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NEW RESULTS IN EXTENDED THERMODYNAMICS

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Preface

In ordinary thermodynamics fluids are studied through the system of equations constituted by the conservation laws of mass, momentum and energy, closed by the Navier-Stokes and Fourier equations. This system is valid to describe situations in which field are smooth. When step gradients or rapid changes occurs they are no more adequate to describe the physical situation.

To overcome this difficulty, Liu and Müller proposed a new approach to thermodynamics, in which further variables and balance equations are considered and the closure is given by imposing the galilean relativity and the entropy principle. The new theory is called "Extended Thermodynamics". Very important contributes to it has been given by Ruggeri. For example, the general structure of the system has been considered, looking in particular what happens when a change of frame occurs. Moreover, it has been shown that a particular set of independent variables can be chosen such that the system converts into a symmetric hyperbolic one with all the important mathematical properties as the well posedness of the Cauchy problem and the continuous dependence on initial data but especially the fact that the velocities of propagation of shock waves are finite. To solve equations, finding the exact solutions was very hard; so, recently, Pennisi and Ruggeri proposed a new methodology to impose the galilean relativity principle that overcomes these difficulties and leads to more elegant equations, easier to solve.

During this three years I have worked in the direction of proving that the new methodology can be applied to the many moments case, to materials different from ideal gases and also in the relativistic case and non only in the classical one; furthermore I have been able to find the exact solutions for many of these problems. This will be the subject of the following chapters.

The first chapter is a brief introduction on Extended Thermodynamics.
In chapter 2 I will show the new methodology to close the system, recently proposed by Pennisi and Ruggeri, and we will see some applications as the 5, the 13 and the 14 moments case for ideal gases. The last of these is a model obtained by Pennisi and myself.

In chapter 3 we will see a further application of the new methodology to cases with more complex fluids.
In chapter 4 we will see the results of my publications regarding the case with an arbitrary but fixed number of moments for ideal gases and I will also show what happens when we consider its subsystems.
In chapter 5 we will analyze my results for the case with an arbitrary but fixed number of moments but in a relativistic context and I will also show what happens when we consider its subsystems.
In chapter 6 we will do the non relativistic limit of the relativistic equations.
In chapter 7 we will consider a new kind of system of balance equations that is suggested from the non relativistic limit of the relativistic case.

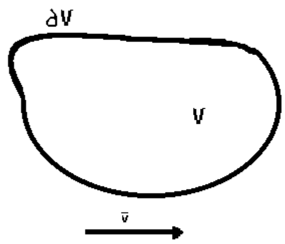
Contents

Chapter 1

Extended Thermodynamics

1.1 The balance equations

The aim of Extended Thermodynamics is to determine N variables $F_{i_1 \ldots i_n}$ with n=0, ..., N, called moments, in all points of the body and at all times. In order to achieve this objective, we need field equations, that generally are based on balance equations of mechanics and thermodynamics.
Let's consider a material volume V with fixed surface ∂V that moves with velocity $\overline{v}$.



Let's consider now a generic quantity Ψ referred to a property of the body, such that its macroscopic contribute for the volume V is

$$\Psi_V = \int_{V(t)} \psi dV.$$

If the volume changes in time and we want to evaluate the rate of change we must do the derivative with respect to time. We have

$$\frac{d}{dt} \int_{V(t)} \psi dV = \int_V (\dot{\psi} + \psi div\ \overline{v}) dV,$$

or $\int_V \frac{\partial \psi}{\partial t} + v_i \partial_i \psi + \psi \partial_i \overline{v} dV = \int_V \frac{\partial \psi}{\partial t} + \int_{\partial V} \psi \overline{v} \cdot \overline{n} d\Sigma.$

The change could be due to an escape of material out of the surface, or to a inner source or to an incoming quantity of material trough the surface. If we call ϕ_i the flux out of the surface, σ the inner production and S the supply from outside we have that

$$\int_V \frac{\partial \psi}{\partial t} + \int_{\partial V} \psi v_i n_i d\Sigma = - \int_{\partial V} \phi_i n_i d\Sigma + \int_V \sigma dV + \int_V S dV.$$

In local form the balance equation becomes

$$\frac{\partial \psi}{\partial t} + \frac{\partial (\psi v_i + \phi_i)}{\partial x_i} = \sigma + S. \tag{1.1}$$

The conservation laws of mass, momentum and energy are special cases of the equation above and can be obtained by identifying quantities as suggested in the following table

Ψ	ψ	ϕ_i	σ	S
Mass	F	0	0	0
Momentum	F_i	$-t_{ij}$	0	f_j
...	...	...	...	...
N-order moment	$F_{i_1 \dots i_N}$	$F_{i_1 \dots i_{N+1}}$	$S_{i_1 \dots i_N}$	$S_{i_1 \dots i_N}$

where t_{ij} is the stress tensor, f_i is the specific body force, q_i is the heat flux and r is the radiative supply.
If we substitute the elements of a row with the corresponding in the balance equation (1.1) we obtain

$$\frac{\partial F_{i_1 \dots i_n}}{\partial t} + \frac{\partial F_{i_1 \dots i_n k}}{\partial x_k} = S_{i_1 \dots i_n}. \tag{1.2}$$

If we consider the index n going from 0 to a fixed value N, eq. (1.2) converts into the following system of quasi-linear partial differential equations:

$$\begin{cases} \partial_t F + \partial_i F_i = 0 \\ \partial_t F_i + \partial_i F_{ik} = 0 \\ \dots \\ \partial_t F_{i_1 \dots i_N} + \partial_i F_{i_1 \dots i_N k} = S_{i_1 \dots i_N} \end{cases} \tag{1.3}$$

We can see that the flux in the first equation is the independent variable in the second equation. The same thing happens for all equations except for the last, so that $F_{i_1 \cdots i_N}$ remains unknown and the system is not closed. In the contest of Extended Thermodynamics the closure of the system is obtained by imposing the entropy principle and that of galilean relativity. We will see it in details in the following sections.

1.2 The entropy inequality

From now on let's call $\mathbf{u}$ the vector that includes all the independent variables. The entropy principle requires that the entropy inequality

$$h^A_{,A} = \Sigma \geq 0$$

holds for all "thermodynamical process" , i.e. for every solution of the system of balance equations (1.3). When A=0, $h^A_{,A}$ is the partial derivative with respect to time of the entropy density h^0, while, when A=1,2,3, $h^A_{,A}$ is the partial derivative with respect to space of the entropy flux h^i. Σ is the entropy production.
The quantities appearing in the equation above are all functions of the independent variables $\mathbf{u}$.
Furthermore h^0 must be a concave function, i.e.

$$\frac{\partial^2 h^0}{\partial \mathbf{u} \partial \mathbf{u}} \quad \sim \quad \text{negative defined,} \tag{1.4}$$

In [1] Liu proved that there exist $\mathbf{\Lambda}$, functions of the independent variables, called "Lagrange Multipliers", such that the following conditions are equivalent

$$\begin{cases} h^A_{,A} \geq 0 \\ \text{for every thermodynamical process.} \end{cases}$$

$$\begin{cases} h^A_{,A} - \mathbf{\Lambda} \cdot (\mathbf{F}^A_{,A} - \Pi) \geq 0 \\ \text{for every value of the independent variables.} \end{cases} \tag{1.5}$$

By using the chain rule for the derivation of composite functions the inequality (1.5) can be written in the form

$$\left(\frac{\partial h^A}{\partial \mathbf{u}} - \mathbf{\Lambda} \cdot \frac{\partial F^A}{\partial \mathbf{u}} \right) \cdot \mathbf{u}_{,A} + \mathbf{\Lambda} \cdot \mathbf{\Pi} \geq 0.$$

Such inequality must hold for all $\mathbf{u}$, in particular, for all $\mathbf{u}_{,A}$. So it follows that if we want that the inequality isn't violated for some value of $\mathbf{u}_{,A}$, all their coefficients must be equal to zero, i.e.

$$\frac{\partial h^A}{\partial \mathbf{u}} = \mathbf{\Lambda} \cdot \frac{\partial \mathbf{F^A}}{\partial \mathbf{u}} \qquad \text{and} \qquad \mathbf{\Lambda} \cdot \mathbf{\Pi} \geq 0. \tag{1.6}$$

Eq. $(1.6)_2$ is called residual inequality. Eq. $(1.6)_1$ can be written also as

$$dh^A = \mathbf{\Lambda} \cdot d\mathbf{F}^A. \tag{1.7}$$

What said until now is independent from the choice of $\mathbf{u}$. Without loosing of generality we can choose $\mathbf{u} = \mathbf{F}^0$, and, by putting it into eq. (1.7), for $A = 0$, we have

$$\frac{\partial h^0}{\partial \mathbf{u}} = \mathbf{\Lambda}$$

that, after a further differentiation, becomes

$$\frac{\partial^2 h^0}{\partial \mathbf{u} \partial \mathbf{u}} = \frac{\partial \mathbf{\Lambda}}{\partial \mathbf{u}}.$$

We notice that $\frac{\partial \mathbf{\Lambda}}{\partial \mathbf{u}}$ is negative defined (because the left hand side satisfy the condition (1.4)) and symmetric, so the functions that renders the change of variables from $\mathbf{u}$ to $\mathbf{\Lambda}$ is locally and globally invertible and we can take $\mathbf{\Lambda}$ as independent variables.
To make the calculation simpler it is convenient to introduce

$$h'^A = \mathbf{\Lambda} \cdot \mathbf{F}^A - h^A \tag{1.8}$$

that is called potential. In this way eq. (1.7) converts into

$$dh'^A = \mathbf{F}^A \cdot d\mathbf{\Lambda},$$

from which

$$\mathbf{F}^A = \frac{\partial h'^A}{\partial \mathbf{\Lambda}}. \tag{1.9}$$

By putting the above equation into (1.8) it follows that

$$h^A = -h'^A + \mathbf{\Lambda} \cdot \frac{\partial h'^A}{\partial \mathbf{\Lambda}}. \tag{1.10}$$

So all the costitutive functions $\mathbf{F}^A(\mathbf{\Lambda})$ and $h^A(\mathbf{\Lambda})$ can be obtained from $h'^A(\mathbf{\Lambda})$, that's why it is called "potential".
There remains the further inequality

$$\Sigma = \Lambda \cdot \Pi(\Lambda) \geq 0, \tag{1.11}$$

that says that the entropy production Σ is a non-negative function.
It is possible to prove (see [2]) that conditions (1.9), (1.10) and (1.11) are

necessary and sufficient. Moreover, from the integrability condition for h'^A, given by eq. (1.9) we have that

$$\frac{\partial \mathbf{F}^A}{\partial \boldsymbol{\Lambda}} \qquad \text{must be symmetric.}$$

Let's recall that the hyperbolic system (1.3) is also symmetric if we choose the Lagrange Multipliers as independent variables. This special role played by the Lagrange multipliers (or "main field") has been found by Ruggeri and Strumia in [3]. This is a generalization of the symmetric hyperbolic systems studied by Friendrichs in [4].

1.3 The Galilean relativity principle

Let's study now the behavior of the constitutive quantities appearing in the balance equations and in the supplementary entropy law when we consider a change of frame.
In classical (non relativistic) thermodynamics we can consider three different type of change of frame:

$$\begin{array}{lll} \text{Rotation of coordinates} & x_i^* = O_{ij}x_j, & t^* = t \\ \text{Galilean transformations} & x_i^* = O_{ij}x_i + c_i t, & t^* = t \\ \text{Euclidean transformation} & x_i^* = O_{ij}(t)x_j + b_i(t), & t^* = t. \end{array} \tag{1.12}$$

A quantity is named tensor of order A if its components in two different frames are related by the following equation

$$T^*_{i_1 i_2 \ldots i_A} = O_{i_1 j_1} \ldots O_{i_A j_A} T_{j_1 j_2 \ldots j_A}. \tag{1.13}$$

Obviously, scalars and vectors are included because they are tensors of order 0 and 1 respectively.
In the case of Euclidean or Galilean transformation they are called objective and Galilean tensors respectively. In case of rotation of coordinates they are called only tensors.
The velocity v_i transforms according to the following rules

$$\begin{array}{ll} v_i^* = O_{ij}v_j & \text{for rotations,} \\ v_i^* = O_{ij}v_j + c_i & \text{for Galilean transformations,} \\ v_i^* = O_{ij}(t)v_j + \dot{O}_{ij}(t)x_j + \dot{b}_i(t) & \text{for Euclidean transformations,} \end{array}$$

obtained by taking the derivative with respect to time of eqs. (1.12); so velocity is a tensor, neither Galilean nor objective.

In general the vector $\mathbf{u}$ of the independent variables may have between its n components some functions of the velocity v_i of the particles. The remaining (n-3) variables are postulated to be Galilean tensors, and are called "non convective part" of $\mathbf{u}$ and will be indicated with $\mathbf{w}$. So $\mathbf{w}$ represents that (n-3) components of $\mathbf{u}$ which aren't the velocity.
The presence of the velocity v_i can be noticed easily in the fluxes $\mathbf{F}^i$ and h^i, that, by separating the convective and the non convective parts, appear as follows:

$$\begin{aligned} \mathbf{F}^i &= \mathbf{F}^0 v^i + \mathbf{G}^i, \\ h^i &= h^0 v^i + \varphi^i, \end{aligned} \tag{1.14}$$

where $\mathbf{G}^i$ and φ^i are the non convective fluxes.
This doesn't mean that $\mathbf{F}^0$, $\mathbf{G}^i$, $\mathbf{\Pi}$ or h^0, φ^i and Σ are necessarily independent of v_i, in fact we can write the constitutive equations as

$$\begin{array}{ll} \mathbf{F}^0 = \mathbf{F}^0(\mathbf{v},\mathbf{w}), & \mathbf{G}^i = \mathbf{G}^i(\mathbf{v},\mathbf{w}), \\ \mathbf{\Pi} = \mathbf{\Pi}(\mathbf{v},\mathbf{w}), & h^0 = h^0(\mathbf{v},\mathbf{w}), \\ \varphi^i = h^i(\mathbf{v},\mathbf{w}), & \Sigma = \Sigma(\mathbf{v},\mathbf{w}), \end{array} \tag{1.15}$$

If eqs. (1.15) are the constitutive equations in a particular Galilean frame, in an other frame they will be

$$\begin{array}{ll} \mathbf{F}^{*0} = \mathbf{F}^0(\mathbf{v}^*,\mathbf{w}^*) & \mathbf{G}^{*i} = \mathbf{G}^i(\mathbf{v}^*,\mathbf{w}^*), \\ \mathbf{\Pi}^* = \mathbf{\Pi}(\mathbf{v}^*,\mathbf{w}^*), & h^{*0} = h^0(\mathbf{v}^*,\mathbf{w}^*), \\ \varphi^{*i} = h^i(\mathbf{v}^*,\mathbf{w}^*), & \Sigma^* = \Sigma(\mathbf{v}^*,\mathbf{w}^*). \end{array}$$

The Galilean relativity principle says that the field equations must have the same form in all Galileanly equivalent frames (i.e. frames related by Galilean transformations).
It follows that the constitutive functions are invariant, while both their values and the values of their variables can change. In fact the field equations can't have the same form in all Galileanly equivalent frames if it don't happens before for the constitutive functions.
The Galilean relativity principle imposes that balance equations and entropy law must have the same form in two galilean equivalent frames, i.e.

$$\mathbf{\Pi}(\mathbf{v},\mathbf{w}) = \frac{\partial \mathbf{F}^0(\mathbf{v},\mathbf{w})}{\partial t} + \frac{\partial \mathbf{F}^0(\mathbf{v},\mathbf{w})v^i + \mathbf{G}^i(\mathbf{v},\mathbf{w})}{\partial x_i} \tag{1.16}$$

converts into

$$\mathbf{\Pi}(\mathbf{v}^*,\mathbf{w}^*) = \frac{\partial \mathbf{F}^0(\mathbf{v}^*,\mathbf{w}^*)}{\partial t^*} + \frac{\partial \mathbf{F}^0(\mathbf{v}^*,\mathbf{w}^*)v^{i*} + \mathbf{G}^i(\mathbf{v}^*,\mathbf{w}^*)}{\partial x_i^*}, \tag{1.17}$$

and

$$\Sigma(\mathbf{v},\mathbf{w}) = \frac{\partial h^0(\mathbf{v},\mathbf{w})}{\partial t} + \frac{\partial h^0(\mathbf{v},\mathbf{w})v^i + \varphi^i(\mathbf{v},\mathbf{w})}{\partial x_i} \tag{1.18}$$

converts into

$$\Sigma(\mathbf{v}^*,\mathbf{w}^*) = \frac{\partial h^0(\mathbf{v}^*,\mathbf{w}^*)}{\partial t^*} + \frac{\partial h^0(\mathbf{v}^*,\mathbf{w}^*)v^{i*} + \varphi^i(\mathbf{v}^*,\mathbf{w}^*)}{\partial x_i^*}. \tag{1.19}$$

Ruggeri, in [5], studied the Galilean invariance through the decomposition of quantities into their internal and non-convective parts.
We said that the (n-3) fields $\mathbf{w}$ are components of a Galilean tensor. If $O_{ij} = \delta_{ij}$ we have that $\mathbf{w} = \mathbf{w}^*$ and

$$\frac{\partial}{\partial x_i^*} = \frac{\partial}{\partial x_i} \qquad \text{and} \qquad \frac{\partial}{\partial t^*} = \frac{\partial}{\partial t} - c_i \frac{\partial}{\partial x_i}.$$

So eq. (1.19) becomes

$$\Sigma(\mathbf{v}+\mathbf{c},\mathbf{w}) = \frac{\partial h^0(\mathbf{v}+c,\mathbf{w})}{\partial t} + \frac{\partial h^0(\mathbf{v}+\mathbf{c},\mathbf{w})v^i + \varphi^i(\mathbf{v}+\mathbf{c},\mathbf{w})}{\partial x_i} \tag{1.20}$$

that must be equivalent to eq. (1.18).
Comparing eqs. (1.18) and (1.20) we have

$$\begin{aligned} h^0(\mathbf{v}+c,\mathbf{w}) &= h^0(\mathbf{v},\mathbf{w}), \\ \varphi^i(\mathbf{v}+\mathbf{c},\mathbf{w}) &= \varphi^i(\mathbf{v},\mathbf{w}), \\ \Sigma(\mathbf{v}+\mathbf{c},\mathbf{w}) &= \Sigma(\mathbf{v},\mathbf{w}). \end{aligned}$$

That equations must hold true for every $\mathbf{v}$ e $\mathbf{c}$. So h^0, φ^i and Σ must be independent from $\mathbf{v}$, i.e.

$$h^0 = h^0(\mathbf{w}), \qquad \varphi^i = \varphi^i(\mathbf{w}), \qquad \Sigma = \Sigma(\mathbf{w}). \tag{1.21}$$

Analogously we can write (1.17) as

$$\mathbf{\Pi}(\mathbf{v}+\mathbf{c},\mathbf{w}) = \frac{\partial \mathbf{F}^0(\mathbf{v}+\mathbf{c},\mathbf{w})}{\partial t} + \frac{\partial \mathbf{F}^0(\mathbf{v}+\mathbf{c},\mathbf{w})v^i + \mathbf{G}^i(\mathbf{v}+\mathbf{c},\mathbf{w})}{\partial x_i}, \tag{1.22}$$

that must be equivalent to eq. (1.16). We can't compare directly the two equations because they are system and not single equations. The system preserve the Galilean invariance even if densities, fluxes and productions depend on velocity although they depend in a certain way.

The equivalence between (1.16) and (1.22) must be linear. So there exists a non singular matrix $\mathbf{X}(\mathbf{c})$ of order $n \times n$ such that

$$\begin{aligned}
\mathbf{F}^0(\mathbf{v}+\mathbf{c},\mathbf{w}) &= \mathbf{X}(\mathbf{c})\mathbf{F}^0(\mathbf{v},\mathbf{w}),\\
\mathbf{G}^i(\mathbf{v}+\mathbf{c},\mathbf{w}) &= \mathbf{X}(\mathbf{c})\mathbf{G}^i(\mathbf{v},\mathbf{w}),\\
\mathbf{\Pi}(\mathbf{v}+\mathbf{c},\mathbf{w}) &= \mathbf{X}(\mathbf{c})\mathbf{\Pi}(\mathbf{v},\mathbf{w}).
\end{aligned}$$

This must be true for every choice of $\mathbf{v}$ e $\mathbf{c}$, in particular when $\mathbf{v} = \mathbf{0}$ and $\forall \mathbf{c}$

$$\begin{aligned}
\mathbf{F}^0(\mathbf{c},\mathbf{w}) &= \mathbf{X}(\mathbf{c})\mathbf{F}^0(\mathbf{0},\mathbf{w}),\\
\mathbf{G}^i(\mathbf{c},\mathbf{w}) &= \mathbf{X}(\mathbf{c})\mathbf{G}^i(\mathbf{0},\mathbf{w}),\\
\mathbf{\Pi}(\mathbf{c},\mathbf{w}) &= \mathbf{X}(\mathbf{c})\mathbf{\Pi}(\mathbf{0},\mathbf{w}),
\end{aligned} \tag{1.23}$$

where $\mathbf{F}^0(\mathbf{0},\mathbf{w})$, $\mathbf{G}^i(\mathbf{0},\mathbf{w})$ and $\mathbf{\Pi}(\mathbf{0},\mathbf{w})$ represents quantities calculated in the frame where the body is at rest. We will call them "internal quantities" and we will write them as $\hat{\mathbf{F}}^0$, $\hat{\mathbf{G}}^i$ and $\hat{\mathbf{\Sigma}}$, so that eq. (1.23) converts into

$$\begin{aligned}
\mathbf{F}^0(\mathbf{c},\mathbf{w}) &= \mathbf{X}(\mathbf{c})\hat{\mathbf{F}}^0(\mathbf{w}),\\
\mathbf{G}^i(\mathbf{c},\mathbf{w}) &= \mathbf{X}(\mathbf{c})\hat{\mathbf{G}}^i(\mathbf{w}),\\
\mathbf{\Pi}(\mathbf{c},\mathbf{w}) &= \mathbf{X}(\mathbf{c})\hat{\mathbf{\Pi}}(\mathbf{w}).
\end{aligned} \tag{1.24}$$

Before putting the above decomposition into the field equations (1.16), let's recall some properties of the matrix $\mathbf{X}$:

$$\frac{\partial \mathbf{X}}{\partial v_r}(\mathbf{v}) = \mathbf{A}^r\mathbf{X}(\mathbf{v}) = \mathbf{X}(\mathbf{v})\mathbf{A}^r, \tag{1.25}$$

where

$$\mathbf{A}^r = \frac{\partial \mathbf{X}}{\partial v_r}(0),$$

and

$$\mathbf{A}^r\mathbf{A}^s = \mathbf{A}^s\mathbf{A}^r, \tag{1.26}$$

See [5] for details.
By using the above properties and the material temporal derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x_i},$$

we have:

$$\mathbf{X}\left\{\frac{d\hat{\mathbf{F}}^0}{dt} + \hat{\mathbf{F}}^0\frac{\partial v_i}{\partial x_i} + \frac{\partial \hat{\mathbf{G}}^i}{\partial x_i} + \mathbf{A}^r\left\{\hat{\mathbf{F}}^0\frac{dv_r}{dt} + \hat{\mathbf{G}}^i\frac{\partial v_r}{\partial x_i}\right\} - \hat{\mathbf{\Pi}}\right\} = 0.$$

So we have obtained an alternative form of the balance equations

$$\frac{d\hat{\mathbf{F}}^0}{dt} + \hat{\mathbf{F}}^0 \frac{\partial v_i}{\partial x_i} + \frac{\partial \hat{\mathbf{G}}^i}{\partial x_i} + \mathbf{A}^r \left\{ \hat{\mathbf{F}}^0 \frac{dv_r}{dt} + \hat{\mathbf{G}}^i \frac{\partial v_r}{\partial x_i} \right\} = \hat{\mathbf{\Pi}}.$$

This last equations is easier to handle than (1.16). Once $\mathbf{A}^r$ is known, even $\mathbf{X}(\mathbf{v})$ it is, so only the internal quantities $\hat{\mathbf{F}}^0$, $\hat{\mathbf{G}}^i$ and $\hat{\mathbf{\Pi}}$ remains as constitutive functions and $\mathbf{v}$ is no more a variable.
Let's apply the decomposition also to the the entropy inequality; in particular we want to separate internal and non-convective parts.
Remember that from (1.21) it follows that $h^0 = \hat{h}^0$, $\varphi^i = \hat{\varphi}^i$, $\Sigma = \hat{\Sigma}$ and, by using eqs. (1.14) and (1.24), eq. (1.7) can be wrote as

$$\begin{cases} dh^0 = \mathbf{\Lambda} d(\mathbf{X}\hat{\mathbf{F}}^0) & \text{per A=0,} \\ d(h^0 v^i + \varphi^i) = \mathbf{\Lambda} d[\mathbf{X}(\hat{\mathbf{F}}^0 v^i + \hat{\mathbf{G}}^i)] & \text{per A=i.} \end{cases}$$

Now, from (1.25) we have

$$d\mathbf{X} = \mathbf{X}\mathbf{A}^r dv_r,$$

so that he last two equations become

$$\begin{aligned} dh^0 - \mathbf{\Lambda X} d\hat{\mathbf{F}}^0 &= \mathbf{\Lambda X A}^r \hat{\mathbf{F}}^0 dv_r, \\ d\varphi^i - \mathbf{\Lambda X} d\hat{\mathbf{G}}^i &= \big(\mathbf{\Lambda X A}^r \hat{\mathbf{G}}^i - [\hat{h}_0 - \mathbf{\Lambda X}\hat{\mathbf{F}}^0]\delta^{ri}\big) dv_r. \end{aligned}$$

This equations must hold for every value of $d\mathbf{v}$, so

$$\begin{aligned} dh^0 &= \mathbf{\Lambda X} d\hat{\mathbf{F}}^0, \\ \mathbf{\Lambda X A}^r \hat{\mathbf{F}}^0 &= 0, \\ d\varphi^i &= \mathbf{\Lambda X} d\hat{\mathbf{G}}^i, \\ \mathbf{\Lambda X A}^r \hat{\mathbf{G}}^i &= [\hat{h}_0 - \mathbf{\Lambda X}\hat{\mathbf{F}}^0]\delta^{ri}. \end{aligned} \tag{1.27}$$

Furthermore h^0, $\hat{\mathbf{F}}^0$, φ^i and $\hat{\mathbf{G}}^i$ don't depend on $\mathbf{v}$, so from $(1.27)_{1,3}$ it follows that $\mathbf{\Lambda X}$ don't depend on $\mathbf{v}$ too. If we write

$$\hat{\mathbf{\Lambda}} = \mathbf{\Lambda X}, \tag{1.28}$$

where $\hat{\mathbf{\Lambda}}$ are called "internal Lagrange multipliers", eq. (1.27) becomes

$$\begin{aligned} dh^0 &= \hat{\mathbf{\Lambda}} d\hat{\mathbf{F}}^0, \\ \hat{\mathbf{\Lambda}} \mathbf{A}^r \hat{\mathbf{F}}^0 &= 0, \\ d\varphi^i &= \hat{\mathbf{\Lambda}} d\hat{\mathbf{G}}^i, \\ \hat{\mathbf{\Lambda}} \mathbf{A}^r \hat{\mathbf{G}}^i &= [\hat{h}_0 - \hat{\mathbf{\Lambda}}\hat{\mathbf{F}}^0]\delta^{ri}. \end{aligned} \tag{1.29}$$

Finally, by using $(1.24)_3$ and (1.28), eq. (1.11) can be wrote as

$$\hat{\mathbf{\Lambda}}\hat{\mathbf{\Pi}} \geq 0. \tag{1.30}$$

Eqs. (1.29) and (1.30) represent the restriction imposed to the system by the entropy inequality, after having divided the contribution of the non convective parts and of the velocity. These equations contain only internal quantities.
Eq. $(1.29)_{1,3}$ stand for eq. $(1.6)_1$, while (1.30) substitutes eq. $(1.6)_2$; eqs. $(1.29)_{2,4}$ are additional conditions deriving from the relativity principle.
Eq. $(1.29)_1$ can be named Gibbs equation of the extended thermodynamics and it is useful to determine the value of the Lagrange multipliers at equilibrium. Moreover there are analogous conditions, see $(1.29)_3$, for the entropy flux.
Before concluding this section I want only show how the matrix X is in detail:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ v_{k_1} & \delta^{h_1}_{k_1} & 0 & 0 & \cdots & 0 \\ v_{k_1}v_{k_2} & 2\delta^{h_1}_{(k_1}v_{k_2)} & \delta^{h_1}_{(k_1}\delta^{h_2}_{k_2)} & \cdots & \cdots & 0 \\ v_{k_1}v_{k_2}v_{k_3} & 3\delta^{h_1}_{(k_1}v_{k_2}v_{k_3)} & 3\delta^{h_1}_{(k_1}\delta^{h_2}_{k_2}v_{k_3)} & \delta^{h_1}_{k_1}\delta^{h_2}_{k_2}\delta^{h_3}_{k_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{k_1}v_{k_2}\cdots v_{k_n} & \binom{n}{1}\delta^{h_1}_{(k_1}v_{k_2}\ldots v_{k_n)} & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ v_{k_1}v_{k_2}\cdots v_{k_N} & \binom{N}{1}\delta^{h_1}_{(k_1}v_{k_2}\ldots v_{k_n)} & \cdots & \cdots & \cdots & \delta^{h_1}_{k_1}\delta^{h_2}_{k_2}\cdots\delta^{h_N}_{k_N} \end{pmatrix} \tag{1.31}$$

A complete and general procedure to prove that the form of the matrix is the above one can be found in [5]. After having split each tensor into its convective and non-convective parts, we have that some non-convective tensor are functions of other non-convective one; restrictions on their generality is imposed by eq. (1.13) which must be satisfied both by the functions and also by their arguments. How to impose these restrictions is explained in [6], [7], [8], [9], [10], [11], [12].
By applying this method to a general tensorial density we find

$$F_{i_1\ldots i_l} = \sum_{h=0}^{l} \binom{l}{h} \hat{F}_{(i_1\ldots i_h}v_{i_{h+1}}\cdots v_{i_l)}. \tag{1.32}$$

The same decomposition can be applied to $\mathbf{G}^i$ and $\mathbf{\Pi}$. In every case with a physical meaning F e F^k are the mass and momentum densities, while $\frac{1}{2}F_{kk}$ is the energy density, so the non-convective flux G^i and the internal part of F^k disappears because don't exist fluxes of mass and the momentum density doesn't have a part independent of velocity. Moreover Π, Π_k, Π_{kk} are all zero.

1.4 The 13 moments case

Let's consider now the particular case of Extended Thermodynamics of monoatomic gases, viscous and heat conductor with 13 moments. This has been the first case that have been studied (see [13]). The 13 moments are

$$\begin{array}{ll} F=\rho & \text{mass density,} \\ F_i=\rho v_i & \text{momentum density,} \\ F_{ij} & \text{flux momentum density,} \\ \frac{1}{2}F_{ppi} & \text{flux energy density.} \end{array} \tag{1.33}$$

The appropriate balance equations are:

$$\begin{cases} \frac{\partial F}{\partial t}+\frac{\partial F_k}{\partial x_k}=0 \\ \frac{\partial F_i}{\partial t}+\frac{\partial F_{ik}}{\partial x_k}=0 \\ \frac{\partial F_{ij}}{\partial t}+\frac{\partial F_{ijk}}{\partial x_k}=S_{<ij>} \\ \frac{\partial F_{ppi}}{\partial t}+\frac{\partial F_{ppik}}{\partial x_k}=S_{ppi}, \end{cases} \tag{1.34}$$

where all the tensors are symmetric and S_{ij} has zero trace because the trace of eq. $(1.34)_3$ is the conservation of energy. To close the system we need constitutive relations for the quantities

$$F_{<ijk>}, \qquad F_{ppik}, \qquad S_{<ij>} \quad \text{and} \quad S_{ppi}, \tag{1.35}$$

where $F_{<ijk>}$ is the traceless part of F_{ijk}.
In Extended Thermodynamics the constitutive quantities (1.35) everywhere and in each instant depend on the values of (1.33) in that point and time, so that we have

$$\begin{aligned} F_{<ijk>} &= \hat{F}_{<ijk>}(F,F_i,F_{ij},F_{ppi}), \\ F_{ppik} &= \hat{F}_{ppik}(F,F_i,F_{ij},F_{ppi}), \\ S_{<ij>} &= \hat{S}_{<ij>}(F,F_i,F_{ij},F_{ppi}), \\ S_{ppi} &= \hat{S}_{ppi}(F,F_i,F_{ij},F_{ppi}). \end{aligned} \tag{1.36}$$

If we know the constitutive functions $\hat{F}_{<ijk>}$ through $\hat{S}_{<ppi>}$ it is possible to eliminate $F_{<ijk>}$, F_{ppik}, $S_{<ij>}$ e S_{ppi} by using eq. (1.36), obtaining a quasi-linear system of partial differential equations. The entropy inequality

$$\frac{\partial h}{\partial t}+\frac{\partial h_i}{\partial x_i}=\Sigma \geq 0 \tag{1.37}$$

must be satisfied for each thermodynamical process, i.e. for every solution of the field equations. The entropy density and the entropy flux are constitutive quantities:

$$\begin{aligned} h &= \hat{h}(F, F_i, F_{ij}, F_{ppi}), \\ h_i &= \hat{h}_i(F, F_i, F_{ij}, F_{ppi}). \end{aligned} \tag{1.38}$$

The entropy density is a non-convective quantity, while the flux can be decomposed into an internal and a convective part, as follows

$$h_i = hv_i + \varphi_i. \tag{1.39}$$

In such a way the entropy inequality can be wrote as follows:

$$\varrho \left(\frac{h}{\varrho}\right)^{\bullet} + \frac{\partial \varphi_i}{\partial x_i} = \Sigma \geq 0,$$

where the dot stands for the Lagrangian derivative $\frac{\partial}{\partial t} + v_k \frac{\partial}{\partial x_k}$ and $(1.34)_1$ has been used.

This inequality must hold for each Euclidean frame so that h e Σ must be objective scalars while φ_i must be an objective vector.

1.4.1 The Lagrange multipliers.

By introducing the Lagrange multipliers $\mathbf{\Lambda}$ the entropy inequality converts into

$$\begin{aligned} &\frac{\partial h}{\partial t} + \frac{\partial h_i}{\partial x_i} - \mathbf{\Lambda}\left(\frac{\partial F}{\partial t} + \frac{\partial F_k}{\partial x_k}\right) - \mathbf{\Lambda}_i\left(\frac{\partial F_i}{\partial t} + \frac{\partial F_{ik}}{\partial x_k}\right) - \\ &-\mathbf{\Lambda}_{ij}\left(\frac{\partial F_{ij}}{\partial t} + \frac{\partial F_{ijk}}{\partial x_k} - S_{<ij>}\right) - \mathbf{\Lambda}_{qqi}\left(\frac{\partial F_{ppi}}{\partial t} + \frac{\partial F_{ppik}}{\partial x_k} - S_{ppi}\right) \geq 0, \end{aligned}$$

according to Liu's theorem; the Lagrange multipliers are function of F, F_i, F_{ij}, e F_{ppi} in a way that depends on the material. If we consider the constitutive relations (1.36) and (1.38) for h, h_i, $F_{<ijk>}$ e F_{ppik}, we obtain an expression that is linear with respect to the derivatives

$$\frac{\partial F}{\partial t}, \frac{\partial F_i}{\partial t}, \frac{\partial F_{ij}}{\partial t}, \frac{\partial F_{ppi}}{\partial t}, \frac{\partial F}{\partial x_k}, \frac{\partial F_i}{\partial x_k}, \frac{\partial F_{ij}}{\partial x_k}, \frac{\partial F_{ppi}}{\partial x_k}.$$

The inequality must hold for arbitrary values of that derivatives, so that their coefficients must be zero. So that we obtain

$$\begin{aligned} dh &= \mathbf{\Lambda} dF + \mathbf{\Lambda}_i dF_i + \mathbf{\Lambda}_{ij} dF_{ij} + \mathbf{\Lambda}_{ppi} dF_{qqi}, \\ dh_k &= \mathbf{\Lambda} dF_k + \mathbf{\Lambda}_i dF_{ik} + \mathbf{\Lambda}_{ij} dF_{ijk} + \mathbf{\Lambda}_{ppi} dF_{qqik}, \end{aligned} \tag{1.40}$$

plus the residual inequality

$$\Sigma = \mathbf{\Lambda}_{ij} S_{<ij>} + \mathbf{\Lambda}_{ppi} S_{ppi} \geq 0 \tag{1.41}$$

that represents the non negative entropy production.
By applying eq. (1.24) to our quantities we obtain the following decomposition into convective and non-convective parts (The quantities ρ... stands for $\hat{F}$...):

$$\begin{array}{lllllll} F & = & \rho & & & & \\ F_i & = & & +\rho v_i & & & \\ F_{ij} & = & \rho_{ij} & + & +\rho v_i v_j & & \\ F_{ijk} & = & \rho_{ijk} & +3\rho_{(ij} v_{k)} & + & +\rho v_i v_j v_k & . \\ F_{ppij} & = & \rho_{ppij} & +4\rho_{(ijp} v_{j)} & +6\rho_{(ip} v_p v_{j)} & +\rho v^2 v_i v_j & \\ S_{<ij>} & = & s_{<ij>} & & & & \\ S_{ppi} & = & s_{ppi} & +2 s_{<ip>} v_p & & & \end{array} \tag{1.42}$$

$\rho \ldots$ and $s \ldots$, h and φ_k are galilean tensors, so their components in two frames are linked by the following relations

$$\begin{aligned} \rho_{i_1 i_2 \ldots i_n} &= O_{i_1 j_1} \cdots O_{i_n j_n} \rho^*_{j_1 j_2 \ldots j_n}, \\ s_{i_1 i_2 \ldots i_n} &= O_{i_1 j_1} \cdots O_{i_n j_n} s^*_{j_1 j_2 \ldots j_n}, \\ h &= h^*, \\ \varphi_k &= O_{kj} \varphi^*_j \end{aligned} \tag{1.43}$$

Let's impose that ρ..., s..., h and ϕ_k be objective tensors and not only galilean, so they satisfy eq. (1.43) for an Euclidean transformation like this

$$x^*_i = O_{ij}(t) x_j + b_i(t).$$

1.4.2 The Galilean Relativity principle

Let's consider now eq. (1.40). From eqs. (1.39) and (1.42) we now explicit the dependence of F and h_i on velocity. By putting them on eqs. (1.40), after long calculations we have:

$$\begin{aligned} dh &= \lambda d\rho + \lambda_{ij} d\rho_{ij} + \lambda_{qqi} d\rho_{ppi}, \\ d\varphi_k &= +\lambda_i d\rho_{ik} + \lambda_{ij} d\rho_{ijk} + \lambda_{qqi} d\rho_{ppik}, \end{aligned} \tag{1.44}$$

$$\begin{aligned} & \lambda_t \rho + \lambda_{qqi}(2\rho_{it} + \rho_{pp}\delta_{it}) = 0, \\ & 2\lambda_{tj}\rho_{jk} + \lambda_{qqi}(2\rho_{itk} + \rho_{ppk}\delta_{it}) = (h - \lambda\rho - \lambda_{ij}\rho_{ij}\lambda_{qqi}\rho_{ppi})\delta_{tk}, \end{aligned} \tag{1.45}$$

where the new quantities λ have been used. They represent the internal part, not depending on velocity, of the Lagrange multipliers. They can be obtained by the previous $\mathbf{\Lambda}$ through:

$$
\begin{aligned}
\lambda &= \Lambda + \Lambda_i v_i + \Lambda_{ij} v_i v_j + \Lambda_{ppi} v^2 v_i, \\
\lambda_i &= \Lambda_i + 2\Lambda_{ij} v_j + \Lambda_{ppj}(v^2 \delta_{ji} + 2 v_j v_i), \\
\lambda_{ij} &= \Lambda_{ij} + \Lambda_{ppt} 3 v_{(t} \delta_{ij)}, \\
\lambda_{ppi} &= \Lambda_{ppi}.
\end{aligned} \tag{1.46}
$$

as shown in the previous section.
From eq. (1.44) we can see that λ, λ_{ij} e λ_{qqi} are partial derivatives of the objective scalar h with respect to the objective quantities ϱ..., so also they must be objective tensors. From (1.44) follows also that λ_i is an objective vector. For the same reasons the λ... must be isotropic functions of ρ, ρ_{ij}, ρ_{ppi}.

1.4.3 The scalar and the vector potential

Let's suppose that $\rho, \rho_{ik}, \rho_{ppi}$ are independent variables through the Lagrange multipliers $\lambda, \lambda_{ij}, \lambda_{ppi}$.
Let's define the scalar and the vector potential h' and h'_k as follows:

$$
\begin{aligned}
h' &= -h + \lambda\rho + \lambda_{ij}\rho_{ij} + \lambda_{qqi}\rho_{ppi}, \\
h'_k &= -\varphi_k + \lambda_{ij}\rho_{ijk} + \lambda_{qqi} d\rho_{ppik}.
\end{aligned} \tag{1.47}
$$

In such a way we can rewrite eqs. (1.44) and $(1.45)_2$ as

$$
\begin{aligned}
dh' &= \rho d\lambda + \rho_{rs} d\lambda_{rs} + \rho_{ppn} d\lambda_{qqn}, \\
dh'_k &= \left[\lambda_{qqe} a_{ei} \frac{\partial \rho_{ij}}{\partial \lambda}\right] d\lambda + \left[\rho_{rsk} + \lambda_{qqe} a_{ei} \frac{\partial \rho_{ik}}{\partial \lambda_{rs}}\right] d\lambda_{rs} + \\
&+ \left[\rho_{ppnk} + \lambda_{qqe} a_{ei} \frac{\partial \rho_{ik}}{\partial \lambda_{qqn}}\right] d\lambda_{qqn},
\end{aligned} \tag{1.48}
$$

$$
2\lambda_{tj}\rho_{jk} + \lambda_{qqi}(2\rho_{itk} + \rho_{ppk}\delta_{it}) = -h'\delta_{tk}. \tag{1.49}
$$

λ_t has been eliminated by using eq. $(1.45)_1$ and the new tensor

$$
a_{ei} = \frac{1}{\rho}(2\rho_{ei} + \rho_{pp}\delta_{ei}) \tag{1.50}
$$

has been inserted.

1.4.4 Equilibrium

The Lagrange multipliers are a very important mathematical feature to solve thermodynamical problems but they can't be easily measured except for their values at equilibrium.
Equilibrium is defined as the process in which:

1. Productions $s_{<ij>}$ and s_{ppi} are equal to zero,
2. $s_{<ij>}$ and s_{ppi} near this process are invertible with respect to $\lambda_{<ij>}$ and λ_{ill},
3. Near this process the entropy production have a proper relative minimum at equilibrium.

From eq. (1.41) and condition 1 it follows that the entropy production have a minimum at equilibrium, and its value is 0.
Condition 3 requires that this value is also a proper relative minimum. By taking λ, λ_{ll}, $s_{<ij>}$ ed s_{ppi} as independent variables it follows that, at equilibrium, the partial derivatives of Σ with respect to $s_{<ij>}$ and s_{ppi} must be zero, so the following necessary conditions hold

$$\begin{aligned} \lambda_{<ij>} \mid_E &= 0, \\ \lambda_{qqi} \mid_E &= 0, \end{aligned} \tag{1.51}$$

where E stands for equilibrium. What about the other Lagrange multipliers? After defining the internal energy

$$\varrho_{ii} = 2\varrho\varepsilon \tag{1.52}$$

we have, from eq. $(1.44)_1$

$$dh_E = \frac{2}{3}\lambda_{ii} \mid_E d(\varrho\varepsilon) + \lambda_E d\varrho. \tag{1.53}$$

By comparing it with the Gibbs equation we find that

$$\lambda_E = -\frac{g}{T}, \qquad \lambda_{ii} \mid_E = \frac{3}{2}\frac{1}{T}. \tag{1.54}$$

where T is the absolute temperature and

$$g = \varepsilon - T\frac{h_E}{\rho} + \frac{p_E}{\varrho} \tag{1.55}$$

is the chemical potential.
Eqs. (1.51) and (1.54) give the value at equilibrium of all Lagrange multipliers. Blending together eqs. (1.53) and (1.54) we have

$$dh_E = \frac{1}{T}d(\varrho\varepsilon) - \frac{g}{T}d\varrho. \tag{1.56}$$

Leaving the noisy calculations we say only that taking as variables T and z with $z = \frac{\varrho}{T^{\frac{3}{2}}}$ the solution of the condition above is given by

$$\frac{dh_E}{\varrho} = \frac{3}{2}\int\left(\frac{F'}{z} - \frac{5}{3}\frac{F}{z^2}\right)dz + \zeta \tag{1.57}$$

where F(z) is an arbitrary function and ζ is a constant. Now it is possible to find the value of $\frac{g}{T}$ that is

$$\frac{g}{T} = \frac{F}{z} + \int\frac{F}{z^2}dz - \zeta \tag{1.58}$$

1.4.5 Determination of the potential at equilibrium

The potentials h' and h'_k, defined by eq. (1.47), are isotropic scalars and vectorial functions of λ, λ_{ij} e λ_{ppi}. Let's write their polynomial representation (by using the representation theorems) up to third order with respect to equilibrium:

$$\begin{aligned} h' &= h'_E + h'_1\lambda_{<ij>}\lambda_{<ij>} + h'_2\lambda_{ppi}\lambda_{qqi} + \\ &\quad + h'_3\lambda_{<in>}\lambda_{<nj>}\lambda_{<ji>} + h'_4\lambda_{<kj>}\lambda_{ppk}\lambda_{qqj} + O(4), \\ h'_k &= \varphi_1\lambda_{ppk} + \varphi_2\lambda_{<ke>}\lambda_{ppe}. \end{aligned} \tag{1.59}$$

All coefficients h'... e φ... are functions of (1.54). From eq. $(1.48)_1$, by using eqs. (1.54) and (1.55), after soma algebra, we obtain

$$\begin{aligned} \rho &= \frac{\partial h'}{\partial\lambda} = \rho_E - \frac{\partial h'_1}{\partial g/T}\lambda_{<ij>}\lambda_{<ij>} - \frac{h'_2}{\partial g/T}\lambda_{ppi}\lambda_{qqi} + O(3), \\ \frac{1}{3}\rho_{pp} &= \frac{\partial h'}{\partial\lambda_{pp}} = p_E + \frac{2}{3}\frac{\partial h'_1}{\partial 1/T}\lambda_{<ij>}\lambda_{<ij>} + \frac{2}{3}\frac{h'_2}{\partial 1/T}\lambda_{ppi}\lambda_{qqi} + O(3), \\ \rho_{<ij>} &= \frac{\partial h'}{\partial\lambda_{<ij>}} = 2h'_1\lambda_{<ij>} + 3h'_3(\lambda_{<ik>}\lambda_{<kj>} - \frac{1}{3}\lambda_{<ln>}\lambda_{<ln>}\delta_{ij}) + \\ &\quad + h'_4\lambda_{qq<i}\lambda_{j>pp} + O(3), \\ \rho_{ppi} &= \frac{\partial h'}{\partial\lambda_{ppi}} = 2h'_2\lambda_{ppi} + 2h'_4\lambda_{<ik>}\lambda_{kpp} + O(3). \end{aligned} \tag{1.60}$$

From eq. $(1.60)_{1,2}$ we see that the density ρ and the pressure $p = \frac{1}{3}\rho_{pp}$ differ from their values at equilibrium ρ_E and p_E to within second order terms. Thanks to eq. (1.60), eq. (1.50) gives

$$a_{ei} = \frac{1}{\rho}(4h_1' \lambda_{<ei>} + 5p_E \delta_{ei}) + O(2) \tag{1.61}$$

that permits to obtain $(1.48)_2$, and by using $(1.59)_2$, also:

$$\begin{aligned}
&0 = \left(\frac{\partial \varphi_1}{\partial g/T} - 5pT \right) \lambda_{qqk} + \left(\frac{\partial \varphi_2}{\partial g/T} - 10 \frac{p}{\rho} \frac{\partial h_1'}{\partial g/T} - 4h_1' T \right) \lambda_{<kl>} \lambda_{qql} + O(3) \\
&0 = \rho_{<rsk>} + O(2) \\
&0 = \left(\frac{\partial \varphi_1}{\partial 1/T} - h_2' + \frac{25}{2} \frac{p^2 T}{\rho} \right) \lambda_{ppk} + \left(\frac{\partial \varphi_2}{\partial 1/T} - h_4' - 10 \frac{p}{\rho} \frac{\partial h_1'}{\partial 1/T} \right) \lambda_{<lk>} \lambda_{ppl} - \\
&-10 h_1' \frac{pT}{\rho} \lambda_{<lk>} \lambda_{ppl} + O(3) \\
&0 = \left(\varphi_2 - \frac{4}{5} h_2' - 10 \frac{p}{\rho} h_1' \right) \cdot \left(\lambda_{qqs} \delta_{kr} + \lambda_{qqr} \delta_{ks} - \frac{2}{3} \lambda_{qqs} \delta_{rs} \right) + O(2) \\
&\rho_{ppnk} = \varphi_1 \delta_{nk} + \varphi_2 \lambda_{<nk>} + O(2).
\end{aligned} \tag{1.62}$$

Thanks to (1.61) the trace of (1.49) is

$$2\lambda_{<kj>} \frac{\partial h'}{\partial \lambda_{<kj>}} + 2\lambda_{ll} \frac{\partial h'}{\partial \lambda_{ll}} + 3\lambda_{qqi} \frac{\partial h'}{\partial \lambda_{<ill>}} = -3h', \tag{1.63}$$

that is a partial linear differential equation. To solve it we do a change of unknown function (from h' to H) and of independent variables, such that

$$h' = \lambda_{ll}^{-\frac{3}{2}} H(\lambda, \lambda_{ll}^{-1} \lambda_{<kj>}, \lambda_{ll}^{-\frac{3}{2}} \lambda_{ill}).$$

Substituting it into eq. (1.63), we obtain $\frac{\partial H}{\partial \lambda_{ll}} = 0$. So, the general solution of (1.63) is

$$h' = \lambda_{ll}^{-\frac{3}{2}} H(\lambda, \lambda_{ll}^{-1} \lambda_{<kj>}, \lambda_{ll}^{-\frac{3}{2}} \lambda_{ill}). \tag{1.64}$$

At equilibrium the above equation is certainly satisfied by our results, because conditions have yet been imposed at equilibrium, although with other variables. It remains to impose it far from equilibrium, but before it, let's consider the traceless part of (1.49), i.e.

$$\begin{aligned}
0 = {} & 2(h_1' + pT)\lambda_{<tk>} + (4h_1' T + 3h_3') \cdot \left(\lambda_{<tj>} \lambda_{<jk>} - \frac{1}{3} \lambda_{<mn>} \lambda_{<mn>} \delta_{tk} \right) + \\
& + \left(\frac{18}{5} h_2' T + h_4' \right) \lambda_{qq<k} \lambda_{t>pp} + O(3).
\end{aligned}$$

from which we have

$$h_1' = -pT, \qquad h_3' = \frac{4}{3}pT^2 \qquad \text{and} \qquad h_4' = -\frac{18}{5}h_2'T. \tag{1.65}$$

From these and $(1.62)_{1,3}$, we have, for the derivatives of the variables of φ_1 and φ_2

$$\begin{aligned} \frac{\partial\varphi_1}{\partial g/T} &= 5pT, \qquad \frac{\partial\varphi_1}{\partial 1/T} = h_2' - \frac{25}{2}\frac{p^2T}{\rho}, \\ \frac{\partial\varphi_2}{\partial g/T} &= -14pT^2, \qquad \frac{\partial\varphi_2}{\partial 1/T} = h_4' + 45\frac{p^2T^2}{\rho}. \end{aligned} \tag{1.66}$$

while, from $(1.62)_4$ and $(1.65)_3$ we obtain

$$h_2' = \frac{5}{4}\left(\varphi_2 + 10\frac{p^2T}{\rho}\right) \qquad \text{and} \qquad h_4' = -\frac{9}{2}\left(\varphi_2 + 10\frac{p^2T}{\rho}\right)T. \tag{1.67}$$

By eliminating h_2' and h_4' from (1.66) and by using (1.67) it is possible to obtain

$$\frac{\partial\varphi_1}{\partial g/T} = -\frac{5}{14}\frac{\partial\varphi_2/T}{\partial g/T}, \qquad \frac{\partial\varphi_1}{\partial 1/T} = -\frac{5}{14}\frac{\partial\varphi_2/T}{\partial 1/T},$$

that can be integrated giving

$$\varphi_1 = -\frac{5}{14}\frac{\varphi_2}{T} + C.$$

On the other hand we have, from (1.66) and (1.57)

$$\begin{aligned} \frac{\partial\varphi_1}{\partial g/T} &= 5T^{\frac{7}{2}}F(z), \\ \frac{\partial\varphi_2}{\partial g/T} &= -14T^{\frac{9}{2}}F(z) \qquad \text{or by (1.58)} \\ \frac{\partial\varphi_1}{\partial z} &= 5T^{\frac{7}{2}}\frac{FF'}{z}, \\ \frac{\partial\varphi_1}{\partial z} &= -14T^{\frac{9}{2}}\frac{FF'}{z}. \end{aligned} \tag{1.68}$$

By integrating we have

$$\begin{aligned} \varphi_1 &= 5T^{\frac{7}{2}}\left(\int \frac{FF'}{z}dz + c\right) + C, \\ \varphi_2 &= -14T^{\frac{9}{2}}\left(\int \frac{FF'}{z}dz + c\right). \end{aligned} \tag{1.69}$$

In this way all coefficients of h' are known from eq. (1.65), (1.67) and (1.58) thanks to (1.68)

$$
\begin{aligned}
h'_1 &= -T^{\frac{7}{2}}F(z), \\
h'_2 &= T^{\frac{9}{2}}\left[\frac{25}{2}\frac{F^2(z)}{z} - \frac{35}{2}\left(\int \frac{FF'}{z}dz + c\right)\right], \\
h'_3 &= \frac{4}{3}T^{\frac{9}{2}}F(z), \\
h'_4 &= T^{\frac{11}{2}}\left[-45\frac{F^2(z)}{z} + 63\left(\int \frac{FF'}{z}dz + c\right)\right].
\end{aligned}
\tag{1.70}
$$

It is easy to verify that these functions satisfy automatically condition (1.64). Let's notice that all the coefficients of h'... and φ... that determine the potential are known except for the function F(z), that is known explicitly in particular cases.

1.4.6 Determination of the constitutive functions.

For the constitutive functions appearing in the balance equations we have

$$
\begin{aligned}
\rho_{<rsk>} &= O(2), \\
\rho_{ppnk} &= \left[5T^{\frac{7}{2}}\left(\int \frac{FF'}{z}dz + c\right) + C\right]\delta_{nk} - \\
&\quad - 14T^{\frac{9}{2}}\left(\int \frac{FF'}{z}dz + c\right)\lambda_{<nk>} + O(2), \\
S_{<ij>} &= s\lambda_{<ij>} + O(2), \\
S_{ppi} &= t\lambda_{ppi} + O(2).
\end{aligned}
\tag{1.71}
$$

We can use eqs. (1.60), (1.70) and (1.65) to substitute in (1.71) the Lagrange multipliers with the starting variables $\rho_{<ij>}$ and ρ_{ppi}, because, until the second order terms O(2),

$$
\begin{aligned}
\lambda_{<ij>} &= -\frac{1}{2pT}\rho_{<ij>} - \frac{1}{2T^{\frac{7}{2}}F(z)}\rho_{<ij>}, \\
\lambda_{ppi} &= \frac{1}{2h'_2}\rho_{ppi} = \frac{1}{5T^{\frac{9}{2}}\left[5\frac{F^2}{z} - 7\left(\int \frac{FF'}{z}dz + c\right)\right]}\rho_{ppi}.
\end{aligned}
$$

So the constitutive relations (1.71) can be written as

$$\rho_{<ijk>} = O(2),$$
$$\rho_{ppnk} = \left[5T^{\frac{7}{2}}\left(\int \frac{FF'}{z}dz + c\right) + C\right]\delta_{ij} + 7\frac{T}{F}\left(\int \frac{FF'}{z}dz + c\right)\rho_{<ij>} + O(2),$$
$$S_{<ij>} = -\frac{s}{2T^{\frac{7}{2}}F(Z)}\rho_{<ij>} + O(2),$$
$$S_{ppi} = \frac{t}{5T^{\frac{9}{2}}[5\frac{F^2(z)}{z} - 7\left(\int \frac{FF'}{z}dz + c\right)]}\rho_{ppi} + O(2). \tag{1.72}$$

For the entropy flux and density we recall that, from eq. (1.47) we have

$$\begin{aligned} h' &= -h' + \lambda\rho + \lambda_{ij}\rho_{ij} + \lambda_{qqi}\rho_{ppi}, \\ h'_k &= -\varphi'_k + \lambda_{ij}\rho_{ijk} + \lambda_{pi}d\rho_{ppik}. \end{aligned}$$

Thanks to (1.59), (1.57), to the coefficients (1.69) and (1.70), we have until the second order terms,

$$\begin{aligned} h &= \rho\left[\frac{3}{2}\int\left(\frac{F'}{z} - \frac{5}{3}\frac{F}{z^2}\right)dz + \zeta\right] - \frac{\rho_{<ij>}\rho_{<ij>}}{4T^{\frac{7}{2}}F(z)} - \\ &\quad - \frac{\rho_{ppi}\rho_{qqi}}{10T^{\frac{7}{2}}F(z)[-5TF/z + 7T/F(z)\left(\int \frac{FF'}{z}dz + c\right)]}, \\ \varphi_k &= \frac{1}{T}\frac{1}{2}\rho_{ppk} - \frac{2}{5}\frac{1}{T^{\frac{7}{2}}F(z)}\rho_{<kj>}\frac{1}{2}\rho_{ppj}. \end{aligned} \tag{1.73}$$

Eqs. (1.72) and (1.73) are the final results obtained from the entropy principle. We conclude noticing that $\rho_{<ijk>}$, ρ_{ppnk}, h and φ_k are fully determined except for the constants c, C and ζ, as we know the function F(z). This determines the thermal equation of state and can be known from experiments or from statistical mechanics. These results have been obtained for the first time by Liu & Müller in [13].

