = \neg\varphi(a)$ is complemented.*

Proof.
(1) Clearly, φ restricts to an isomorphism between the sublattices of $\mathbf{L}$ and $\mathbf{M}$ with respective universes $\mathcal{K}(\mathbf{L})$ and $\mathcal{K}(\mathbf{M})$. Now, let $a \in L$. Since $a \wedge \neg a = \bot^{\mathbf{L}}$, $\varphi(a) \wedge \varphi(\neg a) = \bot^{\mathbf{M}}$, whence $\varphi(\neg a) \leqslant \neg\varphi(a)$. For the converse inequality, $\varphi(a) \wedge \neg\varphi(a) = \bot^{\mathbf{M}}$ implies $a \wedge \varphi^{-1}(\neg\varphi(a)) = \bot^{\mathbf{L}}$, which means $\varphi^{-1}(\neg\varphi(a)) \leqslant \neg a$ and thus $\neg\varphi(a) \leqslant \varphi(\neg a)$.

(2) If $\mathbf{L}$ is compactly Stonean, then by (1), given $a \in \mathcal{K}(\mathbf{M})$,

$$\neg\varphi(a) \vee \neg\neg\varphi(a) = \varphi(\neg a \vee \neg\neg a) = \varphi(\top^{\mathbf{L}}) = \top^{\mathbf{M}}.$$

(3) Immediate from the preceding items. □

[3] See Section 1.1.

2.2.2 Filters in Integral Residuated Lattices

Let $\mathbf{A}$ be a residuated lattice. A *multiplicative filter* F of $\mathbf{A}$ is a filter of its lattice reduct that is closed under multiplication. A subset $X \subseteq A$ (not necessarily a filter) is *normal* provided that it is closed under all conjugations,[4] that is for all $b \in X$ and $a \in A$, $\rho_a(b) = (ab/a) \wedge e$ and $\lambda_a(b) = (a\backslash ba) \wedge e$ are in X. As we mentioned in Section 1.5, $\mathcal{RL}$ is a 1-regular variety, and hence congruences of an RL $\mathbf{A}$ correspond to ideals of $\mathbf{A}$, which coincide with convex normal subalgebras of $\mathbf{A}$. If $\mathbf{A}$ is integral, these coincide, in turn, with normal multiplicative filters of $\mathbf{A}$.

Now, let $\mathbf{A}$ be an IRL. If $X \subseteq A$, we denote by $\uparrow_{\mathbf{A}} X$ (respectively, $\langle X\rangle_{\mathbf{A}}$, $N_{\mathbf{A}}(X)$) the lattice filter (respectively multiplicative filter, normal multiplicative filter) generated in $\mathbf{A}$ by X. Subscripts will only be dropped when $\mathbf{A}$ is understood; on the other hand, braces will be invariably omitted if $X = \{a\}$ is a singleton. $\mathrm{LF}(\mathbf{A})$, $\mathrm{MF}(\mathbf{A})$, $\mathrm{NF}(\mathbf{A})$ will respectively refer to the lattices of lattice filters, multiplicative filters and normal multiplicative filters (hereafter shortened to *normal filters*) of $\mathbf{A}$. With a mild abuse of notation, the same labels will sometimes be employed for the universes of such lattices. We set:

$$F \vee^L G = \uparrow(F \cup G); \quad F \vee^M G = \langle F \cup G\rangle; \quad \text{and} \quad F \vee^N G = N(F \cup G).$$

However, since the focus of the present chapter is on multiplicative filters, we will often write $F \vee G$ for $F \vee^M G$. $\mathrm{LF}(\mathbf{A})$, $\mathrm{MF}(\mathbf{A})$, and $\mathrm{NF}(\mathbf{A})$ are algebraic and distributive (hence relatively pseudo-complemented) lattices; the result for $\mathrm{MF}(\mathbf{A})$ is proved in [15]. In the case of integral residuated lattices, Lemma 1.4 takes the following form:

Lemma 2.4. $\langle X\rangle = \{a \in A : (b_1 \cdots b_k)^n \leqslant a$, *for some* $b_1, \ldots, b_k \in X$ *and* $n \in \mathbb{N}\}$.

An *iterated conjugation map* is a composition $\gamma = \gamma_1 \circ \cdots \circ \gamma_n$, where each γ_i is a right conjugate or a left conjugate by an element $a_i \in A$. If $X \subseteq A$, we denote by Γ the set of all iterated conjugation maps on $\mathbf{A}$, and by $\widehat{X}$ the submonoid of the corresponding reduct of $\mathbf{A}$ generated by the set $\{\gamma(a) : a \in X, \gamma \in \Gamma\}$. With this notation at hand, we notice notice that the following result follows from Lemma 1.6 in the case of integral residuated lattices (see also [72, Proposition 4.24]):

Lemma 2.5. $N(X) = \{a \in A : b \leqslant a$, *for some* $b \in \widehat{X}\}$.

We now introduce a technique for defining multiplicative filters out of arbitrary subsets of the universe of a given IRL $\mathbf{A}$. Given $X \subseteq A$, by the integrality of $\mathbf{A}$, the

[4]See Section 1.5.

polar $X^\perp$ of X is the set

$$\{y \in A : x \vee y = e \text{ for every } x \in X\}.$$

Again, whenever $X = \{a\}$ is a singleton, we will shorten $\{a\}^\perp$ to $a^\perp$ and call the latter set a *principal polar*. In case $\mathbf{A}$ is distributive, we have that [66, Lemma 8 and Corollary 9]:

Lemma 2.6. *For all $X \subseteq A$, $X^\perp \in \mathrm{MF}(\mathbf{A})$. Moreover, $X^\perp$ is the pseudo-complement of $\uparrow X$ in $\mathrm{LF}(\mathbf{A})$ (respectively, the pseudo-complement of $\langle X \rangle$ in $\mathrm{MF}(\mathbf{A})$).*

On the other hand, given an arbitrary $X \subseteq A$, $X^\perp$ need not be a *normal* filter of $\mathbf{A}$; if it is, then it is the pseudo-complement of $N(X)$ in $\mathrm{NF}(\mathbf{A})$.

Lemma 2.7. $(\uparrow a)^\perp = \langle a \rangle^\perp = a^\perp$.

Proof. Use Lemmas 2.4 and 2.5 above. □

2.2.3 Projectable Integral Residuated Lattices

Recall from Section 1.9 that there are two rightful candidates to be the generalization of the concept of projectability for arbitrary e-cyclic residuated lattices, according to the structural and lattice-theoretic characterizations of projectability for ℓ-groups: We say that an e-cyclic residuated lattice $\mathbf{A}$ is $\vee$-*projectable* if for every $a \in A$,

$$L = a^\perp \vee^{\mathcal{C}(\mathbf{L})} a^{\perp\perp},$$

where $\mathcal{C}(\mathbf{A})$ is the Heyting algebra of the convex subalgebras of $\mathbf{A}$; and $\mathbf{A}$ is $\boxplus$-*projectable* if for every $a \in A$,

$$L = a^\perp \boxplus a^{\perp\perp}.$$

In [66], the equivalence between $\boxplus$-projectability and $\vee$-projectability, which holds for the cases of ℓ-groups and negative cones of ℓ-groups, has been extended to the class $\mathcal{A}$ of IRLs satisfying that quasi-equation:

$$x \vee y \approx e \rightarrow xy \approx x \wedge y. \tag{2.1}$$

Throughout this subsection, unless otherwise specified, we will assume that $\mathbf{A}$ is a member of $\mathcal{A}$. Thus, for the members of $\mathcal{A}$, $\boxplus$-projectability coincides with $\vee$-projectability – which is in general weaker – and therefore, it can be "captured" by the filter lattice of the underlying lattice-structure.

Lemma 2.8. *If $\mathbf{A}$ is in the class $\mathcal{A}$ and is $\boxplus$-projectable, then:*

1. $a^{\perp} \in \mathrm{NF}(\mathbf{A})$;
2. $\mathbf{A} = a^{\perp} \vee^{L} a^{\perp\perp} = a^{\perp} \vee^{M} a^{\perp\perp}$.

Lemma 2.2 can be put to good use by applying it to the lattice $\mathrm{LF}(\mathbf{A})$ of lattice filters of our $\mathbf{A} \in \mathcal{A}$. In fact, if $\mathbf{A}$ is projectable, then the compact elements of the lattice $\mathrm{LF}(\mathbf{A})$ of the lattice filters of $\mathbf{A}$ are its principal filters, whereby $\mathrm{LF}(\mathbf{A})$ is compactly Stonean. This implies that each interval $[\{e\}, \uparrow a]$ in the sublattice of principal lattice filters of $\mathbf{A}$ is a Stonean lattice. In light of the order reversing isomorphism between the lattice reduct of $\mathbf{A}$ and the sublattice of principal filters in $\mathrm{LF}(\mathbf{A})$, then, for all $a \in H$ the order dual of each interval $[a, e]$ is a Stonean lattice. In sum:

Theorem 2.9. [66, Theorem 20] $\mathbf{A}$ *is projectable if and only if the order dual of each interval* $[a, e]$*, for* $a \in A$*, is a Stonean lattice.*

In particular, if $\mathbf{A}$ is an IGMV algebra, we get something more. Every member x of any such interval is a fixpoint of the mapping $f_a(x) = a/(x \backslash a)$, whence the interval is self-dual in the order-theoretic sense. It follows that every interval $[a, e]$, and therefore any arbitrary interval $[a, b]$ (see [8, [Section 8.7, Theorem 13]]), is a Stonean lattice. Thus, following [8, Theorem 10, p. 176], $(A, \wedge, \vee)$ is a relative Stonean lattice and, as such, it can be expanded to a relatively pseudo-complemented lattice, actually a positive Gödel algebra. In conclusion,

Lemma 2.10. *For an IGMV algebra* $\mathbf{A}$*, the following are equivalent:*

1. $\mathbf{A}$ *is projectable;*
2. *The lattice* $(A, \wedge, \vee)$ *can be expanded to a relatively pseudo-complemented lattice, actually a positive Gödel algebra.*

2.2.4 GMV algebras

As we mentioned in Section 1.3, nucleus retracts of negative cones of ℓ-groups are IGMV algebras. In this section, we sketch the construction in [41, Section 3] by means of which Galatos and Tsinakis establish the converse, namely that every IGMV algebra is a nucleus retract of the negative cone of an ℓ-group. This representation theorem is subsequently lifted [41, Section 4] to a full-fledged categorical equivalence between the categories of IGMV algebras and of negative cones of ℓ-groups endowed with a nucleus. This result will be briefly summarized as well.

The first part of the construction relies on an idea by Bosbach, aimed at identifying the purely implicational subreducts of negative cones of ℓ-groups ([14], [13]). A *cone algebra* is an algebra $\mathbf{C} = (C, \backslash, /, e)$, of type $(2, 2, 0)$, that satisfies the identities

C1 $(x\backslash y)\backslash(x\backslash z) \approx (y\backslash x)\backslash(y\backslash z)$
C2 $e\backslash x \approx x$
C3 $x\backslash(y/z) \approx (x\backslash y)/z$
C4 $x\backslash x \approx e$

as well as their mirror images (in the RL sense). The variety of cone algebras will be sometimes referred to as $\mathcal{CA}$. It is easily seen that the $(\backslash, /, e)$-reducts of IGMV algebras are cone algebras. Bosbach shows that the converse holds true too. More precisely:

Proposition 2.11. [13] *Every cone algebra can be embedded into the $(\backslash, /, e)$-reduct of an appropriate member of $\mathcal{LG}^-$.*

Proof. (Sketch). The target negative cone is obtained as a union of an ascending chain $\{\mathbf{C}_n : n < \omega\}$ of cone algebras, each of which is a subalgebra of its successor. Products in the target algebra are constructed stepwise, in such a way that each $\mathbf{C}_{n+1}$ contains products of members of $\mathbf{C}_n$, until all products are finally available in the directed union of the $\mathbf{C}_i$'s.

In greater detail, we proceed as follows. Given a cone algebra $\mathbf{C}$ and elements $(a,b),(c,d)$ in C^2, let

$$(a,b)\backslash\!\!\backslash(c,d) = (b\backslash(a\backslash c), ((a\backslash c)\backslash b)\backslash((c\backslash a)\backslash d)),$$
$$(d,c)/\!\!/(b,a) = ((d/(a/c))/(b/(c/a)), (c/a)/b).$$

The rationale for this definition is given by the fact that $\mathcal{LG}^-$ satisfies the identity

$$xy\backslash zw \approx (y\backslash(x\backslash z)) \cdot (((x\backslash z)\backslash y)\backslash((z\backslash x)\backslash w)) \tag{2.2}$$

and its mirror image, whence the Cartesian product operation, so to speak, acts as an *ersatz* for the RL product and $\backslash\!\!\backslash$, $/\!\!/$ can be viewed as residuals of sorts. Now, the relation

$$\Theta = \{((a,b),(c,d)) : (a,b)\backslash\!\!\backslash(c,d) = (e,e) = (c,d)/\!\!/(a,b)\}$$

is a congruence on $\mathbf{C}^2$, and

$$s(\mathbf{C}) = \mathbf{C}^2/\Theta$$

is a cone algebra containing $\mathbf{C}$ as a subalgebra, via the embedding $\varphi(a) = [(a,e)]_\Theta$. To attain our target negative cone, we run this construction over and over again, letting $\mathbf{C}_0 = \mathbf{C}$ and $\mathbf{C}_{n+1} = s(\mathbf{C}_n)$. In this way, in each $\mathbf{C}_i$, (2.2) is satisfied by the elements of C_j, for every $j \leqslant i-1$. The directed union $\overline{\mathbf{C}} = \bigcup\{\mathbf{C}_n : n < \omega\}$ is a cone algebra that still contains $\mathbf{C}$ as a subalgebra. Moreover, it is the $(\backslash, /, e)$-reduct of the negative cone

$$\overline{\mathbf{C}} = (\overline{C}, \wedge, \vee, \cdot, \backslash^{\overline{\mathbf{C}}}, /^{\overline{\mathbf{C}}}, e^{\overline{\mathbf{C}}}),$$

where $ab = [(a,b)]_\Theta$, $a \vee b = a/^{\overline{\mathbf{C}}}(b\backslash^{\overline{\mathbf{C}}}a)$ and $a \wedge b = (a/^{\overline{\mathbf{C}}}b)b$. $\square$

We make a note of the fundamental fact that every element of $\overline{C}$ can be written as a product of members of C, and proceed to outline the proof of the representation theorem for IGMV algebras. Hereafter, we find convenient to use the term *dense nucleus* for a nucleus on the negative cone $\mathbf{G}^-$ whose image G^-_γ generates $\mathbf{G}^-$ as a monoid.

Theorem 2.12. [41, Theorem 3.12] *An IRL is a GMV algebra if and only if it is the retract of a dense nucleus on the negative cone of some ℓ-group.*

Proof. (Sketch) We are going to prove only the forward direction. For the converse, we refer the reader to [41, Theorem 3.4]. Let $\mathbf{A} = (A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \cdot^{\mathbf{A}}, \backslash^{\mathbf{A}}, /^{\mathbf{A}}, e^{\mathbf{A}})$ be an IGMV algebra. The crucial observation, here, is that its implicative reduct $(A, \backslash^{\mathbf{A}}, /^{\mathbf{A}}, e^{\mathbf{A}})$ is a cone algebra, whence by Proposition 2.11 it can be embedded into the corresponding reduct of a RL $\mathbf{G}^- \in \mathcal{LG}^-$ which is generated by A as a monoid. All we need for our claim to hold true is some nucleus γ that makes the nucleus retract $\mathbf{G}^-_\gamma$ isomorphic to $\mathbf{A}$. To this effect, let $a = \prod^{\mathbf{G}^-}_{j\leqslant n} a_j \in G^-$, where each $a_j \in A$, and define

$$\gamma(a) = \gamma\left(\prod^{\mathbf{G}^-}_{j\leqslant n} a_j\right) = \prod^{\mathbf{A}}_{j\leqslant n} a_j.$$

This map is well-defined and is actually a nucleus on $\mathbf{G}^-$. Clearly, the universe of the nucleus retract $\mathbf{G}^-_\gamma$ coincides with $\mathbf{A}$, and it can be seen that the operations in both structures coincide with one another. In particular, $a \cdot^{\mathbf{G}^-_\gamma} b = \gamma(a \cdot^{\mathbf{G}^-} b) = a \cdot^{\mathbf{A}} b$. □

The preceding representation theorem can be actually viewed as just part of a more general categorical equivalence. The categories in point are $\boldsymbol{IGMV}$, the category whose objects are IGMV algebras and whose morphisms are RL homomorphisms, and $\boldsymbol{LG}^-_*$, the category whose objects are expansions of negative cones by a dense nucleus γ, and whose morphisms are RL homomorphisms that preserve γ.

Theorem 2.13. *The categories $\boldsymbol{IGMV}$ and $\boldsymbol{LG}^-_*$ are equivalent.*

Proof. (Sketch). Let $(\mathbf{K}, \gamma^{\mathbf{K}})$ be an object in $\boldsymbol{LG}^-_*$; we let $\Gamma(\mathbf{K}, \gamma^{\mathbf{K}}) = \mathbf{K}_{\gamma^{\mathbf{K}}}$. Moreover, if $f : (\mathbf{K}, \gamma^{\mathbf{K}}) \to (\mathbf{L}, \gamma^{\mathbf{L}})$ is a morphism in $\boldsymbol{LG}^-_*$, we define $\Gamma(f)$ as the restriction of f to $K_{\gamma^{\mathbf{K}}}$. We prove in turn each of the following items:

- Γ *is a well-defined functor.* $\Gamma(\mathbf{K}, \gamma^{\mathbf{K}})$ is an object in $\boldsymbol{IGMV}$ because nucleus retracts of negative cones of ℓ-groups are IGMV algebras (Theorem 2.12). It can be easily checked that $\Gamma(f)$ is a morphism in $\boldsymbol{IGMV}$, essentially because f commutes with the nuclei $\gamma^{\mathbf{K}}, \gamma^{\mathbf{L}}$. It is immediate that Γ preserves composition of arrows and the identity morphism.

- Γ *is full*. Every object in $\boldsymbol{IGMV}$ is the Γ-image of an object in $\boldsymbol{LG}_*^-$ by Theorem 2.12. It takes a lot more work to show that Γ is surjective on morphisms; however, by using a variation on Cignoli and Mundici's technique of *good sequences* [23, Chapter 2], it is possible to prove that whenever we are given objects $(\mathbf{K}, \gamma^{\mathbf{K}}), (\mathbf{L}, \gamma^{\mathbf{L}})$ in $\boldsymbol{LG}_*^-$ and a homomorphism f from $\mathbf{K}_{\gamma^{\mathbf{K}}}$ to $\mathbf{L}_{\gamma^{\mathbf{L}}}$, there exist a unique RL homomorphism $\overline{f} : \mathbf{K} \to \mathbf{L}$ such that

$$f \circ \gamma^{\mathbf{K}} = \gamma^{\mathbf{L}} \circ \overline{f},$$

  whence the claim follows.
- Γ *is faithful*. Since γ is assumed to be dense, $\mathbf{K} = \mathbf{L}$ whenever $\mathbf{K}_{\gamma^{\mathbf{K}}} = \mathbf{L}_{\gamma^{\mathbf{L}}}$ and, for $f, g : (\mathbf{K}, \gamma^{\mathbf{K}}) \to (\mathbf{L}, \gamma^{\mathbf{L}})$, $f = g$ in case $f \restriction K_{\gamma^{\mathbf{K}}} = g \restriction K_{\gamma^{\mathbf{K}}}$.

This much suffices for our main claim. □

2.3 Projectable IGMV Algebras and Projectable ℓ-groups

The results in Sections 2.2.3 and 2.2.4 suggest a very natural conjecture to the effect that suitable analogues of Theorems 2.12 and 2.13 continue to hold for projectable IGMV algebras. More precisely, it seems plausible to surmise that such algebras – which, by virtue of Lemma 2.10, coincide with IGMV algebras that admit a positive Gödel implication – are nucleus retracts of negative cones of projectable ℓ-groups, and that the corresponding categories are equivalent to each other. In this section, we will see that both statements actually hold, if appropriately qualified. Namely, the equivalence between the categories of IGMV algebras and of negative cones of ℓ-groups restricts to an equivalence of the respective full subcategories whose objects are the projectable members, and whose morphisms are γ-preserving RL homomorphisms.

In greater detail, let $\boldsymbol{PLG}_*^-$ be the category whose objects are negative cones of projectable ℓ-groups equipped with a dense nucleus γ, and whose arrows are their γ-preserving RL homomorphisms; analogously, let $\boldsymbol{PGMV}$ will be the category whose objects are projectable GMV algebras and whose arrows are their RL homomorphisms. We will prove in this section that:

Theorem 2.14. *$\boldsymbol{PLG}_*^-$ and $\boldsymbol{PGMV}$ are equivalent.*

If we want the signature of our algebras to include the Gödel implication, and our category morphisms to preserve it – turning projectable IGMV algebras and negative cones of projectable ℓ-groups into *varieties*, so as to profit from the well-known advantages yielded by this move – the exact relationship between the resulting categories is not as simple as that, although we will defer to the next section a detailed investigation of the problem.

Let $\mathbf{M} = (M, \wedge, \vee, \cdot, \backslash, /, e)$ be a projectable GMV algebra. The construction of Theorem 2.12 vouches for the existence of an ℓ-group $\mathbf{G}$, and of a dense nucleus γ on its negative cone $\mathbf{G}^-$, such that $\mathbf{M}$ is isomorphic to $\mathbf{G}^-_\gamma$. The next useful Lemma shows that, in an appropriate sense, G^-_γ is *dense* in G^-.

Lemma 2.15. *Let a be a member of G such that $a < e$. Then there exists $b \in G^-_\gamma$ such that $a \leqslant b < e$.*

Proof. Since γ is dense, we know that for some $x_1, \ldots, x_n$ we have that $a = \prod^{\mathbf{G}^-}_{j \leqslant n} \gamma(x_j)$. For some k, $\gamma(x_k) < e$ (otherwise $a = \prod^{\mathbf{G}^-}_{j \leqslant n} \gamma(x_j) = e$, a contradiction). Pick such a k. Then

$$a = \prod_{j \leqslant n}^{\mathbf{G}^-} \gamma(x_j) \leqslant \gamma(x_k) < e. \qquad \square$$

Lemma 2.16. *Let $\mathbf{G}^-$ be the negative cone of an ℓ-group, and let γ be a dense nucleus on $\mathbf{G}^-$ with image $\mathbf{G}^-_\gamma$. The lattices $\mathrm{MF}(\mathbf{G}^-)$ and $\mathrm{MF}(\mathbf{G}^-_\gamma)$ of multiplicative filters of $\mathbf{G}^-$ and $\mathbf{G}^-_\gamma$, respectively, are isomorphic. The isomorphism is given by the mutually inverse maps $\varphi(F) = \langle F \rangle_{\mathbf{G}^-}$ and $\psi(H) = \gamma[H] = H \cap G^-_\gamma$.*

Proof. Let $F, H \in \mathrm{MF}(\mathbf{G}^-_\gamma)$. Now, if $\langle F \rangle_{\mathbf{G}^-} = \langle H \rangle_{\mathbf{G}^-}$ and $a \in F$, then $a \in \langle F \rangle_{\mathbf{G}^-} = \langle H \rangle_{\mathbf{G}^-}$, whence there exist $h_1, \ldots, h_n \in H$ such that $\prod^{\mathbf{G}^-}_{j \leqslant n} h_j \leqslant a$. So

$$\prod_{j \leqslant n}^{\mathbf{G}^-_\gamma} h_j = \gamma\Big(\prod_{j \leqslant n}^{\mathbf{G}^-} h_j\Big) \leqslant \gamma(a) = a,$$

and thus $a \in H$.

For surjectivity, it suffices to show that an arbitrary multiplicative filter J of $\mathbf{G}^-$ is such that $J = \langle \gamma[J] \rangle_{\mathbf{G}^-}$. For the nontrivial direction, let $a \in J$. Since γ is dense, $a = \prod^{\mathbf{G}^-}_{i \leqslant m} h_i$, for some $h_1, \ldots, h_m \in G^-_\gamma$; so, for every $i \leqslant m$, $a \leqslant h_i$, and hence $\gamma(a) \leqslant \gamma(h_i) = h_i$. Thus, for every $i \leqslant m$, $h_i \in \langle \gamma[J] \rangle_{\mathbf{G}^-}$, and therefore $a = \prod^{\mathbf{G}^-}_{i \leqslant m} h_i \in \langle \gamma[J] \rangle_{\mathbf{G}^-}$. $\square$

In the next Lemma we make a note of some interesting properties of generated filters and of the mappings φ and ψ in Lemma 2.16. In the interests of readability, we write $F^{\perp_\gamma}$ in place of $F^{\perp_{\mathbf{G}^-_\gamma}}$, and $F^{\perp}$ in place of $F^{\perp_{\mathbf{G}^-}}$. Also, we let $\langle X \rangle_\gamma$ stand for $\langle X \rangle_{\mathbf{G}^-_\gamma}$ and $\langle X \rangle$ for $\langle X \rangle_{\mathbf{G}^-}$.

Lemma 2.17. *Let $\mathbf{G}^-$ be the negative cone of a projectable ℓ-group, and let $\mathbf{G}^-_\gamma$ be a nucleus retract of it, with γ a dense nucleus.*

1. *For any $a \in G^-$, $\psi(\langle a \rangle) = \langle \gamma(a) \rangle_\gamma$;*

2. *For any* $a \in G^-_\gamma$, $\varphi(\langle a\rangle_\gamma) = \langle a\rangle$;
3. *For any* $a \in G^-_\gamma$, $\varphi(a^{\perp_\gamma}) = a^\perp$;
4. *For any* $a \in G^-$, $\psi(a^\perp) = \gamma(a)^{\perp_\gamma}$; *and*
5. *If* $a \in G^-_\gamma$, $\varphi(a^{\perp_\gamma})$ *is a complemented element in* $\mathrm{MF}(\mathbf{G}^-)$, *its complement being* $(a^{\perp_\gamma})^\perp$.

Proof. (1) For the nontrivial direction, let $x \in \psi(\langle a\rangle) = \langle a\rangle \cap G^-_\gamma$. Thus $x \geqslant a^n$, for some $n \in \mathbb{N}$. It follows that $x = \gamma(x) \geqslant \gamma(a^n) = \gamma(a) \cdot_{\mathbf{G}^-_\gamma} \cdots \cdot_{\mathbf{G}^-_\gamma} \gamma(a)$, whence our claim follows.

(2) From (1), by applying the isomorphism φ on both sides.

(3) By Proposition 2.3.(1) and item (2),

$$\varphi(a^{\perp_\gamma}) = \varphi(\langle a\rangle_\gamma^{\perp_\gamma}) = \varphi(\langle a\rangle_\gamma)^\perp = \langle a\rangle^\perp = a^\perp.$$

(4) By Proposition 2.3.(1) and item (1),

$$\psi(a^\perp) = \psi(\langle a\rangle^\perp) = (\psi(\langle a\rangle))^{\perp_\gamma} = (\langle\gamma(a)\rangle_\gamma)^{\perp_\gamma} = \gamma(a)^{\perp_\gamma}.$$

(5) From Proposition 2.3.(1)-(3). □

Lemma 2.18. *An IRL* $\mathbf{M}$ *is a projectable GMV algebra if and only if it is a retract of a dense nucleus on the negative cone* $\mathbf{G}^-$ *of some projectable ℓ-group.*

Proof. In view of the previous Lemma and in virtue of Theorem 2.12 we confine ourselves to proving the left to right direction. Let $\mathbf{M}$ be a projectable GMV algebra. We use the construction in Theorem 2.12 to obtain an ℓ-group $\mathbf{G}$, and a nucleus γ on its negative cone $\mathbf{G}^-$, such that $\mathbf{M}$ is isomorphic to $\mathbf{G}^-_\gamma$. It remains to show that $\mathbf{G}$ is projectable. Now, by Lemma 2.10, $\mathbf{G}^-_\gamma$ is projectable, and this property is witnessed by its lattice of multiplicative filters; namely, for all $a \in G^-_\gamma$,

$$a^{\perp_\gamma} \vee a^{\perp_\gamma\perp_\gamma} = G^-_\gamma.$$

Now, recall that for our claim to hold, it suffices to show that $\mathbf{G}^-$ is projectable, namely that for all $a \in G^-$,

$$a^\perp \vee a^{\perp\perp} = G^-.$$

This much will suffice, because the map that sends convex subalgebras of an ℓ-group to convex subalgebras of its negative cone is an isomorphism. Let $a \in G^-$. Then

$$\begin{aligned}
\psi(a^\perp \vee a^{\perp\perp}) &= \psi(a^\perp) \vee \psi(a^{\perp\perp}) && \psi \text{ preserves joins}\\
&= \psi(a^\perp) \vee \psi(a^\perp)^{\perp_\gamma} && \text{Proposition 2.3.(1)}\\
&= \gamma(a)^{\perp_\gamma} \vee \gamma(a)^{\perp_\gamma\perp_\gamma} = G^-_\gamma, && \text{Lemma 2.17.(4)}
\end{aligned}$$

whence our conclusion follows given that ψ is an isomorphism. □

We now proceed to the proof of Theorem 2.14.

Proof of Theorem 2.14. Lemma 2.10 and Lemma 2.18 imply that $\boldsymbol{PLG_*^-}$ and $\boldsymbol{PGMV}$ are full subcategories of the categories $\boldsymbol{LG_*^-}$ and $\boldsymbol{IGMV}$, respectively. So, the functor Γ in Theorem 2.13 restricts to a full and faithful functor from $\boldsymbol{PLG_*^-}$ to $\boldsymbol{PGMV}$, whence our claim follows. $\square$

Corollary 2.19. *The categories of projectable MV algebras and projectable unital Abelian ℓ-groups are equivalent.*

2.4 Introducing the Categories $\boldsymbol{GLG^-}$ and $\boldsymbol{GGMV}$

As already observed, it is natural to give an equational characterization of projectability by including in the signature the operation symbol for the Gödel implication. If so, our category morphisms should obviously preserve the additional operation, but the morphisms in both $\boldsymbol{PLG_*^-}$ and $\boldsymbol{PGMV}$ fall short of this *desideratum*. As we shall see in this section, however, imposing this further constraint upon our arrows will downgrade the previous equivalence to an *adjunction*.

Thus, in what follows, we will deal with projectable IGMV algebras in the signature expanded by an additional binary operation symbol $\rightarrow$, which denotes the relative pseudo-complement whose existence is guaranteed by Lemma 2.10. To distinguish these algebras from their $\rightarrow$-free counterparts we need a special label, provided via the next definition.

Definition 2.20. A *Gödel GMV algebra* is an algebra $\mathbf{M} = (M, \wedge, \vee, \cdot, \backslash, /, \rightarrow, e)$ of type $(2,2,2,2,2,2,0)$ such that:

1. $(M, \wedge, \vee, \cdot, \backslash, /, e)$ is an IGMV algebra;
2. $(M, \wedge, \vee, \rightarrow, e)$ is a positive Gödel algebra.

The labels $\mathcal{GGMV}$ and $\mathcal{GLG}^-$ will henceforth stand for the varieties of Gödel GMV algebras and of Gödel negative cones (Gödel GMV algebras whose RL reducts are negative cones of ℓ-groups), respectively.

Theorem 2.21. *Any Gödel GMV algebra $\mathbf{M} = (M, \wedge, \vee, \cdot, \backslash, /, \rightarrow, e)$ is the retract of a dense nucleus*[5] *of some Gödel negative cone.*

[5] Notice that by nucleus, here, we mean a nucleus on the RL reduct of $\mathbf{A}$; it should be pointed out that, by [41, Corollary 3.7], such nuclei also preserve meets.

Proof. The claim follows from Lemma 2.18 if we can show that $\to^{\mathbf{M}}$ coincides with the relative pseudo-complement in $\mathbf{G}^-_\gamma$, whose existence is guaranteed by the fact that $\mathbf{G}^-$ is projectable. However, if $a, b \in G^-_\gamma$, $a \to^{\mathbf{M}} b$ is a closed element in that $\gamma(b) = b \leqslant a \to^{\mathbf{M}} b$, and closed elements form a lattice filter of $\mathbf{M}$. Since it is the largest x such that $a \wedge x \leqslant b$, in particular it is the largest closed element with that property. In sum,

$$a \to^{\mathbf{M}} b = \max\{\gamma(x) : a \wedge \gamma(x) \leqslant b\} = a \to^{\mathbf{G}^-_\gamma} b. \qquad \square$$

The preceding proof also yields:

Corollary 2.22. *The $\{\backslash, /, \to, e\}$-reduct of any Gödel GMV algebra is a subreduct of a Gödel negative cone.*

For our purposes, the following generalization (for which see e.g. [94]) of the usual concept of free algebra over a set of free generators will come in handy.

Definition 2.23. Let $\mathcal{K}$ and $\mathcal{K}'$ be classes of algebras of respective signatures ν and ν', with $\nu' \subseteq \nu$. The algebra $\mathbf{K} \in \mathcal{K}$ is a *$\mathcal{K}$-free extension* over $\mathbf{A} \in \mathcal{K}'$ in case:

(1) $\mathbf{A}$ is a ν'-subreduct of $\mathbf{K}$;
(2) the subalgebra of $\mathbf{K}$ generated by A is $\mathbf{K}$; and
(3) every homomorphism of $\mathbf{A}$ to the ν'-reduct of any $\mathbf{C} \in \mathcal{K}$ can be extended to a unique homomorphism of $\mathbf{K}$ to $\mathbf{C}$.

$$\begin{tikzcd} \mathbf{K}{\restriction}\nu' \arrow[dr, dashed, "\bar{f}"] & \\ \mathbf{A} \arrow[u, hook, "i"] \arrow[r, "f"'] & \mathbf{C}{\restriction}\nu' \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathbf{K} \arrow[d, dashed, "\exists!\bar{f}"] \\ \mathbf{C} \end{tikzcd}$$

To make terminology less cumbersome, we will refer to the $\mathcal{K}$-free extension over $\mathbf{A}$ as "the free K over $\mathbf{A}$." Thus, for example, the $\mathcal{LG}^-$-free extension over $\mathbf{A} \in \mathcal{CA}$ will be described as the free negative cone over $\mathbf{A}$.

Lemma 2.24. *Let $\mathbf{M}$ be a Gödel GMV algebra, and let $\mathbf{A}$ and $\mathbf{B}$ be its $\{\backslash, /, \to, e\}$-reduct[6] and its $\{\backslash, /, e\}$-reduct, respectively. Then:*

1. *The free Gödel negative cone $\mathbf{K}$ over $\mathbf{A}$ exists.*
2. *The RL subreduct $\mathbf{L}$ generated by B in $\mathbf{K}$ is the free negative cone over $\mathbf{B}$.*

[6] The $\{\backslash, /, \to, e\}$-subreducts of Gödel negative cones are easily seen to be axiomatized by the axioms of cone algebras together with the axioms for Hilbert algebras; see, for example, [33].

Proof.
(1) Let $\mathbf{F}$ be the free Gödel negative cone over the set A. Further, let θ be the congruence relation on $\mathbf{F}$ generated by all pairs $(a\backslash^{\mathbf{F}}b, a\backslash^{\mathbf{A}}b)$, $(a/^{\mathbf{F}}b, a/^{\mathbf{A}}b)$, and $(a \to^{\mathbf{F}} b, a \to^{\mathbf{A}} b)$, for all $a, b \in A$. Let $\mathbf{K}$ be the quotient algebra $\mathbf{F}/\theta$. Consider the $\{\backslash, /, \to, e\}$-homomorphism $i : \mathbf{A} \to \mathbf{K}$ that sends any element $a \in A$ to its equivalence class $[a]_\theta$. In view of Corollary 2.22, i is injective and hence $\mathbf{A}$ can be identified with its image under i. A direct check shows that $\mathbf{K}$ is the free Gödel negative cone extension of $\mathbf{A}$.

(2) It is a consequence of the following result of [97, Corollary 3.15]: If a cone algebra $\mathbf{A}$ is a subreduct of a negative $\mathbf{G}^-$, then the subalgebra of $\mathbf{G}^-$ generated by $\mathbf{A}$ is the free extension of $\mathbf{A}$. □

Remark 2.25. *Retaining the notation of the foregoing lemma,* $\mathbf{A}$ *is a subreduct of* $\mathbf{K}$ *and moreover, by* [41, Theorem 3.4.(5)], *the RL reduct of* $\mathbf{M}$ *is contained in* $\mathbf{L}$ *as a lattice filter – actually, it is the image of a dense nucleus* γ *on* $\mathbf{L}$*. Note that* $\mathbf{L}$ *is a projectable GMV algebra by Theorem 2.21; hence, it can be equipped with a Gödel implication* $\to$ *(Theorem 2.21), which extends* $\to^{\mathbf{A}}$ *but is not necessarily the restriction to* L *of* $\to^{\mathbf{K}}$*. Therefore,* $\mathbf{A}$ *is included in the Gödel negative cone* $\bar{\mathbf{L}}$ *that expands* $\mathbf{L}$ *by* $\to$*. Namely, we are in the situation depicted in the figure below. Since* $\mathbf{K}$ *is the free Gödel negative cone over* $\mathbf{A}$, there exists a unique $\mathcal{GLG}^-$ homomorphism β making the diagram commutative. *Actually,* β *is idempotent, whence* $\bar{\mathbf{L}}$ *is an RL retract of* $\mathbf{K}$ *in the usual, universal algebraic sense. Therefore, any Gödel GMV algebra* $\mathbf{M}$ *uniquely determines a pair* $(\mathbf{K}, \beta\gamma)$*, where* $\mathbf{K}$ *is the free Gödel negative cone over* $\mathbf{A}$*, with* β *and* γ *as in the preceding sentences.*

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "i"] \arrow[dr, "i"'] & \mathbf{K} \arrow[d, dashed, "\exists!\beta"] \\
& \mathbf{L}
\end{tikzcd}$$

We now define the categories we wish to investigate:

- $GGMV$ is the category whose objects are Gödel GMV algebras and whose arrows are their algebra homomorphisms.
- GLG^- is the category whose objects are the pairs $(\mathbf{K}, \beta\gamma)$ such that $\mathbf{K}$ is a Gödel negative cone, β is an idempotent endomorphism on $\mathbf{K}$ and γ is a dense nucleus on its image;[7] and its morphisms are mappings $f : (\mathbf{K}_1, \beta_1\gamma_1) \to (\mathbf{K}_2, \beta_2\gamma_2)$ such that f is a $\mathcal{GLG}^-$-homomorphisms that satisfies $f\gamma_1\beta_1 = \gamma_2 f\beta_1$, as shown in the

[7] It should be noted here that in the definition of GLG^- we do not assume that $\mathbf{K}$ is the free Gödel negative cone over a Gödel GMV algebra, and β, γ need not be the special mappings discussed in Remark 2.25.

next diagram:

$$\begin{tikzcd}
\mathbf{K}_1 \arrow[r, "f"] \arrow[d, "\beta_1"'] & \mathbf{K}_2 \arrow[d, "\beta_2"] \\
\bar{\mathbf{L}}_1 \arrow[r, "f\restriction\bar{\mathbf{L}}_1"] \arrow[d, "\gamma_1"'] & \bar{\mathbf{L}}_2 \arrow[d, "\gamma_2"] \\
\mathbf{M}_1 \arrow[r, "f\restriction \operatorname{im}\gamma_1"] & \mathbf{M}_2
\end{tikzcd}$$

It is implicit in the previous definition that $f[L_1] \subseteq L_2$, although there is no assumption in the diagram above that f preserves the Gödel implication in $\bar{\mathbf{L}}_1$, because, as already noted, $\bar{\mathbf{L}}_1$ need not be a subalgebra of $\mathbf{K}_1$. The condition $f\gamma_1\beta_1 = \gamma_2 f \beta_1$ expresses the commutativity of the diagram below:

$$\begin{tikzcd}
\bar{\mathbf{L}}_1 \arrow[r, "f\restriction\bar{\mathbf{L}}_1"] \arrow[d, "\gamma_1"'] & \bar{\mathbf{L}}_2 \arrow[d, "\gamma_2"] \\
\mathbf{M}_1 \arrow[r, "f\restriction \operatorname{im}\gamma_1"] & \mathbf{M}_2
\end{tikzcd}$$

Let $f : \mathbf{M}_1 \to \mathbf{M}_2$ be a homomorphism of Gödel GMV algebras, and let $\mathbf{K}_1$ and $\mathbf{K}_2$ be the free Gödel negative cones over the $\{\backslash, /, \to, e\}$-reducts $\mathbf{A}_1$ and $\mathbf{A}_2$ of $\mathbf{M}_1$ and $\mathbf{M}_2$, respectively. Observe that f, as such, restricts to a homomorphism between these reducts. By Lemma 2.24, $\mathbf{A}_1, \mathbf{A}_2$ respectively embed into the appropriate reducts of the free Gödel negative cones $\mathbf{K}_1$ and $\mathbf{K}_2$:

$$\begin{tikzcd}
\mathbf{A}_1 \arrow[rr, "i"] \arrow[dr, "f"] \arrow[ddr, bend right, "i\circ f"'] & & \mathbf{K}_2 \arrow[ddl, dashed, bend left, "\exists!\bar{f}"] \\
& \mathbf{A}_2 \arrow[d, "i"] & \\
& \mathbf{K}_2 &
\end{tikzcd}$$

Since $\mathbf{K}_1$ is the free Gödel negative cone over $\mathbf{A}_1$, f extends to a unique homomorphism $\bar{f} : \mathbf{K}_1 \to \mathbf{K}_2$. We call $\bar{f}$ the *free extension* of f.

Equipped with this notion, we introduce two assignments $\mathcal{F} : \boldsymbol{GGMV} \to \boldsymbol{GLG}^-$ and $\mathcal{G} : \boldsymbol{GLG}^- \to \boldsymbol{GGMV}$, with an eye to showing that they are well-defined functors and that they form an adjoint pair between the categories $\boldsymbol{GLG}^-$ and $\boldsymbol{GGMV}$.

- Given an object $\mathbf{M}$ in $\boldsymbol{GGMV}$, $\mathcal{F}(\mathbf{M})$ is the pair $(\mathbf{K}, \beta\gamma)$ determined as in Remark 2.25, and given a morphism $f : \mathbf{M}_1 \to \mathbf{M}_2$ in $\boldsymbol{GGMV}$, $\mathcal{F}(f)$ is the free extension $\bar{f}$ of f.
- Given an object $(\mathbf{K}, \beta\gamma)$ in $\boldsymbol{GLG}^-$, $\mathcal{G}(\mathbf{K}, \beta\gamma)$ is the algebra $\gamma[\beta[\mathbf{K}]]$, and given a morphism $f : (\mathbf{K}_1, \beta_1\gamma_1) \to (\mathbf{K}_2, \beta_2\gamma_2)$ in $\boldsymbol{GLG}^-$, $\mathcal{G}(f)$ is $f\restriction \operatorname{im}\gamma_1$.