1.5 The kinetic approach

Kinetic theory allow us to find the following solution for our problem:

$$\begin{aligned} h'^0 &= \int F\left(\sum_{n=0}^{N}\lambda_{i_1\ldots i_n}c^{i_1}\ldots c^{i_n}\right)d\mathbf{c}, \\ h'^i &= \int F\left(\sum_{n=0}^{N}\lambda_{i_1\ldots i_n}c^{i_1}\ldots c^{i_n}\right)c^i d\mathbf{c}, \end{aligned}$$

that depends on an single variable arbitrary function F(x). This quantities satisfy all the property that a solution must satisfy. By deriving the relations before, we have

$$\frac{\partial h'^0}{\partial \lambda_{i_1 \ldots i_n}} = \int F' c^{i_1} \ldots c^{i_n} d\mathbf{c} \quad = \quad \frac{\partial h'^{i_n}}{\partial \lambda_{i_1 \ldots i_{n-1}}}$$

that coincides with eq. (1.9).
Recall that, in the kinetic theory, moments are defined as

$$F_{i_1 \ldots i_n} = \int m c_{i_1} \cdots c_{i_n} f(\mathbf{x}, \mathbf{c}, t) d\mathbf{c},$$

where the distribution function $f(\mathbf{x}, \mathbf{c}, t)$ describes the density of molecules in the point $\mathbf{x}$, at time t with velocity $\mathbf{c}$.
Furthermore they satisfy the condition that the flux in a balance equation must be the independent variable in the following equation, as in (1.3).
Concerning the concavity of h'^0 we can notice that the first element of the quadratic form $Q = \delta \mathbf{\Lambda} \delta \mathbf{u}$ is

$$Q = \int F'' \cdot \left(\sum_{n=0}^{N} \lambda_{i_1 \ldots i_n} c^{i_1} \ldots c^{i_n} \right)^2 d\mathbf{c}$$

that is negative defined, as we assume $F''(x) < 0$.
Even wrote in such way our quantities respect the galilean invariance but we will not report the proof because it is present in literature and is not the matter of this work.

1.6 The subsystems

Sometimes to describe a physical situation a lot of variables are necessary, for example when rapid changes in its characteristics or step gradients occur. Instead, where conditions are smoother it can be described sufficiently well by using less variables and equations.
Recently Boillat & Ruggeri [14] found important results on subsystems obtained after reducing the number of equations, as the fact that subsystems inherit symmetry, hyperbolicity and the fact that the domain of the characteristic speeds of the subsystem is included in that of the main system.
Let's suppose that some between the n components of mean field, that of the Lagrange multipliers $\mathbf{\Lambda}(x^D)$, be constant. In this way a corresponding

number of balance equations is too big. Let's see how to eliminate the superfluous equations.
Remember that the main system have the following shape

$$\left(\frac{\partial h'^A}{\partial \mathbf{\Lambda}}\right)_A = \mathbf{\Pi} \tag{1.74}$$

(remeber eqs. (1.3) and (1.9)).
It is not restrictive to suppose that the vector $\mathbf{\Lambda}$, that have n components, can be divided into $\mathbf{l}$ with (n-m) components and $\mathbf{L}$ with m components. Furthermore it is non restrictive to divide the vector production $\mathbf{\Pi}$, that have n components, into $\mathbf{p}$ with (n-m) components and $\mathbf{P}$ with m components. So even the system (1.74) can be divided into two partial systems:

$$\left(\frac{\partial h'^A(\mathbf{L}, \mathbf{l})}{\partial \mathbf{L}}\right)_{,A} = \mathbf{P}(\mathbf{L}, \mathbf{l}) \qquad \text{and} \qquad \left(\frac{\partial h'^A(\mathbf{L}, \mathbf{l})}{\partial \mathbf{l}}\right)_{,A} = \mathbf{p}(\mathbf{L}, \mathbf{l}). \tag{1.75}$$

Obviously there are $2^n - 2$ possibilities for such decomposition if we exclude the trivial cases m=0 e m=n.
Let's suppose now that the m-n components of $\mathbf{l}$ are constant, i.e. $\mathbf{l} = \mathbf{l}^* = const$. So we can ignore system $(1.75)_2$ and we consider only system $(1.75)_1$ to determine $\mathbf{L}$. Let's write it as follows:

$$\frac{\partial^2 h'^A(\mathbf{L}, \mathbf{l}^*)}{\partial \mathbf{L} \partial \mathbf{L}} \cdot \mathbf{L}_{,A} = \mathbf{P}(\mathbf{L}, \mathbf{l}^*) \tag{1.76}$$

and call it "main subsystem". The field components survived to the restriction are called "main field components". The definition of main subsystem can be generalized to the case where the vector $\mathbf{l}^*$ isn't constant but it is function of x^D. In such case the subsystem depends explicitly on x^D; see Boillat & Ruggeri [14].
The symmetric system (1.76) is also hyperbolic; in fact the quadratic form

$$Q = \frac{\partial^2 h'^0}{\partial \mathbf{\Lambda} \partial \mathbf{\Lambda}} \delta \mathbf{\Lambda} \delta \mathbf{\Lambda}$$

was negative defined. But it can be written as

$$Q = \frac{\partial^2 h'^0}{\partial \mathbf{L} \partial \mathbf{L}} \delta \mathbf{L} \delta \mathbf{L} + 2 \frac{\partial^2 h'^0}{\partial \mathbf{L} \partial \mathbf{l}} \delta \mathbf{L} \delta \mathbf{l} + \frac{\partial^2 h'^0}{\partial \mathbf{l} \partial \mathbf{l}} \delta \mathbf{l} \delta \mathbf{l}$$

and $Q < 0$ every time that $\delta \mathbf{L}$ and $\delta \mathbf{l}$ aren't all equal to zero.
In particular, for $\delta \mathbf{l} = 0$ we have

$$Q = \frac{\partial^2 h'^0}{\partial \mathbf{L} \partial \mathbf{L}} \delta \mathbf{L} \delta \mathbf{L}$$

negative defined. Furthermore we saw that h'^A restricted to the subsystem is $h'^A(\mathbf{L}, \mathbf{l}^*)$, i.e the restriction of h'^A to l^*. The same thing happens for the entropy density h^A; in fact eq. (1.10) applied to the subsystem, i.e. with $\mathbf{L}$ instead of $\mathbf{\Lambda}$, gives

$$h^A = -h'^A + \mathbf{L}\frac{\partial h'^A}{\partial \mathbf{L}} \tag{1.77}$$

while, if we apply it to the whole system and then we calculate it in $\mathbf{l} = \mathbf{l}^*$, we find

$$h^A(\mathbf{L}, \mathbf{l}^*) = -h'^A + \mathbf{L}\frac{\partial h'^A}{\partial \mathbf{L}} + \mathbf{l}^*\frac{\partial h'^A}{\partial \mathbf{l}}_{|l=l^*}.$$

Exploiting h'^A and putting that expression into eq. (1.77) we find

$$h^A = h^A(\mathbf{L}, \mathbf{l}^*) - l^*\frac{\partial h'^A}{\partial \mathbf{l}}_{|l=l^*}.$$

It follows that the solution of the main subsystem (1.76) satisfy a balance equation like $\overline{h}^A_{,A} = \overline{\Sigma}$, with

$$\begin{aligned} \overline{h}^A(\mathbf{L}) &= h^A(\mathbf{L}, \mathbf{l}^*) - \mathbf{l}^* \cdot \frac{\partial h'^A(\mathbf{L}, \mathbf{l}^*)}{\partial \mathbf{l}} |_{\mathbf{l}=\mathbf{l}^*}, \\ \text{and} \qquad \overline{\Sigma}(\mathbf{L}) &= \mathbf{L} \cdot \mathbf{P}(\mathbf{L}, \mathbf{l}^*). \end{aligned}$$

That's why we call, $\overline{h}^A$ and $\overline{\Sigma}$ "sub-entropy flux" and "sub-entropy production" respectively.
Moreover it is not restrictive to suppose that the sub-entropy production is non negative. In such a way we will have the entropy inequality for the subsystem.
Let's prove now that system (1.76) is symmetric and hyperbolic.
If we consider the subsystem (1.76) we can see that the symmetry of coefficients matrices is guaranteed by their Hessian character. The concavity of $h'^0(\mathbf{\Lambda})$ with respect to $\mathbf{\Lambda}$ imply the concavity of $h'^0(\mathbf{L}, \mathbf{l}^*)$ with respect to $\mathbf{L}$. And, since $h'^0(\mathbf{L}, \mathbf{l}^*)$ is concave in $\mathbf{L}$, so is the sub-entropy $\overline{h}^0$ with respect to $\frac{\partial h'^0(\mathbf{L}, \mathbf{l}^*)}{\partial \mathbf{L}}$. In order that the subsystem inherits symmetry, hyperbolicity and all the other properties it is necessary that the subsystem is a main one, i.e. the constraints must be on the Lagrange multipliers. Boillat & Ruggeri gave in [14] an example in which the hyperbolicity was lost when a component of the field vector $\mathbf{u}$ rather then $\mathbf{\Lambda}$ was constrained.
The requirement of Galilean invariance permits to generate a particular order for the n equations of the main system and a ceratin simplification of the

matrix $\mathbf{X}(\mathbf{v})$ that describes the change of components between two different frames. Without reporting all the noisy calculations we can say only that, after ordering the system we can eliminate one equation after the other starting from the end of the system and requiring at each step that the residual system is galilean invariant. In such a way we obtain a matrix $\mathbf{X}$ lower triangular, in which all the entries up on the main diagonal are zero.

So we have a good criteria to order the system: move the equations in a way that the last equation/s can be removed without loosing the galilean invariance for the remaining system.

This methodology will be used in the following chapters to obtain, for example, the 13 moments case as a subsystem of the 14 moments one, or the subsystem with N-1 fields instead of N for the case with an arbitrary but fixed number of variables.

Chapter 2

The new methodology

In the previous chapter we have seen how the closure of the system of balance equations can be obtained by imposing the entropy principle and that of Galilean relativity.

By using the mean field as independent variables, the entropy principle is equivalent to some conditions on entropy density h and entropy flux ϕ_k; after that, it gives the constitutive functions. But when we impose the objectivity principle too, the calculations become much complicated because the non convective components of the mean field aren't independent (in particular λ_i^I).

Here we want to show how the new methodology proposed by Pennisi and Ruggeri in [15] enable to avoid these difficulties by considering all the Lagrange multipliers as independent variables. After that, they substitute in the results the value of λ_i^I implicitly defined by considering the derivative of h' with respect to λ_i^I equal to zero and they arrive to the same results obtained with the other methodology but with easier calculations.

2.1 Comparison between different methods

Let's consider the case with an arbitrary but fixed number of moments, whose appropriate equations are:

$$\partial_t F^A + \partial_k F^{Ak} = P^A. \tag{2.1}$$

After the insertion of the Lagrange multipliers, the equations describing the entropy principle are:

$$dh = \lambda_A dF^A \quad ; \quad d\phi^k = \lambda_A dF^{Ak} \quad ; \quad \lambda_A P^A \geq 0. \tag{2.2}$$

A stands for $i_1 \cdots i_n$ and n goes from 0 to a fixed N.
Let us compare now 3 ways to impose the entropy principle and that of material objectivity. We shall refer to the following table.

Entropy principle	Material Objectivity
$\begin{cases} F^A = \frac{\partial h'}{\partial \lambda_A} \\ F^{Ak} = \frac{\partial \phi'^k}{\partial \lambda_A} \end{cases}$	$\begin{cases} \lambda_A \, A^{rA}{}_C \, F^C = 0 \\ \lambda_A \, A^{rA}{}_C \, F^{Ck} + h' \delta^{kr} = 0 \end{cases}$
$\begin{cases} m^A = \frac{\partial \hat{h}'}{\partial \hat{\lambda}_A} \\ \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_A} = m^{Ak} + m^{ik} \frac{\partial \hat{\lambda}_i}{\partial \hat{\lambda}_A} \quad \text{with } A \neq i_1 \end{cases}$	$\begin{cases} \hat{\lambda}_A \, A^{rA}{}_C \, m^C = 0 \\ \hat{\lambda}_A \, A^{rA}{}_C \, m^{Ck} + h' \delta^{kr} = 0 \end{cases}$
$\begin{cases} \frac{\partial H}{\partial \hat{\lambda}_{Ak}} = \frac{\partial H^k}{\partial \hat{\lambda}_A} \quad \text{with } n = 0, \cdots, N-1 \\ \text{Symmetry of } \frac{\partial H^k}{\partial \hat{\lambda}_{i_1 \cdots i_N}} \\ \frac{\partial H}{\partial \hat{\lambda}_i} = 0 \text{ defines } \hat{\lambda}_i = \hat{\lambda}_i(\hat{\lambda}, \hat{\lambda}_{i_1 i_2}, \cdots, \hat{\lambda}_{i_1 \cdots i_N}) \\ \hat{h}' \text{ and } \hat{\phi}'^k \text{ are H and } H^k \text{ calculated in} \\ \text{the above value of the function } \hat{\lambda}_i. \end{cases}$	$\begin{cases} \hat{\lambda}_A \, A^{rA}{}_C \, \frac{\partial H}{\partial \hat{\lambda}_C} = 0 \\ \hat{\lambda}_A \, A^{rA}{}_C \, \frac{\partial H^k}{\partial \hat{\lambda}_C} + H \delta^{kr} = 0 \end{cases}$

Let us consider firstly the second row. It is possible to change independent variables, from F^A to m^A, v^i with the law

$$F^A = X^A{}_B(\vec{v}) m^B \quad , \quad m^i = 0 \, ,$$

where $X^A{}_B(\vec{v})$ is the matrix (1.31).
The material objectivity imposes that h, $\phi^k - hv^k$ and $m^{i_1 \cdots i_N k}$ are tensorial functions of m^B and don't depend on v_i. Now, by defining $\hat{\lambda}_A = X^B{}_A \lambda_B$ and using the property

$$\frac{\partial X^A{}_B}{\partial v_r} = A^{rB}{}_C X^C{}_A = X^B{}_C A^{rC}{}_A \, , \tag{2.3}$$

the two equations in the second column of the table express the fact that h and $\phi^k - hv^k$ don't depend on v_i. If we change again independent variables, from $\vec{v}$, m^B to $\vec{v}$, $\hat{\lambda}_A$ (except for $\hat{\lambda}_i$), what remains of the entropy principle is expressed in the first column, with

$$\hat{h}' = \hat{\lambda}_A m^A - h \quad \text{and} \quad \hat{\phi}'^k = \hat{\lambda}_A m^{Ak} - (\phi^k - hv^k) \, .$$

From the second of these it follows that $m^{i_1 \cdots i_N k}$ doesn't depend on v_i, so we don't have to impose it.

Let consider now the first row. If we change independent variables, from F^A to λ_A, the entropy principle becomes expressed by the 2 equations in the first column, with

$$h' = \lambda_A F^A - h \quad \text{and} \quad \phi'^k = \lambda_A F^{Ak} - \phi'^k \,.$$

Moreover, as consequence of eq. (2.3), the eqs. in the second row, second column of the table can be rewritten also in the form in the first row, second column.

Therefore, the first and second row of the table are two equivalent ways to impose the same conditions, because from the first row immediately follows the second one, also the fact that h, $\phi^k - hv^k$ and $m^{i_1\cdots i_N k}$ don't depend on v_i, when we take v^i and m^B as independent variables.

Now let's explain what the third row contains. Let H and H^k be functions of all $\hat{\lambda}_A$ (included $\hat{\lambda}_i$) satisfying the conditions of column 1 and 2, with H a convex function. After that, let us define the function $\hat{\lambda}_i = \hat{\lambda}_i(\hat{\lambda}\,,\, \hat{\lambda}_{i_1 i_2}\, \cdots\,,\, \hat{\lambda}_{i_1\cdots i_N})$ from $\frac{\partial H}{\partial \hat{\lambda}_i} = 0$; well, $\hat{\lambda}_i$ and the functions H and H^k calculated on such expression of $\hat{\lambda}_i$ satisfy exactly the conditions in row 2 of the table, which restrict $\hat{\lambda}_i$, $\hat{h}'$ and $\hat{\phi}'^k$.

In fact, by using the chain rule for the derivation of composite functions, we have:

$$\frac{\partial \hat{h}'}{\partial \hat{\lambda}_A} = \frac{\partial H}{\partial \hat{\lambda}_A} + \overbrace{\frac{\partial H}{\partial \hat{\lambda}_i}}^{0} \cdot \frac{\partial \hat{\lambda}_i}{\partial \hat{\lambda}_A} = m^A.$$

The equation above defines m^A, and it is equivalent to the first equation in the second row and first column of the table. With the same procedure it is possible to prove that the first equation in the second column is equivalent to the respective in the second row.

Let's consider the second equation in row 2, column 1, distinguishing the cases with $n = 2, \cdots, N-1$, $n = 0$, $n = N$, i.e., respectively

$$\begin{aligned} \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_A} &= \frac{\partial H^k}{\partial \hat{\lambda}_A} + \frac{\partial H^k}{\partial \hat{\lambda}_i} \cdot \frac{\partial \hat{\lambda}_i}{\partial \hat{\lambda}_A} = \\ &= \frac{\partial H}{\partial \hat{\lambda}_{Ak}} + \frac{\partial H}{\partial \hat{\lambda}_{ik}} \cdot \frac{\partial \hat{\lambda}_i}{\partial \hat{\lambda}_A} = m^{Ak} + m^{ik} \frac{\partial \hat{\lambda}_i}{\partial \hat{\lambda}_A}\,; \end{aligned}$$

$$\begin{aligned} \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}} &= \frac{\partial H^k}{\partial \hat{\lambda}} + \frac{\partial H^k}{\partial \hat{\lambda}_i} \cdot \frac{\partial \hat{\lambda}_i}{\partial \hat{\lambda}} = \\ &= \frac{\partial \not{H}}{\partial \hat{\lambda}_k} + \frac{\partial H}{\partial \hat{\lambda}_{ik}} \cdot \frac{\partial \hat{\lambda}_i}{\partial \hat{\lambda}} = \not{m}^k + m^{ik} \frac{\partial \hat{\lambda}_i}{\partial \hat{\lambda}}\,; \end{aligned}$$

$$\begin{aligned}\frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_A} &= \frac{\partial H^k}{\partial \hat{\lambda}_A} + \frac{\partial H^k}{\partial \hat{\lambda}_i} \cdot \frac{\partial \hat{\lambda}_i}{\partial \hat{\lambda}_A} = \\ &= \frac{\partial H^k}{\partial \hat{\lambda}_A} + \frac{\partial H}{\partial \hat{\lambda}_{ik}} \cdot \frac{\partial \hat{\lambda}_i}{\partial \hat{\lambda}_A} = \frac{\partial H^k}{\partial \hat{\lambda}_A} + m^{ik} \frac{\partial \hat{\lambda}_i}{\partial \hat{\lambda}_A} ;\end{aligned}$$

So the second equation in row 2, column 1 is proved with

$$m^{i_1 i_2 \cdots i_N k} = \frac{\partial H^k}{\partial \hat{\lambda}_{i_1 i_2 \cdots 1_N}}$$

which is symmetric, as requested.
It remains now to prove the second equation in row 2, column 2 of the table. It follows from

$$\frac{\partial H^k}{\partial \hat{\lambda}_C} = \frac{\partial H}{\partial \hat{\lambda}_{Ck}} = m^{Ck}$$

as it can be seen from the first equations in row 2 and 3, column 1, and from the fact that

$$A^{rA}{}_C = 0 \quad \text{for}\, C = i_1 i_2 \cdots i_N .$$

This follows from

$$X^A_{i_1 i_2 \cdots i_N} = \begin{cases} 0 & \text{if} \quad A \neq i_1 i_2 \cdots i_N \\ 1 & \text{if} \quad A = i_1 i_2 \cdots i_N \end{cases} \quad \text{and}\, A^{rA}_B = \left(\frac{\partial X^A_B}{\partial v_r} \right)_{v_s = 0} ,$$

so that

$$A^{rA}_{i_1 i_2 \cdots i_N} = \left(\frac{\partial X^A_{i_1 i_2 \cdots i_N}}{\partial v_r} \right)_{v_s = 0} = 0 \, .$$

We note that the first 4 equations in row 3 of the table, are nothing more than the compatibility conditions between the eqs. in the row 1, except that now H, H^k, m^B and $\hat{\lambda}_A$ replace h', φ'^k, F^A and λ_A respectively.
We have proved that the three different methods are equivalent, so they leads to the same result: they unable us to find the expressions of the potentials h' and φ'^k.
After that, from

$$F^A = \frac{\partial h'}{\partial \lambda_A} \tag{2.4}$$

we can find $\lambda_A = \lambda_A(F^B)$, and then these are substituted in h', ϕ'^k and $F^{i_1 \cdots i_N k}$ to find the constitutive functions

$$\begin{aligned} h' &= h' \left[\lambda_A \left(F^B \right) \right] , \\ \phi'^k &= \phi'^k \left[\lambda_A \left(F^B \right) \right] , \\ F^{i_1 \cdots i_N k} &= F^{i_1 \cdots i_N k} \left[\lambda_A \left(F^B \right) \right] . \end{aligned}$$

Substituting $F^B = X^B{}_C(\vec{v})m^C$, in these eqs., they become

$$\begin{aligned} h' &= h'\left\{\lambda_A\left[F^B\left(m^C,\vec{v}\right)\right]\right\}\,, \\ \phi'^k &= \phi'^k\left\{\lambda_A\left[F^B\left(m^C,\vec{v}\right)\right]\right\}\,, \\ F^{i_1\cdots i_N k} &= F^{i_1\cdots i_N k}\left\{\lambda_A\left[F^B\left(m^C,\vec{v}\right)\right]\right\}\,. \end{aligned} \tag{2.5}$$

We want to calculate these in $\vec{v}=\vec{0}$; but, firstly, let us note that

$$\begin{aligned} h' &= \lambda_A F^A - h = \hat{\lambda}_A m^A - h = \hat{h}' \\ \phi'^k &= \lambda_A F^{Ak} - \phi^k = \lambda_A\left(F^A v^k + X^A{}_B\, m^{Bk}\right) - \phi^k = \\ &= \hat{\lambda}_A m^A v^k + \hat{\lambda}_A m^{Ak} - \phi^k = \\ &= \hat{\lambda}_A m^A v^k + \hat{\phi}'^k - h v^k = \hat{h}' v^k + \hat{\phi}'^k\,; \end{aligned}$$

from which it follows that h', ϕ'^k, F^B and $F^{i_1\cdots i_N k}$ calculated in $\vec{v}=\vec{0}$ become $\hat{h}'$, $\hat{\phi}'^k$, m^B and $m^{i_1\cdots i_N k}$ respectively. Therefore, eqs. (2.5) calculated in $\vec{v}=\vec{0}$ become

$$\begin{aligned} \hat{h}' &= h'\left\{\lambda_A\left[m^B\right]\right\}\,, \\ \hat{\phi}'^k &= \phi'^k\left\{\lambda_A\left[m^B\right]\right\}\,, \\ m^{i_1\cdots i_N k} &= m^{i_1\cdots i_N k}\left\{\lambda_A\left[m^B\right]\right\} \end{aligned}$$

and the functions $\lambda_A\left[m^B\right]$ are the solutions of $(2.4)_1$ i.e., $F^A = \frac{\partial h'}{\partial \lambda_A}$, but calculated in $\vec{v}=\vec{0}$; in other words they have to be obtained from

$$m^A = \frac{\partial h'}{\partial \lambda_A} \quad \text{with} \quad m^i = 0\,,$$

so obtaining the last 2 propositions in row 3 of the table.

- It is important to note that with this method, what we have done for eqs. (1), can be done also for whatever of its subsystems.
- We see also that we can consider directly the problem

$$\begin{cases} m^A = \frac{\partial \hat{h}'}{\partial \hat{\lambda}_A} \\ \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_A} = m^{Ak} \end{cases} \qquad \begin{cases} \hat{\lambda}_A\, A^{rA}{}_C\, m^C = 0\,; \\ \hat{\lambda}_A\, A^{rA}{}_C\, m^{Ck} + h'\delta^{kr} = 0\,; \end{cases}$$

in all the independent variables $\hat{\lambda}_A$, and with all its compatibility conditions, but $m^i = 0$ doesn't mean that $\hat{h}'$ doesn't depend on $\hat{\lambda}_i$; it is simply the relation which will give, in the next step, $\hat{\lambda}_i$ in terms of the other $\hat{\lambda}_A$. Similarly, $m^i = 0$ doesn't mean that $\hat{\phi}'^k$ doesn't depend on $\hat{\lambda}$; it has to be considered simply as an equivalent way to consider the condition $\frac{\partial \hat{h}'}{\partial \hat{\lambda}_i} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}}$.

2.2 The 13 moments case

Let's see the application of the new methodology to the 13 moments case, described by the system (1.34).
In order to exploit the entropy principle and that of material objectivity with the present approach, we have to consider the eqs.

$$m = \frac{\partial \hat{h}'}{\partial \hat{\lambda}}, \qquad m^i = \frac{\partial \hat{h}'}{\partial \hat{\lambda}_i}, \qquad m^{ij} = \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ij}}, \qquad m^{ill} = \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ill}}, \tag{2.6}$$

$$m^k = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}}, \qquad m^{ki} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_i}, \qquad m^{kij} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{ij}}, \qquad m^{kll} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{ij}}\delta_{ij},$$

$$\begin{aligned} 0 &= \hat{\lambda}_j \frac{\partial \hat{h}'}{\partial \hat{\lambda}} + 2\hat{\lambda}_{ij}\frac{\partial \hat{h}'}{\partial \hat{\lambda}_i} + \hat{\lambda}_{ill}\left(2\frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ij}} + \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{rs}}\delta_{rs}\delta^i_j\right), \\ 0 &= \hat{\lambda}_j \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}} + 2\hat{\lambda}_{ij}\frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_i} + \hat{\lambda}_{ill}\left(2\frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{ij}} + \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{rs}}\delta_{rs}\delta^i_j\right) + \hat{h}'\delta^k_j\,. \end{aligned} \tag{2.7}$$

So we have to consider the eqs. (2.7) and the compatibility conditions between eqs. (2.6), i.e.,

$$\frac{\partial \hat{h}'}{\partial \hat{\lambda}_i} = \frac{\partial \hat{\phi}'^i}{\partial \hat{\lambda}}, \qquad \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ij}} = \frac{\partial \hat{\phi}'^i}{\partial \hat{\lambda}_j}, \qquad \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ill}} = \frac{\partial \hat{\phi}'^i}{\partial \hat{\lambda}_{rs}}\delta_{rs}, \tag{2.8}$$
$$\frac{\partial \hat{\phi}'^{[i}}{\partial \hat{\lambda}_{j]k}} = 0, \qquad \frac{\partial \hat{\phi}'^{[i}}{\partial \hat{\lambda}_{j]ll}} = 0.$$

To impose all these conditions, let us consider the Taylor's expansion of $\hat{\phi}'^i$ around the state of equilibrium, where $\hat{\lambda}_i = 0$, $\hat{\lambda}_{ill} = 0$, $\hat{\lambda}_{jk} = \frac{1}{3}\hat{\lambda}_{ll}\delta_{jk}$, which we will indicate with the index "c"; we find

$$\begin{aligned} \hat{\phi}'^i = & \sum_{n=1}^{\infty}\sum_{p+q+r=n} \frac{1}{p!q!r!}\phi^{ii_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}_{p,q,r}(\hat{\lambda}, \hat{\lambda}_{ll}) \cdot \hat{\lambda}_{i_1}\cdots\hat{\lambda}_{i_p}\hat{\lambda}_{j_1 ll}\cdots\hat{\lambda}_{j_q ll}\cdot \\ & \left(\hat{\lambda}_{k_1h_1} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_1h_1}\right)\cdots\left(\hat{\lambda}_{k_rh_r} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_rh_r}\right), \end{aligned} \tag{2.9}$$

with

$$\phi^{ii_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}_{p,q,r} = \left(\frac{\partial^n \hat{\phi}'^i}{\partial \hat{\lambda}_{i_1}\cdots\partial\hat{\lambda}_{i_p}\partial\hat{\lambda}_{j_1ll}\cdots\partial\hat{\lambda}_{j_qll}\partial\hat{\lambda}_{k_1h_1}\cdots\partial\hat{\lambda}_{k_rh_r}}\right)_c \tag{2.10}$$

because of the symmetry of eq. $(2.8)_2$ we can exchange the index i in the tensor (2.10) with whatever of the indices $i_1 \cdots i_p$; the same thing can be done also with the indexes $j_1 \cdots j_q$ and $k_1 \cdots k_r$ or $h_1 \cdots h_r$, thanks to eqs. $(2.8)_5$ and $(2.8)_4$ respectively. So i can be exchanged with whatever of the indexes and $\phi^{ii_1 \cdots i_p j_1 \cdots j_q k_1 h_1 \cdots k_r h_r}_{p,q,r}$ is a completely symmetric tensor. Moreover it depends only on the scalars $\hat{\lambda}$, $\hat{\lambda}_{ll}$, so it is zero if $p+q+2r+1$ is odd, while it is of the type

$$\phi^{ii_1 \cdots i_p j_1 \cdots j_q k_1 h_1 \cdots k_r h_r}_{p,q,r} = \phi_{p,q,r}(\hat{\lambda}\,,\ \hat{\lambda}_{ll})\delta^{(ii_1} \cdots \delta^{k_r h_r)}\,, \tag{2.11}$$

if $p+q+2r+1$ is even. So it is determined except for scalar functions.
The same result can be achieved also for $\hat{h}'$; its Taylor' s expansion around the equilibrium state "c" is

$$\begin{aligned} \hat{h}' \;=\; & \sum_{n=0}^{\infty} \sum_{p+q+r=n} \frac{1}{p!q!r!} h^{i_1 \cdots i_p j_1 \cdots j_q k_1 h_1 \cdots k_r h_r}_{p,q,r}(\hat{\lambda}\,,\ \hat{\lambda}_{ll}) \hat{\lambda}_{i_1} \cdots \hat{\lambda}_{i_p} \hat{\lambda}_{j_1 ll} \cdots \hat{\lambda}_{j_q ll} \cdot \\ & \left(\hat{\lambda}_{k_1 h_1} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_1 h_1}\right) \cdots \left(\hat{\lambda}_{k_r h_r} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_r h_r}\right)\,, \end{aligned} \tag{2.12}$$

with

$$h^{i_1 \cdots i_p j_1 \cdots j_q k_1 h_1 \cdots k_r h_r}_{p,q,r} = \left(\frac{\partial^n \hat{h}'}{\partial \hat{\lambda}_{i_1} \cdots \partial \hat{\lambda}_{i_p} \partial \hat{\lambda}_{j_1 ll} \cdots \partial \hat{\lambda}_{j_q ll} \partial \hat{\lambda}_{k_1 h_1} \cdots \partial \hat{\lambda}_{k_r h_r}} \right)_c \tag{2.13}$$

The derivatives of eq. $(2.8)_1$ with respect to $\hat{\lambda}_{hk}$ and with respect to $\hat{\lambda}_{kll}$ are

$$\frac{\partial^2 \hat{h}'}{\partial \hat{\lambda}_i \partial \hat{\lambda}_{hk}} = \frac{\partial^2 \hat{\phi}'^i}{\partial \hat{\lambda} \partial \hat{\lambda}_{hk}}\,;\ \frac{\partial^2 \hat{h}'}{\partial \hat{\lambda}_i \partial \hat{\lambda}_{kll}} = \frac{\partial^2 \hat{\phi}'^i}{\partial \hat{\lambda} \partial \hat{\lambda}_{kll}}\,,$$

and the derivative of $(2.8)_2$ with respect to $\hat{\lambda}_{kll}$ is

$$\frac{\partial^2 \hat{h}'}{\partial \hat{\lambda}_{ji} \partial \hat{\lambda}_{kll}} = \frac{\partial^2 \hat{\phi}'^i}{\partial \hat{\lambda}_j \partial \hat{\lambda}_{kll}}\,.$$

Thanks to $(2.8)_{4,5}$, it follows

$$\frac{\partial^2 \hat{h}'}{\partial \hat{\lambda}_{[i} \partial \hat{\lambda}_{h]k}} = 0\,;\ \frac{\partial^2 \hat{h}'}{\partial \hat{\lambda}_{[i} \partial \hat{\lambda}_{k]ll}} = 0\,,\ \frac{\partial^2 \hat{h}'}{\partial \hat{\lambda}_{j[i} \partial \hat{\lambda}_{k]ll}} = 0\,,$$

from which we obtain that $h^{i_1 \cdots i_p j_1 \cdots j_q k_1 h_1 \cdots k_r h_r}_{p,q,r}$ is a symmetric tensor with respect to every couple of indices, and it depends only on scalars, so it is zero if $p+q+2r$ is odd, while it is of the type

$$h^{i_1 \cdots i_p j_1 \cdots j_q k_1 h_1 \cdots k_r h_r}_{p,q,r} = h_{p,q,r}(\hat{\lambda}\,,\ \hat{\lambda}_{ll})\delta^{(i_1 i_2} \cdots \delta^{k_r h_r)}\,, \tag{2.14}$$

if $p+q+2r$ is even. So it is determined except for scalar functions too. These properties don't exhaust the consequences of eqs. (2.7) and eq. (2.8) but it is easier now to impose them.
After some algebra (see Appendix of [15] for details) conditions (2.7) and (2.8) convert into

$$\begin{aligned} \phi_{p,q,r} &= 3^r \frac{p+q+2}{p+q+2r+2} \frac{\partial^r}{\partial \hat{\lambda}_{ll}^r} \phi_{p,q,0}, \\ h_{p,q,r} &= 3^r \frac{p+q+1}{p+q+2r+1} \frac{\partial^r}{\partial \hat{\lambda}_{ll}^r} h_{p,q,0}, \end{aligned} \tag{2.15}$$

so that the unknown $\phi_{p,q,r}$ and $h_{p,q,r}$ are determined in terms of $\phi_{p,q,0}$ and $h_{p,q,0}$. We will determine these last functions by the use of the following infinite matrix

$$\begin{pmatrix} h_{000} & \phi_{010} & h_{020} & \phi_{030} & h_{040} & \phi_{050} & h_{060} & \vdots \\ \phi_{100} & h_{110} & \phi_{120} & h_{130} & \phi_{140} & h_{150} & \phi_{160} & \vdots \\ h_{200} & \phi_{210} & h_{220} & \phi_{230} & h_{240} & \phi_{250} & h_{260} & \vdots \\ \phi_{300} & h_{310} & \phi_{320} & h_{330} & \phi_{340} & h_{350} & \phi_{360} & \vdots \\ h_{400} & \phi_{410} & h_{420} & \phi_{430} & h_{440} & \phi_{450} & h_{460} & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \end{pmatrix} \tag{2.16}$$

In particular, starting from an arbitrary function $\tilde{h}_{0,0}(\hat{\lambda})$, the functions $\tilde{h}_{0,q}(\hat{\lambda})$ (with q even) are obtained from eq.

$$\frac{\partial}{\partial \hat{\lambda}} \tilde{h}_{0,q}(\hat{\lambda}) = -\frac{81}{8} \frac{(q-1)^2}{q+1} (9q^2-1) \tilde{h}_{0,q-2}(\hat{\lambda}) \tag{2.17}$$

except for a family of constants arising from integration. After that, the elements in the first line of the matrix, but in column odd, are determined by

$$h_{0,q,0} = \tilde{h}_{0,q}(\hat{\lambda})(\hat{\lambda}_{ll})^{-\frac{3q+3}{2}} . \tag{2.18}$$

The elements in the first line of the matrix, but in column even, are determined by

$$\phi_{0,j,0} = -\frac{2}{3} \frac{1}{3j+4} \tilde{h}_{0,j+1}(\hat{\lambda})(\hat{\lambda}_{ll})^{-\frac{3j+4}{2}} + c_j , \tag{2.19}$$

where c_j is another family of constants arising from integration. In this way, all the elements in the first line of the matrix are determined. Those of the other lines are expressed as

$$h_{1,j,0} = 9 \frac{j}{j+2} \frac{\partial^2}{\partial \hat{\lambda}_{ll}^2} h_{0,j-1,0} \quad \forall \text{ j odd}, \tag{2.20}$$

$$h_{i,j,0} = 3^{i/2} \frac{j+1}{i+j+1} \frac{\partial^i}{\partial \hat{\lambda}^{i/2} \partial \hat{\lambda}_{ll}^{i/2}} h_{0,j,0} \quad \forall \text{ i and j even },\tag{2.21}$$

$$\phi_{i,j,0} = 3^{(i+1)/2} \frac{j+1}{i+j+2} \frac{\partial^i}{\partial \hat{\lambda}^{(i-1)/2} \partial \hat{\lambda}_{ll}^{(i+1)/2}} h_{0,j,0} \quad \forall \text{ j even and i odd},\tag{2.22}$$

$$h_{i,j,0} = 3^{(i-1)/2} \frac{j+2}{i+j+1} \frac{\partial^{i-1}}{\partial \hat{\lambda}^{(i-1)/2} \partial \hat{\lambda}_{ll}^{(i-1/2}} h_{1,j,0} \quad \forall \text{ j and i odd},\tag{2.23}$$

$$\phi_{i,j,0} = 3^{i/2} \frac{j+2}{i+j+2} \frac{\partial^{i-1}}{\partial \hat{\lambda}^{(i-2)/2} \partial \hat{\lambda}_{ll}^{i/2}} h_{1,j,0} \quad \forall \text{ j odd and i} \geq 2 \text{ even}.\tag{2.24}$$

Now, if we use the method described in the last 2 lines of row 3 of the table, we find the solution of the problem of row 2. Truncating it up to fourth order with respect to equilibrium, we find the results already present in literature [13], [16] which are limited to this order.
In the following section we will use an iterative procedure to give another proof of the equivalence of the two methodology. I have published it in [17]

2.2.1 The iterative procedure

First of all it is necessary to rewrite equations in the second row of the table in a more explicit way in order to make the treatment simpler. Remembering that $m_i = 0$, the first column is

$$m = \frac{\partial \hat{h}'}{\partial \hat{\lambda}}, \, m^{ij} = \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ij}} \Leftrightarrow \begin{cases} m^{ll} = 3 \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ll}} \\ m^{<ij>} = \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{<rs>}} \delta^r_{<i} \delta^s_{j>} \end{cases}, \, m^{ill} = \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ill}}\tag{2.25}$$

$$\begin{aligned} \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}} &= m^{rk} \frac{\partial \hat{\lambda}_r}{\partial \hat{\lambda}} \\ \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{ij}} &= m^{rk} \frac{\partial \hat{\lambda}_r}{\partial \hat{\lambda}_{ij}} + m^{kij} \Leftrightarrow \begin{cases} \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{ll}} = m^{rk} \frac{\partial \hat{\lambda}_r}{\partial \hat{\lambda}_{ll}} + \frac{1}{3} m_{kll} \\ \left(\frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{<rs>}} - m^{ak} \frac{\partial \hat{\lambda}_a}{\partial \hat{\lambda}_{<rs>}} \right) \delta^{<i}_r \delta^{j>}_s + \frac{1}{3} m^{kll} \delta^{ij} = m^{ijk} \end{cases} \\ \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{ill}} &= m^{rk} \frac{\partial \hat{\lambda}_r}{\partial \hat{\lambda}_{ill}} + m^{ikll} \end{aligned}\tag{2.26}$$

while the second column is

$$\begin{aligned} 0 &= \hat{\lambda}_j \frac{\partial \hat{h}'}{\partial \hat{\lambda}} + \hat{\lambda}_{ill} \left(2 \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ij}} + \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{rs}} \delta_{rs} \delta^i_j \right), \\ 0 &= 2 \hat{\lambda}_{ij} m^{ik} + \hat{\lambda}_{ill} \left(2 m^{ikj} + m_{kll} \delta^{ij} \right) + \hat{h}' \delta^k_j. \end{aligned}\tag{2.27}$$

Firstly we notice that the two results coincide at thermodynamical equilibrium. In fact eqs. (2.25) calculated in this state become:

$$m = \frac{\partial \hat{h}_0'}{\partial \hat{\lambda}}\,,\; m_{ll} = \frac{\partial \hat{h}_0'}{\partial \hat{\lambda}_{ll}}\,,\; m_{<ij>} = 0\,,\; m_{ill} = 0$$

because $\hat{\lambda}$ and $\hat{\lambda}_{ll}$ are the only variables at equilibrium and there are no linear terms in $\hat{h}'$, for the representation theorems. Similarly eqs. $(2.26)_{1,2}$ amount to identities, eq. $(2.26)_3$ to $m_{ijk} = 0$, while eq. $(2.26)_4$ gives m_{ikll} at equilibrium, without conditions on $\hat{\phi}'^k$. Finally eq. $(2.27)_1$ becomes an identity and eq. $(2.27)_2$ yields $\frac{2}{9}\hat{\lambda}_{ll} m^{ll}\delta^k_j + 0 + 0 + \hat{h}_0'\delta^k_j = 0$. Integrating this last equation we find

$$\hat{h}_0' = \hat{\lambda}_{ll}^{-\frac{3}{2}} H_0(\hat{\lambda})$$

which coincides with eq. (19) of [13]. So the two approaches coincides at thermodynamical equilibrium, because they give the same result for $\hat{h}_0'$ and $\hat{\phi}_0'^k$ (which is zero).
Let us suppose now to know $\hat{h}'$ and $\hat{\phi}'^k$ up to order N, with respect to equilibrium.
Equation $(2.27)_1$ at order n and for $n = 0, 1, ..., N$ gives $\hat{\lambda}_i$ at order n (we know everything except $\hat{\lambda}_i$ that we can obtain from the equation).
The trace of eq. $(2.27)_2$ yields

$$0 = 2\hat{\lambda}_{ll}\frac{\partial \hat{h}'}{\hat{\lambda}_{ll}} + 2\hat{\lambda}_{<ij>}\frac{\partial \hat{h}'}{\hat{\lambda}_{<ij>}} + 3\hat{\lambda}_{ill}\frac{\partial \hat{h}'}{\hat{\lambda}_{ill}} + 3\hat{h}'$$

which is a partial differential equation for $\hat{h}'$, whose solution is

$$\hat{h}' = \hat{\lambda}_{ll}^{-\frac{3}{2}} H\left(\hat{\lambda}, \hat{\lambda}_{<ij>}\hat{\lambda}_{ll}^{-1}, \hat{\lambda}_{ill}\hat{\lambda}_{ll}^{-\frac{3}{2}}\right)$$

Equation $(2.27)_2$ is

$$0 = \left(\frac{2}{3}\hat{\lambda}_{ll}\frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ll}} + \hat{h}'\right)\delta^k_j + 2\frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ll}}\hat{\lambda}_{<jk>} + \overbrace{\frac{2}{3}\hat{\lambda}_{ll}\frac{\partial \hat{h}'}{\partial \hat{\lambda}_{<rs>}}\delta^{<j}_r\delta^{k>}_s}$$
$$+2\hat{\lambda}_{ill}\left(\frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{<rs>}} - \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ll}}\frac{\partial \hat{\lambda}_k}{\partial \hat{\lambda}_{<rs>}} - \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{<bc>}}\delta^{<a}_b\delta^{k>}_c\frac{\partial \hat{\lambda}_a}{\partial \hat{\lambda}_{<rs>}}\right)\delta^{<i}_r\delta^{j>}_s$$
$$+2\hat{\lambda}_{<ij>}\frac{\partial \hat{h}'}{\partial \hat{\lambda}_{<rs>}}\delta^{<i}_r\delta^{k>}_s + \frac{5}{3}\hat{\lambda}_{jll}\frac{\partial \hat{h}'}{\partial \hat{\lambda}_{kll}} \quad .$$

Its symmetric trace-less part, at order N, needs the knowledge of $\hat{h}'$, $\hat{\phi}'^k$ and λ_i up to order N (which we already have) except for the overbraced term (in which $\hat{h}'$ appears at order N+1); than we use this equation to obtain this term, i.e.,

$$\frac{2}{3}\hat{\lambda}_{ll}\frac{\partial \overset{N+1}{\hat{h}'}}{\partial\hat{\lambda}_{<rs>}}\delta_r^{<j}\delta_s^{k>} \quad ,$$

from which we can obtain $\overset{N+1}{\hat{h}'}$ except for an arbitrary function $\overset{N+1}{h^*}(\hat{\lambda}, \hat{\lambda}_{ill}\hat{\lambda}_{ll}^{-\frac{3}{2}})$. The integrability condition between $(2.26)_1$ and $(2.26)_{2a}$ is

$$\frac{\partial m^{rk}}{\partial\hat{\lambda}_{ll}}\frac{\partial\hat{\lambda}_r}{\partial\hat{\lambda}} = \frac{\partial m^{rk}}{\partial\hat{\lambda}}\frac{\partial\hat{\lambda}_r}{\partial\hat{\lambda}_{ll}} + \frac{1}{3}\frac{\partial m^{kll}}{\partial\hat{\lambda}}$$

or

$$\frac{\partial^2\hat{h}'}{\partial\hat{\lambda}_{ll}^2}\frac{\partial\hat{\lambda}_k}{\partial\hat{\lambda}} + \frac{\partial^2\hat{h}'}{\partial\hat{\lambda}_{ll}\partial\hat{\lambda}_{<ab>}}\delta_a^{<r}\delta_b^{k>}\frac{\partial\hat{\lambda}_r}{\partial\hat{\lambda}} = \frac{\partial^2\hat{h}'}{\partial\hat{\lambda}\partial\hat{\lambda}_{ll}}\frac{\partial\hat{\lambda}_k}{\partial\hat{\lambda}_{ll}} + \frac{\partial^2\hat{h}'}{\partial\hat{\lambda}\partial\hat{\lambda}_{<ab>}}\delta_a^{<r}\delta_b^{k>}\frac{\partial\hat{\lambda}_r}{\partial\hat{\lambda}} + \frac{1}{3}\frac{\partial^2\hat{h}'}{\partial\hat{\lambda}\partial\hat{\lambda}_{ill}}.$$

The terms of this equation are known at order N except for the last one (remember that $\lambda_r = 0$ at order 0) which we then obtain from this equation, i.e. $\frac{\partial^2 \overset{N+1}{\hat{h}'}}{\partial\hat{\lambda}\partial\hat{\lambda}_{ill}}$.

Calculating this expression in $\hat{\lambda}_{<rs>} = 0$ we find $\frac{\partial^2}{\partial\hat{\lambda}\partial\hat{\lambda}_{ill}}\overset{N+1}{h^*}(\hat{\lambda}, \hat{\lambda}_{ill}\hat{\lambda}_{ll}^{-\frac{3}{2}})$. Integrating with respect to $\hat{\lambda}_{ill}$ we find $\frac{\partial}{\partial\hat{\lambda}}\overset{N+1}{h^*}(\hat{\lambda}, \hat{\lambda}_{ill}\hat{\lambda}_{ll}^{-\frac{3}{2}})$ except for a function which doesn't depend on $\hat{\lambda}_{ill}$; but this function is 0 because it doesn't depend on $\hat{\lambda}_{<rs>}$ and on $\hat{\lambda}_{ill}$ so that it is of order 0, while $\overset{N+1}{h^*}$ is of order N+1. So we have found $\frac{\partial}{\partial\hat{\lambda}}\overset{N+1}{h^*}(\hat{\lambda}, \hat{\lambda}_{ill}\hat{\lambda}_{ll}^{-\frac{3}{2}})$. Integrating with respect to $\hat{\lambda}$ we find $\overset{N+1}{h*}$ except for a function $\overset{N+1}{h^{**}}(\hat{\lambda}_{ill}\hat{\lambda}_{ll}^{-\frac{3}{2}})$; because $\overset{N+1}{h^{**}}$ is of order N+1 we have

$$\overset{N+1}{h^{**}} = \begin{cases} 0 & \text{if N+1 is odd,} \\ \overset{N+1}{c}(\hat{\lambda}_{ill}\hat{\lambda}^{ill}\hat{\lambda}_{ll}^{-3})^{\frac{N+1}{2}} & \text{if N+1 is even.} \end{cases} \tag{2.28}$$

So $\overset{N+1}{h^{**}}$ has been found except for a constant of integration $\overset{N+1}{c}$ which arises only when N+1 is even.
Now we already know the solutions of our equations obtained with the second approach which we call h'_1 and $\phi_1'^k$. If $\hat{h}'$ coincides with h'_1 up to order N,

and $\hat{\phi}'^k$ coincides with $\phi_1'^k$ also up to order N, we have found that $\hat{h}'$ and h_1' coincides also at order N+1, except for a function like (2.28); but also h_1' has been determined except for a function like (2.28) with arbitrary constant $\overset{N+1}{c}_1$, instead of $\overset{N+1}{c}$. So it suffices to choose $\overset{N+1}{c}_1 = \overset{N+1}{c}$ to make coincide $\hat{h}'$ and h_1' also up to order N+1. At the other hand $\overset{N+1}{c}$ and $\overset{N+1}{c}_1$ are both arbitrary constant and than it is not necessary to do any choice: $\overset{N+1}{\hat{h}'}$ and $\overset{N+1}{\hat{h}'}_1$ coincides at all.
After that, eq. $(2.27)_1$ at order N+1 gives $\hat{\lambda}_i$ at this order.
Let's now turn our attention to $\hat{\phi}'^k$ and consider eq. $(2.26)_{2b}$ and $(2.26)_3$; they define m^{ijk} and m^{ikll}, but the symmetry conditions have to be satisfied:

$$\left[\frac{\partial \hat{\phi}'^{[k}}{\partial \hat{\lambda}_{<rs>}} - \frac{\hat{\lambda}_a}{\partial \hat{\lambda}_{<rs>}}\left(\frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ll}}\delta^{a[k} + \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{<bc>}}\delta_b^{<a}\delta_c^{[k>}\right)\right]\delta_r^{<i]}\delta_s^{j>} + \frac{1}{3}\delta^{j[i}\frac{\partial \hat{h}'}{\partial \hat{\lambda}_{k]ll}} = 0$$

$$\frac{\partial \hat{\phi}'^{[k}}{\partial \hat{\lambda}_{i]ll}} - \left(\frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ll}}\delta^{a[k} + \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{<bc>}}\delta_b^{<a}\delta_c^{[k>}\right)\frac{\partial \hat{\hat{\lambda}}_a}{\partial \hat{\lambda}_{i]ll}} = 0 \quad . \tag{2.29}$$

These, at order N, give $\frac{\partial \overset{N+1}{\hat{\phi}'^{[k}}}{\partial \hat{\lambda}_{<rs>}}\delta_r^{<i]}\delta_s^{j>}$ and $\frac{\partial \overset{N+1}{\hat{\phi}'^{[k}}}{\partial \hat{\lambda}_{i]ll}}$ as function of known quantities; because also $\overset{N+1}{\hat{\phi}_1'^k}$ satisfies these same conditions, substituting from (2.29) the corresponding quantities with $\phi_1'^k$ instead of $\hat{\phi}'^k$, we find

$$\frac{\partial \psi^{[k}}{\partial \hat{\lambda}_{<rs>}}\delta_r^{<i]}\delta_s^{j>} = 0, \quad \frac{\partial \psi^{[k}}{\partial \hat{\lambda}_{i]ll}} = 0 \tag{2.30}$$

with $\psi^k = \overset{N+1}{\hat{\phi}'^k} - \overset{N+1}{\hat{\phi}_1'^k}$.
Now, because ψ^k is of order N+1, it can be expressed in the form

$$\psi^k = \sum_{r=0}^{N+1}\psi_r^{ka_1b_1\ldots a_rb_rc_1\ldots c_{N+1-r}}\hat{\lambda}_{<a_1b_1>}\ldots\hat{\lambda}_{<a_rb_r>}\hat{\lambda}_{c_1ll}\hat{\lambda}_{c_{N+1-r}ll} \tag{2.31}$$

with $\psi_r^{ka_1b_1\ldots a_rb_rc_1\ldots c_{N+1-r}}$ a tensor depending only on $\hat{\lambda}$ and $\hat{\lambda}_{ll}$.
Eq. (2.31) doesn't change if we put $\psi_r^{kp_1q_1\ldots p_rq_r(c_1\ldots c_{N+1-r})}P_{p_1q_1}^{(a_1b_1}\ldots P_{p_rq_r}^{a_rb_r)}$ instead of $\psi_r^{ka_1b_1\ldots a_rb_rc_1\ldots c_{N+1-r}}$, where $P_{p_iq_i}^{a_ib_i} = \delta_{p_i}^{(a_i}\delta_{q_i}^{b_i)} - \frac{1}{3}\delta_{p_iq_i}\delta^{a_ib_i}$ and the symmetrization is done treating a_ib_i as a single index. In other words we can still keep eq. (2.31) but with $\psi_r^{ka_1b_1\ldots a_rb_rc_1\ldots c_{N+1-r}}$ symmetric with respect to two generic indexes c_i and c_j, which remains the same exchanging any two

couples of indexes $a_i b_i$ and $a_j b_j$, and that gives 0 when we contract it with $\forall$ $\delta_{a_i b_i}$. After that eqs. (2.30) become

$$r\psi_r^{[ki]ja_2b_2...a_rb_rc_1...c_{N+1-r}} = 0 \quad \text{for r=1,...N+1}$$
$$\psi_r^{k_1a_1b_1a_2b_2...a_rb_rc_1...c_{N+1-r}}\delta_{k_1}^{[k}\delta_{c_{N+1-r}}^{i]} = 0 \quad \text{for r=0,...N} \quad .$$

In other words $\psi_r^{ka_1b_1a_2b_2...a_rb_rc_1...c_{N+1-r}}$ is a tensor symmetric with respect to $\forall$ couple of its indexes, so that

$$\psi_r^{ka_1b_1a_2b_2...a_rb_rc_1...c_{N+1-r}} = \begin{cases} 0 & \text{if N-r is odd,} \\ \psi_r(\hat{\lambda}, \hat{\lambda}_{ll})\delta^{(ka_1}\delta^{b_1a_2}...\delta^{c_{N-r}c_{N-r+1})} & \text{if N-r is even.} \end{cases} \tag{2.32}$$

Now, if $r \geq 1$ eq. (2.32) contracted with $\delta_{a_1a_2}$ gives

$$0 = \begin{cases} 0 & \text{if N-r is odd,} \\ \psi_r \frac{N+3+r}{N+1+r}\delta^{(ka_2}\delta^{b_2a_3}...\delta^{c_{N-r}c_{N-r+1})} & \text{if N-r is even.} \end{cases} \Rightarrow \psi_r = 0$$

So $\psi_r^{ka_1b_1a_2b_2...a_rb_rc_1...c_{N+1-r}} = 0$ if $r \geq 1$.
Thanks to this results, eq. (2.31) becomes

$$\psi^k = \psi_0^{kc_1...c_{N+1}}\hat{\lambda}_{c_1ll}...\hat{\lambda}_{c_{N+1}ll} = \begin{cases} 0 & \text{if N is odd,} \\ \psi_0(\hat{\lambda}, \hat{\lambda}_{ll})\lambda^{kll}(\hat{\lambda}_{rll}\hat{\lambda}^{rll})^{\frac{N}{2}} & \text{if N is even.} \end{cases}$$

Now, eq. $(2.26)_1$ is

$$\frac{\hat{\phi}^{k'}}{\partial\hat{\lambda}} = \frac{\partial\hat{h}'}{\partial\hat{\lambda}_{ll}}\frac{\partial\hat{\lambda}_k}{\partial\hat{\lambda}} + \frac{\partial\ \hat{h}'}{\partial\hat{\lambda}_{<ab>}}\delta_a^{<r}\delta_b^{k>}\frac{\partial\hat{\lambda}_r}{\partial\hat{\lambda}}$$

Because $\hat{\lambda}_r = 0$ at order $0 \Rightarrow \frac{\partial \overset{N+1}{\hat{\phi}'^k}}{\partial\hat{\lambda}}$ is function of known quantities, so that $\frac{\partial\psi_k}{\partial\hat{\lambda}} = 0$; consequently, we have that ψ_0 doesn't depend on $\hat{\lambda}$. Now we can repeat all the considerations from the beginning of this section until eq. (2.29), but with N+1 instead of N and noticing that the results don't change if we add at $\overset{N+1}{\hat{\phi}'^k}$ a term like $\psi_0(\hat{\lambda}_{ll})\hat{\lambda}^{kll}(\hat{\lambda}_{rll}\hat{\lambda}^{rll})^{\frac{N}{2}}$. Then we find that $\overset{N+2}{\hat{h}'} = \overset{N+2}{h_1'}$.
Let's pass to eq. $(2.26)_{2a}$, i.e.

$$\frac{\partial\hat{\phi}'^k}{\partial\hat{\lambda}_{ll}} = \frac{\partial\hat{h}'}{\partial\hat{\lambda}_{ll}}\frac{\partial\hat{\lambda}_k}{\partial\hat{\lambda}_{ll}} + \frac{\partial\hat{h}'}{\partial\hat{\lambda}_{<ab>}}\delta_a^{<r}\delta_b^{k>}\frac{\partial\hat{\lambda}_r}{\partial\hat{\lambda}_{ll}} + \frac{1}{3}\frac{\partial\hat{h}'}{\partial\hat{\lambda}_{kll}}$$

which, at the order N+1 gives $\frac{\partial \overset{N+1}{\hat{\phi}'^k}}{\partial \hat{\lambda}_{ll}}$ as function of known quantities $\Rightarrow$ we have that $\frac{\partial \psi_k}{\partial \hat{\lambda}_{ll}} = 0$; in other words ψ_0 is a constant. So $\overset{N+1}{\hat{\phi}'^k}$ and $\overset{N+1}{\phi_1'^k}$ differ for the term $\psi_0 \hat{\lambda^{kll}} (\hat{\lambda_{rll} \lambda^{rll}})^{\frac{N}{2}}$ (present only when N is even); but in $\overset{N+1}{\hat{\phi}'^k}$ were already present a term like this with an arbitrary constant coefficient. So we can affirm, without loss of generality, that $\overset{N+1}{\hat{\phi}'^k} = \overset{N+1}{\phi_1'^k}$. This completes the proof that $\hat{h}' = h_1'$ and $\overset{N+1}{\hat{\phi}'^k} = \overset{N+1}{\phi_1'^k}$.

2.3 The 14 moments case

In [18] I have applied the new methodology to the 14 moments model. We have also shown that the 13 moments case can be obtained from the present one by using the method of subsystems. Let's report here a part of the paper. The 14 moments model was firstly investigated by Kremer [19], up to second order with respect to equilibrium; here we want to exploit it up to whatever order. The appropriate balance equations for this model read:

$$\begin{aligned}
&\partial_t F + \partial_k F^k = 0 \\
&\partial_t F^i + \partial_k F^{ik} = 0 \\
&\partial_t F^{ij} + \partial_k F^{ijk} = P^{<ij>} \\
&\partial_t F^{ill} + \partial_k F^{illk} = P^{ill} \\
&\partial_t F^{iill} + \partial_k F^{iillk} = P^{iill},
\end{aligned} \tag{2.33}$$

where the independent variables are F, F^i, F^{ij}, F^{ill}, F^{iill}, which are symmetric tensors. See also ref. [2] for further details. The right hand sides of eqs. $(2.33)_{1,2}$ are zero, such as the trace of that in eq. $(2.33)_3$ for the conservation laws of mass, momentum and energy. As usual, we have to add the entropy law and to impose the galilean relativity to close the system (2.33). By introducing the Lagrange multipliers, the entropy principle for these equations is

$$\begin{aligned}
&dh = \lambda dF + \lambda_i dF^i + \lambda_{ij} dF^{ij} + \lambda_{ill} dF^{ill} + \lambda_{iill} dF^{iill} \\
&d\phi^k = \lambda dF^k + \lambda_i dF^{ki} + \lambda_{ij} dF^{kij} + \lambda_{ill} dF^{kill} + \lambda_{iill} dF^{kiill} \\
&\sigma = \lambda_{ij} P^{<ij>} + \lambda_{ill} P^{ill} + \lambda_{iill} P^{iill} \geq 0.
\end{aligned} \tag{2.34}$$

and the potential have the following form

$$\begin{aligned}
&h' = \lambda F + \lambda_i F^i + \lambda_{ij} F^{ij} + \lambda_{ill} F^{ill} + \lambda_{iill} F^{iill} - h \\
&\phi'^k = \lambda F^k + \lambda_i F^{ki} + \lambda_{ij} F^{kij} + \lambda_{ill} F^{kill} + \lambda_{iill} F^{kiill} - \phi^k.
\end{aligned} \tag{2.35}$$

By differentiating eqs. (2.35) and using eqs. $(2.34)_{1,2}$ we obtain

$$dh' = Fd\lambda + F^i d\lambda_i + F^{ij} d\lambda_{ij} + F^{ill} d\lambda_{ill} + F^{iill} d\lambda_{iill}$$
$$d\phi'^k = F^k d\lambda + F^{ki} d\lambda_i + F^{kij} d\lambda_{ij} + F^{kill} d\lambda_{ill} + F^{kiill} d\lambda_{iill}. \quad (2.36)$$

2.3.1 The Galilean relativity principle and the entropy principle

We know that, if we have a change of Galileanly equivalent frames with relative velocity $\underline{v}$, the densities convert as:

$$\begin{aligned} F &= m \\ F_i &= m_i + mv_i \\ F_{ij} &= m_{ij} + 2m_{(i}v_{j)} + mv_iv_j \\ F_{ill} &= m_{ill} + m_{ll}v_i + 2m_{il}v_l + m_iv^2 + 2m_lv_lv_i + mv^2v_i \\ F_{iill} &= m_{iill} + 4m_{ill}v_i + 2m_{ll}v^2 + 4m_{li}v_iv_l + 4m_lv_lv^2 + mv^4; \end{aligned} \quad (2.37)$$

here the $m_{\dots}$ are the tensors corresponding to $F_{\dots}$ in the second reference frame. Moreover we have

$$\begin{aligned} F_k &= Fv_k + m_k \\ F_{ik} &= F_iv_k + m_{ik} + m_kv_i \\ F_{ijk} &= F_{ij}v_k + m_{ijk} + 2m_{k(i}v_{j)} + m_kv_iv_j \\ F_{illk} &= F_{ill}v_k + m_{illk} + m_{kll}v_i + 2m_{kil}v_l + m_{ki}v^2 + 2m_{kl}v_lv_i + m_kv^2v_i \\ F_{iillk} &= F_{iill}v_k + m_{iillk} + 4m_{kill}v_i + 2m_{kll}v^2 + 4m_{kli}v_lv_i + 4m_{kl}v_lv^2 + m_kv^4 \\ h &= \hat{h} \\ \phi^k &= \hat{h}v_k + \hat{\phi}^k. \end{aligned} \quad (2.38)$$

The first two of these, as the trace of the third and fourth ones, are identities, while what remains is the transformation law of the dependent variables. Substituting the relations above into eq. $(2.34)_1$ and defining

$$\begin{aligned} &\hat{\lambda} = \lambda + \lambda_iv_i + \lambda_{ij}v_iv_j + \lambda_{ipp}v^2v_i + \lambda_{ppqq}v^4 \\ &\hat{\lambda}_i = \lambda_i + 2\lambda_{ij}v_j + 2\lambda_{jpp}v_jv_i + \lambda_{ipp}v^2 + 4\lambda_{ppqq}v^2v_i \\ &\hat{\lambda}_{ij} = \lambda_{ij} + \lambda_{hpp}v_h\delta_i^j + 2\lambda_{ipp}v_j + 2\lambda_{ppqq}v^2\delta_i^j + 4\lambda_{ppqq}v_iv_j \\ &\hat{\lambda}_{ill} = \lambda_{ipp} + 4\lambda_{ppqq}v_i \\ &\hat{\lambda}_{ppqq} = \lambda_{ppqq} \end{aligned} \quad (2.39)$$

we have

$$d\hat{h} = \hat{\lambda}dm + \hat{\lambda}_idm_i + \hat{\lambda}_{ij}dm_{ij} + \hat{\lambda}_{ill}dm_{ill} + \hat{\lambda}_{iill}dm_{iill}. \quad (2.40)$$

For eq. $(2.38)_6$ we note that eq. (2.40) is the counterpart of eq. $(2.34)_1$ in the second frame; this allows us to see that eqs. (2.39) are the transformation rules for the Lagrange multipliers.
Similarly, by substituting eqs. (2.38) in eq. $(2.34)_2$, we find

$$d\hat{\phi}^k = \hat{\lambda} dm_k + \hat{\lambda}_i dm_{ki} + \hat{\lambda}_{ij} dm_{kij} + \hat{\lambda}_{ill} dm_{kill} + \hat{\lambda}_{iill} dm_{kiill} \tag{2.41}$$

which is the counterpart of eq. $(2.34)_2$ in other frame.
The counterparts of eqs. (2.35) in the second frame are

$$\begin{aligned} \hat{h}' &= m\hat{\lambda} + m_i\hat{\lambda}_i + m_{ij}\hat{\lambda}_{ij} + m_{ill}\hat{\lambda}_{ill} + m_{iill}\hat{\lambda}_{iill} - \hat{h} \\ \hat{\phi}'^k &= m_k\hat{\lambda} + m_{ki}\hat{\lambda}_i + m_{kij}\hat{\lambda}_{ij} + m_{kill}\hat{\lambda}_{ill} + m_{kiill}\hat{\lambda}_{iill} - \hat{\phi}^k; \end{aligned} \tag{2.42}$$

differentiating them and using eqs. $(2.38)_{6,7}$, (2.40) and (2.41) we obtain respectively

$$\begin{aligned} d\hat{h}' &= md\hat{\lambda} + m_i d\hat{\lambda}_i + m_{ij} d\hat{\lambda}_{ij} + m_{ill} d\hat{\lambda}_{ill} + m_{iill} d\hat{\lambda}_{iill}, \\ d\hat{\phi}'^k &= m_k d\hat{\lambda} + m_{ki} d\hat{\lambda}_i + m_{kij} d\hat{\lambda}_{ij} + m_{kill} d\hat{\lambda}_{ill} + m_{kiill} d\hat{\lambda}_{iill}. \end{aligned}$$

Taking their derivatives with respect to the various components of the main field we have

$$\begin{aligned} m &= \frac{\partial \hat{h}'}{\partial \hat{\lambda}}, \quad m_i = \frac{\partial \hat{h}'}{\partial \hat{\lambda}_i}, \quad m_{ij} = \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ij}}, \quad m_{ill} = \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ill}}, \quad m_{iill} = \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{iill}}, \\ m_k &= \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}}, \quad m_{ki} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_i}, \quad m_{kij} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{ij}}, \quad m_{kill} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{ill}}, \quad m_{kiill} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{iill}}. \end{aligned} \tag{2.43}$$

Comparing the correspondent terms in the two rows of eq. (2.43) we obtain the following compatibility conditions:

$$\begin{aligned} &\frac{\partial \hat{h}'}{\partial \hat{\lambda}_k} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}}, \qquad \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ki}} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_i}, \qquad \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ill}} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{ij}} \delta_i^j, \\ &\frac{\hat{\phi}'^{k]}}{\partial \hat{\lambda}_{i[j}} = 0, \qquad \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{kkll}} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{ill}} \delta_i^k, \qquad \frac{\partial \hat{\phi}'^{k]}}{\partial \hat{\lambda}_{ll[i}} = 0. \end{aligned} \tag{2.44}$$

By substituting h, $\hat{h}$, ϕ^k and $\hat{\phi}^k$ from eqs. $(2.35)_1$, $(2.42)_1$, $(2.35)_2$, $(2.42)_2$ into eqs. $(2.38)_{6,7}$, these become

$$h' = \hat{h}', \qquad \phi'^k = \hat{\phi}'^k + \hat{h}' v^k, \tag{2.45}$$

where $(2.38)_{1-5}$ and (2.39) have been used.
Now, from eqs. (2.45) we see that h' and ϕ'^k are composite functions of $\hat{h}'$

and $\hat{\phi}'^k$ and of eqs. (2.39); but h' and ϕ'^k depend only on λ, λ_i, λ_{ij}, λ_{ill}, λ_{iill} and not on v_h. In other words, the derivative of h' and ϕ'^k with respect to v_h, through the above mentioned composite functions, must be zero, i.e.

$$\frac{\partial h'}{\partial v_h} = 0 = m\hat{\lambda}_h + 2m_i\hat{\lambda}_{ih} + \hat{\lambda}_{ipp}\left(m_{ll}\delta_i^h + 2m_{ih}\right) + 4m_{hll}\hat{\lambda}_{ppqq} \tag{2.46}$$

$$\frac{\partial \phi'^k}{\partial v_h} = 0 = m_k\hat{\lambda}_h + 2m_{ki}\hat{\lambda}_{ih} + \hat{\lambda}_{ipp}\left(m_{kll}\delta_i^h + 2m_{kih}\right) + 4m_{khll}\hat{\lambda}_{ppqq} + \delta_h^k h' \tag{2.47}$$

where (2.43) and (2.39) have been used.
The entropy principle and that of material objectivity reduce in imposing eqs. (2.46), (2.47) and (2.44). We want to impose these conditions up to whatever order with respect to thermodynamical equilibrium. As said in chapter 1, this is defined as the state where all the components of the main field, except $\hat{\lambda}$ and $\hat{\lambda}_{ij} = \frac{1}{3}\hat{\lambda}_{ll}\delta_{ij}$, amounts to zero. To avoid an excessive quantity of indexes, we will do later the expansion with respect to $\hat{\lambda}_{ppqq}$. The expansion of the tensor $\hat{\phi}'^i$ with respect to the other variables is

$$\begin{aligned}\hat{\phi}'^i = \quad & \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\frac{1}{p!q!r!}\phi_{p,q,r}^{ii_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}\hat{\lambda}_{i_1}\cdots\hat{\lambda}_{i_p}\hat{\lambda}_{j_1 ll}\cdots\hat{\lambda}_{j_q ll}\cdot \\ & \left(\hat{\lambda}_{k_1h_1} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_1h_1}\right)\cdots\left(\hat{\lambda}_{k_rh_r} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_rh_r}\right)\end{aligned} \tag{2.48}$$

$$\begin{aligned}\text{with} \quad & \phi_{p,q,r}^{ii_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}(\hat{\lambda}, \hat{\lambda}_{ll}, \hat{\lambda}_{ppqq}) = \\ & = \left(\frac{\partial^{p+q+r}\hat{\phi}'^i}{\partial\hat{\lambda}_{i_1}\cdots\partial\hat{\lambda}_{i_p}\partial\hat{\lambda}_{j_1ll}\cdots\partial\hat{\lambda}_{j_qll}\partial\hat{\lambda}_{k_1h_1}\cdots\partial\hat{\lambda}_{h_rk_r}}\right)_{eq}.\end{aligned} \tag{2.49}$$

Now, from the compatibility conditions $(2.44)_2$, $(2.44)_6$ and $(2.44)_4$ we see that we can exchange the index i respectively with each other index taken from $i_1\cdots i_p$, $j_1,\cdots j_q$ and $h_1\cdots h_r$ or $k_1\cdots k_r$, so $\phi_{p,q,r}^{ii_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}$ is a symmetric tensor with respect to any couple of indexes.
Moreover $\phi_{p,q,r}^{ii_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}$ depends only on scalars, so that

$$\begin{cases}\phi_{p,q,r}^{ii_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r} = 0 & \text{if p+q+2r+1 is odd} \\ \phi_{p,q,r}^{ii_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r} = \phi_{p,q,r}(\hat{\lambda}, \hat{\lambda}_{ll}, \hat{\lambda}_{ppqq})\delta^{ii_1}\cdots\delta^{k_rh_r} & \text{if p+q+2r+1 is even,}\end{cases} \tag{2.50}$$

so that $\phi_{p,q,r}^{ii_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}$ is known except for a scalar function.
Similarly, for the tensor $\hat{h}'$ we have

$$\hat{h}' = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\frac{1}{p!q!r!}h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}\hat{\lambda}_{i_1}\cdots\hat{\lambda}_{i_p}\hat{\lambda}_{j_1 ll}\cdots\hat{\lambda}_{j_q ll}\cdot \left(\hat{\lambda}_{k_1 h_1}-\frac{1}{3}\hat{\lambda}_{ll}\delta_{k_1 h_1}\right)\cdots\left(\hat{\lambda}_{k_r h_r}-\frac{1}{3}\hat{\lambda}_{ll}\delta_{k_r h_r}\right) \tag{2.51}$$

with $$h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}(\hat{\lambda},\hat{\lambda}_{ll},\hat{\lambda}_{ppqq}) = = \left(\frac{\partial^{p+q+r}\hat{h}'}{\partial\hat{\lambda}_{i_1}\cdots\partial\hat{\lambda}_{i_p}\partial\hat{\lambda}_{j_1 ll}\cdots\partial\hat{\lambda}_{j_q ll}\partial\hat{\lambda}_{k_1 h_1}\cdots\partial\hat{\lambda}_{h_r k_r}}\right)_{eq}. \tag{2.52}$$

Taking the derivatives with respect to $\hat{\lambda}_{jll}$ of the compatibility conditions $(2.44)_1$ and $(2.44)_2$ and using $(2.44)_6$ we see that we can exchange every index taken from $j_1,\cdots j_q$ with each other. Similarly, taking the derivative of eq. $(2.44)_1$ with respect to $\hat{\lambda}_{rs}$ and using eq. $(2.44)_4$ we see that we can exchange every index taken from $i_1,\cdots,i_p$ with each other. Consequently, $h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}$ is a symmetric tensor with respect to any couple of indexes; moreover it depends only on scalars, so that

$$\begin{cases} h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r} = 0 & \text{if p+q+2r is odd} \\ h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r} = h_{p,q,r}(\hat{\lambda},\hat{\lambda}_{ll},\hat{\lambda}_{ppqq})\delta^{i_1 i_2}\cdots\delta^{k_r h_r} & \text{if p+q+2r is even.} \end{cases} \tag{2.53}$$

In other words, also $h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}$ is known except for a scalar function. We want to avoid to use eqs. (2.49) and (2.52) in the sequel. To this end we note that we can consider

$$\frac{\partial^{p+q+r}\hat{h}'}{\partial\hat{\lambda}_{i_1}\cdots\partial\hat{\lambda}_{i_p}\partial\hat{\lambda}_{j_1 ll}\cdots\partial\hat{\lambda}_{j_q ll}\partial\hat{\lambda}_{k_1 h_1}\cdots\partial\hat{\lambda}_{h_r k_r}}$$

and $$\frac{\partial^{p+q+r}\hat{\phi}'^k}{\partial\hat{\lambda}_{i_1}\cdots\partial\hat{\lambda}_{i_p}\partial\hat{\lambda}_{j_1 ll}\cdots\partial\hat{\lambda}_{j_q ll}\partial\hat{\lambda}_{k_1 h_1}\cdots\partial\hat{\lambda}_{h_r k_r}}$$

depending on $\hat{\lambda}_{ab}$ as composite functions through $\hat{\lambda}_{<ab>} = \left(\delta_a^i\delta_b^j - \frac{1}{3}\delta^{ij}\delta_{ab}\right)\hat{\lambda}_{ij}$ and $\hat{\lambda}_{ll}$. With this in mind let us take their derivatives with respect to $\hat{\lambda}_{ab}$, after that contract them with δ_{ab} and calculate the result at equilibrium; we find

$$h_{p,q,r+1}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r ab}\delta_{ab} = 3\frac{\partial}{\partial\hat{\lambda}_{ll}}h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r} \tag{2.54}$$

$$\text{and} \quad \phi_{p,q,r+1}^{ii_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r ab}\delta_{ab} = 3\frac{\partial}{\partial \hat{\lambda}_{ll}}\phi_{p,q,r}^{ii_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}. \tag{2.55}$$

An interesting consequence of eq. (2.54) can be observed as follows.
Let us take the derivative of $\hat{h}'$ with respect to $\hat{\lambda}_{ij}$ taking into account that $\hat{\lambda}_{ij} = \frac{1}{3}\hat{\lambda}_{ll}\delta_{ij} + \hat{\lambda}_{<ij>}$:

$$\frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ij}} = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\frac{1}{p!q!r!}\frac{\partial h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}}{\partial \hat{\lambda}_{ll}}\hat{\lambda}_{i_1}\cdots\hat{\lambda}_{i_p}\hat{\lambda}_{j_1 ll}\cdots\hat{\lambda}_{j_q ll}\cdot$$
$$\left(\hat{\lambda}_{k_1h_1} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_1h_1}\right)\cdots\left(\hat{\lambda}_{k_rh_r} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_rh_r}\right)\delta_{ij} +$$
$$+\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=1}^{\infty}\frac{r}{p!q!r!}h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}\hat{\lambda}_{i_1}\cdots\hat{\lambda}_{i_p}\hat{\lambda}_{j_1 ll}\cdots\hat{\lambda}_{j_q ll}\cdot$$
$$\left(\hat{\lambda}_{k_1h_1} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_1h_1}\right)\cdots\left(\hat{\lambda}_{k_{r-1}h_{r-1}} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_{r-1}h_{r-1}}\right)\left(\delta_{h_r}^{i}\delta_{k_r}^{j} - \frac{1}{3}\delta_{h_rk_r}\delta^{ij}\right),$$

which, by using eq. (2.54), becomes

$$\begin{aligned}
\frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ij}} \;=\; & \Bigg[\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\frac{1}{p!q!r!}\frac{1}{3}h_{p,q,r+1}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r ab}\delta_{ab}\hat{\lambda}_{i_1}\cdots\hat{\lambda}_{i_p}\cdot \\
& \hat{\lambda}_{j_1 ll}\cdots\hat{\lambda}_{j_q ll}\left(\hat{\lambda}_{k_1h_1} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_1h_1}\right)\cdots\left(\hat{\lambda}_{k_rh_r} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_rh_r}\right)\delta^{ij} + \\
- \;& \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=1}^{\infty}\frac{r}{p!q!r!}\frac{1}{3}h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}\hat{\lambda}_{i_1}\cdots\hat{\lambda}_{i_p}\hat{\lambda}_{j_1 ll}\cdots\hat{\lambda}_{j_q ll}\cdot \\
& \left(\hat{\lambda}_{k_1h_1} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_1h_1}\right)\cdots\left(\hat{\lambda}_{k_{r-1}h_{r-1}} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_{r-1}h_{r-1}}\right)\delta_{h_rk_r}\Bigg]\delta^{ij} + \\
+ \;& \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=1}^{\infty}\frac{r}{p!q!r!}h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}\hat{\lambda}_{i_1}\cdots\hat{\lambda}_{i_p}\hat{\lambda}_{j_1 ll}\cdots\hat{\lambda}_{j_q ll}\cdot \\
& \left(\hat{\lambda}_{k_1h_1} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_1h_1}\right)\cdots\left(\hat{\lambda}_{k_{r-1}h_{r-1}} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_{r-1}h_{r-1}}\right)\delta_{h_r}^{i}\delta_{k_r}^{j};
\end{aligned}$$

We note that the term in square brackets amounts to zero as can be easily proved by substituting r=R+1 in the second sum. What remains can be written as

$$\frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ij}} = \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{<ij>}},$$

where the derivative in the right hand side has been taken without considering that the components of $\hat{\lambda}_{<ij>}$ aren't independent because restricted by $\hat{\lambda}_{<ij>}\delta^{ij} = 0$. Proceeding similarly with $\hat{\phi}'^k$ and using eq. (2.55) we find that

$$\frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{ij}} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{<ij>}}.$$

After that, we see that eq. (2.49) and (2.52) become consequences of eqs. (2.48) and (2.51) so that they can be forgotten. But, instead of them, we have to impose eqs. (2.54) and (2.55).
Expliciting eq. (2.54) by means of eq. (2.53) we have

$$h_{p,q,r+1} = 3\frac{p+q+2r+1}{p+q+2r+3}\frac{\partial h_{p,q,r}}{\partial \hat{\lambda}_{ll}}, \tag{2.56}$$

from which

$$h_{p,q,r} = 3^r \frac{p+q+1}{p+q+2r+1}\frac{\partial^r h_{p,q,0}}{\partial \hat{\lambda}_{ll}^r}, \tag{2.57}$$

as it can be seen by using the iterative procedure.
Similarly, expliciting eq. (2.55) by means of eq. (2.50), we have

$$\phi_{p,q,r+1} = 3\frac{p+q+2r+2}{p+q+2r+4}\frac{\partial \phi_{p,q,r}}{\partial \hat{\lambda}_{ll}}, \tag{2.58}$$

from which

$$\phi_{p,q,r} = 3^r \frac{p+q+2}{p+q+2r+2}\frac{\partial^r \phi_{p,q,0}}{\partial \hat{\lambda}_{ll}^r}, \tag{2.59}$$

that can be proved using the iterative procedure.
If we introduce the quantities

$$\begin{cases} k_{p,q} = h_{p,q,0} & \text{if p+q is even} \\ k_{p,q} = \phi_{p,q,0} & \text{if p+q is odd,} \end{cases} \tag{2.60}$$

we note that $\hat{h}'$ and $\hat{\phi}'^k$ are known if we know all the terms of the infinity matrix $k_{p,q}$; so our aim is to find $k_{p,q}$. We have also to impose the compatibility conditions (2.44) and the conditions (2.46) and (2.47) expressing the Galilean relativity principle.
Let us begin by investigating the conditions (2.44).

2.3.2 Exploitation of the conditions (2.44)

Now let's impose conditions (2.44) on our tensors. We notice that equations $(2.44)_{4,6}$ are already satisfied because the tensors $\phi_{p,q,r}^{ii_1\cdots i_p j_1\cdots j_q h_1 k_1\cdots h_r k_r}$ are symmetric, so there remains to impose eqs. $(2.44)_{1,2,3,5}$.

- Eq. $(2.44)_1$, by using (2.48), (2.50), (2.52) and (2.53), becomes

$$h_{p+1,q,r} = \frac{\partial \phi_{p,q,r}}{\partial \hat{\lambda}} \tag{2.61}$$

which, for r=0 reads

$$h_{p+1,q,0} = \frac{\partial \phi_{p,q,0}}{\partial \hat{\lambda}} \tag{2.62}$$

and, for the other values of r is consequence of (2.57), (2.59), (2.62). This last one, by using (2.60), can be written also as

$$k_{p+1,q} = \frac{\partial k_{p,q}}{\partial \hat{\lambda}} \quad \text{with p+q+1 even.} \tag{2.63}$$

In other words, the elements with p+q+1 even of the matrix $k_{p+1,q}$ can be expressed in terms of that of the same column but previous row.

- Let us impose now eq. $(2.44)_2$, using eqs. (2.48), (2.50), (2.52) and (2.53); we obtain

$$h_{p,q,r+1} = \phi_{p+1,q,r} \tag{2.64}$$

which, by using eqs. (2.57) and (2.59) is equivalent to

$$\phi_{p+1,q,0} = 3\frac{p+q+1}{p+q+3}\frac{\partial h_{p,q,0}}{\partial \hat{\lambda}_{ll}} \tag{2.65}$$

and this, by using (2.60), becomes

$$k_{p+1,q} = 3\frac{p+q+1}{p+q+3}\frac{\partial k_{p,q}}{\partial \hat{\lambda}_{ll}} \quad \text{with p+q even.} \tag{2.66}$$

Using (2.63) or (2.66) we can express all the elements of the matrix $k_{p,q}$ in terms of those in the same column and previous row. Iterating this procedure each element can be expressed in terms of the elements in the first row of the matrix. In fact joining eqs. (2.63) and (2.66) we obtain

$$\begin{cases} k_{p,q} = 3^{\frac{p}{2}} \frac{q+1}{p+q+1} \frac{\partial^p}{\partial \hat{\lambda}_{ll}{}^{\frac{p}{2}} \partial \hat{\lambda}^{\frac{p}{2}}} k_{0,q} & \text{with p and q even,} \\ k_{p,q} = 3^{\frac{p-1}{2}} \frac{q+2}{p+q+1} \frac{\partial^p}{\partial \hat{\lambda}_{ll}{}^{\frac{p-1}{2}} \partial \hat{\lambda}^{\frac{p+1}{2}}} k_{0,q} & \text{with p and q odd,} \\ k_{p,q} = 3^{\frac{p}{2}} \frac{q+2}{p+q+2} \frac{\partial^p}{\partial \hat{\lambda}_{ll}{}^{\frac{p}{2}} \partial \hat{\lambda}^{\frac{p}{2}}} k_{0,q} & \text{with p even and q odd,} \\ k_{p,q} = 3^{\frac{p+1}{2}} \frac{q+1}{p+q+2} \frac{\partial^p}{\partial \hat{\lambda}_{ll}{}^{\frac{p+1}{2}} \partial \hat{\lambda}^{\frac{p-1}{2}}} k_{0,q} & \text{with p odd and q even.} \end{cases} \tag{2.67}$$

- Finally, let us consider eqs. $(2.44)_{3,5}$. Using eqs. (2.48), (2.50), (2.52) and (2.53) they become respectively

$$h_{p,q+1,r} = \frac{p+q+2r+4}{p+q+2r+2} \phi_{p,q,r+1} \tag{2.68}$$

and

$$\frac{\partial h_{p,q,r}}{\partial \hat{\lambda}_{kkll}} = \frac{p+q+2r+3}{p+q+2r+1} \phi_{p,q+1,r}. \tag{2.69}$$

By using eqs. (2.57), (2.59) and finally (2.60) the above equations transform respectively into

$$k_{p,q+1} = 3 \frac{\partial k_{p,q}}{\partial \hat{\lambda}_{ll}} \quad \text{with p+q+1 even} \tag{2.70}$$

and

$$k_{p,q+1} = \frac{p+q+1}{p+q+3} \frac{\partial k_{p,q}}{\partial \hat{\lambda}_{aabb}} \quad \text{with p+q+1 odd.} \tag{2.71}$$

In other words with eqs. (2.70) and (2.71) each element of the matrix $k_{p,q}$ can be written in terms of the element in the same row and previous column. But we already know, by eqs. (2.67), each row of the matrix $k_{p,q}$ in terms of the first one; so we have to investigate the compatibility of these two results. By substituting eqs. (2.67) into eqs. (2.70) and (2.71) we obtain a series of equations for the first row of the matrix $k_{p,q}$, i.e.,

$$\begin{cases} k_{0,q+1} = 3 \frac{\partial}{\partial \hat{\lambda}_{ll}} k_{0,q} & \text{q odd,} \\ 9 \frac{q+1}{q+3} \frac{\partial^2}{\partial \hat{\lambda}_{ll}^2} k_{0,q} = \frac{\partial}{\partial \hat{\lambda}} k_{0,q+1} & \text{q even,} \\ \frac{q+1}{q+3} \frac{\partial}{\partial \hat{\lambda}_{aabb}} k_{0,q} = k_{0,q+1} & \text{q even,} \\ \frac{\partial}{\partial \hat{\lambda}_{aabb}} \frac{\partial}{\partial \hat{\lambda}} k_{0,q} = 3 \frac{\partial}{\partial \hat{\lambda}_{ll}} k_{0,q+1} & \text{q odd,} \end{cases} \tag{2.72}$$

and other equations which are consequences of these last ones. Now eqs. $(2.72)_1$ and $(2.72)_3$ give each element $k_{0,q}$ in terms of $k_{0,0}$, i.e.,

$$k_{0,q} = 3^{\frac{q}{2}} \frac{1}{q+1} \frac{\partial^q}{\partial \hat{\lambda}_{ll}^{\frac{q}{2}} \partial \hat{\lambda}_{aabb}^{\frac{q}{2}}} k_{0,0} \quad \text{with q even} \tag{2.73}$$

$$k_{0,q} = 3^{\frac{q-1}{2}} \frac{1}{q+2} \frac{\partial^q}{\partial \hat{\lambda}_{ll}^{\frac{q-1}{2}} \partial \hat{\lambda}_{aabb}^{\frac{q+1}{2}}} k_{0,0} \quad \text{with q odd.} \tag{2.74}$$

Through these two equations is possible to express a generic element in the first row in terms of the first element in the same first row of the matrix.
Eq. $(2.72)_{2,4}$ remain to be imposed. The first one of these with q=0 and by use of (2.73) reads

$$9 \frac{\partial^2}{\partial \hat{\lambda}_{ll}^2} k_{0,0} = \frac{\partial}{\partial \hat{\lambda}} \frac{\partial}{\partial \hat{\lambda}_{aabb}} k_{0,0}, \tag{2.75}$$

which is a condition on $k_{0,0}$. After that eq. $(2.72)_2$ for the other values of q is a consequence of eq. (2.75).
At last, eq. $(2.72)_4$ with use of (2.73) and (2.74) becomes equivalent to its value for q=1, i.e.,

$$9 \frac{\partial^3}{\partial \hat{\lambda}_{ll}^2 \partial \hat{\lambda}_{aabb}} k_{0,0} = \frac{\partial^3}{\partial \hat{\lambda} \partial \hat{\lambda}_{aabb}^2} k_{0,0},$$

which is eq. (2.75) differentiated with respect to $\hat{\lambda}_{aabb}$; so it is sufficient to impose eq. (2.75).
We can now substitute eqs. (2.73) and (2.74) into eqs. (2.67) which now become

$$\begin{cases} k_{p,q} = 3^{\frac{p+q}{2}} \frac{1}{p+q+1} \frac{\partial^{p+q}}{\partial \hat{\lambda}_{ll}{}^{\frac{p+q}{2}} \partial \hat{\lambda}^{\frac{p}{2}} \partial \hat{\lambda}_{aabb}{}^{\frac{q}{2}}} k_{0,0} & \text{with p and q even,} \\ k_{p,q} = 3^{\frac{p+q-2}{2}} \frac{1}{p+q+1} \frac{\partial^{p+q}}{\partial \hat{\lambda}_{ll}{}^{\frac{p+q-2}{2}} \partial \hat{\lambda}^{\frac{p+1}{2}} \partial \hat{\lambda}_{aabb}{}^{\frac{q+1}{2}}} k_{0,0} & \text{with p and q odd,} \\ k_{p,q} = 3^{\frac{p+q-1}{2}} \frac{1}{p+q+2} \frac{\partial^{p+q}}{\partial \hat{\lambda}_{ll}{}^{\frac{p+q-1}{2}} \partial \hat{\lambda}^{\frac{p}{2}} \partial \hat{\lambda}_{aabb}{}^{\frac{q+1}{2}}} k_{0,0} & \text{with p even and q odd,} \\ k_{p,q} = 3^{\frac{p+q+1}{2}} \frac{1}{p+q+2} \frac{\partial^{p+q}}{\partial \hat{\lambda}_{ll}{}^{\frac{p+q+1}{2}} \partial \hat{\lambda}^{\frac{p-1}{2}} \partial \hat{\lambda}_{aabb}{}^{\frac{q}{2}}} k_{0,0} & \text{with p odd and q even.} \end{cases} \tag{2.76}$$

In this way all the elements of the matrix $k_{p,q}$ are determined in terms of $k_{0,0}$ which is restricted, until now, only by eq. (2.75). Another restriction will be found in the next section.

2.3.3 Exploitation of the conditions (2.46) and (2.47)

There remains now to impose eqs. (2.46) and (2.47), but we can see that (2.46) is a consequence of (2.47) and (2.44). In fact

- the derivative of (2.46) with respect to $\hat{\lambda}_k$ is equal to the derivative of (2.47) with respect to λ, thanks to (2.43), $(2.44)_1$,
- the derivative of (2.46) with respect to $\hat{\lambda}_{kb}$ is exactly the derivative of (2.47) with respect to λ_b, thanks to (2.43), $(2.44)_{2,1}$,
- the derivative of (2.46) with respect to $\hat{\lambda}_{kll}$ is exactly the derivative of (2.47) with respect to λ_{ab}, contracted after derivation by δ_{ab}, thanks to (2.43), $(2.44)_{3,2}$,
- the derivative of (2.46) with respect to $\hat{\lambda}_{kkll}$ is exactly the derivative of (2.47) with respect to λ_{ill}, contracted after derivation by δ_{ki}, thanks to (2.43), $(2.44)_5$.

Consequently, eq. (2.46) needs to be imposed only for $\hat{\lambda}_k = 0$, $\hat{\lambda}_{ab} = 0$, $\hat{\lambda}_{kll} = 0$ and $\hat{\lambda}_{kkll} = 0$, and in this case it is an identity. So it remains to impose only eq. (2.47). To this end it is useful to use the identity

$$\begin{aligned} \frac{\partial^r}{\partial \hat{\lambda}_{k_1 h_1} \cdots \partial \hat{\lambda}_{k_r h_r}} \left(\hat{\lambda}_{ij} \frac{\partial \phi'^k}{\partial \hat{\lambda}_i} \right) &= \hat{\lambda}_{ij} \frac{\partial^{r+1} \phi'^k}{\partial \hat{\lambda}_i \partial \hat{\lambda}_{k_1 h_1} \cdots \partial \hat{\lambda}_{k_r h_r}} \\ &+ r \delta_{j(k_1} \frac{\partial^r \phi'^k}{\partial \hat{\lambda}_{h_1} \partial \hat{\lambda}_{k_2 h_2} \cdots \partial \hat{\lambda}_{k_r h_r)}} \end{aligned}$$

whose proof can be found in the Appendix of [15] and holds also if, in our case, $\hat{\phi}'^k$ depends on the further independent variable $\hat{\lambda}_{aabb}$.
Let us take now the derivative of eq. (2.47) with respect to $\hat{\lambda}_{i_1} \cdots \hat{\lambda}_{i_p}$, $\hat{\lambda}_{j_1 ll} \cdots \hat{\lambda}_{j_q ll}$, $\hat{\lambda}_{k_1 h_1} \cdots \hat{\lambda}_{k_r h_r}$. If we calculate it at equilibrium and we use eqs. (2.49) and (2.52) we obtain

$$\begin{aligned} & p \delta_{h(i_1} \frac{\partial}{\partial \hat{\lambda}} \phi^{i_2 \cdots i_p) k j_1 \cdots j_q h_1 k_1 \cdots h_r k_r}_{p-1,q,r} + \frac{2}{3} \hat{\lambda}_{ll} \phi^{k h i_1 \cdots i_p j_1 \cdots j_q h_1 k_1 \cdots h_r k_r}_{p+1,q,r} + \\ + \; & 2r \delta_{h(k_1} \phi^{h_1 k_2 \cdots h_r k_r) k i_1 \cdots i_p j_1 \cdots j_q}_{p+1,q,r-1} + 2q \phi^{k h i_1 \cdots i_p j_1 \cdots j_q h_1 k_1 \cdots h_r k_r}_{p,q-1,r+1} + \\ + \; & q \delta_{h(j_1} \phi^{j_2 \cdots j_q) k i_1 \cdots i_p h_1 k_1 \cdots h_r k_r ab}_{p,q-1,r+1} \delta_{ab} + 4 \hat{\lambda}_{aabb} \phi^{k i_1 \cdots i_p j_1 \cdots j_q h h_1 k_1 \cdots h_r k_r}_{p,q+1,r} + \\ + \; & h^{i_1 \cdots i_p j_1 \cdots j_q h_1 k_1 \cdots h_r k_r}_{p,q,r} \delta^{hk} = 0. \end{aligned} \tag{2.77}$$

To evaluate this condition it will be useful to do the following considerations:

1) Let $\psi^{\cdots}$ be a symmetric tensor; it is easy to prove that

$$\begin{aligned}\delta^{h(i_1}\psi^{i_2\cdots i_p j_1\cdots j_q e_1\cdots e_s k)} &= \frac{p}{p+q+s+1}\delta^{h(i_1}\psi^{i_2\cdots i_p)j_1\cdots j_q e_1\cdots e_s k} + \\ &+ \frac{q}{p+q+s+1}\delta^{h(j_1}\psi^{j_2\cdots j_q)i_1\cdots i_p e_1\cdots e_s k} + \\ &+ \frac{s}{p+q+s+1}\delta^{h(e_1}\psi^{e_2\cdots e_s)i_1\cdots i_p j_1\cdots j_q k} + \\ &+ \frac{1}{p+q+s+1}\delta^{hk}\psi^{i_1\cdots i_p j_1\cdots j_q e_1\cdots e_s}.\end{aligned}$$

2) Moreover we have

$$\phi^{j_2\cdots j_q k i_1\cdots i_p h_1 k_1\cdots h_r k_r ab}_{p,q-1,r+1}\delta_{ab} = \phi_{p,q-1,r+1}\frac{q+p+2r+3}{q+p+2r+1}\delta^{(j_2\cdots j_q k i_1\cdots i_p h_1 k_1\cdots h_r k_r)}.$$

3) Finally, we can express everything in terms of the scalar $h_{p,q,r}$ using the following relations:

$$\begin{cases} \frac{\partial}{\partial\hat{\lambda}}\phi_{p-1,q,r} = h_{p,q,r} & \text{from eq. (2.61)},\\ \phi_{p+1,q,r-1} = h_{p,q,r}, \quad \phi_{p+1,q,r} = h_{p,q,r+1} & \text{from eq. (2.64)},\\ \phi_{p,q-1,r+1} = \frac{p+q+2r+1}{p+q+2r+3}h_{p,q,r} & \text{from eq. (2.68)},\\ \phi_{p,q+1,r} = \frac{p+q+2r+1}{p+q+2r+3}\frac{\partial}{\partial\hat{\lambda}_{aabb}}h_{p,q,r} & \text{from eq. (2.69)}.\end{cases}$$

All these results allow to rewrite eq. (2.77) as

$$0 = h_{p,q,r}(p+q+2r+1)\delta^{h(i_1}\delta^{i_2\cdots h_r k_r k)} + \frac{2}{3}\hat{\lambda}_{ll}\delta^{khi_1\cdots h_r k_r}h_{p,q,r+1} +$$
$$+2q\frac{p+q+2r+1}{p+q+2r+3}\delta^{khi_1\cdots h_r k_r}h_{p,q,r} + 4\hat{\lambda}_{aabb}\frac{p+q+2r+1}{p+q+2r+3}\frac{\partial h_{p,q,r}}{\partial\hat{\lambda}_{aabb}}\delta^{hki_1\cdots h_r k_r},$$

where the notation $\delta^{e_1 e_2\cdots e_{2s}} = \delta^{(e_1 e_2}\cdots\delta^{e_{2s-1}e_{2s})}$ has been used; the result is equivalent to

$$\begin{aligned}0 &= (p+q+2r+1)h_{p,q,r} + \frac{2}{3}\hat{\lambda}_{ll}h_{p,q,r+1} + \\ &+ \frac{p+q+2r+1}{p+q+2r+3}\left(2qh_{p,q,r} + 4\hat{\lambda}_{aabb}\frac{\partial h_{p,q,r}}{\partial\hat{\lambda}_{aabb}}\right).\end{aligned}$$

This equation, by using eqs. (2.57) and (2.60), becomes

$$0 = (p+3q+2r+3)\frac{\partial^r}{\partial\hat{\lambda}^r_{ll}}k_{p,q} + 2\hat{\lambda}_{ll}\frac{\partial^{r+1}}{\partial\hat{\lambda}^{r+1}_{ll}}k_{p,q} + 4\hat{\lambda}_{aabb}\frac{\partial^r}{\partial\hat{\lambda}^r_{ll}}\frac{\partial k_{p,q}}{\partial\hat{\lambda}_{aabb}}$$

with p+q even. We note that if this relation holds until a fixed r taking its derivative with respect to $\hat{\lambda}_{ll}$ we obtain that it holds also with r+1 replacing

r. Therefore, it suffices to impose this relation for the lower value of r, i.e for r=0. In this case it becomes

$$0 = (p + 3q + 3)k_{p,q} + 2\hat{\lambda}_{ll}\frac{\partial}{\partial\hat{\lambda}_{ll}}k_{p,q} + 4\hat{\lambda}_{aabb}\frac{\partial k_{p,q}}{\partial\hat{\lambda}_{aabb}}, \tag{2.78}$$

with p+q even.
Let us firstly analyze the case with p and q even. Putting eq. $(2.67)_1$ into (2.78) we have

$$0 = (p + 3q + 3)\frac{\partial^p}{\partial\hat{\lambda}_{ll}^{\frac{p}{2}}\partial\hat{\lambda}^{\frac{p}{2}}}k_{0,q} + 2\hat{\lambda}_{ll}\frac{\partial^{p+1}}{\partial\hat{\lambda}_{ll}^{\frac{p}{2}+1}\partial\hat{\lambda}^{\frac{p}{2}}}k_{0,q} + 4\hat{\lambda}_{aabb}\frac{\partial^p}{\partial\hat{\lambda}_{ll}^{\frac{p}{2}}\partial\hat{\lambda}^{\frac{p}{2}}}\frac{\partial k_{0,q}}{\partial\hat{\lambda}_{aabb}}.$$

We note that if this relation holds until a fixed p taking its derivative with respect to $\hat{\lambda}_{ll}$ and then with respect to $\hat{\lambda}$, we obtain that it holds also with p+2 replacing p (p must be even). Therefore, it suffices to impose this relation for the lower even value of p, i.e for p=0.
In this case it becomes

$$0 = (3q + 3)k_{0,q} + 2\hat{\lambda}_{ll}\frac{\partial}{\partial\hat{\lambda}_{ll}}k_{0,q} + 4\hat{\lambda}_{aabb}\frac{\partial k_{0,q}}{\partial\hat{\lambda}_{aabb}}, \tag{2.79}$$

that is (2.78) calculated in p=0.
By using eq. (2.73) we see that eq. (2.79) becomes

$$0 = (3q + 3)\frac{\partial^q}{\partial\hat{\lambda}_{ll}^{\frac{q}{2}}\partial\hat{\lambda}_{aabb}^{\frac{q}{2}}}k_{0,0} + 2\hat{\lambda}_{ll}\frac{\partial^{q+1}}{\partial\hat{\lambda}_{ll}^{\frac{q}{2}+1}\partial\hat{\lambda}_{aabb}^{\frac{q}{2}}}k_{0,0} + 4\hat{\lambda}_{aabb}\frac{\partial^q}{\partial\hat{\lambda}_{ll}^{\frac{q}{2}}\partial\hat{\lambda}_{aabb}^{\frac{q}{2}}}\frac{\partial k_{0,0}}{\partial\hat{\lambda}_{ccgg}}.$$

We note that if this relation holds until a fixed q taking its derivative with respect to $\hat{\lambda}_{ll}$ and then with respect to $\hat{\lambda}_{aabb}$, we obtain that it holds also with q+2 replacing q (q must be even). Therefore, it suffices to impose this relation for the lower even order of q, i.e for q=0. In this case it becomes

$$0 = 3k_{0,0} + 2\hat{\lambda}_{ll}\frac{\partial}{\partial\hat{\lambda}_{ll}}k_{0,0} + 4\hat{\lambda}_{aabb}\frac{\partial k_{0,0}}{\partial\hat{\lambda}_{aabb}}, \tag{2.80}$$

that is (2.78) calculated in p=0, q=0.
There remains the case with p and q odd. We will see that it will give only identities. In fact, putting eq. $(2.67)_2$ into (2.78), this becomes

$$\begin{aligned} 0 &= (p + 3q + 3)\frac{\partial^p}{\partial\hat{\lambda}_{ll}^{\frac{p-1}{2}}\partial\hat{\lambda}^{\frac{p+1}{2}}}k_{0,q} + \\ &+ 2\hat{\lambda}_{ll}\frac{\partial^{p+1}}{\partial\hat{\lambda}_{ll}^{\frac{p-1}{2}+1}\partial\hat{\lambda}^{\frac{p+1}{2}}}k_{0,q} + 4\hat{\lambda}_{aabb}\frac{\partial^p}{\partial\hat{\lambda}_{ll}^{\frac{p-1}{2}}\partial\hat{\lambda}^{\frac{p+1}{2}}}\frac{\partial k_{0,q}}{\partial\hat{\lambda}_{aabb}}. \end{aligned}$$

We note that if this relation holds until a fixed p taking its derivative with respect to $\hat{\lambda}_{ll}$ and then with respect to $\hat{\lambda}$, we obtain that it holds also with p+2 replacing p (p must be odd). Therefore, it suffices to impose this relation for the lower odd value of p, i.e p=1. In this case it becomes

$$0 = (3q+4)\frac{\partial}{\partial\hat{\lambda}}k_{0,q} + 2\hat{\lambda}_{ll}\frac{\partial^2}{\partial\hat{\lambda}\partial\hat{\lambda}_{ll}}k_{0,q} + 4\hat{\lambda}_{aabb}\frac{\partial^2 k_{0,q}}{\partial\hat{\lambda}\partial\hat{\lambda}_{aabb}}. \tag{2.81}$$

This relation, by using eq. (2.74) becomes

$$\begin{aligned} 0 &= (3q+4)\frac{\partial^q}{\partial\hat{\lambda}_{ll}^{\frac{q-1}{2}}\partial\hat{\lambda}_{aabb}^{\frac{q+1}{2}}}\frac{\partial}{\partial\hat{\lambda}}k_{0,0} + \\ &+ 2\hat{\lambda}_{ll}\frac{\partial^{q+1}}{\partial\hat{\lambda}_{ll}^{\frac{q-1}{2}+1}\partial\hat{\lambda}_{aabb}^{\frac{q+1}{2}}}\frac{\partial}{\partial\hat{\lambda}}k_{0,0} + 4\hat{\lambda}_{aabb}\frac{\partial^q}{\partial\hat{\lambda}_{ll}^{\frac{q-1}{2}}\partial\hat{\lambda}_{aabb}^{\frac{q+1}{2}}}\frac{\partial^2 k_{0,0}}{\partial\hat{\lambda}_{ccgg}\partial\hat{\lambda}}. \end{aligned}$$

We note that if this relation holds until a fixed q, taking its derivative with respect to $\hat{\lambda}_{ll}$ and then with respect to $\hat{\lambda}_{aabb}$, we obtain that it holds also with q+2 replacing q (q must be odd). Therefore, it suffices to impose this relation for the lower odd value of q, i.e q=1. In this case it becomes

$$0 = 7\frac{\partial^2}{\partial\hat{\lambda}\partial\hat{\lambda}_{ppqq}}k_{0,0} + 2\hat{\lambda}_{ll}\frac{\partial^3}{\partial\hat{\lambda}\partial\hat{\lambda}_{ll}\partial\hat{\lambda}_{ppqq}}k_{0,0} + 4\hat{\lambda}_{ppqq}\frac{\partial^3 k_{0,0}}{\partial\hat{\lambda}\partial\hat{\lambda}_{ppqq}^2},$$

which is a consequence of (2.80) because it is its second derivative with respect to $\hat{\lambda}$ and $\hat{\lambda}_{ppqq}$. In this way, we have seen that the conditions (2.46) and (2.47) give only the restriction (2.80) for $k_{0,0}$ and many identities.
So we have that every element of the matrix $k_{p,q}$ can be expressed as function of $k_{0,0}$ and this is restricted only by eqs. (2.75) **and** (2.80)**.**
Let us conclude by exploiting these conditions and let us do it by using the expansion of $k_{0,0}$ around the state with $\hat{\lambda}_{ppqq} = 0$, i.e.,

$$k_{0,0} = \sum_{s=0}^{\infty}\frac{1}{s!}k_s(\hat{\lambda}, \hat{\lambda}_{ll})\hat{\lambda}_{ppqq}^s. \tag{2.82}$$

Using (2.82), eq. (2.75) becomes

$$9\frac{\partial^2 k_s}{\partial\hat{\lambda}_{ll}^2} = \frac{\partial k_{s+1}}{\partial\hat{\lambda}}, \tag{2.83}$$

while eq. (2.80) transforms into

$$0 = 3\sum_{s=0}^{\infty}\frac{1}{s!}k_s\hat{\lambda}_{ppqq}^s + 2\hat{\lambda}_{ll}\sum_{s=0}^{\infty}\frac{1}{s!}\frac{\partial k_s}{\partial\hat{\lambda}_{ll}}\hat{\lambda}_{ppqq}^s + 4\sum_{s=1}^{\infty}\frac{1}{(s-1)!}k_s\hat{\lambda}_{ppqq}^s$$

i.e. $$\begin{cases} 3k_0 + 2\hat{\lambda}_{ll}\frac{\partial k_0}{\partial \hat{\lambda}_{ll}} = 0 & \text{for s=0,} \\ 3k_s + 2\hat{\lambda}_{ll}\frac{\partial k_s}{\partial \hat{\lambda}_{ll}} + 4sk_s = 0 & \text{for } s \geq 1; \end{cases}$$

but the relation for s=0 is contained in the other equation, so that they can be written as

$$(3+4s)k_s + 2\hat{\lambda}_{ll}\frac{\partial k_s}{\partial \hat{\lambda}_{ll}} = 0 \quad \forall s \geq 0,$$

whose solution is

$$k_s = \hat{\lambda}_{ll}^{-\frac{3+4s}{2}} \widetilde{k}_s(\hat{\lambda}). \tag{2.84}$$

This allows to rewrite eq. (2.83) as

$$\frac{\partial \widetilde{k}_{s+1}}{\partial \hat{\lambda}} = \widetilde{k}_s \frac{9}{4}(3+4s)(5+4s). \tag{2.85}$$

In this way we have found that $\widetilde{k}_0(\hat{\lambda})$ is an arbitrary single-variable function, while the other functions $\widetilde{k}_{s+1}(\hat{\lambda})$ are determined by (2.85), except for a numerable family of constants arising from integration.

2.3.4 The 13 moments model as a subsystem of the 14 moments one

To verify that the 13 moments case is a subsystem of the 14 moments one we will show that the relations obtained in the previous section for the scalar functions $j_{0,q}$ are satisfied by the value of $k_{0,q}$ found here but considering $\hat{\lambda}_{ppqq} = 0$. Firstly we have to rewrite the expressions of $k_{0,q}$. Substituting eq. (2.82) into eq. (2.73) we have

$$\begin{aligned} k_{0,q} &= 3^{\frac{q}{2}} \frac{1}{q+1} \sum_{s=0}^{\infty} \frac{1}{s!} \frac{\partial^{\frac{q}{2}} k_s}{\partial \hat{\lambda}_{ll}^{\frac{q}{2}}} \frac{\partial^{\frac{q}{2}} \hat{\lambda}_{ppqq}^s}{\partial \hat{\lambda}_{ppqq}^{\frac{q}{2}}} = \\ &= 3^{\frac{q}{2}} \frac{1}{q+1} \sum_{s=\frac{q}{2}}^{\infty} \frac{1}{s!} \frac{\partial^{\frac{q}{2}} k_s}{\partial \hat{\lambda}_{ll}^{\frac{q}{2}}} s(s-1)\cdots(s-\frac{q}{2}+1)\hat{\lambda}_{ppqq}^{s-\frac{q}{2}} \end{aligned}$$

If we calculate this for $\hat{\lambda}_{ppqq} = 0$, only the term for $s = \frac{q}{2}$ remains, so our relations become

$$k_{0,q} = 3^{\frac{q}{2}} \frac{1}{q+1} \frac{\partial^{\frac{q}{2}} k_{\frac{q}{2}}}{\partial \hat{\lambda}_{ll}^{\frac{q}{2}}} \quad \text{with q even.}$$

Substituting eq. (2.82) into eq. (2.74), still making the previous considerations, we have

$$k_{0,q} = 3^{\frac{q-1}{2}} \frac{1}{q+2} \frac{\partial^{\frac{q-1}{2}} k_{\frac{q+1}{2}}}{\partial \hat{\lambda}_{ll}^{\frac{q-1}{2}}} \quad \text{with q odd.}$$

Now using eqs. (2.84) we obtain

$$k_{0,q} = \begin{cases} 3^{\frac{q}{2}} \frac{1}{q+1} \left(-\frac{1}{2}\right)^{\frac{q}{2}} \eta(3+2q, 3q+1) \hat{\lambda}_{ll}^{-\frac{3+3q}{2}} \tilde{k}_{\frac{q}{2}} & \text{for q even,} \\ 3^{\frac{q-1}{2}} \frac{1}{q+2} \left(-\frac{1}{2}\right)^{\frac{q-1}{2}} \eta(5+2q, 3q+2) \hat{\lambda}_{ll}^{-\frac{4+3q}{2}} \tilde{k}_{\frac{q+1}{2}} & \text{for q odd.} \end{cases} \tag{2.86}$$

where $\eta(a, b) = a(a-2)(a-4)\cdots(b+2)b$.
Comparing this result with the corresponding one for $h_{0,q,0}$ and $\phi_{0,q,0}$ (i.e. eqs. (2.18) and (2.19)), we find that they are the same, except for identifying

$$\widetilde{h}_{0,q}(\hat{\lambda}) = \left(-\frac{3}{2}\right)^{\frac{q}{2}} \frac{1}{q+1} \eta(3+2q, 3q+1) \widetilde{k}_{\frac{q}{2}}(\hat{\lambda}) \tag{2.87}$$

and for setting $c_q = 0$.
It is easy to verify that with $\widetilde{h}_{0,q}$ given by eq. (2.87), the condition (2.17) becomes exactly the present eq. (2.85), except for substituting q=2s+2, and viceversa. All the other results for the 13 moments model, can be obtained by substituting $\hat{\lambda}_{aabb} = 0$ in the present ones except for the new restriction $c_q = 0$.
In other words, for the 13 moments model, the solution was found except for two families of constants, one arising from integration of eq. (2.17) and another constituted by the constants c_q appearing in eq. (2.19). This second family of constants doesn't appear if the 13 moments model is obtained as a subsystem of the 14 moments one.