Lemma 2.26. *$\mathcal{F}$ is a functor between the categories $\boldsymbol{GGMV}$ and $\boldsymbol{GLG^-}$.*

Proof. We already noticed that $\mathcal{F}(\mathbf{M})$ is an object in $\boldsymbol{GLG^-}$. Next, take any morphism $f : \mathbf{M}_1 \to \mathbf{M}_2$, and let $\mathbf{A}_i, \mathbf{B}_i$ $(i \in \{1,2\})$ be, respectively, the $(\backslash, /, \to, e)$-reducts and $(\backslash, /, e)$-reducts of $\mathbf{M}_i$. Observe that f restricts to a homomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$, which in turn extends to a homomorphism $f^* : \mathbf{L}_1 \to \mathbf{L}_2$, where $\mathbf{L}_i$ $(i \in \{1,2\})$ is the free negative cone over $\mathbf{B}_i$. We claim that $\bar{f}{\restriction}_{\mathbf{L}_1}$ coincides with f^*. By [41, Theorem 11], $\mathbf{L}_1$ is generated by $\mathbf{B}_1$ as a monoid. Therefore, for any $a \in L_1$, $a = \prod_{i\leqslant m}^{\mathbf{L}_1} a_i$, with $a_i \in B_1$, for any $i \leqslant m$. Thus,

$$f^*(a) = f^*\left(\prod_{i\leqslant m}^{\mathbf{L}_1} a_i\right) = \prod_{i\leqslant m}^{\mathbf{L}_2} f^*(a_i) = \prod_{i\leqslant m}^{\mathbf{L}_2} f(a_i) = \prod_{i\leqslant m}^{\mathbf{K}_2} f(a_i),$$

since $\mathbf{L}_2$ is an RL subalgebra of $\mathbf{K}_2$. Moreover, since $\bar{f}$ extends f,

$$\bar{f}(a) = \bar{f}\left(e\prod_{i\leqslant m}^{\mathbf{K}_1} a_i\right) = \prod_{i\leqslant m}^{\mathbf{K}_2} \bar{f}(a_i) = \prod_{i\leqslant m}^{\mathbf{K}_2} f(a_i),$$

whence our claim follows. Now, since β_1 is onto, all we have to show is that the diagram below is commutative.

$$\begin{tikzcd}
\mathbf{L}_1 \arrow[r, "\bar{f}\restriction \mathbf{L}_1"] \arrow[d, "\gamma_1"'] & \mathbf{L}_2 \arrow[d, "\gamma_2"] \\
\mathbf{M}_1 \arrow[r, "f\restriction \operatorname{im}\gamma_1"] & \mathbf{M}_2
\end{tikzcd}$$

Let $a \in L_1$. There exist $a_1, \dots, a_m \in A_1$ such that $a = \prod_{i\leqslant m}^{\mathbf{L}_1} a_i$. So,

$$\begin{aligned}
f\gamma_1(a) &= f\gamma_1\left(\prod_{i\leqslant m}^{\mathbf{L}_1} a_i\right) = f\left(\prod_{i\leqslant m}^{\mathbf{M}_1} a_i\right) = \prod_{i\leqslant m}^{\mathbf{M}_2} f(a_i) = \gamma_2\left(\prod_{i\leqslant m}^{\mathbf{L}_2} f(a_i)\right) \\
&= \gamma_2\bar{f}\left(\prod_{i\leqslant m}^{\mathbf{L}_1} a_i\right) = \gamma_2\bar{f}(a).
\end{aligned}$$

Thus indeed $f{\restriction}\operatorname{im}\gamma_1 \circ \gamma_1$ equals $\gamma_2 \circ \bar{f}{\restriction}\mathbf{L}_1$. Finally, it is also easy to check that $\mathcal{F}$ preserves compositions. Therefore $\mathcal{F}$ is a functor between the categories $\boldsymbol{GGMV}$ and $\boldsymbol{GLG^-}$. □

Lemma 2.27. *$\mathcal{G}$ is a functor from $\boldsymbol{GLG^-}$ to $\boldsymbol{GGMV}$.*

Proof. By Theorem 2.21, $\mathcal{G}(\mathbf{K}, \beta\gamma)$ is an object in $\boldsymbol{GGMV}$. Moreover, by the commutativity requirement $f\gamma_1\beta_1 = \gamma_2 f \beta_1$, $f{\restriction}\operatorname{im}\gamma$ is a $\boldsymbol{GGMV}$-morphism and, in particular, it preserves the Gödel implication. □

Theorem 2.28. *$\mathcal{F}$ and $\mathcal{G}$ are adjoint functors.*

Proof. Let $\mathbf{M}$, $\widetilde{\mathbf{K}}$ be objects in the categories $\boldsymbol{GGMV}$ and $\boldsymbol{GLG}^-$, respectively. We want to show that there is a bijective correspondence between $\boldsymbol{GLG}^-(\mathcal{F}(\mathbf{M}), \widetilde{\mathbf{K}})$ and $\boldsymbol{GGMV}(\mathbf{M}, \mathcal{G}(\widetilde{\mathbf{K}}))$, that is natural in both coordinates. As regards injectivity, let g, h be distinct morphisms in $\boldsymbol{GGMV}(\mathbf{M}, \mathcal{G}(\widetilde{\mathbf{K}}))$. If $\bar{g}$ is the free extension of g, as observed in the proof of Lemma 2.26, $\mathcal{F}(g)\restriction\mathbf{M} = \bar{g}\restriction\mathbf{M} = g$ and $\mathcal{F}(h) = \bar{h}\restriction\mathbf{M} = h$. Since g, h are assumed to be distinct, $\mathcal{F}(g) \neq \mathcal{F}(h)$. Now, let $g \in \boldsymbol{GLG}^-(\mathcal{F}(\mathbf{M}), \widetilde{\mathbf{K}})$. Let $\mathcal{F}(\mathbf{M}) = \widetilde{\mathbf{K}}$. Notice that, since $\widetilde{\mathbf{K}}$ is free over the $\{/, \backslash, \to, e\}$-reduct of $\mathbf{M}$, both β and γ are uniquely determined up to isomorphism. By the results in [41], there exists a uniquely determined $\boldsymbol{GLG}^-$-homomorphism $\bar{g}$ between the negative cone $\mathbf{L}$ associated to $\mathbf{M}$ and $\mathbf{L}'$ that makes the diagram

$$\begin{tikzcd} \widetilde{\mathbf{K}} \arrow[r, "\bar{g}"] \arrow[d, "\gamma\circ\beta"'] & \widetilde{\mathbf{K}}' \arrow[d, "\gamma'\circ\beta'"] \\ \mathbf{M} \arrow[r, "g"] & \mathbf{M}' \end{tikzcd}$$

commutative. Arguing as in Lemma 2.26, it is easy to see that $\bar{g}$ is a morphism from $\widetilde{\mathbf{K}}$ to $\widetilde{\mathbf{K}}'$, whence $\mathcal{F}$ is onto.

A routine verification shows that the stablished bijection between the hom-sets $\boldsymbol{GLG}^-(\mathcal{F}(\mathbf{M}), \widetilde{\mathbf{K}}')$ and $\boldsymbol{GGMV}(\mathbf{M}, \mathcal{G}(\widetilde{\mathbf{K}}'))$ is natural in both $\mathbf{M}$ and $\mathbf{K}'$. Namely, the following diagram commutes for $g \in \boldsymbol{GGMV}(\mathbf{M}_1, \mathbf{M}_2)$ and $f \in \boldsymbol{GLG}^-(\widetilde{\mathbf{K}}_1, \widetilde{\mathbf{K}}_2)$:

$$\begin{tikzcd} \boldsymbol{GLG}^-(\mathcal{F}(\mathbf{M}_2, \widetilde{\mathbf{K}}_1) \arrow[r, "\Phi_{\mathbf{M}_2,\mathbf{K}_1}"] \arrow[d, "\mathcal{F}(g)\circ(\,)\circ f"'] & \boldsymbol{GGMV}(\mathbf{M}_2, \mathcal{G}(\widetilde{\mathbf{K}}_1) \arrow[d, "g\circ(\,)\circ\mathcal{G}(f)"] \\ \boldsymbol{GLG}^-(\mathcal{F}(\mathbf{M}_1, \widetilde{\mathbf{K}}_2) \arrow[r, "\Phi_{\mathbf{M}_1,\mathbf{K}_2}"] & \boldsymbol{GGMV}(\mathbf{M}_1, \mathcal{G}(\widetilde{\mathbf{K}}_2) \end{tikzcd}$$ □

Chapter 3

Lateral Completions, Projectable Hulls, and Orthocompletions

3.1 Introduction

Inspired from the idea that a good deal of significant properties of lattice-ordered groups are independent from the group structure, but are essentially lattice-theoretical, from the 1960's, Paul Conrad and his group started a widespread research programme. The leitmotif of this programme was capturing relevant features of ℓ-groups by looking into the algebraic features of the lattices of convex ℓ-subgroups. Conrad's programme produced outstanding contributions and opened to new research perspectives to the theory of ℓ-groups. Of particular relevance to this chapter are the results on lateral completions of ℓ-groups. Lateral completions were considered by Stone in [87] where it is shown that an Archimedean ℓ-group can be embedded in the laterally complete ℓ-group of almost finite continuous functions on a Stone space. In 1949, Lorenzen proved in [67] that an ℓ-group can be embedded as an ℓ-subgroup in an unrestricted cardinal sum of totally ordered groups (and, therefore, laterally complete) if and only if it is representable. The same year, Nakano [75] considers orthogonality in the context of vector lattices. One of the problems he discussed is that of constructing an extension of a conditionally complete vector lattice in which every orthogonal subset (of the extension) has a supremum, and proved that, in case a vector lattice $\mathbf{L}$ is conditionally complete, then it admits a unique lateral completion $\mathbf{H}$. A remarkably elegant proof of this result is due to Pinkser [81, 93]. Nakano's ideas were generalized by Amemiya in [3] to the case of arbitrary vector lattices. Afterwards, in 1963, Jakubik [51] showed that $\mathbf{H}$ is completely determined by the lattice reduct of $\mathbf{G}$. Nakano's construction was extended also by P. Bernau in [9] and applied to representable ℓ-groups in [9]. In 1968, P. Conrad J. Harvey and C. Holland capitalized on the results above and proved in [30] that any abelian ℓ-group can be embedded into a laterally complete abelian ℓ-group of real-valued functions. As a consequence, each of the main embedding theorems for ℓ-groups is in fact an embedding into a laterally complete ℓ-group. A year later, in 1969, Conrad showed that, if an ℓ-group is representable, then it admits a unique minimal lateral completion [29].

Taking advantage of ideas from functional analysis, Conrad's programme led to major outcomes in describing lateral completions of ℓ-groups. A natural continuation of such consists in widening the program to the more general framework of *residuated lattices*, that encompass ℓ-groups and also include MV algebras, Heyting algebras, and several other algebraic structures of leading importance to algebraic logic.

The leading idea in this chapter is to construct, for any given residuated lattice, an orthocomplete extension such that the former is dense in the latter. This extension is obtained as the direct limit of a family of residuated lattices that are constructed using maximal partitions of the algebra of polars of the original residuated lattice. The restriction of semilinearity (the corresponding property to representability for ℓ-groups) seems to be essential at this point.

The structure of the present chapter is as follows: In Section 3.2 we start by studying the partitions of the Boolean algebra of polars of a residuated lattice – which we will simply call *partitions of a residuated lattice*, with a meek abuse of terminology. We will show that they form a join-semilattice, and thus, a directed poset. We will use them to define a directed system of algebras in Section 3.4, provided a semilinear residuated lattice $\mathbf{L}$. In Section 3.3, we will explain a general method to obtain the direct limit of a directed family of algebras. We will use this construction to calculate, given a semilinear residuated lattice $\mathbf{L}$, the direct limit of the direct family of algebras induced by the directed poset of partitions of polars of $\mathbf{L}$. We will prove that this limit, denoted $\mathcal{O}(\mathbf{L})$, enjoys many interesting properties. In particular, $\mathbf{L}$ is densely embeddable in $\mathcal{O}(\mathbf{L})$ (Theorem 3.14), and furtheremore it is laterally complete (Theorem 3.21). As a consequence, we obtain one of the main results of this chapter:

Theorem A (Corollary 3.22)**.** *Every e-cyclic semilinear residuated lattice* $\mathbf{L}$ *is densely embeddable in a laterally complete lattice that belongs to the variety generated by* $\mathbf{L}$*.*

In Section 3.5 we continue our study of $\mathcal{O}(\mathbf{L})$. We devote this section to the proof that $\mathcal{O}(\mathbf{L})$ is also projectable, (Theorem 3.27) and hence obtain that every semilinear residuated lattice is densely embeddable in a projectable residuated lattice. But a further analysis leads to an improvement of this result: $\mathcal{O}(\mathbf{L})$ is actually strongly projectable (Theorem 3.31), which combined with Theorems 3.14 and 3.21 gives us the following result:

Theorem B (Corollary 3.33)**.** *Every e-cyclic semilinear residuated lattice* $\mathbf{L}$ *is densely embeddable in a* $\boxplus$*-orthocomplete lattice that belongs to the variety generated by* $\mathbf{L}$*.*

We also introduce in this section the lattice $\mathcal{O}_{<\omega}(\mathbf{L})$, which is contained in $\mathcal{O}(\mathbf{L})$, but in general smaller. While $\mathcal{O}(\mathbf{L})$ is laterally complete, as we mentioned, $\mathcal{O}_{<\omega}(\mathbf{L})$ might fail this property. Nonetheless, $\mathbf{L}$ is also densely embeddable in $\mathcal{O}_{<\omega}(\mathbf{L})$, which is strongly projectable. Lastly, in Section 3.6, we look for minimal extensions, which we call *hulls*, of $\mathbf{L}$, containing $\mathbf{L}$ densely, and being laterally complete, projectable, strongly projectable, and orthocomplete respectively. We show the existence and uniqueness of projectable and strongly projectable hulls of semilinear residuated lattices; and in the special cases of GMV algebras, we also prove the existence and uniqueness of laterally complete hulls and orthocomplete hulls.

Theorem C (Theorem 3.53)**.** *Every e-cyclic semilinear residuated lattice* $\mathbf{L}$ *has a strongly* $\boxplus$*-projectable hull and a* $\boxplus$*-projectable hull in the variety generated by* $\mathbf{L}$*; and every semilinear GMV algebra* $\mathbf{L}$ *has laterally complete hull and an* $\boxplus$*-orthocomplete hull in the variety generated by* $\mathbf{L}$*.*

We end this chapter proving that $\mathcal{O}_{<\omega}(\mathbf{L})$ is actually the unique, up to isomorphims, strongly projectable hull of $\mathbf{L}$ (Theorem 3.55).

3.2 Partitions

We introduce in this section the notion of a partition of a complete Boolean algebra and study the particular case of Boolean algebras of polars of a residuated lattice, which will be essential for this entire chapter. Recall that a *partition* of a set X is a nonempty set $\mathcal{C} \subseteq \mathcal{P}(X)$ such that $\emptyset \notin \mathcal{C}$, for every pair of different elements $A, B \in \mathcal{C}$, $A \cap B = \emptyset$, and $\bigcup \mathcal{C} = X$. Our notion of partition generalizes this one to arbitrary complete Boolean algebras. Indeed, a partition of a set X is just a partition of the Boolean algebra $\mathcal{P}(X)$.

We say that two elements a, b of a Boolean algebra are *disjoint* if and only if $a \wedge b = \bot$. A word of caution is in order. According to Definition 3.15, we say that two (negative) elements x, y of an e-cyclic residuated lattices are disjoint if $x \vee y = e$, which in the particular case of IRLs, is exactly the dual notion of the one that we have just defined. This should not lead to confusion, as the Boolean algebras in which we are interested are the algebras of polars of residuated lattices. So, both notions of disjointness will be clearly separated by the context. In Lemma 3.20 we will see what is the connection between these two homonymous concepts.

Definition 3.1. Let $\mathbf{B} = \langle B, \wedge, \vee, \neg, \bot, \top \rangle$ be a non-trivial complete Boolean algebra. A *partition* of $\mathbf{B}$ is a maximal set of disjoint elements of $B \setminus \{\bot\}$, that is to say, it is a set $\mathcal{C} \subseteq B$ such that

(1) $\bot \notin \mathcal{C}$,
(2) for every $c, d \in \mathcal{C}$, if $c \neq d$ then $c \wedge d = \bot$, and
(3) if $a \in B$ is such that $a \neq \bot$, then there exists $c \in \mathcal{C}$ such that $a \wedge c \neq \bot$.

The following result is an immediate consequence of the preceding definition:

Lemma 3.2. *A subset $\mathcal{C} \subseteq B$ is a partition of $\mathbf{B}$ if and only if it satisfies the following conditions:*

(1) $\bot \notin \mathcal{C}$,
(2) for every $c, d \in \mathcal{C}$, if $c \neq d$ then $c \wedge d = \bot$, and
(3) $\bigvee \mathcal{C} = \top$.

Further, any subset $\mathcal{C}$ of B that satisfies conditions (1) and (2) can be extended to a partition, for instance $\mathcal{C} \cup \{\neg(\bigvee \mathcal{C})\}$.

The set $\mathbb{D}$ of partitions of $\mathbf{B}$ can be ordered in the following manner: given two partitions $\mathcal{C}$ and $\mathcal{A}$, we say that $\mathcal{A}$ is a *refinement* of $\mathcal{C}$, and write $\mathcal{C} \preccurlyeq \mathcal{A}$, if for every $a \in \mathcal{A}$ there exists a (necessarily unique) $c \in \mathcal{C}$ such that $a \leqslant c$. It is easily checked that $\preccurlyeq$ is a partial order on $\mathbb{D}$. We in fact prove that $\langle \mathbb{D}, \preccurlyeq \rangle$ is a join semilattice, and hence any two partitions have a least common refinement. Indeed let $\mathcal{C}, \mathcal{D}$ be partitions. We claim that

$$\mathcal{A} = \{c \wedge d \neq \bot : c \in \mathcal{C},\ d \in \mathcal{D}\} \tag{3.1}$$

is their join in $\langle \mathbb{D}, \preccurlyeq \rangle$. Let us first verify that $\mathcal{A}$ is actually a partition. Observe that $\bot \notin \mathcal{A}$ by definition. If $a = c \wedge d$ and $a' = c' \wedge d'$ are in $\mathcal{A}$, with $c, c' \in \mathcal{C}$ and $d, d' \in \mathcal{D}$, and $a \neq a'$, then $c \neq c'$ or $d \neq d'$, and in either case $a \wedge a' = (c \wedge c') \wedge (d \wedge d') = \bot$. And finally, if $a \in B$ is such that $a \neq \bot$, then there exists $c \in \mathcal{C}$ such that $a \wedge c \neq \bot$, by the maximality of $\mathcal{C}$, and therefore there exists $d \in \mathcal{D}$ such that $(a \wedge c) \wedge d \neq \bot$, by the maximality of $\mathcal{D}$. Thus, we have found $c \in \mathcal{C}$ and $d \in \mathcal{D}$ such that $c \wedge d \neq \bot$, and therefore $c \wedge d \in \mathcal{A}$, and $a \wedge (c \wedge d) \neq \bot$, which proves the maximality of $\mathcal{A}$. Lastly, $\mathcal{A}$ is clearly a refinement of $\mathcal{C}$ and $\mathcal{D}$, and any other refinement of $\mathcal{C}$ and $\mathcal{D}$ must also be a refinement of $\mathcal{A}$.

Lemma 3.3. *Let* $\mathbf{B}$ *be a complete Boolean algebra and* $\mathcal{C}, \mathcal{A}$ *be partitions of* $\mathbf{B}$*. Then the following are equivalent:*

1. $\mathcal{C} \preccurlyeq \mathcal{A}$*;*
2. *for every* $c \in \mathcal{C}$*,* $\{a \in \mathcal{A} : a \leqslant c\}$ *is a partition of the Boolean algebra* $[\bot, c]$*;*
3. *for every* $c \in \mathcal{C}$*,* $c = \bigvee\{a \in \mathcal{A} : a \leqslant c\}$*; and*
4. *for every* $c \in \mathcal{C}$*,* $\neg c = \bigwedge\{\neg a : a \in \mathcal{A}, a \leqslant c\}$*.*

Proof. $(1 \Rightarrow 2)$: Let $c \in \mathcal{C}$ and let $\mathcal{D} = \{a \in \mathcal{A} : a \leqslant c\}$. Obviously $\bot \notin \mathcal{D}$ and if $a, b \in \mathcal{D}$ and $a \neq b$, then $a \wedge b = \bot$, since $\mathcal{A}$ is a partition. Now, by Lemma 3.2 all we need to show is that $c = \bigvee \mathcal{D}$, which is true by virtue of the distributivity law, and the facts that $\mathcal{C}$ is a refinement of $\mathcal{A}$ and $\bigvee \mathcal{A} = \top$, by hypothesis.

$(2 \Rightarrow 1)$: Consider $b \in \mathcal{A}$. Then $b \neq \bot$, and therefore $b \wedge c \neq \bot$, for some $c \in \mathcal{C}$. Obviously $b \wedge c \in [\bot, c]$, and since $\{a \in \mathcal{A} : a \leqslant c\}$ is a partition of $[\bot, c]$, there exists $a \in \mathcal{A}$ such that $a \leqslant c$ and $a \wedge b \wedge c \neq \bot$. Thus, $a \wedge b \neq \bot$, and since $\mathcal{A}$ is a partition, $b = a \leqslant c$. We have established that $\mathcal{C} \preccurlyeq \mathcal{A}$.

$(2 \Leftrightarrow 3)$: This equivalence is an immediate consequence of Lemma 3.2.

$(3 \Leftrightarrow 4)$: This equivalence follows from two facts: (i) complementation in a Boolean algebra is a dual order-automorphism; and (ii) arbitrary joins and meets in $[\bot, c]$ coincide with those in $\mathbf{B}$. □

Given an e-cyclic residuated lattice $\mathbf{L}$, we denote by $\mathbb{D}(\mathbf{L})$ the join-semilattice of partitions of the Boolean algebra $\mathrm{Pol}(\mathbf{L})$ of polars of $\mathbf{L}$.[1] Recall that if $\mathbf{L}$ is semilinear then each polar of $\mathbf{L}$ is a convex normal subalgebra, by Theorem 1.13. Thus the following corollary is an immediate consequence of Lemma 3.3.

Corollary 3.4. *Let* $\mathbf{L}$ *be an e-cyclic residuated lattice and* $\mathcal{C}, \mathcal{A}$ *be partitions of* $\mathrm{Pol}(\mathbf{L})$*. Then the following are equivalent:*

1. $\mathcal{C} \preccurlyeq \mathcal{A}$*;*
2. *for every* $C \in \mathcal{C}$*,* $\{A \in \mathcal{A} : A \subseteq C\}$ *is a partition of the Boolean algebra* $[\{e\}, C]$*;*
3. *for every* $C \in \mathcal{C}$*,* $C = \bigvee^{\mathrm{Pol}(\mathbf{L})}\{A \in \mathcal{A} : A \subseteq C\}$*; and*
4. *for every* $C \in \mathcal{C}$*,* $C^{\perp} = \bigcap\{A^{\perp} : A \in \mathcal{A}, A \subseteq C\}$*.*

If moreover $\mathbf{L}$ *is semilinear, then the previous conditions are equivalent to*

5. *The homomorphism* $f : \mathbf{L}/C^{\perp} \to \prod\{\mathbf{L}/A^{\perp} : A \in \mathcal{A}, A \subseteq C\}$*, defined by* $f([a]_{C^{\perp}}) = ([a]_{A^{\perp}} : A \in \mathcal{A}, A \subseteq C)$ *provides a subdirect representation of* $\mathbf{L}/C^{\perp}$ *in terms of the algebras* $\{\mathbf{L}/A^{\perp} : A \in \mathcal{A}, A \subseteq C\}$*.*

[1] See Section 1.7

Proof. The equivalence of the first four conditions follows from Lemma 3.3. Further, in view of Theorem 1.13, all polars of $\mathbf{L}$ are normal. Hence, 4 and 5 are equivalent. $\square$

Let $\mathbf{L}$ be an e-cyclic semilinear residuated lattice. As we mentioned already, in view of Theorem 1.13, all polars of $\mathbf{L}$ are normal, and hence for every $C^{\perp} \in \mathrm{Pol}(L)$ one can form the quotient algebra $\mathbf{L}/C^{\perp}$. For every partition $\mathcal{C}$ of $\mathrm{Pol}(\mathbf{L})$, we define the product $\mathbf{L}_{\mathcal{C}} = \prod_{C \in \mathcal{C}} \mathbf{L}/C^{\perp}$. We will see that if $\mathcal{C}$ and $\mathcal{A}$ are two partitions such that $\mathcal{C} \preccurlyeq \mathcal{A}$ then we can define an injective homomorphism $\phi_{\mathcal{C}\mathcal{A}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{L}_{\mathcal{A}}$. The family of homomorphisms of residuated lattices $\{\phi_{\mathcal{C}\mathcal{A}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{L}_{\mathcal{A}} : \mathcal{C} \preccurlyeq \mathcal{A}\}$ satisfies a compatibility property, namely, given three partition $\mathcal{A} \preccurlyeq \mathcal{B} \preccurlyeq \mathcal{C}$, we have

$$\phi_{\mathcal{C}} = id_{\mathcal{C}} \quad \text{and} \quad \phi_{\mathcal{B}\mathcal{C}} \circ \phi_{\mathcal{A}\mathcal{B}} = \phi_{\mathcal{A}\mathcal{C}}.$$

Recall that $\mathbb{D}(\mathbf{L})$ is an join-semilattice, and in particular a directed set. Thus, we can form the direct limit of this family and obtain a residuate lattice $\mathcal{O}(\mathbf{L})$ that will contain all the algebras $\mathbf{L}_{\mathcal{C}}$ in a minimal way. Next section is devoted to the construction of the direct limit of any family of compatible homomorphisms and its basic properties.

3.3 Direct Limits

The direct limit of a directed family of algebras of the same signature is usually obtained as a suitable homomorphic image of the coproduct of this family. In this section, we describe an alternative construction of the direct limit that is briefly discussed in [24, (p. 114)] and [46, (Exercises 32 and 33, pp. 155-156)], and [29]. In the sequel we consider exclusively direct limits of algebras and algebra homomorphisms.

Recall that a partially ordered set $(I, \leqslant)$ is said to be a *directed set* if for any $i, j \in I$ there is a $k \in I$ such that $i, j \leqslant k$. Let $\mathcal{K}$ be a category of algebras and algebra homomorphisms, $(I, \leqslant)$ a directed set, and $\{\mathbf{A}_i : i \in I\}$ a family of objects of $\mathcal{K}$. A family $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j : i, j \in I,\ i \leqslant j\}$ of homomorphisms in $\mathcal{K}$ is a *directed system* for $\{\mathbf{A}_i : i \in I\}$ if for every $i \in I$, $f_{ii} = id_{\mathbf{A}_i}$, and for $i \leqslant j \leqslant k$, $f_{jk} \circ f_{ij} = f_{ik}$, that is to say, the diagrams

$$\begin{tikzcd}
\mathbf{A}_i \arrow[rr, "f_{ik}"] \arrow[dr, "f_{ij}"'] & & \mathbf{A}_k \\
& \mathbf{A}_j \arrow[ur, "f_{jk}"'] &
\end{tikzcd}$$

commute.[2] Given a directed system $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j : i, j \in I,\ i \leqslant j\}$ in $\mathcal{K}$, a family of

[2] More formally, one can think of a directed system in a category $\mathcal{K}$ as a functor $F : \mathcal{I} \to \mathcal{K}$, where the directed set $\mathcal{I} = (I, \leqslant)$ is regarded as a category.

homomorphisms $\{\phi_i : \mathbf{A}_i \to \mathbf{A} : i \in I\}$ is said to be *compatible* with it provided the equation $\phi_j \circ f_{ij} = \phi_i$ holds, for all $i \in I$. Such a family is called a *direct limit* of the directed system if it is "minimal" among the families of homomorphisms compatible with it, in the sense that it satisfies the following universal property: for any family $\{\psi_i : \mathbf{A}_i \to \mathbf{B} : i \in I\}$ compatible with $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j : i, j \in I,\ i \leqslant j\}$, there exists a unique $\psi : \mathbf{A} \to \mathbf{B}$ rendering the following diagram commutative, for all $i \in I$:

$$\begin{tikzcd} \mathbf{A} \arrow[r, dashed, "\exists!\psi"] & \mathbf{B} \\ \mathbf{A}_i \arrow[u, "\phi_i"] \arrow[ur, "\psi_i"'] & \end{tikzcd} \tag{3.2}$$

It is very easy to see that direct limits are unique up to a unique isomorphism, in the sense that whenever $\{\phi_i : \mathbf{A}_i \to \mathbf{A} : i \in I\}$ and $\{\psi_i : \mathbf{A}_i \to \mathbf{B} : i \in I\}$ are direct limits of the same system, then there exists a unique isomorphism $\psi : \mathbf{A} \to \mathbf{B}$ rendering commutative the diagram (3.2). Very often, the common target of the homomorphisms of the direct limit of a system is also called the *direct limit* of the system.

Intuitively, the elements of the direct limit are determined by "approximations," which are elements in the algebras of the system. Thanks to the compatibility law of the system and the property of being directed, those approximations can be chosen in algebras with arbitrarily large index. Thus, we intend to represent the elements of the limit by sequences of elements in the algebras such that, from one index on, they respect the compatibility law of the system. The behavior of the sequences "before" this index is irrelevant. Thus, two sequences such that from one index on are identical should be considered the same element in the limit. Formally, let $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j : i, j \in I,\ i \leqslant j\}$ be a directed system in a class $\mathcal{K}$ of algebras of the same signature, and consider the set T of *threads* in $\prod_{i \in I} A_i$:

$$T = \big\{a \in \textstyle\prod_{i \in I} A_i : \exists k \forall j \geqslant k,\ a_j = f_{kj}(a_k)\big\}. \tag{3.3}$$

In the definition of T, and in the sequel, we write a_i instead of $a(i)$, for $a \in \prod_{i \in I} A_i$ and $i \in I$. We define the following binary relation $\sim$ on T, for all $a, b \in T$:

$$a \sim b \quad \Leftrightarrow \quad \exists k \forall j \geqslant k,\ a_j = b_j. \tag{3.4}$$

It can be readily proven the following result.

Lemma 3.5. *The set T is the universe of a subalgebra $\mathbf{T}$ of $\prod_{i \in I} \mathbf{A}_i$, and moreover $\sim$ is a congruence of $\mathbf{T}$.*

Given a directed system $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j : i, j \in I,\ i \leqslant j\}$ and the set T of threads as defined in (3.3), we call $i \in I$ a *witness* of $a \in T$, or just a *witness* for a, if for every $k \geqslant i$, $a_k = f_{ik}(a_i)$. By the very definition of T, every thread has a witness and the set of witnesses of a thread is closed upwards.

Now we fix an arbitrary element $u \in \prod_{i \in I} \mathbf{A}_i$, and define the map $\phi_i : A_i \to T$ as follows for all $a \in A_i$:

$$\phi_i(a)_j = \begin{cases} f_{ij}(a) & \text{if } i \leqslant j; \\ u_j & \text{otherwise.} \end{cases} \tag{3.5}$$

One can easily verify that for each $a \in T$, and each witness i for a, $a \sim \phi_i(a_i)$. The map ϕ_i defined in (3.5) induces a map $\overline{\phi}_i : A_i \to T/{\sim}$ defined, for all $a \in A_i$, by:

$$\overline{\phi}_i(a) = [\phi_i(a)]_{\sim}. \tag{3.6}$$

In what follows, we denote by $\mathbf{A}$ the quotient $\mathbf{T}/{\sim}$. The next result shows that $\{\overline{\phi}_i : \mathbf{A}_i \to \mathbf{A} : i \in I\}$ is the direct limit of the directed system $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j : i, j \in I,\ i \leqslant j\}$. We sketch its proof for the convenience of the reader.

Proposition 3.6. *Given a directed system $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j : i, j \in I,\ i \leqslant j\}$, the family of homomorphisms $\{\overline{\phi}_i : \mathbf{A}_i \to \mathbf{A} : i \in I\}$ just defined above is the direct limit of $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j : i, j \in I,\ i \leqslant j\}$. That is, $\mathbf{A}$ has the universal property:*

$$\begin{tikzcd}
\mathbf{A}_i \arrow[rr, "f_{ij}"] \arrow[dr, bend right, "\overline{\phi}_i"] \arrow[ddr, bend right=40, "\psi_i"'] & & \mathbf{A}_j \arrow[dl, bend left, "\overline{\phi}_j"'] \arrow[ddl, bend left=40, "\psi_j"] \\
& \mathbf{A} \arrow[d, dashed, "\exists!\psi"'] & \\
& \mathbf{B} &
\end{tikzcd}$$

for any family $\{\psi_i : \mathbf{A}_i \to \mathbf{B} : i \in I\}$ of homomorphisms compatible with $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j : i, j \in I,\ i \leqslant j\}$, there is a unique $\psi : \mathbf{A} \to \mathbf{B}$ such that, for every $i \in I$, $\psi \circ \overline{\phi}_i = \psi_i$.

Proof. We leave the reader to verify that the system $\{\overline{\phi}_i : \mathbf{A}_i \to \mathbf{T}/{\sim} : i \in I\}$ is indeed a family of homomorphisms compatible with the directed system $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j : i, j \in I,\ i \leqslant j\}$

Suppose that $a, b \in T$ are such that $a \sim b$, and let i, j be witnesses for a, b, respectively, and k such that a and b agree from k on. Let's consider any $r \geqslant i, j, k$, which exists since I is a directed set. Thus, a and b agree on r and therefore

$$\psi_i(a_i) = \psi_r(f_{ir}(a_i)) = \psi_r(a_r) = \psi_r(b_r) = \psi_r(f_{jr}(b_j)) = \psi_j(b_j).$$

Therefore, we can define the map $\psi : \mathbf{A} \to \mathbf{B}$ in the following way: for every $[a]_\sim \in \mathbf{A}$,

$$\psi([a]_\sim) = \psi_i(a_i),$$

where i is any witness for a.

Let σ be an n-ary operation symbol in the signature and $a^1, \ldots, a^n \in T$ with common witness k. Then, it can be easily seen that k is also a witness for $\sigma^{\mathbf{T}}(a^1, \ldots, a^n)$, and hence:

$$\begin{aligned}\psi(\sigma^{\mathbf{A}}([a^1]_\sim, \ldots, [a^n]_\sim)) &= \psi([\sigma^{\mathbf{T}}(a^1, \ldots, a^n)]_\sim) = \psi_k(\sigma^{\mathbf{T}}(a^1, \ldots, a^n)_k) \\ &= \psi_k(\sigma^{\mathbf{A}_k}(a^1_k, \ldots, a^n_k)) = \sigma^{\mathbf{B}}(\psi_k(a^1_k), \ldots, \psi_k(a^n_k)) \\ &= \sigma^{\mathbf{B}}(\psi([a^1]_\sim), \ldots, \psi([a^n]_\sim)).\end{aligned}$$

That ψ renders the diagram commutative is a direct consequence of the fact that, for every $i \in I$, and every $a \in \mathbf{A}_i$, i is a witness for $\phi_i(a)$. As regards the uniqueness, note that if i is a witness of $a \in T$, then $a \sim \phi_i(a_i)$, and therefore if $\psi' : \mathbf{A} \to \mathbf{B}$ is a map rendering commutative the diagram, then

$$\psi'([a]_\sim) = \psi'([\phi_i(a_i)]_\sim) = \psi'(\overline{\phi}_i(a_i)) = \psi_i(a_i) = \psi([a]_\sim). \qquad \square$$

We define now a concept that will be very useful in the next section.

Definition 3.7. If $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j : i, j \in I,\ i \leqslant j\}$ is a directed system, $i \in I$, and $x \in \mathbf{A}_i$, we call x a *proxy* of $\overline{\phi}_i(x)$ at i.

Note that if $[a]_\sim \in \mathbf{A}$, and i is a witness for a, then a_i is a proxy of $[a]_\sim$ at i. Consequently, every element of the limit has a proxy at some index i, and the set of indices where a particular element has a proxy is closed upwards. Moreover, if $i \leqslant j$, $x \in \mathbf{A}_i$ and $y = f_{ij}(x)$, then x is a proxy of an element $s \in \mathbf{A}$ at i if and only if y is a proxy of s at j.

We note for future reference the following result:

Lemma 3.8. *If all the homomorphisms of a directed system of algebras are embeddings, then the homomorphisms of the direct limit are also embeddings.*

Under the assumptions of the preceding lemma, whenever an element of the direct limit $\mathbf{A}$ has a proxy in $i \in I$, this proxy is unique. Note also that, as a consequence of Proposition 3.6, the direct limit of a directed system is the quotient of a subalgebra of the product of the algebras of the system. Thus, varieties are closed under direct limits. In fact, it can be shown that the same result holds for quasivarieties.[3]

[3] Actually, a stronger result can be proved. Namely, given a set of quasi-equations Π and a directed

3.4 Lateral Completeness of $\mathcal{O}(\mathbf{L})$

We devote this section to the construction of a laterally complete extension, $\mathcal{O}(\mathbf{L})$, of an arbitrary e-cyclic residuated lattice $\mathbf{L}$, provided that it is semilinear. This extension $\mathcal{O}(\mathbf{L})$ will satisfy another very important property, namely $\mathbf{L}$ is densely[4] embeddable in $\mathcal{O}(\mathbf{L})$. Thus, the main result of this section, Corollary 3.22, is that every semilinear e-cyclic residuated lattice is densely embeddable in a laterally complete one. Moreover, we will construct $\mathcal{O}(\mathbf{L})$ as a limit of algebras that belong to the variety generated by $\mathbf{L}$, and therefore, $\mathcal{O}(\mathbf{L})$ stays in the same variety.

As we mentioned before, every semilinear residuated lattice is embeddable in a laterally complete one, which belongs to the same variety. Indeed, this is a consequence of the fact that, by definition, every semilinear residuated lattice is a subdirect product of chains, and in a chain there are no infinite sets of disjoint elements (in fact, no set with more that two elements can be disjoint), and therefore chains are laterally complete. Moreover, lateral completeness is preserved by products. Thus, the dense-embeddability of $\mathbf{L}$ into $\mathcal{O}(\mathbf{L})$ is a quite significant requirement. Intuitively, it is a restriction on the size of $\mathcal{O}(\mathbf{L})$.

We will split the main result into two parts. Surprisingly enough, the proof that $\mathbf{L}$ is densely embeddable in $\mathcal{O}(\mathbf{L})$, which is the statement of Theorem 3.14, is much simpler than the proof that $\mathcal{O}(\mathbf{L})$ is laterally complete, Theorem 3.21. Thus, most of the burden of this section will be into the proof of the lateral completeness of $\mathcal{O}(\mathbf{L})$.

Let $\mathbf{L}$ be an e-cyclic semilinear residuated lattice. In view of Theorem 1.13, all polars of $\mathbf{L}$ are normal, and hence for every $C \in \mathrm{Pol}(L)$ one can form the quotient algebra $\mathbf{L}/C^{\perp}$. For every partition $\mathcal{C}$ of $\mathrm{Pol}(\mathbf{L})$, we define the product $\mathbf{L}_{\mathcal{C}} = \prod_{C\in\mathcal{C}} \mathbf{L}/C^{\perp}$. If $\mathcal{C}$ and $\mathcal{A}$ are two partitions with $\mathcal{C} \preccurlyeq \mathcal{A}$, we define a homomorphism $\phi_{\mathcal{C}\mathcal{A}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{L}_{\mathcal{A}}$ in the following manner (see diagram below): for every $A \in \mathcal{A}$, we chose the unique $C \in \mathcal{C}$ such that $A \subseteq C$. Then, $C^{\perp} \subseteq A^{\perp}$, whence there exists a homomorphism $f_{CA} : \mathbf{L}/C^{\perp} \to \mathbf{L}/A^{\perp}$. Composing with the canonical projection $\pi_C : \mathbf{L}_{\mathcal{C}} \to \mathbf{L}/C^{\perp}$, we obtain a homomorphism $f_{CA}\pi_C : \mathbf{L}_{\mathcal{C}} \to \mathbf{L}/A^{\perp}$. Then, by the co-universal property of the product $\mathbf{L}_{\mathcal{A}}$, there exists a unique homomorphism $\phi_{\mathcal{C}\mathcal{A}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{L}_{\mathcal{A}}$ such that for all

system of algebras indexed on I, if the set $F \subseteq I$ of indeces of the algebras satisfying Π is cofinal in I, that is, for every index $i \in I$ there is another index $j \in F$ such that $i \leqslant j$ and Π valid in the algebra indexed by j, then the direct limit also satisfies Π.

[4]See Definition 3.13.

$A \in \mathcal{A}$, $\pi_A \phi_{\mathcal{C}\mathcal{A}} = f_{CA}\pi_C$, where $\pi_A : \mathbf{L}_{\mathcal{A}} \to \mathbf{L}/A^{\perp}$ is the canonical projection.

$$\begin{tikzcd} \mathbf{L}_{\mathcal{C}} \arrow[r, dashed, "\phi_{\mathcal{C}\mathcal{A}}"] \arrow[d, two heads, "\pi_C"'] & \mathbf{L}_{\mathcal{A}} \arrow[d, two heads, "\pi_A"] \\ \mathbf{L}/C^{\perp} \arrow[r, "f_{CA}"'] & \mathbf{L}/A^{\perp} \end{tikzcd}$$

We can describe $\phi_{\mathcal{C}\mathcal{A}}$ as follows: Every element $x \in \mathbf{L}_{\mathcal{C}}$ is the form $x = ([x_C]_{C^{\perp}} : C \in \mathcal{C})$, with $x_C \in L$. Then, $\phi_{\mathcal{C}\mathcal{A}}(x) = ([y_A]_{A^{\perp}} : A \in \mathcal{A})$, where for every $A \in \mathcal{A}$, $y_A = x_C$, for the unique $C \in \mathcal{C}$ such that $A \subseteq C$. Recall that the ordered set $\mathbb{D}(\mathbf{L})$ of all partitions is a join-semilattice, hence any two partitions have a common refinement. It can be easily shown, by using for instance the previous description, that $\{\phi_{\mathcal{C}\mathcal{A}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{L}_{\mathcal{A}} : \mathcal{C} \preccurlyeq \mathcal{A}\}$ is a directed system. We denote the direct limit of this system by $\mathcal{O}(\mathbf{L})$. Our objective in this section is to prove that $\mathcal{O}(\mathbf{L})$ is laterally complete and that $\mathbf{L}$ is densely embeddable into it (refer to Definitions 3.13 and 3.15 below).

Let us specialize the discussion of the preceding section to the construction of the direct limit of $\{\phi_{\mathcal{C}\mathcal{A}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{L}_{\mathcal{A}} : \mathcal{C} \preccurlyeq \mathcal{A}\}$. We start with the subalgebra $\mathbf{T}$ of $\prod_{\mathcal{C}\in\mathbb{D}(\mathbf{L})} \mathbf{L}_{\mathcal{C}}$ whose universe is the set $T = \{l \in \prod_{\mathcal{B}\in\mathbb{D}(\mathbf{L})} \mathbf{L}_{\mathcal{B}} : \exists \mathcal{C}\, \forall \mathcal{A} \succcurlyeq \mathcal{C},\ l_{\mathcal{A}} = \phi_{\mathcal{C}\mathcal{A}}(l_{\mathcal{C}})\}$. Then we obtain $\mathcal{O}(\mathbf{L})$ as the quotient of $\mathbf{T}$ by the congruence $\sim$ defined by: $l \sim k$ if there exists a partition $\mathcal{C}$ such that for any refinement $\mathcal{A}$ of it, $l_{\mathcal{A}} = k_{\mathcal{A}}$. Then, the elements of $\mathcal{O}(\mathbf{L})$ are the equivalence classes of the elements $l = (l_{\mathcal{C}} : \mathcal{C} \in \mathbb{D}(\mathbf{L})) \in T$. As we have noted in the previous section, for every partition $\mathcal{C}$, there exists a homomorphism $\overline{\phi}_{\mathcal{C}} : \mathbf{L}_{\mathcal{C}} \to \mathcal{O}(\mathbf{L})$. More specifically, first we fix an element of $\prod_{\mathcal{B}\in\mathbb{D}(\mathbf{L})} \mathbf{L}_{\mathcal{B}}$, in this case we conveniently choose the identity element e. We then define, for every $x \in \mathbf{L}_{\mathcal{C}}$, $\overline{\phi}_{\mathcal{C}}(x) = [\phi_{\mathcal{C}}(x)]_{\sim}$, where $\phi_{\mathcal{C}}(x) \in \prod_{\mathcal{C}\in\mathbb{D}(\mathbf{L})} \mathbf{L}_{\mathcal{C}}$ is such that

$$\phi_{\mathcal{C}}(x)_{\mathcal{A}} = \begin{cases} \phi_{\mathcal{C}\mathcal{A}}(x) & \text{if } \mathcal{C} \preccurlyeq \mathcal{A} \\ e_{\mathcal{C}} & \text{otherwise.} \end{cases} \tag{3.7}$$

Furthermore, since all the homomorphisms $\phi_{\mathcal{C}\mathcal{A}}$ are embeddings (see Lemma 3.12), the same is true for the homomorphisms $\overline{\phi}_{\mathcal{C}}$ by Lemma 3.8.

Given an element $x \in \mathbf{L}_{\mathcal{C}}$, $x = ([x_C]_{C^{\perp}} : C \in \mathcal{C})$, it will be very important to distinguish the polars C such that $[x_C]_{C^{\perp}} \neq [e]_{C^{\perp}}$, which we will call the support of x, from the rest. In Lemma 1.9, we gave some conditions characterizing $[a]_H = [e]_H$ in the case of a normal convex subalgebra H. If H is moreover a polar, then we can extend this lemma as follows.

Lemma 3.9. *If $\mathbf{L}$ is an e-cyclic residuated lattice and $H \in \mathrm{Pol}(\mathbf{L})$ is normal, then the following statements are equivalent:*

(i) $[a]_H = [e]_H$,

(ii) $a \in H$,

(iii) $C[a] \subseteq H$,

(iv) $C[a] \cap H^\perp = \{e\}$,

(v) $a^{\perp\perp} \cap H^\perp = \{e\}$.

Sometimes we will define some concepts depending on an element $x \in \mathbf{L}_\mathcal{C}$, $x = ([x_C]_{C^\perp} : C \in \mathcal{C})$, by means of its representatives x_C. We will need the following lemma to prove that those definitions are actually independent of the choice of representatives.

Lemma 3.10. *Let* $\mathbf{L}$ *be an e-cyclic residuated lattice,* $H \in \mathrm{Pol}(\mathbf{L})$ *be a normal convex subalgebra, and* $a, b \in L$*. If* $[a]_{H^\perp} = [b]_{H^\perp}$ *then* $a^{\perp\perp} \cap H = b^{\perp\perp} \cap H$*.*

Proof. Suppose that $[a]_{H^\perp} = [b]_{H^\perp}$. Then, $[|a|]_{H^\perp} = [|b|]_{H^\perp}$, and therefore by [10, Lemma 4.11], there exists $c \in H^\perp$ such that $|a|c \leqslant b$ and $|b|c \leqslant a$, and hence, for any $d \in b^\perp$:

$$|c| = e \cdot |c| = (|d| \vee |b|)|c| = |d||c| \vee |b||c| \leqslant |d| \vee |b|c \leqslant |d| \vee |a|.$$

Therefore, for any $h \in H$, $e = |c| \vee |h| \leqslant |d| \vee |a| \vee |h|$, whence, $|h| \vee |d| \in a^\perp$. If moreover $h \in a^{\perp\perp}$, then $|h| \vee |d| \in a^{\perp\perp}$, and therefore $|h| \vee |d| = e$. Thus, for any $h \in a^{\perp\perp} \cap H$, we have proved that $h \in b^{\perp\perp}$, as we wanted. □

The converse of the implication of the previous lemma is not true in general. In order to provide a counterexample it suffices to consider a pair of distinct elements a, b in an integral semilinear residuated lattice $\mathbf{L}$ such that $a^{\perp\perp} = b^{\perp\perp}$ and let $H = L$ (and therefore $H^\perp = \{e\}$). For example, this is the case for the three-element Gödel algebra $\mathbf{L}$ with $a = 0 < b = \frac{1}{2} < e = 1$.

Definition 3.11. Let $\mathbf{L}$ be an e-cyclic semilinear residuated lattice, and let $\mathcal{O}(\mathbf{L})$ be the direct limit of $\{\phi_{\mathcal{C}\mathcal{A}} : \mathbf{L}_\mathcal{C} \to \mathbf{L}_\mathcal{A} : \mathcal{C} \preccurlyeq \mathcal{A}\}$. Given a partition $\mathcal{C}$ of $\mathrm{Pol}(\mathbf{L})$ and an element $x = ([x_C]_{C^\perp} : C \in \mathcal{C}) \in \mathbf{L}_\mathcal{C}$, we define the *support* of x at $\mathcal{C}$ to be the set $\mathrm{Supp}(x) = \{C \in \mathcal{C} : [x_C]_{C^\perp} \neq [e]_{C^\perp}\}$.

It is clear from the definition that, for any $x \in \mathbf{L}_\mathcal{C}$, x is equal to the identity element $e_\mathcal{C}$ of $\mathbf{L}_\mathcal{C}$ if and only if $\mathrm{Supp}(x) = \emptyset$.

Lemma 3.12. *Let* $\mathbf{L}$ *be an e-cyclic semilinear residuated lattice and let* $\mathcal{C}, \mathcal{A}$ *two partitions with* $\mathcal{C} \preccurlyeq \mathcal{A}$*. For every* $C \in \mathcal{C}$ *let* $\mathcal{A}_C = \{A \in \mathcal{A} : A \subseteq C\}$*. Then:*

1. *For all* $x \in \mathbf{L}_\mathcal{C}$*,* $C \in \mathrm{Supp}(x)$ *if and only if* $A \in \mathrm{Supp}(\phi_{\mathcal{C}\mathcal{A}}(x))$*, for some* $A \in \mathcal{A}_C$*; and*
2. $\phi_{\mathcal{C}\mathcal{A}}$ *is injective.*

Proof. Both (1) and (2) follow directly from Corollary 3.4.(5). Indeed, let $x = ([x_C]_{C^\perp} : C \in \mathcal{C}) \in \mathbf{L}_\mathcal{C}$ and let $y = ([y_A]_{A^\perp} : A \in \mathcal{A}) = \phi_{\mathcal{C}\mathcal{A}}(x) \in \mathbf{L}_\mathcal{A}$. As we noted above, if $C \in \mathcal{C}$ and $A \in \mathcal{A}_C$, then we can choose $y_A = x_C$. Thus, for any $C \in \mathcal{C}$, since $\mathbf{L}/C^\perp$ is a subdirect product of the algebras in $\{\mathbf{L}/A^\perp : A \in \mathcal{A}_C\}$, $[x_C]_{C^\perp} \neq [e]_{C^\perp}$ if and only if $[y_A]_{A^\perp} \neq [e]_{A^\perp}$ for some $A \in \mathcal{A}_C$. □

As noted in Lemma 3.12, $\phi_{\mathcal{C}\mathcal{A}}$ is injective whenever $\mathcal{C} \preccurlyeq \mathcal{A}$. Whence, for the particular case of the trivial partition $\{L\}$, we have $\mathbf{L}_{\{L\}} = \mathbf{L}/L^\perp = \mathbf{L}/\{e\} \cong \mathbf{L}$. Therefore, there exists an embedding of $\mathbf{L}$ into $\mathcal{O}(\mathbf{L})$, more specifically the composition of the isomorphism $\mathbf{L} \cong \mathbf{L}/\{e\}$ with the embedding $\overline{\phi}_{\{L\}}$. In Theorem 3.14 below, we prove that this embedding is dense in the sense of the next definition.

Definition 3.13. An embedding $\phi : \mathbf{L} \to \mathbf{L}'$ between residuated lattices is *dense* if for every $p \in L'$, $p < e$, there exists $a \in L$ such that $p \leqslant \phi(a) < e$.

Recall that every element of $\mathcal{O}(\mathbf{L})$ has a proxy at some partition $\mathcal{C}$. That is, given an element $p \in \mathcal{O}(\mathbf{L})$ there exists a partition $\mathcal{C}$ and an element $x \in \mathbf{L}_\mathcal{C}$ such that $\overline{\phi}_\mathcal{C}(x) = p$. Moreover, due to the fact that the homomorphisms are embeddings, the proxies of p at a given partition are unique, if they exist. Obviously, an element of $\mathcal{O}(\mathbf{L})$ is different from the identity if and only if all its proxies, at the different partitions at which they exist, are different from the identity.

Theorem 3.14. *Any e-cyclic semilinear residuated lattice* $\mathbf{L}$ *can be densely embedded into* $\mathcal{O}(\mathbf{L})$.

Proof. As was noted above, the map $\overline{\phi} : \mathbf{L} \xrightarrow{\cong} \mathbf{L}_{\{L\}} \xrightarrow{\overline{\phi}_{\{L\}}} \mathcal{O}(\mathbf{L})$ is an embedding of $\mathbf{L}$ into $\mathcal{O}(\mathbf{L})$. For every $a \in L$, $\overline{\phi}(a) = [\overline{a}]_\sim$, where $\overline{a} = (\overline{a}_\mathcal{C} : \mathcal{C} \in \mathbb{D}(\mathbf{L}))$, and for every partition $\mathcal{C}$, $\overline{a}_\mathcal{C} = ([a]_{C^\perp} : C \in \mathcal{C})$.

In order to establish the density of $\overline{\phi}$, consider $p \in \mathcal{O}(\mathbf{L})$ is such that $p < e_{\mathcal{O}(\mathbf{L})}$. Let $x = ([x_C]_{C^\perp} : C \in \mathcal{C})$ be a proxy of p at some partition $\mathcal{C}$. Then, for every $C \in \mathcal{C}$, $[x_C]_{C^\perp} \leqslant [e]_{C^\perp}$, and hence there is no loss of generality to assume that all the representatives x_C are negative. Since $p \neq e_{\mathcal{O}(\mathbf{L})}$, there exists a $C \in \mathcal{C}$ such that $[x_C]_{C^\perp} \neq [e]_{C^\perp}$. Therefore, by Lemma 3.9.(v), $x_C^{\perp\perp} \cap C \neq \{e\}$ and we can choose an element $b \in x_C^{\perp\perp} \cap C$, $b < e$. By the convexity of the polars, $a = x_C \vee b \in x_C^{\perp\perp} \cap C$. If $a = e$, then $b \in x_C^\perp$, and therefore $b = e$, contradicting the hypothesis that $b < e$. Hence, $x_C \leqslant a < e$.

Since $a \in C$, $a^{\perp\perp} \subseteq C$, and hence for every $D \in \mathcal{C}$, $C \neq D$ implies $a^{\perp\perp} \cap D = \{e\}$, and therefore $[a]_{D^\perp} = [e]_{D^\perp}$. Thus, $\overline{a}_\mathcal{C} = ([a]_{C^\perp} : C \in \mathcal{C})$ has only one component different from the identity, which is $[a]_{C^\perp}$, and moreover $x_C \leqslant a$ implies $[x_C]_{C^\perp} \leqslant [a]_{C^\perp}$. Hence $x \leqslant \overline{a}_\mathcal{C} < e$, and then $p = \overline{\phi}_\mathcal{C}(x) \leqslant \overline{\phi}_\mathcal{C}(\overline{a}_\mathcal{C}) = \overline{\phi}(a) < e_{\mathcal{O}(\mathbf{L})}$. □

Definition 3.15. Two elements $a, b < e$ of a residuated lattice $\mathbf{L}$ are said to be *disjoint* if $a \vee b = e$. An non-empty subset $X \subseteq L$ is called *disjoint* provided any two distinct elements of it are disjoint. A residuated lattice is said to be *laterally complete* if all disjoint subsets of it have an infimum.

Remark 3.16. *Let $\{x_\lambda : \lambda \in \Lambda\}$ be a nonempty family of elements of $\mathbf{L}_{\mathcal{C}}^-$, for some partition $\mathcal{C}$, which have pairwise disjoint supports:* $\mathrm{Supp}(x_\lambda) \cap \mathrm{Supp}(x_\mu) = \emptyset$ *if $\lambda \neq \mu$. Then the meet $\bigwedge_\Lambda^{\mathbf{L}_{\mathcal{C}}} x_\lambda$ exists. Actually, it is trivial if the family has only one element, and otherwise $\bigwedge_\Lambda^{\mathbf{L}_{\mathcal{C}}} x_\lambda = z = ([z_C]_{C^\perp} : C \in \mathcal{C})$, where*

$$z_C = \begin{cases} (x_\lambda)_C & \textit{if } C \in \mathrm{Supp}(x_\lambda), \textit{for some (unique) } \lambda \in \Lambda; \\ e & \textit{otherwise.} \end{cases}$$

In the remainder of the section we prove that, given a family of disjoint elements $S \subseteq \mathcal{O}(\mathbf{L})$, there exists a partition $\mathcal{E}$ such that (i) every element of the disjoint family has a proxy at $\mathcal{E}$, (ii) the proxies of these elements have disjoint support, and (iii) their meet is a proxy of the meet of S. We start by proving two fairly technical lemmas, which will be used in the following proofs. The intuition behind them is that, under certain conditions involving partitions and the supports of the elements, we can move proxies around and choose them in a "canonical way."

Lemma 3.17. *Let $\mathbf{L}$ be an e-cyclic semilinear residuated lattice and let $\mathcal{C}, \mathcal{A}$ be two partitions such that $\mathcal{C} \preccurlyeq \mathcal{A}$. Then, whenever $y \in \mathbf{L}_{\mathcal{A}}$ and* $\mathrm{Supp}(y) \subseteq \mathcal{C}$*, then there is a (unique) $x \in \mathbf{L}_{\mathcal{C}}$ such that $\phi_{\mathcal{C}\mathcal{A}}(x) = y$. Furthermore,* $\mathrm{Supp}(x) = \mathrm{Supp}(y)$.

Proof. Let $y = ([y_A]_{A^\perp} : A \in \mathcal{A}) \in \mathbf{L}_{\mathcal{A}}$ such that $\mathrm{Supp}(y) \subseteq \mathcal{C}$. For every $C \in \mathcal{C}$, we define $x_C = y_C$ if $C \in \mathrm{Supp}(y)$, and $x_C = e$ otherwise, and set $x = ([x_C]_{C^\perp} : C \in \mathcal{C}) \in \mathbf{L}_{\mathcal{C}}$. Then obviously $\mathrm{Supp}(x) = \mathrm{Supp}(y)$.

We claim that $\phi_{\mathcal{C}\mathcal{A}}(x) = y$, which will establish the statement of the lemma. Let $\phi_{\mathcal{C}\mathcal{A}}(x) = ([t_A]_{A^\perp} : A \in \mathcal{A})$. Recall that for each $A \in \mathcal{A}$, we can choose $t_A = x_C$, where C is the unique element in $\mathcal{C}$ such that $A \subseteq C$. Consider A and C as in the preceding sentence. If $C \in \mathrm{Supp}(y)$, which by assumption is a subset of $\mathcal{C}$, then $A \subseteq C \in \mathcal{A}$ implies $A = C \in \mathrm{Supp}(y)$. Thus, if $A \notin \mathrm{Supp}(y)$, then $C \notin \mathrm{Supp}(y) = \mathrm{Supp}(x)$, and therefore $[t_A]_{A^\perp} = [e_A]_{A^\perp} = [y_A]_{A^\perp}$. On the other hand, if $A \in \mathrm{Supp}(y)$, then $C = A$, because $A \in \mathcal{C}$, and therefore $t_A = x_A = y_A$. Thus, we have shown that $\phi_{\mathcal{C}\mathcal{A}}(x) = y$, as required. □

Given a proxy $x \in \mathbf{L}_{\mathcal{C}}$ of an element $p \in \mathcal{O}(\mathbf{L})$, exactly one of the following situations occurs for every $C \in \mathcal{C}$:

(i) $x_C^{\perp\perp} \cap C = \{e\}$,

(ii) $C \subseteq x_C^{\perp\perp}$, or

(iii) $x_C^{\perp\perp} \cap C \neq \{e\}$ and $C \nsubseteq x_C^{\perp\perp}$.

By virtue of Lemma 3.9, (i) is equivalent to $[x_C]_{C^\perp} = [e]_{C^\perp}$, that is, $C \in \mathrm{Supp}(x)$, while (ii) implies that $C \notin \mathrm{Supp}(x)$. The next lemma shows that (iii) is avoidable in the sense that proxies can be chosen so that their coordinates satisfy either (i) or (ii).

Definition 3.18. Let $\mathbf{L}$ be an e-cyclic semilinear residuated lattice and $\mathcal{C}$ a partition of the Boolean algebra of polars of $\mathbf{L}$. An element $x \in \mathbf{L}_\mathcal{C}$ is said to be *canonical* if for every $C \in \mathrm{Supp}(x)$, $C \subseteq x_C^{\perp\perp}$.

Notice that canonicity is a well-defined notion, that is, it does not depend on the representatives: if $[a]_{C^\perp} = [b]_{C^\perp}$ then, by virtue of Lemma 3.10, $a^{\perp\perp} \cap C = b^{\perp\perp} \cap C$, and therefore $C \subseteq a^{\perp\perp}$ if and only if $C \subseteq b^{\perp\perp}$. It is also important to note, and easy to prove, that canonicity is preserved by refinements, in the sense that if $x \in \mathbf{L}_\mathcal{C}$ is canonical and $\mathcal{C} \preccurlyeq \mathcal{A}$, then $\phi_{\mathcal{C}\mathcal{A}}(x)$ is also canonical.

Lemma 3.19. *Let $\mathbf{L}$ be an e-cyclic semilinear residuated lattice, and let $\mathcal{O}(\mathbf{L})$ be the direct limit of the family $\{\phi_{\mathcal{C}\mathcal{A}} : \mathbf{L}_\mathcal{C} \to \mathbf{L}_\mathcal{A} : \mathcal{C} \preccurlyeq \mathcal{A}\}$. Consider an arbitrary partition $\mathcal{C}$ and an element $p \in \mathcal{O}(\mathbf{L})$. Then:*

1. *If x is a proxy of p at $\mathcal{C}$, then there is a refinement $\mathcal{A}$ of $\mathcal{C}$ such that $y = \phi_{\mathcal{C}\mathcal{A}}(x)$ is canonical.*
2. *If x is a proxy of p at $\mathcal{C}$ and $\mathcal{B}$ is any partition such that $\mathrm{Supp}(x) \subseteq \mathcal{B}$, then p has a proxy z at $\mathcal{B}$ and $\mathrm{Supp}(z) = \mathrm{Supp}(x)$. Moreover, if x is canonical, then so is z.*

Proof.

1. Let x be a proxy of p at $\mathcal{C}$ and consider the set $\mathcal{E}_x = \{x_C^{\perp\perp} \cap C : C \in \mathrm{Supp}(x)\}$. Since $\mathcal{C}$ is a disjoint family of polars, so is $\mathcal{E}_x$. Moreover $\{e\} \notin \mathcal{E}_x$, and therefore it can be extended to a partition $\bar{\mathcal{E}}_x$. Consider the common refinement $\mathcal{A} = \mathcal{C} \vee \bar{\mathcal{E}}_x$ (see Equation (3.1)) of both $\mathcal{C}$ and $\bar{\mathcal{E}}_x$. Notice that $\mathcal{E}_x \subseteq \mathcal{A}$. Indeed, if $E \in \mathcal{E}_x$, then there is $C \in \mathcal{C}$ such that $\{e\} \neq E = x_C^{\perp\perp} \cap C$, whence $E = C \cap E \in \mathcal{C} \vee \bar{\mathcal{E}}_x$.

Let $y = \phi_{\mathcal{C}\mathcal{A}}(x)$. As usual, we choose the representatives of y as follows: $y_A = x_C$, where for every $A \in \mathcal{A}$, C is the unique polar such that $A \subseteq C \in \mathcal{C}$. In order to prove the canonicity of y, consider an arbitrary $A \in \mathcal{A}$. If $A \in \mathcal{E}_x$, then there is $C \in \mathrm{Supp}(x)$ such that $A = x_C^{\perp\perp} \cap C \subseteq x_C^{\perp\perp} = y_A^{\perp\perp}$. If $A \notin \mathcal{E}_x$, let C the unique polar such that $A \subseteq C \in \mathcal{C}$. If $C \notin \mathrm{Supp}(x)$, then $y_A^{\perp\perp} \cap A \subseteq x_C^{\perp\perp} \cap C = \{e\}$, whence $A \notin \mathrm{Supp}(y)$. If $C \in \mathrm{Supp}(x)$, then $x_C^{\perp\perp} \cap C$ and A are two distinct moments of $\mathcal{A}$ (since $A \notin \mathcal{E}_x$), and so $(X_C^{\perp\perp}C) \cap A = \{e\}$. Hence

$$y_A^{\perp\perp} \cap A = x_C^{\perp\perp} \cap (C \cap A) = (X_C^{\perp\perp}C) \cap A = \{e\},$$

showing that $A \notin \mathrm{Supp}(y)$. Notice that we have actually proven that $\mathrm{Supp}(y) = \mathcal{E}_x$, and for every $A \in \mathrm{Supp}(y)$, $A \subseteq y_A^{\perp\perp}$.

2. Suppose now that x is a proxy of p at $\mathcal{C}$ and that $\mathcal{B}$ is a partition such that $\mathrm{Supp}(x) \subseteq \mathcal{B}$. Consider $\mathcal{A} = \mathcal{B} \vee \mathcal{C}$ and $y = \phi_{\mathcal{C}\mathcal{A}}(x)$, where we choose the representatives of y as usual. Since $\mathrm{Supp}(x) \subseteq \mathcal{B}$ and $\mathrm{Supp}(x) \subseteq \mathcal{C}$, then obviously $\mathrm{Supp}(x) \subseteq \mathcal{A}$, whence it follows that $\mathrm{Supp}(y) = \mathrm{Supp}(x)$. By virtue of Lemma 3.17, there is $z \in \mathbf{L}_{\mathcal{B}}$ such that $\phi_{\mathcal{B}}(z) = y$ and $\mathrm{Supp}(z) = \mathrm{Supp}(y) = \mathrm{Supp}(x)$. Moreover, if x is canonical then y is canonical, and by the way we construct z, we deduce also the canonicity of z. □

The next lemma is the missing piece that we need to prove Theorem 3.21. We have already seen that we can choose proxies in a canonical way and that, under certain conditions, we can move them from one partition to another. Informally, we could say that the "information" carried by an element lays in the coordinates of its support, and that we can innocuously "move" it from its original partition to another, as far as the new partition contains its support. What we prove in the next lemma is that being disjoint is also a property that depends on the support of the elements. Namely, two negative elements are disjoint if and only if their supports are disjoint, in the sense that they form a disjoint set of polars.

Lemma 3.20. *Let $\mathbf{L}$ be an e-cyclic semilinear residuated lattice, $p, q < e$ in $\mathcal{O}(\mathbf{L})$, and x and y canonical proxies of p and q at some partitions $\mathcal{C}$ and $\mathcal{D}$, respectively. Then, p and q are disjoint elements of $\mathcal{O}(\mathbf{L})$ if and only if* $\mathrm{Supp}(x) \cup \mathrm{Supp}(y)$ *is a disjoint set of polars of* $\mathbf{L}$.

Proof. Without loss of generality, we can assume that all the representatives of x and y are negative, since $p, q < e$. Let $\mathcal{A} = \mathcal{C} \vee \mathcal{D}$, $s = \phi_{\mathcal{C}\mathcal{A}}(x)$ and $t = \phi_{\mathcal{B}\mathcal{A}}(y)$, where the representatives of s and t are chosen in the usual way.

Suppose that $C \in \mathcal{C}$ and $D \in \mathcal{D}$ are such that $A = C \cap D \neq \{e\}$. Notice that $(s_A \vee t_A)^{\perp\perp} = (x_C \vee y_D)^{\perp\perp} = x_C^{\perp\perp} \cap y_D^{\perp\perp}$, in virtue of Lemma 1.12. One can easily see that the result follows from the fact that:

$$A \in \mathrm{Supp}(s \vee t) \quad \Leftrightarrow \quad C \in \mathrm{Supp}(x) \text{ and } D \in \mathrm{Supp}(y).$$

The implication $(\Rightarrow)$ can be readily obtained. From the implication $(\Leftarrow)$ we need to use the canonicity of x and y. Indeed, if $C \in \mathrm{Supp}(x)$ and $D \in \mathrm{Supp}(y)$, then $C \subseteq x_C^{\perp\perp}$ and $D \subseteq y_D^{\perp\perp}$, and hence $\{e\} \neq A = C \cap D \subseteq x_C^{\perp\perp} \cap y_D^{\perp\perp} = (s_A \vee t_A)^{\perp\perp}$. □

We have now all the tools we need to prove that $\mathcal{O}(\mathbf{L})$ is actually laterally complete. The idea is the following: given a disjoint set of negative elements of $\mathcal{O}(\mathbf{L})$, we can conveniently choose canonical proxies for them, whose supports will be disjoint to one another. Therefore, we can collect all the supports and complete this to a partition, in which we can also find proxies of the elements of original family, whose infimum exists and is a proxy of the infimum of the original family.

Theorem 3.21. *If* $\mathbf{L}$ *is an e-cyclic semilinear residuated lattice, then* $\mathcal{O}(\mathbf{L})$ *is laterally complete.*

Proof. Let $\{p_\lambda : \lambda \in \Lambda\}$ be a disjoint subset of $\mathcal{O}(\mathbf{L})$, and for every $\lambda \in \Lambda$, let x_λ be a canonical proxy of p_λ at some partition $\mathcal{C}_\lambda$. Then, by Lemma 3.20, the set $\bigcup_\Lambda \mathrm{Supp}(x_\lambda)$ is a disjoint set of polars of $\mathbf{L}$ and can be extended to a partition $\mathcal{E}$. Now, for every $\lambda \in \Lambda$, $\mathcal{E}$ is a partition containing $\mathrm{Supp}(x_\lambda)$, and then by virtue of Lemma 3.19, p_λ has a canonical proxy x'_λ at $\mathcal{E}$ and $\mathrm{Supp}(p_\lambda, \mathcal{E}) = \mathrm{Supp}(x_\lambda)$. It follows that the supports of the elements x'_λ at $\mathcal{E}$ are all disjoint, and therefore their meet $z = \bigwedge_\Lambda x'_\lambda$ in $\mathbf{L}_\mathcal{E}$ exists, by Remark 3.16.

We complete the proof by showing that $\bigwedge_\Lambda p_\lambda$ exists and z is its proxy at $\mathcal{E}$. Since $z \leqslant x'_\lambda$ for all $\lambda \in \Lambda$, $\overline{\phi}_\mathcal{E}(z) \leqslant \overline{\phi}_\mathcal{E}(x'_\lambda) = p_\lambda$. Suppose now that $q \in \mathcal{O}(\mathbf{L})$ is a lower bound of $\{p_\lambda : \lambda \in \Lambda\}$, let y be a proxy of q at some partition $\mathcal{C}$, and let $\mathcal{A}$ be a refinement of $\mathcal{E}$ and $\mathcal{C}$. Set $y_\lambda = \phi_{\mathcal{E}\mathcal{A}}(x'_\lambda)$, for every $\lambda \in \Lambda$. It is not dif and only ificult to see that $\bigwedge_\Lambda y_\lambda$ exists in $\mathbf{L}_\mathcal{A}$, and actually $\bigwedge_\Lambda y_\lambda = \phi_{\mathcal{E}\mathcal{A}}(z)$: Obviously, $\phi_{\mathcal{E}\mathcal{A}}(z) \leqslant y_\lambda$, for every $\lambda \in \Lambda$. Suppose now that $s \in \mathbf{L}_\mathcal{A}$ and for every $\lambda \in \Lambda$, $s \leqslant y_\lambda$. Fix $A \in \mathcal{A}$ and let $E \in \mathcal{E}$ be the unique element in $\mathcal{E}$ such that $A \subseteq E$. Since all the supports of the x'_λ are disjoint, then there is at most one $\lambda_0 \in \Lambda$ such that $[(x'_{\lambda_0})_E]_{E^\perp} \neq [e]_{E^\perp}$, in which case $[z_E]_{E^\perp} = [(x'_{\lambda_0})_E]_{E^\perp}$, and therefore $[s_A]_{A^\perp} \leqslant [(x'_{\lambda_0})_A]_{A^\perp} = [z_A]_{A^\perp}$. Otherwise, $[z_E]_{E^\perp} = [e]_{E^\perp}$, whence $[s_A]_{A^\perp} \leqslant [z_A]_{A^\perp}$. Thus, it follows that $s \leqslant \phi_{\mathcal{E}\mathcal{A}}(z)$.

Further, for every $\lambda \in \Lambda$,

$$\overline{\phi}_\mathcal{A}(\phi_{\mathcal{C}\mathcal{A}}(y) \wedge y_\lambda) = \overline{\phi}_\mathcal{C}(y) \wedge \overline{\phi}_\mathcal{A}(y_\lambda) = q \wedge p_\lambda = q = \overline{\phi}_\mathcal{A}(\phi_{\mathcal{C}\mathcal{A}}(y)).$$

Therefore, due to the injectivity of $\overline{\phi}_\mathcal{A}$, $\phi_{\mathcal{C}\mathcal{A}}(y) \wedge y_\lambda = \phi_{\mathcal{C}\mathcal{A}}(y)$, that is to say, $\phi_{\mathcal{C}\mathcal{A}}(y) \leqslant y_\lambda$. This implies that $\phi_{\mathcal{C}\mathcal{A}}(y) \leqslant \bigwedge_\Lambda y_\lambda = \phi_{\mathcal{E}\mathcal{A}}(z)$, and therefore $q = \overline{\phi}_\mathcal{A}(\phi_{\mathcal{C}\mathcal{A}}(y)) \leqslant \overline{\phi}_\mathcal{A}(\phi_{\mathcal{E}\mathcal{A}}(z)) = \overline{\phi}_\mathcal{E}(z)$. This establishes the proof of $\overline{\phi}_\mathcal{E}(z) = \bigwedge_\Lambda p_\lambda$. □

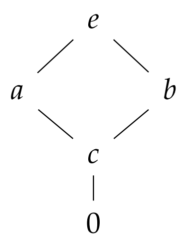
Finally, we have the main result of the section:

Corollary 3.22. *Every e-cyclic semilinear residuated lattice* $\mathbf{L}$ *is densely embeddable in a laterally complete lattice that belongs to the variety generated by* $\mathbf{L}$.

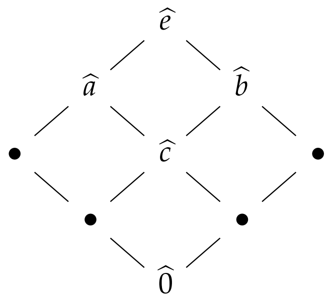
Proof. It is an immediate consequence of Theorems 3.14 and 3.21, and the fact that $\mathcal{O}(\mathbf{L})$ is a direct limit of products of quotients of $\mathbf{L}$. □

As we already mentioned, $\mathcal{O}(\mathbf{L})$ cannot be "much larger" than $\mathbf{L}$, since $\mathbf{L}$ is dense in $\mathcal{O}(\mathbf{L})$. We could then inquire into the minimality of $\mathcal{O}(\mathbf{L})$. That is, we can ask whether $\mathcal{O}(\mathbf{L})$ is the smallest laterally complete residuated lattice in which $\mathbf{L}$ is densely embeddable? The answer is no in general, and it is not dif and only ificult to find a counterexample:

Example 3.23. Consider the Heyting algebra $\mathbf{L}$ given by the following Hasse diagram:



It can be easily seen that $\mathbf{L}$ is an integral semilinear residuated lattice (Gödel algebra). The Boolean algebra of polars of $\mathbf{L}$ is $\mathrm{Pol}(\mathbf{L}) = \{\{e\}, a^{\perp\perp}, b^{\perp\perp}, L\}$, with $a^{\perp\perp} = \{e, a\}$ and $b^{\perp\perp} = \{e, b\}$. Hence, the set of partitions of $\mathrm{Pol}(\mathbf{L})$ is $\mathbb{D}(\mathbf{L}) = \{\{L\}, \{a^{\perp\perp}, b^{\perp\perp}\}\}$. Let us denote the non trivial partition of $\mathbf{L}$ by $\mathcal{C}$. $\mathbb{D}(\mathbf{L})$ is a directed set with a top element, namely $\mathcal{C}$, and therefore the limit of the directed system $\{\phi_{\{L\}\mathcal{C}} : \mathbf{L}_{\{L\}} \to \mathbf{L}_{\mathcal{C}}\}$ is $\mathbf{L}_{\mathcal{C}}$ itself. It is not dif and only ificult to see that $\mathbf{L}/a^{\perp}$ is a chain with three elements $[0]_{a^{\perp}} < [a]_{a^{\perp}} < [e]_{a^{\perp}}$, and analogously $\mathbf{L}/b^{\perp}$. Then $\mathcal{O}(\mathbf{L})$ is the Heyting algebra:



where we have named the images of the embedding of $\mathbf{L}$ into $\mathcal{O}(\mathbf{L})$. We note that since $\mathbf{L}$, being finite, is laterally complete, the theory developed in this section does not produce a "minimal" laterally complete extension. We devote Section 3.6 to prove the existence of minimal such extensions in the class of GMV algebras.

Notice that the preceding algebra $\mathbf{L}$ is not projectable,[5] while $\mathcal{O}(\mathbf{L})$ is. This is actually an instance of a general result, which will be discussed in the next section, and the

[5] See Definition 3.24.

main reason why the algebra $\mathbf{L}$ of the precedent example cannot be $\mathcal{O}(\mathbf{L})$.

3.5 Projectablility of $\mathcal{O}(\mathbf{L})$ and $\mathcal{O}_{<\omega}(\mathbf{L})$

As the title of the section suggests, we are going to study the projectability of the lattice $\mathcal{O}(\mathbf{L})$, which we constructed in the last section, provided an e-cyclic residuated lattice $\mathbf{L}$. We start by showing that $\mathcal{O}(\mathbf{L})$ is projectable, and therefore every e-cyclic semilinear residuated lattice can be densely embeddable in a projectable lattice that belongs to the variety generated by $\mathbf{L}$. We actually show that there is another projectable residuated lattice, denoted by $\mathcal{O}_{<\omega}(\mathbf{L})$, which is also obtained as a direct limit of products of quotients of $\mathbf{L}$, densely contains $\mathbf{L}$, and is in general smaller than $\mathcal{O}(\mathbf{L})$. Towards the end of the section we are going to prove that, actually, both $\mathcal{O}(\mathbf{L})$ and $\mathcal{O}_{<\omega}(\mathbf{L})$ are strongly projectable.

We recall first the definitions of projectability and strong projectability for arbitrary e-cyclic residuated lattices. Remember that we have two pair of notions of projectability, corresponding to the respective lattice-theoretical and structural characterizations of projectability for ℓ-groups:

Definition 3.24. An e-cyclic residuated lattice $\mathbf{L}$ is $\vee$*-projectable* if every principal polar is a complemented element of $\mathcal{C}(\mathbf{L})$. That is, for all $a \in L$,

$$L = a^{\perp} \vee^{\mathcal{C}(\mathbf{L})} a^{\perp\perp}.$$

It is called *strongly* $\vee$*-projectable* if for every convex subalgebra $H \in \mathcal{C}(\mathbf{L})$,

$$\mathbf{L} = H^{\perp} \vee^{\mathcal{C}(\mathbf{L})} H^{\perp\perp}.$$

An $\mathbf{L}$ is $\boxplus$*-projectable* if for every $a \in L$,

$$L = a^{\perp} \boxplus a^{\perp\perp}.$$

It is called *strongly* $\boxplus$*-projectable* if for every convex subalgebra $H \in \mathcal{C}(\mathbf{L})$,

$$\mathbf{L} = H^{\perp} \boxplus H^{\perp\perp}.$$

Lastly, $\mathbf{L}$ is said to be *($\boxplus$-) $\vee$-orthocomplete* if it is both laterally complete and strongly ($\boxplus$-) $\vee$-projectable.

Recall that $\mathbf{L} = \mathbf{B} \boxplus \mathbf{C}$ means that $\mathbf{L}$ is the *internal cardinal product* of the subalgebars $\mathbf{B}$ and $\mathbf{C}$, i.e. every $a \in L$ can be written uniquely as a product bc, for some $b \in B$ and some $c \in C$, and moreover, $a_1 = b_1 c_1 \leqslant^{\mathbf{L}} b_2 c_2 = a_2$ if and only if $b_1 \leqslant^{\mathbf{B}} b_2$ and $c_1 \leqslant^{\mathbf{C}} c_2$.

Remark 3.25. *It is easy to see that (strong) $\boxplus$-projectability implies (strong) $\vee$-projectability, since for every $H, K \in \mathcal{C}(\mathbf{L})$, $H \boxplus K \subseteq H \vee^{\mathcal{C}(\mathbf{L})} K$. Therefore, sometimes we will drop the prefix '$\boxplus$-' and say that $\mathbf{L}$ is* projectable, *instead of $\boxplus$-projectable. Also, notice that $\mathbf{L}$ is strongly $\vee$-projectable if and only if the Boolean algebra of polars* $\mathrm{Pol}(L)$ *is a sublattice of $\mathcal{C}(\mathbf{L})$, that is $\mathcal{C}(\mathbf{L})$ is a Stonean lattice.*[6]

In what follows, given an e-cyclic semilinear residuated lattice $\mathbf{L}$ and $S \subseteq \mathcal{O}(\mathbf{L})$, we denote by S^* the polar of S in $\mathcal{O}(\mathbf{L})$.

We start by describing the principal polars of $\mathcal{O}(\mathbf{L})$ in terms of the support of the canonical proxies. More specifically, given two elements $p, q \in \mathcal{O}(\mathbf{L})$, we will characterize when $q \in p^*$ and when $q \in p^{**}$ in terms of the supports of suitable proxies of p and q, what will be very useful to prove the projectability of $\mathcal{O}(\mathbf{L})$.

Lemma 3.26. *Let $\mathbf{L}$ be an e-cyclic semilinear residuated lattice, and let $p, q < e$ be elements of $\mathcal{O}(\mathbf{L})$. Further, let x, y be canonical proxies of p, q at a partition $\mathcal{C}$.*

(i) $\mathrm{Supp}(y) \subseteq \mathcal{C} \setminus \mathrm{Supp}(x)$ *if and only if $q \in p^*$; and*

(ii) $\mathrm{Supp}(y) \subseteq \mathrm{Supp}(x)$ *if and only if $q \in p^{**}$.*

Proof.

(i) This is an immediate consequence of Lemma 3.20. By the canonicity of x and y, p and q are disjoint if and only if $\mathrm{Supp}(x) \cup \mathrm{Supp}(y)$ is disjoint, which is equivalent to $\mathrm{Supp}(y) \subseteq \mathcal{C} \setminus \mathrm{Supp}(x)$.

(ii) Suppose that $\mathrm{Supp}(y) \subseteq \mathrm{Supp}(x)$ and let $r \in \mathcal{O}(\mathbf{L})$ be disjoint to p, that is, $r \in p^*$. (Without loss of generality, we can assume that r is negative.) Consider z to be a canonical proxy of r at some partition $\mathcal{B}$. We have that $\mathrm{Supp}(x) \cup \mathrm{Supp}(z)$ is disjoint by Lemma 3.20. But then, $\mathrm{Supp}(y) \subseteq \mathrm{Supp}(x)$ implies that $\mathrm{Supp}(y) \cup \mathrm{Supp}(z)$ is also disjoint, and therefore q and r are disjoint. Since $r \in p^*$ is arbitrary, we obtain that $q \in p^{**}$.

For the other implication, suppose that there exists $C_0 \in \mathrm{Supp}(y)$ such that $C_0 \notin \mathrm{Supp}(x)$. We define $z = ([z_C]_{C^\perp} : C \in \mathcal{C})$ in $\mathbf{L}_{\mathcal{C}}$ in the following manner:

$$z_C = \begin{cases} y_{C_0} & \text{if } C = C_0, \\ e & \text{otherwise}, \end{cases}$$

and take $r = \overline{\phi}_{\mathcal{C}}(z)$. Clearly, z is a canonical proxy of r and furthermore $q \leqslant r < e$, and therefore q is not disjoint to r. But, $\mathrm{Supp}(z) = \{C_0\} \subseteq \mathcal{C} \setminus \mathrm{Supp}(x)$, and hence $r \in p^*$, by (i). Therefore, $q \notin p^{**}$. □

[6]See Section 1.6.

The idea behind the proof of the projectability of $\mathcal{O}(\mathbf{L})$ is to observe that, if x is a canonical proxy of a generic element p at some partition $\mathcal{C}$, then $\mathbf{L}_{\mathcal{C}}$ can be decomposed as follows:

$$\mathbf{L}_{\mathcal{C}} = \prod_{C \in \mathcal{C} \setminus \mathrm{Supp}(x)} \mathbf{L}/C^{\perp} \times \prod_{C \in \mathrm{Supp}(x)} \mathbf{L}/C^{\perp}.$$

Thus, for any element $q \in \mathcal{O}(\mathbf{L})$ with a proxy y in $\mathbf{L}_{\mathcal{C}}$, we can construct two elements $y_1, y_2 \in \mathbf{L}_{\mathcal{C}}$, using the aforementioned decomposition of $\mathbf{L}_{\mathcal{C}}$, in such a way that $y = y_1 \cdot y_2$. By construction, y_1 and y_2 will be proxies of elements $q_1, q_2 \in \mathcal{O}(\mathbf{L})$ such that $q_1 \in p^*$ and $q_2 \in p^{**}$.

Theorem 3.27. *If* $\mathbf{L}$ *is an e-cyclic semilinear residuated lattice, then* $\mathcal{O}(\mathbf{L})$ *is* $\boxplus$*-projectable.*

Proof. Let $p, q \in \mathcal{O}(\mathbf{L})$ be arbitrary elements. Consider a partition $\mathcal{C}$ that has canonical proxies x and y for p and q, respectively. Let us define $z = ([z_C]_{C^{\perp}} : C \in \mathcal{C})$ and $t = ([t_C]_{C^{\perp}} : C \in \mathcal{C})$ in $\mathbf{L}_{\mathcal{C}}$ as follows:

$$z_C = \begin{cases} e & \text{if } C \in \mathrm{Supp}(x), \\ y_C & \text{otherwise} \end{cases} \qquad t_C = \begin{cases} y_C & \text{if } C \in \mathrm{Supp}(x), \\ e & \text{otherwise} \end{cases}$$

Set $q_1 = \overline{\varphi}_{\mathcal{C}}(z)$ and $q_2 = \overline{\varphi}_{\mathcal{C}}(t)$. Then, obviously, z and t are canonical,

$$q = \overline{\varphi}_{\mathcal{C}}(y) = \overline{\varphi}_{\mathcal{C}}(z \cdot t) = \overline{\varphi}_{\mathcal{C}}(z) \cdot \overline{\varphi}_{\mathcal{C}}(t) = q_1 \cdot q_2,$$

$\mathrm{Supp}(z) \subseteq \mathcal{C} \setminus \mathrm{Supp}(x)$, and $\mathrm{Supp}(t) \subseteq \mathrm{Supp}(x)$. It follows, by virtue of Lemma 3.26, that $q_1 \in p^*$ and $q_2 \in p^{**}$, as was to be proved.

In order to establish the uniqueness of the the decomposition of q as a product of an element in p^* and an element in p^{**}, suppose that we have two such decompositions:

$$q_1 \cdot q_2 = q = q_1' \cdot q_2'.$$

Let $\mathcal{C}$ be a partition that contains canonical proxies x, y, z, t, z', and t' for the elements p, q_1, q_2, q_1', and q_2', respectively. Hence, $z \cdot t = y = z' \cdot t'$, because proxies are unique at each partition. Note that, since $q_1, q_1' \in p^*$, for every $C \in \mathrm{Supp}(x)$, $[z_C]_{C^{\perp}} = [e]_{C^{\perp}} = [z'_C]_{C^{\perp}}$, by Lemma 3.26. And analogously, for every $C \in \mathcal{C} \setminus \mathrm{Supp}(x)$, $[t_C]_{C^{\perp}} = [e]_{C^{\perp}} = [t'_C]_{C^{\perp}}$, since $q_2, q_2' \in p^{**}$, and therefore:

$$\begin{aligned} [z_C]_{C^{\perp}} &= [z_C]_{C^{\perp}} \cdot [e]_{C^{\perp}} = [z_C]_{C^{\perp}} \cdot [t_C]_{C^{\perp}} = [y_C]_{C^{\perp}} = [z'_C]_{C^{\perp}} \cdot [t'_C]_{C^{\perp}} \\ &= [z'_C]_{C^{\perp}} \cdot [e]_{C^{\perp}} = [z'_C]_{C^{\perp}}. \end{aligned}$$

Hence, $z = z'$. Analogously, $t = t'$, and therefore $q_1 = q_1'$ and $q_2 = q_2'$.

Lastly, we have to prove that if $q_1 \cdot q_2 \leqslant q_1' \cdot q_2'$, with $q_1, q_1' \in p^*$ and $q_2, q_2' \in p^{**}$, then $q_1 \leqslant q_1'$ and $q_2 \leqslant q_2'$. Arguing as and retaining the notations of the preceding paragraph, we can show that $t \leqslant t'$ and $z \leqslant z'$ in $\mathcal{C}$. Hence $q_1 \leqslant q_1'$ and $q_2 \leqslant q_2'$ in $\mathcal{O}(\mathbf{L})$. □

Thus, we obtain the result that we announced:

Corollary 3.28. *Every e-cyclic semilinear residuated lattice* $\mathbf{L}$ *is densely embeddable in a projectable residuated lattice that belongs to the variety generated by* $\mathbf{L}$*.*

Proof. It is an immediate consequence of Theorems 3.14 and 3.27, and the fact that $\mathcal{O}(\mathbf{L})$ is a direct limit of products of quotients of $\mathbf{L}$. □

Given an e-cyclic semilinear residuated lattice, we denote by $\mathbb{D}_{<\omega}(\mathbf{L})$ the set of all finite partitions of $\mathrm{Pol}(\mathbf{L})$. If $\mathcal{C}, \mathcal{D} \in \mathbb{D}_{<\omega}(\mathbf{L})$, then the refinement of $\mathcal{C}$ and $\mathcal{D}$ is also finite (see Equation (3.1)), and thus the set $\mathbb{D}_{<\omega}(\mathbf{L})$ is also a directed set. Let $\mathcal{O}_{<\omega}(\mathbf{L})$ denote the direct limit of the directed system $\{\phi_{\mathcal{C}\mathcal{A}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{L}_{\mathcal{A}} : \mathcal{C} \preccurlyeq \mathcal{A} \text{ in } \mathbb{D}_{<\omega}(\mathbf{L})\}$. Notice that, since $\mathbb{D}_{<\omega}(\mathbf{L})$ is a subposet of $\mathbb{D}(\mathbf{L})$, then $\mathcal{O}_{<\omega}(\mathbf{L})$ is embeddable in $\mathcal{O}(\mathbf{L})$.

We note that the only point in the previous section where the use of infinite partitions was needed was in the proof of the lateral completeness of $\mathcal{O}(\mathbf{L})$, since the set $\{p_\lambda : \lambda \in \Lambda\}$ chosen at the beginning of the proof could be infinite, in which case the partition $\mathcal{E}$ constructed in the proof could be infinite. Thus, all we proved for $\mathcal{O}(\mathbf{L})$ is also true for $\mathcal{O}_{<\omega}(\mathbf{L})$, except for Theorem 3.21. Lemma 3.26 is also true if we take $p, q \in \mathcal{O}_{<\omega}(\mathbf{L})$. Therefore, we also have the following result.

Theorem 3.29. *If* $\mathbf{L}$ *is an e-cyclic semilinear residuated lattice, then* $\mathbf{L}$ *can be densely embedded in* $\mathcal{O}_{<\omega}(\mathbf{L})$*, which is projectable and belongs to the variety generated by* $\mathbf{L}$*.*

We can actually improve Theorem 3.27. Indeed, we can prove that both $\mathcal{O}(\mathbf{L})$ and $\mathcal{O}_{<\omega}(\mathbf{L})$ are strongly projectable. But first, we need a description of all the polars of $\mathcal{O}(\mathbf{L})$ (and $\mathcal{O}_{<\omega}(\mathbf{L})$) in terms of the supports of the elements. Given a set $X \subseteq L$, we will abuse the notation and write X^* for $(\overline{\phi}(X))^*$. This is harmless, since all we are doing is identifying elements of $\mathbf{L}$ with their images inside $\mathcal{O}(\mathbf{L})$. (See Theorem 3.14.)

Lemma 3.30. *Let* $\mathbf{L}$ *be an e-cyclic semilinear residuated lattice,* $A \in \mathrm{Pol}(\mathbf{L})$*, and* $p \in \mathcal{O}(\mathbf{L})$*. Consider a partition* $\mathcal{C}$ *such that it is a refinement of the partition* $\{A, A^\perp\}$ *and* $\mathbf{L}_{\mathcal{C}}$ *contains a canonical proxy* x *of* p*. Then:*

1. $p \in A^*$ *if and only if every* $C \in \mathrm{Supp}(x)$ *satisfies* $C \subseteq A^\perp$*.*
2. $p \in A^{**}$ *if and only if every* $C \in \mathrm{Supp}(x)$ *satisfies* $C \subseteq A$*.*

Proof. Notice that, without loss of generality, we can choose $p < e_{\mathcal{O}(\mathbf{L})}$, and also that for every $C \in \mathcal{C}$ there are only two mutually exclusive possibilities: $C \subseteq A$ or $C \subseteq A^{\perp}$. Consider an element $a \in A^{-}$ and its image $\overline{a} = \overline{\phi}(a)$ inside $\mathcal{O}(\mathbf{L})$. Thus, $\overline{a}$ has proxies in every partition, and in particular in $\mathcal{C}$: $\overline{a}_{\mathcal{C}} = ([a]_{C^{\perp}} : C \in \mathcal{C})$. Since $a \in A$, for every $C \in \mathcal{C}$ such that $C \subseteq A^{\perp}$, we have that $a^{\perp\perp} \cap C = \{e\}$, and hence $[a]_{C^{\perp}} = [e]_{C^{\perp}}$. We can proceed now to the prove of the statements:

(i) ($\Leftarrow$) If for every $C \in \mathrm{Supp}(x)$, $C \subseteq A^{\perp}$, then for every $C \in \mathcal{C}$, we have:

$$[x_C]_{C^{\perp}} \vee [a]_{C^{\perp}} = \begin{cases} [e]_{C^{\perp}} \vee [a]_{C^{\perp}} & \text{if } C \subseteq A \\ [x_C]_{C^{\perp}} \vee [e]_{C^{\perp}} & \text{if } C \subseteq A^{\perp} \end{cases} = [e]_{C^{\perp}}.$$

Therefore, x and $\overline{a}_{\mathcal{C}}$ are disjoint, whence p and $\overline{a}$ are disjoint too. Since $a \in A^{-}$ was arbitrary, we have proved that $p \in A^{*}$.