2.3.5 The comparison with the kinetic approach

The solution of our conditions proposed by the kinetic approach, see [2] and [20], is

$$\begin{aligned} h' &= \int F(\lambda + \lambda_i c^i + \lambda_{ij} c^i c^j + \lambda_{ill} c^i c^2 + \lambda_{aabb} c^4) dc_1 dc_2 dc_3 \\ \phi'^k &= \int F(\lambda + \lambda_i c^i + \lambda_{ij} c^i c^j + \lambda_{ill} c^i c^2 + \lambda_{aabb} c^4) c^k dc_1 dc_2 dc_3, \end{aligned}$$

(where F is related with the distribution function at equilibrium), and it is easy to see that it satisfies the conditions (2.44), (2.46), (2.47). We can now see that it is a particular case of our general solution. In fact eqs. (2.49) and (2.52) now become

$$\begin{aligned}\phi^{ii_1\cdots i_pj_1\cdots j_qk_1h_1\cdots k_rh_r}_{p,q,r} &= \int F^{(p+q+r)}(\lambda+\frac{1}{3}\lambda_{ll}c^2+\lambda_{aabb}c^4)\\ &\quad c^ic^{i_1}\cdots c^{i_p}c^{j_1}\cdots c^{j_q}c^{2q}c^{h_1}c^{k_1}\cdots c^{h_r}c^{k_r}dc_1dc_2dc_3,\\ h^{i_1\cdots i_pj_1\cdots j_qk_1h_1\cdots k_rh_r}_{p,q,r} &= \int F^{(p+q+r)}(\lambda+\frac{1}{3}\lambda_{ll}c^2+\lambda_{aabb}c^4)\\ &\quad c^{i_1}\cdots c^{i_p}c^{j_1}\cdots c^{j_q}c^{2q}c^{h_1}c^{k_1}\cdots c^{h_r}c^{k_r}dc_1dc_2dc_3,\end{aligned}$$

and it is easy to see that eqs. (2.54) and (2.55) are satisfied.
Eqs. (2.50) and (2.53) hold with

$$\begin{aligned}\phi_{p,q,r} &= \frac{4\pi}{p+q+2r+2}\int_0^\infty F^{(p+q+r)}(\lambda+\frac{1}{3}\lambda_{ll}c^2+\lambda_{aabb}c^4)c^{p+3q+2r+3}dc,\\ h_{p,q,r} &= \frac{4\pi}{p+q+2r+1}\int_0^\infty F^{(p+q+r)}(\lambda+\frac{1}{3}\lambda_{ll}c^2+\lambda_{aabb}c^4)c^{p+3q+2r+2}dc.\end{aligned}$$

Eqs. (2.57) and (2.59) are consequences of these. The definitions (2.60) now become

$$\begin{aligned}k_{p,q} &= \frac{4\pi}{p+q+1}\int_0^\infty F^{(p+q)}(\lambda+\frac{1}{3}\lambda_{ll}c^2+\lambda_{aabb}c^4)c^{p+3q+2}dc \quad \text{if p+q is even},\\ k_{p,q} &= \frac{4\pi}{p+q+2}\int_0^\infty F^{(p+q)}(\lambda+\frac{1}{3}\lambda_{ll}c^2+\lambda_{aabb}c^4)c^{p+3q+3}dc \quad \text{if p+q is odd}.\end{aligned}$$

From these it follows

$$k_{0,0} = 4\pi\int_0^\infty F(\lambda+\frac{1}{3}\lambda_{ll}c^2+\lambda_{aabb}c^4)c^2dc$$

and it is not difficult to see that eqs. (2.75) and (2.76) are satisfied.
Proof of eq. (2.80) needs an integration by parts, as follows

$$\begin{aligned}0 &= 3\int_0^\infty Fc^2dc+\frac{2}{3}\lambda_{ll}\int_0^\infty F'c^4dc+4\lambda_{aabb}\int_0^\infty F'c^6dc=\\ &= 3\int_0^\infty Fc^2dc+\int_0^\infty\left(\frac{dF}{dc}\right)c^3dc = 3\int_0^\infty Fc^2dc+\left|Fc^3\right|_0^\infty-\int_0^\infty 3Fc^2dc\end{aligned}$$

which is satisfied because

$$\lim_{c\to\infty}Fc^3=0.$$

We can now see that eq. (2.82) holds with

$$k_s = 4\pi \int_0^\infty F^{(s)}(\lambda + \frac{1}{3}\lambda_{ll}c^2)c^{4s+2}dc,$$

of which eq. (2.83) is an easy consequence.
By using the change of the integration variables $c = \eta\lambda_{ll}^{-\frac{1}{2}}$, we obtain eq. (2.84) with

$$\widetilde{k}_s = 4\pi \int_0^\infty F^{(s)}(\lambda + \frac{1}{3}\eta^2)\eta^{4s+2}d\eta. \tag{2.88}$$

Proof of eq. (2.84) needs two integrations by part, as follows

$$\begin{aligned}
\frac{d}{d\lambda}\widetilde{k}_{s+1} &= 4\pi \int_0^\infty F^{(s+2)}(\lambda + \frac{1}{3}\eta^2)\eta^{4s+6}d\eta = \\
&= \left|4\pi F^{(s+1)}(\lambda + \frac{1}{3}\eta^2)\frac{3}{2}\eta^{4s+5}\right|_0^\infty + \\
&- \int_0^\infty 6\pi(4s+5)F^{(s+1)}(\lambda + \frac{1}{3}\eta^2)\eta^{4s+4}d\eta = \\
&= \left|-6\pi(4s+5)F^{(s)}(\lambda + \frac{1}{3}\eta^2)\frac{3}{2}\eta^{4s+3}\right|_0^\infty + \\
&- \int_0^\infty -9\pi(4s+5)(4s+3)F^{(s)}(\lambda + \frac{1}{3}\eta^2)\eta^{4s+2}d\eta = \\
&= \frac{9}{4}(4s+3)(4s+5)\widetilde{k}_s.
\end{aligned}$$

Consequently, the kinetic approach suggest to take

$$\widetilde{k}_0(\lambda) = 4\pi \int_0^\infty F(\lambda + \frac{1}{3}\eta^2)\eta^2 d\eta,$$

which is only a change from our arbitrary function $\widetilde{k}_0(\lambda)$ to the arbitrary function F; moreover it considers only a particular solution of the eqs. (2.85), i.e., eq. (2.88). In this way the numerable family of arbitrary constants arising from integration of eq. (2.85) doesn't appear in the kinetic approach. Then the macroscopic approach here considered is more general than the kinetic one.

Chapter 3

Dense gases and macromolecular fluids

In previous chapters we have considered ideal gases but this case doesn't take into account the interactions between atoms and molecules. In this chapter we will apply the methodology described in the previous chapter to less restrictive model describing more complex fluids. In particular we will find the solutions of interesting models as that for dense gases and macromolecular fluids with 13 and 14 moments.

3.1 The 13 moments case

For dense gases and macromolecular fluids the symmetry of system (1.34) is lost, so that the appropriate equations are

$$\begin{aligned} \partial_t F &+ \partial_k F_k = 0\,, \\ \partial_t F_i &+ \partial_k G_{ik} = 0\,, \\ \partial_t F_{ij} &+ \partial_k G_{ijk} = P_{<ij>}\,, \\ \partial_t F_{ill} &+ \partial_k G_{illk} = P_{ill}\,, \end{aligned} \tag{3.1}$$

with $F_{ij} = F_{ji}$, $G_{ijk} = G_{jik}$, $P_{<ij>} = P_{<ji>}$, and this last tensor, with P_{ill} are the production terms. If we consider also the conditions $G_{ik} = F_{ik}$, $G_{ill} = F_{ill}$ we came back to the case of ideal gases.
The entropy law reads

$$\begin{aligned} dh &= \lambda dF + \lambda_i dF_i + \lambda_{ij} dF^{ij} + \lambda_{ill} dF^{ill}\,, \\ d\phi_k &= \lambda dF_k + \lambda_i dG_{ik} + \lambda_{ij} dG_{ijk} + \lambda_{ill} dG_{illk}\,, \end{aligned} \tag{3.2}$$

plus the residual inequality which we leave out for the sake of brevity.
By taking λ, λ_i, λ_{ij}, λ_{ill} as independent variables, and defining the potentials

$$\begin{aligned} \tilde{h} &= \lambda F + \lambda_i F^i + \lambda_{ij} F^{ij} + \lambda_{ill} F^{ill} - h \,, \\ \tilde{\phi}_k &= \lambda F_k + \lambda_i G_{ik} + \lambda_{ij} G_{ijk} + \lambda_{ill} G_{illk} - \phi_k \,, \end{aligned} \tag{3.3}$$

eqs. (3.2) become

$$F = \frac{\partial \tilde{h}}{\partial \lambda}, \quad F^i = \frac{\partial \tilde{h}}{\partial \lambda_i}, \quad F^{ij} = \frac{\partial \tilde{h}}{\partial \lambda_{ij}}, \quad F^{ill} = \frac{\partial \tilde{h}}{\partial \lambda_{ill}} \,, \tag{3.4}$$

$$\frac{\partial \tilde{\phi}_k}{\partial \lambda} = \frac{\partial \tilde{h}}{\partial \lambda^k}, \quad G_{ik} = \frac{\partial \tilde{\phi}_k}{\partial \lambda_i}, \quad G_{ijk} = \frac{\partial \tilde{\phi}_k}{\partial \lambda_{ij}}, \quad G_{illk} = \frac{\partial \tilde{\phi}_k}{\partial \lambda_{ill}} \,. \tag{3.5}$$

In order to impose the principle of material objectivity, let us consider the following change of independent variables

$$\begin{aligned} F &= m \\ F_i &= m v_i \\ F_{ij} &= m v_i v_j + m_{ij} \\ F_{ill} &= m_{ill} + m_{ll} v_i + 2 m_{il} v_l + m v^2 v_i \end{aligned} \tag{3.6}$$

and of constitutive functions

$$\begin{aligned} G_{ik} &= m v_i v_k + M_{ik} \,, \\ G_{ijk} &= F_{ij} v_k + 2 v_{(i} M_{j)k} + M_{ijk} \,, \\ G_{illk} &= F_{ill} v_k + v^2 M_{ik} + 2 v_i v_l M_{lk} + v_i M_{llk} + 2 v_l M_{lik} + M_{illk} \,. \end{aligned} \tag{3.7}$$

The principle of material objectivity implies that h, $\phi_k - h v_k$, M_{ik}, M_{ijk}, M_{illk}, M_i don't depend on v_i. Imposing this condition for h and $\phi_k - h v_k$ we obtain

$$\begin{aligned} 0 &= F \lambda_a + 2 \lambda_{ia} F_i + \lambda_{ill} (F_{ll} \delta_{ia} + 2 F_{ia}) \,, \\ 0 &= F_k \lambda_a + 2 \lambda_{ia} G_{ik} + \lambda_{ill} (G_{llk} \delta_{ia} + 2 G_{iak}) + \\ &+ (\lambda F + \lambda_i F_i + \lambda_{ij} F_{ij} + \lambda_{ill} F_{ill} - h) \delta_{ka} \,; \end{aligned} \tag{3.8}$$

where eqs (3.2) have been used. The independence of M_{ik}, M_{ijk}, M_{illk}, M_i on v_i follows as consequence. In fact, eqs. (3.2) now become

$$\begin{aligned} dh &= \lambda^I dm + \lambda^I_{ij} dm_{ij} + \lambda^I_{ill} dm_{ill} \\ d(\phi_k - h v_k) &= \lambda^I_i dM_{ik} + \lambda^I_{ij} dM_{ijk} + \lambda^I_{ill} dM_{illk} \end{aligned} \tag{3.9}$$

with

$$\begin{aligned}
\lambda^I &= \lambda + \lambda_i v_i + \lambda_{ij} v_i v_j + \lambda_{ill} v_i v^2 \, , \\
\lambda^I_i &= \lambda_i + 2\lambda_{ai} v_a + \lambda_{ill} v^2 + 2\lambda_{all} v_a v_i \, , \\
\lambda^I_{ij} &= \lambda_{ij} + \lambda_{all} v_a \delta_{ij} + 2\lambda_{ll(i} v_{j)} \, , \\
\lambda^I_{ill} &= \lambda_{ill} \, .
\end{aligned}$$

from eq. $(3.9)_1$ we see that λ^I, λ^I_{ij}, λ^I_{ill} don't depend on v_i (because $\frac{\partial h}{\partial m} = \lambda^I$ but h and m don't depend on v_i, similarly for the others); but eq. $(3.8)_1$ can be written also as

$$0 = m\lambda^I_a + \lambda^I_{ill}(m_{ll}\delta_{ia} + 2m_{ia}) \, , \tag{3.10}$$

so that also λ^I_i doesn't depend on v_i. By defining h' and ϕ'_k from

$$\begin{aligned}
h &= \lambda^I m + \lambda^I_{ij} m_{ij} + \lambda^I_{ill} m_{ill} - h' \\
\phi_k - h v_k &= \lambda^I_i M_{ik} + \lambda^I_{ij} M_{ijk} + \lambda^I_{ill} M_{illk} - \phi'_k
\end{aligned}$$

the eqs. (3.9) become

$$\begin{aligned}
dh' &= m d\lambda^I + m_{ij} d\lambda^I_{ij} + m_{ill} d\lambda^I_{ill} \\
d\phi'_k &= M_{rk} d\lambda^I_r + M_{ijk} d\lambda^I_{ij} + M_{illk} d\lambda^I_{ill}
\end{aligned}$$

from which by taking λ^I, λ^I_{ij}, λ^I_{ill} as independent variables, it follows

$$\begin{aligned}
m &= \frac{\partial h'}{\partial \lambda^I}, \quad m_{ij} = \frac{\partial h'}{\partial \lambda^I_{ij}}, \quad m_{ill} = \frac{\partial h'}{\partial \lambda^I_{ill}} \, , \qquad (3.11) \\
\frac{\partial \phi'_k}{\partial \lambda^I} &= M_{rk} \frac{\partial \lambda^I_r}{\partial \lambda^I}, \quad \frac{\partial \phi'_k}{\partial \lambda^I_{ij}} = M_{rk} \frac{\partial \lambda^I_r}{\partial \lambda^I_{ij}} + M_{ijk}, \quad \frac{\partial \phi'_k}{\partial \lambda^I_{ill}} = M_{rk} \frac{\partial \lambda^I_r}{\partial \lambda^I_{ill}} + M_{illk} \, .
\end{aligned}$$

moreover, the sum of eq. $(3.8)_1$, pre-multiplied by $-v_k$, and of eq. $(3.8)_2$ becomes

$$0 = 2\lambda^I_{ia} M_{ik} + \lambda^I_{ill}(M_{llk}\delta_{ia} + 2M_{iak}) + h'\delta_{ka} \, , \tag{3.12}$$

or, by using $(3.11)_{4,6}$,

$$0 = \left[2\lambda^I_{ra} - \lambda^I_{all} \frac{\partial \lambda^I_r}{\partial \lambda^I_{ij}} \delta_{ij} - 2\lambda^I_{ill} \frac{\partial \lambda^I_r}{\partial \lambda^I_{ia}} \right] M_{rk} + \lambda^I_{all} \frac{\partial \phi'_k}{\partial \lambda^I_{ij}} \delta_{ij} + 2\lambda^I_{ill} \frac{\partial \phi'_k}{\partial \lambda^I_{ia}} + h'\delta_{ka} \tag{3.13}$$

From this relation we see that M_{rk} doesn't depend on v_i; let us prove this by the iterative procedure on the order respect to the state with $\lambda^I_{ra} = \frac{1}{3}\lambda^I_{ll}\delta_{ra,}$, $\lambda^I_{<ra>} = 0$, $\lambda^I_{all} = 0$. Equation (3.13) at the order N gives

$$\frac{2}{3}\lambda^I_{ll}\left(M_{ak}\right)^N + \sum_{q=0}^{N-1}(M_{rk})^q \left[2\lambda^I_{ra} - \lambda^I_{all}\frac{\partial\lambda^I_r}{\partial\lambda^I_{ij}}\delta_{ij} - 2\lambda^I_{ill}\frac{\partial\lambda^I_r}{\partial\lambda^I_{ia}}\right]^{N-q}$$

as a function of quantities not depending on v_i. (here $(\cdots)^q$ denotes the expression of $(\cdots)$ at the order q). For example, for $N = 0$, we obtain that M^0_{ak} doesn't depend on v_i; by assuming, via the iterative procedure, that also $(M_{ak})^q$ satisfies this property for $q \leq N-1$, it follows that also $(M_{ak})^N$ satisfies it. After that, $(3.11)_{6,7,8}$ show that also M_{ijk}, M_{illk} and M_j don't depend on v_i. In this way we have proved that entropy principle and the principle of material objectivity amount simply to conditions $(3.11)_4$, (3.10) and (3.12).
In order to solve the conditions (3.10)-(3.12), let us firstly consider another mathematical problem: we look for two functions $h^*(\lambda^I,\ \lambda^I_i,\ \lambda^I_{ij},\ \lambda^I_{ill})$ and $\phi^*_k(\lambda^I,\ \lambda^I_i,\ \lambda^I_{ij},\ \lambda^I_{ill})$ that satisfy the subsequent

$$m = \frac{\partial h^*}{\partial\lambda^I}, \quad m_{ij} = \frac{\partial h^*}{\partial\lambda^I_{ij}}, \quad m_{ill} = \frac{\partial h^*}{\partial\lambda^I_{ill}}, \tag{3.14}$$

$$\frac{\partial\phi^*_k}{\partial\lambda^I} = \frac{\partial h^*}{\partial\lambda^I_k}, \quad \frac{\partial\phi^*_k}{\partial\lambda^I_i} = M_{ik}, \quad \frac{\partial\phi^*_k}{\partial\lambda^I_{ij}} = M_{ijk}, \quad \frac{\partial\phi^*_k}{\partial\lambda^I_{ill}} = M_{illk}, \tag{3.15}$$

$$\begin{aligned} 0 &= \frac{\partial h^*}{\partial\lambda^I}\lambda^I_a + 2\frac{\partial h^*}{\partial\lambda^I_i}\lambda^I_{ia} + \lambda^I_{ill}\left(\frac{\partial h^*}{\partial\lambda^I_{rs}}\delta_{rs}\delta_{ia} + 2\frac{\partial h^*}{\partial\lambda^I_{ia}}\right), \\ 0 &= \frac{\partial\phi^*_k}{\partial\lambda^I}\lambda^I_a + 2\frac{\partial\phi^*_k}{\partial\lambda^I_i}\lambda^I_{ia} + \lambda^I_{ill}\left(\frac{\partial\phi^*_k}{\partial\lambda^I_{rs}}\delta_{rs}\delta_{ia} + 2\frac{\partial\phi^*_k}{\partial\lambda^I_{ia}}\right) + h^*\delta_{ka}. \end{aligned} \tag{3.16}$$

After that, we consider λ^I_i implicitly defined by the equation $0 = \frac{\partial h^*}{\partial\lambda^I_i}$. Well, h^* and ϕ^*_k calculated in this value of λ^I_i are exactly the functions h' and ϕ'_k (respectively) satisfying the eqs. (3.10)-(3.12). So let us begin with the mathematical problem (3.14)-(3.16).
We look for a solution, of the conditions $(3.15)_1$-(3.16), of the type

$$\phi^*_k = \phi^0_k + \phi^*_{0k}(\lambda^I_i, \lambda^I_{ij}, \lambda^I_{ill}, \lambda^I_{ppqq}) \tag{3.17}$$

where h^* and ϕ^0_k are the functions $\hat{h}'$ and $\hat{\phi'^k}$ of chapter 2 satisfying eqs. (2.6) and (2.7).

But all relations are certainly satisfied if $\phi_{0k}^{0*}=0$, because in this case they are nothing else than the corresponding ones in ideal gases. There we have found the solution (3.17) with $\phi_{0k}^{*}=0$. Obviously, eq. (3.17) satisfies the conditions $(3.16)_2$ iff

$$0 \;=\; 2\frac{\partial \phi_{0k}^{*}}{\partial \lambda_i^I}\lambda_{ia}^I + \left(2\frac{\partial \phi_{0k}^{*}}{\partial \lambda_{ia}^I} + \frac{\partial \phi_{0k}^{*}}{\partial \lambda_{rs}^I}\delta_{rs}\delta_{ia}\right)\lambda_{ill}^I; \tag{3.18}$$

let us impose this with an expansion with respect to the state s where $\lambda_i^I=0$, $\lambda_{<ia>}^I=0$, $\lambda_{ill}^I=0$. The symbol ϕ_{0k}^{N*} denotes the expression of ϕ_{0k}^{*} of order N with respect to this state. Obviously, we have $\phi_{0k}^{0*}=0$ because at the order 0, ϕ_{0k}^{*} may depend only on λ_{ll}^I. We shall see that, by imposing eq. (3.18) at order N, we find ϕ_{0k}^{N+1*} except for terms not depending on λ_i^I which, on the other hand , can be also found with the representation theorems. In fact, eq. (3.18) at the order zero gives

$$0=\frac{2}{3}\lambda_{ll}^I\frac{\partial \phi_{0k}^{1*}}{\partial \lambda_a^I}$$

from which ϕ_{0k}^{1*} doesn't depend on λ_a^I. But we have already seen that $\phi_{0k}^{0*}=0$ so that up to the order 1, we have that ϕ_{0k}^{*} is given by

$$\phi_{0k}^{1*} \;=\; f_1(\lambda_{ll}^I)\lambda_{kll}^I\,, \tag{3.19}$$

with f_1 arbitrary function. Eq. (3.18) at the order 1 is

$$0 \;=\; \frac{2}{3}\lambda_{ll}^I\frac{\partial \phi_{0k}^{2*}}{\partial \lambda_a^I} + 2\lambda_{<ia>}^I\cancel{\frac{\partial \phi_{0k}^{1*}}{\partial \lambda_i^I}} + \left(2\cancel{\frac{\partial \phi_{0k}^{1*}}{\partial \lambda_{<rs>}^I}}\delta_{<i}^r\delta_{a>}^s + 5\cancel{\frac{\partial \phi_{0k}^{0*}}{\partial \lambda_{ll}^I}}\,\delta_{ia}\right)\lambda_{ill}^I$$

from which

$$\phi_{0k}^{2*} \;=\; f_2(\lambda_{ll}^I)\lambda_{<ki>}^I\lambda_{ill}^I\,, \tag{3.20}$$

with f_2 arbitrary function. Eq. (3.18) at the order 2 is

$$0 \;=\; \frac{2}{3}\lambda_{ll}^I\frac{\partial \phi_{0k}^{3*}}{\partial \lambda_a^I} + 2\lambda_{<ia>}^I\frac{\partial \phi_{0k}^{2*}}{\partial \lambda_i^I} + \left(2\frac{\partial \phi_{0k}^{2*}}{\partial \lambda_{<rs>}^I}\delta_{<i}^r\delta_{a>}^s + 5\frac{\partial \phi_{0k}^{1*}}{\partial \lambda_{ll}^I}\delta_{ia}\right)\lambda_{ill}^I$$

from which

$$\begin{aligned}\phi_{0k}^{3*} \;=\;& -\frac{3}{2}f_2(\lambda_{ll}^I)^{-1}(\lambda_{rll}^I\lambda_{rll}^I)\lambda_k^I - \frac{1}{2}(f_2+15f_1')(\lambda_{ll}^I)^{-1}(\lambda_r^I\lambda_{rll}^I)\lambda_{kll}^I + \\ &+\; \left[f_3(\lambda_{ll}^I)(\lambda_{rll}^I\lambda_{rll}^I) + f_4(\lambda_{ll}^I)(tr(\lambda_{<rs>}^I)^2)\right]\lambda_{kll}^I + f_5(\lambda_{ll}^I)(\lambda_{<kr>}^I)^2\lambda_{rll}^I\,;\end{aligned} \tag{3.21}$$

with $f_3, \;f_4, \;f_5$ arbitrary function. Eq. (3.18) at the order 3 gives

$$
\begin{aligned}
\phi^{4*}_{0k} &= -\frac{3}{2} f_5 (\lambda^I_{ll})^{-1} (\lambda^I_{<rs>} \lambda^I_{rll} \lambda^I_{sll}) \lambda^I_k + \\
&+ \frac{3}{2} \left[f_2 + 15 f_1')(\lambda^I_{ll})^{-2} - (4 f_4 + f_5)(\lambda^I_{ll})^{-1} \right] \cdot (\lambda^I_{<rs>} \lambda^I_r \lambda^I_{sll}) \lambda^I_{kll} + \\
&+ \frac{3}{2} (\lambda^I_{ll})^{-2} (3 f_2 - f_5 \lambda^I_{ll})(\lambda^I_{rll} \lambda^I_{rll}) \lambda^I_{<ka>} \lambda^I_a + \\
&+ \frac{1}{2} (f_5 - 15 f_2')(\lambda^I_{ll})^{-1} (\lambda^I_r \lambda^I_{rll}) \lambda^I_{<ks>} \lambda^I_{sll} + \text{ terms not depending on } \lambda_i
\end{aligned}
\tag{3.22}
$$

and so on.

3.2 The 14 moments case

The appropriate equations for this model are

$$
\begin{aligned}
\partial_t F &+ \partial_k F_k = 0\,, \\
\partial_t F_i &+ \partial_k G_{ik} = 0\,, \\
\partial_t F_{ij} &+ \partial_k G_{ijk} = P_{<ij>}\,, \\
\partial_t F_{ill} &+ \partial_k G_{illk} = P_{ill}\,, \\
\partial_t F_{iill} &+ \partial_k G_{iillk} = P_{iill}\,,
\end{aligned}
\tag{3.23}
$$

where F, F_i, F_{ij},F_{ill}, F_{iill} are the independent variables and they are completely symmetric tensors. $P_{<ij>}$, P_{ill}, P_{iill} are the productions and they are completely symmetric too. G_{ik},G_{ijk}, G_{illk}, G_{iillk} are the constitutive functions and are symmetric with respect to all indexes except for the index k. The tensors $G_{...}$ and $F_{...}$ are related by the following law

$$
G_{i_1 \cdots i_n k} = F_{i_1 \cdots i_n} v_k + H_{i_1 \cdots i_n k}. \tag{3.24}
$$

A we already now, the entropy law is equivalent to the assumption of the existence of the Lagrange multipliers λ, λ_i, λ_{ij}, λ_{ill}, λ_{iill} such that

$$
\begin{aligned}
dh &= \Lambda dF + \Lambda_i dF_i + \Lambda_{ij} dF^{ij} + \Lambda_{ill} dF^{ill} + \Lambda_{iill} dF^{iill}\,, \\
d\phi_k &= \Lambda dF_k + \Lambda_i dG_{ik} + \Lambda_{ij} dG_{ijk} + \Lambda_{ill} dG_{illk} + \Lambda_{iill} dG_{iillk}
\end{aligned}
$$

plus the residual inequality that we will not consider in this treatment.
Under a change of Galileanly equivalent frames the independent variables satisfy the conditions above

$$
F_{i_1 i_2 \ldots i_n} = \sum_{k=0}^{n} \binom{n}{k} m_{(i_1 i_2 \ldots i_k} v_{i_{k+1} \ldots i_n)} \tag{3.25}
$$

that , in our particular case becomes

$$
\begin{aligned}
F &= m, \qquad (3.26)\\
F_i &= mv_i + m_i,\\
F_{ij} &= mv_iv_j + m_{ij} + 2m_{(i}v_{j)}\ ,\\
F_{ill} &= m_{ill} + m_{ll}v_i + 2m_{il}v_l + mv^2v_i + m_iv^2 + 2m_lv_iv_l\ ,\\
F_{iill} &= m_{iill} + mv^4 + 4m_iv_iv^2 + 2m_{ii}v^2 + 4m_{il}v_iv_l + 4m_{iil}v_l.
\end{aligned}
$$

while for the constitutive functions holds

$$
\begin{aligned}
G_{ik} &= mv_iv_k + m_iv_k + M_kv_i + M_{ik}\ ,\\
G_{ijk} &= mv_iv_jv_k + m_{ij}v_k + 2m_{(i}v_{j)}v_k + M_kv_iv_j + 2M_{k(i}v_{j)} + M_{ijk}\ ,\\
G_{illk} &= m_{ill}v_k + m_{ll}v_iv_k + 2m_{il}v_lv_k + mv^2v_iv_k + m_iv^2v_k +\\
&+ 2m_lv_iv_lv_k + M_{ik}v^2 + 2M_{lk}v_iv_l + M_{llk}v_i + 2M_{ilk}v_l +\\
&+ M_kv_iv^2 + M_{illk},\\
G_{iillk} &= m_{iill}v_k + mv^4v_k + 4m_iv_iv^2v_k + 2m_{ii}v^2v_k + 4m_{il}v_iv_lv_k +\\
&+ 4m_{iil}v_lv_k + M_kv^4 + 4M_{ik}v_iv^2 + 2M_{iik}v^2 + 4M_{ilk}v_iv_l +\\
&+ 4M_{iilk}v_l + M_{iillk}.
\end{aligned}
$$

see [21] for details. The new variables m, m_i, m_{ij}, m_{ijk} and M, M_i, M_{ij}, M_{ijk} satisfy the same symmetry property of $F_{i_1\ldots i_n}$ and $G_{i_1\ldots i_n k}$ respectively. If we put the equations for independent variables and constitutive functions into eqs. (3.25) they become

$$
\begin{aligned}
dh &= \lambda dm + \lambda_i dm_i + \lambda_{ij}dm_{ij} + \lambda_{ill}dm_{ill} +\\
&+ \lambda_{iill}dm_{iill} + (\lambda_i m + 2\lambda_{ij}m_j + \lambda_{jll}m_{ll}\delta_{ij} +\\
&+ 2\lambda_{jll}m_{ij} + 4\lambda_{ppqq}m_{ill})dv_i \qquad (3.27)
\end{aligned}
$$

$$
\begin{aligned}
d(\phi_k) &= (\lambda dm + \lambda_i dm_i + \lambda_{ij}dm_{ij} + \lambda_{ill}dm_{ill}\\
&+ \lambda_{iill}dm_{iill})v_k + \lambda dM_k + \lambda_i dM_{ik} + \lambda_{il}dM_{ilk} +\\
&+ \lambda_{ill}dM_{illk} + \lambda_{iill}dM_{iillk} + (\lambda m + \lambda_i mi + \lambda_{ij}m_{ij}\\
&+ \lambda_{ill}m_{ill} + \lambda_{ppqq}m_{iill})v_kdv_i + (\lambda_i M_k + 2\lambda_{ij}M_{jk}\\
&+ \lambda_{ipp}M_{llk} + 2\lambda_{lpp}M_{ilk} + 4\lambda_{ppqq}M_{illk})dv_i, \qquad (3.28)
\end{aligned}
$$

with

$$
\begin{aligned}
\lambda &= \Lambda + \Lambda_iv_i + \Lambda_{ij}v_iv_j + \Lambda_{ill}v_iv^2 + \Lambda_{ppqq}v^4\ ,\\
\lambda_i &= \Lambda_i + 2\Lambda_{ij}v_j + \Lambda_{ill}v^2 + 2\Lambda_{jpp}v_iv_j + 4\Lambda_{ppqq}v_iv^2\ ,\\
\lambda_{ij} &= \Lambda_{ij} + \Lambda_{hpp}v_h\delta_{il} + 2\Lambda_{(ipp}v_{j)} + 4\Lambda_{ppqq}v_jv_i + 2\Lambda_{hhpp}v^2\delta_{ij}\ ,\\
\lambda_{ill} &= \Lambda_{ipp} + 4\Lambda_{hhpp}v_i\\
\lambda_{iill} &= \Lambda_{ppqq}
\end{aligned}
$$

By imposing that h, $\phi_k - hv_k$, M_{ik}, M_{ijk}, M_{illk}, M_i don't depend on velocity we obtain

$$\begin{aligned}
\frac{\partial h}{\partial v_i} &= 0 = m\lambda_i + 2\lambda_{ij}m_j + \lambda_{ipp}(m_{ll}\delta_{hj} + 2m_{ij}) + 4\lambda_{ppqq}m_{ill} \\
\frac{\partial(\phi^k - hv_k)}{\partial v_i} &= 0 = M_k\lambda_i + 2M_{jk}\lambda_{ij} + M_{llk}\lambda_{ipp} + 2M_{ilk}\lambda_{lpp} + \\
&+ \ 4\lambda_{ppqq}M_{illk} + \tilde{h'}\delta_{ik}. \qquad (3.29)
\end{aligned}$$

In such a way eqs. (3.27) and (3.28) becomes respectively:

$$\begin{aligned}
dh^I &= \lambda dm + \lambda_i dm_i + \lambda_{il}dm_{il} + \lambda_{ill}dm_{ill} + \lambda_{iill}dm_{iill} \\
d\phi_k^I &= \lambda dM_k + \lambda_i dM_{ik} + \lambda_{il}dM_{ilk} + \lambda_{ill}dM_{illk} + \lambda_{ppqq}dM_{iillk}
\end{aligned}$$

with $\phi_k^I = \phi_k - hv_k$. If we define

$$\begin{aligned}
h' &= \lambda m + \lambda_i m_i + \lambda_{il}m_{il} + \lambda_{ill}m_{ill} + \lambda_{iill}m_{iill} - h^I \\
\phi_k' &= \lambda M_k + \lambda_i M_{ik} + \lambda_{il}M_{ilk} + \lambda_{ill}M_{illk} + \lambda_{ppqq}M_{iillk} - \phi_k
\end{aligned}$$

eqs (3.30) becomes

$$\begin{aligned}
dh' &= md\lambda + m_i d\lambda_i + m_{il}d\lambda_{il} + m_{ill}d\lambda_{ill} + m_{iill}d\lambda_{iill} \\
d\phi_k' &= M_k d\lambda + M_{ik}d\lambda_i + M_{ilk}d\lambda_{il} + M_{illk}d\lambda_{ill} + M_{iillk}d\lambda_{ppqq}
\end{aligned}$$

from which, by taking λ^I, λ_{ij}^I,λ_{ij}^I λ_{iill}^I as independent variables and by taking the partial derivatives with respect to them, it follows

$$\begin{aligned}
m &= \frac{\partial h'}{\partial \lambda}, \quad m_i = \frac{\partial h'}{\partial \lambda_i}, \quad m_{il} = \frac{\partial h'}{\partial \lambda_{il}}, \\
m_{ill} &= \frac{\partial h'}{\partial \lambda_{ill}}, \quad m_{iill} = \frac{\partial h'}{\partial \lambda_{iill}}\,, \qquad (3.30) \\
M_k &= \frac{\partial \phi_k'}{\partial \lambda}, \quad M_{ik} = \frac{\partial \phi_k'}{\partial \lambda_i}, \quad M_{ilk} = \frac{\partial \phi_k'}{\partial \lambda_{il}}, \\
M_{illk} &= \frac{\partial \phi_k'}{\partial \lambda_{ill}}, \quad M_{iillk} = \frac{\partial \phi_k'}{\partial \lambda_{iill}}. \qquad (3.31)
\end{aligned}$$

Remembering that $M_k = m_k$ it is possible to compare the corresponding terms obtaining the following compatibility condition:

$$\frac{\partial \phi_k'}{\partial \lambda} = \frac{\partial h'}{\partial \lambda_k}\,. \qquad (3.32)$$

So, to solve our problem we have to find h' and ϕ'_k such that they satisfy equations (3.29), (3.30) and (3.32). To this end we look for a solution of the type

$$\begin{aligned} h' &= \int f(\lambda + \lambda_i c_i + \lambda_{ij} c_i c_j + \lambda_{ill} c_i c^2 + \lambda_{ppll} c^4) d\underline{c}, \\ \phi'_k &= \phi'_{0k} + \widetilde{\phi}_k(\lambda, \lambda_i, \lambda_{ij}, \lambda_{ill}, \lambda_{ppll}), \\ \phi'_{0k} &= \int f(\cdots) c_k d\underline{c}, \end{aligned} \tag{3.33}$$

where c_i are the integration variables in the phase-space.
It is easy to see that all our conditions are satisfied if $\widetilde{\phi}_k = 0$, because in this case h' and ϕ'_k are the corresponding ones for ideal gases and with the kinetic approach. For this reason we call the present solution a "kinetic type" solution. Obviously (3.33) satisfy the present conditions iff

$$2\lambda_{ij} \frac{\partial \tilde{\phi}_k}{\partial \lambda_j} + \lambda_{jll} \left(\frac{\partial \tilde{\phi}_k}{\partial \lambda_{rs}} \delta_{rs} \delta_{ij} + 2 \frac{\partial \tilde{\phi}_k}{\partial \lambda_{ij}} \right) + 4\lambda_{ppqq} \frac{\partial \tilde{\phi}_k}{\partial \lambda_{ill}} = 0. \tag{3.34}$$

We want to find the expression of $\widetilde{\phi}'_k$ up to third order with respect to thermodynamical equilibrium. By using the representation theorems we have

$$\tilde{\phi}_k = a\lambda_i + b\lambda_{ij}\lambda_j + c\lambda_{ij}^2 \lambda i + d\lambda_{ill} + e\lambda_{ill}\lambda_{ij} + f\lambda_{ij}^2 \lambda_{ill}$$

where a, b, c, d, e, f are all scalar functions of the Lagrange multipliers to which we can apply newly the representation theorems. Finally we have that

$$\begin{aligned} \tilde{\phi}_k &= \lambda_k \big[a_1 + a_2 \lambda_{ppqq} + a_3 {\lambda_{ppqq}}^2 + \\ &+ a_4 \lambda_{<ij>} \lambda_{<ij>} + a_5 \lambda_p \lambda_p + a_6 \lambda_{pll} \lambda_{pll} + a_7 \lambda_p \lambda_{pll} \big] + \\ &+ \lambda_{<kj>} \lambda_j \big[b_1 + b_2 \lambda_{ppqq} \big] + c_1 \lambda_{<kh>} \lambda_{<hj>} \lambda_j + \\ &+ \big[d_1 + d_2 \lambda_{ppqq} + d_3 {\lambda_{ppqq}}^2 + d_4 \lambda_{<ij>} \lambda_{<ij>} + d_5 \lambda_p \lambda_p + \\ &+ d_6 \lambda_{pll} \lambda_{pll} + d_7 \lambda_p \lambda_{pll} \big] \lambda_{kll} + \big[e_1 + e_2 \lambda_{ppqq} \big] \lambda_{ill} \lambda_{<ik>} + \\ &+ f_1 \lambda_{<kh>} \lambda_{<hj>} \lambda_{jll} \end{aligned} \tag{3.35}$$

with a_1 ... f_1 are functions of λ^I_{ll}.
Starting from eq. (3.35) we can consider the different order with respect to equilibrium. At zero order eq. (3.35) becomes:

$$\tilde{\phi}^k_0 = 0 \tag{3.36}$$

To determine $\tilde{\phi}_1^k$, $\tilde{\phi}_2^k$, $\tilde{\phi}_3^k$ we must put the expression of $\tilde{\phi}^k$ into eq. (3.34) finding:

$$\begin{aligned} & 2\lambda_{ij}\frac{\partial\tilde{\phi}_k}{\partial\lambda_j} + \lambda_{jll}\Big\{5\frac{\partial\tilde{\phi}_k}{\partial\lambda_{ll}}\delta_{ij} + 2\frac{\partial\tilde{\phi}_k}{\partial\lambda_{<ab>}}\Big[\frac{1}{2}\delta_{ia}\delta_{jb} + \\ + & \frac{1}{2}\delta_{ja}\delta_{ib} - \frac{1}{3}\delta_{ab}\delta_{ij}\Big]\Big\} + 4\lambda_{ppqq}\frac{\partial\tilde{\phi}_k}{\partial\lambda_{ill}} = 0 \end{aligned} \tag{3.37}$$

into which we put the expression of $\tilde{\phi}_k$ found in (3.35) and we cut at different orders. Let's start by considering eq. (3.37) at order 0

$$\frac{2}{3}\lambda_{ll}\delta_{ki}a_1(\lambda_{ll}) = 0$$

that gives $a_1 = 0$, so, from (3.35) we find

$$\tilde{\phi}_1^k = \lambda_{kll}d_1(\lambda_{ll}) \tag{3.38}$$

where d_1 is an arbitrary function of λ_{ll}.
Going on in an similar way we find, for order 1:

$$2\lambda_{ppqq}\delta_{ik}\left(\frac{1}{3}\lambda_{ll}a_2 + 2d_1\right) + \frac{2}{3}\lambda_{ll}\lambda_{<ik>}b_1 = 0$$

that brings to

$$\begin{aligned} \frac{1}{3}\lambda_{ll}a_2 + 2d_1 &= 0\,, \\ b_1 &= 0 \end{aligned}$$

from which we can find a_2. From eq. (3.35) at order 2 we find:

$$\tilde{\phi}_2^k = \lambda_k a_2\lambda_{ppqq} + \lambda_{kll}d_2\lambda_{ppqq} + \lambda_{ill}\lambda_{<ik>}e_1(\lambda_{ll}) \tag{3.39}$$

where d_2, e_1 are arbitrary functions of λ_{ll}.
Finally we consider order 2 finding:

$$\begin{aligned} 0 = & 2{\lambda_{ppqq}}^2\delta_{ik}\left(\frac{1}{3}\lambda_{ll}a_3 + 2d_2\right) + 2\lambda_{<ik>}\lambda_{ppqq}\left(\frac{1}{3}\lambda_{ll}b_2 + 2e_1 + a_2\right) + \\ + & \frac{2}{3}\lambda_{ll}\delta_{ij}a_4\lambda_{<ij>}\lambda_{<ij>} + \frac{2}{3}\lambda_{ll}\delta_{ik}a_5\lambda_p\lambda_p + \frac{2}{3}\lambda_{ll}\delta_{ik}\lambda_{pll}\lambda_{pll}a_6 + \\ + & \frac{2}{3}\lambda_{ll}\delta_{ik}\lambda_{pll}\lambda_p a_7 + \frac{4}{3}\lambda_{ll}\delta_{ij}\lambda_k\lambda_j a_5 + \frac{2}{3}\lambda_{ll}\delta_{ip}\lambda_{pll}\lambda_k a_7 + \\ + & \frac{2}{3}\lambda_{ll}\delta_{ij}\lambda_{<kh>}\lambda_{<hj>}c_1 + \frac{4}{3}\lambda_{ll}\delta_{ip}\lambda_{kll}\lambda_{pll}d_5 + \frac{2}{3}\lambda_{ll}\delta_{ip}\lambda_{kll}\lambda_{pll}d_7 + \\ + & \frac{1}{2}\delta_{ip}\lambda_{jll}\delta_{jk}e_1\lambda_{pll} + \frac{1}{2}\delta_{ik}\lambda_{jll}e_1\lambda_{jll} - \frac{1}{3}\delta_{ij}\lambda_{kll}e_1\lambda_{jll} + \\ + & \frac{1}{2}\delta_{jk}\lambda_{jll}e_1\lambda_{ill} + \frac{1}{2}\delta_{ik}\lambda_{jll}e_1\lambda_{jll} - \frac{1}{3}\delta_{ij}\lambda_{kll}e_1\lambda_{jll} + 5\delta_{ij}\lambda_{jll}d_1'\lambda_{kll} \end{aligned}$$

from which

$$\begin{aligned} \frac{1}{3}\lambda_{ll}a_3 + 2d_2 &= 0 & \qquad \frac{1}{3}\lambda_{ll}b_2 + 2e_1 + a_2 = 0 \\ \frac{2}{3}\lambda_{ll}a_6 + \frac{2}{3}e_1 &= 0 & \qquad \frac{2}{3}\lambda_{ll}d_7 + 5d_1' + \frac{2}{3}e_1 = 0, \end{aligned} \tag{3.40}$$

that allow us to determine $a_2,\ a_3,\ a_4,\ a_6,\ d_7$.
$a_2 = a_5 = a_7 = c_1 = d_5$ are all equal zero.
From eq. (3.35) at order 3 we obtain:

$$\begin{aligned} \tilde{\phi}_3^k &= \lambda_k(a_3\lambda_{ppqq}^2 + a_6\lambda_{pll}\lambda_{pll}) + \lambda_{<kj>}\lambda_j\lambda_{ppqq}b_2 + \qquad (3.41) \\ &+ \lambda_{kll}(d_3\lambda_{ppqq}^2 + d_4\lambda_{<ij>}\lambda_{<ij>} + d_6\lambda_{pll}\lambda_{pll} + d_7\lambda_{pll}\lambda_p) + \\ &+ \lambda_{ill}\lambda_{<ik>}\lambda_{ppqq}e_2\lambda_{jll}\lambda_{<kh>}\lambda_{<hj>}f_1 \end{aligned}$$

where $d_3,\ d_4,\ d_6,\ e_2,\ f_1$ are arbitrary functions of λ_{ll}.

3.2.1 The 13 moments case as subsystem of the 14 moments one

The 13 moments case can be obtained as subsystem of the present one by taking $\lambda_{ppqq}^I = 0$.
In fact, by substituting $\lambda_{ppqq}^I = 0$ into eqs. (3.36), (3.38), (3.39) and (3.41) we find, respectively:

$$\begin{aligned} \tilde{\phi}_0^k &= 0, \\ \tilde{\phi}_1^k &= \lambda_{kll}d_1(\lambda_{ll}), \\ \tilde{\phi}_2^k &= \lambda_{ill}\lambda_{<ik>}e_1(\lambda_{ll}), \\ \tilde{\phi}_3^k &= a_6\lambda_{pll}\lambda_{pll}\lambda_k + d_4\lambda_{<ij>}\lambda_{<ij>}\lambda_{kll} + d_6\lambda_{pll}\lambda_{pll}\lambda_{kll} + \\ &+ d_7\lambda_{pll}\lambda_p\lambda_{kll} + \lambda_{jll}\lambda_{<kh>}\lambda_{<hj>}f_1. \end{aligned}$$

They coincides with eqs. (3.19), (3.20) and (3.21) for the 13 moments case, except for the following identifications:

$$\begin{aligned} a_3 &= -6\frac{d_2}{\lambda_{ll}} \quad \text{with } d_2 \text{ arbitrary function of } \lambda_{ll} \\ a_2 &= -6\frac{d_1}{\lambda_{ll}} \quad \text{with } d_1 \text{ arbitrary function of } \lambda_{ll} \\ a_6 &= -6\frac{e_1}{\lambda_{ll}} \quad \text{with } e_1 \text{ arbitrary function of } \lambda_{ll} \\ d_7 &= (-5d_1' - \frac{2}{3}e_1)\frac{3}{2\lambda_{ll}} \quad \text{with } d_1 \text{ and } e_1 \text{ arbitrary function of } \lambda_{ll}. \end{aligned}$$

Chapter 4

The many moments case

In this chapter we will consider the macroscopic approach to Extended Thermodynamics with an arbitrary but fixed number of moments, and in particular we will find the expression of the constitutive functions appearing in the balance equations up to whatever order with respect to thermodynamical equilibrium.

4.1 The balance equations

The balance equations of this Extended Thermodynamics with an arbitrary number of moments are

$$\partial_t F^{i_1 \dots i_n} + \partial_k F^{i_1 \dots i_n k} = S_{i_1 \dots i_n} \quad \text{for } n = 0, \cdots, N, \tag{4.1}$$

where N and M are two given numbers such that $M < N$, $M + N$ odd, and we call F the tensor $F_{i_1 \dots i_n}$ when n=0.
The entropy principle for this system, by using Liu's theorem ensures the existence of the Lagrange Multipliers $\lambda_{i_1 \dots i_n}$ with $n = 0, \cdots N$ such that

$$\begin{aligned} dh &= \lambda_{i_1 \dots i_n} dF^{i_1 \dots i_n} \\ dh^k &= \lambda_{i_1 \dots i_n} dF^{k i_1 \dots i_n} \end{aligned} \tag{4.2}$$

where h is the entropy density and h^k its flux.
Now in eq. (4.1), the various tensors are symmetric and $F_{i_1 \dots i_N k}$ and $S_{i_1 \dots i_N}$ are supposed to be functions of the previous one, in order to obtain a closed system. In particular F, F_i, F_{ll}, F_{ill} denote the densities of mass, momentum, energy, and energy flux respectively. In this way eqs. (4.1) for $n = 0, 1$, and the trace of eqs. (4.1) for n=2 are the conservation laws of mass, momentum and energy; obviously to this end it is necessary to assume that S=0, $S_i = 0$

and $S_{ll} = 0$.
Eq. (4.1) can be rewrote in a more compact form using a 4-dimensional notation in a space that we suppose to be Euclidean (nothing will change if the space is pseudo-Euclidean with -+++ signature, so we have chosen the simpler case).
In particular, let us define the symmetric tensors $M^{\alpha_1 \dots \alpha_{N+1}}$ and $S^{\alpha_1 \dots \alpha_N}$ as follows:

1. the Greek indexes go from 0 to 3,
2. $M^{i_1 \dots i_n 0 \dots 0} = F_{i_1 \dots i_n}$ for n=0,...,N+1
3. $S^{i_1 \dots i_n 0 \dots 0} = S_{i_1 \dots i_n}$ for n=0,...,N.

In that way the balance equations (4.1) can be simply wrote as

$$\partial_\alpha M^{\alpha \alpha_1 \dots \alpha_N} = S^{\alpha_1 \dots \alpha_N}, \tag{4.3}$$

where ∂_α for $\alpha = 0$ means the partial derivative with respect to time.
The entropy principle for this equations converts into

$$dH^\alpha = L_{\alpha_1 \cdots \alpha_N} dM^{\alpha_1 \cdots \alpha_N \alpha}, \quad L_{\alpha_1 \cdots \alpha_N} S^{\alpha_1 \cdots \alpha_N} \geq 0 \tag{4.4}$$

where $L_{\alpha_1 \cdots \alpha_N}$ are the Lagrange Multipliers, H^0 is the entropy density and H^i its flux. Let's introduce the potentials

$$H'^\alpha = -H^\alpha + L_{\alpha_1 \cdots \alpha_N} M^{\alpha_1 \cdots \alpha_N \alpha} \tag{4.5}$$

and take the Lagrange Multipliers as independent variables. In this way eq. $(4.4)_1$ becomes $dH'^\alpha = M^{\alpha_1 \cdots \alpha_N \alpha} dL_{\alpha_1 \cdots \alpha_N}$, from which

$$M^{\alpha_1 \alpha_2 \dots \alpha_{N+1}} = \frac{\partial H'^{\alpha_{N+1}}}{\partial L_{\alpha_1 \dots \alpha_N}} \tag{4.6}$$

In this way the tensors appearing in the balance equations (4.3) are found as functions of the parameters $L_{\alpha_1 \dots \alpha_N}$, called also mean field, as soon as H'^α is known. Obviously $L_{\alpha_1 \dots \alpha_N}$ is symmetric. By substituting (4.6) into eq. (4.3) this takes the symmetric form

$$\frac{\partial^2 H'^{\alpha_{N+1}}}{\partial L_{\beta_1 \cdots \beta_N} \partial L_{\alpha_1 \cdots \alpha_N}} \partial_{\alpha_{N+1}} L_{\beta_1 \cdots \beta_N} = S^{\alpha_1 \cdots \alpha_N},$$

so that hyperbolicity is ensured provided that H'^α is a convex function of the mean field. By eliminating these parameters from eqs. (4.6) we obtain

$F_{i_1 \ldots i_{N+1}}$ again, as function of F, F_i, ..., $F_{i_1 \ldots i_N}$. If we want a model in which some among eqs. (4.1) is present only by means of one of its traces, it can be obtained from the present model with the method of the subsystems [2]. Note that eq. (4.6) for $\alpha_1\alpha_2 \ldots \alpha_{N+1} = i_1 \ldots i_n i_{n+1} 0 \ldots 0$ and for $\alpha_1\alpha_2 \ldots \alpha_{N+1} = i_1 \ldots i_n 0 \ldots 0 i_{n+1}$ gives respectively

$$F_{i_1 \ldots i_n i_{n+1}} = \frac{\partial H'^0}{L_{i_1 \ldots i_n i_{n+1}}} \qquad , \qquad F_{i_1 \ldots i_n i_{n+1}} = \frac{\partial H'^{n+1}}{L_{i_1 \ldots i_n}} \tag{4.7}$$

as in the 3-dimensional notation.
So, to impose eq. (4.6) we have to find the more general expression of H'^α such that $M^{\alpha_1\alpha_2 \ldots \alpha_{N+1}}$ is symmetric. We will refer to this as "the symmetry condition".