($\Rightarrow$) Suppose that there is $C \in \mathrm{Supp}(x)$ such that $C \subseteq A$, and consider $a \in C$, $a < e$. By the canonicity of x and our choice of C, we have that $a^{\perp\perp} \subseteq C \subseteq x_C^{\perp\perp} \cap A$, and hence

$$(a \vee x_C)^{\perp\perp} \cap C = a^{\perp\perp} \cap x_C^{\perp\perp} \cap C = a^{\perp\perp} \neq \{e\}.$$

Therefore, x and $\overline{a}_{\mathcal{C}}$ are not disjoint, and thus p and $\overline{a}$ are not disjoint either. Since $a \in A$, we have obtained that $p \notin A^{*}$.

(ii) Is a consequence of (i) and the fact that $A^{\perp *} = A^{**}$. □

Now we can prove the strong projectability of $\mathcal{O}(\mathbf{L})$.

Theorem 3.31. *Let* $\mathbf{L}$ *be an e-cyclic semilinear residuated lattice. Then* $\mathcal{O}(\mathbf{L})$ *is strongly* $\boxplus$*-projectable.*

Proof. Let $B \in \mathrm{Pol}(\mathcal{O}(\mathbf{L}))$ be an arbitrary polar, and $A = \{a \in L : \overline{a} \in B\}$. It can be shown that $A \in \mathrm{Pol}(\mathbf{L})$ and $B = A^{**}$. (See Proposition 3.42.) Let $p \in \mathcal{O}(\mathbf{L})$ and $\mathcal{C}$ a partition of $\mathbf{L}$ such that it refines $\{A, A^{\perp}\}$ and p has a canonical proxy in $\mathbf{L}_{\mathcal{C}}$. We define, for every $C \in \mathcal{C}$:

$$z_C = \begin{cases} e & \text{if } C \subseteq A^{\perp}, \\ x_C & \text{if } C \subseteq A, \end{cases} \qquad t_C = \begin{cases} x_C & \text{if } C \subseteq A^{\perp}, \\ e & \text{if } C \subseteq A. \end{cases}$$

Thus, taking $z = ([z_C]_{C^{\perp}} : C \in \mathcal{C})$, $t = ([t_C]_{C^{\perp}} : C \in \mathcal{C})$, we can easily see that both z and t are canonical and $zt = x$. Thus, if $q_1 = \overline{\phi}_{\mathcal{C}}(z)$ and $q_2 = \overline{\phi}_{\mathcal{C}}(t)$, we have

$$p = \overline{\phi}_{\mathcal{C}}(x) = \overline{\phi}_{\mathcal{C}}(z \cdot t) = \overline{\phi}_{\mathcal{C}}(z) \cdot \overline{\phi}_{\mathcal{C}}(t) = q_1 \cdot q_2.$$

Moreover, in view of Lemma 3.30, $q_1 \in A^{**} = B$ and $q_2 \in A^{*} = B^{*}$. The proofs of the uniqueness and other properties of q_1 and q_2 go along the same lines of the corresponding ones the proof of Theorem 3.27. □

We readily obtain the following results:

Corollary 3.32. *If* $\mathbf{L}$ *is an e-cyclic semilinear residuated lattice, then* $\mathcal{O}(\mathbf{L})$ *is* $\boxplus$*-orthocomplete.*

Proof. It is an immediate consequence of Theorems 3.21 and 3.31. □

Corollary 3.33. *Every e-cyclic semilinear residuated lattice* $\mathbf{L}$ *is densely embeddable in a* $\boxplus$*-orthocomplete lattice that belongs to the variety generated by* $\mathbf{L}$*.*

Proof. It is an immediate consequence of Theorem 3.14 and Corollary 3.32. □

To end this section, we notice that in the proofs of the results leading to Theorem 3.31, we need not assume at any time that the partitions had to be infinite. Thus, the proofs work as they are even if we restricts ourselves to finite partitions. Whence, we obtain the following improvement of Theorem 3.29:

Theorem 3.34. *If* $\mathbf{L}$ *is an e-cyclic semilinear residuated lattice, then* $\mathbf{L}$ *can be densely embedded in* $\mathcal{O}_{<\omega}(\mathbf{L})$*, which is strongly* $\boxplus$*-projectable and belongs to the variety generated by* $\mathbf{L}$*.*

3.6 Laterally Complete Hulls and Projectable Hulls

We have seen that every e-cyclic semilinear residuated lattice $\mathbf{L}$ can be densely embedded in a residuated lattice that is simultaneously laterally complete and strongly projectable. But, we cannot assure that there is an extension of $\mathbf{L}$ which is minimal with respect to any of those properties, in the sense of the following definition.

Definition 3.35. A *laterally complete hull* of a residuated lattice $\mathbf{L}$ is a laterally complete residuated lattice $\mathbf{H}$ containing $\mathbf{L}$ as subalgebra, such that (i) no proper subalgebra of $\mathbf{H}$ containing $\mathbf{L}$ is laterally complete; and (ii) $\mathbf{L}$ is dense in $\mathbf{H}$.

In order to obtain the existence of laterally complete hulls, we have to restrict ourselves to the varieties of GMV algebras. Recall from section 1.2 that the variety $\mathcal{GMV}$ of GMV algebras is a very important variety of residuated lattices, which includes MV algebras (the equivalent algebraic semantics of Łukasievic's infinite valued logic) as well as ℓ-groups. Many and very interesting properties have been shown for this variety. We start by recalling some of them that we will need to prove our results.

Definition 3.36. A residuated lattice is a *GMV algebra* if it satisfies the equations:

$$x/((x \vee y)\backslash x) \approx x \vee y \approx (x/(x \vee y))\backslash x. \tag{GMV}$$

Or equivalently, the equations:

$$x/(y\backslash x \wedge e) \approx x \vee y \approx (x/y \wedge e)\backslash x. \tag{GMV'}$$

The conditions of the next lemma will be used in the sequel without explicit reference.

Lemma 3.37. [41, Lemmas 2.7, 2.9 and Theorem 5.2] *Every GMV algebra*

1. *satisfies the identities* $x/x \approx e \approx x\backslash x$*;*
2. *is e-cyclic;*
3. *has a distributive lattice reduct; and*
4. *satisfies both prelinearity laws* (LP) *and* (LP).[7]

Combining Lemma 3.37 with Theorem 1.13, we obtain:

Proposition 3.38. *A variety* $\mathcal{V}$ *of GMV algebras is semilinear if and only if for every* $\mathbf{L} \in \mathcal{V}$*, all (principal) polars in* $\mathbf{L}$ *are normal.*

We are going to prove that, given a GMV algebra $\mathbf{L}$, $\mathcal{O}(\mathbf{L})$ contains a minimal subalgebra containing $\mathbf{L}$ which is laterally complete, namely, the intersection of all laterally complete subalgebras of $\mathcal{O}(\mathbf{L})$ containing $\mathbf{L}$. Of course, in each one of these subalgebas, every set of disjoint elements will have an infimum. But, in order to show that the intersection of all of them is laterally complete, we have to make sure that every disjoint set of the intersection will have the same infimum in all of the subalgebras. This is what we prove in the next lemma.

Lemma 3.39. *Let* $\mathbf{L}$ *be a dense subalgebra of a GMV algebra* $\mathbf{H}$*. For any subset* X *of* L^-*, if* $\bigwedge^{\mathbf{L}} X$ *exists, then so does* $\bigwedge^{\mathbf{H}} X$ *and they are equal.*

Proof. Let $a = \bigwedge^{\mathbf{L}} X$. Then, a is a lower bound of X in $\mathbf{L}$, and therefore in $\mathbf{H}$. Suppose that b is a lower bound of X in $\mathbf{H}$. Then clearly $a \leqslant a \vee b \leqslant e$, since all elements of X are negative, and therefore $a = e\backslash a \leqslant (a \vee b)\backslash a \leqslant a\backslash a = e$.

Suppose that $(a \vee b)\backslash a < e$. Then by the density of $\mathbf{L}$ in $\mathbf{H}$, there exists a $c \in L$ such that

$$a \leqslant (a \vee b)\backslash a \leqslant c < e.$$

Hence, $a/c \leqslant a/((a \vee b)\backslash a) = a \vee b$, and therefore a/c is a lower bound of X. Now, since $a/c \in L$, we obtain $a/c \leqslant \bigwedge^{\mathbf{L}} X = a$. Whence $e = a\backslash a \leqslant (a/c)\backslash a = (a/c \wedge e)\backslash a = a \vee c$, since a/c is negative. But $a \leqslant c$, and therefore $c = a \vee c = e$, against the choice of $c < e$. Therefore, $(a \vee b)\backslash a = e$, which implies $b \leqslant a \vee b \leqslant a$. Since b is an arbitrary lower bound of X in $\mathbf{H}$, we deduce that $\bigwedge^{\mathbf{H}} X$ exists and $\bigwedge^{\mathbf{H}} X = a$, as we wanted to prove. $\square$

[7] See section 1.8.

Corollary 3.40. *If* $\mathbf{H}$ *is a GMV algebra and* $\{\mathbf{L}_i : i \in I\}$ *is a nonempty family of subalgebras of* $\mathbf{H}$ *that are laterally complete and dense in* $\mathbf{H}$*, then* $\bigcap_I \mathbf{L}_i$ *is laterally complete.*

Proof. Let $\mathbf{L} = \bigcap_I \mathbf{L}_i$. In order to prove that $\mathbf{L}$ is laterally complete, suppose that $X \subseteq L^-$ is a disjoint subset. Then, for every $i \in I$, $X \subseteq B_i^-$, and therefore $\bigwedge^{\mathbf{B}_i} X$ exists. Since $\mathbf{L}_i$ is dense in $\mathbf{H}$, by Lemma 3.39, $\bigwedge^{\mathbf{H}} X$ exists and $\bigwedge^{\mathbf{H}} X = \bigwedge^{\mathbf{L}_i} X \in L_i$. Thus, $\bigwedge^{\mathbf{H}} X$ is in every $\mathbf{L}_i$, and hence $\bigwedge^{\mathbf{L}} X$ exists and coincides with $\bigwedge^{\mathbf{H}} X$. □

We could prove already that every GMV algebra has a laterally complete hull inside $\mathcal{O}(\mathbf{L})$. But we also want to prove that this is unique up to isomorphism. In order to do so, we first have to study what is the relation between the polars of $\mathbf{L}$ and the polars of $\mathcal{O}(\mathbf{L})$. Indeed, we will see that the Boolean algebras of polars of $\mathbf{L}$ and $\mathcal{O}(\mathbf{L})$ are isomorphic. But actually, the only property we really need to prove that is that $\mathbf{L}$ is densely embeddable in $\mathcal{O}(\mathbf{L})$. So, we will prove it for any pair of algebras $\mathbf{L}$ and $\mathbf{H}$ such that $\mathbf{L}$ is densely embeddable in $\mathbf{H}$.

Let $\mathbf{L}, \mathbf{H}$ be e-cyclic residuated lattices such that $\mathbf{L}$ is a subalgebra of $\mathbf{H}$. Define the order-homomorphisms $\mu : \mathcal{C}(\mathbf{L}) \to \mathcal{C}(\mathbf{H})$ and $\nu : \mathcal{C}(\mathbf{H}) \to \mathcal{C}(\mathbf{L})$ as follows: for all $A \in \mathcal{C}(\mathbf{L})$ and $B \in \mathcal{C}(\mathbf{H}), \mu(A) = \mathrm{C}_{\mathbf{H}}[A]$, the convex subalgebra of $\mathbf{H}$ generated by A, and $\nu(B) = B \cap L$. We first note that (μ, ν) is an adjunction, since for all $A \in \mathcal{C}(\mathbf{L})$ and every $B \in C(\mathbf{H})$,

$$A \subseteq \nu(B) \quad \Leftrightarrow \quad A \subseteq B \cap L \quad \Leftrightarrow \quad A \subseteq B \quad \Leftrightarrow \quad \mathrm{C}_{\mathbf{H}}[A] \subseteq B \quad \Leftrightarrow \quad \mu(A) \subseteq B.$$

Furthermore, ν is surjective, and hence μ is injective. Indeed, let $A \in \mathcal{C}(\mathbf{L})$ and let $S = \{h : h \in H,\ a \leqslant h \leqslant e, \text{ for some } a \in A\}$. If $B = \mathrm{C}_{\mathbf{H}}[S]$, then $S = B^-$ (see Lemma 1.4) and $\nu(B) = B \cap L = A$. Lastly, μ preserves finite meets since for every $A_1, A_2 \in \mathcal{C}(\mathbf{L})$ and every $x \in \mathrm{C}_{\mathbf{H}}[A_1] \cap \mathrm{C}_{\mathbf{H}}[A_2]$, if $x \leqslant e$ then there are $a_i \in A_i$ such that $a_i \leqslant x$, for $i = 1, 2$, and therefore $a_1 \vee a_2 \leqslant x \leqslant e$, whence $x \in \mathrm{C}_{\mathbf{H}}[A_1 \cap A_2]$, and the other inclusion is trivial. Thus, μ is an injective lattice homomorphism preserving arbitrary joins.

When an e-cyclic residuated lattice $\mathbf{H}$ is an extension of $\mathbf{L}$, we use $(\,)^*$ to denote the polars of $\mathbf{H}$ and $(\,)^\perp$ to denote those of $\mathbf{L}$. In the event $\mathbf{L}$ is dense in $\mathbf{H}$, we can say more about the adjunction (μ, ν):

Lemma 3.41. *Let* $\mathbf{L}$ *and* $\mathbf{H}$ *be two e-cyclic residuated lattices such that* $\mathbf{L}$ *is a dense subalgebra of* $\mathbf{H}$*. The following hold with respect to the adjunction* (μ, ν) *defined above:*

(i) For every $B \in \mathcal{C}(\mathbf{H}), B^* = (\mu\nu(B))^*$.
(ii) For every $A \in \mathcal{C}(\mathbf{L}), \nu(\mu(A)^*) = (\nu\mu(A))^\perp$.

(iii) The map ν preserves pseudo-complements, that is, for every $B \in \mathcal{C}(\mathbf{H})$, $\nu(B^*) = \nu(B)^\perp$.

Proof.
(i) Let $B \in \mathcal{C}(\mathbf{H})$. We need to show that $B^* = (\mu\nu(B))^*$. For the left-to-right inclusion, notice that $\mu\nu(B) \cap B^* \subseteq B \cap B^* = \{e\}$, and so $B^* \subseteq (\mu\nu(B))^*$. For the other inclusion, suppose that $D \in \mathcal{C}(\mathbf{H})$ is such that $\mu\nu(B) \cap D = \{e\}$. Then,

$$\{e\} = \nu(\mu\nu(B) \cap D) = \nu\mu\nu(B) \cap \nu(D) = \nu(B) \cap \nu(D) = \nu(B \cap D) = (B \cap D) \cap L.$$

Since $\mathbf{L}$ is dense in $\mathbf{H}$, we have that $B \cap D = \{e\}$, whence $D \subseteq B^*$. Therefore, $(\mu\nu(B))^* \subseteq B^*$.

(ii) Let $A \in \mathcal{C}(\mathbf{L})$. We need to prove that $\nu(\mu(A)^*) = (\nu\mu(A))^\perp$. One inclusion is just the observation that $\nu(\mu(A)^*) \cap \nu(\mu(A)) = \nu(\mu(A)^* \cap \mu(A)) = \nu(\{e\}) = \{e\}$, whence $\nu(\mu(A)^*) \subseteq (\nu\mu(A))^\perp$. For the other inclusion, suppose that $D \in \mathcal{C}(\mathbf{L})$ is such that $\nu\mu(A) \cap D = \{e\}$. Then, $\mu(A) \cap \mu(D) = \mu\nu\mu(A) \cap \mu(D) = \mu(\nu\mu(A) \cap D) = \mu(\{e\}) = \{e\}$, and hence $\mu(D) \subseteq \mu(A)^*$. Since ν is onto, we have $D = \nu\mu(D) \subseteq \nu(\mu(A)^*)$.

(iii) Let $B \in \mathcal{C}(\mathbf{H})$. We have in view of the preceding discussion that $\nu(B^*) = \nu((\mu\nu(B))^*) = (\nu(\mu\nu(B)))^\perp = \nu(B)^\perp$, as we wanted to prove. □

The reason why we introduced the adjunction (μ, ν) is because we want to prove that, provided $\mathbf{L}$ is dense in $\mathbf{H}$, where $\mathbf{L}$ and $\mathbf{H}$ are e-cyclic semilinear residuated lattices, the Boolean algebras of polars of $\mathbf{L}$ and $\mathbf{H}$ are isomorphic. This will allow us to define an isomorphisms between the directed sets of partitions, which ultimately we will use to define an embedding from $\mathcal{O}(\mathbf{L})$ into $\mathcal{O}(\mathbf{H})$, in Proposition 3.44.

Proposition 3.42. *If $\mathbf{L}$ is a dense subalgebra of an e-cyclic residuated lattice $\mathbf{H}$, then the Boolean algebras $\mathrm{Pol}(\mathbf{L})$ and $\mathrm{Pol}(\mathbf{H})$ are isomorphic.*

Proof. In light of Lemma 3.41.(ii), $\nu(B) \in \mathrm{Pol}(\mathbf{L})$, for each $B \in \mathrm{Pol}(\mathbf{H})$. Let $\hat{\nu} : \mathrm{Pol}(\mathbf{H}) \to \mathrm{Pol}(\mathbf{L})$ be defined by $\hat{\nu}(B) = \nu(B)$, for all $B \in \mathrm{Pol}(\mathbf{H})$. Note that $\hat{\nu}$ is surjective, since ν is surjective and preserves pseudocomplements. We claim that $\hat{\mu} : \mathrm{Pol}(\mathbf{L}) \to \mathrm{Pol}(\mathbf{H})$, defined by $\hat{\mu}(A) = (\mu(A^\perp))^*$ is the left adjoint of $\hat{\nu}$. Note first that if $A \in \mathrm{Pol}(\mathbf{L})$, there exists $C \in \mathrm{Pol}(\mathbf{H})$ such that $A = \nu(C)$. Then, by invoking both conditions of Lemma 3.41, we get $\hat{\mu}(A) = (\mu(A^\perp))^* = (\mu(\nu(C)^\perp))^* = (\mu\nu(C^*))^* = C^{**} = C$.

Now let $A \in \mathrm{Pol}(\mathbf{L})$ and $B \in \mathrm{Pol}(\mathbf{H})$. We need to prove that $\hat{\mu}(A) \subseteq B$ if and only if $A \subseteq \hat{\nu}(B)$. Suppose first that $\hat{\mu}(A) \subseteq B$ and let $C \in \mathrm{Pol}(\mathbf{H})$ such that $A = \nu(C)$. Then $C \subseteq B$, and so $A = \nu(C) \subseteq B$. On the other hand, if $A \subseteq B$, then $\nu(B)^\perp \subseteq A^\perp$, which, combined with Lemma 3.41.(iii), implies that $\mu\nu(B^*) \subseteq \mu(A^\perp)$. Then another application of Lemma 3.41.(ii) yields $\hat{\mu}(A) = (\mu(A^\perp))^* \subseteq (\mu\nu(B^*))^* = B^{**} = B$. We have verified that $(\hat{\mu}, \hat{\nu})$ is an adjunction, and hence, in particular, that $\hat{\mu}$ is injective. Lastly, $\hat{\mu}$ is surjective, since for $B \in \mathrm{Pol}(\mathbf{H}), \hat{\mu}(\nu(B)) = (\mu(\nu(B)^{\perp}))^{*} = B^{**} = B$ (with the equalities been direct consequences of Lemma 3.41). We have shown that $\hat{\mu}$ is a lattice (and hence a Boolean) isomorphism with inverse $\hat{\nu}$. □

Corollary 3.43. *If* $\mathbf{L}$ *is a dense subalgebra of an e-cyclic residuated lattice* $\mathbf{H}$*, the map*

$$\mathcal{C} \mapsto \overline{\mathcal{C}} = \{\hat{\mu}(C) : C \in \mathcal{C}\}$$

is an order isomorphism from the join-semilattice $\langle \mathbb{D}(\mathbf{L}), \preccurlyeq \rangle$ *of partitions of* $\mathrm{Pol}(\mathbf{L})$ *to the the join-semilattice* $\langle \mathbb{D}(\mathbf{H}), \preccurlyeq \rangle$ *of partitions of* $\mathrm{Pol}(\mathbf{H})$*.*

Proof. Using the fact that $\hat{\mu} : \mathrm{Pol}(\mathbf{L}) \to \mathrm{Pol}(\mathbf{H})$ and $\hat{\nu} : \mathrm{Pol}(\mathbf{H}) \to \mathrm{Pol}(\mathbf{L})$ are isomorphisms, and Lemma 3.2, it is easy to see that the map is well defined, and actually a bijection. If $\mathcal{C} \preccurlyeq \mathcal{A}$, and $E \in \overline{\mathcal{A}}$, then there exists a unique $A \in \mathcal{A}$ such that $\hat{\mu}(A) = E$, and a unique $C \in \mathcal{C}$ such that $A \subseteq C$. Therefore $E = \hat{\mu}(A) \subseteq \hat{\mu}(C) \in \overline{\mathcal{C}}$, and it is straightforward that $\hat{\mu}(C)$ is the only element of $\overline{\mathcal{C}}$ containing E. That is, $\overline{\mathcal{C}} \preccurlyeq \overline{\mathcal{A}}$. □

Proposition 3.44. *Let* $\mathbf{L}$ *be a dense subalgebra of an e-cyclic residuated lattice* $\mathbf{H}$*, and let* $\alpha : \mathbf{L} \to \mathcal{O}(\mathbf{L})$*,* $\beta : \mathbf{H} \to \mathcal{O}(\mathbf{H})$ *be the canonical embeddings. Then there is an embedding* $\tau : \mathcal{O}(\mathbf{L}) \to \mathcal{O}(\mathbf{H})$ *rendering commutative the following diagram:*

$$\begin{array}{ccc} \mathbf{H} & \xrightarrow{\beta} & \mathcal{O}(\mathbf{H}) \\ {\scriptstyle i}\big\uparrow & & \big\uparrow{\scriptstyle \tau} \\ \mathbf{L} & \xrightarrow[\alpha]{} & \mathcal{O}(\mathbf{L}) \end{array} \tag{3.8}$$

Proof. By Proposition 3.42, there exists an isomorphism $\hat{\mu} : \mathrm{Pol}(\mathbf{L}) \to \mathrm{Pol}(\mathbf{H})$, with inverse $\hat{\nu}$. If $C \in \mathrm{Pol}(\mathbf{L})$, then the assignment $f_{C^{\perp}} : \mathbf{L}/C^{\perp} \to \mathbf{H}/\hat{\mu}(C)^{*}$ – defined by $f_{C^{\perp}}([a]_{C^{\perp}}) = [a]_{\hat{\mu}(C)^{*}}$, for all $a \in L$ – is an injective homomorphism. Indeed, just note that $\hat{\mu}(C)^{*} \cap L = \hat{\mu}(C^{\perp}) \cap L = \hat{\nu}\hat{\mu}(C^{\perp}) = C^{\perp}$. This produces the family of homomorphisms $\{f_{C^{\perp}}\pi_{C^{\perp}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{H}_{\hat{\mu}(C)^{*}} : C \in \mathcal{C}\}$. Therefore, recalling that $\overline{\mathcal{C}} = \{\hat{\mu}(C) : C \in \mathcal{C}\}$, the co-universal property of the product $\mathbf{H}_{\overline{\mathcal{C}}}$ induces a homomorphism $\tau_{\mathcal{C}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{H}_{\overline{\mathcal{C}}}$ such that $\pi_{\hat{\mu}(C)^{*}}\tau_{\mathcal{C}} = f_{C^{\perp}}\pi_{C^{\perp}}$, for all $C \in \mathcal{C}$. Note that, for every $x = ([x_C]_{C^{\perp}} : C \in \mathcal{C})$ in $\mathbf{L}_{\mathcal{C}}$, $\tau_{\mathcal{C}}(x) = ([x_C]_{\hat{\mu}(C)^{*}} : \hat{\mu}(C) \in \overline{\mathcal{C}})$. Further, $\tau_{\mathcal{C}}$ is an embedding, since each $f_{C^{\perp}}$, $C \in \mathcal{C}$, is an embedding.

It can be readily seen that for every $\mathcal{C} \preccurlyeq \mathcal{A}$ in $\mathbb{D}(\mathbf{L})$, $\overline{\mathcal{C}} \preccurlyeq \overline{\mathcal{A}}$ (by Corollary 3.43), and the bottom square of the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}(\mathbf{L}) \arrow[rrr, dashed, "\tau"] & & & \mathcal{O}(\mathbf{H}) \\
& \mathbf{L}_{\mathcal{C}} \arrow[ul, bend left, "\overline{\phi}_{\mathcal{C}}"'] \arrow[r, "\tau_{\mathcal{C}}"] \arrow[d, "\phi_{\mathcal{C}\mathcal{A}}"'] & \mathbf{H}_{\overline{\mathcal{C}}} \arrow[ur, bend right, "\overline{\phi}_{\overline{\mathcal{C}}}"] \arrow[d, "\phi_{\overline{\mathcal{C}}\,\overline{\mathcal{A}}}"] & \\
& \mathbf{L}_{\mathcal{A}} \arrow[uul, bend left, "\overline{\phi}_{\mathcal{A}}"] \arrow[r, "\tau_{\mathcal{A}}"] & \mathbf{H}_{\overline{\mathcal{A}}} \arrow[uur, bend right, "\overline{\phi}_{\overline{\mathcal{A}}}"'] &
\end{tikzcd}$$

Therefore there exists a unique τ rendering the whole diagram commutative. Furthermore, τ is an embedding, since if $p, q \in \mathcal{O}(\mathbf{L})$ are such that $\tau(p) = \tau(q)$, and x, y are proxies of p, q at $\mathcal{C}$, then $\overline{\phi}_{\overline{\mathcal{C}}}\tau_{\mathcal{C}}(x) = \tau(\overline{\phi}_{\mathcal{C}}(x)) = \tau(p) = \tau(q) = \tau(\overline{\phi}_{\mathcal{C}}(y)) = \overline{\phi}_{\overline{\mathcal{C}}}\tau_{\mathcal{C}}(y)$. The equality $\overline{\phi}_{\overline{\mathcal{C}}}\tau_{\mathcal{C}}(x) = \overline{\phi}_{\overline{\mathcal{C}}}\tau_{\mathcal{C}}(y)$ shows that $\tau_{\mathcal{C}}(x)$ and $\tau_{\mathcal{C}}(y)$ are proxies of $\tau(p)$ in $\overline{\mathcal{C}}$. It follows that $\tau_{\mathcal{C}}(x) = \tau_{\mathcal{C}}(y)$, and hence $x = y$ by the injectivity of $\tau_{\mathcal{C}}$.

Finally, taking α and β the embeddings of $\mathbf{L}$ and $\mathbf{H}$ into $\mathcal{O}(\mathbf{L})$ and $\mathcal{O}(\mathbf{H})$, respectively, we readily see that the following diagram commutes, where i is the inclusion of $\mathbf{L}$ into $\mathbf{H}$:

$$\begin{tikzcd}
\mathbf{H} \arrow[r, "\cong"'] \arrow[rr, bend left, "\beta"] & \mathbf{H}_{\{H\}} \arrow[r, "\overline{\phi}_{\{H\}}"'] & \mathcal{O}(\mathbf{H}) \\
\mathbf{L} \arrow[u, hook, "i"] \arrow[r, "\cong"] \arrow[rr, bend right, "\alpha"'] & \mathbf{L}_{\{L\}} \arrow[u, "\tau_{\{L\}}"] \arrow[r, "\overline{\phi}_{\{L\}}"] & \mathcal{O}(\mathbf{L}) \arrow[u, "\tau"']
\end{tikzcd}$$

□

Now we have all we need to prove one of the main results of this section, namely that every GMV algebra possesses a laterally complete hull, which is unique up to isomorphism.

Theorem 3.45. *Any semilinear GMV algebra* $\mathbf{L}$ *has a unique, up to isomorphism, semilinear laterally complete hull that belongs to the variety generated by* $\mathbf{L}$*.*

Proof. Let $\mathbf{L}$ be a semilinear GMV algebra. In view of Theorems 3.14 and 3.21, $\mathcal{O}(\mathbf{L})$ is laterally complete and the isomorphic copy $\alpha[\mathbf{L}]$ of $\mathbf{L}$ under the canonical embedding $\alpha : \mathbf{L} \to \mathcal{O}(\mathbf{L})$ is a dense subalgebra of $\mathcal{O}(\mathbf{L})$. It is clear that any subalgebra of $\mathcal{O}(\mathbf{L})$ that contains $\alpha[\mathbf{L}]$ is a dense subalgebra. Let $\mathbf{K}$ be the intersection of all (necessarily dense) subalgebras of $\mathcal{O}(\mathbf{L})$ that are laterally complete and contain $\alpha[\mathbf{L}]$. Hence, $\mathbf{K}$ is laterally complete by Corollary 3.40. Further, $\mathcal{O}(\mathbf{L})$ belongs to the variety generated by $\mathbf{L}$, and so does $\mathbf{K}$, being a subalgebra of $\mathcal{O}(\mathbf{L})$. Combining these facts, we conclude that $\mathbf{L}$ has a laterally complete hull $\mathbf{K}$ that belongs to the variety generated by $\mathbf{L}$. In particular, it is a GMV algebra.

Suppose that $\mathbf{H}$ is another algebra that belongs to the variety generated by $\mathbf{L}$ and is a laterally complete hull of $\mathbf{L}$. Hence, it is a GMV algebra and we can apply Proposition 3.44 to find an embedding τ that renders Diagram (3.8) commutative. Note that, since $\mathbf{L}$ is dense in $\mathbf{H}$ and β is a dense embedding, we have that $\beta[\mathbf{L}]$ is dense in $\mathcal{O}(\mathbf{H})$. Hence $\tau[\mathcal{O}(\mathbf{L})]$ is dense in $\mathcal{O}(\mathbf{H})$, since $\beta[\mathbf{L}] = \tau\alpha[\mathbf{L}] \leqslant \tau[\mathcal{O}(\mathbf{L})] \leqslant \mathcal{O}(\mathbf{H})$. Therefore, $\tau[\mathcal{O}(\mathbf{L})]$ and $\beta[\mathbf{H}]$ are both laterally complete and dense in $\mathcal{O}(\mathbf{H})$, and hence $\tau[\mathcal{O}(\mathbf{L})] \cap \beta[\mathbf{H}]$ is laterally complete by Corollary 3.40. Therefore, $\beta[\mathbf{H}] \cap \tau[\mathcal{O}(\mathbf{L})] = \beta[\mathbf{H}]$, since $\beta[\mathbf{H}]$ is a laterally complete hull of $\beta[\mathbf{L}]$, and $\beta[\mathbf{H}] \cap \tau[\mathcal{O}(\mathbf{L})]$ is a lateral complete subalgebra of $\beta[\mathbf{H}]$ containing $\beta[\mathbf{L}]$. Thus, $\beta[\mathbf{H}] \leqslant \tau[\mathcal{O}(\mathbf{L})]$, and we can take $\mathbf{H}' = \tau^{-1}\beta[\mathbf{H}] \leqslant \mathcal{O}(\mathbf{L})$. Then, $\alpha[\mathbf{L}] = \tau^{-1}\beta[\mathbf{L}] \leqslant \mathbf{H}'$ and $\mathbf{H}'$ is laterally complete, and therefore $\mathbf{K} \leqslant \mathbf{H}'$. But, since $\mathbf{H}$ is a laterally complete hull of $\mathbf{L}$, then $\mathbf{H}'$ is a laterally complete hull of $\alpha[\mathbf{L}]$, and therefore $\mathbf{H}' = \mathbf{K}$. Hence, $\mathbf{H} \cong \mathbf{K}$. □

We proceed now to prove that every polar of a laterally complete GMV algebra is principal. The idea of the proof is the following: we prove that (i) the polar of a set of negative elements of a GMV algebra coincides with the polar of its infimum, in case it exits; (ii) if X is a maximal disjoint subset of a polar C, then $C = X^{\perp\perp}$; and (iii) every polar contains a maximal disjoint set. We will prove this results in a series of lemmas, which we will use to prove finally that any laterally complete projectable GMV algebra is strongly projectable, which is the content of Propostion 3.50.

First, we start with a characterization of the disjointness in GMV algebras:

Lemma 3.46. *If $\mathbf{A}$ is a GMV algebra and $a, b \in A^-$, then $a \vee b = e$ if and only if $a\backslash b \wedge e = b$.*

Proof. First note that $a\backslash b \wedge e = a\backslash b \wedge b\backslash b = (a \vee b)\backslash b$. Therefore, if $a \vee b = e$, then $a\backslash b \wedge e = (a \vee b)\backslash b = e\backslash b = b$. Conversely, if $a\backslash b \wedge e = b$, then $a \vee b = b/((a \vee b)\backslash b) = b/((a\backslash b) \wedge (b\backslash b)) = b/((a\backslash b) \wedge e) = b/b = e$. □

Lemma 3.47. *Let $\mathbf{A}$ is a GMV algebra and let $X \subseteq A^-$ such that $\bigwedge X = a$ exists. Then $X^\perp = a^\perp$.*

Proof. Obviously, if $|y| \vee a = e$, then for every $x \in X$, $|y| \vee x = e$, since $a \leqslant x \leqslant e$. Thus, by Equation (1.2), $a^\perp \subseteq X^\perp$. In order to prove the other inclusion, let us suppose that $y \in X^\perp$. Then, again by Equation (1.2), $|y| \vee x = e$, for all $x \in X$, and so, $|y|\backslash x \wedge e = x$ by Lemma 3.46. Hence

$$|y|\backslash a \wedge e = (|y|\backslash \bigwedge X) \wedge e = \Big(\bigwedge_{x\in X} (|y|\backslash x)\Big) \wedge e = \bigwedge_{x\in X} ((|y|\backslash x) \wedge e) = \bigwedge_{x\in X} x = a,$$

which implies, again by Lemma 3.46, that $|y| \vee a = e$. Thus $X^\perp \subseteq a^\perp$, by Equation (1.2). □

Lemma 3.48. *Let* $\mathbf{L}$ *be an e-cyclic residuated lattice,* C *a polar of* $\mathbf{L}$*, and* X *a maximal disjoint subset of the residuated lattice* $\mathbf{C}$*. Then* $C = X^{\perp\perp}$ *in* $\mathcal{C}(\mathbf{L})$*.*

Proof. Since X is a maximal disjoint subset of $\mathbf{C}$, $C \cap X^{\perp} = \{e\}$, and so $C \subseteq X^{\perp\perp}$. Then $C^{\perp} \vee X^{\perp\perp} = L$, where the join $\vee$ is taken in the Boolean algebra of polars. On the other hand, $X \subseteq C$ implies that $X^{\perp\perp} \subseteq C^{\perp\perp} = C$. Thus $C = X^{\perp\perp}$ as was to be shown. □

Lemma 3.49. *If* $\mathbf{A}$ *is an e-cyclic residuated lattice, then every polar of* $\mathbf{L}$ *contains a maximal disjoint subset.*

Proof. The proof is an standard application of the Zorn Lemma. If C is a polar of $\mathbf{L}$, we consider the poset $\mathcal{X}$ of all disjoint subsets of C ordered by inclusion. Then, the union of every chain of $\mathcal{X}$ is obviously disjoint, and therefore, by Zorn Lemma, $\mathcal{X}$ possesses a maximal element. If X is a such an element, then $X^{\perp} \cap C = \{e\}$, because otherwise there would be some $a \in C \cap X^{\perp}$ and $a < e$ disjoint to all the elements of X. But then $X \cup \{a\}$ would be in $\mathcal{X}$, contradicting the maximality of X. □

Proposition 3.50. *If a GMV algebra is laterally complete and (*$\boxplus$*-)* $\vee$*-projectable, then it is (*$\boxplus$*-)* $\vee$*-orthocomplete.*

Proof. Let $\mathbf{L}$ a laterally complete and projectable GMV algebra. We need to prove that $\mathbf{L}$ is strongly projectable. Let C be a polar of $\mathbf{A}$, and $X \subseteq C$ a maximal disjoint subset of C. Since $\mathbf{A}$ is laterally complete, the meet $a = \bigwedge^{\mathbf{A}} X$ exists, and by virtue of Lemmas 3.47 and 3.48, $C = X^{\perp\perp} = a^{\perp\perp}$, whence the result follows. □

To end this section and chapter, we prove that every e-cyclic semilinear residuated lattice possesses minimal extensions that are $\boxplus$-projectable and strongly $\boxplus$-projectable, respectively (not necessarily minimal with respect to (strongly) $\vee$-projectablility), and every GMV algebra possesses a minimal $\boxplus$-orthocomplete extension, according to the following definitions. Moreover, these extensions are are unique up to isomorphisms.

Definition 3.51. A *(strongly) (*$\boxplus$*-)* $\vee$*-projectable hull* of a residuated lattice $\mathbf{L}$ is a (strongly) ($\boxplus$-) $\vee$-projectable residuated lattice $\mathbf{H}$ containing $\mathbf{L}$ as subalgebra, such that (i) no proper subalgebra of $\mathbf{H}$ containing $\mathbf{L}$ is (strongly) ($\boxplus$-) $\vee$-projectable; and (ii) $\mathbf{L}$ is dense in $\mathbf{H}$. Analogously, we define an *(*$\boxplus$*-)* $\vee$*-orthocomplete hull* of a residuated lattice.

In order to prove our final results, we first show that $\boxplus$-projectability and strong $\boxplus$-projectability are properties that are preserved under intersection of subalgebras:

Lemma 3.52. *Let* $\mathbf{A}$ *be a (strongly)* $\boxplus$*-projectable e-cyclic residuated lattice,* $\mathbf{B}$ *a dense subalgebra of* $\mathbf{A}$*, and* $\{\mathbf{H}_i : i \in I\}$ *a family of (strongly)* $\boxplus$*-projectable subalgebras of* $\mathbf{A}$ *that contain* $\mathbf{B}$*. Then,* $\mathbf{H} = \bigcap_{i \in I} \mathbf{H}_i$ *is (strongly)* $\boxplus$*-projectable.*

Proof. To begin with, notice that since for every $i \in I$, $\mathbf{B} \subseteq \mathbf{H}_i$ and $\mathbf{B}$ is dense in $\mathbf{A}$, then so are $\mathbf{H}_i$, for every $i \in I$, and $\mathbf{H} = \bigcap_{i \in I} \mathbf{H}_i$.

Fix an arbitrary $i \in I$. Since $\mathbf{H}_i$ is a dense subalgebra of $\mathbf{A}$, $\hat{v}_i : \mathrm{Pol}(\mathbf{A}) \to \mathrm{Pol}(\mathbf{H}_i)$ determined by $\hat{v}_i(F) = F \cap H_i$ is an isomorphism of Boolean algebras, by virtue of Theorem 3.42. Obviously, if $X \subseteq H_i$, we have that $\hat{v}_i(X^{\perp^{\mathbf{A}}}) = X^{\perp^{\mathbf{A}}} \cap H_i = X^{\perp^{\mathbf{H}_i}}$, even though X is not a polar. Therefore, $X^{\perp^{\mathbf{A}}\perp^{\mathbf{A}}} \cap H_i = \hat{v}_i(X^{\perp^{\mathbf{A}}\perp^{\mathbf{A}}}) = \hat{v}_i(X^{\perp^{\mathbf{A}}})^{\perp^{\mathbf{H}_i}} = X^{\perp_i^{\mathbf{H}}\perp^{\mathbf{H}_i}}$.

We consider an arbitrary element $h \in H$ and will see that $\mathbf{H} = h^{\perp^{\mathbf{H}}} \boxplus h^{\perp^{\mathbf{H}}\perp^{\mathbf{H}}}$, the case of strong $\boxplus$-projectability being entirely analogous. Every $x \in H$ admits a unique decomposition $x = x_1 x_2$ as an element of $\mathbf{H}_i = h^{\perp^{\mathbf{H}_i}} \boxplus h^{\perp^{\mathbf{H}_i}\perp^{\mathbf{H}_i}}$, since $\mathbf{H}_i$ is $\boxplus$-projectable. But, $h^{\perp^{\mathbf{H}_i}} = h^{\perp^{\mathbf{A}}} \cap H_i$ and $h^{\perp^{\mathbf{H}_i}\perp^{\mathbf{H}_i}} = h^{\perp^{\mathbf{A}}\perp^{\mathbf{A}}} \cap H_i$, as we mentioned before. Therefore, $x = x_1 x_2$ is the unique decomposition of x in $\mathbf{A} = h^{\perp^{\mathbf{A}}} \boxplus h^{\perp^{\mathbf{A}}\perp^{\mathbf{A}}}$.

Since $i \in I$ was arbitrarily chosen, then all the decompositions of x as an element of $\mathbf{H}_i = h^{\perp^{\mathbf{H}_i}} \boxplus h^{\perp^{\mathbf{H}_i}\perp^{\mathbf{H}_i}}$, for every $i \in I$, actually coincide among them, as they coincide with the decomposition of x as an element of $\mathbf{A} = h^{\perp^{\mathbf{A}}} \boxplus h^{\perp^{\mathbf{A}}\perp^{\mathbf{A}}}$, whence $x_1, x_2 \in H = \bigcap_I H_i$. Therefore, $x_1 \in h^{\perp^{\mathbf{A}}} \cap H = h^{\perp^{\mathbf{H}}}$, and $x_2 \in h^{\perp^{\mathbf{A}}\perp^{\mathbf{A}}} \cap H = h^{\perp^{\mathbf{H}}\perp^{\mathbf{H}}}$. It is also obvious now that if $x = x_1 x_2$ and $y = y_1 y_2$ are unique decompositions of x and y as elements of $\mathbf{H} = h^{\perp^{\mathbf{H}}} \boxplus h^{\perp^{\mathbf{H}}\perp^{\mathbf{H}}}$, then $x \leqslant y$ if and only if $x_1 \leqslant y_1$ and $x_2 \leqslant y_2$. □

Theorem 3.53. *Every e-cyclic semilinear residuated lattice* $\mathbf{L}$ *has a strongly* $\boxplus$*-projectable hull and a* $\boxplus$*-projectable hull in the variety generated by* $\mathbf{L}$*; and every semilinear GMV algebra* $\mathbf{L}$ *has laterally complete hull an* $\boxplus$*-orthocomplete hull in the variety generated by* $\mathbf{L}$*.*

Proof. By Theorem 3.29, if $\mathbf{L}$ is an e-cyclic semilinear residuated lattice, then it can be densely embedded in the strongly $\boxplus$-projectable residuated lattice $\mathcal{O}_{<\omega}(\mathbf{L})$. Therefore, by Lemma 3.52, $\mathbf{L}$ has a $\boxplus$-projectable hull and a strongly $\boxplus$-projectable hull, which are the intersection of all the (strongly) $\boxplus$-projectable subalgebras of $\mathcal{O}_{<\omega}(\mathbf{L})$ containing $\mathbf{L}$.

If moreover $\mathbf{L}$ is a GMV algebra, then in view of Corollary 3.33 $\mathbf{L}$ is densely embeddable in an $\boxplus$-orthocompete GMV algebra, namely $\mathcal{O}(\mathbf{L})$. Therefore, by Corollary 3.40 and Lemma 3.52, $\mathbf{L}$ has an $\boxplus$-orthocomplete hull, which is the intersection of all the $\boxplus$-orthocomplte subalgebras of $\mathcal{O}(\mathbf{L})$ containing $\mathbf{L}$.

An argument similar to the one in the proof of Theorem 3.45 shows that these hulls are unique up to isomorphism. □

To end this section, and the chapter, we provide a description of the strongly $\boxplus$-projectable hull of an e-cyclic semilinear residuated lattice $\mathbf{L}$. Indeed, we can show that it is nothing else but $\mathcal{O}_{<\omega}(\mathbf{L})$! Notice that this also proves that the strongly $\boxplus$-projectable hull of $\mathbf{L}$ is unique, up to isomorphisms. We prove a technical lemma first.

Lemma 3.54. *If $\mathcal{C} = \{C_1, \ldots, C_n\}$ is a partition of a $\boxplus$-projectable residuated lattice* $\mathbf{H}$*, and $a_i \in C_i$, for $i = 1, \ldots, n$, are such that $a_1 \cdots a_n = e$, then for all i, $a_i = e$.*

Proof. We proceed by induction in n. If $n = 1$, there is nothing to prove. Suppose then that $n > 1$. Since $\mathcal{C}$ is a partition and $a_i \in C_i$, we have that for every $i \neq j$, $a_i \in C_j^{\perp}$. Therefore,

$$e = a_1 \cdots a_n = a_1 \cdot (a_2 \cdots a_n) \in C_1 \boxplus C_1^{\perp},$$

whence we obtain that $a_1 = e$ and $a_2 \cdots a_n = e$. By the induction hypothesis, $a_2 = \cdots = a_n = e$, as was to be proved. $\square$

Theorem 3.55. *Let* $\mathbf{L}$ *be an e-cyclic semilinear residuated lattice. Then $\mathcal{O}_{<\omega}(\mathbf{L})$ is the strongly $\boxplus$-projectable hull of* $\mathbf{L}$*.*

Proof. We only have to show that if $\mathbf{L}$ is densely embeddable in a strongly $\boxplus$-projectable residuated lattice $\mathbf{H}$ (without loss of generality, we can assume that $\mathbf{L}$ is a subalgebra of $\mathbf{H}$), then $\mathcal{O}_{<\omega}(\mathbf{L})$ is also embeddable in $\mathbf{H}$.

In order to do so, we start by defining for every $C \in \mathrm{Pol}(\mathbf{L})$ a homomorphism $f_C : \mathbf{L} \to \mathbf{H}$, using the decomposition $\mathbf{H} = C^* \boxplus C^{**}$: for every $x \in L$, $f_C(x) = x_1$ is the unique element of C^* such that there is $x_2 \in C^{**}$ such that $x = x_1 \cdot x_2$. The map f_C is well defined and a homomorphism. We notice that if $x \in C$, then $f_C(x) = e_{\mathbf{H}}$, and hence $C \subseteq \ker f_C$, whence we obtain a homomorphism $\widetilde{f_C} : \mathbf{L}/C \to \mathbf{H}$.

Consider now a partition $\mathcal{C}$ of $\mathbf{L}$, and the map $\psi_{\mathcal{C}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{H}$ determined by:

$$\psi_{\mathcal{C}}([x_1]_{C_1^{\perp}}, \ldots, [x_n]_{C_n^{\perp}}) = \widetilde{f}_{C_1^{\perp}}([x_1]) \cdots \widetilde{f}_{C_n^{\perp}}([x_n]).$$

$\psi_{\mathcal{C}}$ is trivially a homomorphism and moreover, it is injective by virtue of Lemma 3.54. This defines a family $\{\psi_{\mathcal{C}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{H} : \mathcal{C} \in \mathbb{D}_{<\omega}(\mathbf{L})\}$ of injective homomorphisms, which moreover is compatible with the system $\{\phi_{\mathcal{C}\mathcal{A}} : \mathcal{C} \preccurlyeq \mathcal{A},\ \mathcal{C}, \mathcal{A} \in \mathbb{D}_{<\omega}(\mathbf{L})\}$, in the sense that for every $\mathcal{C} \preccurlyeq \mathcal{A}$ in $\mathbb{D}_{<\omega}(\mathbf{L})$, $\psi_{\mathcal{A}}\phi_{\mathcal{C}\mathcal{A}} = \psi_{\mathcal{C}}$. Thus, there is a unique homomorphism $\psi : \mathcal{O}_{<\omega}(\mathbf{L}) \to \mathbf{L}$ rendering commutative the diagram:

$$\begin{tikzcd} \mathcal{O}_{<\omega}(\mathbf{L}) \arrow[rr, hook, dashed, "\psi"] & & \mathbf{H} \\ & \mathbf{L}_{\mathcal{C}} \arrow[ul, hook, "\phi_{\mathcal{C}}"] \arrow[ur, hook, "\psi_{\mathcal{C}}"'] & \end{tikzcd}$$

Since all the involved homomorphisms are injective, we have that $\psi_{\mathcal{C}}$ is an embedding of $\mathcal{O}_{<\omega}(\mathbf{L})$ into $\mathbf{H}$, as we wanted to prove. $\square$

Chapter 4

The Failure of the Amalgamation Property

4.1 Introduction

The word "amalgamation" refers to the process of combining a pair of algebras in such a way as to preserve a common subalgebra. This is made precise in the following definitions. Let $\mathcal{K}$ be a class of algebras of the same signature. A *V-formation* in $\mathcal{K}$ is a quintuple $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and i, j are embeddings of $\mathbf{A}$ into $\mathbf{B}, \mathbf{C}$, respectively. Given a V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{K}$, $(\mathbf{D}, h, k)$ is said to be an *amalgam* of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{K}$ if $\mathbf{D} \in \mathcal{K}$ and h, k are embeddings of $\mathbf{B}, \mathbf{C}$, respectively, into $\mathbf{D}$ such that the compositions hi and kj coincide:

$$\begin{tikzcd}
 & \mathbf{C} \arrow[dr, hook, dashed, "k"] & \\
\mathbf{A} \arrow[ur, hook, "j"] \arrow[dr, hook, "i"'] & & \mathbf{D} \\
 & \mathbf{B} \arrow[ur, hook, dashed, "h"'] &
\end{tikzcd}$$

$\mathcal{K}$ has the *amalgamation property* (AP) if each V-formation in $\mathcal{K}$ has an amalgam in $\mathcal{K}$.

Amalgamations were first considered for groups by Schreier [86] in the form of amalgamated free products. The general form of the AP was first formulated by Fraïsse [37], and the significance of this property to the study of algebraic systems was further demonstrated in Jónsson's pioneering work on the topic [54–58]. The added interest in the AP for algebras of logic is due to its relationship with various syntactic interpolation properties. We refer the reader to [71] for relevant references and an extensive discussion of these relationships; see also [72] and [69].

There are no results to date of non-commutative varieties of residuated lattices enjoying the AP. The variety $\mathcal{S}em\mathcal{RL}$ of *semilinear*[1] residuated lattices, i.e., the variety generated by all totally ordered residuated lattices, seems like a natural candidate for enjoying this property, since most varieties that have a manageable representation theory and satisfy the AP are semilinear. An indication that this may not be the case comes from the fact that the variety $\mathcal{R}ep\mathcal{LG}$ of representable lattice-ordered groups fails the AP.

Indeed, we devote this chapter to the study of the AP in the varieties $\mathcal{S}em\mathcal{RL}$ and $\mathcal{S}em\mathcal{C}an\mathcal{RL}$ of semilinear and cancellative semilinear residuated lattices, respectively. One of the main results of this study is the following theorem:

Theorem A (See Theorem 4.8.)**.** *The varieties $\mathcal{S}em\mathcal{RL}$ and $\mathcal{S}em\mathcal{C}an\mathcal{RL}$ fail the AP.*

In addition, we prove that the much larger variety $\mathcal{U}$ of residuated lattices with distributive lattice reduct and satisfying the identity $x(y \wedge z)w \approx xyw \wedge xzw$ also fails the AP. In fact, we show that any subvariety of this variety fails the AP, as log as its intersection with the variety of ℓ-groups fails the AP.

Theorem B (See Theorem 4.7)**.** *Let $\mathcal{V}$ be a variety of residuated lattices satisfying the following equations:*

(1) $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge w)$,

(2) $x(y \wedge z)w \approx xyw \wedge xzw$.

If $\mathcal{V} \cap \mathcal{LG}$ fails the AP, then so does $\mathcal{V}$.

There are two key ingredients in the proofs of these results. First, the fact that the specific V-formations that demonstrate the failure of the AP for the variety $\mathcal{R}ep\mathcal{LG}$ of representable lattice-ordered groups ([82], [11]; see Theorem 4.4 and [11, Theorem B]) also demonstrate its failure for $\mathcal{S}em\mathcal{RL}$ and $\mathcal{S}em\mathcal{C}an\mathcal{RL}$. The second key element in the proofs is the fact that each algebra in these varieties has a representation in terms of residuated maps of a chain ([78], [6]; see Lemma 4.5). In Section 4.3 we present parts of the original proof of the failure of the AP for $\mathcal{R}ep\mathcal{LG}$, while in Section 4.4 we prove the main results of this chapter.

4.2 Basic Notions

In this section we briefly recall basic facts about the varieties of residuated lattices that we will need throughout this chapter.

[1] Also referred as *representable* residuated lattices in the literature.

As we mention in Section 1.2, an element $a \in L$ is said to be *invertible* if $(e/a)a = e = a(a\backslash e)$. This is of course true if and only if a has a (two-sided) inverse a^{-1}, in which case $e/a = a^{-1} = a\backslash e$. The residuated lattices in which every element is invertible are therefore precisely the ℓ-groups.

The *negative cone* of a residuated lattice **L** is the residuated lattice whose universe is the set $L^- = \{x \in L : x \leqslant e\}$, and whose operations are the corresponding restrictions of those of **L**, except for the residuals, which are given by:

$$x\backslash^- y = (x\backslash y) \wedge e \quad \text{and} \quad y/^- x = (y/x) \wedge e$$

where $\backslash$ and $/$ denote the residuals in **L**.

Given a class $\mathcal{V}$ of residuated lattices, we denote the class of the negative cones of algebras of $\mathcal{V}$ by $\mathcal{V}^-$. We state the following result from [7, Theorem 7.1] for future reference:

Lemma 4.1. *If $\mathcal{V}$ is a variety of ℓ-groups, then $\mathcal{V}^-$ is a variety of residuated lattices. Moreover, $\mathcal{V}$ and $\mathcal{V}^-$ are isomorphic as categories.*

Recall that the varieties of representable ℓ-groups and semilinear residuated lattices, denoted by $\mathcal{R}ep\mathcal{LG}$ and $\mathcal{S}em\mathcal{RL}$, respectively are the varieties generated by the totally ordered structures. They can be axiomatized relative to $\mathcal{LG}$ and $\mathcal{RL}$ by either of the equations:

$$\lambda_u((x \vee y)\backslash x) \vee \rho_v((x \vee y)\backslash y) \approx e, \tag{SL1}$$

$$\lambda_u(x/(x \vee y)) \vee \rho_v(y/(x \vee y)) \approx e, \tag{SL2}$$

which in the case of ℓ-groups simplify to the single equation $(x^{-1}yx \vee y^{-1}) \wedge e \approx e$.

A residuated lattice **L** is *cancellative* if multiplication is cancellative, in the sense that **L** satisfies the quasi-equations:

$$xz \approx yz \Rightarrow x \approx y \quad \text{and} \quad zx \approx zy \Rightarrow x \approx y.$$

Although defined by quasi-equations, the class $\mathcal{C}an\mathcal{RL}$ of calculative residuated lattices is actually a variety, as it is shown in [7] that a residuated lattice is cancellative if and only if it satisfies the equations:

$$xy/y \approx x \approx y\backslash yx. \tag{Can}$$

Finally, recall the definition of a residuated map that we will need in Section 4.4. Given two partially ordered sets **P** and **Q**, a map $f : P \to Q$ is *residuated* if there exists a map $f^* : Q \to P$ such that for any $a \in P$ and any $b \in Q$, $f(a) \leqslant^{\mathbf{Q}} b$ iff $a \leqslant^{\mathbf{P}} f^*(b)$. In this case, we say that f and f^* form a *residuated pair*, and that f^* is a *residual* of f. Recall from Lemma 1.1 of Section 1.1 the basic properties of residuated maps.

4.3 Failure of the AP for Representable ℓ-groups

Our proof of the failure of the AP for $\mathcal{SemRL}$ extends the techniques used in [82] to establish the failure of the AP in $\mathcal{RepLG}$. We therefore start with a review of the latter.

Let us first note that, for every positive integer n, any representable ℓ-group satisfies the quasi-equation

$$x^n = y^n \Rightarrow x = y. \tag{$\star$}$$

The plan is to construct a V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{RepLG}$, with $\mathbf{A}, \mathbf{B}, \mathbf{C}$ totally ordered and i, j inclusions. For a given positive integer n, the ℓ-groups $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ will contain elements a, b, c, respectively, such that $b^n = c^n \in A$, but $a^b = bab^{-1}$ and $a^c = cac^{-1}$ are distinct elements of A. Therefore, the V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ cannot be amalgamable in $\mathcal{RepLG}$, since otherwise the amalgam would contain distinct elements b and c such that $b^n = c^n$, falsifying ($\star$).

To construct the V-formation in question, let us first consider a totally ordered group $\mathbf{A}$, its group of order-automorphisms of the underlying total order of $\mathbf{A}$, and $\alpha \in \mathcal{A}ut(\mathbf{A})$ such that $\alpha \neq id$. The *cyclic extension* of $\mathbf{A}$ by α is the totally ordered group $\mathbf{A}(\alpha)$ whose universe is

$$\{(a, \alpha^n) : a \in A \text{ and } n \in \mathbb{Z}\}$$

with operations defined by

$$(a, \alpha^m)(b, \alpha^n) = (a\alpha^m(b), \alpha^{m+n}),$$

and order[2] defined by

$$(a, \alpha^n) \leqslant (b, \alpha^m) \quad \Leftrightarrow \quad n < m \text{ or } n = m \text{ and } a \leqslant b.$$

Notice that $(a, \alpha^m)(\alpha^{-m}(a^{-1}), \alpha^{-m}) = (a\alpha^m(\alpha^{-m}(a^{-1})), \alpha^{m-m}) = (1, id)$, and therefore

$$(a, \alpha^m)^{-1} = (\alpha^{-m}(a^{-1}), \alpha^{-m}).$$

It is convenient to regard $\mathbf{A}$ as a subalgebra $\mathbf{A}(\alpha)$ by identifying a with (a, id), as given $a, b \in A$, $(a, id)(b, id) = (a\, id(b), id) = (ab, id)$ and $(a, id)^{-1} = (a^{-1}, id)$. We also identify $(1, \alpha)$ with α, and notice that for ever $n \in \mathbb{Z}$, $(1, \alpha)^n = (1, \alpha^n)$. With

[2] Note that since $\alpha \neq id$, there is $x \in A$ such that $x < \alpha(x)$ or $\alpha(x) < x$. In the first case, we would have the chain $\alpha^{-n}(x) < \cdots < x < \cdots < \alpha^n(x) < \alpha^{n+1}(x)$, for every $n \geqslant 1$, and the reverse chain in the second case. Thus, either way, the unique $n \in \mathbb{N}$ such that $\alpha^n = id$ is $n = 0$, and therefore the order in $\mathbf{A}(\alpha)$ is well defined.

this identifications, we could obtain a simple expression for the conjugate a^α in $\mathbf{A}(\alpha)$. Indeed, $a^\alpha = \alpha(a)$, since

$$a^\alpha = (1,\alpha)(a,id)(1,\alpha)^{-1} = (1\alpha(a),\alpha)(1,\alpha^{-1}) = (\alpha(a)\alpha(1),\alpha^{1-1}) = (\alpha(a),id).$$

Let now α, β, γ be order-automorphisms of the totally ordered group $\mathbf{A}$ such that, for a positive integer n, $\alpha = \beta^n = \gamma^n$, but $\beta \neq \gamma$. Note that $\mathbf{A}(\alpha)$ is a subalgebra of both $\mathbf{A}(\beta)$ and $\mathbf{A}(\gamma)$. Note further that for all $a \in A$, $a^\beta = \beta(a)$ in $\mathbf{A}(\beta)$ and $a^\gamma = \gamma(a)$ in $\mathbf{A}(\gamma)$, and, of therefore a^β and a^γ are elements of $\mathbf{A}$.

We claim that any subvariety of $\mathcal{R}ep\mathcal{LG}$ that contains $\mathbf{A}(\beta)$ and $\mathbf{A}(\gamma)$ fails the AP. More specifically, the V-formation $(\mathbf{A}(\alpha), \mathbf{A}(\beta), \mathbf{A}(\gamma), i, j)$ – with i, j inclusions – does not have an amalgam in $\mathcal{R}ep\mathcal{LG}$. Indeed, in any ℓ-group $\mathbf{G}$ that contains $\mathbf{A}(\alpha)$ and $\mathbf{A}(\beta)$ as ℓ-subgroups, the equality $\beta^n = \gamma^n$ is satisfied. On the other hand, there is $a \in A$ such that $\beta(a) \neq \gamma(a)$, and hence there is $a \in A$ such that a^γ and a^β are distinct elements of A and hence of $\mathbf{G}$. This shows that $\beta \neq \gamma \in \mathbf{G}$, and so $\mathbf{G}$ cannot be representable.

Thus, we have the following result from [82] (see also [44]):

Lemma 4.2. *Let $\mathbf{A}$ be a totally ordered group, and let α, β, γ be distinct order automorphisms of the lattice-reduct of $\mathbf{A}$ such that $\alpha = \beta^n = \gamma^n$, for some integer $n \geqslant 2$. Then any subvariety of $\mathcal{R}ep\mathcal{LG}$ that contains $\mathbf{A}(\beta)$ and $\mathbf{A}(\gamma)$ fails the AP.*

There is a natural way of constructing totally ordered groups such as $\mathbf{A}(\alpha), \mathbf{A}(\beta)$ and $\mathbf{A}(\gamma)$. Let us look at the case $n = 2$. Set $I = \mathbb{Z} \overleftarrow{\times} \mathbb{Z}$, ordered anti-lexicogaphically, and define $\bar{\beta}, \bar{\gamma} \in \mathcal{A}ut(\mathbf{I})$ by:

$$\bar{\beta}(x,y) = \begin{cases} (x+1, y+1), & \text{if } y \text{ is even;} \\ (x, y+1), & \text{otherwise.} \end{cases}$$

$$\bar{\gamma}(x,y) = \begin{cases} (x, y+1), & \text{if } y \text{ is even;} \\ (x+1, y+1), & \text{otherwise.} \end{cases}$$

The following lemma follows straightforwardly from the definitions.

Lemma 4.3. *Let I, $\bar{\beta}, \bar{\gamma}$ as above. Then:*

(i) $\bar{\beta} \neq \bar{\gamma}$;

(ii) $\bar{\beta}^2 = \bar{\gamma}^2 = \bar{\alpha}$;

(iii) *for all $a \in I$, $\bar{\alpha}(a) > a$.*

Set $\mathbf{A}$ to be $\overleftarrow{\bigoplus}_{i \in I} \mathbb{Z}_i$ the direct sum of copies of the integers, anti-lexicogaphically ordered, indexed by I. That is, $(x_i)_{i \in I} < (y_i)_{i \in I}$ if there is $i \in I$ such that $x_i < y_i$ and for every $j \in I$, if $j > i$ then $x_j = y_j$. Let us remark that each order automorphism $\bar{\delta}$ of I determines an automorphism δ on $\mathbf{A}$ defined by

$$\delta((x_i)_{i \in I}) = \big(x_{\bar{\delta}(i)}\big)_{i \in I}.$$

As a consequence, both $\bar{\beta}, \bar{\gamma}$ induce distinct order-automorphisms β, γ of the underlying chain of $\mathbf{A}$ such that $\beta^2 = \gamma^2$. In light of the preceding discussion, any subvariety of $\mathcal{R}ep\mathcal{LG}$ that contains $\mathbf{A}(\beta)$ and $\mathbf{A}(\gamma)$ fails the AP.

The careful analysis in [82] shows much more than the failure of the AP for $\mathcal{R}ep\mathcal{LG}$. It can be proved – refer to [82] for details – that the totally ordered groups constructed in the preceding paragraph actually belong to $\mathcal{M}$, the variety[3] of ℓ-groups generated by the wreath product $\mathbb{Z}\,\mathtt{wr}\,\mathbb{Z}$, where $\mathbb{Z}$ denotes the ℓ-group of integers. This leads to the following result of [82].

Theorem 4.4. *If $\mathcal{V}$ is a subvariety of $\mathcal{R}ep\mathcal{LG}$ containing $\mathcal{M}$, then $\mathcal{V}$ fails the AP.*

It should be noted that, by results in [36], the interval $[\mathcal{M}, \mathcal{R}ep\mathcal{LG}]$ is uncountable. Thus, in light of Theorem 4.4, there are uncountably many subvarieties of $\mathcal{R}ep\mathcal{LG}$ that fail the AP.

4.4 Failure of the AP for Semilinear Residuated Lattices

In this section, we prove the main results of this chapter. Namely, we show that the variety $\mathcal{S}em\mathcal{RL}$ of semilinear residuated lattices and the variety $\mathcal{S}em\mathcal{C}an\mathcal{RL}$ of semilinear cancellative residuated lattices fail the AP. In addition, we prove that the variety $\mathcal{U}$ consisting of all residuated lattices that have a distributive lattice reduct and satisfy the identity $x(y \wedge z)w \approx xyw \wedge xzw$ also fails the AP.

We start with the definition of an ℓ-monoid. An *ℓ-monoid* is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, e)$ of type $(2,2,2,0)$ such that

(i) $(L, \wedge, \vee)$ is a lattice;
(ii) $(L, \cdot, e)$ is a monoid; and
(iii) $\mathbf{L}$ satisfies the following equations:

$$x\,(y \vee z)\,w \approx xyw \vee xzw \quad \text{and} \quad x\,(y \wedge z)\,w \approx xyw \wedge xzw.$$

Homomorphisms of ℓ-monoids are referred to as *ℓ-homomorphisms*, and, in the preceding equations and in what follows, we use plain juxtaposition "xy" in place of "$x \cdot y$" as usual.

[3] $\mathcal{M}$ is a cover of the variety of Abelian ℓ-groups [70].

Now, given any chain $\mathbf{\Omega}$, the set $\mathcal{R}es(\mathbf{\Omega})$ of all residuated maps on $\mathbf{\Omega}$ is (the universe of) a monoid with respect to function composition, and a lattice with respect to pointwise join and meet; moreover, it is easily checked that $\mathcal{R}es(\mathbf{\Omega})$ is the universe of an ℓ-monoid whose lattice reduct is *distributive*. By abuse of notation, we denote such an ℓ-monoid by the same label $\mathcal{R}es(\mathbf{\Omega})$. Also $\mathcal{A}ut(\mathbf{\Omega})$, the set of all *order-automorphisms* of $\mathbf{\Omega}$, is the universe of an ℓ-monoid which is actually an ℓ-group. We make use of the following result in [78] (see also [6]), which generalizes Holland's Embedding Theorem ([48]).

Theorem 4.5. *A residuated lattice* $\mathbf{A}$ *embedded as an* ℓ*-monoid into* $\mathcal{R}es(\mathbf{\Omega})$*, for some chain* $\mathbf{\Omega}$*, if and only if it satisfies the equations*

(1) $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge w)$*; and*
(2) $x(y \wedge z)w \approx xyw \wedge xzw$.

This representation afforded by the preceding result will play a key role in the proofs of the results below.

Lemma 4.6. *If* $\mathbf{D}$ *is a residuated lattice that satisfies the equations*

(1) $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge w)$*; and*
(2) $x(y \wedge z)w \approx xyw \wedge xzw$,

then the set $Inv(\mathbf{D})$ *of invertible elements of* $\mathbf{D}$ *is the universe of a subalgebra – which we denote by the same symbol – of* $\mathbf{D}$*. Moreover,* $Inv(\mathbf{D})$ *is an* ℓ*-group.*

Proof. Let $Inv(\mathbf{D})$ be the set of invertible elements of $\mathbf{D}$. We have to prove that it is closed under the operations of $\mathbf{D}$. It is obvious that $Inv(\mathbf{D})$ is closed under products and contains e. Further, it is easy to see that if $a, b \in Inv(\mathbf{D})$, then $a \backslash b = a^{-1}b$ and $a/b = ab^{-1}$. Let us just verify the equality $a \backslash b = a^{-1}b$, as the other is analogous. It suffices noticing that for every $c \in D$,

$$c \leqslant a \backslash b \quad \Leftrightarrow \quad ac \leqslant b \quad \Leftrightarrow \quad a^{-1}ac \leqslant a^{-1}b \quad \Leftrightarrow \quad c \leqslant a^{-1}b.$$

As for the lattice operations, we note first that in virtue of Theorem 4.5 there exists a chain $\mathbf{\Omega}$ such that $\mathbf{D}$ is an ℓ-submonoid of $\mathcal{R}es(\mathbf{\Omega})$. Since $\mathbf{\Omega}$ is a chain, $\mathcal{R}es(\mathbf{\Omega})$ is distributive and the product distributes over joins and meets. Moreover, the invertible elements of $\mathcal{R}es(\mathbf{\Omega})$ are the order automorphisms of $\mathbf{\Omega}$, and given an order automorphism $a : \mathbf{\Omega} \to \mathbf{\Omega}$, it is easy to see that $a \wedge a^{-1} \leqslant e \leqslant a \vee a^{-1}$. Therefore, for every pair of invertible elements $g, h \in \mathcal{R}es(\mathbf{\Omega})$, $g^{-1}h \wedge h^{-1}g \leqslant e \leqslant g^{-1}h \vee h^{-1}g$. Thus, if $g, h \in Inv(\mathbf{D})$, then

$$\begin{aligned}(g^{-1} \vee h^{-1})(g \wedge h) &= g^{-1}(g \wedge h) \vee h^{-1}(g \wedge h) \\ &= (g^{-1}g \wedge g^{-1}h) \vee (h^{-1}g \wedge h^{-1}h) \\ &= (e \wedge g^{-1}h) \vee (h^{-1}g \wedge e) \\ &= e \wedge (g^{-1}h \vee h^{-1}g) = e,\end{aligned}$$

and analogously, $(g^{-1} \wedge h^{-1})(g \vee h) = e$, which shows that $(g \wedge h)$ and $(g \vee h)$ are invertible, as we wanted to prove. Therefore, $Inv(\mathbf{D})$ is the universe of a subalgebra of $\mathbf{D}$, and obviously every element in $Inv(\mathbf{D})$ has an inverse, and hence it is an ℓ-group. $\square$

It is already known (see [79], [11] or [83]) that the variety of all ℓ-groups fails the AP. We remark that [83] contains an improved presentation of the original proof in [79], while the recent paper [11] shows that the ℓ-groups $\mathbb{Z}\overleftarrow{\times}\mathbb{Z}$ and $\mathbb{Z}^n$, for $n \geqslant 3$, are not an *amalgamation base* of $\mathcal{LG}$. This means that there exist V-formations $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ – with $\mathbf{A} = \mathbb{Z}\overleftarrow{\times}\mathbb{Z}$ or $\mathbb{Z}^n, n \geqslant 3$ – that do not have an amalgam in $\mathcal{LG}$.

We can use these results to prove that any variety of residuated lattices satisfying equations (1) and (2) of the previous lemma and containing the variety of ℓ-groups fails the AP. More generally, we have:

Theorem 4.7. *Let $\mathcal{V}$ be a variety of residuated lattices satisfying the following equations:*

(1) $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge w)$,
(2) $x(y \wedge z)w \approx xyw \wedge xzw$.

If $\mathcal{V} \cap \mathcal{LG}$ fails the AP, then so does $\mathcal{V}$.

Proof. Let $\mathbf{B}$, $\mathbf{C}$ in $\mathcal{V} \cap \mathcal{LG}$, and $\mathbf{A}$ a common subalgebra. Suppose that a V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ has an amalgam $(h, k, \mathbf{D})$ in $\mathcal{V}$. We may assume that all maps i, j, h and k are inclusions. Then, by Lemma 4.6, $Inv(\mathbf{D})$ is a subalgebra of $\mathbf{D}$, which obviously contains $\mathbf{B}$ and $\mathbf{C}$, because every element of $B \cup C$ is invertible in $\mathbf{D}$. Furthermore $Inv(\mathbf{D})$ is an ℓ-group which is also in $\mathcal{V}$. Hence, $Inv(\mathbf{D})$ would be an amalgam in $\mathcal{V} \cap \mathcal{LG}$ of the V-formation, which does not exist in general. $\square$

As a consequence of this theorem and Theorem 4.4, we obtain the last two results of this chapter:

Theorem 4.8. *The varieties $\mathcal{S}em\mathcal{RL}$ and $\mathcal{S}em\mathcal{C}an\mathcal{RL}$ fail the AP.*

Proof. The variety $\mathcal{LG} \cap \mathcal{SemRL}$ is the variety $\mathcal{RepLG}$, which we know by Theorem 4.4 that fails the AP. Likewise for $\mathcal{SemCanRL}$. □

Corollary 4.9. *The varieties $\mathcal{LG}^-$ and $\mathcal{RepLG}^-$ fail the AP.*

Proof. As was noted above, the varieties $\mathcal{LG}$ and $\mathcal{RepLG}$ fail the AP. Thus, the result follows from Lemma 4.1.

Another way of proving the failure of the AP for $\mathcal{RepLG}^-$ directly is the following: Let us first note that Condition ($\star$) holds in $\mathcal{RepLG}^-$. Suppose that there exists a $\mathbf{D}$ in $\mathcal{RepLG}^-$ amalgamating the negative cones of $\mathbf{A}(\alpha)$, $\mathbf{A}(\beta)$, $\mathbf{A}(\gamma)$. Consider the elements $((0), \beta^{-1})$, $((0), \gamma^{-1})$, defined as above. Clearly, $((0), \beta^{-1})$, $((0), \gamma^{-1})$, $((0), \alpha^{-1})$ are in the negative cone. Moreover, $((0), \beta^{-1})^2 = ((0), \gamma^{-1})^2 = ((0), \alpha^{-1})$. However, $((0), \beta^{-1}) \neq ((0), \gamma^{-1})$. This contradicts Condition ($\star$), which is impossible. □

Chapter 5

Recognizable Elements of Residuated Lattices

5.1 Introduction.

The notions of a language, a finite state device, and a grammar, which are fundamental in Computer Science, have proved to be very closely related. If we try to determine which words belong to a language over an alphabet Σ, it might be possible to do it in some of the following ways. If there is a finite mechanical device, which is usually formalized as a finite state automaton, discerning the words that belong to the language from those that do not, we say that the language is *recognizable*. If there is a regular expression, which is an expression recursively defined in a specific way describing the language, we say that the language is *regular*. This concept was introduced by Kleene [60] during his investigations on the electronic models of the nervous systems. More specifically, the set of *regular languages* over an alphabet Σ is the smallest set that contains the full language Σ^* and the singletons $\{w\}$, for every word $w \in \Sigma^*$, and is closed under finite intersections and unions, complementation, complex multiplication of subsets of Σ^*, and the 'closure operation' $(\,)^*$. Kleene proved that the regular languages are exactly the recognizable languages. Another way of describing a language is by using some grammar, which roughly speaking is a mechanical way of obtaining all the words of the language by using a set of some specific rules for rewriting words. Further results from Chomsky and Miller [22] show the link between finite automata and grammars, namely the languages recognized by finite state automata are the same as the languages given by grammars of type 3, also called *regular grammars*.

Myhill [74] (see also [76]) proved an intrinsic characterization of regular languages in terms of the finiteness index of a certain syntactic equivalence relation between words. The main result of this chapter can be seen precisely as a generalization of this result, and it is proven in Section 5.4.

In Section 5.2 we start revising the notions of an automaton and a language recognized by an automaton. We note that a language L on an alphabet Σ is recognized by a finite state automaton if and only if there is a surjective monoid homomorphism $\varphi : \Sigma^* \to \mathbf{M}$ onto some finite monoid $\mathbf{M}$ containing some set $T \subseteq M$ such that $L = \varphi^{-1}(T)$, which is the content of Theorem 5.5. We notice that the direct image map $\overline{\varphi} : \mathcal{P}(\Sigma^*) \to \mathcal{P}(M)$ is residuated, with residual φ^{-1}. And furthermore, $\mathcal{P}(\Sigma^*)$ has a natural structure of residuated lattice, the action of Σ^* on $\mathbf{M}$ induced by φ can be extended to an action of $\mathcal{P}(\Sigma^*)$ on $\mathcal{P}(S)$, that is, to a $\mathcal{P}(\Sigma^*)$-module, and the fact that L is recognizable can be then expressed in terms of the residuation.

This motivates the notion of a recognizable element of a residuated lattice, which we investigate in Section 5.4, but first in Section 5.3 we develop the basics of the theory of modules over residuated lattices. Previous researchers have explored the concept of a module over a quantale, which essentially is an action of a quantale on a complete lattice. Such structures provide a suitable algebraic framework for extending the concept of a recognizable language (see [64]), and also for the study of some fundamental aspects of Algebraic Logic (see [42]). Here we consider the possibility of extending these ideas by letting the scalars come from an arbitrary residuated lattice and replacing the complete lattice by any partially ordered set. Indeed, the present chapter is a natural sequel of Hoseung Lee's Ph.D. dissertation [64], which includes a number of joint results with C. Tsinakis.

We notice that given a residuated lattice $\mathbf{R} = \langle R, \wedge, \vee, \cdot, e, \backslash, / \rangle$, it acts over itself by left multiplication, giving rise to an $\mathbf{R}$-module that we denote by $\mathbb{R} = \langle \mathbf{R}, \cdot \rangle$. We define for every element a of a residuated lattice $\mathbf{R}$ a special closure operator γ_a on $\mathbb{R}$, and describe its basic properties in Proposition 5.25. This closure operator turns out to be crucial in deciding whether the element a is recognizable.

In Section 5.4, as we mentioned before, we introduce the notion of a recognizable element of a residuated lattice. We then prove that this is the correct abstraction of the notion of a recognizable language to the context of residuated lattices, by showing that a language over an alphabet Σ is recognizable by a finite state automaton if and only if it is recognizable as an element of the residuated lattice $\mathcal{P}(\Sigma^*)$, see Proposition 5.29. Next we prove Theorem 5.31, which is the main result of the section and of this chapter. It is a characterization of the recognizable elements of residuated lattices in the following terms:

Theorem A (Theorem 5.31)**.** *Let* $\mathbf{R}$ *be a residuated lattice and* $a \in R$*. The following are equivalent:*

(i) The element a *is recognizable.*

(ii) There exists a structural closure operator γ *on* $\mathbb{R}$ *with finite image such that* $\gamma(a) = a$*.*

(iii) The image $\{x\backslash a : x \in R\}$ *of the closure operator* γ_a *is finite.*

(iv) The set $\{a/x : x \in R\}$ *is finite.*

Finally we devote Section 5.5 to two interesting problems. Our results shed light on them and could lead to their eventual resolution. We look for a Kleene's-like characterization of the recognizable elements of a residuated lattice. According to Kleene's Theorem, regular languages are exactly the languages recognized by a finite state automata. Therefore, in order to provide an appropriate generalization of this result for an arbitrary residuated lattice $\mathbf{R}$, we have to study the structure of the set of recognizable elements inside $\mathbf{R}$. We find that, whenever $\mathbf{R}$ has a top element, and only in this case, the set of recognizable elements is nonempty, contains the top element, and it is closed under (finite) meets and residuation. We also find that in the case $\mathbf{R} = \mathcal{P}(\mathbf{M})$, for some monoid $\mathbf{M}$, it is also closed under complementation and (finite) unions, although it may not contain all the singletons in general.

The second and last problem that we study in this section is the following: we notice that every recognizable language is recognized by a module whose poset reduct is indeed a Boolean algebra. This is not the general case for recognizable elements of residuated lattices. We provide conditions under which we can assure that a particular element can be recognized by a *Boolean module*, that is to say, a module whose poset reduct is a Boolean algebra.

5.2 Background and Motivation

One of the goals of this chapter is to define the concept of a *recognizable element* in an arbitrary residuated lattice. Since we are borrowing this term from the area of logic and computation, it will be helpful to give here a brief overview of the notion of a *recognizable language*, which also will provide us with the motivating example for our work. The reader is directed to [49] and [50] for a more detailed treatment of the subject.

Let Σ be any set. We shall refer to Σ as an *alphabet* and the elements of Σ as *letters*. We use the symbol Σ^* to denote the collection of all finite sequences (including the empty sequence) of letters from Σ. The members of Σ^* are called *words* and we normally write a word as a string of adjacent letters. Thus, if a, b, and c are letters, the word $\langle abbca \rangle$

will more often be written *abbca*. The empty sequence, also called the *empty word*, is denoted by ε. Clearly, Σ^* forms a monoid – in fact the free monoid over Σ – under the operation of concatenation with ε in the role of the identity (for words $w, v \in \Sigma^*$, we write wv for the concatenation of w followed by v).

As with natural human languages, we often consider most random strings of letters to be gibberish and only a selected subset of Σ^* will form a *language*. In examples coming from the realm of mathematics, languages are usually constituted by the so-called well-formed expressions. For instance, this is the case of the language of the formulas of classical logic, that are normally taken to be the well-formed expressions over the language $\Sigma = V \cup \{\wedge, \vee, \rightarrow, \neg, \top, \bot\} \cup \{\,),(\,\}$, where V is a set of variables. But, there are also other examples that appeal for a more general definition of a language. For instance, we could identify the set $\mathbb{Z}[X]$ of polynomials in one variable and integer coefficients with a language over the set of symbols $\Sigma = \mathbb{Z}$, using the uniqueness of the expression of a polynomial as a sum of monomials. Indeed, under this identification, $\mathbb{Z}[X] = \mathbb{Z}^*$.

Definition 5.1. A *language* over the alphabet Σ is a subset $L \subseteq \Sigma^*$. The *full language* over Σ is Σ^*.

Now, this definition of a language might seem too general, as we have the intuition that languages are usually generated following some mechanical rules. Suppose that L is a language over some alphabet Σ. One wonders if it is possible to decide, in finitely many steps of some automated process, whether a given word $w \in \Sigma^*$ is, or is not, a member of L. For example, such a task is carried out by compilers when parsing code written in some programming language. In our context, this leads to the idea of a *finite state automaton*.

Definition 5.2. A *finite state automaton* is a triple $\langle S, \Sigma, \star, i, F\rangle$ consisting of the following five components:

- a finite set S, called the set of *states*,
- a finite[1] set Σ, the *alphabet*,
- an *action* of Σ^* on S, that is, a binary map $\star : \Sigma^* \times S \to S$ satisfying the properties:
  - (associativity) for all $w, v \in \Sigma^*$ and $s \in S$, $w \star (v \star s) = (wv) \star s$, and
  - (identity) for all $s \in S$, $\varepsilon \star s = s$.

[1] Usually, the alphabet is taken to be finite, because of the applications of automata to Computer Science, but none of the results that we mention here depend on the finiteness of the alphabet.

- a state $i \in S$, called the *initial state*, and
- a set $F \subseteq S$, called the set of *final states*.

If one thinks of S as actual states of some mechanical device and Σ^* as potential instructions supplied to that device, then we think of the action as carrying out the instructions by transforming the device from one state into another; we are attempting to capture the internal workings of a computer in an algebraic setting. As an example, consider the finite state automaton $\langle S, \Sigma, \star, 1, \{3\} \rangle$ where $S = \{1, 2, 3\}$, $\Sigma = \{a, b\}$, and $\star : \Sigma^* \times S \to S$ is the map implicitly defined by the table:

$\star$	1	2	3
a	2	2	3
b	1	3	3

It is clear that any function $\Sigma \times S \to S$ extends uniquely to an action of Σ^* over S. One can depict this action as shown in Diagram (5.1), and thus this diagram can be used to represent the automaton, just by marking the initial state and set of final states in some way. If $w = aabb$ and $v = baab$, for example, then $w \star 1 = 2$ and $v \star 1 = 3$.

$$\begin{tikzcd} 1 \arrow[loop above, "b"] \arrow[r, "a"] & 2 \arrow[loop above, "a"] \arrow[r, "b"] & 3 \arrow[loop above, "b"] \arrow[loop right, "a"] \end{tikzcd} \tag{5.1}$$

Definition 5.3. Given a finite state automaton $\langle S, \Sigma, \star, i, F \rangle$, we say that a language $L \subseteq \Sigma^*$ is *recognized* by this automaton if for any word $w \in \Sigma^*$, $w \in L$ if and only if $w \star i \in F$. A language L is *recognizable* if there exists some finite state automaton that recognizes L.

Example 5.4. Letting $L = \{wbav : w, v \in \Sigma^*\}$, one can easily see that L is recognized by the automaton of our previous example, if we set $i = 1$ and $F = \{3\}$, and therefore, L is recognizable.

It is well known that there is a bijective correspondence between actions of a monoid $\mathbf{M}$ on a set S and monoid homomorphisms from $\mathbf{M}$ to the monoid $\mathrm{End}(S)$ of endomaps of S. Thus, we obtain the following characterization of recognizable languages, which can be found in [80].

Theorem 5.5. *A language L is recognizable if and only if there exist a finite monoid $\mathbf{M}$, a (surjective) monoid homomorphism $\varphi : \Sigma^* \to \mathbf{M}$, and a subset $T \subseteq M$ such that $L = \varphi^{-1}(T)$.*

Proof. If L is recognizable by $\langle S, \Sigma, \star, i, F\rangle$, then consider the monoid $\mathbf{M} \subseteq \mathrm{End}(S)$ with $M = \{\lambda_w : w \in \Sigma^*\}$, where $\lambda_w(s) = w \star s$, for every $s \in S$. Since S is finite, hence $\mathbf{M}$ is finite and the map $\lambda : \Sigma^* \to \mathbf{M}$ determined by $\lambda : w \mapsto \lambda_w$ is a surjective monoid homomorphism. Now consider the set $T = \lambda[L] = \{\lambda_w : w \in L\}$, and let see that $L = \lambda^{-1}(T)$. It is clear that $L \subseteq \lambda^{-1}(T)$. In order to see the other inclusion, suppose that $v \in \lambda^{-1}(T)$. Hence, there exists $w \in L$ such that $\lambda_v = \lambda_w$, whence we have $v \star i = \lambda_v(i) = \lambda_w(i) = w \star i \in T$, and therefore $v \in L$.

For the other implication, suppose that there is a monoidal homomorphism $\varphi : \Sigma^* \to \mathbf{M}$, and a subset $T \subseteq M$ such that $L = \varphi^{-1}(T)$. Hence, we can define an action $\star$ of Σ^* on M by $w \star x = \varphi(w) \cdot x$. Thus, if M is finite, then $\langle M, \Sigma, \star, e_{\mathbf{M}}, T\rangle$ is a finite state automaton, and moreover

$$w \in L \quad\Leftrightarrow\quad w \in \varphi^{-1}(T) \quad\Leftrightarrow\quad \varphi(w) \in T \quad\Leftrightarrow\quad \varphi(w) \cdot e_{\mathbf{M}} \in T \quad\Leftrightarrow\quad w \star e_{\mathbf{M}} \in T.$$

Therefore L is recognized by $\langle M, \Sigma, \star, e_{\mathbf{M}}, T\rangle$. □

The previous discussion implies, in particular, that $\overline{\varphi}(L) = \overline{\varphi}(\varphi^{-1}(T))$ and $L = \varphi^{-1}(T) = \varphi^{-1}(\overline{\varphi}(L))$, where $\overline{\varphi}$ and φ^{-1} are the direct and inverse images maps between the lattices $\mathcal{P}(\Sigma^*)$ and $\mathcal{P}(M)$. Notice that $\overline{\varphi}$ is a residuated map and φ^{-1} is its residual. Pursuing this line of thought leads to the following observations:

- We can extend the action $\star$ of Σ^* over S to an action of $\mathcal{P}(\Sigma^*)$ on $\mathcal{P}(S)$ by:

$$A * X = \{w \star s : w \in \Sigma^*,\ s \in X\}.$$

- This extended action $* : \mathcal{P}(\Sigma^*) \times \mathcal{P}(S) \to \mathcal{P}(S)$ is residuated in both coordinates.
- $L = \varphi^{-1}(T)$ translates in terms of the residuals of $*$.

This indicates that we can capture the concept of recognizability of languages in terms of residuation. We explain in more details these concepts in the next section.

5.3 Residuation, Residuated Lattices, and Modules over Residuated Lattices

Recall from Section 1.1 that given two partially ordered sets $\mathbf{P}$ and $\mathbf{Q}$, a map $f : P \to Q$ is *residuated* if there exists another map $g : Q \to P$, called the *residual* of f, so that for all $p \in P$ and $q \in Q$,

$$f(p) \leqslant q \quad\Leftrightarrow\quad p \leqslant g(q).$$

Residuals are uniquely determined. We say that (f, g) is a *residuated pairs*.

These are the very basic properties of residuated pairs:

Remark 5.6. *If $f : \mathbf{P} \to \mathbf{Q}$ is a residuated map with residual g, then among the many well-understood properties of f and g there are the following:*

(i) *Both f and g are order-preserving.*
(ii) *f preserves arbitrary existing joins and g preserves arbitrary existing meets.*
(iii) *$\gamma = g \circ f$ is a closure operator on $\mathbf{P}$ with associated closure system $P_\gamma = \{g(x) : x \in Q\}$.*
(iv) *$\delta = f \circ g$ is an interior operator on $\mathbf{Q}$ with associated interior system given by $Q_\delta = \{g(x) : x \in Q\}$.*
(v) *$f \circ g \circ f = f$ and $g \circ f \circ g = g$.*
(vi) *The corresponding restrictions of f and g determine an order-isomorphism and its inverse between $\mathbf{P}_\gamma$ and $\mathbf{Q}_\delta$.*

Recall also that a *residuated lattice* is a structure $\mathbf{R} = \langle R, \wedge, \vee, \cdot, \backslash, /, e\rangle$ comprising monoidal and lattice structures over the same underlying set R, and such that the product $\cdot$ is residuated in both coordinates with residuals $\backslash$ and $/$. This means that for every $a, b, c \in R$,

$$a \cdot b \leqslant c \quad \Leftrightarrow \quad b \leqslant a\backslash c \quad \Leftrightarrow \quad a \leqslant c/b.$$

The following properties of residuated lattices – which essentially are the only ones that we are going to need for this chapter – can be readily proven.

Remark 5.7. *If $\mathbf{R}$ is a residuated lattice, then the following hold for every $a, b, c \in R$:*

(i) *$(a/b) \cdot b \leqslant a$ and $b \cdot (b\backslash a) \leqslant a$.*
(ii) *$e \leqslant a/a$ and $e \leqslant a\backslash a$.*
(iii) *$a/e = a = e\backslash a$.*
(iv) *$(a/b)/c = a/(cb)$ and $c\backslash(b\backslash a) = (bc)\backslash a$.*
(v) *The operation $\cdot$ is order-preserving in both coordinates, while $/$ and $\backslash$ are order-preserving in their numerators and order-reversing in their denominators.* $\square$

Example 5.8. One of the prototypical examples of a residuated lattice is a frame. A *frame* is a complete lattice $\mathbf{F}$ in which the meet operation $\wedge$ distributes over arbitrary joins. Therefore, $\wedge$ is residuated in both coordinates, and since it is commutative, left and right residuals coincide, and are often denoted by $\to$. This operation is determined by $b \to c = \bigvee\{x \in F : b \wedge x \leqslant c\}$. Thus, in a frame,

$$a \wedge b \leqslant c \quad \Leftrightarrow \quad a \leqslant b \to c.$$

The induced residuated lattice is $\mathbf{F} = \langle F, \wedge, \vee, \wedge, \to, \leftarrow, \top\rangle$. But, we will in practice omit the operation $\leftarrow$ because, as we noted, both residuals coincide. We can do that whenever the product is commutative.