4.2 The Galilean relativity principle

Now we impose also the principle of galilean invariance. This has been exploited in [1], [2], [5], [22] for a generic system of balance laws; here we will apply these results to our system, taking care of converting them in the present 4-dimensional notation, so obtaining further conditions. To impose this principle, it is firstly necessary to know how our variables transform under a change of Galileanly equivalent frames Σ and Σ'. This problem has been studied by Ruggeri in [5] and we have only to write its results in our 4-dimensional form. This is easily achieved in the kinetic model because the kinetic counterpart of $M^{\alpha_1 \cdots \alpha_{N+1}}$ is

$$M^{\alpha_1 \cdots \alpha_{N+1}} = \int f c^{\alpha_1} \cdots c^{\alpha_{N+1}} d\underline{c} \tag{4.8}$$

with $c^0 = 1$, $d\underline{c} = dc^1 dc^2 dc^3$ and f is the distribution function. Consequently, in Σ' we have

$$m^{\alpha_1 \cdots \alpha_{N+1}} = \int f c'^{\alpha_1} \cdots c'^{\alpha_{N+1}} dc'$$

and, if v^i is the constant velocity of each point of Σ' with respect to Σ, we have $c^\alpha = c'^\alpha + v^\alpha$, with $v^0 = 0$. It follows that

$$M^{\alpha_1 \ldots \alpha_{N+1}} = \sum_{i=0}^{N+1} \begin{pmatrix} N+1 \\ i \end{pmatrix} v^{(\alpha_1} \ldots v^{\alpha_i} m^{\alpha_{i+1} \ldots \alpha_{N+1})}$$

or

$$M^{\alpha_1 \ldots \alpha_{N+1}} = \sum_{i=0}^{N+1} \begin{pmatrix} N+1 \\ i \end{pmatrix} v^{(\alpha_1} \ldots v^{\alpha_i} m^{\alpha_{i+1} \ldots \alpha_{N+1})\beta_1 \ldots \beta_i} t_{\beta_1} \ldots t_{\beta_i} \tag{4.9}$$

with $t_\mu \equiv (1,0,0,0)$ for our previous notation. We obtain the transformation of $M^{\alpha_1 \cdots \alpha_N 0}$ (which was the initial independent variable) multiplying eq. (4.9) by $t_{\alpha_{N+1}}$ so finding

$$M^{\alpha_1 \dots \alpha_N 0} = X^{\alpha_1 \dots \alpha_N}_{\beta_1 \dots \beta_N}(\underline{v}) m^{\beta_1 \dots \beta_N 0} \tag{4.10}$$

with

$$X^{\alpha_1 \dots \alpha_N}_{\beta_1 \dots \beta_N} = \sum_{i=0}^{N} \left(\begin{array}{c} N \\ i \end{array} \right) t_{(\beta_1} \dots t_{\beta_i} v^{(\alpha_1} \dots v^{\alpha_i} \delta^{\alpha_{i+1}}_{\beta_{i+1}} \dots \delta^{\alpha_N)}_{\beta_N)} \tag{4.11}$$

where we have taken into account of $v^0 = 0$, of the identity $\binom{N+1}{i} \frac{N+1-i}{N+1} = \binom{N}{i}$ and that the term with i=N+1 gives a null contribution. Comparison between (4.10) and (4.11) with (4.9) shows that $X^{\alpha_1 \dots \alpha_N}_{\beta_1 \dots \beta_N}$ could be obtained from $X^{\alpha_1 \dots \alpha_{N+1}}_{\beta_1 \dots \beta_{N+1}}$ simply replacing N+1 with N. From eq. (4.11) it follows also

$$X^{\alpha_1 \dots \alpha_N \alpha}_{\beta_1 \dots \beta_N \beta} = X^{\alpha_1 \dots \alpha_N}_{(\beta_1 \dots \beta_N} t_{\beta)} v^\alpha + X^{\alpha_1 \dots \alpha_N}_{(\beta_1 \dots \beta_N} \delta^\alpha_{\beta)}. \tag{4.12}$$

Similarly, H^α transforms according to the rule

$$H^\alpha = h^0 v^\alpha + h^\alpha, \tag{4.13}$$

of which $H^0 = h^0$ is a component.
Eqs. (4.9) and (4.13) have been obtained with the kinetic model only for the sake of simplicity; it is obvious that they hold also in the macroscopic case. The transformation rule of the Lagrange multipliers can be obtained now from $(4.4)_1$ with $\alpha = 0$, i.e.

$$dh^0 = dH^0 = L_{\alpha_1 \cdots \alpha_N} dM^{\alpha_1 \cdots \alpha_N 0} = L_{\alpha_1 \cdots \alpha_N} X^{\alpha_1 \dots \alpha_N}_{\beta_1 \dots \beta_N} dm^{\beta_1 \cdots \beta_N 0}$$

where (4.13) and (4.10) have been used. In other words we have

$$dh^0 = l_{\beta_1 \cdots \beta_N} dm^{\beta_1 \cdots \beta_N 0} \tag{4.14}$$

with

$$l_{\alpha_1 \dots \alpha_N} = X^{\beta_1 \dots \beta_N}_{\alpha_1 \dots \alpha_N} L_{\beta_1 \dots \beta_N}$$

i.e.

$$l_{\alpha_1 \dots \alpha_N} = \sum_{i=0}^{N} \left(\begin{array}{c} N \\ i \end{array} \right) t_{(\alpha_1} \dots t_{\alpha_i} v^{\beta_1} \dots v^{\beta_i} L_{\alpha_{i+1} \dots \alpha_N) \beta_1 \dots \beta_i}. \tag{4.15}$$

A consequence of this result can be obtained from (4.5) with $\alpha = 0$ and written in the frame Σ', i.e., $h'^0 = -h^0 + l_{\alpha_1\cdots\alpha_N} m^{\alpha_1\cdots\alpha_N 0}$; it follows $dh'^0 = m^{\alpha_1\cdots\alpha_N 0} dl_{\alpha_1\cdots\alpha_N}$ from which

$$m^{\alpha_1\cdots\alpha_N 0} = \frac{\partial h'^0}{\partial l_{\alpha_1\cdots\alpha_N}} \tag{4.16}$$

as in Σ. Moreover, from (4.5), (4.13), (4.9), (4.12), (4.15) and again (4.5) and (4.13) it follows

$$\begin{aligned} H'^\alpha &= -h^0 v^\alpha - h^\alpha + L_{\alpha_1\cdots\alpha_N} X^{\alpha_1\ldots\alpha_N\alpha}_{\beta_1\ldots\beta_N\beta} m^{\beta_1\cdots\beta_N\beta} \\ &= -h^0 v^\alpha - h^\alpha + l_{\beta_1\cdots\beta_N} m^{\beta_1\cdots\beta_N 0} v^\alpha + l_{\beta_1\cdots\beta_N} m^{\beta_1\cdots\beta_N\alpha} \end{aligned}$$

i.e.,

$$H'^\alpha = h'^0 v^\alpha + h'^\alpha \tag{4.17}$$

which is similar to (4.13).
We are now ready to consider the Galilean relativity principle. It imposes that the following diagram is commutative

$L_{\alpha_1\cdots\alpha_N}$	$\longrightarrow$	$l_{\gamma_1\cdots\gamma_N} = X^{\alpha_1\ldots\alpha_N}_{\gamma_1\ldots\gamma_N} L_{\alpha_1\cdots\alpha_N}$
$\downarrow$		$\downarrow$
$M^{\beta_1\cdots\beta_{N+1}}(L_{\alpha_1\cdots\alpha_N})$ $H'^\alpha(L_{\alpha_1\cdots\alpha_N})$		
$\vert\vert$		
$X^{\beta_1\ldots\beta_{N+1}}_{\delta_1\ldots\delta_{N+1}} m^{\delta_1\cdots\delta_{N+1}}(X^{\alpha_1\ldots\alpha_N}_{\gamma_1\ldots\gamma_N} L_{\alpha_1\cdots\alpha_N})$ $v^\alpha t_\delta h'^\delta(\cdots) + h'^\alpha(\cdots)$	$\longleftarrow$	$m^{\delta_1\cdots\delta_{N+1}}(X^{\alpha_1\cdots\alpha_N}_{\gamma_1\cdots\gamma_N} L_{\alpha_1\cdots\alpha_N})$ $h'^\delta(X^{\alpha_1\cdots\alpha_N}_{\gamma_1\cdots\gamma_N} L_{\alpha_1\cdots\alpha_N})$

In other words, we must have

$$\begin{aligned} &H'^\alpha(L_{\alpha_1\cdots\alpha_N}) = v^\alpha t_\delta h'^\delta(X^{\alpha_1\ldots\alpha_N}_{\gamma_1\ldots\gamma_N}(\underline{v}) L_{\alpha_1\cdots\alpha_N}) + h'^\alpha(X^{\alpha_1\ldots\alpha_N}_{\gamma_1\ldots\gamma_N}(\underline{v}) L_{\alpha_1\cdots\alpha_N}) \\ &M^{\beta_1\cdots\beta_{N+1}}(L_{\alpha_1\cdots\alpha_N}) = X^{\beta_1\ldots\beta_{N+1}}_{\delta_1\ldots\delta_{N+1}}(\underline{v}) m^{\delta_1\cdots\delta_{N+1}}(X^{\alpha_1\ldots\alpha_N}_{\gamma_1\ldots\gamma_N}(\underline{v}) L_{\alpha_1\cdots\alpha_N}) \end{aligned} \tag{4.18}$$

Eq. $(4.18)_2$, by using eqs. (4.12) and (4.16) becomes

$$\begin{aligned} M^{\beta_1\cdots\beta_N\alpha} &= X^{\beta_1\ldots\beta_N}_{\delta_1\ldots\delta_N} m^{\delta_1\cdots\delta_N 0} v^\alpha + X^{\beta_1\ldots\beta_N}_{\delta_1\ldots\delta_N} m^{\delta_1\cdots\delta_N\alpha} = \\ &= X^{\beta_1\ldots\beta_N}_{\delta_1\ldots\delta_N} \frac{\partial h'^0}{\partial l_{\delta_1\cdots\delta_N}} v^\alpha + X^{\beta_1\ldots\beta_N}_{\delta_1\ldots\delta_N} m^{\delta_1\cdots\delta_n\alpha} \end{aligned}$$

Now the derivative of $(4.18)_1$ with respect to $L^{\beta_1\cdots\beta_N}$ is

$$\frac{\partial H'^{\alpha}}{\partial L^{\beta_1\cdots\beta_N}} = M^{\beta_1\cdots\beta_N\alpha} = v^{\alpha}\frac{\partial h'^{0}}{\partial l_{\gamma_1\cdots\gamma_N}} X^{\beta_1\cdots\beta_N}_{\gamma_1\cdots\gamma_N}(\underline{v}) + \frac{\partial h'^{\alpha}}{\partial l_{\gamma_1\cdots\gamma_N}} X^{\beta_1\cdots\beta_N}_{\gamma_1\cdots\gamma_N}(\underline{v}).$$

It follows that eq. $(4.18)_2$, holds iff

$$\frac{\partial h'^{\alpha}}{\partial l_{\gamma_1\cdots\gamma_N}} X^{\beta_1\cdots\beta_N}_{\gamma_1\cdots\gamma_N} = m^{\delta_1\cdots\delta_N\alpha} X^{\beta_1\cdots\beta_N}_{\gamma_1\cdots\gamma_N}$$

i.e.

$$m^{\gamma_1\cdots\gamma_N\alpha} = \frac{\partial h'^{\alpha}}{\partial l_{\gamma_1\cdots\gamma_N}} \tag{4.19}$$

which is the counterpart of eq. (4.6) in the frame Σ'.
There remains to impose eq. $(4.18)_1$.
It becomes an identity when calculated in $\underline{v} = 0$ (see eqs. (4.17) and (4.11) to this regard) so that it holds iff its derivative with respect to v_j is satisfied, i.e.,

$$\begin{aligned} 0 &= \frac{\partial h'^{0}}{\partial l_{\gamma_1\cdots\gamma_N}} \frac{\partial l_{\gamma_1\cdots\gamma_N}}{\partial v_j} && \text{for } \alpha = 0, \\ 0 &= h'^{0}\delta^{\alpha}_{j} + \frac{\partial h'^{\alpha}}{\partial l_{\gamma_1\cdots\gamma_N}} \frac{\partial l_{\gamma_1\cdots\gamma_N}}{\partial v_j} && \text{for } \alpha = 0,1,2,3. \end{aligned} \tag{4.20}$$

The second of this has been obtained by taking into account also eq. $(4.20)_1$; on the other hand, this is included in $(4.20)_2$ with $\alpha = 0$. Eq. $(4.20)_2$, by using eq. $(4.15)_2$ now becomes

$$h'^{0}\delta^{\alpha}_{j} + \frac{\partial h'^{\alpha}}{\partial l_{\alpha_1\ldots\alpha_N}} \sum_{i=1}^{N} \binom{N}{i} i \cdot t_{(\alpha_1}\ldots t_{\alpha_i} v^{\beta_1}\ldots v^{\beta_{i-1}} L_{\alpha_{i+1}\ldots\alpha_N)\beta_1\ldots\beta_{i-1}j} = 0.$$

We remove the symmetrization with respect to $\alpha_1\cdots\alpha_N$ which is not necessary because of the contraction with $\frac{\partial h'^{\alpha}}{\partial l_{\alpha_1\ldots\alpha_N}}$ which is symmetric; for the same reason we can exchange α_i and α_N and then reintroduce the symmetrization with respect to $\alpha_1\ldots\alpha_{N-1}$, obtaining so

$$h'^{0}\delta^{\alpha}_{j} + t_{\alpha_N}\frac{\partial h'^{\alpha}}{\partial l_{\alpha_1\ldots\alpha_N}} \sum_{i=1}^{N} \binom{N}{i} i \cdot t_{(\alpha_1}\ldots t_{\alpha_{i-1}} v^{\beta_1}\ldots v^{\beta_{i-1}} L_{\alpha_i\ldots\alpha_{N-1})\beta_1\ldots\beta_{i-1}j} = 0.$$

We replace i with i+1 and we have

$$h'^0\delta_j^\alpha + \frac{\partial h'^\alpha}{\partial l_{\alpha_1\ldots\alpha_N}} t_{\alpha_N} \cdot$$
$$\sum_{i=0}^{N-1} \left(\begin{array}{c} N \\ i+1 \end{array} \right) (i+1) \cdot t_{(\alpha_1}\ldots t_{\alpha_i} v^{\beta_1}\ldots v^{\beta_i} L_{\alpha_{i+1}\ldots\alpha_{N-1})\beta_1\ldots\beta_i j} = 0$$

or

$$h'^0\delta_j^\alpha + \tfrac{\partial h'^\alpha}{\partial l_{\alpha_1\ldots\alpha_N}} t_{\alpha_N} \cdot$$
$$\textstyle\sum_{i=0}^{N-1} N \left(\begin{array}{c} N-1 \\ i \end{array} \right) \cdot t_{(\alpha_1}\ldots t_{\alpha_i} v^{\beta_1}\ldots v^{\beta_i} L_{\alpha_{i+1}\ldots\alpha_{N-1})\beta_1\ldots\beta_i j} = 0. \qquad (4.21)$$

But, by using eq. (4.15) we have

$$\begin{aligned} l_{\alpha\cdots\alpha_{N-1}j} &= \sum_{i=1}^{N} \frac{i}{N} \binom{N}{i} t_j t_{(\alpha_1} \cdots t_{\alpha_{i-1}} v^{\beta_1} \cdots v^{\beta_i} L_{\alpha_i\cdots\alpha_{N-1})\beta_1\cdots\beta_i} + \\ &+ \sum_{i=0}^{N-1} \frac{N-i}{N} \binom{N}{i} t_{(\alpha_1} \cdots t_{\alpha_i} v^{\beta_1} \cdots v^{\beta_i} L_{\alpha_{i+1}\cdots\alpha_{N-1})j\beta_1\cdots\beta_i} \\ &= \sum_{i=0}^{N-1} \binom{N-1}{i} t_{(\alpha_1} \cdots t_{\alpha_i} v^{\beta_1} \cdots v^{\beta_i} L_{\alpha_{i+1}\cdots\alpha_{N-1})j\beta_1\cdots\beta_i} \qquad (4.22) \end{aligned}$$

because $t_j = 0$. This allows to rewrite eq. (4.21) as

$$0 = h'^\mu t_\mu \delta_j^\alpha + N \frac{\partial h'^\alpha}{\partial l_{\alpha_1\ldots\alpha_N}} t_{\alpha_N} l_{\alpha_1\ldots\alpha_{N-1}j}. \qquad (4.23)$$

Until now we have obtained that the entropy principle jointly with the galilean relativity principle amounts to say that

1. eqs. (4.6) are invariant under changes of galileanly equivalent observers (see eq. (4.19)),
2. the further condition (4.23) must hold.

For the sake of completeness, we note that eq. $(4.18)_1$ might be satisfied also with H^α and h^α, i.e.

$$H^\alpha(L_{\alpha_1\cdots\alpha_N}) = v^\alpha t_\delta h^\delta(X^{\alpha_1\cdots\alpha_N}_{\gamma_1\cdots\gamma_N} L_{\alpha_1\cdots\alpha_N}) + h'^\alpha(X^{\alpha_1\cdots\alpha_N}_{\gamma_1\cdots\gamma_N} L_{\alpha_1\cdots\alpha_N}).$$

But this is a consequence of (4.18) as it can be seen running over backwards the above passages which allowed to obtain eq. (4.17) from eq. (4.13). Moreover, in [1] and [5] it has be proved that the conditions here obtained are the same of the following approach:

1. consider eqs. (4.9), (4.13) and (4.17) but with $v_i = \frac{F_i}{F}$, instead of an arbitrary constant v_i; in this way $m^{\alpha_1 \cdots \alpha_{N+1}}$, h^α and h'^α become the non-convective parts of $M^{\alpha_1 \cdots \alpha_{N+1}}$, H^α and H'^α, respectively,

2. impose the conditions (4.19) and (4.23) but considering $l_{\gamma_1 \cdots \gamma_N}$ independent variables,

3. consider eqs. (4.19) with $\alpha = 0$ and $m^{i0 \cdots 0}$ as definition of $l_{\gamma_1 \cdots \gamma_N} = l_{\gamma_1 \cdots \gamma_N}(m^{\alpha_1 \cdots \alpha_N 0})$, and substitute this in the expressions of $m^{i_1 \cdots i_{N+1}}$, h^α and h'^α so obtaining the closure in terms of the non-convective quantities $m^{\alpha_1 \cdots \alpha_N 0}$.

In any case, we have to impose (4.19) and (4.23); in other words we have to find the quadri-vector $h'^{\alpha_{N+1}}$ such that the right hand side of eq. (4.19) is symmetric and for which eq. (4.23) holds; after that eq. (4.19) gives $m^{\beta_1 \cdots \beta_N \beta_{N+1}}$. In this way we will find the required closure satisfying the entropy principle and that of galilean relativity. This will be done in the next section.

4.3 Exploitation of the conditions (4.19) and (4.23)

We want now to impose eqs. (4.19) and (4.23) up to whatever order with respect to thermodynamical equilibrium. This is defined as the state where

$$l_{\beta_1 \cdots \beta_N} = \lambda t_{\beta_1} \cdots t_{\beta_N} + \frac{1}{3} \lambda_{ll} h_{(\beta_1 \beta_2} t_{\beta_3} \cdots t_{\beta_N)} \tag{4.24}$$

holds, with $h_{\beta\gamma} = \delta_{\beta\gamma} - t_\beta t_\gamma = diag(0,1,1,1)$,

$$\lambda = t^{\beta_1} \cdots t^{\beta_N} l_{\beta_1 \cdots \beta_N} \qquad \lambda_{ll} = \binom{N}{2} h^{\beta_1 \beta_2} t^{\beta_3} \cdots t^{\beta_N} l_{\beta_1 \cdots \beta_N}. \tag{4.25}$$

We can consider the Taylor expansion for h'^α

$$h'^\alpha = \sum_{k=0}^{\infty} \frac{1}{k!} A^{\alpha B_1 \cdots B_k} \tilde{l}_{B_1} \cdots \tilde{l}_{B_k}, \tag{4.26}$$

with

$$\tilde{l}_{\beta_1 \cdots \beta_N} = l_{\beta_1 \cdots \beta_N} - \lambda t_{\beta_1} \cdots t_{\beta_N} - \frac{1}{3} \lambda_{ll} h_{(\beta_1 \beta_2} t_{\beta_3} \cdots t_{\beta_N)}, \tag{4.27}$$

$$A^{\alpha B_1\cdots B_k} = \left(\frac{\partial^k h'^\alpha}{\partial l_{B_1}\cdots\partial l_{B_k}}\right)_{eq} \tag{4.28}$$

where the multi-index notation $B_i = \beta_i^1 \cdots \beta_i^N$ has been used. Thanks to eq. (4.19) we can exchange α with each other index taken from those included in any B_i. So it is possible to exchange every index with all the others, i.e., $A^{\alpha B_1\cdots B_k}$ is symmetric with respect to any couple of indexes. We note that there are 2 compatibility conditions between eqs. (4.26) and (4.28); they can be obtained as follows: let us consider the tensor $\frac{\partial^k h'^\alpha}{\partial l_{B_1}\cdots l_{B_k}}$ as function of $\tilde{l}_B, \lambda, \lambda_{ll}$, and take the derivatives with respect to $l_{\beta_1\cdots\beta_N}$, calculating the result at equilibrium; we find

$$A^{\alpha B_1\cdots B_k\beta_1\cdots\beta_N} = \left(\frac{\partial^{k+1} h'^\alpha}{\partial l_{B_1}\cdots\partial l_{B_k}\partial\tilde{l}_{\gamma_1\cdots\gamma_N}}\right)_{eq}\frac{\partial\tilde{l}_{\gamma_1\cdots\gamma_N}}{\partial l_{\beta_1\cdots\beta_N}} +$$
$$+\left(\frac{\partial^{k+1} h'^\alpha}{\partial l_{B_1}\cdots\partial l_{B_k}\partial\lambda}\right)_{eq}\frac{\partial\lambda}{\partial l_{\beta_1\cdots\beta_N}} + \left(\frac{\partial^{k+1} h'^\alpha}{\partial l_{B_1}\cdots\partial l_{B_k}\partial\lambda_{ll}}\right)_{eq}\frac{\partial\lambda_{ll}}{\partial l_{\beta_1\cdots\beta_N}}.$$

If we multiply this by $t_{\beta_1}\cdots t_{\beta_N}$ and by $h_{\beta_1\beta_2}t_{\beta_3}\cdots t_{\beta_N}$ we find, respectively

$$\begin{cases} A^{\alpha B_1\cdots B_k\beta_1\cdots\beta_N} t_{\beta_1}\cdots t_{\beta_N} = \frac{\partial}{\partial\lambda}A^{\alpha B_1\cdots B_k} \\ A^{\alpha B_1\cdots B_k\beta_1\cdots\beta_N} h_{\beta_1\beta_2}t_{\beta_3}\cdots t_{\beta_N} = 3\frac{\partial}{\partial\lambda_{ll}}A^{\alpha B_1\cdots B_k}, \end{cases} \tag{4.29}$$

where we have taken into account that from eqs. (4.25) and (4.27) it follows

$$\begin{aligned} \frac{\partial\lambda}{\partial l_{\beta_1\cdots\beta_N}} &= t^{\beta_1}\cdots t^{\beta_N} \qquad\qquad \frac{\partial\lambda_{ll}}{\partial l_{\beta_1\cdots\beta_N}} = \binom{N}{2} h^{(\beta_1\beta_2}t^{\beta_3}\cdots t^{\beta_N)} \\ \frac{\partial\tilde{l}_{\gamma_1\cdots\gamma_N}}{\partial l_{\beta_1\cdots\beta_N}} &= g^{(\beta_1}_{\gamma_1}\cdots g^{\beta_N)}_{\gamma_N} - t^{\beta_1}\cdots t^{\beta_N}t_{\gamma_1}\cdots t_{\gamma_N} + \\ &- \frac{1}{3}\binom{N}{2} h^{(\beta_1\beta_2}t^{\beta_3}\cdots t^{\beta_N)}h_{(\gamma_1\gamma_2}t_{\gamma_3}\cdots t_{\gamma_N)}, \end{aligned}$$

from which

$$\begin{aligned} &\frac{\partial\lambda}{\partial l_{\beta_1\cdots\beta_N}}t_{\beta_1}\cdots t_{\beta_N} = 1 &&\frac{\partial\lambda}{\partial l_{\beta_1\cdots\beta_N}}h_{\beta_1\beta_2}t_{\beta_3}\cdots t_{\beta_N} = 0 \\ &\frac{\partial\lambda_{ll}}{\partial l_{\beta_1\cdots\beta_N}}t_{\beta_1}\cdots t_{\beta_N} = 0 &&\frac{\partial\lambda_{ll}}{\partial l_{\beta_1\cdots\beta_N}}h_{\beta_1\beta_2}t_{\beta_3}\cdots t_{\beta_N} = 3 \\ &\frac{\partial\tilde{l}_{\gamma_1\cdots\gamma_N}}{\partial l_{\beta_1\cdots\beta_N}}t_{\beta_1}\cdots t_{\beta_N} = 0 &&\frac{\partial\tilde{l}_{\gamma_1\cdots\gamma_N}}{\partial l_{\beta_1\cdots\beta_N}}h_{\beta_1\beta_2}t_{\beta_3}\cdots t_{\beta_N} = 0. \end{aligned}$$

It will be useful in the sequel to note a consequence of the condition (4.29). By using also eq. (4.26) we have

$$\begin{aligned}
\frac{\partial h'^{\alpha}}{\partial l_{\beta_1\cdots\beta_N}} &= \sum_{k=1}^{\infty}\frac{1}{(k-1)!}A^{\alpha B_1\cdots B_{k-1}\gamma_1\cdots\gamma_N}\tilde{l}_{B_1}\cdots\tilde{l}_{B_{k-1}} \\
&\quad \left(g_{\gamma_1}^{(\beta_1}\cdots g_{\gamma_N}^{\beta_N)} - t^{\beta_1}\cdots t^{\beta_N}t_{\gamma_1}\cdots t_{\gamma_N} + \right. \\
&- \left.\frac{1}{3}\binom{N}{2}h^{(\beta_1\beta_2}t^{\beta_3}\cdots t^{\beta_N)}h_{(\gamma_1\gamma_2}t_{\gamma_3}\cdots t_{\gamma_N)}\right) \\
&+ \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\partial}{\partial\lambda}A^{\alpha B_1\cdots B_k}\right)\tilde{l}_{B_1}\cdots\tilde{l}_{B_k}t^{\beta_1}\cdots t^{\beta_N} + \\
&+ \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\partial}{\partial\lambda_{ll}}A^{\alpha B_1\cdots B_k}\right)\tilde{l}_{B_1}\cdots\tilde{l}_{B_k}h^{(\beta_1\beta_2}t^{\beta_3}\cdots t^{\beta_N)}\binom{N}{2} = \\
&= \sum_{k=1}^{\infty}\frac{1}{(k-1)!}A^{\alpha B_1\cdots B_{k-1}\beta_1\cdots\beta_N}\tilde{l}_{B_1}\cdots\tilde{l}_{B_{k-1}},
\end{aligned}$$

where conditions (4.29) have been used in the last passage. So we have proved that derivation of eq. (4.26) with respect to $l_{\beta_1\cdots\beta_N}$ is equivalent to its derivation with respect to $\tilde{l}_{\beta_1\cdots\beta_N}$, but considering independent the components of this tensor, except for the symmetry. Proceeding with the subsequent derivatives and calculating the result at equilibrium, we find eq. (4.28). In other words we can forget eq. (4.28) but we have to retain eqs. (4.29). We have then to transform eqs. (4.19), (4.23) and (4.29) in conditions for the tensor $A^{\alpha B_1\cdots B_k}$; the above mentioned symmetry of this tensor ensures that eq. (4.19) is satisfied. Before imposing eqs. (4.23) and (4.29), we note that the most general expression for a symmetric tensor depending on the scalars λ, λ_{ll} and on t^{α} is

$$A^{\alpha_1\alpha_2\cdots\alpha_{Nk+1}} = \sum_{s=0}^{\left[\frac{Nk+1}{2}\right]}\binom{Nk+1}{2s}g_{k,2s}(\lambda,\lambda_{ll})h^{(\alpha_1\alpha_2}\cdots h^{\alpha_{2s-1}\alpha_{2s}}t^{\alpha_{2s+1}}\cdots t^{\alpha_{Nk+1})} \tag{4.30}$$

where the binomial factor has been introduced for later convenience. Thanks to this, eqs. (4.29) become

$$\begin{cases} g_{k+1,2s} = \frac{\partial}{\partial\lambda}g_{k,2s} \\ \\ g_{k+1,2s+2} = \frac{2s+1}{2s+3}3\frac{\partial}{\partial\lambda_{ll}}g_{k,2s}. \end{cases} \quad \text{for } s=0,\cdots,\left[\tfrac{Nk+1}{2}\right] \tag{4.31}$$

There remains to consider eq. (4.23); thanks to eq. (4.26), (4.24) and (4.30), its value at equilibrium is

$$0 = g_{0,0} + \frac{2}{3}\lambda_{ll} g_{1,2}$$

which, thanks to eq. $(4.31)_2$, becomes

$$0 = g_{0,0} + \frac{2}{3}\lambda_{ll}\frac{\partial}{\partial \lambda_{ll}} g_{0,0}.$$

Its solution is

$$g_{0,0} = \lambda_{ll}^{-\frac{3}{2}} G_{0,0}(\lambda), \tag{4.32}$$

with $G_{0,0}(\lambda)$ an arbitrary single variable function.
But eq. (4.23) is equivalent to its value at equilibrium, and to its r^{th} derivatives with respect to l_{B_i} calculated at equilibrium, for all values of r. The r^{th} derivatives of eq. (4.23) with respect to l_{B_i} is

$$\begin{aligned} 0 &= \delta^{\alpha}_{j} \frac{\partial^r h'^{\mu} t_{\mu}}{\partial l_{B_1} \cdots \partial l_{B_r}} + N \frac{\partial^{r+1} h'^{\alpha}}{\partial l_{B_1} \cdots \partial l_{B_r} \partial l_{\alpha_1 \cdots \alpha_N}} t_{\alpha_N} l_{\alpha_1 \cdots \alpha_{N-1} j} + \\ &+ N r t_{\alpha_N} \frac{\partial^r h'^{\alpha}}{\partial l_{\alpha_1 \cdots \alpha_N} \partial l_{(B_1} \cdots \partial l_{B_{r-1}}} \frac{\partial l_{\alpha_1 \cdots \alpha_{N-1} j}}{\partial l_{B_r)}}, \end{aligned} \tag{4.33}$$

where the indicated symmetrization is treated as the multi-index B_i was a single index. The eq. (4.33) can be easily proved with the iterative procedure. Now we have to calculate this expression at equilibrium. Let us evaluate each single term of this relation.
• Thanks to eqs. (4.26) and (4.24), we have for the first term

$$\delta^{\alpha}_{j} \left(\frac{\partial^r h'^{\mu} t_{\mu}}{\partial l_{B_1} \cdots \partial l_{B_r}} \right)_{eq} = \delta^{\alpha}_{j} A^{\mu B_1 \cdots B_r} t_{\mu}.$$

• The second term at equilibrium, thanks to eq. (4.24), is

$$\begin{aligned} &\left(N \frac{\partial^{r+1} h'^{\alpha}}{\partial l_{B_1} \cdots \partial l_{B_r} \partial l_{\alpha_1 \cdots \alpha_N}} t_{\alpha_N} l_{\alpha_1 \cdots \alpha_{N-1} j} \right)_{eq} = \\ &= N A^{\alpha B_1 \cdots B_r \alpha_1 \cdots \alpha_N} t_{\alpha_N} \frac{1}{3} \lambda_{ll} \frac{2}{N} h_{j(\alpha_1} t_{\alpha_2} \cdots t_{\alpha_{N-1})} \end{aligned}$$

The symmetrization in the right hand side can be omitted because the term is contracted with a symmetric tensor. Now we use eq. (4.30). We see that the terms containing the factor t^{α_1} gives zero contribute, so that the above expression can be written as

$$\sum_{s=1}^{\left[\frac{N(r+1)+1}{2}\right]} g_{r+1,2s} \binom{N(r+1)+1}{2s} \frac{2s}{N(r+1)+1}$$
$$h^{\alpha_1(\alpha_2} \cdots h^{\alpha_{2s-1}\alpha_{2s}} t^{\alpha_{2s+1}} \cdots t^{\alpha_{N(r+1)}} t^{\alpha)} t_{\alpha_N} \cdot \frac{1}{3} \lambda_{ll} 2 h_{j\alpha_1} t_{\alpha_2} \cdots t_{\alpha_{N-1}}$$

where the indexes in $B_1 \cdots B_r$ and α_N are included into the α_i; after the contraction with $t_{\alpha_2} \cdots t_{\alpha_N}$ this expression becomes

$$\sum_{s=1}^{\left[\frac{Nr+2}{2}\right]} \binom{Nr+1}{2s-1} \frac{2}{3} \lambda_{ll} g_{r+1,2s} h_j^{(\gamma_2} \cdots h^{\gamma_{2s-1}\gamma_{2s}} t^{\gamma_{2s+1}} \cdots t^{\gamma_{Nr+1}} t^{\alpha)}$$

where the indexes $\gamma^{\cdot}$ represent $B_1 \cdots B_r$.

• Let us evaluate now the contribute of the last term in eq. (4.33), i.e.

$$\begin{aligned} & Nr t_{\alpha_N} \left(\frac{\partial^r h'^{\alpha}}{\partial l_{\alpha_1 \cdots \alpha_N} \partial l_{B_1} \cdots \partial l_{B_{r-1}}} \right)_{eq} \frac{\partial l_{\alpha_1 \cdots \alpha_{N-1} j}}{\partial l_{B_r}} = \\ & = Nr t_{\alpha_N} A^{\alpha B_1 \cdots B_{r-1} \alpha_1 \cdots \alpha_N} g_{\alpha_1}^{(\beta_1^r} \cdots g_{\alpha_{N-1}}^{\beta_{N-1}^r} h_j^{\beta_N^r)} \\ & = Nr t_{\alpha_N} A^{\alpha \alpha_N B_1 \cdots B_{r-1} (\beta_1^r \cdots \beta_{N-1}^r} h_j^{\beta_N^r)} \end{aligned}$$

where we have exploited $B_r = \beta_1^r \cdots \beta_N^r$. We can now prove that

$$Nr t_{\alpha_N} \left(\frac{\partial^r h'^{\alpha}}{\partial l_{\alpha_1 \cdots \alpha_N} \partial l_{(B_1} \cdots \partial l_{B_{r-1}}} \right)_{eq} \frac{\partial l_{\alpha_1 \cdots \alpha_{N-1} j}}{\partial l_{B_r)}}$$

is symmetric with respect to two generic indexes β_i^s and β_q^t, with $s \leq t = 1, \cdots, r$. In fact it can be written as

$$\begin{aligned} & \sum_{k=1}^{r} N t_{\alpha_N} \frac{\partial^r h'^{\alpha}}{\partial l_{\alpha_1 \cdots \alpha_N} \partial l_{B_1} \cdots \partial l_{B_{k-1}} \partial l_{B_{k+1}} \cdots \partial l_{B_r}} \frac{\partial l_{\alpha_1 \cdots \alpha_N j}}{\partial B_k} = \\ & = \sum_{k=1,\cdots,r}^{k \neq s, k \neq t} N t_{\alpha_N} A^{\alpha \alpha_N B_1 \cdots B_{k-1} B_{k+1} \cdots B_r (\beta_1^k \cdots \beta_{N-1}^k} h_j^{\beta_N^k)} + \\ & + N t_{\alpha_N} A^{\alpha \alpha_N B_1 \cdots B_{s-1} B_{s+1} \cdots B_{t-1} \beta_1^t \cdots \beta_N^t B_{t+1} \cdots B_r (\beta_1^s \cdots \beta_{N-1}^s} h_j^{\beta_N^s)} + \\ & + N t_{\alpha_N} A^{\alpha \alpha_N B_1 \cdots B_{s-1} \beta_1^s \cdots \beta_N^s B_{s+1} \cdots B_{t-1} B_{t+1} \cdots B_r (\beta_1^t \cdots \beta_{N-1}^t} h_j^{\beta_N^t)}. \end{aligned}$$

The first of these terms is clearly symmetric with respect to β_i^s and β_q^t, while the sum of the last two is

$$t_{\alpha_N}A^{\alpha\alpha_N B_1\cdots B_{s-1}B_{s+1}\cdots B_{t-1}\beta_1^t\cdots\beta_k^t\cdots\beta_N^t B_{t+1}\cdots B_r\beta_1^s\cdots\beta_{i-1}^s\beta_{i+1}^s\cdots\beta_N^s}h_j^{\beta_i^s}+$$
$$+t_{\alpha_N}A^{\alpha\alpha_N B_1\cdots B_{s-1}\beta_1^s\cdots\beta_i^s\cdots\beta_N^s B_{s+1}\cdots B_{t-1}B_{t+1}\cdots B_r\beta_1^t\cdots\beta_{k-1}^t\beta_{k+1}^t\beta_N^t}h_j^{\beta_k^t}+$$
$$+\text{terms like } t_{\alpha_N}A^{\alpha\alpha_N\beta_i^s\cdots\beta_k^t\cdots}h_j^{\cdot}$$

that is obviously symmetric with respect to β_i^s and β_k^t.
Consequently our tensor is symmetric with respect to every couple of indexes taken between $B_1\cdots B_r$, so that it can be expressed as

$$\begin{aligned}&Nrt_{\alpha_N}A^{\alpha\alpha_N(\beta_1^1\cdots\beta_N^1\cdots\beta_1^r\cdots\beta_{N-1}^r}h_j^{\beta_N^r)}=\\
=&\sum_{s=0}^{\left[\frac{Nr}{2}\right]}2s\binom{Nr}{2s}g_{r,2s}h^{\alpha(\gamma_2}\cdots h^{\gamma_{2s-1}\gamma_{2s}}t^{\gamma_{2s+1}}\cdots t^{\gamma_{Nr}}h_j^{\gamma_{Nr+1})}\\
+&\sum_{s=0}^{\left[\frac{Nr}{2}\right]}(Nr-2s)\binom{Nr}{2s}g_{r,2s}t^{\alpha}h^{(\gamma_2\gamma_3}\cdots h^{\gamma_{2s}\gamma_{2s+1}}t^{\gamma_{2s+2}}\cdots t^{\gamma}_{Nr}h_j^{\gamma_{Nr+1})}\end{aligned}\tag{4.34}$$

Here we have calculated firstly $t_{\alpha_N}A^{\alpha\alpha_N\beta_1^1\cdots\beta_N^1\cdots\beta_1^r\cdots\beta_{N-1}^r}$ by using eq. (4.30) and then distinguishing the terms in which α is index of an $h^{\cdot\cdot}$ from those in which it is an index of a $t^{\cdot}$; finally we have multiplied the result times $h_j^{\gamma_{Nr+1}}$ and symmetrized with respect to $\gamma_2\cdots\gamma_{Nr+1}$.
Until now we have finished to evaluate the three terms of eq. (4.33) calculated at equilibrium; so it becomes

$$\begin{aligned}0=&\sum_{s=1}^{\left[\frac{Nr+2}{2}\right]}g_{r+1,2s}\binom{Nr+1}{2s-1}\frac{2}{3}\lambda_{ll}h_j^{(\gamma_2}\cdots h^{\gamma_{2s-1}\gamma_{2s}}t^{\gamma_{2s+1}}\cdots t^{\gamma_{Nr+1}}t^{\alpha)}+\\
+&\sum_{s=1}^{\left[\frac{Nr+2}{2}\right]}g_{r+1,2s}\binom{Nr+1}{2s-1}\frac{2}{3}\lambda_{ll}h_j^{(\gamma_2}\cdots h^{\gamma_{2s-1}\gamma_{2s}}t^{\gamma_{2s+1}}\cdots t^{\gamma_{Nr+1}}t^{\alpha)}+\\
+&\sum_{s=0}^{\left[\frac{Nr}{2}\right]}2s\binom{Nr}{2s}g_{r,2s}h^{\alpha(\gamma_2}\cdots h^{\gamma_{2s-1}\gamma_{2s}}t^{\gamma_{2s+1}}\cdots t^{\gamma_{Nr}}h_j^{\gamma_{Nr+1})}+\\
+&\sum_{s=0}^{\left[\frac{Nr}{2}\right]}(Nr-2s)\binom{Nr}{2s}g_{r,2s}t^{\alpha}h^{(\gamma_2\gamma_3}\cdots h^{\gamma_{2s}\gamma_{2s+1}}t^{\gamma_{2s+2}}\cdots t^{\gamma_{Nr}}h_j^{\gamma_{Nr+1})}=\end{aligned}$$

$$= \sum_{s=0}^{\left[\frac{Nr}{2}\right]} (Nr+1) \binom{Nr}{2s} g_{r,2s} h^{(\alpha\gamma_2} \cdots h^{\gamma_{2s-1}\gamma_{2s}} t^{\gamma_{2s+1}} \cdots t^{\gamma_{Nr}} h_j^{\gamma_{Nr+1})} +$$

$$+ \sum_{s=0}^{\left[\frac{Nr}{2}\right]} \binom{Nr+1}{2s+1} \frac{2}{3} \lambda_{ll} g_{r+1,2s+2} h_j^{(\gamma_2} \cdots h^{\gamma_{2s+1}\gamma_{2s+2}} t^{\gamma_{2s+3}} \cdots t^{\gamma_{Nr+1}} t^{\alpha)} \quad (4.35)$$

where in the second term we have changed the summation index s according to s=S+1.

Note that this equation is automatically symmetric. In [34] was proved that $\frac{\partial \phi_{[k}}{\partial v_{i]}} = 0$ is an identity for the case of 13 moments; here we find that this property is valid also for an arbitrary number of moments.

So we have proved that eq. (4.33) amounts to

$$0 = (Nr+1) \binom{Nr}{2s} g_{r,2s} + \binom{Nr+1}{2s+1} \frac{2}{3} \lambda_{ll} g_{r+1,2s+2} \quad , \text{ i.e.,}$$

$$g_{r,2s} + \frac{2}{3} \lambda_{ll} \frac{1}{2s+1} g_{r+1,2s+2} = 0 \quad \text{for } s = 0, \cdots, \left[\frac{Nr}{2}\right]. \quad (4.36)$$

Consequently, all our conditions are equivalent to the scalar eqs. (4.31), (4.32) and (4.36) which are constraints on the scalars $g_{r,2s}$ of the expansion (4.30). It remains to exploit them. For $s = 0, \cdots, \left[\frac{Nr}{2}\right]$ we can substitute $g_{r+1,2s+2}$ from eq. (4.31) into eq. $(4.36)_2$ which now becomes

$$\frac{2}{2s+3} \lambda_{ll} \frac{\partial}{\partial \lambda_{ll}} g_{r,2s} + g_{r,2s} = 0 \quad (4.37)$$

whose solution is

$$g_{r,2s} = \lambda_{ll}^{-\frac{2s+3}{2}} G_{r,2s}(\lambda) \quad \text{for } s = 0, \cdots, \left[\frac{Nr}{2}\right]. \quad (4.38)$$

In this way eq. $(4.31)_2$ is exhausted, except for $s = \frac{Nr+1}{2}$ but only for the case with Nr odd.

If Nr is even eq. (4.38) holds for all $g_{r,2s}$, while if Nr is odd the validity of eq. (4.38) is not still proved for $g_{r,Nr+1}$. But for Nr odd we can use eqs. (4.31) with $k = r$, $s = \frac{Nr+1}{2}$, i.e.,

$$\begin{cases} \frac{\partial}{\partial \lambda} g_{r,Nr+1} = g_{r+1,Nr+1} \\ \frac{\partial}{\partial \lambda_{ll}} g_{r,Nr+1} = \frac{Nr+4}{Nr+2} \frac{1}{3} g_{r+1,Nr+3}. \end{cases} \quad (4.39)$$

In the right hand sides we can use eq. (4.38) because $\frac{Nr+1}{2} \leq \left[\frac{N(r+1)}{2}\right]$ and $\frac{Nr+3}{2} \leq \left[\frac{N(r+1)}{2}\right]$ hold, except for the trivial cases N=1,2. In this way the system (4.39) becomes

$$\begin{cases} \frac{\partial}{\partial\lambda} g_{r,Nr+1} = \lambda_{ll}^{-\frac{Nr+4}{2}} G_{r+1,Nr+1}(\lambda) \\ \frac{\partial}{\partial\lambda_{ll}} g_{r,Nr+1} = \frac{Nr+4}{Nr+2}\frac{1}{3}\lambda_{ll}^{-\frac{Nr+6}{2}} G_{r+1,Nr+3}(\lambda). \end{cases} \tag{4.40}$$

The integrability conditions for this system gives

$$G'_{r+1,Nr+3} = \frac{-3}{2}(Nr+2)G_{r+1,Nr+1}. \tag{4.41}$$

After that the system (4.40) can be integrated and gives

$$g_{r,Nr+1} = \lambda_{ll}^{-\frac{Nr+4}{2}} G_{r,Nr+1}(\lambda) + c_{r,Nr+1}, \tag{4.42}$$

with

$$G_{r,Nr+1} = -\frac{2}{3}\frac{1}{Nr+2} G_{r+1,Nr+3}, \tag{4.43}$$

while $c_{r,Nr+1}$ is an arbitrary constant arising from integration. So eq. (4.38) is a valid solution also in the case Nr odd and $s = \frac{Nr+1}{2}$, except to add the arbitrary constant $c_{r,Nr+1}$.
Now we can see that this constant doesn't occur in eq. $(4.31)_1$ (because the right hand side is differentiated, while in the left hand side and in the case N(k+1) odd, we have $2s \leq 2\left[\frac{Nk+1}{2}\right]$ from which $2s < N(k+1)+1$). Nor it occurs in eqs. (4.32), (4.38), (4.41), (4.43) and (4.36) (the proof for this last equation amounts to verify that $\left[\frac{Nr}{2}\right] < \left[\frac{Nr+1}{2}\right]$ for Nr odd and $\left[\frac{Nr}{2}\right] + 1 < \left[\frac{N(r+1)+1}{2}\right]$ for N(r+1) odd; obviously, in both of them we have N odd. If r is odd too, we have to verify only the first one, i.e. $\frac{Nr-1}{2} < \frac{Nr+1}{2}$, which is an identity; if r is even, we have to verify only the second one, i.e. $\frac{Nr}{2} + 1 < \frac{N(r+1)+1}{2}$ which is true, at least for $N > 1$).
On the other hand, the contribute of this constant to the tensor $A^{\alpha_1\alpha_2\cdots\alpha_{Nk+1}}$ is $h^{(\alpha_1\alpha_2}\cdots h^{\alpha_{N_k}\alpha_{N_k+1})}\cdot c_{k,Nk+1}$, as it can be seen from eq. (4.30).
The contribute of all these constants to h'^{α} follows from eq. (4.26) and reads

$$\sum_{r=0}^{\infty}\frac{1}{(2r+1)!}c_{2r+1,N(2r+1)+1}$$
$$h^{\alpha(\beta_1^1}\cdots h^{\beta_{N-1}^1\beta_N^1}\cdots h^{\beta_N^{N-1}\beta_1^N}\cdots h^{\beta_{N-1}^N\beta_N^N)}\cdot l_{\beta_1^1\cdots\beta_N^1}\cdots l_{\beta_1^N\cdots\beta_N^N}, \tag{4.44}$$

where we have put $k = 2r+1$.
It is easy to verify that this additional term satisfies identically the symmetry conditions for eq. (4.19) and (4.23) (in fact t_{α_N} is contracted with an h^{α_N},

for this additional term). In other words, we can assume eq. (4.38) for all $g_{r,2s}$ (also for $s = \left[\frac{Nr+1}{2}\right]$), except that, in the case with N odd, we have to add to $h^{'\alpha}$ the additional term (4.44).
Let's then substitute from eq. (4.38) into eq. $(4.31)_1$ and (4.36); so they become

$$G_{k+1,2s} = G'_{k,2s} \quad \text{for } s = 0, \cdots, \left[\frac{Nk+1}{2}\right], \tag{4.45}$$

$$G_{r+1,2s+2} = -3\frac{2s+1}{2}G_{r,2s} \quad \text{for } s = 0, \cdots \left[\frac{Nr}{2}\right]. \tag{4.46}$$

But this last equation holds also for $s = 0, \cdots, \left[\frac{Nr+1}{2}\right]$; this is obvious when Nr is even, while it is just eq. (4.43) when Nr is odd (remember that we have eq. (4.43) only for the case with Nr odd).
After that, we see that eq. (4.32) is contained in (4.38) for r=s=0, while eq. (4.41), by using eq. (4.43), becomes $G'_{N,Nr+1} = G_{r+1,Nr+1}$ which is just eq. (4.45) with k=r and $s = \left[\frac{Nr+1}{2}\right]$ (remember that eq. (4.41) holds only for Nr odd).
There remain eqs. (4.45) and (4.46). To this end, let us define $H_{r,s}$ from

$$G_{r,2s} = \left(\frac{-3}{2}\right)^r \frac{(2s)!}{2^s s!} H_{r,s}. \tag{4.47}$$

In this way eqs. (4.45) and (4.46) become

$$H_{r+1,s+1} = H_{r,s}, \qquad H'_{r,s} = \frac{-3}{2}H_{r+1,s} \quad \text{for } s = 0, \cdots, \left[\frac{Nr+1}{2}\right]. \tag{4.48}$$

Eq. $(4.48)_1$ suggests to define $H_{r,s}$ also for $s > \left[\frac{Nr+1}{2}\right]$. In fact, let h be a number such that $s + h \leq \left[\frac{N(r+h)+1}{2}\right]$ (for example, $h = \left[\frac{2s-Nr+1}{N-2}\right]$); we can define $H_{r,s} = H_{r+h,s+h}$. In this way eq. $(4.48)_1$ holds for all r and s. Regarding eq. $(4.48)_2$ we have

$$H'_{r,s} = H'_{r+h,s+h} = \frac{-3}{2}H_{r+h+1,s+h} = \frac{-3}{2}H_{r+1,s};$$

in other word, also $(4.48)_2$ holds for all r and s.
After that,

- if $r \geq s$ we have

$$H_{r,s} = H_{r-s,0} = \left(\frac{-2}{3}\right)^{r-s} \frac{d^{r-s}H_{0,0}}{d\lambda^{r-s}} \tag{4.49}$$

- if $r < s$ we have

$$H_{r,s} = H_{0,s-r}. \tag{4.50}$$

In this way $H_{r,s}$ is known except for $H_{0,p}$.
On the other hand, it is easy to see that (4.49) and (4.50) satisfy eq. $(4.48)_1$. Regarding $(4.48)_2$, we see that

- if $r \geq s \;\Rightarrow\; r+1 \geq s$, we have to use eq. (4.49) for both sides of eq. $(4.48)_2$ and it becomes an identity,
- if $r = s-1$, we have to use eq. (4.50) for the left hand side of eq. $(4.48)_2$ and eq. (4.49) for the right hand side. The result is $H'_{0,1} = \frac{-3}{2} H_{0,0}$,
- if $r < s-1$, we have to use eq. (4.50) for both sides of eq. $(4.48)_2$ which becomes $H'_{0,s-r} = \frac{-3}{2} H_{0,s-r-1}$.

In conclusion, $H_{0,0}$ is arbitrary and $H_{0,p}$ is defined by

$$H'_{0,p} = \frac{-3}{2} H_{0,p-1}, \tag{4.51}$$

except for a constant arising from integration. after that, eq. (4.49) and (4.50) give all the other functions $H_{r,s}$.

4.4 The kinetic approach

Let us now search a solution, for conditions (4.19) and (4.23), of the form

$$h'^{\alpha} = \int F\left(l_{\beta_1 \cdots \beta_N} c'^{\beta_1} \cdots c'^{\beta_N}\right) c'^{\alpha} d\underline{c'} \tag{4.52}$$

where F is an arbitrary single variable function; it is related to the distribution function, but this relation doesn't affect the following considerations, so that we choose to omit it.
The symmetry for the left hand side of eq. (4.19) is certainly ensured; remembering that $c'^0 = 1$, eq. (4.23) becomes

$$0 = \int \frac{\partial}{\partial c'_j} \left[F\left(l_{\beta_1 \cdots \beta_N} c'^{\beta_1} \cdots c'^{\beta_N}\right) c'^{\alpha}\right] d\underline{c'} \tag{4.53}$$

which is certainly true. The expansion of eq. (4.52) with respect to equilibrium is

$$h'^{\alpha} = \sum_{k=0}^{\infty} \frac{1}{k!} \int F^{(k)}\left(\lambda + \frac{1}{3}\lambda_{ll} c'^2\right) c'^{\alpha} c'^{B_1} \cdots c'^{B_k} d\underline{c'} \tilde{l}_{B_1} \cdots \tilde{l}_{B_k}$$

where eqs. (4.25), (4.27) and the multi-index notation have been used. Then we have obtained eq. (4.26) with

$$\begin{aligned} A^{\alpha B_1\cdots B_k} &= \frac{\partial^k B^{\alpha B_1\cdots B_k}}{\partial \lambda_k}, \\ B^{\alpha B_1\cdots B_k} &= \int F\left[\lambda + \frac{1}{3}\lambda_{ll}c'^2\right] c'^{\alpha} c'^{B_1}\cdots c'^{B_k} d\underline{c}'. \end{aligned} \tag{4.54}$$

It is easy to verify that eqs. (4.29) are satisfied with this expression. The integral in eq. $(4.54)_2$ can be calculated with a well known procedure. To reach faster the result, let us consider the tensor

$$B^{\beta_1\cdots\beta_s\beta_{s+1}\cdots\beta_r} h^{\gamma_1}_{\beta_1}\cdots h^{\gamma_s}_{\beta_s} t_{\beta_{s+1}}\cdots t_{\beta_r} =$$
$$\int F\left[\lambda + \frac{1}{3}\lambda_{ll}c'^2\right] c'^{\beta_1}\cdots c'^{\beta_s} h^{\gamma_1}_{\beta_1}\cdots h^{\gamma_s}_{\beta_s} d\underline{c}'.$$

The above tensor depends only on scalar quantities and is symmetric, so it is equal to

$$\begin{cases} 0 & \text{if s is odd,} \\ g_s(\lambda,\lambda_{ll}) h^{(\gamma_1\gamma_2}\cdots h^{\gamma_{s-1}\gamma_s)} & \text{if s is even.} \end{cases}$$

To know $g_s(\lambda,\lambda_{ll})$ it suffices to multiply both members by $h_{\gamma_1\gamma_2}\cdots h_{\gamma_{s-1}\gamma_s}$ obtaining

$$\int_0^{\infty} F\left[\lambda + \frac{1}{3}\lambda_{ll}c'^2\right] c'^{s+2}\left(\int_0^{\pi}\sin\theta d\theta\right)\left(\int_0^{2\pi} d\phi\right) dc' = g_s(\lambda,\lambda_{ll})(s+1)$$

where we have changed the integration variables according to the rule

$$\begin{array}{lll} c'^1 = c'\sin\theta\cos\phi, & c'^2 = c'\sin\theta\sin\phi, & c'^3 = c'\cos\theta \\ c' \in [0,+\infty[, & \theta\in[0,\pi], & \phi\in[0,2\pi[. \end{array}$$

We obtain

$$g_s(\lambda,\lambda_{ll}) = \frac{4\pi}{s+1}\int_0^{\infty} F\left[\lambda + \frac{1}{3}\lambda_{ll}c'^2\right] c'^{s+2}dc' = (\lambda_{ll})^{-\frac{s+3}{2}}\, G_s(\lambda)$$

with

$$G_s(\lambda) = \frac{4\pi}{s+1}\int_0^{\infty} F\left[\lambda + \frac{1}{3}\eta^2\right]\eta^{s+2}d\eta, \qquad \eta = \sqrt{\lambda_{ll}}c'. \tag{4.55}$$

For the sequel it will be useful to note that

$$G'_s(\lambda) = \frac{4\pi}{s+1}\int_0^\infty \left\{\frac{d}{d\eta}F\left[\lambda + \frac{1}{3}\eta^2\right]\right\}\eta^{s+1}d\eta\frac{3}{2} =$$
$$= 4\pi\frac{-3}{2}\int_0^\infty F\left[\lambda+\frac{1}{3}\eta^2\right]\eta^s d\eta = -\frac{3}{2}(s-1)G_{s-2} \tag{4.56}$$

provided that $F\eta^{s+1}$ is infinitesimal for η going to infinity. After that, we have

$$\begin{aligned} B^{\gamma_1\cdots\gamma_r} &= B^{\beta_1\beta_2\cdots\beta_r}\left(h^{\gamma_1}_{\beta_1}+t_{\beta_1}t^{\gamma_1}\right)\left(h^{\gamma_2}_{\beta_2}+t_{\beta_2}t^{\gamma_2}\right)\cdots\left(h^{\gamma_r}_{\beta_r}+t_{\beta_r}t^{\gamma_r}\right) = \\ &= \sum_{s=0}^{r}\binom{r}{s}B^{\beta_1\cdots\beta_r}h^{(\gamma_1}_{\beta_1}\cdots h^{\gamma_s}_{\beta_s}t_{\beta_{s+1}}t^{\gamma_{s+1}}\cdots t_{\beta_r}t^{\gamma_r)} = \\ &= \sum_{q=0}^{\left[\frac{r}{2}\right]}\binom{r}{2q}h^{(\gamma_1\gamma_2}\cdots h^{\gamma_{2q-1}\gamma_{2q}}t^{\gamma_{2q+1}}\cdots t^{\gamma_r)}\lambda_{ll}^{-\frac{2q+3}{2}}G_{2q}(\lambda). \end{aligned}$$

This allows to rewrite eq. (4.54) as

$$A^{\alpha B_1\cdots B_k} = \frac{\partial^k B^{\alpha B_1\cdots B_k}}{\partial\lambda^k} =$$
$$\sum_{q=0}^{\left[\frac{kN+1}{2}\right]}\binom{kN+1}{2q}h^{(\gamma_1\gamma_2}\cdots h^{\gamma_{2q-1}\gamma_{2q}}t^{\gamma_{2q+1}}\cdots t^{\gamma_{kN}}t^{\alpha)}\lambda_{ll}^{-\frac{2q+3}{2}}G^{(k)}_{2q}(\lambda).$$

This result confirms eq. (4.30) also in the kinetic case, but with $g_{k,2s}(\lambda,\lambda_{ll}) = \lambda_{ll}^{-\frac{2s+3}{2}}G^{(k)}_{2s}(\lambda)$, and it is easy to see that these functions $g_{k,2s}$ satisfy eq. (4.31), as consequence of eq. (4.56). Also eqs. (4.38) and (4.42) are confirmed, with $G_{r,2s}(\lambda) = G^{(r)}_{2s}(\lambda)$, $c_{r,Nr+1} = 0$. In this way we see that the additional term (4.44) is not present in the kinetic approach. Moreover, the matrix $H_{r,s}$ defined in eq. (4.47) becomes, in this approach

$$H_{r,s} = \left[\frac{-2}{3}\right]^r\frac{2^s s!}{(2s)!}G^{(r)}_{2s}(\lambda),$$

and eq. (4.51) becomes a consequence of eq. (4.56). But the constants arising from integration of eq. (4.51) are not present in the kinetic approach, because all the functions $G_s(\lambda)$ are defined by (4.55) in terms of the single variable function F.

4.5 On subsystems

We aim to obtain now the model with N-1 instead of N through the method of subsystems. To this end we firstly need the relation between the 4-dimensional Lagrange multipliers and the 3-dimensional ones. The first of these are defined by eq. (4.4), from which we obtain

$$\begin{aligned} dH^\alpha &= l^N_{\alpha_1\cdots\alpha_N} dM_N^{\alpha\alpha_1\cdots\alpha_N} = l_N^{\alpha_1\cdots\alpha_N} dM_N^{\alpha\beta_1\cdots\beta_N} g_{\alpha_1\beta_1}\cdots g_{\alpha_N\beta_N} = \\ &= l_N^{\alpha_1\cdots\alpha_N} dM_N^{\alpha\beta_1\cdots\beta_N}(h_{\alpha_1\beta_1}+t_{\alpha_1}t_{\beta_1})\cdots(h_{\alpha_N\beta_N}+t_{\alpha_N}t_{\beta_N}) = \\ &= \sum_{r=0}^{N}\binom{N}{r} l_N^{\alpha_1\cdots\alpha_r\alpha_{r+1}\cdots\alpha_N} dM_N^{\alpha\beta_1\cdots\beta_r\beta_{r+1}\cdots\beta_N} \\ &\quad h_{\alpha_1\beta_1}\cdots h_{\alpha_r\beta_r} t_{\alpha_{r+1}}\cdots t_{\alpha_N} t_{\beta_{r+1}}\cdots t_{\beta_N} = \\ &= \sum_{r=0}^{N} \lambda^N_{j_1\cdots j_r} dF_N^{\alpha j_1\cdots j_r} \end{aligned}$$

$$\text{with} \quad \lambda^N_{j_1\cdots j_r} = \binom{N}{r} l_N^{\alpha_1\cdots\alpha_r\alpha_{r+1}\cdots\alpha_N} h_{\alpha_1 j_1}\cdots h_{\alpha_r j_r} t_{\alpha_{r+1}}\cdots t_{\alpha_N}$$

$$\text{and} \quad F_N^{\alpha j_1\cdots j_r} = M_N^{\alpha\beta_1\cdots\beta_r\beta_{r+1}\cdots\beta_N} h^{j_1}_{\beta_1}\cdots h^{j_r}_{\beta_r} t_{\beta_{r+1}}\cdots t_{\beta_N}. \tag{4.57}$$

Eq. $(4.57)_1$ gives the 3-dimensional Lagrange multipliers in terms of the 4-dimensional ones. We have introduced the index N to remember that we are considering the model with N as maximum order of moments. In this way it will be distinguished from that with N-1 instead of N.
The inverse of eq. $(4.57)_1$ is

$$\begin{aligned} l_N^{\alpha_1\cdots\alpha_N} &= l_N^{\beta_1\cdots\beta_N} g^{\alpha_1}_{\beta_1}\cdots g^{\alpha_N}_{\beta_N} = l_N^{\beta_1\cdots\beta_N}(h^{\alpha_1}_{\beta_1}+t^{\alpha_1}t_{\beta_1})\cdots(h^{\alpha_N}_{\beta_N}+t^{\alpha_N}t_{\beta_N}) = \\ &= \sum_{s=0}^{N}\binom{N}{s} l_N^{\beta_1\cdots\beta_s\cdots\beta_N} h^{(\alpha_1}_{\beta_1}\cdots h^{\alpha_s}_{\beta_s} t^{\alpha_{s+1}}\cdots t^{\alpha_N)} t_{\beta_{s+1}}\cdots t_{\beta_N} = \\ &= \sum_{s=0}^{N} \lambda_N^{(\alpha_1\cdots\alpha_s} t^{\alpha_{s+1}}\cdots t^{\alpha_N)}. \end{aligned} \tag{4.58}$$

The model with N-1 instead of N can be obtained as subsystem of the above one by taking

$$\begin{aligned} \lambda_N^{\alpha_1\cdots\alpha_N} &= 0, \\ \lambda_N^{\alpha_1\cdots\alpha_s} &= \lambda_{N-1}^{\alpha_1\cdots\alpha_s} \quad \text{for } s=0,\cdots,N-1. \end{aligned} \tag{4.59}$$

We have now to express these relations in terms of the 4-dimensional Lagrange multipliers; to this end we see that

$$l_N^{\alpha_1\cdots\alpha_N} = \sum_{s=0}^{N-1} \lambda_{N-1}^{(\alpha_1\cdots\alpha_s} t^{\alpha_{s+1}} \cdots t^{\alpha_N)}, \tag{4.60}$$

while eq. $(4.57)_1$, with N-1 instead of N, is

$$\lambda_{j_1\cdots j_r}^{N-1} = \binom{N-1}{r} l_{N-1}^{\alpha_1\cdots\alpha_r\alpha_{r+1}\cdots\alpha_{N-1}} h_{\alpha_1 j_1} \cdots h_{\alpha_r j_r} t_{\alpha_{r+1}} \cdots t_{\alpha_{N-1}}. \tag{4.61}$$

Then, by substituting eq. (4.61) in eq. (4.60) we find

$$l_N^{\alpha_1\cdots\alpha_N} = \sum_{s=0}^{N-1} t^{(\alpha_{s+1}} \cdots t^{\alpha_N} h_{\gamma_1}^{\alpha_1} \cdots h_{\gamma_s}^{\alpha_s)} l_{N-1}^{\gamma_1\cdots\gamma_s\gamma_{s+1}\cdots\gamma_{N-1}} t_{\gamma_{s+1}} \cdots t_{\gamma_{N-1}} \binom{N-1}{s}$$

from which

$$l_N^{\alpha_1\cdots\alpha_N} = l_{N-1}^{(\alpha_1\cdots\alpha_{N-1}} t^{\alpha_N)}, \tag{4.62}$$

because

$$\begin{aligned} l_{N-1}^{\alpha_1\cdots\alpha_{N-1}} t^{\alpha_N} &= l_{N-1}^{\gamma_1\cdots\gamma_{N-1}} t^{\alpha_N} (h_{\gamma_1}^{\alpha_1} + t^{\alpha_1} t_{\gamma_1}) \cdots (h_{\gamma_{N-1}}^{\alpha_{N-1}} + t^{\alpha_{N-1}} t_{\gamma_{N-1}}) = \\ &= \sum_{s=0}^{N-1} \binom{N-1}{s} l_{N-1}^{\gamma_1\cdots\gamma_s\cdots\gamma_{N-1}} h_{\gamma_1}^{(\alpha_1} \cdots h_{\gamma_s}^{\alpha_s} t_{\gamma_{s+1}} t^{\alpha_{s+1}} \cdots t_{\gamma_{N-1}} t^{\alpha_{N-1})} t^{\alpha_N}. \end{aligned}$$

Now, from eq. (4.24), we have

$$l_{N-1\ eq.}^{\alpha_1\cdots\alpha_{N-1}} = \lambda t^{\alpha_1} \cdots t^{\alpha_{N-1}} + \frac{1}{3} \lambda_{ll} h^{(\alpha_1\alpha_2} t^{\alpha_3} \cdots t^{\alpha_{N-1})}.$$

This and eq. (4.24) yield

$$l_{N\ eq.}^{\alpha_1\cdots\alpha_N} = l_{N-1\ eq.}^{(\alpha_1\cdots\alpha_{N-1}} t^{\alpha_N)},$$

that is, eq. (4.62) holds also when we calculate it at equilibrium. The deviation of eq. (4.62) from its value at equilibrium is

$$\tilde{l}_N^{\alpha_1\cdots\alpha_N} = \tilde{l}_{N-1}^{(\alpha_1\cdots\alpha_{N-1}} t^{\alpha_N)}; \tag{4.63}$$

in other words, eq. (4.62) holds when we substitute the Lagrange multipliers with their deviation with respect to equilibrium. We can now substitute eq. (4.63) into eq. (4.26); in this way we find the counterpart of (4.26) with N-1 instead of N. To this end we have to contract an index of each $B_1, \cdots, B_k$ with a $t_{\cdot}$; in other words, we have to contract the expression (4.30) with $t_{\alpha_{(N-1)k+2}} \cdots t_{\alpha_{Nk+1}}$. It is easy to verify that in this way eq. (4.30) remains unchanged except that now N-1 replaces N; obviously this is true also for $g_{k,2s}$ where s now goes from 0 to $\left[\frac{(N-1)k+1}{2}\right]$. This property is transferred to $G_{k,2s}$ for eq. (4.38) and to $H_{r,s}$ for eq. (4.47). But $H_{r,s}$ is defined by eqs. (4.49) and (4.50) in terms of $H_{0,p}$ which are determined by eq. (4.51). Therefore, the family of constants arising by integrating eq. (4.51), is inherited also by the subsystem.

We have only to notice that from eq. (4.50) it follows that $H_{0,p}$ is useful for $H_{r,r+p}$ which, for eq. (4.47) is useful for $G_{r,2(r+p)}$. It follows that $H_{0,p}$ is present in the subsystem when $r+p \leq \left[\frac{(N-1)r+1}{2}\right]$, that is $p \leq \left[\frac{(N-3)r+1}{2}\right]$, while for the initial system was useful when $p \leq \left[\frac{(N-2)r+1}{2}\right]$. Now, for a fixed value of p, it is always possible to find r such that both of the previous inequalities are satisfied. The only difference is that in the subsystem, $H_{0,p}$ occurs only in terms of higher order with respect to equilibrium, than in the initial system. This is true, provided that $N > 3$, that is if neither the system, nor the subsystem are the 10 moments model.

But what happens to the other family of constants, that is for the supplementary term (4.44)?

If N is even, the model has not this term and, consequently, it cannot be inherited by the subsystem.

If N is odd, this term is present; but when we substitute eq. (4.62) in (4.44) we obtain zero because each $t_{\cdot}$ is contracted with a projector $h^{\cdot\cdot}$. We expected this result because, with N odd, we have N-1 even in which case the term (4.44) is not present. We may conclude that the other family of constants, or the supplementary term (4.44), disappears in the subsystem. Only the other family of constants is inherited.

This can be seen also from the following viewpoint: the family of constants arising from integration of eq. (4.51), in the case N=3, will perpetuate also for the subsequent values of N; equivalently, we can say that the closure in the model with a generic $N > 3$ is exactly determined in terms of that with N=3, except for the supplementary term (4.44).

Chapter 5

The Relativistic E.T. : The many moments case

The first paper on Relativistic Extended Thermodynamics has been produced by I-S. Liu, I. Müller and T. Ruggeri [35]. It was obtained with 14 independent variables satisfying the following system of quasi-linear partial differential equations

$$\partial_\alpha V^\alpha = 0, \qquad \partial_\alpha T^{\alpha\beta} = 0, \qquad \partial_\alpha A^{\alpha\beta\gamma} = 0, \tag{5.1}$$

and then by imposing the entropy principle and the relativity principle.
V^α is the particle flux vector, $T^{\alpha\beta}$ is the energy-momentum tensor and $A^{\alpha\beta\gamma}$ represents the tensor of fluxes. Let's consider the counterpart of the above variables in statistical mechanics. They are defined as moments of the distribution function $f(x^\alpha, p^\alpha)$,

$$V^\alpha = \int f p^\alpha dP, \qquad T^{\alpha\beta} = \int f p^\alpha p^\beta dP, \qquad A^{\alpha\beta\gamma} = \int f p^\alpha p^\beta p^\gamma dP, \tag{5.2}$$

where p^α is the four-momentum of the particle so that we have $p^\alpha p_\alpha = -m^2$ and $dP = \sqrt{-g}\frac{dp_1 dp_2 dp_3}{p_0}$ is the invariant element of the momentum space; m is the particle mass. From eqs. (5.2) the following "trace condition" holds:

$$A^{\alpha\beta\gamma} g_{\beta\gamma} = -m^2 V^\alpha. \tag{5.3}$$

The closure proposed is covariant, complete, and the resulting system is hyperbolic. Here the exact general solution for the **many moments** case is found, satisfying all these conditions up to whatever order. Extension to very many moments is needed in order to improve on the results of ordinary thermodynamics, as shown in [2], page 197. The present results are included in [23]; its title is suggested by one of the possible physical applications, the

study of electron beams, as in [24] and [25] for the 14 moments case. However, the treatment is general and the result can be applied to every fluid.
For the case with many moments we have to choose an even number M and an odd number N; after that the equations are

$$\begin{cases} \partial_\alpha A^{\alpha\alpha_1\ldots\alpha_M} = I^{\alpha_1\ldots\alpha_M}, \\ \partial_\alpha B^{\alpha\alpha_1\ldots\alpha_N} = I^{\alpha_1\ldots\alpha_N}. \end{cases} \tag{5.4}$$

Equations involving lower order tensors are already included in (5.4) because of the following trace conditions (5.5).
All the tensors appearing in the above equations are symmetric and M+N is odd in order to obtain independent equations. The counterparts of these variables in statistical mechanics are

$$A^{\alpha\alpha_1\ldots\alpha_M} = \int f p^\alpha p^{\alpha_1} \cdots p^{\alpha_M} dP, \qquad B^{\alpha\alpha_1\ldots\alpha_N} = \int f p^\alpha p^{\alpha_1} \cdots p^{\alpha_N} dP,$$

from which the trace conditions follows

$$\begin{aligned} -m^2 A^{\alpha\alpha_1\ldots\alpha_{M-2}} &= A^{\alpha\alpha_1\ldots\alpha_M} g_{\alpha_{M-1}\alpha_M}, \\ -m^2 B^{\alpha\alpha_1\ldots\alpha_{N-2}} &= B^{\alpha\alpha_1\ldots\alpha_N} g_{\alpha_{N-1}\alpha_N}. \end{aligned} \tag{5.5}$$

Let us define the maximum trace of a tensor as the trace of the trace ... of the trace of this tensor, so many times as possible. The maximum traces of $I^{\alpha_1\ldots\alpha_M}$ and of $I^{\alpha_1\ldots\alpha_N}$ are zero, so that the maximum traces of eqs. (5.4) are the conservation laws of mass and of momentum-energy.
Now there are less independent components in the eqs. (5.4) than in the variables $A^{\alpha\alpha_1\ldots\alpha_M}$ and $B^{\alpha\alpha_1\ldots\alpha_N}$, so that relations between these variables are needed. We will investigate the "closure problem" by using the procedure used for the macroscopic approach, so we impose the supplementary conservation law

$$\partial_\alpha h^\alpha = \sigma \geq 0, \tag{5.6}$$

that must hold true for all the solutions of the system (5.4). It amounts in assuming the existence of the Lagrange Multipliers $\lambda_{\alpha_1\cdots\alpha_M}$ and $\mu_{\alpha_1\cdots\alpha_N}$, such that

$$dh^\alpha = \lambda_{\alpha_1\cdots\alpha_M} dA^{\alpha\alpha_1\ldots\alpha_M} + \mu_{\alpha_1\cdots\alpha_N} dB^{\alpha\alpha_1\ldots\alpha_N}, \tag{5.7}$$

$$\lambda_{\alpha_1\cdots\alpha_M} I^{\alpha_1\ldots\alpha_M} + \mu_{\alpha_1\cdots\alpha_N} I^{\alpha_1\ldots\alpha_N} \geq 0,$$

where h^α is the entropy-(entropy flux) 4-vector.
We introduce now the potential

$$h'^\alpha = -h^\alpha + \lambda_{\alpha_1\cdots\alpha_M} A^{\alpha\alpha_1\ldots\alpha_M} + \mu_{\alpha_1\cdots\alpha_N} B^{\alpha\alpha_1\ldots\alpha_N}, \tag{5.8}$$

and take the Lagrange Multipliers as independent variables. In this way eq. $(5.7)_1$ becomes

$$dh^{'\alpha} = A^{\alpha\alpha_1\ldots\alpha_M} d\lambda_{\alpha_1\cdots\alpha_M} + B^{\alpha\alpha_1\ldots\alpha_N} d\mu_{\alpha_1\cdots\alpha_N},$$

from which

$$A^{\alpha\alpha_1\ldots\alpha_M} = \frac{\partial h^{'\alpha}}{\partial\lambda_{\alpha_1\cdots\alpha_M}}, \qquad B^{\alpha\alpha_1\ldots\alpha_N} = \frac{\partial h^{'\alpha}}{\partial\mu_{\alpha_1\cdots\alpha_N}}. \tag{5.9}$$

In this way the tensors appearing in the balance equations (5.4) are found as functions of the parameters $\lambda_{\alpha_1\ldots\alpha_M}$ and $\mu_{\alpha_1\ldots\alpha_N}$, as soon as $h^{'\alpha}$ is known. So it remains to find the exact general expression for $h^{'\alpha}$ such that both members of eq. (5.9) are symmetric and, as usual, we will find them through their Taylor expansion with respect to equilibrium.

5.1 Definition of equilibrium and properties

Equilibrium is defined as the state described by the independent variables λ and μ_α such that

$$\begin{aligned} \lambda_{\beta_1\cdots\beta_M} &= \lambda g_{(\beta_1\beta_2}\cdots g_{\beta_{M-1}\beta_M)}(-m^2)^{-\frac{M}{2}}, \\ \mu_{\beta_1\cdots\beta_N} &= \mu_{(\beta_1}g_{\beta_2\beta_3}\cdots g_{\beta_{N-1}\beta_N)}(-m^2)^{-\frac{N-1}{2}}, \end{aligned} \tag{5.10}$$

from which it follows

$$\begin{aligned} \lambda &= 2\frac{(M-1)!!}{(M+2)!!}\lambda_{\alpha_1\cdots\alpha_M} g^{\alpha_1\alpha_2}\cdots g^{\alpha_{M-1}\alpha_M}(-m^2)^{\frac{M}{2}}, \\ \mu_\alpha &= 8\frac{N!!}{(N+3)!!}\mu_{\alpha\alpha_1\cdots\alpha_{N-1}} g^{\alpha_1\alpha_2}\cdots g^{\alpha_{N-2}\alpha_{N-1}}(-m^2)^{\frac{N-1}{2}}. \end{aligned} \tag{5.11}$$

The physical meaning of this definition is evident when we substitute eqs. (5.10) into eq. $(5.7)_1$; by using also the trace conditions (5.5), eq. $(5.7)_1$ becomes

$$dh^\alpha = \lambda dA^\alpha + \mu_{\alpha_1} dB^{\alpha\alpha_1}$$

which is still eq. $(5.7)_1$, but in the case M=0, N=1, i.e. in the case we consider only the conservation laws of mass and of momentum-energy.
It is also evident that eqs. (5.11) are identities if M=0, N=1. (Use firstly the identity $(M-1)!! = \frac{(M+1)!!}{M+1}$).
Eqs. (5.9) now become

$$A^\alpha = \frac{\partial h^{'\alpha}}{\partial\lambda}, \qquad B^{\alpha\alpha_1} = \frac{\partial h^{'\alpha}}{\partial\mu_{\alpha_1}}$$

and the symmetry condition on the second of these is surely satisfied; in fact, from the representation theorems [36], [37] we have that

$$h'^{\alpha} = H(\lambda, \gamma)\mu^{\alpha} \tag{5.12}$$

with $\gamma = \sqrt{-\mu^{\alpha}\mu_{\alpha}}$ and $H(\lambda, \gamma)$ is an arbitrary function. It follows

$$A^{\alpha} = \frac{\partial H}{\partial \lambda}\mu^{\alpha}, \tag{5.13}$$

$$B^{\alpha\alpha_1} = -\frac{1}{\gamma}\frac{\partial H}{\partial \gamma}\mu^{\alpha}\mu^{\alpha_1} + Hg^{\alpha\alpha_1} \tag{5.14}$$

which is surely symmetric.
We stress now that the function $H(\lambda, \gamma)$ has to be arbitrary, if we want that our equations may be applied to all materials. In fact, eqs. (5.12)-(5.14) and eq. (5.8) yield

$$h^{\alpha} = -nsu^{\alpha}, \quad A^{\alpha} = nu^{\alpha}, \quad B^{\alpha\beta} = eu^{\alpha}u^{\beta} + ph^{\alpha\beta} \tag{5.15}$$

with

$$u^{\alpha} = \frac{\mu^{\alpha}}{\gamma}, \qquad h^{\alpha\beta} = g^{\alpha\beta} + u^{\alpha}u^{\beta} \quad \text{(projector)}, \qquad p = H \quad \text{(pressure)},$$
$$n = \gamma\frac{\partial H}{\partial \lambda} \quad \text{(particle density)}, \qquad e = -H - \gamma\frac{\partial H}{\partial \gamma} \quad \text{(energy density)},$$
$$s = -\lambda - \gamma\frac{\frac{\partial H}{\partial \gamma}}{\frac{\partial H}{\partial \lambda}} \quad \text{(entropy density)}. \tag{5.16}$$

From (5.16) it follows $d\left(\frac{e}{n}\right) + pd\left(\frac{1}{n}\right) = \frac{1}{\gamma}ds$ which, compared with the Gibbs relation

$$Tds = d\left(\frac{e}{n}\right) + pd\left(\frac{1}{n}\right) \tag{5.17}$$

identifies γ as $\frac{1}{T}$, with T the absolute temperature. Now $(5.16)_4$ can be used to change variables from γ, λ to γ, p; by substituting its solution $\lambda = \lambda(\gamma, p)$ into eqs. $(5.16)_{3,5}$ we obtain the state functions $n = n(\gamma, p)$ and $e = e(\gamma, p)$. Vice versa, if these state functions are assigned, by substituting p from $(5.16)_4$ into $(5.16)_{3,5}$, these become

$$\frac{\partial H}{\partial \lambda} = \frac{1}{\gamma}n[\gamma, H(\lambda, \gamma)],$$
$$\frac{\partial H}{\partial \lambda} = -\frac{1}{\gamma}e[\gamma, H(\lambda, \gamma)] - \frac{1}{\gamma}H(\lambda, \gamma) \tag{5.18}$$

which are differential equations for the determination of the function $H(\lambda,\gamma)$. Therefore, if we want that our field equations may be used for all materials, i.e., for all possible state functions $n = n(\gamma, p)$ and $e = e(\gamma, p)$, then the function $H(\lambda,\gamma)$ must be arbitrary.
Note also that the integrability condition for eqs. (5.18) is $(e+p)n_p - \gamma n_\gamma = ne_p$. But this is not a new condition on the state function because it is the same integrability condition for the equations

$$\frac{\partial s}{\partial p} = \gamma \frac{\partial}{\partial p}\left(\frac{e}{n}\right) + \gamma p \frac{\partial}{\partial p}\left(\frac{1}{n}\right) \qquad \text{and} \qquad \frac{\partial s}{\partial \gamma} = \gamma \frac{\partial}{\partial \gamma}\left(\frac{e}{n}\right) + \gamma p \frac{\partial}{\partial \gamma}\left(\frac{1}{n}\right)$$

which are equivalent to the Gibbs relation reported above in eq. (5.17).
We conclude this section showing that an expression for $h^{'\alpha}$ at equilibrium is

$$h^{'\alpha} = \int F(\lambda, \mu_\nu p^\nu) p^\alpha dp \tag{5.19}$$

where the function F(X,Y) is related to the distribution function at equilibrium by

$$\frac{\partial}{\partial X} F(X,Y) = f_{eq}(X,Y). \tag{5.20}$$

The expression (5.19) will be useful in the sequel and is equivalent to (5.12) with

$$H(\lambda,\gamma) = -\frac{4\pi}{\gamma} m^3 \int_0^\infty F(\lambda, \gamma m \cosh\rho) \sinh^2\rho \; d\rho. \tag{5.21}$$

which gives the relation between H and F.
It follows that F(X,Y) is arbitrary, because $H(\lambda,\gamma)$ is arbitrary, and (5.19) is the most general expression for $h^{'\alpha}$ at equilibrium. A particular case follows from (5.20) and from

$$f_{eq}(X,Y) = \frac{\frac{w}{h^3}}{e^{\frac{X+Y}{k}} \pm 1}$$

which is the Jüttner distribution function at equilibrium (see [38] and [39]), where k is the Boltzmann constant, the upper and lower signs refer to Fermions and Bosons, respectively, h is the Plank's constant and w is equal to 2s+1 for particles with spin $\frac{sh}{2\pi}$.

5.2 The entropy principle

To impose eqs. (5.9) we have to find the most general expression of h'^{α} such that the left hand sides are symmetric. We will refer to this as "the symmetry condition". An exact particular solution of this condition is

$$h_1'^{\alpha} = \int F(\lambda_{\alpha_1 \ldots \alpha_M} p^{\alpha_1} \ldots p^{\alpha_M}, \mu_{\beta_1 \ldots \beta_N} p^{\beta_1} \ldots p^{\beta_N}) p^{\alpha} dP, \tag{5.22}$$

as it can be easily verified, where the function F(X,Y) has been determined in the previous section (eqs. (5.18) – (5.21)) in terms of the state functions at equilibrium. This solution is more general than the corresponding one in the kinetic approach [40], where the particular case $F(X,Y) = F(X+Y)$ is considered, but it is not still the most general one. We aim here to find this most general solution.
To this end let us note that the symmetry condition for eqs. (5.9) reads

$$\frac{\partial h'^{[\alpha}}{\partial \lambda_{\alpha_1]\alpha_2 \cdots \alpha_M}} = 0, \qquad \frac{\partial h'^{[\alpha}}{\partial \mu_{\alpha_1]\alpha_2 \cdots \alpha_N}} = 0;$$

if we subtract from these their expressions with $h_1'^{\alpha}$ instead of h'^{α}, we find

$$\frac{\partial \Delta h'^{[\alpha}}{\partial \lambda_{\alpha_1]\alpha_2 \cdots \alpha_M}} = 0, \qquad \frac{\partial \Delta h'^{[\alpha}}{\partial \mu_{\alpha_1]\alpha_2 \cdots \alpha_N}} = 0. \tag{5.23}$$

with $\Delta h'^{\alpha} = h'^{\alpha} - h_1'^{\alpha}$.
But in the previous section we have found that $(h_1'^{\alpha})_{eq}$ is the most general expression for $(h'^{\alpha})_{eq}$, so that we have

$$(\Delta h'^{\alpha})_{eq} = 0. \tag{5.24}$$

In this way, we have now to find the most general solution of eqs. (5.23) and (5.24), after that we will have

$$h'^{\alpha} = h_1'^{\alpha} + \Delta h'^{\alpha}. \tag{5.25}$$

To exploit eqs. (5.23) and (5.24), let's firstly consider the Taylor expansion of $\Delta h'^{\alpha}$ around equilibrium:

$$\Delta h'^{\alpha} = \sum_{h,k=0}^{\infty} \frac{1}{h!k!} C_{h,k}^{\alpha A_1 \cdots A_h B_1 \cdots B_k} \widetilde{\lambda}_{A_1} \cdots \widetilde{\lambda}_{A_h} \widetilde{\mu}_{B_1} \cdots \widetilde{\mu}_{B_k}, \tag{5.26}$$

where the multi-index notation $A_i = \alpha_{i_1} \cdots \alpha_{i_M}$, $B_j = \alpha_{j_1} \cdots \alpha_{j_N}$ has been used, and, moreover

$$C_{h,k}^{\alpha A_1 \cdots A_h B_1 \cdots B_k} = \left(\frac{\partial^{h+k} \Delta h'^{\alpha}}{\partial \lambda_{A_1} \cdots \partial \lambda_{A_h} \partial \mu_{B_1} \cdots \partial \mu_{B_k}} \right)_{eq.} \tag{5.27}$$

and

$$\begin{cases} \widetilde{\lambda}_{\beta_1\cdots\beta_M} = \lambda_{\beta_1\cdots\beta_M} - \lambda \; g_{(\beta_1\beta_2}\cdots g_{\beta_{M-1}\beta_M)}(-m^2)^{-\frac{M}{2}}, \\ \widetilde{\mu}_{\beta_1\cdots\beta_N} = \mu_{\beta_1\cdots\beta_N} - \mu_{(\beta_1}g_{\beta_2\beta_3}\cdots g_{\beta_{N-1}\beta_N)}(-m^2)^{-\frac{N-1}{2}}, \end{cases} \tag{5.28}$$

denote the deviation of the Lagrange multipliers from their value (5.10) at equilibrium. Note also that their traces are zero, as consequence of eq. (5.11). Because of the symmetry shown by eqs. (5.23) it is possible to exchange the index α with each other index taken from those included in A_i or B_j and moreover each A_i can be exchanged with each other A_s or B_r. So the tensor $C^{\cdots}_{h,k}$ is symmetric with respect to every couple of indexes.
Let's consider $\frac{\partial^{h+k}\Delta {h'}^{\alpha}}{\partial\lambda_{A_1}\cdots\partial\lambda_{A_h}\partial\mu_{B_1}\cdots\partial\mu_{B_k}}$ depending on $\lambda_{\gamma_1\cdots\gamma_M}$ as a composite function through $\widetilde{\lambda}_{\gamma_1\cdots\gamma_M}$ and λ; after that let us take its derivative with respect to $\lambda_{\gamma_1\cdots\gamma_M}$ and calculate the result at equilibrium. We obtain

$$\begin{aligned} C^{\alpha A_1\cdots A_{h+1}B_1\cdots B_k}_{h+1,k} &= \frac{\partial C^{\alpha A_1\cdots A_h B_1\cdots B_k}_{h,k}}{\partial\widetilde{\lambda}_{\beta_1\cdots\beta_M}}\frac{\partial\widetilde{\lambda}_{\beta_1\cdots\beta_M}}{\lambda_{\gamma_1\cdots\gamma_M}} + \frac{\partial C^{\alpha A_1\cdots A_h B_1\cdots B_k}_{h,k}}{\partial\lambda}\frac{\partial\lambda}{\lambda_{\gamma_1\cdots\gamma_M}} = \\ &= \frac{\partial C^{\alpha A_1\cdots A_h B_1\cdots B_k}_{h,k}}{\partial\widetilde{\lambda}_{\beta_1\cdots\beta_M}}\Bigg(g^{(\gamma_1}_{\beta_1}\cdots g^{\gamma_M)}_{\beta_M} - g^{(\gamma_1\gamma_2}\cdots g^{\gamma_{M-1}\gamma_M)}g_{\beta_1\beta_2}\cdots g_{\beta_{M-1}\beta_M}\,\cdot \\ &\quad 2\frac{(M-1)!!}{(M+2)!!}\Bigg) + \frac{\partial C^{\alpha A_1\cdots A_h B_1\cdots B_k}_{h,k}}{\partial\lambda}g^{(\gamma_1\gamma_2}\cdots g^{\gamma_{M-1}\gamma_M)}2\frac{(M-1)!!}{(M+1)!!}(-m^2)^{\frac{M}{2}}, \end{aligned} \tag{5.29}$$

where $(5.11)_1$ and $(5.28)_1$ have been used.
Multiplying both sides by $g_{\gamma_1\gamma_2}\cdots g_{\gamma_{M-1}\gamma_M}$ we obtain

$$C^{\alpha A_1\cdots A_h\gamma_1\cdots\gamma_M B_1\cdots B_k}_{h+1,k}g_{\gamma_1\gamma_2}\cdots g_{\gamma_{M-1}\gamma_M} = \frac{\partial C^{\alpha A_1\cdots A_h B_1\cdots B_k}_{h,k}}{\partial\lambda}(-m^2)^{\frac{M}{2}}. \tag{5.30}$$

Similarly, let's consider $\frac{\partial^{h+k}\Delta {h'}^{\alpha}}{\partial\lambda_{A_1}\cdots\partial\lambda_{A_h}\partial\mu_{B_1}\cdots\partial\mu_{B_k}}$ depending on $\mu_{\gamma\gamma_1\cdots\gamma_{N-1}}$ as a composite function trough $\widetilde{\mu}_{\gamma\gamma_1\cdots\gamma_{N-1}}$ and μ_β; after that let us take its derivative with respect to $\mu_{\gamma\gamma_1\cdots\gamma_{N-1}}$ and calculate the result at equilibrium. We obtain

$$\begin{aligned} C^{\alpha A_1\cdots A_h B_1\cdots B_{k+1}}_{h,k+1} &= \frac{\partial C^{\alpha A_1\cdots A_h B_1\cdots B_k}_{h,k}}{\partial\widetilde{\mu}_{\beta\beta_1\cdots\beta_{N-1}}}\frac{\partial\widetilde{\mu}_{\beta\beta_1\cdots\beta_{N-1}}}{\mu_{\gamma\gamma_1\cdots\gamma_{N-1}}} + \frac{\partial C^{\alpha A_1\cdots A_h B_1\cdots B_k}_{h,k}}{\partial\mu_\beta}\frac{\partial\mu_\beta}{\mu_{\gamma\gamma_1\cdots\gamma_{N-1}}} \\ &= \frac{\partial C^{\alpha A_1\cdots A_h B_1\cdots B_k}_{h,k}}{\partial\widetilde{\mu}_{\beta\beta_1\cdots\beta_{N-1}}}\Big(g^{(\gamma}_{\beta}g^{\gamma_1}_{\beta_1}\cdots g^{\gamma_{N-1})}_{\beta_{N-1}} - g^{(\gamma}_{(\beta}g^{\gamma_1\gamma_2}\cdots g^{\gamma_{N-2}\gamma_{N-1})}g_{\beta_1\beta_2}\cdots g_{\beta_{N-2}\beta_{N-1})} \\ &\quad 8\frac{N!!}{(N+3)!!}\Big) + \frac{\partial C^{\alpha A_1\cdots A_h B_1\cdots B_k}_{h,k}}{\partial\mu_\beta}g^{(\gamma}_{\beta}g^{\gamma_1\gamma_2}\cdots g^{\gamma_{N-2}\gamma_{N-1})}8\frac{N!!}{(N+3)!!}(-m^2)^{\frac{N-1}{2}}, \end{aligned}$$

where $(5.11)_1$ and $(5.28)_2$ have been used.
Multiplying both sides by $g_{\gamma_1\gamma_2}\cdots g_{\gamma_{N-2}\gamma_{N-1}}$ we obtain

$$C_{h,k+1}^{\alpha A_1\cdots A_h B_1\cdots B_k \beta\gamma_1\cdots\gamma_{N-1}} g_{\gamma_1\gamma_2}\cdots g_{\gamma_{N-2}\gamma_{N-1}} = \frac{\partial C_{h,k}^{\alpha A_1\cdots A_h B_1\cdots B_k}}{\partial\mu_\beta}(-m^2)^{\frac{N-1}{2}}. \tag{5.31}$$

So, from eq. (5.27) we have obtained the compatibility conditions (5.30) and (5.31).
Let's proof the vice versa, i.e., that eq. (5.27) is a consequence of all the other equations. To this end it is firstly useful to show a property of $\Delta h'^\alpha$. If we take its derivative with respect to $\lambda_{\gamma_1\cdots\gamma_M}$ we obtain, with passages like that used in eq. (5.29), that

$$\begin{aligned}
\frac{\partial\Delta h'^\alpha}{\partial\lambda_{\gamma_1\cdots\gamma_M}} &= \sum_{h,k=0}^{\infty}\frac{1}{h!k!}\frac{\partial C_{h,k}^{\alpha A_1\cdots A_h B_1\cdots B_k}}{\partial\lambda}\frac{\partial\lambda}{\partial\lambda_{\gamma_1\cdots\gamma_M}}\widetilde{\lambda}_{A_1}\cdots\widetilde{\lambda}_{A_h}\widetilde{\mu}_{B_1}\cdots\widetilde{\mu}_{B_k}+\\
&+\sum_{\substack{h,k=0\\h\neq 0}}^{\infty}\frac{h}{h!k!}C_{h,k}^{\alpha A_1\cdots A_h B_1\cdots B_k}\widetilde{\lambda}_{A_1}\cdots\widetilde{\lambda}_{A_{h-1}}\widetilde{\mu}_{B_1}\cdots\widetilde{\mu}_{B_k}\frac{\partial\widetilde{\lambda}_{A_h}}{\partial\lambda_{\gamma_1\cdots\gamma_M}}=\\
&=\sum_{\substack{h,k=0\\h\neq 0}}^{\infty}\frac{h}{h!k!}C_{h,k}^{\alpha A_1\cdots A_h B_1\cdots B_k}\widetilde{\lambda}_{A_1}\cdots\widetilde{\lambda}_{A_{h-1}}\widetilde{\mu}_{B_1}\cdots\widetilde{\mu}_{B_k}g_{\alpha_1}^{(\gamma_1}\cdots g_{\alpha_M}^{\gamma_M)}=\\
&=\frac{\partial\Delta h'^\alpha}{\partial\widetilde{\lambda}_{\gamma_1\cdots\gamma_M}}
\end{aligned}$$

where we have used eq. (5.30). So we have proved that derivation of eq. (5.26) with respect to $\lambda_{\gamma_1\cdots\gamma_M}$ is equivalent to its derivation with respect to $\widetilde{\lambda}_{\gamma_1\cdots\gamma_M}$.
With analogous passages, but by using eq. (5.31), it is possible to prove that derivation of eq. (5.26) with respect to $\mu_{\gamma_1\cdots\gamma_N}$ is equivalent to its derivation with respect to $\widetilde{\mu}_{\gamma_1\cdots\gamma_N}$.
We are now ready to prove that eq. (5.27) is a consequence of the other equations. To this end, let us take the h^{th} derivative of eq. (5.26) with respect to $\lambda_{\gamma_1\cdots\gamma_M}$, then its k^{th} derivative with respect to $\mu_{\gamma_1\cdots\gamma_N}$ and calculate the result at equilibrium; by using eqs. (5.30), (5.31) and the above mentioned property , we obtain eq. (5.27). In other words we can forget eq. (5.27) and retain only eqs. (5.30) and (5.31).
So it remains to solve eqs. (5.30) and (5.31) in the unknown symmetric tensors $C_{h,k}^{\cdots}$.
We notice that both sides in equation (5.30) and the left-hand side of eq.

(5.31) are symmetric; then we have to impose that the right hand side of this last one is also symmetric. In other words, both the tensors $C^{\cdots}_{h,k}$ and their derivatives with respect to μ_β are symmetric. The tensors satisfying this property are elements of a family $\mathcal{F}$ which will be characterized in the following section. Moreover, interesting properties of this family $\mathcal{F}$ will be shown and they are useful to exploit our eqs. (5.30) and (5.31). The effective exploitation will be accomplished in section 5.4 for the case $N = 1$ and in section 5.5 for the case $N > 1$. We complete the present section simply reporting the results.
For the case $N > 1$, they are

$$C_{h,k}^{\alpha_1\cdots\alpha_{Mh+kN+1}} = \sum_{s=0}^{\left[\frac{Mh+Nk+1}{2}\right]} C_s^{h,k} g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2s-1}\alpha_{2s}}\mu^{\alpha_{2s+1}}\cdots\mu^{\alpha_{Mh+Nk+1})}, \quad (5.32)$$

with

$$\begin{aligned} C_s^{h,k} &= 2^{2\left[\frac{Mh+Nk+1}{2}\right]+\left[\frac{k}{2}\right]-2s}\cdot\frac{\left[\frac{Mh+Nk+1}{2}\right]!}{s!}\frac{1}{(Mh+Nk+1-2s)!}\cdot \\ &\cdot\ \gamma^{-6-Mh-(N+1)k+2s}\cdot\sum_{q=0}^{\frac{Mh+k(N-1)-2}{2}}\frac{\left[Mh+k(N-1)+1+2\left[\frac{k}{2}\right]\right]!!}{\left[hM+k(N-1)-2q-2\right]!!}\cdot \\ &\cdot\ (-m^2)^{\frac{N-1}{2}k+\frac{M}{2}h}\frac{d^h c_{k+Mh,q}}{d\lambda^h}\cdot\frac{\left(q+2+\frac{Mh+(N+1)k}{2}-s\right)!}{(q+2)!}\cdot\left(\frac{1}{\gamma^2}\right)^q, \end{aligned} \quad (5.33)$$

and $c_{k,q} = c_{k,q}(\lambda)$ restricted only by $c_{k,q} = c_{k-1,q}$ for $q = 0, \cdots, \frac{[Mh+(k-1)(N-1)-2]}{2}$.
For the case $N = 1$, and consequently k=0, the results are

$$C_h^{\alpha_1\cdots\alpha_{Mh+1}} = \sum_{s=0}^{\frac{Mh}{2}} C_s^h g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2s-1}\alpha_{2s}}\mu^{\alpha_{2s+1}}\cdots\mu^{\alpha_{Mh+1})}, \quad (5.34)$$

with

$$\begin{aligned} C_s^h &= 2^{Mh-2s}\frac{\left(\frac{Mh}{2}\right)!}{s!}\frac{1}{(Mh+1-2s)!}\cdot\gamma^{-6-Mh+2s}\cdot\sum_{q=0}^{\frac{Mh-2}{2}}\frac{(Mh+1)!!}{(Mh-2q-2)!!}\cdot \\ &\cdot\ (-m^2)^{\frac{M}{2}h}\frac{d^h c_{h,q}}{d\lambda^h}\cdot\frac{\left(q+2+\frac{Mh}{2}-s\right)!}{(q+2)!}\cdot\left(\frac{1}{\gamma^2}\right)^q, \end{aligned} \quad (5.35)$$

and $c_{h,q} = c_{h,q}(\lambda)$ restricted only by $c_{h,q} = c_{h-1,q}$ for $q = 0, \cdots, \frac{[M(h-1)-2]}{2}$.

5.3 On the family $\mathcal{F}$ and its properties

Let us characterize now the family $\mathcal{F}$ of tensorial functions of a scalar λ and of a time-like 4-vector μ_β, which are symmetric together with their derivative with respect to μ_β.
We will prove now that

Proposition 1 *Each element of the family $\mathcal{F}$ can be written as*

$$\phi^{\alpha_1\cdots\alpha_n}(\lambda,\mu_\beta)=\sum_{s=0}^{\left[\frac{n}{2}\right]}\phi^n_s(\lambda,\gamma)g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2s-1}\alpha_{2s}}\mu^{\alpha_{2s+1}}\cdots\mu^{\alpha_n)} \tag{5.36}$$

with the scalars ϕ^n_s satisfying the condition

$$\frac{\partial\phi^n_s}{\partial\gamma}\frac{1}{\gamma}2s+(n-2s+2)(n-2s+1)\phi^n_{s-1}=0 \quad \textit{for } s=1,\cdots\left[\frac{n}{2}\right]. \tag{5.37}$$

(Note that among the terms ϕ^n_s we may call <u>*leading term*</u> *the one with the highest value of s, i.e. $\phi^n_{\left[\frac{n}{2}\right]}$. Once the leading term is known, we can find all the other terms present in eq.* (5.36) *thanks to eq.* (5.37)*).*

Proof. From the representation theorems we know that eq. (5.36) is the most general expression of a symmetric tensorial function depending on λ and μ_β. Taking into account that $\frac{\partial\gamma}{\partial\mu_\beta}=-\frac{\mu^\beta}{\gamma}$, the derivative of $\phi^{\alpha_1\cdots\alpha_n}(\lambda,\mu_\beta)$ with respect to μ_β is

$$\begin{aligned}\frac{\partial\phi^{\alpha_1\cdots\alpha_n}}{\partial\mu_\beta} &= \sum_{s=0}^{\left[\frac{n}{2}\right]}-\frac{\partial\phi^n_s}{\partial\gamma}\frac{\mu^\beta}{\gamma}g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2s-1}\alpha_{2s}}\mu^{\alpha_{2s+1}}\cdots\mu^{\alpha_n)}+\\ &+\sum_{s=0}^{\left[\frac{n}{2}\right]}\phi^n_s g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2s-1}\alpha_{2s}}\mu^{\alpha_{2s+1}}\cdots\mu^{\alpha_{n-1}}g^{\alpha_n)\beta}(n-2s).\end{aligned} \tag{5.38}$$

To be symmetric, the expression above must be equal to its symmetric part with respect to $\alpha_1\cdots\alpha_n\beta$, i.e. to

$$\begin{aligned}&\sum_{s=0}^{\left[\frac{n}{2}\right]}-\frac{\partial\phi^n_s}{\partial\gamma}\frac{1}{\gamma}g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2s-1}\alpha_{2s}}\mu^{\alpha_{2s+1}}\cdots\mu^{\alpha_n}\mu^{\beta)}+\\ +&\sum_{S=1}^{\left[\frac{n}{2}\right]+1}\phi^n_{S-1}g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2S-1}\alpha_{2S}}\mu^{\alpha_{2S+1}}\cdots\mu^{\alpha_n}\mu^{\beta)}(n-2S+2)=\end{aligned}$$

$$
\begin{aligned}
= & -\frac{\partial \phi_0^n}{\partial \gamma}\frac{1}{\gamma}\mu^{\alpha_1}\cdots\mu^{\alpha_n}\mu^{\beta} + \phi^n_{\left[\frac{n}{2}\right]}\left(n-2\left[\frac{n}{2}\right]\right) g^{(\alpha_1\alpha_2}\cdots g^{\alpha_n\beta)} + \\
+ & \sum_{s=1}^{\left[\frac{n}{2}\right]}\left[-\frac{\partial \phi_s^n}{\partial \gamma}\frac{1}{\gamma} + (n-2s+2)\phi^n_{s-1}\right] g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2s-1}\alpha_{2s}}\mu^{\alpha_{2s+1}}\cdots\mu^{\alpha_n}\mu^{\beta)} = \\
= & -\frac{\partial \phi_0^n}{\partial \gamma}\frac{1}{\gamma}\mu^{\alpha_1}\cdots\mu^{\alpha_n}\mu^{\beta} + \phi^n_{\left[\frac{n}{2}\right]}\left(n-2\left[\frac{n}{2}\right]\right) g^{(\alpha_1\alpha_2}\cdots g^{\alpha_n\beta)} + \\
+ & \sum_{s=1}^{\left[\frac{n}{2}\right]}\left[-\frac{\partial \phi_s^n}{\partial \gamma}\frac{1}{\gamma} + (n-2s+2)\phi^n_{s-1}\right]\left[\frac{2s}{n+1} g^{\beta(\alpha_1}\cdots g^{\alpha_{2s-2}\alpha_{2s-1}}\mu^{\alpha_{2s}}\cdots\mu^{\alpha_n)}\right. \\
+ & \left.\frac{n+1-2s}{n+1} g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2s-1}\alpha_{2s}}\mu^{\alpha_{2s+1}}\cdots\mu^{\alpha_n)}\mu^{\beta}\right],
\end{aligned}
$$

where in the second row we have substituted $s = S-1$. In the third row we have reported the term coming from the first row with s=0, and that from the second row with $S = \left[\frac{n}{2}\right]+1$; in the fourth row the remaining terms from first and second row.
Comparing the result with eq. (5.38), we obtain

$$
\begin{aligned}
& \sum_{s=0}^{\left[\frac{n}{2}\right]} -\frac{\partial \phi_s^n}{\partial \gamma}\frac{1}{\gamma} g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2s-1}\alpha_{2s}}\mu^{\alpha_{2s+1}}\cdots\mu^{\alpha_n)} = \\
= & -\frac{\partial \phi_0^n}{\partial \gamma}\frac{1}{\gamma}\mu^{\alpha_1}\cdots\mu^{\alpha_n} + \sum_{s=1}^{\left[\frac{n}{2}\right]}\left[-\frac{\partial \phi_s^n}{\partial \gamma}\frac{1}{\gamma} + (n-2s+2)\phi^n_{s-1}\right]\cdot \\
& \frac{n+1-2s}{n+1} g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2s-1}\alpha_{2s}}\mu^{\alpha_{2s+1}}\cdots\mu^{\alpha_n)}
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{s=0}^{\left[\frac{n}{2}\right]} \phi_s^n (n-2s) g^{\beta(\alpha_1}\cdots g^{\alpha_{2s}\alpha_{2s+1}}\mu^{\alpha_{2s+2}}\cdots\mu^{\alpha_n)} = \\
= & \ \phi^n_{\left[\frac{n}{2}\right]}\left(n-2\left[\frac{n}{2}\right]\right) g^{(\alpha_1\alpha_2}\cdots g^{\alpha_n)\beta} + \\
+ & \sum_{s=0}^{\left[\frac{n}{2}\right]-1}\left[-\frac{\partial \phi^n_{s+1}}{\partial \gamma}\frac{1}{\gamma} + (n-2s)\phi_s^n\right]\frac{2s+2}{n+1} g^{\beta(\alpha_1}\cdots g^{\alpha_{2s}\alpha_{2s+1}}\mu^{\alpha_{2s+2}}\cdots\mu^{\alpha_n)}
\end{aligned}
$$

where we have considered $s = S+1$ and then $S = s$ in the last term.
So we have

$$
\frac{\partial \phi_s^n}{\partial \gamma}\frac{1}{\gamma}2s + (n-2s+2)(n-2s+1)\phi^n_{s-1} = 0 \quad \text{for } s = 1, \cdots \left[\frac{n}{2}\right]
$$

and

$$\frac{\partial \phi^n_{s+1}}{\partial \gamma}\frac{1}{\gamma}(2s+2)+(n-2s)(n-2s-1)\phi^n_s = 0 \quad \text{for } s=0,\cdots\left[\frac{n-2}{2}\right].$$

We can see that the second of these equations coincides with the first one. So the characteristic condition for $\mathcal{F}$ is the above reported eq. (5.37).
This completes the proof of Proposition 1. ■
It will be useful for the sequel to note that, thanks to (5.37), eq. (5.38) becomes

$$\frac{\partial \phi^{\alpha_1\cdots\alpha_n}}{\partial \mu_\beta} = \sum_{s=0}^{\left[\frac{n+1}{2}\right]} \phi^{n+1}_s(\lambda,\gamma) g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2s-1}\alpha_{2s}}\mu^{\alpha_{2s+1}}\cdots\mu^{\alpha_n}\mu^{\beta)} \tag{5.39}$$

with

$$\begin{cases} \phi^{n+1}_0 = -\frac{1}{\gamma}\frac{\partial \phi^n_0}{\partial \gamma}, \\ \phi^{n+1}_s = \frac{n+1}{2s}(n-2s+2)\phi^n_{s-1} \quad \text{for } s=1,\cdots,\left[\frac{n+1}{2}\right]. \end{cases} \tag{5.40}$$

This allows to prove the following

Proposition 2 *If $\phi^{\alpha_1\cdots\alpha_n}\in\mathcal{F}$, it follows that also $\frac{\partial\phi^{\alpha_1\cdots\alpha_n}}{\partial\mu_\beta}\in\mathcal{F}$.*

Proof. Because of eq. (5.39), to demonstrate the theorem it will be sufficient to prove that eq. (5.37) holds true also with n+1 instead of n, i.e.

$$\frac{\partial \phi^{n+1}_s}{\partial \gamma}\frac{1}{\gamma}2s+(n-2s+3)(n-2s+2)\phi^{n+1}_{s-1} = 0 \quad \text{for } s=1,\cdots\left[\frac{n+1}{2}\right]. \tag{5.41}$$

For s=1, thanks to (5.40), it becomes

$$\frac{\partial \phi^n_0}{\partial \gamma}\frac{2}{\gamma}\frac{n}{2}(n+1)+(n+1)n\left(-\frac{1}{\gamma}\right)\frac{\phi^n_0}{\partial \gamma} = 0$$

that is identically satisfied.
Instead, for $s=2,\cdots,\left[\frac{n+1}{2}\right]$, thanks to $(5.40)_2$, it becomes

$$\frac{\partial \phi^n_{s-1}}{\partial \gamma}\frac{2s}{\gamma}\frac{n+1}{2s}(n-2s+2)+(n-2s+3)(n-2s+2)\frac{n+1}{2s-2}(n-2s+4)\phi^n_{s-2} = 0$$

that is eq. (5.37) with s-1 instead of s, after having divided it for 2s-2. ■
It is also important the following

Proposition 3 *If $\phi^{\alpha_1\cdots\alpha_{n+1}}\in\mathcal{F}$, then $\phi^{\alpha_1\cdots\alpha_n}\in\mathcal{F}$ exists such that $\phi^{\alpha_1\cdots\alpha_{n+1}} = \frac{\partial\phi^{\alpha_1\cdots\alpha_n}}{\partial\mu_{\alpha_{n+1}}}$.*

Proof. In fact the integrability condition for the problem

$$\phi^{\alpha_1\cdots\alpha_{n+1}} = \frac{\partial \phi^{\alpha_1\cdots\alpha_n}}{\partial \mu_{\alpha_{n+1}}} \qquad \text{is} \qquad \frac{\partial \phi^{\alpha_1\cdots\alpha_{n-1}[\alpha_n}}{\partial \mu_{\alpha_{n+1}]}} = 0$$

which is surely satisfied because $\phi^{\alpha_1\cdots\alpha_n} \in \mathcal{F}$ and, consequently, it and its derivative with respect to $\mu_{\alpha_{n+1}}$ are symmetric.
We note also that, if $\widetilde{\phi}^{\alpha_1\cdots\alpha_n}$ is a particular solution of this problem, then the general one is

$$\phi^{\alpha_1\cdots\alpha_n} = \begin{cases} \widetilde{\phi}^{\alpha_1\cdots\alpha_n} + \phi(\lambda) g^{(\alpha_1\alpha_2} \cdots g^{\alpha_{n-1}\alpha_n)} & \text{if n is even,} \\ \widetilde{\phi}^{\alpha_1\cdots\alpha_n} & \text{if n is odd,} \end{cases}$$

with $\phi(\lambda)$ a scalar function. ■

Proposition 4 *If* $\phi^{\alpha_1\cdots\alpha_{n+r}} \in \mathcal{F}$*, then* $\phi^{\alpha_1\cdots\alpha_n} \in \mathcal{F}$ *exists such that* $\phi^{\alpha_1\cdots\alpha_{n+r}} = \frac{\partial^r \phi^{\alpha_1\cdots\alpha_n}}{\partial \mu_{\alpha_{n+1}} \cdots \mu_{\alpha_{n+r}}}$.
Moreover, if $\widetilde{\phi}^{\alpha_1\cdots\alpha_n}$ *is a particular solution of this problem, then the general one is*

$$\phi^{\alpha_1\cdots\alpha_n} = \begin{cases} \widetilde{\phi}^{\alpha_1\cdots\alpha_n} + \sum_{i=0}^{\left[\frac{r-1}{2}\right]} \phi_i(\lambda) g^{(\alpha_1\alpha_2} \cdots g^{\alpha_{n+2i-1}\alpha_n+2i)} \mu_{\alpha_{n+1}} \cdots \mu_{\alpha_{n+2i}} \\ \textit{if n is even,} \\ \\ \widetilde{\phi}^{\alpha_1\cdots\alpha_n} + \sum_{i=0}^{\left[\frac{r-2}{2}\right]} \phi_i(\lambda) g^{(\alpha_1\alpha_2} \cdots g^{\alpha_{n+2i}\alpha_n+2i+1)} \mu_{\alpha_{n+1}} \cdots \mu_{\alpha_{n+2i+1}} \\ \textit{if n is odd,} \end{cases}$$

with $\phi_i(\lambda)$ *scalar functions.*

Proof. We can prove this proposition with the iterative procedure.
It holds for r=1 for the Proposition 3.
Let us assume now that it holds for $r = \overline{r}$ and prove that it is satisfied also when $r = \overline{r} + 1$.
If $\phi^{\alpha_1\cdots\alpha_{n+\overline{r}+1}} \in \mathcal{F}$ we can apply this proposition with n+1 instead of n and $\overline{r}$ instead of r. Then we have that $\phi^{\alpha_1\cdots\alpha_{n+1}} \in \mathcal{F}$ exists such that

$$\phi^{\alpha_1\cdots\alpha_{n+\overline{r}+1}} = \frac{\partial^{\overline{r}} \phi^{\alpha_1\cdots\alpha_{n+1}}}{\partial \mu_{\alpha_{n+2}} \cdots \partial \mu_{\alpha_{n+\overline{r}+1}}}. \tag{5.42}$$

But, for the first part of Proposition 3 we have that $\phi^{\alpha_1\cdots\alpha_n} \in \mathcal{F}$ exists such that $\phi^{\alpha_1\cdots\alpha_{n+1}} = \frac{\partial \phi^{\alpha_1\cdots\alpha_n}}{\partial \mu_{\alpha_{n+1}}}$ which, substituted in (5.42) gives

$$\phi^{\alpha_1\cdots\alpha_{n+\overline{r}+1}} = \frac{\partial^{\overline{r}+1} \phi^{\alpha_1\cdots\alpha_n}}{\partial \mu_{\alpha_{n+1}} \cdots \partial \mu_{\alpha_{n+\overline{r}+1}}}. \tag{5.43}$$

So the existence of solutions has been proved.
If $\widetilde{\phi}^{\alpha_1\cdots\alpha_n}$ is a particular of these solutions, we have

$$\phi^{\alpha_1\cdots\alpha_{n+\bar{r}+1}} = \frac{\partial^{\bar{r}+1}\widetilde{\phi}^{\alpha_1\cdots\alpha_n}}{\partial\mu_{\alpha_{n+1}}\cdots\partial\mu_{\alpha_{n+\bar{r}+1}}},$$

which, together with eq. (5.43) implies

$$\frac{\partial^{\bar{r}}}{\partial\mu_{\alpha_{n+1}}\cdots\partial\mu_{\alpha_{n+\bar{r}}}}\left[\frac{\partial}{\partial\mu_{\alpha_{n+\bar{r}+1}}}\left(\phi^{\alpha_1\cdots\alpha_n} - \widetilde{\phi}^{\alpha_1\cdots\alpha_n}\right)\right] = 0.$$

By applying the second part of this Proposition 4, we conclude that

$$\frac{\partial}{\partial\mu_{\alpha_{n+\bar{r}+1}}}\left(\phi^{\alpha_1\cdots\alpha_n} - \widetilde{\phi}^{\alpha_1\cdots\alpha_n}\right) = \begin{cases} \sum_{i=0}^{\left[\frac{\bar{r}-1}{2}\right]}\overline{\phi}_i(\lambda)g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{n+2i}\alpha_{n+2i+1})}\cdot \\ \mu_{\alpha_{n+2}}\cdots\mu_{\alpha_{n+2i+1}} \qquad \text{if n+1 is even,} \\ \\ \sum_{i=0}^{\left[\frac{\bar{r}-2}{2}\right]}\overline{\phi}_i(\lambda)g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{n+2i+1}\alpha_{n+2i+2})}\cdot \\ \mu_{\alpha_{n+2}}\cdots\mu_{\alpha_{n+2i+2}} \qquad \text{if n+1 is odd.} \end{cases}$$

This relation can be integrated and gives

$$\phi^{\alpha_1\cdots\alpha_n} - \widetilde{\phi}^{\alpha_1\cdots\alpha_n} = \begin{cases} \sum_{i=0}^{\left[\frac{\bar{r}-1}{2}\right]}\overline{\phi}_i(\lambda)\frac{1}{2i+1}g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{n+2i}\alpha_{n+2i+1})}\mu_{\alpha_{n+1}}\cdots\mu_{\alpha_{n+2i+1}} \\ \qquad\qquad\qquad \text{if n is odd,} \\ \sum_{i=0}^{\left[\frac{\bar{r}-2}{2}\right]}\overline{\phi}_i(\lambda)\frac{1}{2i+2}g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{n+2i+1}\alpha_{n+2i+2})}\mu_{\alpha_{n+1}}\cdots\mu_{\alpha_{n+2i+2}} \\ +\phi_0(\lambda)g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{n-1}\alpha_n)} \quad \text{if n is even.} \end{cases}$$

So we have obtained the second part of this proposition, with $\bar{r}+1$ instead of r, and

$$\phi_i(\lambda) = \begin{cases} \overline{\phi}_i(\lambda)\frac{1}{2i+1} & \text{if n is odd,} \\ \overline{\phi}_{i-1}(\lambda)\frac{1}{2i} & \text{if n is even (For this case we have put i=I-1).} \end{cases}$$

This completes the proof of Proposition 4. ■

Proposition 5 *If $\phi^{\alpha_1\cdots\alpha_n}\in\mathcal{F}$, then $\phi^{\alpha_1\cdots\alpha_{n+2}}\in\mathcal{F}$ exists such that*

$$\phi^{\alpha_1\cdots\alpha_{n+2}}g_{\alpha_{n+1}\alpha_{n+2}} = \phi^{\alpha_1\cdots\alpha_n}. \tag{5.44}$$

Moreover, its leading term is

$$\phi^{n+2}_{\left[\frac{n+2}{2}\right]} = \gamma^{-2\left(n+3-\left[\frac{n+2}{2}\right]\right)}\frac{(n+1)(n+2)}{2\left[\frac{n+2}{2}\right]}\left[\int\phi^n_{\left[\frac{n}{2}\right]}\gamma^{2\left(n+3-\left[\frac{n+2}{2}\right]\right)-1}d\gamma + f^{n+2}_{\left[\frac{n+2}{2}\right]}(\lambda)\right]. \tag{5.45}$$

with $\phi^n_{\left[\frac{n}{2}\right]}$ leading term of $\phi^{\alpha_1\cdots\alpha_n}$ and $f^{n+2}_{\left[\frac{n+2}{2}\right]}$ an arbitrary function.