Example 5.9. Another standard example of a residuated lattice is the following: Given any monoid $\mathbf{M}$, we can define a residuated lattice $\mathcal{P}(\mathbf{M})$ as follows:

$$\mathcal{P}(\mathbf{M}) = \langle \mathcal{P}(M), \cap, \cup, \cdot, \backslash, /, \{e_{\mathbf{M}}\} \rangle$$

is the residuated lattice with the so-called *complex multiplication*, that is, for every $A, B \in \mathcal{P}(M)$, $A \cdot B = \{a \cdot b : a \in A, b \in B\}$. This product is residuated in both coordinates, and the residuals are determined by:

$$A \backslash C = \{b \in M : ab \in C, \text{ for all } a \in A\} \quad \text{and} \quad C/A = \{b \in M : ba \in C, \text{ for all } a \in A\}.$$

Those two examples are particular cases of *quantales*, which can be just seen as complete residuated lattices. Actually, residuals do not form part of the structure of the quantale, and therefore they are not residuated lattices, strictly speaking. But, since the residuals are uniquely determined by the order and the product, the identification of quantales with complete lattices is innocuous. The main difference is not on the structure itself, but on the morphisms, which in the case of quantales are residuated maps respecting the monoidal structure, but not necessarily the residuals. Quantales have arisen as partially ordered models of linear logic, which turned out to be a certain class of quantales. The precise connection between quantales and linear logic was made in [96]. See [84] for a detailed introduction of the theory of quantales. Abramsky and Vickers used in [1] (see also [2]) the notion of module over a quantale to investigate a variety of process semantics in a uniform algebraic framework. In their work, processes are certain modules over a given quantal of actions. We extend the notion of module over a quantale, allowing scalars to come from an arbitrary residuated lattice. Moreover, we do not require the module to be a complete lattice, but just a partially ordered set.

Definition 5.10. A *module* over a residuated lattice $\mathbf{R}$, or just an $\mathbf{R}$-module, is a pair $\mathbb{P} = \langle \mathbf{P}, * \rangle$ consisting of a a partially ordered set $\mathbf{P} = \langle P, \leqslant \rangle$ and a map $* : \mathbf{R} \times \mathbf{P} \to \mathbf{P}$ satisfying the following three properties:

(i) $e * x = x$, for all $x \in P$,

(ii) $a * (b * x) = (a \cdot b) * x$, for all $a, b \in R$ and $x \in P$,

(iii) $*$ is residuated in both coordinates. That is, there exist two maps $\backslash_* : R \times P \to P$ and $/_* : P \times P \to R$ such that, for every $a \in R$ and $x, y \in P$,

$$a * x \leqslant y \quad \Leftrightarrow \quad x \leqslant a \backslash_* y \quad \Leftrightarrow \quad a \leqslant y /_* x.$$

Example 5.11. Given a monoid $\mathbf{M}$, we saw in Example 5.9 how to construct the residuated lattice $\mathcal{P}(\mathbf{M})$. Now, given a monoid action $\star : M \times S \to S$ of $\mathbf{M}$ on the a set S, we can define a $\mathcal{P}(\mathbf{M})$-module $\langle \mathcal{P}(S), * \rangle$ as follows: the set $\mathcal{P}(S)$ is ordered by inclusion, and the product $* : \mathcal{P}(M) \times \mathcal{P}(S) \to \mathcal{P}(S)$ is given by: $A * X = \{a \star x : a \in A, x \in X\}$. It is easy to see that, indeed, this is a $\mathcal{P}(\mathbf{M})$-module, and that the residuals are given by $A \backslash_* Y = \{x \in S : a \star x \in Y, \text{ for all } a \in A\}$ and $Y /_* X = \{a \in M : a \star x \in Y, \text{ for all } x \in X\}$.

Remark 5.12. *Notice that if $\mathbb{P} = \langle \mathbf{P}, * \rangle$ is an $\mathbf{R}$-module, then it follows immediately from Remark 5.6 that $*$ is order-preserving in both coordinates and $\backslash_*$ and $/_*$ are order-preserving in their numerators and order-reversing in their denominators. Moreover, $*$, preserves existing arbitrary joins in both coordinates, and $\backslash_*$ and $/_*$ preserve existing arbitrary meets in their numerators and transform existing arbitrary joins in the denominators into meets.*

Definition 5.13. Let $\mathbb{P}$ and $\mathbb{Q}$ be $\mathbf{R}$-modules. An $\mathbf{R}$-*morphism* $\varphi : \mathbb{P} \to \mathbb{Q}$ from $\mathbb{P}$ to $\mathbb{Q}$ is a residuated map $\varphi : \mathbf{P} \to \mathbf{Q}$ that preserves scalars; that is, for all $a \in R$ and $x \in P$, $\varphi(a *_{\mathbb{P}} x) = a *_{\mathbb{Q}} \varphi(x)$. An $\mathbf{R}$-module $\mathbb{Q}$ is a *submodule* of $\mathbb{P}$ if $Q \subseteq P$, and the inclusion map is an $\mathbf{R}$-morphism $i : \mathbb{Q} \to \mathbb{P}$. An *isomorphism* is a bijective $\mathbf{R}$-morphism between two $\mathbf{R}$-modules, in which case its inverse is also an $\mathbf{R}$-morphism.

Remark 5.14. *Note that, given an $\mathbf{R}$-module $\mathbb{P}$, a submodule is just an $\mathbf{R}$-module $\mathbb{Q}$ such that $Q \subseteq P$ and the product of $\mathbb{Q}$ is the restriction of the product of $\mathbb{P}$. Therefore, submodules of an $\mathbf{R}$-module are determined by their underlying sets.*

In general, $\mathbf{R}$-morphisms do not respect the residuals, but isomorphisms do.

Proposition 5.15. *If $\varphi : \mathbb{P} \to \mathbb{Q}$ is an isomorphism of $\mathbf{R}$-modules and $a \in R$, and $x, y \in P$, then $x /_{\mathbb{P}} y = \varphi(x) /_{\mathbb{Q}} \varphi(y)$, and $\varphi(a \backslash_{\mathbb{P}} x) = a \backslash_{\mathbb{Q}} \varphi(x)$.*

Proof. First, notice that an isomorphism of $\mathbf{R}$-modules is in particular an isomorphism of posets. Thus, if $b \in R$ is an arbitrary element, then

$$b \leqslant x /_{\mathbb{P}} y \quad \Leftrightarrow \quad b *_{\mathbb{P}} y \leqslant x \quad \Leftrightarrow \quad b *_{\mathbb{Q}} \varphi(y) \leqslant \varphi(x) \quad \Leftrightarrow \quad b \leqslant \varphi(x) /_{\mathbb{Q}} \varphi(y).$$

Which proves that $x /_{\mathbb{P}} y = \varphi(x) /_{\mathbb{Q}} \varphi(y)$. The other equality can be proved in a similar fashion. $\square$

Definition 5.16. Given an $\mathbf{R}$-module $\mathbb{P}$ and an element $p \in P$, we define $\langle p \rangle_{\mathbb{P}}$ to be the submodule of $\mathbb{P}$ with universe $\{a * p : a \in R\}$. We say that $\mathbb{P}$ is *cyclic* if $\mathbb{P} = \langle p \rangle_{\mathbb{P}}$ for some $p \in P$. In this case, p is called a *generator* of $\mathbb{P}$.

Example 5.17. It can be readily seen that any residuated lattice $\mathbf{R}$ has the structure of an $\mathbf{R}$-module $\mathbb{R} = \langle \mathbf{R}, \cdot \rangle$. (Sometimes we say that $\mathbf{R}$ is a module over itself.) Moreover, $\mathbb{R}$ is cyclic, since it is generated by e, that is, $\mathbb{R} = \langle e \rangle_{\mathbb{R}}$. Thus, if $\mathbf{F}$ is a frame, the corresponding $\mathbf{F}$-module is $\mathbb{F} = \langle \mathbf{F}, \wedge \rangle$, and it is cyclic generated by its top element $\top$.

Definition 5.18. A *structural closure operator* on an $\mathbf{R}$-module $\mathbb{P} = \langle \mathbf{P}, * \rangle$ is a closure operator γ on $\mathbf{P}$ – that is, an order-preserving, extensive, and idempotent endomap of $\mathbf{P}$ – such that for every $a \in R$ and $x \in P$,

$$a * \gamma(x) \leqslant \gamma(a * x). \tag{Str.}$$

The property (Str.) is called *structurallity*. In what follows, we often omit the adjective 'structural' when we speak about closure operators on an $\mathbf{R}$-module. Further, we denote by $P_\gamma = \{\gamma(x) : x \in P\}$ the *closure system* associated to γ.

Remark 5.19. *Closure operators on an $\mathbf{R}$-module $\mathbb{P}$ can be ordered point-wise: $\gamma \leqslant \delta$ if and only if for every $x \in P$, $\gamma(x) \leqslant \delta(x)$. It is not difficult to see that $\gamma \leqslant \delta$ if and only if $P_\delta \subseteq P_\gamma$.*

Example 5.20. Recall from Section 1.3 that a *nucleus* on a residuated lattice $\mathbf{R}$ is a closure operator on $\mathbf{R}$ satisfying the inequality,

$$\gamma(a) \cdot \gamma(b) \leqslant \gamma(a \cdot b), \tag{5.2}$$

for all $a, b \in R$. Obviously, every nucleus on a residuated lattice $\mathbf{R}$ is a closure operator on $\mathbb{R}$, but the converse need not be true. We however have:

Proposition 5.21. *The nuclei on a frame are precisely the meet-preserving closure operators on it.* □

As we mentioned in Remark 5.6, a natural example of a closure operator arises by composing residuated maps with their residuals. In the case of $\mathbf{R}$-modules we obtain structural closure operators in the following way: let $\mathbb{P}$ be an $\mathbf{R}$-module, and $p \in P$ an arbitrary element. The map $\gamma_p : R \to R$ defined by $\gamma_p(a) = (a * p)/_* p$ is a closure operator on $\mathbf{R} = \langle R, \leqslant \rangle$. Actually, we can prove that γ_p is a closure operator on $\mathbb{R}$.

Lemma 5.22. *Given an $\mathbf{R}$-module $\mathbb{P}$ and an element $p \in P$, the map γ_p defined above is a closure operator on $\mathbb{R}$, and $R_{\gamma_p} = \{x/_* p : x \in P\}$.*

Proof. Note that for every $a, b \in R$,

$$(a \cdot \gamma_p(b)) * p = a * (\gamma_p(b) * p) = a * \big(((b * p)/_* p) * p\big) = a * (b * p) = (ab) * p,$$

and therefore $a \cdot \gamma_p(b) \leqslant ((ab) * p)/_* p = \gamma_p(ab)$, which shows the structurally of γ_p. The equality $R_{\gamma_p} = \{x/_* p : x \in P\}$ follows from Remark 5.6.(iii), as γ_p is the composition of the residuated $\_ * p$ and its residual $\_/_* p$. □

Given a closure operator γ on an $\mathbf{R}$-module $\mathbb{P}$, the set $P_\gamma = \{\gamma(x) : x \in P\}$ inherits a partial ordering from $\mathbf{P}$, and one can define a scalar product $*_\gamma : R \times P \to P$ by $a *_\gamma x = \gamma(a * x)$. Moreover, for every $a \in R$ and $x, y \in P_\gamma$, $a *_\gamma x \leqslant y$ if and only if $\gamma(a * x) \leqslant y$, which is equivalent to $a * x \leqslant y$, because $y \in P_\gamma$ and γ is expansive and order-preserving . Therefore, $a *_\gamma x \leqslant y$ if and only if $a \leqslant y /_* x$ if and only if $x \leqslant a \backslash_* y$. That is to say, $*_\gamma$ is residuated in both coordinates with residuals the corresponding restrictions of $\backslash_*$ and $/_*$. We also have that $e *_\gamma x = \gamma(e * x) = \gamma(x) = x$, and that $a *_\gamma (b *_\gamma x) = \gamma(a * \gamma(b * x)) \leqslant \gamma(\gamma(a * (b * x))) = \gamma(a * (b * x)) \leqslant \gamma(a * \gamma(b * x)) = a *_\gamma (b *_\gamma x)$, which proves that $(ab) *_\gamma x = \gamma((ab) * x) = \gamma(a * (b * x)) = a *_\gamma (b *_\gamma x)$. Thus, $\mathbb{P}_\gamma = \langle \mathbf{P}_\gamma, *_\gamma \rangle$ is an $\mathbf{R}$-module as well.

Proposition 5.23. *Given an $\mathbf{R}$-module $\mathbb{P}$ and an element $p \in P$, the module $\mathbb{R}_{\gamma_p}$ is isomorphic to $\langle p \rangle_{\mathbb{P}}$, and therefore cyclic.*

Proof. Since the map $\_ * p : \mathbf{R} \to \mathbf{P}$ is residuated with residual $\_ /_* p : P \to R$, then in virtue of Remark 5.6, the restriction $\varphi : \mathbf{R}_{\gamma_p} \to \langle \{a * p : a \in R\}, \leqslant \rangle$ of $\_ * p$ is an isomorphism of partially ordered sets. All we have to prove is that φ also respects the scalars. Given $a \in R$ and $x \in R_{\gamma_p}$, we have

$$\begin{aligned}\varphi(a \cdot_{\gamma_p} x) &= \varphi(\gamma_p(a \cdot x)) = \varphi\big(((a \cdot x) * p) /_* p\big) = \big(((a \cdot x) * p) /_* p\big) * p = (a \cdot x) * p \\ &= a * (x * p) = a * \varphi(x).\end{aligned}$$ □

We have noted that the multiplication of a residuated lattice $\mathbf{R}$ induces an $\mathbf{R}$-module structure $\mathbb{R}$ on $\langle R, \leqslant \rangle$. An $\mathbf{R}$-module structure can also be defined on the dual partial ordering $\mathbf{R}^\partial = \langle R, \leqslant^\partial \rangle$:

Proposition 5.24. *Let $\mathbf{R}$ be a residuated lattice and let $\cdot^{\mathrm{d}} : R \times R \to R$ be defined by $a \cdot^{\mathrm{d}} x = x/a$, for all $a, x \in R$. Then the structure $\mathbb{R}^{\mathrm{d}} = \langle \mathbf{R}^\partial, \cdot^{\mathrm{d}} \rangle$ is an $\mathbf{R}$-module.*

Proof. First note that for every $x \in R$, $e \cdot^{\mathrm{d}} x = x/e = x$, and that for every $a, b, x \in R$,

$$a \cdot^{\mathrm{d}} (b \cdot^{\mathrm{d}} x) = (x/b)/a = x/(ab) = (ab) \cdot^{\mathrm{d}} x.$$

Therefore, it only remains to prove that $\cdot^{\mathrm{d}} : \mathbf{R} \times \mathbf{R}^\partial \to \mathbf{R}^\partial$ is residuated in both coordinates. For every $a, x, y \in R$, we have

$$a \cdot^{\mathrm{d}} x \leqslant^\partial y \quad \Leftrightarrow \quad y \leqslant a \cdot^{\mathrm{d}} x \quad \Leftrightarrow \quad y \leqslant x/a \quad \Leftrightarrow \quad y \cdot a \leqslant x \quad \Leftrightarrow \quad a \leqslant y \backslash x.$$

Therefore, the maps $\backslash_{\mathrm{d}} : R \times R \to R$ and $/_{\mathrm{d}} : R \times R \to R$ determined by $a \backslash_{\mathrm{d}} y = y \cdot a$ and $y /_{\mathrm{d}} x = y \backslash x$ are the residuals of $\cdot^{\mathrm{d}}$, since

$$a \cdot^{\mathrm{d}} x \leqslant^\partial y \quad \Leftrightarrow \quad y \cdot a \leqslant x \quad \Leftrightarrow \quad x \leqslant^\partial y \cdot a \quad \Leftrightarrow \quad x \leqslant^\partial a \backslash_{\mathrm{d}} y$$

and

$$a \cdot^{\mathrm{d}} x \leqslant^{\partial} y \;\Leftrightarrow\; a \leqslant y\backslash x \;\Leftrightarrow\; a \leqslant y/_{\mathrm{d}} x. \qquad \square$$

Thus, given a residuated lattice $\mathbf{R}$ and a particular element $a \in R$, by using the $\mathbf{R}$-module $\mathbb{R}^{\mathrm{d}}$, we can define a closure operator γ_a on $\mathbb{R}$ as follows:

$$\gamma_a(x) = (x \cdot^{\mathrm{d}} a)/_{\mathrm{d}} a = (a/x)\backslash a.$$

Proposition 5.25. *Given a residuated lattice $\mathbf{R}$ and an element $a \in R$, the closure operator γ_a on $\mathbb{R}$ defined above has the following properties:*

(i) $R_{\gamma_a} = \{x\backslash a : x \in R\}$.

(ii) $\gamma_a(a) = a$.

(iii) If γ is a closure operator on $\mathbb{R}$, then $\gamma(a) = a$ if and only if $\gamma \leqslant \gamma_a$.

Proof. (i) By Lemma 5.22, the closure system associated to γ_a is $R_{\gamma_a} = \{x/_{\mathrm{d}} a : x \in R\} = \{x\backslash a : x \in R\}$.

(ii) By (i), $a = e\backslash a \in R_{\gamma_a}$, that is $\gamma_a(a) = a$.

(iii) In virtue of Remark 5.19, it is enough to show that if γ is an operator on $\mathbb{R}$, then $\gamma(a) = a$ if and only if $R_{\gamma_a} \subseteq R_\gamma$. Let suppose that $\gamma(a) = a$. By (i), every element in R_{γ_a} is of the form $x\backslash a$ for some $x \in R$. Since by the structurallty of γ we have that $x \cdot \gamma(x\backslash a) \leqslant \gamma(x \cdot (x\backslash a)) \leqslant \gamma(a) = a$, by the motonicity of γ and the hypothesis, hence $\gamma(x\backslash a) \leqslant x\backslash a \leqslant \gamma(x\backslash a)$. That is, $x\backslash a \in R_\gamma$. For the other implication, simply notice that if $\gamma \leqslant \gamma_a$, then by (ii) and Remark 5.19, $a \in R_{\gamma_a} \subseteq R_\gamma$, and therefore $\gamma(a) = a$. $\square$

5.4 Recognizable Elements in Residuated Lattices

Going back to our discussion about recognizable languages, we recall that a language L in the alphabet Σ is recognizable if and only if there exist a finite state automaton $\langle S, \Sigma, \star, i, F\rangle$ such that for every $w \in \Sigma^*$, $w \in L$ if and only if $w * i \in F$. According to Example 5.11, we can extend the action $\star$ of Σ^* on S to obtain a $\mathcal{P}(\Sigma^*)$-module $\langle \mathcal{P}(S), *\rangle$. Now L, $\{i\}$, and F are in $\mathcal{P}(S)$. Further,

$$F/_*\{i\} = \{w \in \Sigma^* : w \star x \in T, \text{ for all } x \in \{i\}\} = \{w \in \Sigma^* : w \star i \in T\} = L.$$

Thus, the notion of recognizable language can be captured in terms of modules over residuated lattices. We have stablished the following proposition, which also suggests the definition of a recognizable element in a residuated lattice.

Proposition 5.26. *A language L in the alphabet Σ is recognizable by a finite state automaton $\langle S, \Sigma, \star, i, F\rangle$ if and only if $L = F/_{*}\{i\}$, where $/_{*}$ is the residual of the $\mathcal{P}(\Sigma^*)$-module $\langle \mathcal{P}(S), *\rangle$.* □

Definition 5.27. An element a of a residuated lattice $\mathbf{R}$ is said to be *recognizable* provided there exists a finite $\mathbf{R}$-module $\mathbb{P}$ and elements $i, t \in P$ such that $a = t/_{*} i$. If the preceding conditions are satisfied, we also say that a is *recognized* by $\mathbb{P}$, i and t.

Remark 5.28. *If an element of a residuated lattice $\mathbf{R}$ is recognized by $\mathbb{P}$, i and t, then we can always assume that $\mathbb{P}$ is cyclic and generated by i. Indeed, consider the submodule $\langle i\rangle_{\mathbb{P}}$ of $\mathbb{P}$ and let $t' = a * i$. Since $a = t/_{*} i$, we have that $t' = a * i \leqslant t$, and hence $a \leqslant (a * i)/_{*} i = t'/_{*} i \leqslant t/_{*} i = a$. It follows that a is recognized by $\langle i\rangle_{\mathbb{P}}$, i, and t'.*

We prove next that the definition of a recognizable element in a residuated lattice is the correct abstraction of the concept of a recognizable language, in the sense that the recognizable languages in an alphabet Σ are exactly the recognizable elements of the residuated lattice $\mathcal{P}(\Sigma^*)$.

Proposition 5.29. *If L is a language in the alphabet Σ, then L is recognizable as a language if and only if it is recognizable as an element of $\mathcal{P}(\Sigma^*)$.*

Proof. ($\Rightarrow$) In virtue of Proposition 5.26, we have that if L is recognizable by the finite state automaton $\langle S, \Sigma, \star, i, F\rangle$, then L is an element of $\mathcal{P}(\Sigma^*)$ recognized by the module $\mathbb{P} = \langle \mathcal{P}(S), *\rangle$ and the elements $\{i\}$ and F, and since S is finite, then so is $\mathbb{P}$.

($\Leftarrow$) If L is recognizable as an element of $\mathcal{P}(\Sigma^*)$, then there is a finite $\mathcal{P}(\Sigma^*)$-module $\mathbb{P} = \langle \mathbf{P}, *\rangle$ and two elements $i, t \in P$ such that $L = t/_{*} i$. We can define the map $\star : \Sigma^* \times P \to P$ by $w \star x = \{w\} * x$, which can readily be proven to be an action of Σ^* on P. Furthermore,

$$L = t/_{*} i = \max\{A \in \mathcal{P}(\Sigma^*) : A * i \leqslant t\} = \{w \in \Sigma^* : \{w\} * i \leqslant t\} = \{w \in \Sigma^* : w \star i \leqslant t\}$$
$$= \{w \in \Sigma^* : w \star i \in {\downarrow} t\},$$

where ${\downarrow} t = \{x \in P : x \leqslant t\}$. Hence $\langle P, \Sigma, \star, i, {\downarrow} t\rangle$ is a finite state automaton and L is recognized by it. □

Recognizability of elements in a residuated lattice is a notion invariant up to $\mathbf{R}$-isomorphisms. This is an immediate consequence of Proposition 5.15.

Corollary 5.30. *If $\mathbb{P}$ and $\mathbb{Q}$ are two isomorphic $\mathbf{R}$-modules, then an element $a \in R$ is recognized by $\mathbb{P}$ if and only if it is also recognized by $\mathbb{Q}$.* □

The next theorem is an intrinsic characterization of recognizable elements of a residuated lattice. This result in conjunction with Proposition 5.29 indicate that, in order to determine whether a language in an alphabet Σ is recognizable or not, we do not need to look for finite automata or homomorphisms from Σ^* onto finite monoids, but instead we can do it by just analyzing the structure of $\mathcal{P}(\Sigma^*)$ as a residuated lattice. Note that, given a residuated lattice $\mathbf{R}$ and an element $a \in R$, we can consider the module $\mathbb{R}^{\mathrm{d}}$, and as we saw $a = (a \cdot^{\mathrm{d}} a)/_{\mathrm{d}} a$. Therefore, if $\langle a \rangle_{\mathbb{R}^{\mathrm{d}}}$ is finite, then it recognizes a. Moreover, by Proposition 5.23, we now that there is an isomorphism between $\langle a \rangle_{\mathbb{R}^{\mathrm{d}}}$ and $\mathbb{R}_{\gamma_a}$. Therefore, by Corollary 5.30, if $\mathbb{R}_{\gamma_a}$ is finite, then a is recognizable. We prove that this is not only a sufficient condition, but indeed a characterization.

Theorem 5.31. *Let $\mathbf{R}$ be a residuated lattice and $a \in R$. The following are equivalent:*

(i) The element a is recognizable.
(ii) There exists a closure operator γ on $\mathbb{R}$ such that $\gamma(a) = a$ and R_γ is finite.
(iii) The $\mathbf{R}$-module $\mathbb{R}_{\gamma_a}$ is finite. That is, the set $\{x \backslash a : x \in R\}$ is finite.
(iv) The $\mathbf{R}$-module $\langle a \rangle_{\mathbb{R}^{\mathrm{d}}}$ is finite. That is, the set $\{a/x : x \in R\}$ is finite.

Proof. We will show that (i) and (ii) are equivalent, that (ii) and (iii) are equivalent, and that (iii) and (iv) are equivalent as well.

(i) $\Rightarrow$ (ii): Suppose that a is recognized by $\mathbb{P}$, i, and t. Without loss of generality, we assume that $\mathbb{P}$ is cyclic and i is a generator of $\mathbb{P}$. Consider the closure operator γ_i on $\mathbb{R}$. By Proposition 5.23, $\mathbb{R}_{\gamma_i} \cong \langle i \rangle_{\mathbb{P}} = \mathbb{P}$, and therefore R_{γ_i} is finite. Let $b \in R$ such that $b * i = t$. Hence, $\gamma_i(b) = (b * i)/_* i = t/_* i = a$, and thus $a \in R_{\gamma_i}$.

(ii) $\Rightarrow$ (i): Suppose that there is a closure operator γ on $\mathbb{R}$ such that $\mathbb{R}_\gamma$ is finite and $\gamma(a) = a$. Hence, by Proposition 5.25, $\gamma \leqslant \gamma_a$, and therefore $\gamma(e) \leqslant \gamma_a(e) = (a/e)\backslash a = a \backslash a$. Thus, $a \cdot \gamma(e) \leqslant a$, and it follows that $a \leqslant a/\gamma(e) \leqslant a/e = a$, because $e \leqslant \gamma(e)$. Therefore, $a = a/\gamma(e)$, which proves that a is recognized by $\mathbb{R}_\gamma$, $\gamma(e)$, and a. Note that $\mathbb{R}_\gamma$ is cyclic with generator $\gamma(e)$.

(ii) $\Leftrightarrow$ (iii): In virtue of Proposition 5.25, if γ is a closure operator on $\mathbb{R}$ such that $\gamma(a) = a$, then $\gamma \leqslant \gamma_a$, and therefore $R_{\gamma_a} \subseteq R_\gamma$. If furthermore R_γ is finite, then it follows that R_{γ_a} is also finite. The other implication is obvious, since $\gamma_a(a) = a$, as we saw in Proposition 5.25.

(iii) $\Leftrightarrow$ (iv): In virtue of Proposition 5.23, there is an isomorphism between $\mathbb{R}_{\gamma_a}$ and $\langle a \rangle_{\mathbb{R}^{\mathrm{d}}}$, and therefore a bijection between their universes. We saw in Proposition 5.25 that $R_{\gamma_a} = \{x \backslash a : x \in R\}$. Finally, notice that $\langle a \rangle_{\mathbb{R}^{\mathrm{d}}} = \{x \cdot^{\mathrm{d}} a : x \in R\} = \{a/x : x \in R\}$. $\square$

Remark 5.32. *Modules over residuated lattices, as defined above, are sometimes called* left-modules, *since the action of the residuated lattice is on the left, and thus the notion of a recognizable element of a residuated lattice could more accurately be called* left-recognizable. *Analogously, one could define* right-modules, *in which the action of the residuated lattice is on the right, and thereby obtain a notion of a* right-recognizable *element. However, the equivalences of the previous theorem establish that the notions of a left-recognizable and right-recognizable element coincide.*

Last theorem is a generalization of Myhill's Theorem [74]. Given a language L over an alphabet Σ, we can define its *syntactic congruence*, which is actually the congruence on the monoid Σ^*, by

$$w_1 \approx_L w_2 \quad \Leftrightarrow \quad \text{for all } u, v \in \Sigma^*,\ (uw_1v \in L \quad \Leftrightarrow \quad uw_2v \in L).$$

This is the largest monoid congruence on Σ^* that *saturates* L, that is, such that it does not relate words in L with words outside L. We could also define the right-congruence on Σ^* by

$$w_1 \sim_L w_2 \quad \Leftrightarrow \quad \text{for any } v \in \Sigma^*,\ (w_1v \in L \quad \Leftrightarrow \quad w_2v \in L).$$

Myhill's Theorem characterizes the recognizable languages as those for which both $\approx_L$ and $\sim_L$ are of finite index (see [63]).

Theorem 5.33. *For a language L over an alphabet Σ, the following are equivalent:*

(i) *L is recognizable.*
(ii) *$\approx_L$ is of finite index, that is, the quotient $\Sigma^*/\approx_L$ is finite.*
(iii) *$\sim_L$ is of finite index, that is, the quotient $\Sigma^*/\sim_L$ is finite.*

This can be readily proved to be a consequence of Theroem 5.31. We present now a few examples to illustrate the previous discussion.

Example 5.34. Let $\mathbf{R} = \langle \mathbb{N} \cup \{\infty\}, \wedge, \vee, \cdot, \backslash, /, 1 \rangle$, where $\cdot$ is the usual multiplication in the set $R = \mathbb{N} \cup \{\infty\}$ of extended natural numbers, ordered as usual, and in which $\infty \cdot x = x \cdot \infty = \infty$ if $x \neq 0$, and $\infty \cdot 0 = 0 \cdot \infty = 0$. One can verify that this is a residuated lattice and for any $a \in \mathbb{N}$, $x \backslash a \in \{0, 1, 2, \ldots, a\} \cup \{\infty\}$ and $x \backslash \infty = \infty$, for every $x \in R$. Therefore, every element of $\mathbf{R}$ is recognizable.

Example 5.35. Let $\mathbb{Z}$ be the residuated lattice of all integers under the usual addition. Then, no element of $\mathbb{Z}$ is recognizable. In fact, no element of any non-trivial ℓ-group is recognizable. This is so because a non-trivial ℓ-group is infinite, and for a fixed element a of an ℓ-group $\mathbf{G}$, $\{x \backslash a : x \in G\} = \{x^{-1}a : x \in G\} = G$.

Example 5.36. Consider[2] the residuated lattice $\mathcal{P}(\langle \mathbb{N}, +, 0\rangle) = \langle \mathcal{P}(\mathbb{N}), \cap, \cup, +, \backslash, /, \{0\}\rangle$, in which the monoidal operation is defined as follows: $A + B = \{a + b : a \in A, b \in B\}$ for every $A, B \in \mathcal{P}(\mathbb{N})$. All finite members of $\mathcal{P}(\langle \mathbb{N}, +, 0\rangle)$ are recognizable, and so are some of its infinite members.

Let $A \in \mathcal{P}(\mathbb{N})$ be finite and nonempty, and $X \in \mathcal{P}(\mathbb{N})$ an arbitrary element. It follows from the inequality $X + (X\backslash A) \subseteq A$, that if $X \neq \emptyset$, then $X\backslash A \subseteq [0, \max A]$, and $\emptyset\backslash A = \mathbb{N}$. Therefore, $\{X\backslash A : X \in \mathcal{P}(\mathbb{N})\} \subseteq \mathcal{P}([0, \max A]) \cup \{\mathbb{N}\}$. This inequality shows that $\{X\backslash A : X \in \mathcal{P}(\mathbb{N})\}$ is finite, and hence A is recognizable. Furtheremore, for every $X \in \mathcal{P}(\mathbb{N})$, $X\backslash\emptyset = \emptyset$, and therefore $\emptyset$ is also recognizable.

If E and O are the sets of even and odd numbers, respectively, then $\{X\backslash E : X \in \mathcal{P}(\mathbb{N})\} = \{\emptyset, E, O, \mathbb{N}\}$. Therefore, E is recognizable. Notice that this also proves that $\emptyset$, O, and $\mathbb{N}$ are recognizable. We will see in Example 5.43 that all cofinite sets, which are infinite, are also recognizable.

Finally, we give an example of an infinite set which is not recognizable. Let $T = \{n(n+1)/2 : n \in \mathbb{N}\}$ the set of triangular numbers. Then, for every $n \in \mathbb{N}$, $\{n\}\backslash T = \{t - n : t \in T, t \geqslant n\}$. Now, one can verify that $\{n\}\backslash T \neq \{m\}\backslash T$ for $n \neq m$ (just considering the two first elements of each one of these sets), and thus T is not recognizable.

In the remainder of this section we investigate how recognizable elements are affected by certain special maps between residuated lattices. Given two residuated lattices, $\mathbf{R}'$ and $\mathbf{R}$, a residuated monoidal homomorphism $\varphi : \mathbf{R}' \to \mathbf{R}$, and an $\mathbf{R}$-module $\mathbb{P} = \langle \mathbf{P}, * \rangle$, one can define an $\mathbf{R}'$-module $\mathbb{P}' = \langle \mathbf{P}, *' \rangle$ in the following way: for every $a \in R'$, and every $x \in P$, let $a *' x = \varphi(a) * x$. It is easy to see that for all $x \in P$, and $a, b \in R'$ $a *' (b *' x) = (a \cdot' b) *' x$, and $e *' x = x$, since φ preserves products and the neutral element. To show is that $*' : \mathbf{R}' \times \mathbf{P} \to \mathbf{P}$ is residuated in both coordinates, note that if φ^+ is the residual of φ, then for all $a \in R'$ and $x, y \in P$ we have:

$$a *' x \leqslant y \quad\Leftrightarrow\quad \varphi(a) * x \leqslant y \quad\Leftrightarrow\quad \varphi(a) \leqslant y /_* x \quad\Leftrightarrow\quad a \leqslant \varphi^+(y /_* x) \quad\Leftrightarrow\quad x \leqslant \varphi(a) \backslash_* y.$$

Thus, the residuals of $*'$ are given by $y /_{*'} x = \varphi^+(y /_* x)$ and $a \backslash_{*'} y = \varphi(a) \backslash_* y$.

Definition 5.37. Given two residuated lattices, $\mathbf{R}'$ and $\mathbf{R}$, an $\mathbf{R}$-module $\mathbb{P} = \langle \mathbf{P}, * \rangle$, and a residuated monoidal homomorphism $\varphi : \mathbf{R}' \to \mathbf{R}$, we say that the $\mathbf{R}'$-module $\mathbb{P}' = \langle \mathbf{P}, *' \rangle$ is obtained from $\mathbb{P}$ and φ by *restriction of scalars*.

Proposition 5.38. *Let $\varphi : \mathbf{R}' \to \mathbf{R}$ be a residuated monoidal homomorphism $\mathbb{P}$ an $\mathbf{R}$-module, let $\mathbb{P}'$ the $\mathbf{R}'$-module be obtained by restriction of scalars from $\mathbb{P}$ and φ, and let $a' \in R'$, $a \in R$*

[2]This example was suggested by N. Galatos.

be arbitrary elements. If $\mathbb{P}$ recognizes a, then $\mathbb{P}'$ recognizes $\varphi^+(a)$. Moreover, if φ is injective and $\mathbb{P}$ recognizes $\varphi(a')$, then $\mathbb{P}'$ recognizes a'; and if φ is surjective and $\mathbb{P}'$ recognizes a', then $\mathbb{P}$ recognizes $\varphi(a')$.

Proof. In order to prove the first part, notice that if $a \in R$ is recognized by $\mathbb{P}$, and $t, i \in P$, then $a = t/_* i$ and then $\varphi^+(a) = \varphi^+(t/_* i) = t/_{*'} i$. For the second part, we recall that if φ is residuated and injective, then its residual φ^+ is a left inverse, that is, $\varphi^+\varphi(a') = a'$, for every $a' \in R'$. Thus, if $\varphi(a')$ is recognized by $\mathbb{P}$ and $t, i \in P$, then $\varphi(a') = t/_* i$, whence we obtain $a' = \varphi^+\varphi(a') = \varphi^+(t/_* i) = t/_{*'} i$. Analogously, if φ is surjective, then φ^+ is a right inverse, and if a' is recognized by $\mathbb{P}'$ and $t, i \in P$, then $a' = t/_{*'} i = \varphi^+(t/_* i)$, whence we obtain $\varphi(a') = \varphi\varphi^+(t/_* i) = t/_* i$. $\square$

Remark 5.39. *In view of the preceding proposition, recognizability is a notion invariant under isomorphisms of residuated lattices. That is, if φ is an isomorphism between $\mathbf{R}'$ and $\mathbf{R}$, and $a' \in R$, then a' is recognizable if and only if $\varphi(a')$ is recognizable.*

5.5 Regular Elements and Boolean-Recognizability

We will devote this section to the study of two problems, providing some interesting results that might lead to their eventual resolutions. The first one is finding a Kleene-like characterization of the recognizable elements of a residuated lattice, while the second seeks a characterization of those elements of a residuated lattice that are recognizable by Boolean cyclic modules.

A celebrated result due to Kleene establishes that recognizable languages coincide with regular languages. The set of *regular* languages on an alphabet Σ is the smallest set containing the full language Σ^*, the singleton languages $\{w\}$, for every $w \in \Sigma^*$, and is closed under finite intersections and unions, complex multiplication,[3] complementation, and the operation $(\,)^*$. A similar characterization for recognizable elements in a residuated lattice would most likely require an appropriate abstraction of the corresponding terms: whereas intersection, union, and complex multiplication correspond to the meet, join, and multiplication operations of the residuated lattice, respectively, it is not obvious what the proper abstraction of the other operations should be.

As we have already observed, not every residuated lattice has recognizable elements. We will see that Kleene's characterization strongly depends on the fact that the residuated lattice $\mathcal{P}(\Sigma^*)$ is of the form $\mathcal{P}(\mathbf{M})$, and even on the monoidal properties

[3] See Example 5.9.

of Σ^*. Nevertheless, we can prove that whenever a residuated lattice has recognizable elements, the set they form is closed under some operations.

Proposition 5.40. *A residuated lattice* $\mathbf{R}$ *has recognizable elements if and only if it has a top element* $\top$*, in which case the set of recognizable elements of* $\mathbf{R}$ *contains* $\top$*, and is closed under (finite) meets and residuation by arbitrary elements. In other words, given two recognizable elements* $a, b \in R$ *and arbitrary* $c \in R$*, the elements* $a \wedge b, c\backslash a$ *and* a/c *are recognizable.*

Proof. If a residuated lattice $\mathbf{R}$ does not have a top element, then it is easy to see that every closure operator γ on $\mathbf{R}$ would have an infinite associated closed system R_γ, because of the expansiveness of γ, and therefore for every $a \in R$, R_{γ_a} would be infinite, and hence no element of $\mathbf{R}$ would be recognizable. On the other hand, if $\mathbf{R}$ has a top element $\top$, then from $x \cdot \top \leqslant \top$, which is true for every $x \in R$, one could derive that $\top \leqslant x\backslash\top$, and therefore $R_{\gamma_\top} = \{\top\}$, which shows that $\top$ is recognizable.

Note that $x\backslash(a \wedge b) = x\backslash a \wedge x\backslash b$, and hence $R_{\gamma_{a\wedge b}} = \{x\backslash(a \wedge b) : x \in R\} \subseteq \{s \wedge t : s \in R_{\gamma_a},\ t \in R_{\gamma_b}\}$. Since R_{γ_a} and R_{γ_b} are finite by hypotheses, so is $R_{\gamma_{a\wedge b}}$, and therefore $a \wedge b$ is recognizable.

In order to prove that $c\backslash a$ is recognizable, just notice that $c\backslash a \in R_{\gamma_a}$, and therefore $\gamma_a(c\backslash a) = c\backslash a$ and R_{γ_a} is finite, whence in virtue of Theorem 5.31, $c\backslash a$ is recognizable.

Finally, in order to prove that a/c is recognizable, consider the inclusion $\langle a/c\rangle_{\mathbb{R}^d} = \{(a/c)/x : x \in R\} = \{a/(xc) : x \in R\} \subseteq \{a/x : x \in R\} = \langle a\rangle_{\mathbb{R}^d}$, and since $\langle a\rangle_{\mathbb{R}^d}$ is finite, so is $\langle a/c\rangle_{\mathbb{R}^d}$, what shows that a/c is recognizable. $\square$

Remark 5.41. *Proposition 5.40 gives another reason why no element of a non-trivial* ℓ*-group is recognizable (see Example 5.35), since such an algebra is unbounded.*

There is a very straightforward argument why the complement of a recognizable language L over an alphabet Σ is recognizable: if L is recognized by $\langle S, \Sigma, \star\rangle$ with initial state i and set of final states F, then L is the set of all words $w \in \Sigma^*$ such that $w \star i \in F$, and therefore the complement L' of L is the set of all words $w \in \Sigma^*$ such that $w \star i \notin F$, that is to say, L' is recognized by the same automaton $\langle S, \Sigma, \star\rangle$ with the same initial state i and set of final states F', the complement of F. Nevertheless, the modules $\langle \mathcal{P}(S), *\rangle_{\gamma_L}$ and $\langle \mathcal{P}(S), *\rangle_{\gamma_{L'}}$ might look very different. Actually, the property that the set of recognizable elements is closed under complementation is true for every residuated lattice that arises as in Example 5.9.

Proposition 5.42. *Let* $\mathbf{M}$ *be a monoid. Then, the set of recognizable elements of the residuated lattice* $\mathcal{P}(\mathbf{M})$ *is closed under complementation and under (finite) unions.*

Proof. First of all notice that, for every $A \in \mathcal{P}(M)$, $\emptyset \backslash A = M$, and for every $X \neq \emptyset$,

$$X \backslash A = \Big(\bigcup_{x \in X} \{x\}\Big) \backslash A = \bigcap_{x \in X} (\{x\} \backslash A).$$

Thus, it is clear that $\mathcal{P}(M)_{\gamma_A} = \{X \backslash A : X \in \mathcal{P}(M)\}$ is finite if and only if $\big\{\{x\} \backslash A : x \in M\big\}$ is finite. Now, for every $x \in M$, we have:

$$\{x\} \backslash A = \{y \in M : xy \in A\} = \{y \in M : xy \notin A\}' = \{y \in M : xy \in A'\}' = \big(\{x\} \backslash A'\big)'$$

Hence, $\{x\} \backslash A' = \big(\{x\} \backslash A\big)'$, whence by Theorem 5.31 it follows that if A is recognizable, then so is A'. Finally, we only need to notice that the empty union is $\emptyset = M'$, which is therefore recognizable, and that if A and B are recognizable, then $A \cup B = (A' \cap B')'$ which is also recognizable in virtue of Proposition 5.40. $\square$

Example 5.43. Let consider the residuated lattice $\mathcal{P}(\langle \mathbb{N}, +, 0 \rangle)$ of Example 5.36. Since we proved that its finite elements are recognizable, its cofinite elements are also recognizable. As we mentioned before, the lattices $\mathcal{P}(\mathbb{N})_{\gamma_A}$ and $\mathcal{P}(\mathbb{N})_{\gamma_{A'}}$ might look very different. For instance, whereas (see Example 5.54 on page 109) the lattice of $\mathcal{P}(\mathbb{N})_{\gamma_{\{n\}}}$ is that of Diagram (5.4), the lattice of $\mathcal{P}(\mathbb{N})_{\gamma_{\{n\}'}}$ is isomorphic to the Boolean algebra $\mathcal{P}(\{0, 1, \ldots, n\})$.

The fact that singletons $\{w\}$, for $w \in \Sigma^*$, are recognizable languages, not only depends (from our perspective) on the fact that the residuated lattice $\mathcal{P}(\Sigma^*)$ is of the form $\mathcal{P}(\mathbf{M})$, but also on the monoidal properties of Σ^*.

Proposition 5.44. *Let* $\mathbf{M}$ *be a monoid and* $a \in M$ *an arbitrary element, and let consider the residuated lattice* $\mathcal{P}(\mathbf{M})$*. If* a *has a finite number of divisors then* $\{a\}$ *is recognizable. If* $\mathbf{M}$ *is cancellative and* $\{a\}$ *is recognizable, then* a *has a finite number of divisors.*

Proof. As we showed in the proof of Proposition 5.42, for any $A \in \mathcal{P}(\mathbf{M})$, A is recognizable if and only if $\big\{\{x\} \backslash A : x \in M\big\}$ is finite. Given $a, x \in M$, we have that

$$\{x\} \backslash \{a\} = \{y \in M : xy = a\}.$$

Therefore, this set is empty, except when x is a divisor of a, whence it follows that if a has a finite number of divisors, then it is recognizable. For the other implication, observe that, given two different divisors x, x' of an element a, there exist two elements y, y' such that $xy = a = x'y'$, and by the cancellativity of $\mathbf{M}$, we obtain $y \neq y'$. Therefore, the sets $\{x\} \backslash \{a\}$ and $\{x'\} \backslash \{a\}$ are different. Thus, if a has an infinite number of divisors, $\{a\}$ is not recognizable. $\square$

Example 5.45. If $\mathbf{M}$ is an infinite group, then no singleton of $\mathcal{P}(\mathbf{M})$ is recognizable. On the other hand, notice that 0 has an infinite number of divisors in $\langle \mathbb{N}, \cdot, 1 \rangle$, but for every $x \in \mathbb{N}$, $\{x\} \backslash \{0\} = \{y \in \mathbb{N} : xy = 0\} = \{0\}$ if $x \neq 0$, and $\{0\} \backslash \{0\} = \mathbb{N}$. Therefore, $\{0\}$ is recognizable, and since all the positive natural numbers have only a finite number of divisors, it follows that every singleton of $\mathcal{P}(\langle \mathbb{N}, \cdot, 1 \rangle)$ is recognizable.

We next direct our attention to another problem that does not focus on the structure of the set of recognizable elements of a residuated lattice, but on the structure of the modules that recognize them. We notice that, by virtue of Theorem 5.5, given a recognizable language L, there exists a surjective homomorphism of monoids $\varphi : \Sigma^* \to \mathbf{M}$ and a set $T \subseteq M$ such that $\mathbf{M}$ is finite and $L = \varphi^{-1}(T)$. As we mentioned before, this map extends to a residuated map $\overline{\varphi} : \mathcal{P}(\Sigma^*) \to \mathcal{P}(\mathbf{M})$, which is also a homomorphism of monoids and whose residual is φ^{-1}. We can consider the residuated lattice $\mathcal{P}(\mathbf{M})$ as a module $\mathbb{P}$ over itself, and since it is finite (because $\mathbf{M}$ is finite), every of its elements is recognizable. In particular T is recognizable by $\mathbb{P}$ which implies, in light of Proposition 5.38, that $L = \varphi^{-1}(T)$ is recognized by $\mathbb{P}'$, the $\mathcal{P}(\Sigma^*)$-module obtained by $\mathbb{P}$ and $\overline{\varphi}$ by restriction of scalars. Note that since $\mathbb{P}$ is a cyclic $\mathcal{P}(\mathbf{M})$-module and $\overline{\varphi}$ is surjective, $\mathbb{P}'$ is also cyclic, and moreover the lattice reduct of $\mathbb{P}'$ is a Boolean algebra.