Proof. Because of $\phi^{\alpha_1\cdots\alpha_{n+2}} \in \mathcal{F}$, it has the form (5.36) with coefficients satisfying eq. (5.37) and n+2 instead of n, i.e.,

$$\phi^{\alpha_1\cdots\alpha_{n+2}}(\lambda,\mu_\beta) = \sum_{s=0}^{\left[\frac{n+2}{2}\right]} \phi_s^{n+2}(\lambda,\gamma) g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2s-1}\alpha_{2s}}\mu^{\alpha_{2s+1}}\cdots\mu^{\alpha_{n+2})}$$

with

$$\frac{\partial\phi_s^{n+2}}{\partial\gamma}\frac{1}{\gamma}2s+(n-2s+4)(n-2s+3)\phi_{s-1}^{n+2} = 0 \quad \text{for } s=1,\cdots\left[\frac{n+2}{2}\right]. \quad (5.46)$$

Let's substitute this expression of $\phi^{\alpha_1\cdots\alpha_{n+2}}$ in the left hand side of eq. (5.44) and explicit the symmetrization, so obtaining

$$\begin{aligned}
& \phi^{\alpha_1\cdots\alpha_{n+2}} g_{\alpha_{n+1}\alpha_{n+2}} = \\
= & \sum_{s=0}^{\left[\frac{n+2}{2}\right]} \phi_s^{n+2}(\lambda,\gamma) g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2s-1}\alpha_{2s}}\mu^{\alpha_{2s+1}}\cdots\mu^{\alpha_{n+2})} g_{\alpha_{n+1}\alpha_{n+2}} = \\
= & \sum_{s=1}^{\left[\frac{n+2}{2}\right]} \phi_s^{n+2}(\lambda,\gamma)\frac{2s(2s+2)}{(n+2)(n+1)} g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2s-3}\alpha_{2s-2}}\mu^{\alpha_{2s-1}}\cdots\mu^{\alpha_n)} + \\
+ & \sum_{s=1}^{\left[\frac{n+2}{2}\right]} \phi_s^{n+2}(\lambda,\gamma)2\frac{2s(n-2s+2)}{(n+2)(n+1)} g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2s-3}\alpha_{2s-2}}\mu^{\alpha_{2s-1}}\cdots\mu^{\alpha_n)} + \\
+ & \sum_{s=0}^{\left[\frac{n}{2}\right]} \phi_s^{n+2}(\lambda,\gamma)\frac{(n+2-2s)(n+2-2s-1)}{(n+2)(n+1)} \cdot \\
& g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2s-1}\alpha_{2s}}\mu^{\alpha_{2s+1}}\cdots\mu^{\alpha_n)}(-\gamma^2).
\end{aligned}$$

Blending the first two sums and putting S=s-1 we obtain

$$\begin{aligned}
& \phi^{\alpha_1\cdots\alpha_{n+2}} g_{\alpha_{n+1}\alpha_{n+2}} = \sum_{s=0}^{\left[\frac{n}{2}\right]} \left\{ \phi_{s+1}^{n+2} 4\frac{(s+1)(n-s+2)}{(n+2)(n+1)} + \right. \\
& \left. \phi_s^{n+2}(-\gamma^2)\frac{(n+2-2s)(n+1-2s)}{(n+2)(n+1)} \right\} g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2s-1}\alpha_{2s}}\mu^{\alpha_{2s+1}}\cdots\mu^{\alpha_n)}.
\end{aligned}$$

With the use of this expression and of eq. (5.36) for $\phi^{\alpha_1\cdots\alpha_n}$, eq. (5.44) becomes

$$\sum_{s=0}^{\left[\frac{n}{2}\right]} \left\{ \phi_{s+1}^{n+2} 4 \frac{(s+1)(n-s+2)}{(n+2)(n+1)} + \phi_s^{n+2}(-\gamma^2) \frac{(n+2-2s)(n+1-2s)}{(n+2)(n+1)} \right\} \cdot$$

$$g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2s-1}\alpha_{2s}}\mu^{\alpha_{2s+1}}\cdots\mu^{\alpha_n)} = \sum_{s=0}^{\left[\frac{n}{2}\right]} \phi_s^n g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2s-1}\alpha_{2s}}\mu^{\alpha_{2s+1}}\cdots\mu^{\alpha_n)}.$$

i.e.

$$\begin{aligned} \phi_s^n &= \frac{1}{(n+2)(n+1)} \Big[\phi_{s+1}^{n+2} 4(s+1)(n-s+2) + \\ &+ \phi_s^{n+2}(-\gamma^2)(n+2-2s)(n+1-2s) \Big], \end{aligned} \tag{5.47}$$

for $s = 0, \cdots, \left[\frac{n}{2}\right]$.
If we use eqs. (5.37) and (5.46), this expression becomes

$$\begin{aligned} &\frac{\partial \phi_{s+1}^n}{\partial \gamma} = \frac{1}{(n+2)(n+1)} \Big[\frac{\partial \phi_{s+2}^{n+2}}{\partial \gamma} 4(s+2)(n-s+2) + \\ &+ \frac{\partial \phi_{s+1}^{n+2}}{\partial \gamma}(-\gamma^2)(n-2s)(n-1-2s) \Big] \end{aligned} \tag{5.48}$$

that is eq. (5.47) with s+1 instead of s, derivated with respect to γ and with another use of eq. (5.46). So we have proved that if eq. (5.47) holds true for a particular value of s, it will hold true also for all lower values of s; so it suffices to impose eq. (5.47) for the bigger value of s, i.e.,

$$\phi_{\left[\frac{n}{2}\right]}^n = \frac{2\left[\frac{n}{2}\right]+2}{(n+2)(n+1)} \left[2\phi_{\left[\frac{n}{2}\right]}^{n+2} \left(n - \left[\frac{n}{2}\right] + 2 \right) + \gamma \frac{\partial}{\partial \gamma} \phi_{\left[\frac{n+2}{2}\right]}^{n+2} \right], \tag{5.49}$$

where eq. (5.46) has been used. The general solution of this equation is reported in eq. (5.45). ∎
It will be useful also the following

Proposition 6 *If the leading term of* $\phi^{\alpha_1\cdots\alpha_n}$ *is* $\phi_{\left[\frac{n}{2}\right]}^n = f(\lambda)\gamma^{-2(3+p)}$ *with* p *a non negative integer, then the leading term of* $\phi^{\alpha_1\cdots\alpha_n} g_{\alpha_{n-2r+1}\alpha_{n-2r+2}}\cdots g_{\alpha_{n-1}\alpha_n}$ *is*

$$\begin{aligned} \phi_{\left[\frac{n-2r}{2}\right]}^{n-2r} &= \frac{\left(2\left[\frac{n+1}{2}\right]-2r-1\right)!!}{\left(2\left[\frac{n+1}{2}\right]-1\right)!!} f(\lambda)\gamma^{-2(3+p)} \cdot \\ &\quad \eta\left(2\left[\frac{n+1}{2}\right]-2r-2-2p, 2\left[\frac{n+1}{2}\right]-4-2p\right) \end{aligned} \tag{5.50}$$

where, if $a \leq b$, $\eta(a,b)$ denotes the product of all even numbers between a and b, while if $a = b + r$ then $\eta(a,b) = 1$.

Proof. Let us prove eq. (5.50) with the iterative procedure.
In the case r=0 it is an identity.
Let us assume that eq. (5.50) holds up to the integer r and let us prove it with r+1 instead of r.
Let us distinguish the cases with n odd and with n even; for the first one, by applying eq. (5.49) with n-2r-2 instead of n, we obtain

$$\phi^{n-2r-2}_{\frac{n-2r-3}{2}} = \frac{1}{n-2r}\left[(n-2r+3)\phi^{n-2r}_{\frac{n-2r-1}{2}} + \gamma\frac{\partial}{\partial\gamma}\phi^{n-2r}_{\frac{n-2r-1}{2}}\right] = \frac{(n-2r-2)!!}{n!!}\cdot$$
$$f(\lambda)\eta(n-1-2p-2r, n-3-2p)\gamma^{-2(3+p)}(n-2r+3-6-2p)$$

that is eq. (5.50) with r+1 instead of r.
In the case with n even, by applying eq. (5.49) with n-2r-2 instead of n, we obtain

$$\phi^{n-2r-2}_{\frac{n-2r-2}{2}} = \frac{1}{n-2r-1}\left[(n-2r+2)\phi^{n-2r}_{\frac{n-2r}{2}} + \gamma\frac{\partial}{\partial\gamma}\phi^{n-2r}_{\frac{n-2r}{2}}\right] =$$
$$\frac{(n-2r-3)!!}{(n-1)!!}f(\lambda)\eta(n-2-2p-2r, n-4-2p)\gamma^{-2(3+p)}(n-2r-4-2p)$$

that is eq. (5.50) with r+1 instead of r.
■

Proposition 7 *If $\phi^{\alpha_1\cdots\alpha_m} \in \mathcal{F}$ with leading term $f(\lambda)\gamma^{-2(3+p)}$, with p a non negative integer such that $p < m - \left[\frac{m}{2}\right] - 1$ or $p > m - \left[\frac{m}{2}\right] + r - 2$, then $\phi^{\alpha_1\cdots\alpha_{m+2r}} \in \mathcal{F}$ exists such that $\phi^{\alpha_1\cdots\alpha_m\alpha_{m+1}\alpha_{m+2}\cdots\alpha_{m+2r-1}\alpha_{m+2r}}\, g_{\alpha_{m+1}\alpha_{m+2}} \cdots g_{\alpha_{m+2r-1}\alpha_{m+2r}} = \phi^{\alpha_1\cdots\alpha_m}$.*
Moreover the leading term of $\phi^{\alpha_1\cdots\alpha_{m+2r}}$ is

$$\frac{(m+2r)!}{m!}\frac{\left(2\left[\frac{m}{2}\right]\right)!!}{\left(2\left[\frac{m}{2}\right]+2r\right)!!}\frac{\left(2m-2\left[\frac{m}{2}\right]-2p-4\right)!!}{\left(2m-2\left[\frac{m}{2}\right]+2r-2p-4\right)!!}f(\lambda)\gamma^{-2(3+p)} +$$
$$+\sum_{i=0}^{r-1} f_{i,r}(\lambda)\gamma^{-2\left(3+m+i-\left[\frac{m+2}{2}\right]\right)}, \qquad (5.51)$$

with $f_{i,r}(\lambda)$ arbitrary functions.

Proof. Let us prove this proposition with the iterative procedure.
It is easy to verify that eq. (5.51) holds for r=0.
Let us assume that it holds up to the integer r and prove that it holds also

with r+1 instead of r.
Then we have to face the problem

$$\phi^{\alpha_1\cdots\alpha_m\alpha_{m+1}\alpha_{m+2}\cdots\alpha_{m+2r+1}\alpha_{m+2r+2}}g_{\alpha_{m+1}\alpha_{m+2}}\cdots g_{\alpha_{m+2r+1}\alpha_{m+2r+2}}=\phi^{\alpha_1\cdots\alpha_m} \quad (5.52)$$

By defining

$$\phi^{\alpha_1\cdots\alpha_{m+2r}}=\phi^{\alpha_1\cdots\alpha_{m+2r}\alpha_{m+2r+1}\alpha_{m+2r+2}}g_{\alpha_{m+2r+1}\alpha_{m+2r+2}}, \quad (5.53)$$

the problem (5.52) becomes that of the present proposition, which we have assumed holding. Therefore, the leading term of $\phi^{\alpha_1\cdots\alpha_{m+2r}}$ is (5.51) and it remains to face only the problem (5.53); by applying Proposition 5 with n=m+2r we find that the leading term of $\phi^{\alpha_1\cdots\alpha_{m+2r+2}}$ is

$$\gamma^{-2\left(3+m+2r-\left[\frac{m+2r+2}{2}\right]\right)}\frac{(m+2r+1)(m+2r+2)}{2\left[\frac{m+2r+2}{2}\right]}\left[\int\frac{(m+2r)!}{m!}\frac{\left(2\left[\frac{m}{2}\right]\right)!!}{\left(2\left[\frac{m}{2}\right]+2r\right)!!}\right.$$
$$\frac{\left(2m-2\left[\frac{m}{2}\right]-2p-4\right)!!}{\left(2m-2\left[\frac{m}{2}\right]+2r-2p-4\right)!!}f(\lambda)\gamma^{-2p+2m+4r-2\left[\frac{m+2r+2}{2}\right]-1}+$$
$$\left.+\sum_{i=0}^{r-1}f_{i,r}(\lambda)\gamma^{-2i+2r-1}d\gamma+f^{m+2r+2}_{\left[\frac{m+2r+2}{2}\right]}(\lambda)\right]=\gamma^{-6}\frac{(m+2r+1)(m+2r+2)}{2\left[\frac{m+2r+2}{2}\right]}\cdot$$
$$\frac{(m+2r)!}{m!}\frac{\left(2\left[\frac{m}{2}\right]\right)!!}{\left(2\left[\frac{m}{2}\right]+2r\right)!!}\frac{\left(2m-2\left[\frac{m}{2}\right]-2p-4\right)!!}{\left(2m-2\left[\frac{m}{2}\right]+2r-2p-4\right)!!}f(\lambda)\cdot$$
$$\frac{1}{-2p+2m+4r-2\left[\frac{m+2r+2}{2}\right]}\gamma^{-2p}+\sum_{i=0}^{r-1}f_{i,r}(\lambda)\frac{(m+2r+1)(m+2r+2)}{2\left[\frac{m+2r+2}{2}\right]}\cdot\frac{1}{2r-2i}$$
$$\gamma^{-2i-2m-6+2\left[\frac{m+2}{2}\right]}+f^{m+2r+2}_{\left[\frac{m+2r+2}{2}\right]}(\lambda)\frac{(m+2r+1)(m+2r+2)}{2\left[\frac{m+2r+2}{2}\right]}\gamma^{-2\left(m+2r+3-\left[\frac{m+2r+2}{2}\right]\right)}$$

that is eq. (5.51) with r+1 instead of r and

$$f_{i,r+1}=\begin{cases} f_{i,r}\frac{(m+2r+1)(m+2r+2)}{2\left[\frac{m+2r+2}{2}\right]}\frac{1}{2r-2i} & \text{for i=0, ... ,r-1,}\\ f^{m+2r+2}_{\left[\frac{m+2r+2}{2}\right]}(\lambda)\frac{(m+2r+1)(m+2r+2)}{2\left[\frac{m+2r+2}{2}\right]} & \text{for i=r.}\end{cases}$$

This completes the proof. ■
We conclude this section with the

Proposition 8 *If n is an even number, the tensor* $g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{n-1}\alpha_n)}\mu_{\alpha_{n-r+1}}\cdots\mu_{\alpha_n}$ *belongs to* $\mathcal{F}$ *and, moreover,*

$$g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{n-1}\alpha_n)}\mu_{\alpha_{n-r+1}}\cdots\mu_{\alpha_n}=\sum_{s=0}^{\left[\frac{n-r}{2}\right]}\phi^{n-r}_{s,r}g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2s-1}\alpha_{2s}}\mu^{\alpha_{2s+1}}\cdots\mu^{\alpha_{n-r})}$$

(5.54)

with

$$\phi_{s,r}^{n-r} = \begin{cases} 1 & r \leq 1,\ s = \left[\frac{n-r}{2}\right], \\ 0 & r \leq 1,\ 0 \leq s \leq \left[\frac{n-r}{2}\right] - 1, \\ \frac{r!(n-r)!}{(2s+2r-n)!!(2s)!!(n-r-2s)!(n-1)!!} \left(-\gamma^2\right)^{s+r-\frac{n}{2}} & r \geq 2,\ \frac{n}{2} - r \leq s \leq \left[\frac{n-r}{2}\right], \\ 0 & r \geq 2,\ 0 \leq s \leq \frac{n}{2} - r - 1. \end{cases} \tag{5.55}$$

Proof. It is easy to note that the above tensor is an element of $\mathcal{F}$ because it and its derivative with respect to μ_β are manifestly symmetric.
Let us prove (5.55) with the iterative procedure.
It is easy to verify that it holds when r=0.
When r=1 it is a consequence of

$$g^{(\alpha_1\alpha_2} \cdots g^{\alpha_{n-1}\alpha_n)}\mu_{\alpha_n} = g^{(\alpha_1\alpha_2} \cdots g^{\alpha_{n-1})\alpha_n}\mu_{\alpha_n} = g^{(\alpha_1\alpha_2} \cdots g^{\alpha_{n-3}\alpha_{n-2}}\mu^{\alpha_{n-1})}. \tag{5.56}$$

When r=2, by using eq. (5.56) we have

$$g^{(\alpha_1\alpha_2} \cdots g^{\alpha_{n-1}\alpha_n)}\mu_{\alpha_{n-1}}\mu_{\alpha_n} =$$
$$\frac{1}{n-1}\left[g^{(\alpha_1\alpha_2} \cdots g^{\alpha_{n-3}\alpha_{n-2})}\mu^{\alpha_{n-1}} + (n-2)g^{\alpha_{n-1}(\alpha_1} \cdots g^{\alpha_{n-4}\alpha_{n-3}}\mu^{\alpha_{n-2})}\right]\mu_{\alpha_{n-1}}$$
$$= \frac{-\gamma^2}{n-1}g^{(\alpha_1\alpha_2} \cdots g^{\alpha_{n-3}\alpha_{n-2})} + \frac{n-2}{n-1}g^{\alpha_1\alpha_2} \cdots g^{\alpha_{n-5}\alpha_{n-4}}\mu^{\alpha_{n-3}}\mu_{\alpha_{n-2})}$$

from which (5.54) and (5.55) with r=2.
Let us assume now that our proposition holds up to a fixed integer r and let us prove that it holds also with r+1 instead of r. By multiplying eq. (5.54) times $\mu_{\alpha_{n-r}}$ we find

$$g^{(\alpha_1\alpha_2} \cdots g^{\alpha_{n-1}\alpha_n)}\mu_{\alpha_{n-r}} \cdots \mu_{\alpha_n} =$$
$$= \sum_{S=0}^{\left[\frac{n-r}{2}\right]-1} \phi_{S+1,r}^{n-r}\frac{2S+2}{n-r}g^{(\alpha_1\alpha_2} \cdots g^{\alpha_{2S-1}\alpha_{2S}}\mu^{\alpha_{2S+1}} \cdots \mu^{\alpha_{n-r-1})} +$$
$$+ \sum_{s=0}^{\left[\frac{n-r}{2}\right]} \phi_{s,r}^{n-r}\frac{n-r-2s}{n-r}g^{(\alpha_1\alpha_2} \cdots g^{\alpha_{2s-1}\alpha_{2s}}\mu^{\alpha_{2s+1}} \cdots \mu^{\alpha_{n-r-1})}(-\gamma^2),$$

where, in the first summation, we have put s=S+1.
It follows (5.54) with r+1 instead of r and

$$\phi^{n-r-1}_{s,r+1} = \begin{cases} \phi^{n-r}_{s+1,r}\frac{2s+2}{n-r} - \gamma^2\phi^{n-r}_{s,r}\frac{n-r-2s}{n-r} & \text{for } s=0,\cdots,\left[\frac{n-r}{2}\right]-1, \\ & \text{and } s=\left[\frac{n-r}{2}\right] \text{ when r even,} \\ -\gamma^2\phi^{n-r}_{s,r}\frac{n-r-2s}{n-r} & \text{for } s=\frac{n-r-1}{2} \text{ when r odd.} \end{cases}$$

We have taken into account that n-r-2s=0 when $s=\left[\frac{n-r}{2}\right]$ and r is even.
By using this result and eq. (5.55) we have that

• For $0 \le s \le \frac{n}{2}-r-2$, both $\phi^{n-r}_{s+1,r}$ and $\phi^{n}_{s,r}$ are zero, from which $\phi^{n-r-1}_{s,r+1}=0$.

• For $s=\frac{n}{2}-r-1$, we have $\phi^{n-r}_{s,r}=0$ and $\phi^{n-r-1}_{s,r+1}=\frac{(n-r-1)!}{(n-2r-2)!!(n-1)!!}$.

• For $s=\frac{n}{2}-r,\cdots,\left[\frac{n-r}{2}\right]-1$ and for $s=\left[\frac{n-r}{2}\right]$ in the case r even we have

$$\phi^{n-r-1}_{s,r+1} = \frac{(r+1)!\ (n-r-1)!}{(2s+2r+2-n)!!\ (2s)!!\ (n-r-1-2s)!\ (n-1)!!}\,(-\gamma^2)^{s+r+1-\frac{n}{2}}.$$

• For $s=\frac{n-r-1}{2}$ and r odd we have $\phi^{n-r-1}_{s,r+1}=\frac{(r)!\ (n-r-1)!}{(r-1)!!\ (n-r-1)!!\ (n-1)!!}\,(-\gamma^2)^{\frac{r+1}{2}}$.

In this way we have obtained (5.55) with r+1 instead of r as we desired to prove. ■

5.4 The case N=1.

When N=1 only equation (5.30) with k=0 has to be exploited.
We prove now that it amounts to giving the following expression for the leading term of $C^{\alpha A_1\cdots A_h}$

$$C^h_{\frac{M}{2}h} = \gamma^{-6}\sum_{q=0}^{\frac{Mh-2}{2}}(-m^2)^{\frac{M}{2}h}\frac{d^h c_{h,q}(\lambda)}{d\lambda^h}\left(\frac{1}{\gamma^2}\right)^q\frac{(Mh+1)!!}{(Mh-2q-2)!!}$$

with $\qquad c_{h,q}=c_{h-1,q}\quad$ for $q=0,\cdots,\dfrac{M(h-1)-2}{2}$

and $\qquad c_{h,q}\quad$ for $q=\dfrac{M(q-1)}{2},\cdots,\dfrac{Mh-2}{2}$ are arbitrary functions of λ.

(5.57)

Let us prove this with the iterative procedure.
It holds for h=0, because in its right hand side there are no terms, and its left hand side is zero for eqs. (5.24) and (5.26).
Let us apply now the Proposition 7 to eq. (5.30) with k=0, $r = \frac{M}{2}$, m=Mh+1.
We find that

$$C^{h+1}_{\frac{M}{2}(h+1)} = \gamma^{-6} \sum_{q=0}^{\frac{Mh-2}{2}} (-m^2)^{\frac{M}{2}(h+1)} \frac{d^{h+1} c_{h,q}(\lambda)}{d\lambda^{h+1}} \left(\frac{1}{\gamma^2}\right)^q \frac{(Mh+1)!!}{(Mh-2q-2)!!}$$

$$\frac{[M(h+1)+1]!}{(Mh+1)!} \frac{(Mh)!!}{[M(h+1)]!!} \frac{(Mh-2q-2)!!}{[M(h+1)-2q-2]!!} + \sum_{i=0}^{\frac{M-2}{2}} f_{i,\frac{M}{2}} \gamma^{-2(3+\frac{hM}{2}+i)}$$

which is eq. $(5.57)_1$ with h+1 instead of h ($c_{h+1,q} = c_{h,q}$ has to be used), $i = q - \frac{Mh}{2}$,

$$c_{h+1,q} = \begin{cases} c_{h,q} & \text{for } q = 0, \cdots, \left[\frac{Mh-2}{2}\right], \\ \frac{[M(h+1)-2q-2]!!}{[M(h+1)+1]!!} (-m^2)^{-\frac{M}{2}(h+1)} f^*_{q-\frac{Mh}{2},\frac{M}{2}} & \text{for } q = \frac{Mh}{2}, \cdots, \frac{M(h+1)-2}{2} \end{cases}.$$

with $f^*_{q-\frac{Mh}{2},\frac{M}{2}}$ defined by $\frac{d^{h+1}}{d\lambda^{h+1}} f^*_{q-\frac{Mh}{2},\frac{M}{2}} = f_{q-\frac{Mh}{2},\frac{M}{2}}$.
We note that from eq. $(5.57)_2$ it follows

$$c_{h,q} = c_{h-j,q} \qquad \text{for } q = 0, \cdots, \frac{M(h-j)-2}{2}. \tag{5.58}$$

Also this relation can be proved with the iterative procedure. It holds when j=0. From (5.58) and $(5.57)_2$ it follows $c_{h,q} = c_{h-j,q} = c_{h-j-1,q}$ for $q = 0, \cdots, \frac{M(h-j-1)-2}{2}$ that is eq. (5.58) with j+1 instead of j, as we desired to prove.
Let us search now the other coefficients C^h_s. To this end, it is better to prove firstly that from eq. (5.37) with n=Mh+1 it follows

$$\phi^{Mh+1}_{s-r} = (-4)^r \frac{s!}{(s-r)!} \frac{(Mh+1-2s)!}{(Mh+1-2s+2r)!} \frac{\partial^r}{\partial(\gamma^2)^r} \phi^{Mh+1}_s. \tag{5.59}$$

Let us prove this with the iterative procedure.
It holds for r=0. Let us assume that it holds up to the index r. From eq. (5.37) with n=Mh+1 and s-r instead of s, we find

$$\begin{aligned} \phi^{Mh+1}_{s-r-1} &= \frac{-2s+2r}{\gamma} \frac{1}{(Mh+3-2s+2r)(Mh+2-2s+2r)} \frac{\partial}{\partial\gamma} \phi^{Mh+1}_{s-r} = \\ &= -4(s-r) \frac{1}{(Mh+3-2s+2r)(Mh+2-2s+2r)} \frac{\partial}{\partial(\gamma^2)} \phi^{Mh+1}_{s-r} = \\ &= (-4)^{r+1} \frac{s!}{(s-r-1)!} \frac{(Mh+1-2s)!}{(Mh+3-2s+2r)!} \frac{\partial^{r+1}}{\partial(\gamma^2)^{r+1}} \phi^{Mh+1}_s, \end{aligned}$$

where in the last passage eq. (5.59) has been used. In this way we have proved that (5.59) holds when r+1 replaces r; this completes the proof of eq. (5.59) and we use now it to find C_s^h.
If we write eq. (5.59) with $s = \frac{M}{2}h$ and $r = s - s^*$, jointly to eq. (5.57), we find that

$$C_{s^*}^h = (-4)^{\frac{M}{2}h - s^*} \frac{\left(\frac{M}{2}h\right)!}{(s^*)!} \frac{1}{(Mh+1-2s^*)!} \sum_{q=0}^{\frac{Mh-2}{2}} (-m^2)^{\frac{M}{2}h} \cdot$$

$$\cdot \quad \frac{d^h c_{h,q}(\lambda)}{d\lambda^h} \frac{(Mh+1)!!}{(Mh-2-2q)!!} (-1)^{\frac{M}{2}h - s^*} \frac{(q+2+\frac{M}{2}h - s^*)!}{(q+2)!} \left(\frac{1}{\gamma^2}\right)^{q+3+\frac{M}{2}h - s^*},$$

that is, the above mentioned eq. (5.35).

5.5 The case $N > 1$

In this case we have to impose eqs. (5.30) and (5.31). But it will be firstly useful to transform them into more easy equations. To this end we may use the notation $\widetilde{B}_i = \alpha_{i_1} \cdots \alpha_{i_{N-1}}$ (similar to the already used multi-index notation $B_i = \alpha_{i_1} \cdots \alpha_{i_N}$) and Proposition 4; we obtain that the tensor $D_{h,k}^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k}$ exists, such that

$$C_{h,k}^{\alpha A_1 \cdots A_h \beta_1 \widetilde{B}_1 \cdots \beta_k \widetilde{B}_k} = \frac{\partial^k}{\partial \mu_{\beta_1} \cdots \partial \mu_{\beta_k}} D_{h,k}^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k}. \tag{5.60}$$

After that eq. (5.31) becomes

$$\begin{aligned} &\frac{\partial^{k+1}}{\partial \mu_{\beta_1} \cdots \partial \mu_{\beta_k} \partial \mu_\beta} D_{h,k+1}^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k \gamma_1 \cdots \gamma_{N-1}} g_{\gamma_1 \gamma_2} \cdots g_{\gamma_{N-2} \gamma_{N-1}} \\ &= \frac{\partial^{k+1}}{\partial \mu_\beta \partial \mu_{\beta_1} \cdots \partial \mu_{\beta_k}} D_{h,k}^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k} (-m^2)^{\frac{N-1}{2}}. \end{aligned}$$

For the Proposition 4 with n=hM+k(N-1)+1 the general solution of this equation is

$$\begin{aligned} &D_{h,k+1}^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k \gamma_1 \cdots \gamma_{N-1}} g_{\gamma_1 \gamma_2} \cdots g_{\gamma_{N-2} \gamma_{N-1}} = D_{h,k}^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k} (-m^2)^{\frac{N-1}{2}} \\ &+ \sum_{i=0}^{\left[\frac{k-1}{2}\right]} \phi_{i,h,k}(\lambda) g^{(\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k \alpha_{hM+k(N-1)+2} \cdots \alpha_{hM+k(N-1)+2i})} \cdot \\ &\mu_{\alpha_{hM+k(N-1)+2}} \cdots \mu_{\alpha_{hM+k(N-1)+2+2i}}. \end{aligned} \tag{5.61}$$

Let us transform now the unknown tensors $D^{\cdots}_{h,k}$ into the tensors $E^{\cdots}_{h,k}$, according to the following rule

$$D^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k}_{h,k} = E^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k}_{h,k} +$$
$$+ \sum_{i=0}^{\left[\frac{k-2}{2}\right]} \psi_{i,h,k}(\lambda) g^{(\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k \alpha_{hM+k(N-1)+2} \cdots \alpha_{hM+k(N-1)+2i})} \cdot$$
$$\mu_{\alpha_{hM+k(N-1)+2}} \cdots \mu_{\alpha_{hM+k(N-1)+2+2i}}. \tag{5.62}$$

with $\psi_{i,h,k}(\lambda)$ defined with the iterative method by

$$\begin{aligned}
\psi_{i,h,0} &= 0,\\
\psi_{i,h,1} &= 0,\\
\psi_{i,h,k+1} &= \frac{[Mh + k(N-1) + 2i + 4]!!}{[hM + (k+1)(N-1) + 2i + 4]!!} \\
&\quad \frac{[hM + (k+1)(N-1) + 2i + 1]!!}{[hM + k(N-1) + 2i + 1]!!} \left(\psi_{i,h,k}(-m^2)^{\frac{N-1}{2}} + \phi_{i,h,k}\right) \\
&\quad \text{in the case k even and } i = 0, \cdots, \frac{k-2}{2} \\
&\quad \text{and in the case k odd and } i = 0, \cdots, \frac{k-3}{2} \\
\psi_{i,h,k+1} &= \frac{[Mh + k(N-1) + 2i + 4]!!}{[hM + (k+1)(N-1) + 2i + 4]!!} \\
&\quad \frac{[hM + (k+1)(N-1) + 2i + 1]!!}{[hM + k(N-1) + 2i + 1]!!} \phi_{i,h,k} \\
&\quad \text{in the case k odd and } i = \frac{k-1}{2}.
\end{aligned}$$

We note that also $E^{\cdots}_{h,k} \in \mathcal{F}$.
Thanks to this transformation, eq. (5.61) becomes

$$E^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k \gamma_1 \cdots \gamma_{N-1}}_{h,k+1} g_{\gamma_1 \gamma_2} \cdots g_{\gamma_{N-2} \gamma_{N-1}} = E^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k}_{h,k} (-m^2)^{\frac{N-1}{2}}. \tag{5.63}$$

Let us note how many 4-vectors μ_α intervene in the second term in the right hand side of (5.62); they are $2i + 1 \leq 2\left[\frac{k-2}{2}\right] + 1 \leq k - 2 + 1 < k$; therefore eq. (5.61), substituted in (5.60) transforms it into

$$C^{\alpha A_1 \cdots A_h \beta_1 \widetilde{B}_1 \cdots \beta_k \widetilde{B}_k}_{h,k} = \frac{\partial^k}{\partial \mu_{\beta_1} \cdots \partial \mu_{\beta_k}} E^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k}_{h,k}. \tag{5.64}$$

Eqs. (5.63) and (5.64) substitute now eqs. (5.60) and (5.61); consequently, these last one can now be left out!
Eq. (5.64) with k=0 now yields

$$C_{h,0}^{\alpha A_1 \cdots A_h} = E_{h,0}^{\alpha A_1 \cdots A_h} \tag{5.65}$$

which is restricted by eq. (5.30) with k=0. After that eq. (5.63) will give, with an iterative procedure, $E_{h,k+1}^{\cdots}$. Let us firstly deduce, from this procedure, some properties.

Property 1 *The leading term of* $E_{h,0}^{\alpha A_1 \cdots A_h}$ *is* $\left(\frac{1}{\gamma}\right)^6$ *multiplied by a polynomial in the variable* $\frac{1}{\gamma^2}$.

Proof. This property is evident from eq. (5.57). ■

Property 2 *The leading term of* $E_{h,k}^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k}$ *is* $\left(\frac{1}{\gamma}\right)^6$ *multiplied by a polynomial in the variable* $\frac{1}{\gamma^2}$.

Proof. Let us prove this with the iterative procedure.
It holds when k=0, for property 1.
Let us assume it up to the index k. From eq. (5.63), by applying Proposition 7 with m=Mh+(N-1)k+1, $r = \frac{N-1}{2}$, we find that property 2 holds also when k+1 replaces k.
This completes its proof. ■
By substituting now eq. (5.64) into eq. (5.30), this last one becomes

$$0 = \frac{\partial^k}{\partial \mu_{\beta_1} \cdots \partial \mu_{\beta_k}} \left[E_{h+1,k}^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k \gamma_1 \gamma_2 \cdots \gamma_{M-1} \gamma_M} g_{\gamma_1 \gamma_2} \cdots g_{\gamma_{M-1} \gamma_M} - (-m^2)^{\frac{M}{2}} \frac{\partial}{\partial \lambda} E_{h,k}^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k} \right]$$

For Proposition 4 with n=Mh+(N-1)k+1, r=k, it follows that

$$E_{h+1,k}^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k \gamma_1 \gamma_2 \cdots \gamma_{M-1} \gamma_M} g_{\gamma_1 \gamma_2} \cdots g_{\gamma_{M-1} \gamma_M} - (-m^2)^{\frac{M}{2}} \frac{\partial}{\partial \lambda} E_{h,k}^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k} = \\ = \sum_{i=0}^{\left[\frac{k-2}{2}\right]} \phi_i(\lambda) g^{(\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k \alpha_1 \cdots \alpha_{2i+1})} \mu_{\alpha_1} \cdots \mu_{\alpha_{2i+1}}. \tag{5.66}$$

where the notation $g^{\alpha_1 \cdots \alpha_{2n}} = g^{(\alpha_1 \alpha_2} \cdots g^{\alpha_{2n-1} \alpha_{2n})}$ has been used.
But, for the above property 2, jointly with the Propositions 6 and 8, we have that the left hand side of eq. (5.66) has a leading term of the type $\left(\frac{1}{\gamma}\right)^6$

multiplied by a polynomial in the variable $\frac{1}{\gamma^2}$, while its right hand side has a leading term polynomial in γ^2. It follows necessarily that both these leading terms are zero; in other words, both sides of eq. (5.66) are zero. In particular, from the left hand side we obtain

$$E_{h+1,k}^{\alpha A_1\cdots A_h\widetilde{B}_1\cdots\widetilde{B}_k\gamma_1\gamma_2\cdots\gamma_{M-1}\gamma_M} g_{\gamma_1\gamma_2}\cdots g_{\gamma_{M-1}\gamma_M} = (-m^2)^{\frac{M}{2}}\frac{\partial}{\partial\lambda}E_{h,k}^{\alpha A_1\cdots A_h\widetilde{B}_1\cdots\widetilde{B}_k}. \tag{5.67}$$

Therefore, we have to impose only eqs. (5.63) and (5.67); after that, the tensor $C_{h,k}^{\alpha A_1\cdots A_h B_1\cdots B_k}$ defined by eq. (5.64) is the more general solution of eq. (5.30) and (5.31).

5.5.1 A consequence of eqs. (5.63) and (5.67)

We will see now that, as a consequence of eqs. (5.63) and (5.67), we may find the leading term of $E_{h,k}^{\alpha A_1\cdots A_h\widetilde{B}_1\cdots\widetilde{B}_k}$ except for a set of arbitrary functions of the single variable λ. This result is the subsequent (5.71).
To this end let us firstly prove that

$$E^{0,k}_{\frac{N-1}{2}k} = \gamma^{-6}\sum_{q=0}^{\frac{(N-1)k-2}{2}}(-m^2)^{\frac{N-1}{2}k}c_{k,q}(\lambda)\frac{[(N-1)k+1]!!}{[(N-1)k-2q]!!}[(N-1)k-2q]\left(\frac{1}{\gamma^2}\right)^q, \tag{5.68}$$

where the last N-1 functions $c_{k,q}$ are arbitrary functions of λ and the remainder are

$$c_{k,q} = c_{k-1,q}. \tag{5.69}$$

Let us prove it with the iterative procedure.
It holds when k=0, because in this case the right hand side has no terms, while the left hand side is zero for eqs. (5.65) with h=0, (5.24) and (5.26).
Let us apply Proposition 7 to eq. (5.63) with h=0, $r=\frac{N-1}{2}$, m=k(N-1)+1. We find that

$$E^{0,k+1}_{\frac{N-1}{2}(k+1)} = \gamma^{-6}\sum_{q=0}^{\frac{(N-1)k-2}{2}}c_{k,q}\left(\frac{1}{\gamma^2}\right)^q\frac{[(N-1)k+1]!!}{[(N-1)k-2q]!!}[(N-1)k-2q]\cdot$$
$$\cdot(-m^2)^{\frac{N-1}{2}(k+1)}\frac{[(N-1)k+1+N-1]!}{[(N-1)k+1]!}\frac{[(N-1)k]!!}{[(N-1)k+N-1]!!}$$
$$\frac{[(N-1)k-2q-2]!!}{[(N-1)(k+1)-2q-2]!!}+\sum_{i=0}^{\frac{N-3}{2}}f_{i,\frac{N-1}{2}}(\lambda)\gamma^{-[6+k(N-1)+2i]}.$$

Then we have found eq. (5.68) with k+1 instead of k, $i = q - k\frac{N-1}{2}$ and

$$c_{k+1,q} = \begin{cases} c_{k,q} & \text{for } q = 0, \cdots, \frac{(N-1)k-2}{2}, \\ f_{q-k\frac{N-1}{2},\frac{N-1}{2}} & \frac{[(N-1)(k+1)-2q-2]!!}{[(N-1)(k+1)+1]!}(-m^2)^{-\frac{N-1}{2}(k+1)} \\ & \text{for } q = \frac{(N-1)k}{2}, \cdots, \frac{(N-1)(k+1)-2}{2}. \end{cases}$$

This completes the proof.
It will be useful in the sequel also the

Property 3 $c_{k,q} = c_{k-i,q}$ *for* $q = 0, \cdots \frac{(N-1)(k-i)-2}{2}$.

Proof. Let us prove this with the iterative procedure.
It is obvious when i=0.
From eq. (5.69) with k-i instead of k, we have

$$c_{k,q} = c_{k-i,q} = c_{k-i-1,q}$$

for $q = 0, \cdots \frac{(N-1)(k-i-1)-2}{2}$. So the property is valid also when i+1 replaces i. ■

Now, from eq. (5.63) it follows

$$E_{h,k}^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k} = E_{h,k+2}^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k \widetilde{B}_{k+1} \widetilde{B}_{k+2}} g_{\widetilde{B}_{k+1}} g_{\widetilde{B}_{k+2}} (-m^2)^{-(N-1)},$$

and so on, until

$$E_{h,k}^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k} = E_{h,k+Mh}^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k \widetilde{B}_{k+1} \cdots \widetilde{B}_{k+Mh}} g_{\widetilde{B}_{k+1}} \cdots g_{\widetilde{B}_{k+Mh}} (-m^2)^{-\frac{(N-1)Mh}{2}}.$$

By applying h times eq. (5.67), the above equation becomes

$$E_{h,k}^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k} =$$
$$\frac{\partial^h}{\partial \lambda^h} E_{0,k+Mh}^{\alpha A_1 \cdots A_h \widetilde{B}_1 \cdots \widetilde{B}_k \gamma_1 \cdots \gamma_{Mh(N-2)}} g_{\gamma_1 \gamma_2} \cdots g_{\gamma_{Mh(N-2)-1} \gamma_{Mh(N-2)}} (-m^2)^{-\frac{(N-2)Mh}{2}}. \tag{5.70}$$

In this way all the $E_{h,k}^{\cdots}$ are determined in terms of $E_{0,k}^{\cdots}$.
From eqs. (5.70), (5.68) and Proposition 6 with n=Mh+(N-1)k+1+Mh(N-2) and $r = \frac{M}{2}h(N-2)$ we find that

$$E_{\frac{Mh+(N-1)k}{2}}^{h,k} = \gamma^{-6} \sum_{q=0}^{\frac{(N-1)(k+Mh)-2}{2}} (-m^2)^{\frac{N-1}{2}k + \frac{Mh}{2}} \frac{d^h c_{k+Mh,q}}{d\lambda^h} \cdot \left(\frac{1}{\gamma^2}\right)^q$$
$$\frac{[(N-1)(k+Mh)+1]!!}{[(N-1)(k+Mh)-2q-2]!!} \frac{[Mh+(N-1)k+1]!!}{[(Mh+k)(N-1)+1]!!} \cdot$$
$$\cdot \eta[k(N-1)+Mh-2q, (Mh+k)(N-1)-2-2q].$$

This expression can be simplified because, for $q \geq \frac{Mh+k(N-1)}{2}$ we have $Mh + (N-1)k - 2q \leq 0$, $(Mh+k)(N-1) - 2 - 2q \geq 0$, so that at least one, among the factors intervening in $\eta(\cdots,\cdots)$, is zero. Then we can restrict to the values with $q \leq \frac{Mh+k(N-1)-2}{2}$ and, consequently,

$$E^{h,k}_{\frac{Mh+(N-1)k}{2}} = \gamma^{-6} \sum_{q=0}^{\frac{(N-1)k+Mh-2}{2}} (-m^2)^{\frac{N-1}{2}k+\frac{Mh}{2}} \frac{d^h c_{k+Mh,q}}{d\lambda^h} \left(\frac{1}{\gamma^2}\right)^q \frac{[Mh+(N-1)k+1]!!}{[(N-1)k+Mh-2q-2]!!}. \tag{5.71}$$

This agrees with the above used property 2.

5.5.2 Equivalence of eqs. (5.63) and (5.67) with eq. (5.71)

The result (5.71) has been proved as a consequence of eqs. (5.63) and (5.67). We prove now the vice versa, i.e., that eqs. (5.63) and (5.67) become identities when eq. (5.71) is used.
Let us begin with eq. (5.63): For the Proposition 6 with n=hM+(k+1)(N-1)+1 and $r = \frac{N-1}{2}$, and (5.71), the leading term of the left hand side of eq. (5.63) is

$$\gamma^{-6} \sum_{q=0}^{\frac{(N-1)(k+1)+Mh-2}{2}} \frac{[(N-1)(k+1)+Mh+1]!!}{[(N-1)(k+1)+Mh-2q-2]!!} (-m^2)^{\frac{N-1}{2}(k+1)+\frac{Mh}{2}} \frac{d^h c_{k+1+Mh,q}}{d\lambda^h} \left(\frac{1}{\gamma^2}\right)^q \frac{[(N-1)k+Mh+1]!!}{[(N-1)(k+1)+Mh+1]!!} \eta(Mh+k(N-1)-2q, Mh+(k+1)(N-1)-2-2q).$$

If $q \geq \frac{Mh+k(N-1)}{2}$ we have $hM + k(N-1) - 2q \leq 0$, $hM + (k+1)(N-1) - 2 - 2q \geq 0$, so that $\eta(\cdots,\cdots) = 0$; therefore we can limit to values with $a \leq \frac{Mh+k(N-1)}{2} - 1$ and the above leading term becomes equal to the right hand side of eq. (5.71) pre-multiplied by $(-m^2)\frac{N-1}{2}$ (use the property (5.69) with k+1+Mh instead of k, i.e., $c_{k+1+Mh,q} = c_{k+Mh,q}$ for $q = 0, \cdots, \frac{Mh+k(N-1)-2}{2} \leq (N-1)(k+1+Mh)-2$).
Therefore, eq. (5.63) is an identity.
Let us now prove the same thing for (5.67): For the Proposition 6 with n=M(h+1)+k(N-1)+1 and $r = \frac{M}{2}$, and (5.71), the leading term of the left

hand side of eq. (5.67) is

$$\gamma^{-6}\sum_{q=0}^{\frac{(N-1)k+M(h+1)-2}{2}}\frac{[(N-1)k+M(h+1)+1]!!}{[(N-1)k+M(h+1)-2q-2]!!}(-m^2)^{\frac{N-1}{2}k+\frac{M(h+1)}{2}}$$
$$\frac{d^{h+1}c_{k+Mh+M,q}}{d\lambda^{h+1}}\left(\frac{1}{\gamma^2}\right)^q\frac{[(N-1)k+Mh+1]!!}{[(N-1)k+M(h+1)+1]!!}$$
$$\eta(Mh+k(N-1)-2q, M(h+1)+k(N-1)-2-2q).$$

When $q\geq\frac{Mh+k(N-1)}{2}$ we have $\eta(\cdots,\cdots)=0$ so that there remain terms with $q\leq\frac{Mh+k(N-1)-2}{2}$, which is the right hand side of eq. (5.71) multiplied by $(-m^2)^{\frac{M}{2}}$ and derived with respect to λ (use $c_{k+Mh+M,q}=c_{k+Mh,q}$ for $q=0,\cdots,\frac{Mh+k(N-1)-2}{2}$ which holds for property 3 written with i=M and k+Mh+M instead of k because $Mh+k(N-1)-2\leq(N-1)(k+Mh)-2)$. Therefore, eq. (5.67) is an identity.

5.5.3 Determination of the tensor $C_{h,k}^{\alpha A_1\cdots A_h B_1\cdots B_k}$

After having imposed eqs. (5.63) and (5.67), let us now impose eq. (5.64) for the determination of the tensor $C_{h,k}^{\alpha A_1\cdots A_h B_1\cdots B_k}$. To this end, let us firstly note that from eq. $(5.40)_2$ with n odd and $s=\frac{n+1}{2}$ we find that the leading term of $\frac{\partial\phi^{\alpha_1\cdots\alpha_n}}{\partial\mu_\beta}$ is

$$\phi^{n+1}_{\frac{n+1}{2}}=\phi^{n}_{\frac{n-1}{2}}. \tag{5.72}$$

From eq. $(5.40)_2$ with n+1 instead of n and $s=\frac{n+1}{2}$ we find that the leading term of $\frac{\partial^2\phi^{\alpha_1\cdots\alpha_n}}{\partial\mu_{\beta_1}\partial\mu_{\beta_2}}$ is

$$\phi^{n+2}_{\frac{n+1}{2}}=2\frac{n+2}{n+1}\phi^{n+1}_{\frac{n-1}{2}}=-\frac{n+2}{\gamma}\frac{\partial}{\partial\gamma}\phi^{n}_{\frac{n-1}{2}}$$

where in the last passage eq. (5.72) and eq. (5.37) with n+1 instead of n and $s=\frac{n+1}{2}$ have been used.
If $\phi^{n}_{\frac{n-1}{2}}$ depends on γ by means of γ^2, it follows that $\phi^{n+2}_{\frac{n+1}{2}}$ also depends on γ by means of γ^2 and is

$$\phi^{n+2}_{\frac{n+1}{2}}=-2(n+2)\frac{\partial}{\partial\gamma^2}\phi^{n}_{\frac{n-1}{2}} \tag{5.73}$$

It follows that

$$\phi^{n+2r}_{\frac{n+2r-1}{2}}=(-2)^r\frac{(n+2r)!!}{n!}\frac{\partial^r}{\partial(\gamma^2)^r}\phi^{n}_{\frac{n-1}{2}}. \tag{5.74}$$

(In fact, it holds for r=0. Let us assume that it holds up to an integer r. Eq. (5.73) with n+2r instead of n becomes

$$\phi^{n+2r+2}_{\frac{n+2r+1}{2}} = -2(n+2r+2)\frac{\partial}{\partial\gamma^2}\phi^{n+2r}_{\frac{n+2r-1}{2}} = (-2)^{r+1}\frac{(n+2r+2)!!}{n!}\frac{\partial^{r+1}}{\partial(\gamma^2)^{r+1}}\phi^n_{\frac{n-1}{2}},$$

where (5.74) has been used. The result is again (5.74) but with r+1 instead of r; this completes the proof).

From eq. (5.72), with n+2r instead of n, and for eq. (5.73), it follows

$$\phi^{n+2r+1}_{\frac{n+2r+1}{2}} = (-2)^r\frac{(n+2r)!!}{n!!}\frac{\partial^r}{\partial(\gamma^2)^r}\phi^n_{\frac{n-1}{2}}.$$

This result and eq. (5.73) give

$$\phi^{n+k}_{\left[\frac{n+k}{2}\right]} = (-2)^{\left[\frac{k}{2}\right]}\frac{\left(n+2\left[\frac{k}{2}\right]\right)!!}{n!!}\frac{\partial^{\left[\frac{k}{2}\right]}}{\partial(\gamma^2)^{\left[\frac{k}{2}\right]}}\phi^n_{\frac{n-1}{2}}.$$

This relation, with n=Mh+k(N-1)+1, and eq. (5.71) allows to obtain from eq. (5.64) that the leading term of $C^{\alpha A_1\cdots A_h B_1\cdots B_k}_{h,k}$ is

$$C^{h,k}_{\left[\frac{Mh+kN+1}{2}\right]} = 2^{\left[\frac{k}{2}\right]}\gamma^{-6-2\left[\frac{k}{2}\right]}\sum_{q=0}^{\frac{Mh+k(N-1)-2}{2}}\frac{\left(hM+k(N-1)+1+2\left[\frac{k}{2}\right]\right)!!}{[hM+k(N-1)-2q-2]!!}$$
$$(-m^2)^{\frac{N-1}{2}k+\frac{Mh}{2}}\frac{d^h c_{k+Mh,q}}{d\lambda^h}\frac{\left(q+2+\left[\frac{k}{2}\right]\right)!}{(q+2)!}\left(\frac{1}{\gamma^2}\right)^q. \qquad (5.75)$$

This result allows to determine the other coefficients $C^{h,k}_s$.

To this end, it will be useful to note firstly that, from eq. (5.37) with n=Mh+Nk+1 it follows

$$\phi^n_{s-r} = (-4)^r\frac{s!}{(s-r)!}\frac{(Mh+Nk+1-2s)!}{(Mh+Nk+1-2s+2r)!}\frac{\partial^r}{\partial(\gamma^2)^r}\phi^n_s. \qquad (5.76)$$

(In fact, this relation holds for r=0. Let us assume that it also holds up to an index r; from eq. (5.37) with n=Mh+Nk+1 and s-r instead of s we find that

$$\phi^n_{s-r-1} = \frac{-2s+2r}{\gamma}\frac{1}{(Mh+Nk+3-2s+2r)(Mh+Nk+2-2s+2r)}\frac{\partial}{\partial\gamma}\phi^n_{s-r}$$
$$= -4(s-r)\frac{1}{(Mh+Nk+3-2s+2r)(Mh+Nk+2-2s+2r)}\frac{\partial}{\partial(\gamma^2)}\phi^n_{s-r}$$
$$= (-4)^{r+1}\frac{s!}{(s-r-1)!}\frac{(Mh+Nk+1-2s)!}{(Mh+Nk+3-2s+2r)!}\frac{\partial^{r+1}}{\partial(\gamma^2)^{r+1}}\phi^n_s$$

where in the last passage eq. (5.76) has been used. The result is eq. (5.76), with r+1 instead of r, and this completes its proof.).
Eq. (5.76) with $s = \left[\frac{Mh+Nk+1}{2}\right]$ and $r = s - s^*$, jointly with eq. (5.75) shows that

$$C_{s^*}^{h,k} = (-4)^{\left[\frac{Mh+Nk+1}{2}\right]-s^*} \frac{\left[\frac{Mh+Nk+1}{2}\right]!}{s^*!} \frac{\left(Mh+Nk+1-2\left[\frac{Mh+Nk+1}{2}\right]\right)!}{(Mh+Nk+1-2s^*)!}$$
$$(2)^{\left[\frac{k}{2}\right]} \sum_{q=0}^{\frac{mh+k(N-1)-2}{2}} \frac{\left(Mh+k(N-1)+1-2\left[\frac{k}{2}\right]\right)!}{(Mh+k(N-1)-2q-2)!}(-m^2)^{\frac{N-1}{2}k+\frac{Mh}{2}} \frac{d^h c_{k+Mh,q}}{d\lambda^h}$$
$$\frac{\left(q+2+\left[\frac{k}{2}\right]\right)!}{(q+2)!}(-1)^{\left[\frac{Mh+Nk+1}{2}\right]-s^*} \frac{\left(q+2+\left[\frac{k}{2}\right]+\left[\frac{Mh+Nk+1}{2}\right]-s^*\right)!}{\left(q+2+\left[\frac{k}{2}\right]\right)!}$$
$$\left(\frac{1}{\gamma^2}\right)^{q+3+\left[\frac{k}{2}\right]+\left[\frac{Mh+Nk+1}{2}\right]-s^*}$$

from which the above reported eq. (5.33), taking into account that

$$\left(Mh+Nk+1-2\left[\frac{Mh+Nk+1}{2}\right]\right)! = \begin{cases} 0! = 1 & \text{if k is odd,} \\ 1! = 1 & \text{if k is even,} \end{cases}$$

and moreover, that $\left[\frac{Mh+Nk+1}{2}\right] + \left[\frac{k}{2}\right] = \frac{Mh+(N+1)k}{2}$.