Definition 5.46. Given a residuated lattice $\mathbf{R}$, we say that it is *Boolean* if its lattice reduct is so. Also, given an $\mathbf{R}$-module $\mathbb{P}$, we say that $\mathbb{P}$ is *Boolean*[4] if its lattice reduct is so.

Thus, we have stablished the following result.

Proposition 5.47. *Every recognizable language on any alphabet is recognized by a Boolean cyclic module.* □

The questions that follow are completely natural: are all the recognizable elements of residuated lattices recognized by Boolean cyclic modules? If not, which elements are? And in particular, given an element a of a residuated lattice $\mathbf{R}$, when is $\mathbf{R}_{\gamma_a}$ a Boolean module? As we will see, not every recognizable element in a residuated lattice can be recognized by a Boolean cyclic module (see Example 5.63). However, let us analyze a bit further the case of recognizable languages. Since $\overline{\varphi}$ is onto $\mathcal{P}(\mathbf{M})$, because $\varphi : \Sigma^* \to \mathbf{M}$ is surjective, we have that $\gamma = \varphi^{-1}\overline{\varphi}$ is a closure operator on $\mathcal{P}(\Sigma^*)$ such that $\mathcal{P}(\Sigma^*)_\gamma \cong \mathcal{P}(M)$, as lattices. We prove now that this closure operator satisfies the

[4]The reader may be familiar with the concept of a Boolean module as introduced by Brink in [16], but those are not exactly the same kind of structures, as Brink's Boolean modules are modules over relation algebras.

equation

$$\gamma(a \wedge \gamma(b)) = \gamma(a) \wedge \gamma(b), \tag{5.3}$$

which has interesting consequences, as we will see.

Proposition 5.48. *Given any map $\varphi : X \to Y$, and its extension to a residuated map $\overline{\varphi} : \mathcal{P}(X) \to \mathcal{P}(Y)$, for every $A, B \subseteq X$, we have*

$$\overline{\varphi}(\varphi^{-1}\overline{\varphi}(A) \cap B) = \overline{\varphi}(A) \cap \overline{\varphi}(B).$$

Moreover, the closure operator $\gamma = \varphi^{-1}\overline{\varphi}$ satisfies equation (5.3).

Proof. Since $\overline{\varphi} = \overline{\varphi}\varphi^{-1}\overline{\varphi}$, and $\overline{\varphi}(S \cap T) \subseteq \overline{\varphi}(S) \cap \overline{\varphi}(T)$, for any $S, T \subseteq X$, we get $\overline{\varphi}(\varphi^{-1}\overline{\varphi}(A) \cap B) \subseteq \overline{\varphi}(A) \cap \overline{\varphi}(B)$. Conversely, choose $y \in \overline{\varphi}(A) \cap \overline{\varphi}(B)$. Then there exist $a \in A$ and $b \in B$ such that $y = \varphi(a) = \varphi(b)$. Thus, $b \in \varphi^{-1}\overline{\varphi}(A)$, and therefore we get $y = \varphi(b) \in \overline{\varphi}(\varphi^{-1}\overline{\varphi}(A) \cap B)$. Hence, the equation holds. Finally, since φ^{-1} is the residual of a residuated map, hence it preserves meets, and therefore we have

$$\varphi^{-1}\overline{\varphi}(\varphi^{-1}\overline{\varphi}(A) \cap B) = \varphi^{-1}(\overline{\varphi}(A) \cap \overline{\varphi}(B)) = \varphi^{-1}\overline{\varphi}(A) \cap \varphi^{-1}\overline{\varphi}(B),$$

that is, γ satisfies equation (5.3). $\square$

Theorem 5.49. *Let $\mathbf{B}$ be a Boolean algebra and γ a closure operator on $\mathbf{B}$. If γ satisfies equation (5.3) for all $a, b \in B$, then $\mathbf{B}_\gamma$ is also a Boolean algebra.*

Proof. Since $\mathbf{B}$ is a lattice, then $\mathbf{B}_\gamma$ also inherits a structure of lattice, where $x \wedge_\gamma y = x \wedge y$ and $x \vee_\gamma y = \gamma(x \vee y)$. We are going to show that $\mathbf{B}_\gamma$ is actually a bounded complemented distributive lattice.

(i) Bounds: Clearly, by the monotonicity of γ, $\gamma(\bot)$ and $\gamma(\top) = \top$ are the bottom and top elements of $\mathbf{B}_\gamma$.

(ii) Complements: For any element $x \in B_\gamma$, we will see that $\gamma(x')$ is its complement in $\mathbf{B}_\gamma$. Indeed, using equation (5.3):

$$x \wedge_\gamma \gamma(x') = \gamma(x) \wedge_\gamma \gamma(x') = \gamma(x) \wedge \gamma(x') = \gamma(\gamma(x) \wedge x') = \gamma(x \wedge x') = \gamma(\bot).$$

We also have:

$$x \vee_\gamma \gamma(x') = \gamma(x \vee \gamma(x')) \geqslant \gamma(x \vee x') = \gamma(\top) = \top.$$

(iii) Distributivity: If $x, y, z \in \mathbf{B}_\gamma$, then using equation (5.3) we have:

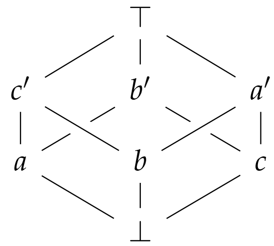
$$\begin{aligned} x \wedge_\gamma (y \vee_\gamma z) &= x \wedge \gamma(y \vee z) = \gamma(x) \wedge \gamma(y \vee z) = \gamma(\gamma(x) \wedge (y \vee z)) = \gamma(x \wedge (y \vee z)) \\ &= \gamma((x \wedge y) \vee (x \wedge z)) = (x \wedge y) \vee_\gamma (x \wedge z) = (x \wedge_\gamma y) \vee_\gamma (x \wedge_\gamma z). \end{aligned}$$

$\square$

Remark 5.50. *Obviously, if* $\mathbf{B}$ *is a complete Boolean algebra, then* $\mathbf{B}_\gamma$ *is also a complete Boolean algebra.*

The converse of Theorem 5.49 is not true. That is, equation (5.3) is a sufficient but not a necessary condition for $\mathbf{B}_\gamma$ to be a Boolean algebra. The following is an example of such a case.

Example 5.51. Let $\mathbf{B}$ the Boolean algebra represented by the diagram:



The map $\gamma : B \to B$ defined by $\gamma(a') = \gamma(b') = \gamma(c') = \gamma(\top) = \top$, $\gamma(a) = a$, $\gamma(b) = b$, $\gamma(c) = c$, $\gamma(\perp) = \perp$, is a closure operator and has a Boolean image, but γ fails equation (5.3), because $\gamma(\gamma(a) \wedge a') = \gamma(a \wedge a') = \gamma(\perp) = \perp$, but $\gamma(a) \wedge \gamma(a') = a \wedge \top = a$.

It is easy to see that the Boolean image $\mathbf{B}_\gamma$ is not necessarily a Boolean subalgebra of $\mathbf{B}$, just because $\gamma(\perp)$ may not be $\perp$. But, the following proposition states that, under the assumption of equation (5.3) for a closure operator γ on a Boolean algebra $\mathbf{B}$, $\gamma(\perp) = \perp$ is an equivalent condition to $\mathbf{B}_\gamma$ being a Boolean subalgebra of $\mathbf{B}$.

Corollary 5.52. *Let* γ *be a closure operator on a Boolean algebra* $\mathbf{B}$ *satisfying equation* (5.3)*. Then, the following are equivalent.*

(i) $\mathbf{B}_\gamma$ *is a Boolean subalgebra of* $\mathbf{B}$.
(ii) $\gamma(\perp) = \perp$.
(iii) $\gamma(a) = \perp$ *if and only if* $a = \perp$.

Proof. (i) $\Leftrightarrow$ (ii): If $\mathbf{B}_\gamma$ is a Boolean subalgebra of $\mathbf{B}$, then clearly $\gamma(\perp) = \perp$. For the reverse direction, assume $\gamma(\perp) = \perp$. It is enough to show that $\mathbf{B}_\gamma$ is closed under complementation. If it is true, then $\mathbf{B}_\gamma$ is also closed under finite joins, because $x \vee y = (x' \wedge y')' \in B_\gamma$, for any $x, y \in B_\gamma$. Therefore, we need to show that for any $x \in B_\gamma$, $\gamma(x') = x'$. But, by the part (ii) of the proof of Theorem 5.49, and by our hypotheses, we know that $x \wedge \gamma(x') = x \wedge_\gamma \gamma(x') = \gamma(\perp) = \perp$, and therefore $\gamma(x') \leqslant x'$, because in a Boolean algebra $a \wedge b = \perp$ implies $b \leqslant a'$. It follows that $\gamma(x') = x'$, as we wanted to prove.

(ii) $\Leftrightarrow$ (iii): This equivalence is trivial. $\square$

Equation (5.3) has an interesting application is the setting of frames. It is well known that given a nucleus γ on a frame $\mathbf{F}$, the image $\mathbf{F}_\gamma$ is also a frame.[5] But, the following proposition states that equation (5.3), which is satisfied by every nucleus, is sufficient to prove this result.

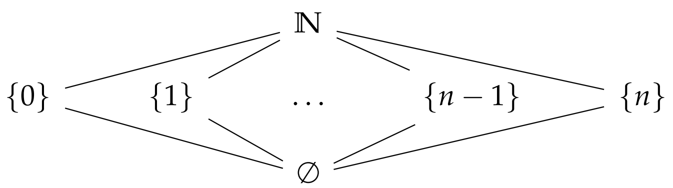
Proposition 5.53. *Let γ be a closure operator on a frame $\mathbf{F}$ satisfying equation (5.3). Then, the image $\mathbf{F}_\gamma$ is also a frame.*

Proof. We know that the image $\mathbf{F}_\gamma$ is a complete lattice and the meets in $\mathbf{F}$ and in $\mathbf{F}_\gamma$ coincide. We need to show that for any $x \in F_\gamma$ and any family $\{x_i : i \in I\} \subseteq F_\gamma$, $x \wedge \bigvee_I^{\mathbf{F}_\gamma} x_i = \bigvee_I^{\mathbf{F}_\gamma}(x \wedge x_i)$. We have:

$$\begin{aligned} x \wedge \bigvee_I^{\mathbf{F}_\gamma} x_i &= x \wedge \gamma\big(\bigvee_I^{\mathbf{F}} x_i\big) = \gamma(x) \wedge \gamma\big(\bigvee_I^{\mathbf{F}} x_i\big) = \gamma\big(\gamma(x) \wedge \bigvee_I^{\mathbf{F}} x_i\big) = \gamma\big(x \wedge \bigvee_I^{\mathbf{F}} x_i\big) \\ &= \gamma\big(\bigvee_I^{\mathbf{F}}(x \wedge x_i)\big) = \bigvee_I^{\mathbf{F}_\gamma}(x \wedge x_i). \end{aligned}$$ □

We next apply equation (5.3) to closure operators of the form γ_a for a recognizable element a of a Boolean residuated lattice. As the example below shows, such operators may not satisfy equation (5.3).

Example 5.54. Consider the residuated lattice $\mathcal{P}(\langle \mathbb{N}, +, 0\rangle)$ of Example 5.36. For any pair of numbers $n, m \in \mathbb{N}$, its easy to compute $\{n\}\backslash\{m\} = \{m\}/\{n\} = \{m-n\}$ if $n \leqslant m$, and otherwise $\{n\}\backslash\{m\} = \{m\}/\{n\} = \emptyset$. Therefore $\gamma_{\{n\}}(\{m\}) = (\{n\}/\{m\})\backslash\{n\} = \{n-m\}\backslash\{n\} = \{n-(n-m)\} = \{m\}$ if $m \leqslant n$, and $\gamma_{\{n\}}(\{m\}) = (\{n\}/\{m\})\backslash\{n\} = \emptyset\backslash\{n\} = \mathbb{N}$ otherwise. Moreover, its not difficult to see that if $A \subseteq \mathbb{N}$ contains more than one number, then $\gamma_{\{n\}}(A) = \mathbb{N}$. Therefore, the closure system associated to $\gamma_{\{n\}}$ is:



(5.4)

which evidently is a Boolean algebra only when $n = 1$. Hence, by Theorem 5.49, for every $n \neq 1$, the closure operator $\gamma_{\{n\}}$ on $\mathcal{P}(\langle \mathbb{N}, +, 0\rangle)$ fails equation (5.3).

Given a frame $\mathbf{F}$, the closure operators on $\mathbb{F}$ of the form γ_a, for some $a \in F$, are special. As we see in the following theorem (see also [39]), they produce Boolean algebras.

[5] This is because if γ is a nucleus on a frame $\mathbf{F}$, then γ is a closure operator on $\mathbb{F}$, and therefore $\mathbb{F}_\gamma = \langle \mathbf{F}_\gamma, \wedge\rangle$ is an $\mathbf{F}$-module. In particular, $\wedge$ is residuated on both coordinates on $\mathbf{F}_\gamma$, and therefore it distributes with respect to arbitrary joins.

Theorem 5.55. *Given a frame* $\mathbf{F}$ *and an arbitrary element* $a \in F$*, the image* $\mathbf{F}_{\gamma_a}$ *is a Boolean algebra.*

Proof. Since γ_a is a closure operator on $\mathbb{F}$, γ_a is a nucleus on $\mathbf{F}$, by virtue of Proposition 5.21. It follows that γ_a satisfies equation (5.3), and thus by Proposition 5.53, $\mathbf{F}_{\gamma_a}$ is also a frame. It remains to prove that $\mathbf{F}_{\gamma_a}$ is complemented. First, notice that for every $a \in F$, $\perp \to a = \top$, and since $\top$ is the neutral element of $\mathbf{F}$ as a residuated lattice, we also have that $\top \to a = a$ and $a \to a = \top$. Therefore the bottom and top elements of $\mathbf{F}_{\gamma_a}$ are $\gamma(\perp) = a$ and $\top$, respectively. We prove now that for every $x \in F_{\gamma_a}$, $x \to a$ is the complement of x in $\mathbf{F}_{\gamma_a}$.

Let $x \in F_{\gamma_a}$ be an arbitrary element. Since $x \wedge (x \to a) \leqslant a$, and a is the bottom element of $\mathbf{F}_{\gamma_a}$ and hence $a \leqslant x \wedge (x \to a)$, we have that $x \wedge (x \to a) = a$. Thus

$$\begin{aligned} x \vee_{\gamma_a} (x \to a) &= \gamma_a(x \vee (x \to a)) = \big((x \vee (x \to a)) \to a\big) \to a \\ &= \big((x \to a) \wedge ((x \to a) \to a)\big) \to a = \big((x \to a) \wedge \gamma_a(x)\big) \to a \\ &= \big((x \to a) \wedge x\big) \to a = a \to a = \top. \end{aligned}$$

□

We have proved that every recognizable element a of a frame $\mathbf{F}$, is recognized by a Boolean cyclic module, namely $\mathbb{F}_{\gamma_a}$. On the other hand, we have seen that it is not true in general that for any residuated lattice $\mathbf{R}$ and any element $a \in R$, $\mathbb{R}_{\gamma_a}$ is a Boolean module, even if $\mathbf{R}$ is Boolean. We close this chapter by providing sufficient conditions for a recognizable element of a Boolean residuated lattice to be recognized by a Boolean cyclic module. We first introduce some additional concepts (see [84]).

Definition 5.56. Let $\mathbf{R}$ be a residuated lattice and let a an arbitrary element of R.

- a is *cyclic* if $a/x = x\backslash a$, for every $x \in R$.
- a is *two-sided* if $a\backslash a = a/a = \top$.
- a is *semiprime* if for every $x \in R$, $x^2 \leqslant a \quad \Rightarrow \quad x \leqslant a$.
- a is *localic* if it is cyclic, two-sided, and semiprime.

We center our attention on localic elements because of the following theorem (see [12]).

Theorem 5.57. *Let* $\mathbf{R}$ *be a complete residuated lattice. Then, there exists a residuated map* $f : \mathbf{R} \to \mathbf{B}$ *of* $\mathbf{R}$ *onto a Boolean algebra* $\mathbf{B}$ *if and only if* $\mathbf{R}$ *contains a localic element.* □

The characterizations of the following lemma will be used extensively.

Lemma 5.58. *Let* $\mathbf{R}$ *be a residuated lattice, and* $a \in R$ *an arbitrary element.*

(i) *a is cyclic if and only if for every* $x, y \in R$, $xy \leqslant a \quad \Leftrightarrow \quad yx \leqslant a$.

(ii) a is two-sided if and only if for every $x \in R$, $xa \leqslant a$ and $ax \leqslant a$.

(iii) if a is cyclic and two-sided, then it is semiprime if and only if for every $x \in R$, $x\backslash a = x^2\backslash a$.

Proof. (i) This is an immediate consequence of residuation.

(ii) For an element $a \in R$, $a\backslash a = \top$ is equivalent to say that for every $x \in R$, $x \leqslant a\backslash a$, or which is the same as for every $x \in R$, $ax \leqslant a$; and analogously for $a/a = \top$ and for every $x \in R$, $xa \leqslant a$.

(iii) If for every $x \in R$, $x\backslash a = x^2\backslash a$, then it is easy to see that it is semiprime, because if $x^2 \leqslant a$, then $e \leqslant x^2\backslash a = x\backslash a$, and therefore $x \leqslant a$. In order to see the other implication suppose that a is cyclic, two-sided and semiprime. Thus, if $y \leqslant x^2\backslash a$, then $x^2 y \leqslant a$, and therefore $xyx \leqslant a$, by the cyclicity of a, which implies that $(yx)^2 = (yx)(yx) \leqslant y(xyx) \leqslant ya \leqslant a$, because a is two-sided. And therefore, $yx \leqslant a$, because it is semiprime, whence we obtain $xy \leqslant a$, again by cyclicity, and hence $y \leqslant x\backslash a$. Now, if $y \leqslant x\backslash a$, then $xy \leqslant a$, and therefore $x^2 y = x(xy) \leqslant xa = a$, because a is two-sided, and hence $y \leqslant x^2\backslash a$. Thus, we have proved that $y \leqslant x^2\backslash a$ if and only if $y \leqslant x\backslash a$, which implies that $x\backslash a = x^2\backslash a$. □

Remark 5.59. *Notice that if a is a cyclic element of a residuated lattice, then γ_a is a nucleus of* $\mathbf{R}$. *First, since γ_a is a structural closure operator on $\mathbb{R}$, hence for every $x, y \in R$, $x\gamma_a(y) \leqslant \gamma_a(xy)$. Now, for $x, y \in R$, using the cyclicity of a we have*

$$y(a/(xy))\gamma_a(x) = y((xy)\backslash a)\gamma_a(x) = y(y\backslash(x\backslash a))\gamma_a(x) \leqslant (x\backslash a)\gamma_a(x) = (a/x)\gamma_a(x) \leqslant a.$$

Thus, using again the cyclicity of a, we have $(a/(xy))\gamma_a(x)y \leqslant a$, whence it follows that $\gamma_a(x)y \leqslant (a/(xy))\backslash a = \gamma_a(xy)$. Now, using this inequality and the structurallity of γ_a we obtain:

$$\gamma_a(x)\gamma_a(y) \leqslant \gamma_a(x\gamma_a(y)) \leqslant \gamma_a(\gamma_a(xy)) = \gamma_a(xy).$$

Theorem 5.60. *Let* $\mathbf{R}$ *be a complete residuated lattice. For every localic element $a \in R$, the module $\mathbb{R}_{\gamma_a}$ is Boolean.*

Proof. To begin with, we are going to show that $\cdot_{\gamma_a}$ and $\wedge$ coincide in $\mathbb{R}_{\gamma_a}$. Indeed $x, y \in R_{\gamma_a}$. We claim that $\gamma_a(x \cdot y) = x \wedge y$. Notice that since a is two-sided, we have $(a/x)(xy) = ((a/x)x)y \leqslant ay \leqslant a$. Thus, $xy \leqslant (a/x)\backslash a = \gamma_a(x) = x$. Analogously, and using the cyclicity of a, we have $(xy)(a/y) = (xy)(y\backslash a) = x(y(y\backslash a)) \leqslant xa \leqslant a$, and again by the cyclicity of a, it follows that $(a/y)(xy) \leqslant a$. Thus, $xy \leqslant (a/y)\backslash a = \gamma_a(y) = y$. Hence, $xy \leqslant x \wedge y$, whence we have $\gamma_a(xy) \leqslant x \wedge y$.

In order to prove the other inequality, suppose that $t \leqslant x \wedge y$. Since $t \leqslant x$, hence $x\backslash a \leqslant t\backslash a$, and thus $t(x\backslash a) \leqslant a$, and by the cyclicity of a, $(x\backslash a)t \leqslant a$. Now, since $t \leqslant y$, hence $t((xy)\backslash a) = t(y\backslash(x\backslash a)) \leqslant y(y\backslash(x\backslash a)) \leqslant x\backslash a$. Therefore, we have

$$\big(((xy)\backslash a)t\big)^2 = \big(((xy)\backslash a)t\big)\big(((xy)\backslash a)t\big) = ((xy)\backslash a)\big((t((xy)\backslash a))t\big) \leqslant ((xy)\backslash a)\big((x\backslash a)t\big) \\ \leqslant ((xy)\backslash a)a \leqslant a.$$

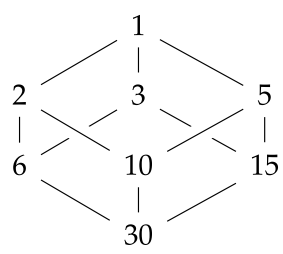
Since a is semiprime, $((xy)\backslash a)t \leqslant a$, and hence by the cyclicity of a, $(a/(xy))t \leqslant a$, which implies $t \leqslant (a/(xy))\backslash a = \gamma_a(xy)$. Thus we have proved that, in particular, $x \wedge y \leqslant \gamma_a(xy)$, as we wanted.

Therefore we have that indeed $\wedge$ and $\cdot_{\gamma_a}$ coincide in $\mathbb{R}_{\gamma_a}$, and since $\cdot_{\gamma_a}$ is residuated in both coordinates, so is $\wedge$, which means that $\wedge$ distributes over arbitrary joins of $\mathbf{R}_{\gamma_a}$. Therefore $\mathbf{R}_{\gamma_a}$ is actually a frame that contains the element a. Now, since the residuals of $\cdot_{\gamma_a}$ are the restictions of the residuals of $\cdot$, the corresponding restriction of γ_a is a closure operator on $\mathbb{R}_{\gamma_a}$, which is actually the identity. By virtue of Theorem 5.55, $\mathbf{R}_{\gamma_a}$ is a Boolean algebra, and this completes the proof. $\square$

The following is an immediate consequence of the preceding theorem.

Corollary 5.61. *Let* $\mathbf{R}$ *be a complete residuated lattice. Every recognizable localic element is recognized by a Boolean cyclic module.* $\square$

Example 5.62. Consider the residuated lattice $\mathbf{R} = \langle \mathbb{N}, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ over the set of natural numbers, where the order is given by division as follows: $n \leqslant m \iff m \mid n$. This is a complete commutative residuated lattice, and therefore we can denote both residuals by $\to$. Notice also that the bottom and the top elements of this residuated lattice are $\bot = 0$ and $\top = 1$. It is easy to see that $m \vee n = \gcd(m,n)$, the greatest common divisor of m and n, and $n \to m = \frac{m}{\gcd(m,n)}$, if m or n are different from 0, and $0 \to 0 = 1$. Therefore, $R_{\gamma_0} = \{0,1\}$, and R_{γ_m} is the set of the divisors of m, if $m \neq 0$. Thus, every element of this complete residuated lattice is recognizable. Moreover, since it is commutative, every element is cyclic, and given any $a, x \in \mathbb{N}$, $a \mid ax = xa$, which means that $ax \leqslant a$ and $xa \leqslant a$, and therefore a is two-sided. Furthermore, every square-free natural number is semiprime, and hence localic. Thus, by Corollary 5.61, every square-free natural number is recognized by a Boolean module. For example, the number 30 is recognized by the Boolean module $\mathbb{R}_{\gamma_{30}}$, whose Hasse diagram is:



The converse of Corollary 5.61 is not true in general, that is, there are elements of complete residuated lattices that are recognized by Boolean cyclic modules, but are not localic. As we saw in Example 5.54, $\{1\}$ is recognized by a four-element Boolean cyclic module. Nevertheless, $\{1\}$ is not two-sided, since $\{1\}\backslash\{1\} = \{0\} \neq \mathbb{N}$, and therefore it is not localic.

Lastly, we show with an example that not every recognizable element can be recognized by a Boolean cyclic module. The question whether any element of a residuated lattice can be recognized by a Boolean module, not necessarily cyclic, is still open.

Example 5.63. Consider the residuated lattice $\mathbf{R} = \langle \mathbb{N} \cup \{\infty\}, \wedge, \vee, \cdot, 1\rangle$ of Example 5.34. As we saw, every element of this residuated lattice is recognizable. But the cyclic $\mathbf{R}$-modules, are all chains, because $\mathbf{R}$ itself is a chain and the cyclic $\mathbf{R}$-modules are always of the form $\mathbb{R}_\gamma$ for some closure operator γ on $\mathbb{R}$. Given an element $n \in R$, and a closure operator γ on $\mathbb{R}$ that fixes n, we have that $\gamma \leqslant \gamma_n$ by Proposition 5.25, and therefore $\mathbf{R}_{\gamma_a} \subseteq \mathbf{R}_\gamma$. Therefore, all we have to do is showing an element $n \in R$ such that R_{γ_n} has more that two elements. For instance, it is easy to see that $R_{\gamma_5} = \{0, 1, 2, 5, \infty\}$, and therefore there is no Boolean cyclic module that recognizes 5.

Chapter 6

Conclusions and Open Problems

We devote this final chapter to concisely summarize the principal results of this dissertation, as well as presenting some of the problems that remain open. We start be recalling that the goal of Chapter 2 was understanding the class of projectable integral GMV algebras from a categorical point of view. We saw that they tantamount to the class of Gödel GMV algebras, which form a variety. The two main results of Chapter 2 describing the category of projectable integral GMV algebras and the variety of Gödel GMV algebras are the following:

Theorem A (See Theorem 2.14.)**.** *The categories of projectable IGMV algebras and of negative cones of projectable ℓ-groups with a nucleus are equivalent.*

Theorem B (See Theorem 2.28.)**.** *There is an adjunction between the categories whose objects are, respectively, Gödel GMV algebras and Gödel negative cones with a retraction and a dense nucleus on the image of the retraction.*

We see that Theorem B is weaker than Theorem A, in the sense that we only have an adjunction, instead of a categorical equivalence. Thus, the first question is obvious:

Problem 1. Is there an equivalence between the categories of Gödel GMV algebras and Gödel negative cones with a retraction and a dense nucleus on the image of the retraction.

And if this equivalence does not exist, then we could ask the following:

Problem 2. Can we modify the structure of the Gödel negative cones with a retraction and dense nucleus in some way in order to obtain an equivalence with the category of Gödel GMV algebras?

Chapter 3 is devoted to finding different kinds of hulls for semilinear residuated lattices. We started by proving that every semilinear e-cyclic residuated lattice $\mathbf{L}$ is densely embeddable in an residuated lattice $\mathcal{O}(\mathbf{L})$ (Theorem 3.14), and furtheremore $\mathcal{O}(\mathbf{L})$ is laterally complete (Theorem 3.21). As a consequence, we obtain one of the main results of this chapter:

Theorem C (Corollary 3.22)**.** *Every e-cyclic semilinear residuated lattice* $\mathbf{L}$ *is densely embeddable in a laterally complete lattice that belongs to the variety generated by* $\mathbf{L}$*.*

Later, we also proved that $\mathcal{O}(\mathbf{L})$ is strongly projectable (Theorem 3.31), which combined with Theorems 3.14 and 3.21 gives us the following result:

Theorem D (Corollary 3.33)**.** *Every e-cyclic semilinear residuated lattice* $\mathbf{L}$ *is densely embeddable in a* $\boxplus$*-orthocomplete lattice that belongs to the variety generated by* $\mathbf{L}$*.*

We also look for minimal extensions, which we call *hulls*, of $\mathbf{L}$, containing $\mathbf{L}$ densely, and being laterally complete, projectable, strongly projectable, and orthocomplete respectively. We show the existence and uniqueness of projectable and strongly projectable hulls of semilinear residuated lattices; and in the special cases of GMV algebras, we also prove the existence and uniqueness of laterally complete hulls and orthocomplete hulls.

Theorem E (Theorem 3.53)**.** *Every e-cyclic semilinear residuated lattice* $\mathbf{L}$ *has a strongly* $\boxplus$*-projectable hull and a* $\boxplus$*-projectable hull in the variety generated by* $\mathbf{L}$*; and every semilinear GMV algebra* $\mathbf{L}$ *has laterally complete hull and an* $\boxplus$*-orthocomplete hull in the variety generated by* $\mathbf{L}$*.*

Lastly, we study the residuated lattice $\mathcal{O}_{<\omega}(\mathbf{L})$, which is contained in $\mathcal{O}(\mathbf{L})$, but in general smaller. While $\mathcal{O}(\mathbf{L})$ is laterally complete, as we mentioned, $\mathcal{O}_{<\omega}(\mathbf{L})$ might fail this property. Nonetheless, $\mathbf{L}$ is also densely embeddable in $\mathcal{O}_{<\omega}(\mathbf{L})$, which is strongly projectable. We prove that actually $\mathcal{O}_{<\omega}(\mathbf{L})$ is the strongly $\boxplus$-projectable hull of $\mathbf{L}$.

Theorem F (Theorem 3.55)**.** *Let* $\mathbf{L}$ *be an e-cyclic semilinear residuated lattice. Then* $\mathcal{O}_{<\omega}(\mathbf{L})$ *is the strongly* $\boxplus$*-projectable hull of* $\mathbf{L}$*.*

There are many open questions concerning the hulls of residuated lattices. To start with, we notice that Theorem F provides a description of the $\boxplus$-projectable hull of an e-cyclic semilinear residuated lattice. We wonder if similar descriptions can be found for other hulls.

Problem 3. Find descriptions of the lateral complete hull and the $\boxplus$-projectable hull of a e-cyclic residuated lattice $\mathbf{L}$ in the manner of Theorem F, if they exists.

Also, we could ask how necessary the semilinearity condition is. Surely, all our techniques need semilinearity, but we wonder if there are other methods that do not use it.

Problem 4. Up to which point the semilinearity condition is necessary in order to find each one of the hulls?

Also, some of the results are proved only for GMV algebras, but we have not found any counterexample to the general results. For instance, Theorem 3.50 is proven only for GMV algebras, but this is because it is based in Lemma 3.47, for which the hypotheses of being a GMV algebra seems very necessary. But maybe another way can be found. Therefore, we do not know the answer to the following:

Problem 5. Given a semilinear e-cyclic residuated lattice which is laterally complete and projectable, is it strongly projectable?

Finally, another very important issue is what is the relation between the two notions of (strong) projectability, and when they coincide.

Problem 6. Characterize the largest class of e-cyclic residuated lattices in which (strong) $\boxplus$-projectability is equivalent to (strong) $\vee$-projectability.

In Chapter 4, we prove the failure of the Amalgamation Property for many varieties of residuated lattices. The main result of this chapter is the following:

Theorem G (See Theorem 4.7)**.** *Let $\mathcal{V}$ be a variety of residuated lattices satisfying the following equations:*

(1) $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge w)$,
(2) $x(y \wedge z)w \approx xyw \wedge xzw$.

If $\mathcal{V} \cap \mathcal{LG}$ fails the AP, then so does $\mathcal{V}$.

And from it we can derive the following two results:

Theorem H (See Theorem 4.8.)**.** *The varieties $\mathcal{SemRL}$ and $\mathcal{SemCanRL}$ fail the AP.*

Theorem I (Corollary 4.9)**.** *The varieties $\mathcal{LG}^-$ and $\mathcal{RepLG}^-$ fail the AP.*

There are many open problems regarding the Amalgamation Property for residuated lattices as the techniques of the Chapter 4 do not seem to be adequate to determine whether the variety $\mathcal{SemIRL}$ of semilinear integral residuated lattices and the variety $\mathcal{SemCanIRL}$ of semilinear cancellative integral residuated lattices fail the AP. Hence, we propose the next two open problems:

Problem 7. Does the variety $\mathcal{SemIRL}$ of semilinear integral residuated lattices fail the AP?

Problem 8. Does the variety $\mathcal{SemCanIRL}$ of semilinear cancellative integral residuated lattices fail the AP?

A substantially harder open problem, which is connected to the long-standing question of embedding a totally ordered group into a divisible one, is the following:

Problem 9. Let $\mathbf{A}$ be an arbitrary (not necessarily commutative) totally ordered group. Do all V-formations of the form $(\mathbb{Z}, \mathbb{Q}, \mathbf{A}, i, j)$ have an amalgam in $\mathcal{RepLG}$? Here, $\mathbb{Z}$ and $\mathbb{Q}$ denote the totally ordered groups of integers and rationals, respectively.

More generally we can ask:

Problem 10. Let $\mathbf{A}, \mathbf{B}$ be arbitrary totally ordered groups. Do all V-formations of the form $(\mathbb{Z}, \mathbf{A}, \mathbf{B}, i, j)$ have an amalgam in $\mathcal{RepLG}$? In other words, is $\mathbb{Z}$ an amalgamation base of $\mathcal{RepLG}$?

As has already been remarked, all subvarieties of $\mathcal{RL}$ that are known to satisfy the AP are commutative.

Problem 11. Is there a non-commutative variety of residuated lattices that satisfies the AP? In particular, does the variety $\mathcal{RL}$ of all residuated lattices satisfy the AP?

Three open problems that may have affirmative answers are the following:

Problem 12. Does the variety $\mathcal{CanCRL}$ of cancellative commutative residuated lattices have the AP?

Problem 13. Does the variety $\mathcal{SemCanCRL}$ of semilinear cancellative commutative residuated lattices have the AP?

Problem 14. Does the variety $\mathcal{SemCRL}$ of semilinear commutative residuated lattices have the AP?

Finally, we introduced in Chapter 5 the notion of a recognizable element of a residuated lattice as an abstraction of the notion of a recognizable language by an automaton. The main result of this chapter is the following internal characterization of recognizable elements of residuated lattices, from which Myhill's Theorem follows.

Theorem J (Theorem 5.31)**.** *Let* $\mathbf{R}$ *be a residuated lattice and* $a \in R$*. The following are equivalent:*

(i) *The element a is recognizable.*

(ii) *There exists a structural closure operator γ on $\mathbb{R}$ with finite image such that $\gamma(a) = a$.*

(iii) *The image $\{x\backslash a : x \in R\}$ of the closure operator γ_a is finite.*

(iv) *The set $\{a/x : x \in R\}$ is finite.*

We end the chapter studying two different problems, on which we make progress but do not have solution yet.

Problem 15. Find a Kleene's-like characterization of the recognizable elements of a residuated lattice.

What we know about this problem is that, whenever $\mathbf{R}$ has a top element, and only in this case, the set of recognizable elements is nonempty, contains the top element, and it is closed under (finite) meets and residuation (see Proposition 5.40). We also found that in the case $\mathbf{R} = \mathcal{P}(\mathbf{M})$, for some monoid $\mathbf{M}$, it is also closed under complementation and (finite) unions, although it may not contain all the singletons in general (see Proposition 5.42).

The second and last problem that we study in this chapter is the following: we notice that every recognizable language is recognized by a module whose poset reduct is indeed a Boolean algebra. This is not the general case for recognizable elements of residuated lattices. Therefore we propose the following problem:

Problem 16. Characterize the elements of a residuated lattice that are recognizable by a Boolean module.

In this direction we prove the following result, but show that this is not a characterization.

Theorem K (Corollary 5.61)**.** *Let $\mathbf{R}$ be a complete residuated lattice. Every recognizable localic element is recognized by a Boolean cyclic module.* □

Appendices

Appendix A

On Some Properties of Directoids

A.1 Introduction

It is superfluous to recall how important partially ordered sets, an in particular directed posets, are for the whole of Mathematics. However, unlike other equally fundamental mathematical structures, such as groups or Boolean algebras, posets and directed posets are *relational structures*, not algebras, whence they do not lend themselves to be the objects of common algebraic constructions like quotient, products, subalgebras, and the like. In fact, insofar as they exist at all for relational structures, these constructions admit of several competing variants, non of which enjoys a universal acclaim, and are generally recognized as more cumbersome and less efficient than the algebraic case. In order to enable such algebraic constructions with ordered sets, J. Ježek and R. Quackenbush – and, independently, Gardner and Parmenter [43] – introduced in [52] the notion of a *directoid*. To every directed poset $\mathbf{A} = \langle A, \leqslant \rangle$ a groupoid $\mathcal{D}(\mathbf{A})$ can be associated in such a way that for all $a, b \in A$, $a \leqslant b$ if and only if $a \sqcup b = b \sqcup a = b$. In the terminology of Ježek and Quackenbush this groupoid is called a *commutative directoid*. Directoids were investigated in detail by several authors; for a survey, see [19].

Here, we study some properties of directoids and some of their expansions by additional signature. This appendix is structured as follows. In Section A.2, after recapping some preliminary notions, we investigate involutive directoids, that correspond to directed posets with an antitank involution, and some of their notable subclasses, including complemented directoids. In Section A.3, we focus on some classes of directors where the binary operation $\sqcup$ has "join-like" properties. In Section A.4, we improve on the direct decomposition theorem for bounded involutive directoids given in [19], by providing a shorter proof; moreover, we present a compact description of central elements of complemented directoids. Finally, in Section A.5, we show that the variety of directoids, as well as its expansions mentioned above, all have the Strong Amalgamation Property.

A.2 Involutive Directoids

Recall that a partially ordered set (or poset) $\mathbf{A} = \langle A, \leqslant \rangle$ is said to be *directed* in case any two $a, b \in A$ have a common upper bound, i.e. in case the *upper corner* $U(a, b) = \{c \in A : a, b \leqslant c\}$ is nonempty. Of course, if $\mathbf{A}$ has a greatest element 1, then it is directed. An *antitone involution* on a poset $\mathbf{A}$ is a unary operation $'$ such that for any $a \in A$, $(a')' = a$, and if $a \leqslant b$ then $b' \leqslant a'$. The element $(a')'$ will be shortened to a'' hereafter. It is evident that, whenever a poset with an antitone involution has a greatest element 1, then it contains a smallest element too, namely, $1'$. In place of $1'$, we denote such an element by 0. Furthermore, observe that if $a \vee b$ exists in A, then the infimum $a' \wedge b' = (a \vee b)'$ also exists in A.

A *directoid* (*commutative directoid*, in the usage of Ježek and Quackenbush) is a groupoid $\mathbf{D} = \langle D, \sqcup \rangle$ that satisfies the following axioms:

(D1) $x \sqcup x \approx x$;
(D2) $x \sqcup y \approx y \sqcup x$;
(D3) $x \sqcup ((x \sqcup y) \sqcup z) \approx (x \sqcup y) \sqcup z$.

If $\mathbf{D} = \langle D, \sqcup \rangle$ is a directoid, the partial order relation $\leqslant$ defined for all $a, b \in D$ by:

$$a \leqslant b \quad \Leftrightarrow \quad a \sqcup b = b$$

will be called the *order induced by* $\sqcup$ on $\mathbf{D}$, or its *induced order*, while the poset $\langle D, \leqslant \rangle$ will be called the *induced poset* of $\mathbf{D}$.

Any directed poset $\mathbf{A} = (A, \leqslant)$ can be turned into a directoid as follows:

- if $a \leqslant b$, then we set $a \sqcup b = b \sqcup a = b$;
- if a and b are incomparable (denoted by $a \parallel b$), then $a \sqcup b = b \sqcup a$ is an arbitrary common upper bound of a and b.

The resulting directoid $\mathcal{D}(\mathbf{A}) = \langle A, \sqcup \rangle$ is such that its induced order coincides with the partial ordering of $\mathbf{A}$. In other words, the directoid fully retrieves the ordering of the original poset. However, it may happen that two incomparable elements $a, b \in A$ have a supremum $a \vee b$ that does not coincide with our choice of $a \sqcup b$. And this is a shortcoming under several respects. It is therefore our aim to prove that, for every directed poset $\mathbf{A} = \langle A, \leqslant \rangle$ that admit an antitone involution, we can get around this difficulty.

An *involutive* directoid is an algebra $\mathbf{D} = \langle D, \sqcup, ' \rangle$ of type $(2,1)$ such that $\langle D, \sqcup \rangle$ is a directoid and $'$ is an antitone involution on the induced poset of $\mathbf{D}$.

Proposition A.1. *The class of involutive directoids is a variety.*

Proof. We only have to prove that the quasi-identity $x \leqslant y \Rightarrow y' \leqslant x'$ can be expressed equationally. Indeed, it can be expressed by the single equation $x' \sqcup (x \sqcup y)' \approx x'$. If we assume the quasi-identity, then since $a \leqslant a \sqcup b$, we have $(a \sqcup b)' \leqslant a'$, and therefore $a' \sqcup (a \sqcup b)' = a'$. For the other implication, if the equation is valid, and $a \leqslant b$, then $a \sqcup b = b$, and therefore $a' \sqcup b' = a' \sqcup (a \sqcup b)' = a'$. That is, $b' \leqslant a'$, as required. □

Recall from [19] that two elements a, b of a directoid $\mathbf{D}$ are said to be *orthogonal* in case $a \leqslant b'$, or equivalently $b \leqslant a'$.

Theorem A.2. *Let* $\mathbf{D} = \langle D, \sqcup, ' \rangle$ *be an involutive directoid and* $\leqslant$ *its induced order. The following conditions are equivalent:*

1. *for all* $a, b \in D$*, if* a, b *are orthogonal then* $a \sqcup b = a \vee b$*;*
2. $\mathbf{D}$ *satisfies the identity:*

$$(((x \sqcup z) \sqcup (y \sqcup z)')' \sqcup (y \sqcup z)') \sqcup z' = z'. \tag{D4}$$

Proof. First, notice that item (1) is equivalent to

$$\text{if } a \leqslant b' \text{ and } a, b \leqslant c \text{ then } a \sqcup b \leqslant c. \tag{A}$$

For, if $a \leqslant b'$ and $a, b \leqslant c$, then $a \sqcup b = a \vee b \leqslant c$. The converse is obvious since, by axiom (D3), $a, b \leqslant a \sqcup b$. Moreover, the identity (D4) is clearly equivalent to

$$((x \sqcup z) \sqcup (y \sqcup z)')' \sqcup (y \sqcup z)' \leqslant z', \tag{B}$$

by the definition induced order..

Hence, to obtain our claim, it suffices to show the equivalence of (A) and (B). Assume (A). Set $a = ((x \sqcup z) \sqcup (y \sqcup z)')'$ and $b = (y \sqcup z)'$. Clearly, $(x \sqcup z) \sqcup (y \sqcup z)' \geqslant (y \sqcup z)'$, i.e. $b \leqslant a'$. Hence, $a \leqslant b'$. Also, $b = \leqslant z'$. Moreover, $a' \geqslant x \sqcup z \geqslant z$. Therefore, $a \leqslant z'$. Thus, by (A) $((x \sqcup z) \sqcup (y \sqcup z)')' \sqcup (y \sqcup z)' \leqslant z'$, which is (B). Conversely, assume (B). Let $x \leqslant y'$ and $x, y \leqslant z$. Then $y \leqslant x'$ and $x', y' \geqslant z'$, i.e. $y \sqcup x' = x'$, $x' \sqcup z' = x'$, and $y' \sqcup z' = y'$. So we obtain:

$$\begin{aligned} x \sqcup y = x'' \sqcup y &= (x' \sqcup y)' \sqcup y = ((x' \sqcup z') \sqcup y)' \sqcup y \\ &= ((x' \sqcup z') \sqcup y'')' \sqcup y = ((x' \sqcup z') \sqcup (y' \sqcup z')')' \sqcup y \\ &= ((x' \sqcup z') \sqcup (y' \sqcup z')')' \sqcup y'' = ((x' \sqcup z') \sqcup (y' \sqcup z')')' \sqcup (y' \sqcup z')' \\ &\leqslant z. \end{aligned}$$

where the last inequality is obtained by (B). □

The previous correspondence assumes a particularly interesting form when the poset in questions bounded, and the type includes two constants denoting the bounds. A case in point is given by *effect algebras*, which play a noteworthy role in quantum logic (see [31] and [34]) – in fact, they can be presented as bounded posets equipped with an antitone involution, such that the supremum $a \vee b$ exists for orthogonal elements a, b. We have that:

Corollary A.3. *Let* $\mathbf{A} = \langle A, \leqslant, ', 0, 1\rangle$ *be a bounded poset with antitone involution, and* $\mathbf{D} = \langle D, \sqcup, ', 0, 1\rangle$ *an algebra of type* $(2, 1, 0, 0)$*. Then:*

(1) If for any pair of orthogonal elements $a, b \in A$, $a \vee b$ *exists, then* $\mathcal{D}(\mathbf{A})$ *satisfies (D2)-(D4) and*

$$x \sqcup 0 \approx x. \tag{D5}$$

(2) If $\mathbf{D}$ *satisfies (D2)-(D5), then its associated order is a bounded poset with antitone involution such that for any orthogonal elements* a, b*, the supremum* $a \vee b$ *exists.*

Proof. (1) Since $\mathbf{A}$ is bounded, it follows that $\mathbf{A}$ is directed. Then, by Theorem A.2, $\mathcal{D}(\mathbf{A})$ satisfies (D2)-(D4), and obviously it also satisfies (D5).

(2) First, let us observe that $\mathbf{D}$ is a directoid, since, putting $y = z = 0$ in (D3), we get for any $a \in D$, $a \sqcup a = a \sqcup (a \sqcup 0) = a \sqcup ((a \sqcup 0) \sqcup 0) = (a \sqcup 0) \sqcup 0 = a \sqcup 0 = a$. Our claim, then, follows from Theorem A.2. □

Corollary A.3 entails that bounded involutive directoids such that $a \vee b$ exists for orthogonal elements a, b, are completely characterized by equations (D2)-(D5), and therefore form a variety of type $(2, 1, 0, 0)$.

Given an involutive directoid $\mathbf{D} = \langle D, \sqcup, '\rangle$, we define a new operation $\sqcap$ as follows:

$$x \sqcap y = (x' \sqcup y')'.$$

It is not difficult to verify (see e.g. [19]) that $\langle D, \sqcap, '\rangle$ is again an involutive directoid, whose induced order is dual to the induced order of $\mathbf{D}$. Moreover, the absorption laws

$$x \sqcap (x \sqcup y) \approx x \quad \text{and} \quad x \sqcup (x \sqcap y) \approx x, \tag{A.1}$$

are satisfied. In fact, $x \leqslant y \;\Leftrightarrow\; y' \leqslant x' \;\Leftrightarrow\; x' \sqcup y' = x' \;\Leftrightarrow\; (x' \sqcup y')' = x'' \;\Leftrightarrow\; x \sqcap y = x$. Therefore, since $x \leqslant x \sqcup y$, we have $x \sqcap (x \sqcup y) = x$. And since $x \sqcap y \leqslant x$, we also have $x \sqcup (x \sqcap y) = x$. Thus, we obtain the following theorem ([19, Theorem 7.8]).

Theorem A.4. *The variety of involutive directoids is congruence distributive, with $\frac{2}{3}$ majority term*

$$M(x,y,z) := ((x \sqcap y) \sqcup (y \sqcap z)) \sqcup (x \sqcap z).$$

Proof. We only prove that $M(x,x,z) = x$, the other conditions being just slight modifications thereof.

$$\begin{aligned} M(x,x,z) &= ((x \sqcap x) \sqcup (x \sqcap z)) \sqcup (x \sqcap z) = (x \sqcup (x \sqcap z)) \sqcup (x \sqcap z) \\ &= x \sqcup (x \sqcap z) = x. \end{aligned}$$

□

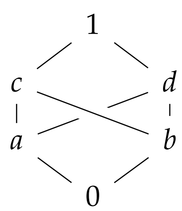
In the absence of involution,Theorem A.4 fails, because semilattices (a subvariety of directoids) satisfy no nontrivial lattice identity (see [38, Theorem 2]).

Let us call a bounded involutive directoid *complemented* in case it satisfies the equation $x \sqcup x' \approx 1$. If this directoid satisfies the equivalent conditions in Theorem A.2, we get that $x \vee x' = x \sqcup x' = 1$, because $x \leqslant x = x''$, i.e., x and x' are orthogonal. Now, all the aforementioned properties are captured by means of identities. That is, the class of complemented directoids satisfying (D4) forms a variety that includes, for example, orthomodular lattices.

A.3 Saturated and Supremal Directoids

We have seen in the previous section that there are directoids where $x \sqcup y = x \vee y$ at least for orthogonal or comparable elements. In this section we show that the classes of directoids where $x \sqcup y$ is minimal in the upper corner $U(x,y)$, or where $x \sqcup y = x \vee y$, in case $x \vee y$ exists, have special significance. To this aim we introduce the following notions: A directoid $\mathbf{D} = \langle D, \sqcup \rangle$ is called *saturated* if $x \sqcup y$ is minimal in $U(x,y)$. $\mathbf{D}$ is *supremal* if $x \sqcup y = x \vee y$ in case $\sup(x,y)$ exists.

Example A.5. Consider the following ordered set:



If we set $a \sqcup b = c$ or $a \sqcup b = d$, and for $\{x,y\} \neq \{a,b\}$ we take $x \sqcup y = x \vee y$, then it is a saturated directoid. However, upon setting $a \sqcup b = 1$, on the same ordered set, the resulting directoid is no longer saturated, since 1 is not minimal in $U(a,b)$, even though it is still trivially supremal, because $a \vee b$ does not exist.

Remark A.6. *Note that every saturated directoid is supremal. In fact, if $x \vee y$ exists, then it is minimal in $U(x,y)$, whence $x \sqcup y = x \vee y$. The previous example shows that the converse is not true.*

Theorem A.7. *A directoid $\mathbf{D} = \langle D, \sqcup \rangle$ is saturated if and only if it satisfies the quasiidentity:*

$$(x \sqcup z \approx z \approx y \sqcup z) \ \& \ (z \sqcup (x \sqcup y) \approx x \sqcup y) \Rightarrow z \approx x \sqcup y. \tag{Q}$$

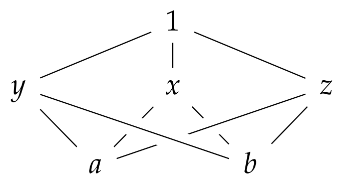
Proof. Assume $\mathbf{D}$ satisfies (Q) and $x, y \leqslant z \leqslant x \sqcup y$. Then, $x \sqcup z = z = y \sqcup z$ and $z \sqcup (x \sqcup y) = x \sqcup y$. Hence, $z = x \sqcup y$. Therefore, $x \sqcup y$ is minimal in $U(x,y)$, i.e. $\mathbf{D}$ is saturated. Conversely, if $\mathbf{D}$ is saturated, and $x \sqcup z = z = y \sqcup z$ and $z \sqcup (x \sqcup y) = x \sqcup y$ hold, then $x, y \leqslant z \leqslant x \sqcup y$. Since $z \in U(x,y)$ and $x \sqcup y$ is minimal in $U(x,y)$, then $x \sqcup y = z$. □

Observe that the quasi-identity (Q) is in fact equivalent to the condition:

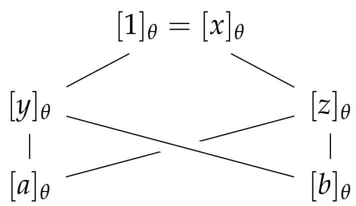
$$x, y \leqslant z \leqslant x \sqcup y \Rightarrow z \approx x \sqcup y.$$

By Theorem A.7, the class of saturated directoids is a quasivariety. The next example shows that it is not a variety, because it is not closed under quotients.

Example A.8. Let $\mathbf{D}$ be the directoid given by the following diagram:



where $a \sqcup b = x$ and $p \sqcup q = p \vee q$ for the remaining elements. Then $\mathbf{D}$ is a saturated directoid. Consider the congruence $\theta(x, 1)$. Then, we obtain the quotient



where $[a]_\theta \sqcup [b]_\theta$ is not minimal in $U\big([a]_\theta, [b]_\theta\big)$.

Note that the variety of join semilattices is a nontrivial class strictly contained in the quasivariety of saturated directoids. For the involutive directoids, we can provide a sufficient condition for saturation formulated in the form of an identity.

Theorem A.9. *Let* $\mathbf{D} = \langle D, \sqcup, ' \rangle$ *be an involutive directoid. If* $\mathbf{D}$ *satisfies*

$$\Big((x \sqcap ((x \sqcup y) \sqcap z)) \sqcup (y \sqcap ((x \sqcup y) \sqcap z))\Big) \sqcup z \approx z \tag{D6}$$

then $\mathbf{D}$ *is saturated.*

Proof. Suppose that for $a, b, c \in D$ $a \sqcup c = c = b \sqcup c$ and $c \sqcup (a \sqcup b) = a \sqcup b$. Then, $c = ((a \sqcap ((a \sqcup b) \sqcap c)) \sqcup (b \sqcap ((a \sqcup b) \sqcap c))) \sqcup c = (a \sqcup b) \sqcup c = a \sqcup b$, whence we get our conclusion. □

Observe that the variety of involutive directoids satisfying (D6) contains all the involutive lattices. We can also characterize the quasivariety of supremal directoids.

Theorem A.10. *A directoid is supremal if and only if it satisfies the quasi-equation*

$$x, y \leqslant w \quad \& \quad w \leqslant x \sqcup y \quad \& \quad x, y, z \leqslant z \quad \Rightarrow \quad w = x \sqcup y.$$

Proof. If a directoid $\mathbf{D}$ satisfies the antecedent of the quasi-equation, the w is the smallest element in $U(x, y)$. Therefore, if it is supremal then $w = x \sqcup y = x \vee y$. And if the quasi-equation itself is satisfied then it is clear that $\mathbf{D}$ is supremal. □

Let us note that the quasi-identity of Theorem A.10 can be easily expressed as a quasi-identity in the language of directoids.

A.4 Decomposition of Complemented Directoids

In [19, Theorem 7.28], the standard direct decomposition theorem for orthomodular lattices (see eg. [17, Theorem 2.7]) is generalized to the effect that an appropriate version of it is shown to hold for bounded involutive directoids. Contextually, a characterization of central elements of bounded involutive directoids is provided. The aim of this section is giving an alternative proof of this result, as well as a simplified description of central elements in case the directoid is complemented. To this aim, we put to good use the tools developed in the theory of *Church algebras* (see [85]).

The key observation motivating the introduction of Church algebras is that many algebras arising in completely different fields of Mathematics – including Heyting algebras, rings with unit, or combinatory algebras – have a term operation q satisfying the fundamental properties of the if-then-else connective: $q(1, x, y) \approx x$ and $q(0, x, y) \approx y$. As simple as they may appear, these properties are enough to yield rather strong results. This motivates the next definitions.

An algebra $\mathbf{A}$ of type ν is a *Church algebra* if there are term-definable elements $0^{\mathbf{A}}, 1^{\mathbf{A}} \in A$ and a term operation $q^{\mathbf{A}}$ such that for all $a, b \; nA$, $q^{\mathbf{A}}(1^{\mathbf{A}}, a, b) = a$ and $q^{\mathbf{A}}(0^{\mathbf{A}}, a, b) = b$. A variety $\mathcal{V}$ of type ν is a Church variety if every member of $\mathcal{V}$ is a Church algebra with respect to the same term $q(x, y, z)$ and the same constants $0, 1$.

Expanding on an idea due to Vaggione (see [91]), we also say that an element e of a Church algebra $\mathbf{A}$ is *central* if the pair $(\theta(e, 0), \theta(e, 1))$ is a pair of complementary factor congruences on $\mathbf{A}$. By $\mathrm{Ce}(\mathbf{A})$ we denote the *center* of $\mathbf{A}$, i.e. the set of central elements of the algebra $\mathbf{A}$.

By defining $x \wedge y = q(x, y, 0)$, $x \vee y = q(x, 1, y)$, and $x^* = q(x, 0, 1)$, we get:

Theorem A.11. [85] *Let* $\mathbf{A}$ *be a Church algebra. Then* $\mathbf{Ce}(\mathbf{A}) = \langle \mathrm{Ce}(\mathbf{A}), \wedge, \vee, {}^*, 0, 1 \rangle$ *is a Boolean algebra with is isomorphic to the Boolean algebra of factor congruences of* $\mathbf{A}$.

Hereafter, it will be clear from the context when the symbols $\wedge, \vee$ will denote the previously defined operations on Church algebras instead of defining lattice meet and join, respectively.

If $\mathbf{A}$ is a Church algebra of type ν and $e \in A$ is a central element, then we define $\mathbf{A}_e = \langle A_e; g_e \rangle_{g \in \nu}$ to be the ν-algebra defined as follows:

$$A_e = \{e \wedge b : b \in A\}; \quad g_e(e \wedge \overline{b}) = e \wedge g(e \wedge \overline{b}).$$

By [65, Theorem 4], we have that:

Theorem A.12. *Let* $\mathbf{A}$ *be a Church algebra of type* ν *and* e *a central element. Then we have:*

1. *For every* n*-ary* $g \in \nu$ *and every sequence of elements* $\overline{b} \in A^n$, $e \wedge g(\overline{b}) = e \wedge g(e \wedge \overline{b})$, *so that the function* $h : A \to A_e$, *defined by* $h(b) = e \wedge b$, *is a homomorphim from* $\mathbf{A}$ *onto* $\mathbf{A}_e$.
2. $\mathbf{A}_e$ *is isomorphic to* $\mathbf{A}/\theta(e, 1)$. *It follows that* $\mathbf{A} \cong \mathbf{A}_e \times \mathbf{A}_{e'}$, *for every central element* e, *as in the Boolean case.*

The if-then-else term that makes orthomodular lattices into Church algebras works, more generally, for bounded involutive directoids:

Proposition A.13. *Bounded involutive directoids form a Church algebra variety with respect to the term* $q(x, y, z) = (x \sqcup z) \sqcap (x' \sqcup y)$.

Proof. If $\mathbf{A}$ is a bounded involutive directoid, and $a, b \in A$, then $q^{\mathbf{A}}(1, a, b) = (1 \sqcup b) \sqcap (1' \sqcup a) = 1 \sqcap (0 \sqcup a) = 1 \sqcap a = a$. And also, $q^{\mathbf{A}}(0, a, b) = (0 \sqcup b) \sqcap (0' \sqcup a) = b \sqcap (1 \sqcup a) = b \sqcap 1 = b$. □

In [19, Chapter 7], central elemetns (in Vaggione's sense) of a bounded involutive directoid $\mathbf{D}$ are described as the members of $C(\mathbf{D}) \cap Is(D)$, namely, those elements e that satisfy the following conditions for all $a, b \in D$:

$$a = (e \sqcap a) \sqcup (e' \sqcap a) \qquad (a \sqcup b) \sqcap e = (a \sqcap e) \sqcup (b \sqcap e)$$
$$(a \sqcap b) \sqcap e = (a \sqcap e) \sqcap (b \sqcap e) \qquad (a \sqcup b) \sqcap e' = (a \sqcap e') \sqcup (b \sqcap e')$$
$$(a \sqcap b) \sqcap e' = (a \sqcap e') \sqcap (b \sqcap e')$$

However, according to [65, Proposition 3.6], the central elements of a Church algebra can also be characterized in a completely general way, as follows.

Proposition A.14 ([65]). *If $\mathbf{A}$ is a Church algebra of type ν and $e \in A$, the following conditions are equivalent:*

(1) e is central;

(4) for all $a, b, \overline{a}, \overline{b} \in A$:

(a) $q(e, a, a) = a$,

(b) $q(e, q(e, a, b), c) = q(e, a, c) = q(e, a, q(e, b, c))$,

(c) $q(e, f(\overline{a}), f(\overline{b})) = f(q(e, a_1, b_1), \ldots, q(e, a_n, b_n))$, for every $f \in \nu$,

(d) $q(e, 1, 0) = e$.

If $\mathbf{A}$ is a bounded involutive directoid, condition (a) says $a = (e \sqcup a) \sqcap (e' \sqcup a)$, for every $a \in A$, or equivalently $a = (e \sqcap a) \sqcup (e' \sqcap a)$, for every $a \in A$.

The first equality of condition (b) says $(e \sqcup c) \sqcap \big(e' \sqcup ((e \sqcup b) \sqcap (e' \sqcup a))\big) = (e \sqcup c) \sqcap (e' \sqcup a)$, for every $a, b, c \in A$. Taking $c = 1$, it is easy to see that this is equivalent to $e' \sqcup ((e \sqcup b) \sqcap (e' \sqcup a)) = e' \sqcup a$, for every $a, b \in A$. The second equality is analogous, and boils down to $e \sqcup ((e' \sqcup b) \sqcap (e \sqcup c)) = e \sqcup c$, for every $b, c \in A$.

Condition (c) is $q(e, 1, 1) = 1$ and $q(e, 0, 0) = 0$ for the constants. These equalities are trivially satisfied for every element $e \in A$. If f is $'$ then $(e \sqcup b') \sqcap (e' \sqcup a') = ((e \sqcup b) \sqcap (e' \sqcup a))'$, that is to say, $(e \sqcup b') \sqcap (e' \sqcup a') = (e' \sqcap b') \sqcup (e \sqcap a')$, for every $a, b \in A$. But this is equivalent to $(e \sqcup b) \sqcap (e' \sqcup a) = (e' \sqcap b) \sqcup (e \sqcap a)$, for every $a, b \in A$. If f is $\sqcup$ then we have $(e \sqcup (b_1 \sqcup b_2)) \sqcap (e' \sqcup (a_1 \sqcup a_2)) = ((e \sqcup b_1) \sqcap (e' \sqcup a_1)) \sqcup ((e \sqcup b_2) \sqcap (e' \sqcup a_2))$.

Finally, condition (d) is trivial, since for every element $q(e, 1, 0) = e$ is always true for every element $e \in A$.

We will use one or the other of these two characterizations of central elements, according to convenience.

WE now focus for a while on *complemented* directoids, for which we show that the later set of conditions can be considerably streamlined. For a start, we need to prove the following lemmas.

Lemma A.15. *If* **A** *is a bounded involutive directoid, then it satisfies:*

(i) $x \sqcup y \approx x \sqcup (x \sqcup y)$,
(ii) $x \approx x \sqcup (x \sqcup (x' \sqcup y))'$.

If it is complemented, it also satisfies:

(iii) $(x \sqcup y) \sqcup y' \approx 1$,

Proof. (i) This is true, since $x \leqslant x \sqcup y$.

(ii) Since $x' \leqslant x' \sqcup y \leqslant x \sqcup (x' \sqcup y)$, we have $(x \sqcup (x' \sqcup y))' \leqslant x$, and therefore $x = x \sqcup (x \sqcup (x' \sqcup y))'$.

(iii) Substituting x by $(x \sqcup y) \sqcup y'$ and y by y' in the previous item, we obtain:

$$\begin{aligned}(x \sqcup y) \sqcup y' &= ((x \sqcup y) \sqcup y') \sqcup (((x \sqcup y) \sqcup y') \sqcup (((x \sqcup y) \sqcup y')' \sqcup y'))' \\ &= ((x \sqcup y) \sqcup y') \sqcup (((x \sqcup y) \sqcup y') \sqcup y')' && \text{(by (ii))} \\ &= ((x \sqcup y) \sqcup y') \sqcup ((x \sqcup y) \sqcup y')' && \text{(by (i))} \\ &= 1. && \square\end{aligned}$$

Now, consider the equations:

$$a = (e \sqcap a) \sqcup (e' \sqcap a), \tag{C1}$$

$$(e \sqcup (b_1 \sqcup b_2)) \sqcap (e' \sqcup (a_1 \sqcup a_2)) = ((e \sqcup b_1) \sqcap (e' \sqcup a_1)) \sqcup ((e \sqcup b_2) \sqcap (e' \sqcup a_2)). \tag{C2}$$

Lemma A.16. *If* **A** *is a complemented directoid and $e \in A$ is satisfy equations (C1) and (C2) for every $a, b, a_1, a_2, b_1, b_2 \in A$, then for every $a, b \in A$,*

1. $e \sqcup (a \sqcup b) = (e \sqcup a) \sqcup (e \sqcup b)$,
2. $e \sqcap (a \sqcap b) = (e \sqcap a) \sqcap (e \sqcap b)$,
3. (a) $e \sqcup a = e \sqcup (e' \sqcap a)$,
   (b) $e \sqcap a = e \sqcap (e' \sqcup a)$,
4. $e \sqcup (a \sqcup (e \sqcap b)) = e \sqcup a$,
5. $e \sqcup (a \sqcup b) = (e \sqcup a) \sqcup b$,
6. $e \sqcap (a \sqcup b) = (e \sqcap a) \sqcup (e \sqcap b)$,
7. $(e \sqcup a)' \sqcup b = (e \sqcup (a' \sqcup b)) \sqcap (e' \sqcup b)$,
8. $(e \sqcup b) \sqcap (e' \sqcup a) = (e' \sqcap b) \sqcup (e \sqcap a)$.

Proof. Taking $b_1 = a$, $b_2 = b$, and $a_1 = 1 = a_2$ in (C2), we obtain:

$$\begin{aligned}e \sqcup (a \sqcup b) &= (e \sqcup (a \sqcup b)) \sqcap (e' \sqcup (1 \sqcup 1)) \\ &= ((e \sqcup a) \sqcap (e' \sqcup 1)) \sqcup ((e \sqcup b) \sqcap (e' \sqcup 1)) \\ &= (e \sqcup a) \sqcup (e \sqcup b).\end{aligned}$$

For (2), we only have to use the De Morgan laws and the fact that e satisfies (C1) and (C2) if and only if e' also satisfies them. In order to prove (3a), observe that:

$$\begin{aligned} e \sqcup a &= e \sqcup ((e \sqcap a) \sqcup (e' \sqcap a)) = (e \sqcup (e \sqcap a)) \sqcup (e \sqcup (e' \sqcap a)) \\ &= e \sqcup (e \sqcup (e' \sqcap a)) = e \sqcup (e' \sqcap a). \end{aligned}$$

(3b) is proved dually. (4) is just:

$$e \sqcup (a \sqcup (e \sqcap b)) = (e \sqcup a) \sqcup (e \sqcup (e \sqcap b)) = (e \sqcup a) \sqcup e = e \sqcup a.$$

For (5), we show that

$$\begin{aligned} (e \sqcup a) \sqcup b &= e \sqcup (b \sqcup (e \sqcup a)) = (e \sqcup b) \sqcup (e \sqcup (e \sqcup a)) = (e \sqcup b) \sqcup (e \sqcup a) \\ &= e \sqcup (a \sqcup b). \end{aligned}$$

As regards (6), it follows from (C1) and (C2) that for any $a, b \in A$:

$$\begin{aligned} e \sqcap (a \sqcup b) &= (e' \sqcap 0) \sqcup (e \sqcap (a \sqcup b)) = (e \sqcup 0) \sqcap (e' \sqcup (a \sqcup b)) \\ &= (e \sqcup (0 \sqcup 0)) \sqcap (e' \sqcup (a \sqcup b)) = ((e \sqcup 0) \sqcap (e' \sqcup a)) \sqcup ((e \sqcup 0) \sqcap (e' \sqcup b)) \\ &= (e \sqcap a) \sqcup (e \sqcap b), \end{aligned}$$

where the last equality uses (3b). For (7), notice that:

$$\begin{aligned} (e \sqcup a)' \sqcup b &= (e \sqcup ((e \sqcup a)' \sqcup b)) \sqcap (e' \sqcup ((e \sqcup a)' \sqcup b)) \\ &= ((e \sqcup (e \sqcup a)') \sqcup b)) \sqcap (e' \sqcup ((e \sqcup a)' \sqcup b)) \\ &= ((e \sqcup a') \sqcup b)) \sqcap (e' \sqcup ((e' \sqcap a') \sqcup b)) \\ &= (e \sqcup (a' \sqcup b)) \sqcap ((e' \sqcup (e' \sqcap a')) \sqcup b) \\ &= (e \sqcup (a' \sqcup b)) \sqcap (e' \sqcup b). \end{aligned}$$

And finally for (8), we substitute in (7) a by a' and b by $e \sqcap b$ in order to obtain:

$$(e \sqcup a')' \sqcup (e \sqcap b) = (e \sqcup (a \sqcup (e \sqcap b))) \sqcap (e' \sqcup (e \sqcap b)),$$

whence using (3) and (4), we obtain the result. □

We are ready to obtain our characterization.

Proposition A.17. *An element e of a complemented directoid* $\mathbf{A}$ *is central if and only if it satisfies (C1) and (C2) for every $a, b, a_1, a_2, b_1, b_2 \in A$:*

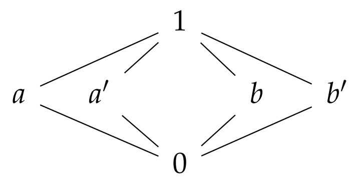
Proof. In the light of the previous lemmas and of the general description of central elements of Church algebras, discussed above, it suffices to prove that an element $e \in A$ satisfying (C2) also satisfies $e \sqcup a = e \sqcup ((e \sqcup a) \sqcap (e' \sqcup b))$ and $e' \sqcup a = e' \sqcup ((e' \sqcup a) \sqcap (e \sqcup b))$. Note that if e satisfies (C2) for every $a, b, a_1, a_2, b_1, b_2\ nA$, then the same equation is true replacing e by e'. Therefore it is enough to prove that (C2) implies $e \sqcup a = e \sqcup ((e \sqcup a) \sqcap (e' \sqcup b))$ for every $a, b \in A$.

Making $b_1 = a$, $b_2 = e$, $a_1 = b$, $a_2 = e$ in (C2), we have

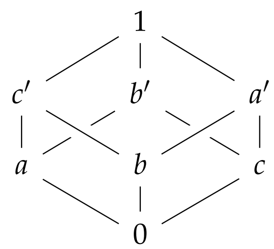
$$(e \sqcup (a \sqcup e)) \sqcap (e' \sqcup (b \sqcup e)) = ((e \sqcup a) \sqcap (e' \sqcup b)) \sqcup ((e \sqcup e) \sqcap (e' \sqcup e)).$$

Using Lemma A.15, we have that $e \sqcup (a \sqcup e) = e \sqcup a$ and that $e' \sqcup (b \sqcup e) = 1$, and therefore $(e \sqcup (a \sqcup e)) \sqcap (e' \sqcup (b \sqcup e)) = (e \sqcup a) \sqcap 1 = e \sqcup a$. For the right-hand of the equation we have, $((e \sqcup a) \sqcap (e' \sqcup b)) \sqcup ((e \sqcup e) \sqcap (e' \sqcup e)) = ((e \sqcup a) \sqcap (e' \sqcup b)) \sqcup (e \sqcap 1) = ((e \sqcup a) \sqcap (e' \sqcup b)) \sqcup e$, as we wanted to prove. □

Example A.18. Conditions (C1) and (C2) are independent. In fact, in the complemented directoid whose Hasse diagram is hereafter reproduced:



every element satisfies (C2), but only 0 and 1 satisfy (C1). On the other hand, every element of the complemented directoid whose Hasse diagram is hereafter reproduced satisfies (C1) bot only 0 and 1 satisfy (C2).



We actually can give a more informative version of Theorem A.11 above:

Proposition A.19. *If* $\mathbf{A}$ *is a bounded involutive directoid and* $\mathrm{Ce}(A)$ *is the set of the central elements of* $\mathbf{A}$*, then* $\mathbf{Ce}(\mathbf{A}) = \langle \mathrm{Ce}(\mathbf{A}), \sqcap, \sqcup, ', 0, 1 \rangle$ *is a Boolean algebra.*

Proof. In virtue of Theorem A.11, $\langle \mathrm{Ce}(A), \wedge, \vee, {}^*, 0, 1 \rangle$ is a Boolean algebra, where $\wedge$, $\vee$ and * are defined as follows:

$$x \wedge y = q(x, y, 0), \qquad x \vee y = q(x, 1, y), \qquad x^* = q(x, 0, 1).$$

The only thing we need to prove is that $\wedge$, $\vee$, and * coincide with $\sqcup$, $\sqcap$, and $'$, respectively. We note that:

$$x \vee y = q(x,1,y) = (x \sqcup y) \sqcap (x' \sqcup 1) = (x \sqcup y) \sqcap 1 = x \sqcup y,$$
$$x^* = q(x,0,1) = (x \sqcup 1) \sqcap (x' \sqcup 0) = 1 \sqcap x' = x'.$$

Therefore, $\vee$ and * coincide with $\sqcup$ and $'$ respectively. And this implies that $\wedge$ and $\sqcap$ also must coincide, because: $x \wedge y = (x^* \vee y^*)^* = (x' \sqcup y')' = x \sqcap y$. □

Remark A.20. *It follows from the previous proposition (and actually, also directly from Proposition A.17) that if* $\mathbf{A}$ *is a complemented directoid and* e *is a central element, then* e' *is also central.*

Now, if $\mathbf{A}$ is a bounded involutive directoid and e is a central element of $\mathbf{A}$, let

$$[0,e] = \langle \{a : a \leqslant e\}, \sqcup, {}^e, 0, e \rangle \quad \text{and} \quad a^e = e \sqcap a'.$$

In the following theorem, we freely avail ourselves of the characterization of central elements in bounded involutive directoids given at the beginning of the section.

Theorem A.21. *Let* $\mathbf{A} = \langle A, \sqcup, ', 0, 1 \rangle$ *be a bounded involutive directoid, and* $e \in A$ *a central element. Then*

$$\mathbf{A} \cong [0,e] \times [0,e'].$$

is a complemented directoid.

Proof. By Theorem A.12 and Proposition A.13, upon observing that for all $a \leqslant e$ we have that $e \wedge a = e \sqcap a$, all we have to prove is the following:

(1) $A_e = \{a : a \leqslant e\}$,
(2) for $a, b \leqslant e$, $a \sqcup b = e \wedge (a \sqcup b)$,
(3) for $a \leqslant e$, $a^e = e \wedge a'$.

For (1), let $a \sqcup e = e$. Then $a = a \sqcap (a \sqcup e) = a \sqcap e = e \wedge a$, whence $a \in A_e$. Conversely, if $a \in A_e$ then for some b we have that $a = e \sqcap (e' \sqcup b)$, and so $e \sqcup a = e \sqcup (e \sqcap (e' \sqcup b)) = e$.

Concerning (2), just notice that

$$\begin{aligned} a \wedge (a \sqcup b) &= e \sqcap (e' \sqcup (a \sqcup b)) = (e \sqcap e') \sqcup (e \sqcap (a \sqcup b)) \\ &= e \sqcap (a \sqcup b) = (e \sqcap a) \sqcup (e \sqcap b) = a \sqcup b \end{aligned}$$

And finally for (3), $e \wedge a' = e \sqcap (e' \sqcup a') = e \sqcap a'$. □

Proposition A.22. *Let* $\mathbf{A}$ *be a complemented directoid,* $e \in \mathrm{Ce}(\mathbf{A})$ *and* $c \in A_e$. *Then,*

$$c \in \mathrm{Ce}(\mathbf{A}) \quad \Leftrightarrow \quad c \in \mathrm{Ce}(\mathbf{A}_e).$$

Moreover, if $c \in \mathrm{Ce}(\mathbf{A}_e)$*, then* $(\mathbf{A}_e)_c = \mathbf{A}_c$.

Proof. ($\Rightarrow$) It is an immediate consequence of the fact that $h : \mathbf{A} \to \mathbf{A}_e$ in Theorem A.12 is an onto homomorphism such that for every $a \in A_e$, $h(a) = a$, and central elements are characterized by equations, in virtue of Proposition A.17.

($\Leftarrow$) Since central elements are characterized by equations, if c_i is a central element of a complemented directoid $\mathbf{A}_i$, for $i = 1, 2$, then $(c_1, c_2) \in \mathrm{Ce}(\mathbf{A}_1 \times \mathbf{A}_2)$. Therefore, if $c \in A_e$, the image of c by the isomorphism of Theorem A.12 is $(c, 0)$. The element 0 is always central, and c is central by hypothesis. Hence $(c, 0)$ is central in $\mathbf{A}_e \times \mathbf{A}_{e'}$. But, this implies that $c \in \mathrm{Ce}(\mathbf{A})$, because $\mathbf{A} \cong \mathbf{A}_e \times \mathbf{A}_{e'}$.

Finally, if we have $c \in \mathrm{Ce}(\mathbf{A}_e)$, we have already proved that $c \in \mathrm{Ce}(\mathbf{A})$, and the only thing we have to check is the definition of the involution ^{c} does not depend of whether we are defining it in terms of ${}'$ or of ^{e}. That is to say, we have to prove that for every $a \leqslant c$, $a' \sqcap c = a^e \sqcap c$. Indeed, $a^e \sqcap c = (a' \sqcap e) \sqcap c = (a' \sqcap c) \sqcap (e \sqcap c) = (a' \sqcap c) \sqcap c = a' \sqcap c$, where we have used Lemma A.16, the fact that $c \leqslant e$, and the dual of Lemma A.15.(i). $\square$

As we have seen, $\mathbf{Ce}(\mathbf{A})$ is a Boolean algebra, and we can consider the set of its atomic elements, which we denote by $\mathrm{At}(\mathbf{A})$. Note that an atomic element of $\mathbf{Ce}(\mathbf{A})$ needs not to be an atomic element of $\mathbf{A}$.

Lemma A.23. *If* $\mathbf{A}$ *is a complemented directoid and* e *is an atomic central element of* $\mathbf{A}$*, then* $\mathrm{At}(\mathbf{A}_{e'}) = \mathrm{At}(\mathbf{A}) - \{e\}$.

Proof. ($\supseteq$) Since e is an atom in the Boolean algebra $\mathbf{Ce}(\mathbf{A})$, for any other atomic central element c of $\mathbf{A}$, $e \sqcap c = 0$, and therefore $e' \sqcup c' = 1$. Hence, $c = c \sqcap 1 = c \sqcap (e' \sqcup c') = (c \sqcap e') \sqcup (c \sqcap c') = (c \sqcap e') \sqcup 0 = c \sqcap e'$, which shows that $c \leqslant e'$. Thus, by Proposition A.22, $c \in \mathrm{Ce}(\mathbf{A}_{e'})$. Moreover, if d is a central element of $\mathbf{A}_{e'}$ such that $d < c$, then d is a central element of $\mathbf{A}$, and since we are assuming that c is atomic central of $\mathbf{A}$, then $d = 0$. Which shows that c is also atomic in $\mathbf{A}_{e'}$.

($\subseteq$) If $c \in \mathrm{At}(\mathbf{A}_{e'})$, then in particular, by Proposition A.22, $c \in \mathrm{Ce}(\mathbf{A})$. If d is a central element of $\mathbf{A}$ such that $d < c$, then we have $d \leqslant e'$, because $c \in A_{e'}$, and therefore $d \in \mathrm{Ce}(\mathbf{A}_{e'})$, again by Proposition A.22. Since by hypothesis c is atomic central in $\mathbf{A}_{e'}$, then $d = 0$. Which shows that c is atomic central in $\mathbf{A}$. Finally, $c \leqslant e'$, and therefore $c \neq e$. Otherwise, we would have $e \leqslant e'$, and hence $e = e \sqcap e' = 0$, which is impossible because e is atomic central. $\square$

Theorem A.24. *If* $\mathbf{A}$ *is a complemented directoid such that* $\mathbf{Ce}(\mathbf{A})$ *is an atomic Boolean algebra with a finitely many atomic elements, then*

$$\mathbf{A} = \prod_{e \in \mathrm{At}(\mathbf{A})} \mathbf{A}_e$$

is a decomposition of $\mathbf{A}$ *as a product of directly indecomposable algebras.*

Proof. In order to proceed with the proof of the theorem, we will use induction on the number of elements of $\mathrm{At}(\mathbf{A})$. If 1 is the only atomic central element of $\mathbf{A}$, then $\mathbf{A}$ is directly indecomposable, and the result follows because $\mathbf{A}_1 = \mathbf{A}$. If there is an atomic central element $e \neq 1$, then $\mathbf{A} = \mathbf{A}_e \times \mathbf{A}_{e'}$ in virtue of Theorem A.21. Since e is an atom, then $\mathrm{Ce}(\mathbf{A}_e) = \{0, e\}$, because if $\mathbf{A}_e$ had another central element, say c, then c would be a central element of $\mathbf{A}$ in virtue of Proposition A.22, and such that $0 < c < e$, contradicting the fact that e is an atom. Therefore, $\mathbf{A}_e$ is directly indecomposable. Now, $\mathrm{At}(\mathbf{A}_{e'}) = \mathrm{At}(\mathbf{A}) - \{e\}$, by Lemma A.23, and by the induction hypothesis, $\mathbf{A}_{e'} = \prod_{c \in \mathrm{At}(\mathbf{A}_{e'})} \mathbf{A}_c$, whence the result readily follows. □

A.5 Strong Amalgamation Property

Recall that a *V-formation* is a tuple $(\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2, i, j)$ such that $\mathbf{A}$, $\mathbf{B}_1$, and $\mathbf{B}_2$ are similar algebras, and $i : \mathbf{A} \to \mathbf{B}_1$ and $j : \mathbf{A}\mathbf{B}_2$ are embeddings. A class $\mathcal{K}$ of similar algebras is said to have the *Amalgamation Property* (AP) if for every V-formation with $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$ (and $A \neq \emptyset$) there exists an algebra $\mathbf{D} \in \mathcal{K}$ and embeddings $h : \mathbf{B}_1 \to \mathbf{D}$ and $k : \mathbf{B}_2 \to \mathbf{D}$ such that $k \circ j = h \circ i$.

$$\begin{tikzcd}
 & \mathbf{B}_1 \arrow[dr, hook, dashed, "h"] & \\
\mathbf{A} \arrow[ur, hook, "i"] \arrow[dr, hook, "j"'] & & \mathbf{D} \\
 & \mathbf{B}_2 \arrow[ur, hook, dashed, "k"'] &
\end{tikzcd} \tag{A.2}$$

In such event, we also say that h and k *amalgamate* the V-formation. $\mathcal{K}$ is said to have the *Strong Amalgamation Property* if moreover $k(B_2) \cap h(B_1) = kj(A) = hi(A)$.

Theorem A.25. *The variety of directoids has the Strong Amalgamation Property.*

Proof. Let us suppose that we have a V-formation like the solid part of Diagram (A.2), and without loss of generality, we will assume that $B_1 \cap B_2 = A$. We are going to give an explicit construction of the amalgam of this V-formation. Let us consider $D =$

$B_1 \cup B_2 \cup \{1\}$, where 1 is a new element. We proceed to define a partial order in D as follows: $x \leqslant 1$, for all $x \in D$, and if $x \in B_i$ and $y \in B_j$:

$$x \leqslant y \quad \Leftrightarrow \quad \begin{cases} i = j \text{ and } x \leqslant^{\mathbf{B}_i} y, \\ or \\ i \neq j,\ x, y \notin A, \text{ and there is } b \in A, \text{ such that } x \leqslant^{\mathbf{B}_i} b \leqslant^{\mathbf{B}_j} y. \end{cases}$$

We show in what follows that $\leqslant$ is a partial ordering of D. First, note that if $x \leqslant y$ and $x, y \in B_i$, then $x \leqslant^{\mathbf{B}_i} y$. Now, $\leqslant$ is obviously reflexive. In order to see that it is antisymmetric, let's suppose that $x \in B_i$, $y \in B_j$, and $x \leqslant y$ and $y \leqslant x$. We will distinguish three different cases:

(1) $i = j$. Then, $x \leqslant^{\mathbf{B}_i} y$ and $y \leqslant^{\mathbf{B}_i} x$, and by the antisymmetry of $\leqslant^{\mathbf{B}_i}$, $x = y$.
(2) $i \neq j$, $x, y \notin A$. Then there are $b_1, b_2 \in A$ such that $x \leqslant^{\mathbf{B}_i} b_1 \leqslant^{\mathbf{B}_j} y$ and $y \leqslant^{\mathbf{B}_j} b_2 \leqslant^{\mathbf{B}_i} x$. In that case, $b_2 \leqslant^{\mathbf{B}_i} x \leqslant^{\mathbf{B}_i} b_1$ and $b_1 \leqslant^{\mathbf{B}_j} y \leqslant^{\mathbf{B}_j} b_2$, whence $b_2 \leqslant^{\mathbf{A}} b_1$ and $b_1 \leqslant^{\mathbf{A}} b_2$, and therefore $b_1 = b_2$. This would imply that $x = b_1 = b_2 = y$, and actually that $x, y \in A$, which is a contradiction. So this case is impossible.
(3) The only remaining case is $x \leqslant 1$ and $1 \leqslant x$. But, $1 \leqslant x$ implies by definition that $x = 1$.

In order to prove the transitivity of $\leqslant$, let's suppose that $x, y, z \in D$ and $x \leqslant y$ and $y \leqslant z$. Obviously, if $z = 1$, then $x \leqslant z$ and there is nothing to prove. We assume then that $z \neq 1$, which implies that $x, y \neq 1$ as well, and distinguish three cases:

(1) If $x, y, z \in B_i$ for some $i = 1, 2$, then $x \leqslant^{\mathbf{B}_i} y \leqslant^{\mathbf{B}_i} z$, and then obviously $x \leqslant^{\mathbf{B}_i} z$, which implies $x \leqslant z$.
(2) $x \in B_i$, $z \in B_j$, $x, z \notin A$. We have different subcases depending of the position of y. If $y \in B_i$ and $y \notin A$, then there exists $b \in A$ such that $y \leqslant^{\mathbf{B}_i} b \leqslant^{\mathbf{B}_j} z$, and therefore $x \leqslant^{\mathbf{B}_i} b \leqslant^{\mathbf{B}_j} z$, which by definition implies $x \leqslant z$. If $y \in A$, then we have $x \leqslant^{\mathbf{B}_i} y \leqslant^{\mathbf{B}_j} z$, which again by definition implies $x \leqslant z$. If $y \in B_j$ and $y \notin A$, then there exists $b \in A$ such that $x \leqslant^{\mathbf{B}_i} b \leqslant^{\mathbf{B}_j} y$ and therefore $x \leqslant^{\mathbf{B}_i} b \leqslant^{\mathbf{B}_j} z$, and hence $x \leqslant z$.
(3) $x, z \in B_i$, $y \in B_j$, and $y \notin A$. If $x \in A$ and $z \notin A$, then there is $b \in A$ such that $y \leqslant^{\mathbf{B}_j} b \leqslant^{\mathbf{B}_i} z$. But then, $x \leqslant^{\mathbf{B}_j} y \leqslant^{\mathbf{B}_j} b$, which implies $x \leqslant^{\mathbf{A}} b$, and therefore $x \leqslant^{\mathbf{B}_i} b \leqslant^{\mathbf{B}_i} z$, and hence $x \leqslant z$. If $x \notin A$ and $z \in A$, then the prove is analog. If $x, z \notin A$, then there are $b_1, b_2 \in A$ such that $x \leqslant^{\mathbf{B}_i} b_1 \leqslant^{\mathbf{B}_j} y \leqslant^{\mathbf{B}_j} b_2 \leqslant^{\mathbf{B}_i} z$. Hence, $b_1 \leqslant^{\mathbf{B}_j} b_2$, which is the same as $b_1 \leqslant^{\mathbf{A}} b_2$, and then $b_1 \leqslant^{\mathbf{B}_i} b_2$. Thus, $x \leqslant^{\mathbf{B}_i} b_1 \leqslant^{\mathbf{B}_i} b_2 \leqslant^{\mathbf{B}_i} z$, and by the transitivity of $\leqslant^{\mathbf{B}_i}$, we obtain $x \leqslant z$.

Thus, we have turn D into a poset. We can readily see that it is directed, because it is bounded above. Then we take the directoid $\mathbf{D} = \mathcal{D}(D, \leqslant)$, which is defined as follows:

$$x \sqcup y = y \sqcup x = \begin{cases} y & \text{if } x \leqslant y, \\ x \sqcup^{\mathbf{B}_i} y & \text{if } x, y \in B_i \text{ and } x \parallel y, \\ 1 & \text{otherwise.} \end{cases}$$

This operation is well defined, because if $x, y \in B_1 \cap B_2 = A$, then $x \sqcup^{\mathbf{B}_1} y = x \sqcup^{\mathbf{A}} y = x \sqcup^{\mathbf{B}_2} y$. Now, it is not difficult to prove that $\mathbf{B}_i$ is a subalgebra of $\mathbf{D}$. Indeed, if $x, y \in B_i$, then it could be that $x \leqslant^{\mathbf{B}_i} y$, $y \leqslant^{\mathbf{B}_i} x$, or $x \parallel y$. In any of those three cases $x \sqcup y = x \sqcup^{\mathbf{B}_i} y$. And as we saw in the first section, $\mathbf{D}$ retains the information relative to the ordering of $(D, \leqslant)$, which can be recovered by stipulating $x \leqslant y$ if and only if $x \sqcup y = y$. By construction, the intersection of $\mathbf{B}_1$ and $\mathbf{B}_2$ as subalgebras of $\mathbf{D}$ is the algebra $\mathbf{A}$. Therefore, we have proven that $\mathbf{D}$ is a strong amalgam of $\mathbf{B}_1$ and $\mathbf{B}_2$. □

Remark A.26. *Note that we needed to add a new element* 1 *to* $B_1 \cup B_2$ *just to assure that* $U(x, y)$ *is nonempty, for every* $x, y \in D$, *in particular when* $x \in B_i$, $y \in B_j$ *and* $x, y \notin A$. *If* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *are algebras with a common subalgebra* $\mathbf{A}$, *in a language with the constant* 1, *which is interpreted as the top element on each of these algebras, then there is no need to add a new element 1 to* $B_1 \cup B_2$. *The construction of the amalgam is otherwise entirely analog.*

Theorem A.27. *The varieties of bounded directoids, involutive directoids, bounded involutive directoids, and complemented directoids have the Strong Amalgamation Property.*

Proof. Essentially, the amalgam of a V-formation in each one of those varieties is found as the amalgam of the $\sqcup$-reducts, although some technical but innocuous modifications are needed in some cases. If we are in a variety of bounded directoids, we have to identify in the amalgam the new top element 1 with the top element of $\mathbf{A}$ (which would be also the top element of $\mathbf{B}_1$ and $\mathbf{B}_2$). Otherwise said, we do not need to add a new element 1. If we are in a variety involutive lattices, then we not only have to add the new element 1 to $B_1 \cup B_2$, but also another new element 0, which should be defined to be the infimum. And the involution * of $\mathbf{D}$ should be defined to be $1^* = 0$, $0^* = 1$, and $x^* = x'^{\mathbf{B}_i}$ if $x \in B_i$. □

As stablished in [59], the epimorphisms of the varieties enjoying the Strong Amalgamation Property can be characterized as the onto homormophisms. Therefore, we have our final result.

Corollary A.28. *In each one of the varieties of directoids, bounded directoids, involutive directoids, bounded involutive, and complemented directoids, the epimorphisms are onto.*

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