5.6 The subsystems

5.6.1 The Subsystems of type 1

The method of subsystems presented in the book [2] can be applied to our case as described in the sequel.
From eq. (5.7) with

$$\mu_{\alpha_1\cdots\alpha_N} = -\frac{1}{m^2}\mu_{(\alpha_1\cdots\alpha_{N-2}}g_{\alpha_{N-1}\alpha_N)} \tag{5.77}$$

and using the trace condition $(5.5)_2$ we obtain $dh^\alpha = \lambda_{\alpha_1\cdots\alpha_M} dA^{\alpha\alpha_1\dots\alpha_M} + \mu_{\alpha_1\cdots\alpha_{N-2}} dB^{\alpha\alpha_1\dots\alpha_{N-2}}$, which is again (5.7) but with N replaced by N-2. But we can obtain a similar model also by starting from the beginning with N-2 instead of N. Let us now compare the two resulting models. We will refer to this as "subsystems of type 1". The other type will be considered in the following subsection. The results of the first type will be published in [41].

From eqs. (5.11_2) and (5.77) we have

$$\mu_\alpha = 8\frac{N!!}{(N+3)!!}\mu_{\alpha\alpha_1\cdots\alpha_{N-1}}g^{\alpha_1\alpha_2}\cdots g^{\alpha_{N-2}\alpha_{N-1}}(-m^2)^{\frac{N-1}{2}} =$$
$$= 8\frac{(N-2)!!}{(N+1)!!}\mu_{\alpha\alpha_1\cdots\alpha_{N-3}}g^{\alpha_1\alpha_2}\cdots g^{\alpha_{N-4}\alpha_{N-3}}(-m^2)^{\frac{N-3}{2}}.$$

i.e. eq. $(5.11)_2$ with N-2 instead of N, so μ_α remains the same even in the subsystem. Thanks to eq. (5.77), eq. (5.22), the particular solution, becomes $\int F(\lambda_{\alpha_1\ldots\alpha_M}p^{\alpha_1}\ldots p^{\alpha_M}, \mu_{\beta_1\ldots\beta_{N-2}}p^{\beta_1}\ldots p^{\beta_{N-2}})p^\alpha dP$, i.e. that with N-2 instead of N. What about the general solution?
For this kind of subsystem eq. $(5.28)_1$ remains the same, while eq. $(5.28)_2$ becomes

$$\widetilde{\mu}_{\beta_1\cdots\beta_N} = -\frac{1}{m^2}\mu_{(\beta_1\cdots\beta_{N-2}}g_{\beta_{N-1}\beta_N)} - \mu_{(\beta_1}g_{\beta_2\beta_3}\cdots g_{\beta_{N-1}\beta_N)}(-m^2)^{-\frac{N-1}{2}}$$
$$= -\frac{1}{m^2}\widetilde{\mu}_{(\beta_1\cdots\beta_{N-2}}g_{\beta_{N-1}\beta_N)}$$

where we have used eq. $(5.28)_2$ with N-2 instead of N. But in eq. (5.26) $\widetilde{\mu}_{\beta_1\cdots\beta_N}$ multiply a symmetric tensor so we can drop the symmetrization. So, defining $B_i = \widetilde{B}_i\beta_{i\ N-1}\beta_{i\ N}$, the tensor $C_{h,k}^{\alpha A_1\cdots A_h\widetilde{B}_1\cdots\widetilde{B}_k}$ of the subsystem is $C_{h,k}^{\alpha A_1\cdots A_h\widetilde{B}_1\beta_{1\ N-1}\beta_{1\ N}\cdots\widetilde{B}_k\beta_{k\ N-1}\beta_{k\ N}}g_{\beta_{1\ N-1}\beta_{1\ N}}\cdots g_{\beta_{k\ N-1}\beta_{k\ N}}(-m^2)^{-k}$. We can apply proposition 6 with n=hM+kN+1, r=k, $p = q+\left[\frac{k}{2}\right]$, and use eq. (5.75), finding that the leading term of $C_{h,k}^{\alpha A_1\cdots A_h\widetilde{B}_1\cdots\widetilde{B}_k}$ is

$$2^{\left[\frac{k}{2}\right]}\gamma^{-6-2\left[\frac{k}{2}\right]}\sum_{q=0}^{\frac{Mh+k(N-1)-2}{2}}\frac{\left(hM+k(N-1)+2\left[\frac{k}{2}\right]+1\right)!!}{(hM+k(N-1)-2q-2)!!}(-m^2)^{\frac{N-3}{2}k+\frac{M}{2}h}$$
$$\frac{d^h c_{k+Mh,q}}{d\lambda^h}\frac{\left(q+2+\left[\frac{k}{2}\right]\right)!}{(q+2)!}\left(\frac{1}{\gamma}\right)^q\frac{\left(2\left[\frac{hM+kN+2}{2}\right]-1-2k\right)!!}{\left(2\left[\frac{hM+kN+2}{2}\right]-1\right)!!}\eta\left(2\left[\frac{hM+kN+2}{2}\right]\right.$$
$$\left.-2q-2\left[\frac{k}{2}\right]-2k-2, 2\left[\frac{Mh+kN+2}{2}\right]-2q-4-2\left[\frac{k}{2}\right]\right). \tag{5.78}$$

But $\left[\frac{hM+kN+2}{2}\right]-\left[\frac{k}{2}\right] = \frac{1}{2}\left[hM+k(N-1)+2\right]$, so if $\frac{Mh+k(N-3)-2}{2}+1 \le q \le \frac{Mh+k(N-1)-2}{2}$ then $hM+k(N-3)-2q \le 0$ and $hM+k(N-1)-2-2q \ge 0$ so $\eta(\ldots,\ldots) = 0$ and we can restrict to values with $q \le \frac{Mh+k(N-3)-2}{2}$ and eq.

(5.78) becomes

$$2^{\left[\frac{k}{2}\right]}\gamma^{-6-2\left[\frac{k}{2}\right]}\sum_{q=0}^{\frac{Mh+k(N-3)-2}{2}}\frac{\left(hM+k(N-3)+2\left[\frac{k}{2}\right]+1\right)!!}{(hM+k(N-3)-2q-2)!!}$$

$$(-m^2)^{\frac{N-3}{2}k+\frac{M}{2}h}\frac{d^h c_{k+Mh,q}}{d\lambda^h}\frac{\left(q+2+\left[\frac{k}{2}\right]\right)!}{(q+2)!}\left(\frac{1}{\gamma}\right)^q, \qquad (5.79)$$

i.e. eq. (5.75) with N-2 instead of N.

$$c_{k,q}^{N-2}=c_{k,q}^{N} \quad \text{for } q=0,\cdots k\frac{N-3}{2}-1; \qquad (5.80)$$

The restriction $c_{k,q}^{N-2}=c_{k-1,q}^{N-2}$ for $q=0,\cdots(k-1)\frac{N-3}{2}-1$, is respected?
For such values of q we can apply eq. (5.80) with k-1 instead of k, i.e. $c_{k,q}^{N}=c_{k-1,q}^{N}$ for $q=0,\cdots(k-1)\frac{N-3}{2}-1$, and this is true for $q=0,\cdots(k-1)\frac{N-1}{2}-1$.
We want prove that all $c_{k,q}^{N}$ present in the model with N intervene also in the model with N-2.
For Property 3 with k+i instead of k we have $c_{k+i,q}^{N}=c_{k,q}^{N}$ for $q=0,\cdots\frac{(N-1)k-2}{2}$, from which, for such values of q and for eq. (5.80) we have $c_{k,q}^{N}=c_{k+i,q}^{N-2}$ provided that $q\leq(k+i)\frac{N-3}{2}-1$. Now, exists a value of i such that $k\frac{N-1}{2}-1\leq(k+i)\frac{N-3}{2}-1$?
Yes, it is sufficient take $i=\left[\frac{2k}{N-3}\right]+1$ and this is possible when $N\geq5$.
Instead, if N=3, in the case N-2 instead of N we mustn't develop with respect to $\widetilde{\mu}$, so only the case k=0 have sense; so the result (5.79) coincide with (5.57), with

$$c_{h,q}^{1}=c_{Mh,q}^{3} \quad \text{for } q=0,\cdots,\frac{Mh-2}{2}. \qquad (5.81)$$

This satisfy the restriction

$$c_{h,q}^{1}=c_{h-1,q}^{1} \quad \text{for } q=0,\cdots,\frac{M(h-1)-2}{2}, \qquad (5.82)$$

in fact with such values of q we can apply eq. (5.81) with h-1 instead of h and eq. (5.82) becomes $c_{Mh,q}^{3}=c_{M(h-1),q}^{3}$ for $q=0,\cdots,\frac{M(h-1)-2}{2}$, and this is certainly respected for property 3 with Mh instead of k and i=M, N=1 or 3, valid for $q=0,\cdots M(h-1)-1$ and so we can apply it for $q=0,\cdots,\frac{M(h-1)-2}{2}$.
And which $c_{k,q}^{3}$ intervene in the model with N=1 instead of 3?
From property 3 with k+1 instead of k and N=3 we have $c_{k+1,q}^{3}=c_{k,q}^{3}$ for $q=0,\cdots,k-1$. To render $c_{k+i,q}^{3}=c_{h,q}^{1}=c_{Mh,q}^{3}$ we have to search the values of i and h such that k+i=Mh and $k-1\leq\frac{Mh-2}{2}$ (that is the highest admissible

value of q in $c^1_{h,q}$). The second of the condition above have certainly solutions, for example $h = \left[\frac{2k}{M}\right]+1$. Once taken this value for h, also the other condition above have solution, i.e. i=Mh-k. Obviously, this can be done except for the case M=0. So we can conclude that the family of arbitrary single variable functions, arising from integration, is preserved in the subsystem except for the case M=0, N=3 that is when the subsystem is constituted only by the conservation laws of mass and momentum-energy.

5.6.2 The Subsystems of type 2

Let us consider now the subsystem obtained by using

$$\lambda_{\alpha_1\cdots\alpha_M} = -\frac{1}{m^2}\lambda_{(\alpha_1\cdots\alpha_{M-2}}g_{\alpha_{M-1}\alpha_M)} \tag{5.83}$$

and the trace condition $(5.5)_1$.

Let us now compare it with the model which can be obtained by starting from the beginning with M-2 instead of M. The results will be published in [42]

From eqs. (5.11_1) and (5.83) we have

$$\begin{aligned}\lambda &= 2\frac{(M-1)!!}{(M+2)!!}\lambda_{\alpha_1\cdots\alpha_M}g^{\alpha_1\alpha_2}\cdots g^{\alpha_{M-1}\alpha_M}(-m^2)^{\frac{M}{2}} = \\ &= 2\frac{(M-3)!!}{(M)!!}\lambda_{\alpha_1\cdots\alpha_{M-2}}g^{\alpha_1\alpha_2}\cdots g^{\alpha_{M-3}\alpha_{M-2}}(-m^2)^{\frac{M-2}{2}},\end{aligned}$$

i.e. eq. $(5.11)_1$ with M-2 instead of M, so λ remains the same even in the subsystem.

Thanks to eq. (5.83), eq. (5.22), the particular solution, becomes

$$\int F(\lambda_{\alpha_1\ldots\alpha_M}p^{\alpha_1}\ldots p^{\alpha_M}, \mu_{\beta_1\ldots\beta_{N-2}}p^{\beta_1}\ldots p^{\beta_{N-2}})p^\alpha dP,$$

i.e. that with M-2 instead of M. For this kind of subsystems eq. $(5.28)_2$ remains the same, while eq. $(5.28)_1$ becomes

$$\begin{aligned}\widetilde{\lambda}_{\beta_1\cdots\beta_M} &= -\frac{1}{m^2}\lambda_{(\beta_1\cdots\beta_{M-2}}g_{\beta_{M-1}\beta_M)} - \lambda g_{(\beta_1\beta_2}\cdots g_{\beta_{M-1}\beta_M)}(-m^2)^{-\frac{M}{2}} = \\ &= -\frac{1}{m^2}\widetilde{\lambda}_{(\beta_1\cdots\beta_{M-2}}g_{\beta_{M-1}\beta_M)}\end{aligned}$$

where we have used eq. $(5.28)_1$ with M-2 instead of M. But in eq. (5.26) $\widetilde{\lambda}_{\beta_1\cdots\beta_M}$ is multiplied by a symmetric tensor, so in the previous relation we can eliminate the symmetrization. So, by defining $\widetilde{A}_i$ from $A_i = \widetilde{A}_i \beta_{i\ M-1}\beta_{i\ M}$, the tensor $C_{h,k}^{\alpha \widetilde{A}_1 \cdots \widetilde{A}_h B_1 \cdots B_k}$ of the subsystem is

$$C_{h,k}^{\alpha \widetilde{A}_1 \cdots \widetilde{A}_h B_1 \cdots B_k \beta_{i\ M-1}\beta_{i\ M} \cdots \beta_{h\ M-1}\beta_{h\ M}} g_{\beta_{i\ M-1}\beta_{i\ M}} \cdots g_{\beta_{h\ M-1}\beta_{h\ M}} (-m^2)^{-h} \tag{5.84}$$

The Subcase $N \geq 3$

Let's apply proposition 6 with n=hM+kN+1, r=h, $p = q + \left[\frac{k}{2}\right]$ and the use of eq. (5.75); we find that the leading term of $C_{h,k}^{\alpha \widetilde{A}_1 \cdots \widetilde{A}_h B_1 \cdots B_k}$ is

$$2^{\left[\frac{k}{2}\right]} \gamma^{-6-2\left[\frac{k}{2}\right]} \sum_{q=0}^{\frac{Mh+k(N-1)-2}{2}} \frac{\left(\frac{hM+k(N-1)+1+2\left[\frac{k}{2}\right]}{}\right)!!}{(hM+k(N-1)-2q-2)!!} (-m^2)^{\frac{N-1}{2}k + \frac{M}{2}h}$$
$$\frac{d^h c_{k+Mh,q}}{d\lambda^h} \frac{\left(q+2+\left[\frac{k}{2}\right]\right)!}{(q+2)!} \left(\frac{1}{\gamma^2}\right)^q \frac{\left(2\left[\frac{hM+kN}{2}\right] - 2h + 1\right)!!}{\left(2\left[\frac{hM+kN}{2}\right] + 1\right)!!} \eta\Bigg(2\left[\frac{hM+kN}{2}\right]$$
$$-2h - 2q - 2\left[\frac{k}{2}\right], 2\left[\frac{hM+kN}{2}\right] - 2 - 2q - \left[\frac{k}{2}\right]\Bigg).$$

But $\left[\frac{hM+kN}{2}\right] = \left[\frac{k}{2}\right] + \frac{hM+k(N-1)}{2}$ and, for $\frac{(M-2)h+k(N-1)-2}{2} + 1 \leq q \leq \frac{Mh+k(N-1)-2}{2}$ we have $h(M-2) + k(N-1) - 2q \leq 0$ and $hM + k(N-1) - 2 - 2q \geq 0$, so that $\eta(...,\ ...) = 0$, so we can limit to the values $q = 0, \cdots, \frac{(M-2)h+k(N-1)-2}{2}$ the summation in the equation above that becomes eq. (5.75) with M-2 instead of M provided that $c_{k+(M-2)h,q}^{M-2} = c_{k+Mh,q}^{M}$ for $q = 0, \cdots, \frac{(M-2)h+k(N-1)-2}{2}$. This last relation, with h=0, becomes $c_{k,q}^{M-2} = c_{k,q}^{M}$ for $q = 0, \cdots, \frac{k(N-1)-2}{2}$. So, not only condition $c_{k,q}^{M-2} = c_{k-1,q}^{M-2}$ for $q = 0, \cdots, \frac{(k-1)(N-1)-2}{2} - 1$ remains satisfied, but all the $c_{h,k}^{M}$ intervene (and they all intervene also in the terms with h=0).

The Subcase $N = 1$

Let's apply proposition 6 with n=hM+1, r=h, p=q, and with the use of eq. (5.57), and we find that the leading term of $C_{h,0}^{\alpha \widetilde{A}_1 \cdots \widetilde{A}_h}$ is

$$\gamma^{-6} \sum_{q=0}^{\frac{Mh-2}{2}} (-m^2)^{\frac{M}{2}h-h} \frac{d^h c_{h,q}}{d\lambda^h} \left(\frac{1}{\gamma^2}\right)^q \frac{(Mh+1)!!}{(Mh-2q-2)!!} \frac{(Mh+1-2h)!!}{(Mh+1)!!} \cdot$$
$$\eta(hM - 2h - 2q, hM - 2 - 2q).$$

But, for $\frac{(M-2)h-2}{2}+1 \leq q \leq \frac{Mh-2}{2}$ we have $hM-2h-2q \leq 0$ and $hM-2-2q \geq 0$, so that $\eta(..., ...) = 0$, so we can limit to the values $q = 0, \cdots, \frac{(M-2)h-2}{2}$ the summation in the equation above that becomes eq. (5.57) with M-2 instead of M provided that

$$c^{M-2}_{h,q} = c^{M}_{h,q} \qquad\qquad q = 0, \cdots, \frac{(M-2)h-2}{2}. \tag{5.85}$$

The restriction $c^{M-2}_{h,q} = c^{M-2}_{h-1,q}$ is satisfied for $q = 0, \cdots, \frac{(M-2)(h-1)-2}{2}$. Such values of q admit to apply eq. (5.85) even with h-1 instead of h, so that the restriction mentioned above becomes $c^{M}_{h,q} = c^{M}_{h-1,q}$; because that is satisfied for $q = 0, \cdots \frac{Mh-2}{2}$ is certainly satisfied for $q = 0, \cdots, \frac{(M-2)(h-1)-2}{2}$ because $\frac{(M-2)(h-1)-2}{2} \leq \frac{Mh-2}{2}$.

We have that all the $c^{M}_{h,q}$ of the model with M intervene also in the model with M-2. In fact, let is $j = \left[\frac{2h}{M-2}\right] + 1$, from eq. (5.58) with h+j instead of h we have $c^{M}_{h+j,q} = c^{M}_{h,q}$ for $q = 0, \cdots, \frac{Mh-2}{2}$, and, for such values of q and thanks to eq. (5.85) we have $c^{M}_{h,q} = c^{M-2}_{h+j,q}$ provided that $\frac{Mh-2}{2} \leq \frac{(M-2)(h+j)-2}{2}$, that is true.
Obviously, this can be done except for the case M=2. But, if M=2, N=1 we have that the subsystem is constituted only by the conservation laws of mass and momentum-energy. So we have found that only in this case the family of arbitrary single variable functions is not preserved in the subsystem.

Chapter 6

The non-relativistic limit of Relativistic E.T.

The non-relativistic limit of Relativistic Extended Thermodynamics with 14 moments can be found in paper [43] by Dreyer and Weiss (see also [2]), which has been widely appreciated. In particular it suggest a particular structure for the classical counterpart of the theory, in particular that developed by Kremer, instead of the previous one with 13 moments. Here we extend their methods for the case with many moments following the macroscopic approach. Also our results predict a particular structure for the classical counterpart with many moments, that will be described in the following chapter. It is noteworthy that in this structure the independent variables are moments of increasing orders; the highest of these is even, as in the kinetic approach. The results are published in [44] and [45].
We consider system (5.4) and the trace conditions (5.5) and we will see that the non-relativistic limit of eqs. (5.4) has the form

$$\begin{cases} \partial_t F^{i_1\cdots i_s} + \partial_k F^{i_1\cdots i_s k} = P^{i_1\cdots i_s} \\ \partial_t F^{i_1\cdots i_r e_1 e_1 \cdots e_{\frac{N+M-1-2r}{2}} e_{\frac{N+M-1-2r}{2}}} + \partial_k F^{i_1\cdots i_r k e_1 e_1 \cdots} = Q^{i_1\cdots i_r} \end{cases} \tag{6.1}$$

for $0 \leq s \leq N-1$, $0 \leq r \leq M-1$. When $M = 2, \quad N = 3$, eqs. (5.4) are the pertinent equations of the 14-moments theory of relativistic extended thermodynamics [35] and eqs. (6.1) are the corresponding equations for the non-relativistic approach [46]. We note that the highest order of moments, among the independent variables, is $M + N - 1$, which is always even; this confirms the same property obtained by the kinetic approach in order to have integrability, i.e., that the integrals involved must be convergent.

6.1 Suggestions from kinetic theory

Because the form of equations (5.4) is suggested from the kinetic theory of gases, it is not restrictive to deduce from this theory the orders of greatness of the moments and productions with respect to c. Meanwhile, we obtain this information also for the entropy and entropy-flux tensor h^α. In particular, we have

$$\begin{array}{l} A^{\alpha_1 \ldots \alpha_N} = \int \widetilde{f}(x^\mu, p^0, p^i) p^{\alpha_1} \cdots p^{\alpha_N} \frac{dp^1 dp^2 dp^3}{p^0} \\ h^\alpha = \int G[\widetilde{f}(x^\mu, p^0, p^i)] p^\alpha \frac{dp^1 dp^2 dp^3}{p^0} \end{array} \tag{6.2}$$

where $\widetilde{f}$ is the relativistic distribution function, $\gamma(u) = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}}$ is the Lorentz factor, $p^\mu = m_0 u^\mu \equiv (m_0 \gamma(u) c, m_0 \gamma(u) u^i)$ is the relativistic momentum particle and G a suitable function of $\widetilde{f}$.
By changing the integration variables from p^i to u^i, we see that the Jacobian of the transformation is $J = \left|\frac{\partial p^i}{\partial u^j}\right| = \left|m_0 \gamma(u) \delta^{ij} + m_0 \frac{\gamma^3}{c^2} u^i u^j\right| = m_0^3 \gamma^5$ and the above integrals (6.2) transform into

$$A^{\alpha_1 \ldots \alpha_s \overbrace{0 \ldots 0}^{N-s}} = m_0^{N+2} c^{N-s-1} \widetilde{F}_N^{i_1 \ldots i_s},\ h^0 = m_0^3 h,\ h^i = \frac{m_0^3}{c} \phi^i$$

with

$$\widetilde{F}_N^{i_1 \ldots i_s} = \int \widetilde{f} \gamma^{N+4} u^{i_1} \cdots u^{i_s} d\underline{u},\ h = \int G(\widetilde{f}) \gamma^5 d\underline{u},\ \phi^i = \int G(\widetilde{f}) \gamma^5 u^i d\underline{u}. \tag{6.3}$$

Now eqs. (5.4) can be written as $\frac{1}{c}\partial_t A^{0\alpha_2 \ldots \alpha_N} + \partial_k A^{k\alpha_2 \ldots \alpha_N} = I^{\alpha_2 \ldots \alpha_N}$, which can be written for $\alpha_2 \ldots \alpha_N = i_1 \ldots i_s 0 \ldots 0$ and becomes the first of the following equations

$$\left\{ \begin{array}{ll} \partial_t \widetilde{F}_N^{i_1 \cdots i_s} + \partial_k \widetilde{F}_N^{k i_1 \cdots i_s} = \widetilde{P}^{i_1 \cdots i_s} & \text{for } 0 \le s \le N-1 \\ \partial_t \widetilde{F}_M^{i_1 \cdots i_r} + \partial_k \widetilde{F}_M^{k i_1 \cdots i_r} = \widetilde{P}_M^{i_1 \cdots i_r} & \text{for } 0 \le r \le M-1 \end{array} \right. \tag{6.4}$$

with

$$\widetilde{P}^{i_1 \ldots i_s} = m_0^{-N-2} c^{-N+s+2} I_N^{i_1 \ldots i_s 0 \ldots 0},\ \widetilde{P}_M^{i_1 \ldots i_r} = m_0^{-M-2} c^{-M+r+2} I_M^{i_1 \ldots i_r 0 \ldots 0}$$

and, obviously, eq.$(6.4)_2$ is the counterpart of eq. $(5.4)_2$, where $B^{\cdots \alpha_M}$ is defined in the same way as $A^{\cdots \alpha_N}$. Similarly, the entropy law $\partial_\alpha h^\alpha = \sigma$ becomes

$$\partial_t h + \partial_k \phi^k = s = m_0^{-3} c \sigma\,.$$

Eqs. (6.4) are still relativistic, although expressed in 3-dimensional form. Their limits as $c \longrightarrow \infty$ don't give independent equations, because from eq. (6.3) it follows that $\lim_{c\to\infty} \widetilde{F}_N^{i_1\ldots i_r} = \lim_{c\longrightarrow\infty} \widetilde{F}_M^{i_1\ldots i_r}$. In other words, $\widetilde{F}_N^{i_1\ldots i_r} - \widetilde{F}_M^{i_1\ldots i_r}$ is higher order infinitesimal with respect to c^{-1}, so that we have to find a suitable linear combination of eqs. (6.4) and multiply the result by an appropriate power of c, before taking the limit. This will be done in the next section.

6.1.1 A new form for the system (6.4).

Let us consider the numbers

$$b_{hr} = (-1)^h \binom{m}{h} \frac{(n+m-h)!}{(n+m)!} \eta\,(N-M-2n, N-M-2n+2h-2) \tag{6.5}$$

where $\eta(a,b)$ denotes the product of all odd numbers between a and b if $a \le b$, while it is 1 if $a > b$; moreover $n = \left[\frac{N-1-r}{2}\right]$, $\quad m = \left[\frac{M-1-r}{2}\right]$. Obviously, $b_{0r} = 1$.

Proposition 9 *: The numbers defined by eq.* (6.5) *satisfy the equations*

$$\sum_{h=0}^{m} b_{hr} c_{n+j-h} = \delta_{j,m+1} b_r, \quad \textit{for } j = 1, \ldots, m+1 \tag{6.6}$$

with

$$c_h = \frac{1}{h!} \eta\,(N-M-2h+2, N-M)\ , \tag{6.7}$$

$$b_r = (-1)^m \frac{n!}{(n+m+1)!} \frac{m!}{(n+m)!} \eta\,(N-M-2n, N-M+2m)\ . \tag{6.8}$$

Let us also consider the numbers

$$a_{kr} = -\sum_{h=0}^{inf\{k,[\frac{M-1-r}{2}]\}} b_{hr} c_{k-h}, \quad \text{for } k = 0, \ldots, \left[\tfrac{N-1-r}{2}\right]. \tag{6.9}$$

After that, let us consider the following linear combination of $\widetilde{F}_M^{i_1\cdots i_r a}$ and of $\widetilde{F}_N^{i_1\cdots i_s a}$:

$$\widetilde{F}^{i_1\cdots i_r a e_1 e_1 \cdots e_{\frac{N+M-1-2r}{2}} e_{\frac{N+M-1-2r}{2}}} = \left[\sum_{q=0}^{[\frac{M-1-r}{2}]} b_{qr} \widetilde{F}_M^{i_1\cdots i_r e_1 e_1 \cdots e_q e_q a} (-2c^2)^{-q} + \right.$$

$$+\sum_{p=0}^{[\frac{N-1-r}{2}]} a_{pr}\widetilde{F}_N^{i_1\cdots i_r e_1 e_1\cdots e_p e_p a}(-2c^2)^{-p}\Bigg]\frac{1}{b_r}\left(-2c^2\right)^{\frac{M+N-1-2r}{2}} \tag{6.10}$$

where the index a has to be omitted if it is zero. Note that this tensor has $N+M-1-r > N-1$ indices (if $a=0$) so that there is no possibility of confusing it with $\widetilde{F}_N^{i_1\cdots i_r}$. The corresponding linear combination of eqs. (6.4) gives eqs. $(6.1)_2$, while eq. $(6.1)_1$ is eq. $(6.4)_1$ except that now the index N has been omitted. Obviously, we also define

$$\widetilde{Q}^{i_1\cdots i_r} = \frac{1}{b_r}\sum_{q=0}^{[\frac{M-1-r}{2}]} b_{qr}(-2c^2)^{\frac{N+M-1-2r-2q}{2}}\widetilde{P}_M^{i_1\cdots i_r e_1 e_1\cdots e_q e_q}+$$

$$+\frac{1}{b_r}\sum_{p=0}^{[\frac{N-1-r}{2}]} a_{pr}(-2c^2)^{\frac{N+M-1-2r-2p}{2}}\widetilde{P}^{i_1\cdots i_r e_1 e_1\cdots e_p e_p} \tag{6.11}$$

where the property $[\frac{N-1-r}{2}]+[\frac{M-1-r}{2}] = \frac{N+M-3-2r}{2}$ has been used (it is a consequence of the fact that $N+M$ is odd). The interesting thing, which we now prove, is that

$$\lim_{c\to\infty}\widetilde{F}^{i_1\cdots i_r a e_1 e_1\cdots e_{\frac{N+M-1-2r}{2}} e_{\frac{N+M-1-2r}{2}}} = \int f u^{i_1}\cdots u^{i_r}u^a (u^2)^{\frac{N+M-1-2r}{2}} d\underline{u}, \tag{6.12}$$

and we indicate this limit by $F^{i_1\cdots i_r a e_1 e_1\cdots e_{\frac{N+M-1-2r}{2}} e_{\frac{N+M-1-2r}{2}}}$; moreover, u^a is 1 if $a=0$, is u^k if $a=k$ and $f=\lim_{c\to\infty} m_0^3\widetilde{f}$, as in [2] (in the sequel the factor m_0^3 does not affect the results, so we will omit it). To prove eq. (6.12), we see that (6.10), by means of (6.3) gives

$$\widetilde{F}^{i_1\cdots i_r a e_1 e_1\cdots e_{\frac{N+M-1-2r}{2}} e_{\frac{N+M-1-2r}{2}}} = \frac{1}{b_r}(-2c^2)^{\frac{N+M-1-2r}{2}}\int \widetilde{f}\gamma^{N+4}u^{i_1}\cdots u^{i_r}u^a$$

$$\cdot\left[\gamma^{M-N}\sum_{q=0}^{[\frac{M-1-r}{2}]} b_{qr}(u^2)^q(-2c^2)^{-q} + \sum_{p=0}^{[\frac{N-1-r}{2}]} a_{pr}(u^2)^p(-2c^2)^{-p}\right] d\underline{u}. \tag{6.13}$$

By inserting the expansion of

$$(\gamma)^{M-N} = \left(1-\frac{u^2}{c^2}\right)^{\frac{N-M}{2}} = \sum_{h=0}^{\infty}\frac{1}{h!}c_h(u^2)^h(-2c^2)^{-h}$$

into the expression between square brackets it becomes

$$\left[c_0 + c_1 u^2(-2c^2)^{-1} + \ldots + c_h (u^2)^h (-2c^2)^{-h}\right] \cdot$$

$$\left[b_{0r} + b_{1r} u^2(-2c^2)^{-1} + \cdots + b_{[\frac{M-1-r}{2}],r}\left(\frac{u^2}{-2c^2}\right)^{[\frac{M-1-r}{2}]}\right] + \sum_{k=0}^{[\frac{N-1-r}{2}]} a_{kr}(u^2)^k$$

$$\cdot(-2c^2)^{-k} = \sum_{k=0}^{\infty} \sum_{h=0}^{\inf\{k,[\frac{M-1-r}{2}]\}} b_{hr} c_{k-h}\left(\frac{u^2}{-2c^2}\right)^k + \sum_{k=0}^{[\frac{N-1-r}{2}]} a_{kr}\left(\frac{u^2}{-2c^2}\right)^k .$$

Now, the tensor for $k \leq [\frac{N-1-r}{2}]$ disappears for eq. (6.9), while those with $[\frac{N-1-r}{2}] + 1 \leq k \leq [\frac{N-1-r}{2}] + [\frac{M-1-r}{2}] = m + n$ disappear for eqs. (6.7) with $j = k - n$ (note that $1 \leq j \leq m$). It remains to consider the terms with $k = m + n + 1$ and those of higher order, i.e.,

$$\left(-2c^2\right)^{-m-n-1}\left[\sum_{h=0}^{m} b_{hr} c_{m+n+1-h}\left(u^2\right)^{m+n+1} + o\left(\frac{1}{c^2}\right)\right] .$$

Inserting this result in eq. (6.13), using eq. (6.7) with $j = m+1$, and taking the limit as $c \longrightarrow \infty$, we obtain eq. (6.12). This completes the proof.
In order to prove the properties (6.7) and (6.8), we first state the following

Lemma 1 *For every $k \in [0, m-1]$ we have* $\sum_{h=0}^{m} (-1)^h \binom{m}{h} h^k = 0$.

Proof. We proceed with the iteration method with respect to k. The property is true when $k = 0$ because, in this case, the first member corresponds to $(-1+1)^m = 0$. If we assume that it is true up to a fixed integer $\overline{k} < m-2$, we have

$$\sum_{h=0}^{m} (-1)^h \binom{m}{h} h^{\overline{k}+1} = \sum_{h=1}^{m} (-1)^h \binom{m}{h} h^{\overline{k}+1} = m \sum_{h=1}^{m} (-1)^h \binom{m-1}{h-1} h^{\overline{k}} =$$

$$- m \sum_{s=0}^{m-1} (-1)^s \binom{m-1}{s} h^{\overline{k}} = 0 ,$$

where in the third passage we have put $h = s + 1$. ■
Now we consider the following functions

$$f(n, m, N-M, j) = \sum_{h=0}^{m} (-1)^h \binom{m}{h} \frac{(n+m-h)!}{n!} \frac{(n+j)!}{(n+j-h)!} \cdot$$
$$\cdot \eta\left(N - M - 2n, N - M - 2n + 2h - 2\right) \cdot$$
$$\cdot \eta\left(N - M - 2n - 2j + 2h + 2, N - M - 2n - 2j + 2m\right) . \qquad (6.14)$$

It is easy to prove the following

Proposition 10 *: "$f(n,m,N-M,1)=0.$"*

Proof. We have

$$f(n,m,N-M,1)=\frac{(n+1)!}{n!}\eta\left(N-M-2n,N-M-2n-2j+2m\right)\cdot$$
$$\sum_{h=0}^{m}(-1)^h\binom{m}{h}\frac{(n+m-h)!}{(n+1-h)!}$$

where the factor $\sum_{h=0}^{m}(-1)^h\binom{m}{h}\frac{(n+m-h)!}{(n+1-h)!}$ is equal to zero both for the previous lemma and because of $\frac{(n+m-h)!}{(n+1-h)!}$ is a polynomial of degree $m-1$ in the variable h. ∎

Proposition 11 $f(n,1,N-M,j)=-\delta_{j,2}(N-M+2)$ *for* $j=1,2.$

Proof. We obtain $f(n,1,N-M,j)=(n+1)\left(N-M-2n-2j+2\right)-\left(N-M-2n\right)(n+j)=\left(N-M-2\right)(1-j)$, with easy calculations. ∎

Proposition 12 *For every* $j=1,\ldots,m,m+1,$ *we have*

$$f(n,m,N-M,j)=\delta_{j,m+1}\eta\left(N-M+2,N-M+2m\right)(-1)^m m!$$

Proof. We proceed with the iteration method with respect to m. The property is true when $m=1$ as consequence of the proposition 2. For $m\geq 2$ we have

$$f(n,m,N-M,j)-\frac{(n+j)}{(n+j-m)}f(n,m,N-M,j-1)=$$
$$=\sum_{h=0}^{m}(-1)^h\binom{m}{h}\frac{(n+m-h)!}{n!}\left(N-M-2n+2h-2\right)\ldots\left(N-M-2n\right)\cdot$$
$$\left[\frac{(n+j)!}{(n+j-h)!}\left(N-M-2n-2j+2m\right)\ldots\left(N-M-2n-2j+2h+2\right)-\right]\cdot$$
$$\left[\frac{(n+j)!}{(n+j-m)(n+j-h-1)!}\left(N-M-2n-2j+2m+2\right)\ldots\right.$$
$$\left.\left(N-M-2n-2j+2h+4\right)\right]=\sum_{h=0}^{m-1}(-1)^h\binom{m}{h}\frac{(n+m-h)!}{n!}\cdot$$
$$\left(N-M-2n+2h-2\right)\ldots\left(N-M-2n\right)\frac{(n+j)!}{(n+j-m)(n+j-h)!}$$
$$\left(N-M-2n-2j+2m\right)\ldots\left(N-M-2n-2j+2h+4\right)\cdot\Big[(n+j-m)\cdot$$
$$\left(N-M-2n-2j+2h+2\right)-(n+j-h)(N-M-2n-2j+2m+2)\Big]=$$

$$= \sum_{h=0}^{m-1} (-1)^h \binom{m-1}{h} \frac{m}{(m-h)} \frac{(n+m-h)!}{(n+1)!} (n+1)\,(N-M-2n+2h-2)\ldots$$
$$(N-M-2n) \frac{(n+j)!}{(n+j-m)(n+j-h)!} (N-M-2n-2j+2m)\ldots$$
$$(N-M-2n-2j+2h+4)\,[(N-M+2)(h-m)] =$$
$$= -m \frac{(n+1)}{(n+j-m)} (N-M+2)\, f(n+1, m-1, N-M+2, j-1) =$$
$$= -m \frac{(n+1)}{(n+j-m)} (N-M+2)\, \delta_{j-1,m}\, (N-M+4)\,(N-M+6)\ldots$$
$$(N-M+2m)\,(-1)^{m-1}\,(m-1)! = \frac{n+1}{n+j-m} \delta_{j-1,m} (N-M+2)(N-M+4)\ldots$$
$$(N-M+2m)\,(-1)^m\, m!$$

for $j = 2, \ldots, m, m+1$.
This relation, when $j = 2$ and using proposition 1, gives us $f(n, m, N-M, 2) = 0$. If we proceed in the same way for the next indexes until $j = m$, we find, always, $f(n, m, N-M, j) = 0$. Instead, when $j = m+1$ the last relation allow us to write $f(n, m, N-M, m+1) = (N-M+2)(N-M+4) \cdot \ldots \cdot (N-M+2m)\,(-1)^m\, m!$, so the proof is complete. ■

Corollary 1

$$f(n, m, N-M, j) = (1-j)\,(2-j) \cdot \ldots \cdot (m-j)\, \eta\,(N-M+2, N-M+2m)$$

Proof. We observe that first and second member in this equation are polynomial of degree m in j (for the first member it is consequence of the fact that $\frac{(n+j)!}{(n+j-h)!} = (n+j)(n+j-1)\ldots(n+j-h+1)$ and this expression has degree h in j while $\eta(N-M-2n-2j+2h+2, N-M-2n-2j+2m)$ has degree $m-h$ in j). Moreover these members give the same results in the $m+1$ different values $j = 1, \ldots, m+1$.
Now we can prove the property 1: to this end it suffices to substitute eq. (6.5) in the left-hand side of eq. (6.6) and to use the definition (6.14); after that the identity
$\frac{\eta(N-M-2n-2j+2h-2, N-M)}{\eta(N-M-2n-2j+2h+2, N-M-2n-2j)} = \eta(N-M-2n-2j+2m+2, N-M)$ has to be used and the proposition 3 to be applied. ■

6.2 The mass, momentum and energy conservation

In this section will be shown how the relativistic conservation laws of mass, momentum and energy are transformed in their classical counterparts, by using the linear combination described in the previous section and then taking the limit for $c \to \infty$. In order to describe the result, we consider eqs. (5.4) and (6.1) for $0 \le s \le N-1$ and $0 \le r \le M-1$. If we start considering only eqs. $(5.4)_1$ with N even, we obtain only the equation $(6.1)_1$ with $\lim_{c\to\infty} P = 0$ (mass conservation) and $\lim_{c\to\infty} P^{i_1} = 0$ (momentum conservation), but loosing energy conservation.

Instead, if we consider also eq. $(5.4)_2$, obviously for M odd, we can prove that P^{ii} is infinitesimal, obtaining in this way energy conservation. Similarly, if we consider only eq. $(5.4)_1$ with N even, we obtain only eq. $(6.1)_1$ with $\lim_{c\to\infty} P = 0$ (mass conservation), but losing momentum and energy conservation. The presence of eq. $(5.4)_2$ with M odd affects also the productions in eq. $(5.4)_1$: we will see that, always as a consequence of eq. $(5.4)_2$, also P^{i_1} and P^{ii} are infinitesimal and by this fact we obtain the momentum and energy conservation. Thus, in a relativistic approach, eqs. $(5.4)_1$ and $(5.4)_2$ cannot be neglected.

6.2.1 The case with N odd and M even

Obviously, in this case is included the 14-moments one. The maximal trace of eq. $(5.4)_1$ gives the mass conservation law; let us express it in terms of the tensor $p^{i_1 \dots i_s}$:

$$
\begin{aligned}
0 &= I_N^{\alpha_2 \dots \alpha_N} g_{\alpha_2\alpha_3} \dots g_{\alpha_{N-1}\alpha_N} = \\
&= I_N^{\alpha_2 \dots \alpha_N} (h_{\alpha_2\alpha_3} - t_{\alpha_2} t_{\alpha_3}) \dots (h_{\alpha_{N-1}\alpha_N} - t_{\alpha_{N-1}} t_{\alpha_N}) = \\
&= \sum_{h=0}^{\frac{N-1}{2}} \binom{\frac{N-1}{2}}{h} (-1)^h I_N^{\alpha_2 \dots \alpha_N} t_{\alpha_2} t_{\alpha_3} \dots t_{\alpha_{2h}} t_{\alpha_{2h+1}} h_{\alpha_{2h+2}\alpha_{2h+3}} \dots h_{\alpha_{N-1}\alpha_N} = \\
&= \sum_{h=0}^{\frac{N-1}{2}} \binom{\frac{N-1}{2}}{h} (-1)^h I_N^{0 \dots 0 e_1 e_1 \dots e_{\frac{N-1-2h}{2}} e_{\frac{N-1-2h}{2}}} = \\
& \sum_{h=0}^{\frac{N-1}{2}} \binom{\frac{N-1}{2}}{h} (-1)^h m_0^{N+2} c^{2h-1} P^{e_1 e_1 \dots e_{\frac{N-1-2h}{2}} e_{\frac{N-1-2h}{2}}}
\end{aligned}
$$

which can be multiplied by c^{2-N} and gives

$$
P = \sum_{h=0}^{\frac{N-3}{2}} \binom{\frac{N-1}{2}}{h} (-1)^{\frac{h+N+1}{2}} c^{2h-N+1} P^{e_1 e_1 \dots e_{\frac{N-1-2h}{2}} e_{\frac{N-1-2h}{2}}} \tag{6.15}
$$

whose non-relativistic limit is

$$
\lim_{c\to\infty} P = 0 \tag{6.16}
$$

which is the mass conservation law for system (6.1). Similarly, the maximal trace of eq. $(5.4)_2$ gives momentum and energy conservation in the relativistic context. It reads: $0 = I_M^{\alpha_2 \dots \alpha_M} g_{\alpha_3\alpha_4} \dots g_{\alpha_{M-1}\alpha_M}$ which, with calculations similar to the ones above, becomes

$$0 = \sum_{h=0}^{\frac{M-2}{2}} \binom{\frac{M-2}{2}}{h} (-1)^h I_M^{\alpha_2 \overbrace{0 \dots 0}^{2h} e_1 e_1 \dots e_{\frac{M-2-2h}{2}} e_{\frac{M-2-2h}{2}}}$$

from which, for $\alpha_2 = 0$ and $\alpha_2 = i_1$ respectively, we obtain

$$\begin{cases} P_M = -\sum_{h=0}^{\frac{M-4}{2}} \binom{\frac{M-2}{2}}{h} (-1)^{h+\frac{M-2}{2}} c^{2h+2-M} P_M^{e_1 e_1 \dots e_{\frac{M-2-2h}{2}} e_{\frac{M-2-2h}{2}}} \\ P_M^{i_1} = -\sum_{h=0}^{\frac{M-4}{2}} \binom{\frac{M-2}{2}}{h} (-1)^{h+\frac{M-2}{2}} c^{2h+2-M} P_M^{i_1 e_1 e_1 \dots e_{\frac{M-2-2h}{2}} e_{\frac{M-2-2h}{2}}} \end{cases} . \tag{6.17}$$

Let us now consider the expression (6.11) of $Q^{i_1 \dots i_r}$, with $r = 2$, and let us compute its trace, thus obtaining:

$$\begin{aligned} (-2c^2)^{-\frac{N+M-3}{2}} Q^{ee} &= \frac{1}{b_2} \sum_{q=0}^{\frac{M-4}{2}} b_{q2} (-2c^2)^{-q} \, P_M^{e_1 e_1 \dots e_q e_q e_{q+1} e_{q+1}} + \\ &+ \frac{1}{b_2} \sum_{p=0}^{\frac{N-3}{2}} a_{p2} (-2c^2)^{-p} \, P^{e_1 e_1 \dots e_p e_p e_{p+1} e_{p+1}} \end{aligned}$$

whose non-relativistic limit is $0 = (b_2)^{-1}(b_{02} \overline{P}_M^{e_1 e_1} + a_{02} \overline{P}^{e_1 e_1})$, where an overlined term denotes its non-relativistic limit. By using the property $a_{02} = -b_{02}$, we obtain

$$\overline{P}_M^{e_1 e_1} = \overline{P}^{e_1 e_1} \quad . \tag{6.18}$$

Note that, in the case $M = 2$, there isn't the term on the left hand side of eq. (6.18), so that this equation is $\overline{P}^{e_1 e_1} = 0$. In other words, we have energy conservation for eq. $(6.1)_1$. Let us also consider the expression (6.11) of $Q^{i_1 \dots i_r}$, with $r = 0$; by writing explicitly the terms with $q = 0$, $p = 0$ and using the expressions $(6.17)_1$ and (6.15) of P_M and P we obtain:

$$\frac{1}{c^{N+M-3}}Q = -\frac{1}{b_0}\, b_{00}(-2)^{\frac{N+M-1}{2}} \sum_{h=0}^{\frac{M-4}{2}} \binom{\frac{M-2}{2}}{h} (-1)^{h+\frac{M-2}{2}} c^{2h+4-M} \cdot$$

$$\cdot P_M^{e_1 e_1 \ldots e_{\frac{M-2-2h}{2}} e_{\frac{M-2-2h}{2}}} + \frac{1}{b_0}\sum_{q=1}^{\frac{M-2}{2}} b_{q0}(-2)^{\frac{N+M-1-2q}{2}} c^{2-2q} P_M^{e_1 e_1 \ldots e_q e_q} +$$

$$\frac{1}{b_0}\sum_{p=1}^{\frac{N-1}{2}} a_{p0}(-2)^{\frac{N+M-1-2p}{2}} c^{2-2p} P^{e_1 e_1 \ldots e_p e_p} + \frac{1}{b_0} a_{00}(-2)^{\frac{N+M-1}{2}} \cdot$$

$$\cdot \sum_{h=0}^{\frac{N-3}{2}} \binom{\frac{N-1}{2}}{h} (-1)^{h+\frac{N+1}{2}} c^{2h-N+3} P^{e_1 e_1 \ldots e_{\frac{N-1-2h}{2}} e_{\frac{N-1-2h}{2}}}$$

whose non-relativistic limit is

$$0 = -\frac{M-2}{2}\frac{1}{b_0} b_{00}(-2)^{\frac{N+M-1}{2}}(-1)^{M-3}\overline{P}_M^{\,e_1 e_1} + \frac{1}{b_0} b_{10}(-2)^{\frac{N+M-3}{2}}\overline{P}_M^{\,e_1 e_1}$$

$$+\frac{1}{b_0} a_{10}(-2)^{\frac{N+M-3}{2}}\overline{P}^{\,e_1 e_1} + \frac{1}{b_0} a_{00}(-2)^{\frac{N+M-1}{2}} \quad \frac{N-1}{2}(-1)^{N-1}\overline{P}^{\,e_1 e_1},$$

which, by using eq. (6.18), becomes

$$0 = \left[b_{00}(M-2)(-1)^{M-3} + b_{10} + a_{10} + a_{00}(N-1)(-1)^N\right]\overline{P}_M^{\,e_1 e_1}$$

that is

$$0 = \left[(M-2)(-1)^{M-3} + b_{10} - (N-M) - b_{10} - (N-1)(-1)^N\right]\overline{P}^{\,e_1 e_1} = \overline{P}^{\,e_1 e_1}.$$

In this way we have obtained energy conservation for the system (6.1). It remains to prove momentum conservation. To this end, let us consider the expression (6.11) of $Q^{i_1 \ldots i_r}$ with $r = 1$; by writing explicitly the term with $q = 0$ and using the expression $(6.17)_2$ of $P_M^{i_1}$, we obtain

$$Q^{i_1}(-2c)^{\frac{3-N-M}{2}} = \frac{1}{b_1}\sum_{q=1}^{\frac{M-2}{2}} b_{q1}(-2c^2)^{-q} P_M^{i_1 e_1 e_1 \ldots e_q e_q}$$

$$+\frac{1}{b_1}\sum_{p=0}^{\frac{N-3}{2}} a_{p1}(-2c^2)^{-p} P^{i_1 e_1 e_1 \ldots e_p e_p} +$$

$$-\frac{1}{b_1} b_{01}\sum_{h=0}^{\frac{M-4}{2}} \binom{\frac{M-2}{2}}{h} (-1)^{h+\frac{M-2}{2}} c^{2h+2-M} P_M^{i_1 e_1 e_1 \ldots e_{\frac{M-2-2h}{2}} e_{\frac{M-2-2h}{2}}}$$

whose non-relativistic limit is $0 = \frac{1}{b_1} a_{01}\overline{P}^{\,i_1}$; but $a_{01} = -b_{01} = -1$, so that it remains $\overline{P}^{\,i_1} = 0$, i.e. momentum conservation for the system (6.1).

6.2.2 The case with N even and M odd

Eqs. (6.15) and (6.17) still hold, but after exchanging M and N, P and P_M, $P_M^{i_1}$ and P^{i_1}, i.e.,

$$
\begin{aligned}
P_M &= \sum_{h=0}^{\frac{M-3}{2}} \binom{\frac{M-1}{2}}{h} (-1)^{h+\frac{M+1}{2}} c^{2h-M+1} P_M^{e_1 e_1 \ldots e_{\frac{M-1-2h}{2}} e_{\frac{M-1-2h}{2}}} \\
P &= -\sum_{h=0}^{\frac{N-4}{2}} \binom{\frac{N-2}{2}}{h} (-1)^{h+\frac{N-2}{2}} c^{2h+2-N} P^{e_1 e_1 \ldots e_{\frac{N-2-2h}{2}} e_{\frac{N-2-2h}{2}}} \\
P^{i_1} &= -\sum_{h=0}^{\frac{N-4}{2}} \binom{\frac{N-2}{2}}{h} (-1)^{h+\frac{N-2}{2}} c^{2h+2-N} P^{i_1 e_1 e_1 \ldots e_{\frac{N-2-2h}{2}} e_{\frac{N-2-2h}{2}}} \qquad (6.19)
\end{aligned}
$$

The non-relativistic limit of $(6.19)_{2,3}$ can be quickly computed and equals $\overline{P} = 0$, $\overline{P}^{i_1} = 0$, i.e., we have mass and momentum conservation for the system (6.1). It remains to prove energy conservation. Now the passages after eqs. (6.17) and until eq. (6.18), of the previous section, can be adapted also to the present case (there is only to substitute the upper values of q and p with $\frac{M-3}{2}$ and $\frac{N-4}{2}$ respectively), so that eq. (6.18) still holds in the present case. Now the expression of $Q^{i_1 \ldots i_r}$ with r=0, by exploiting the terms with q=0, p=0 and using eqs. $(6.19)_{1,2}$, gives

$$
\begin{aligned}
Q(-2c^2)^{\frac{-N-M+3}{2}} &= \frac{1}{b_0} \sum_{q=1}^{\frac{M-1}{2}} b_{q0} (-2c^2)^{-q+1} P_M^{e_1 e_1 \ldots e_q e_q} + \\
&\frac{1}{b_0} \sum_{p=1}^{\frac{N-2}{2}} a_{p0} (-2c^2)^{-p+1} P^{e_1 e_1 \ldots e_p e_p} - \\
&2\frac{1}{b_0} b_{00} \sum_{h=0}^{\frac{M-3}{2}} \binom{\frac{M-1}{2}}{h} (-1)^{h+\frac{M-1}{2}} c^{2h-M+3} P_M^{e_1 e_1 \ldots e_{\frac{M-1-2h}{2}} e_{\frac{M-1-2h}{2}}} + \\
&2\frac{1}{b_0} a_{00} \sum_{h=0}^{\frac{N-4}{2}} \binom{\frac{N-2}{2}}{h} (-1)^{h+\frac{N-2}{2}} c^{2h-N+4} P^{e_1 e_1 \ldots e_{\frac{N-2-2h}{2}} e_{\frac{N-2-2h}{2}}}
\end{aligned}
$$

whose non-relativistic limit is

$$
\begin{aligned}
0 &= \frac{1}{b_0} \left[b_{10} + a_{10} - 2b_{00} \frac{M-1}{2} (-1)^{M-2} + 2a_{00} \frac{N-2}{2} (-1)^{N-3} \right] \overline{P}^{\,e_1 e_1} = \\
&= \frac{1}{b_0} \left[b_{10} - (N-M) - b_{10} + M - 1 + N - 2 \right] \overline{P}^{\,e_1 e_1} = \frac{1}{b_0} [2M-3] \overline{P}^{\,e_1 e_1}
\end{aligned}
$$

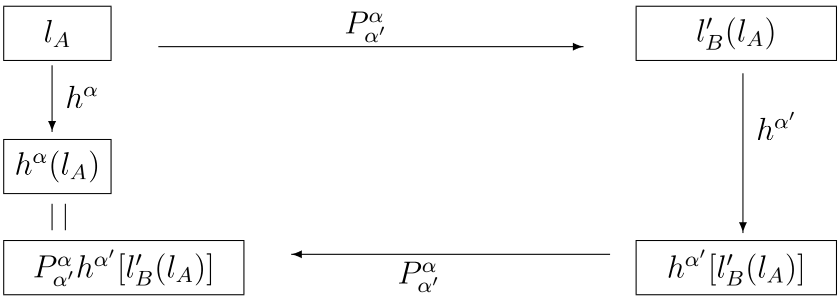
from which $\overline{P}^{\,e_1e_1} = 0$, i.e. we have energy conservation for the system (6.1). In this way all our aims have been accomplished.

6.3 The Einstein's relativity principle

It is well known that, in the relativistic context the relativity principle isn't imposed by separating variables into convective and non convective parts, but by imposing that the costitutive functions satisfy particular conditions; likely to this, the present considerations show that the same results are obtained also in the classical context. The result is achieved by taking the non-relativistic limit of Einstein's Relativity Principle. This fact furnishes further arguments on the naturalness of the work [15]. In particular we will exploit the consequences of the Einstein's relativity principle for the entropy-(entropy flux) tensor h^α and for $A^{\alpha\alpha_2\cdots\alpha_N}$, $B^{\alpha\alpha_2\cdots\alpha_M}$; we will see how they translate into the Galilean's relativity principle when c goes to infinity.

6.3.1 The classical limit of Einstein's relativity principle

Let us denote with l_A a set of independent variables for the system (6.1) and with $l'_B(l_A)$ their expressions after a Lorentz transformation. The Einstein's relativity principle for h^α imposes that both ways in the following diagram give the same result,



i.e. $h^\alpha(l_A) = P^\alpha_{\alpha'} h^{\alpha'}[l'_B(l_A)]$ or, for $\alpha = 0, 1, 2, 3$

$$
\begin{aligned}
m_0^3 h(l_A) &= \gamma m_0^3 h'[l'_B(l_A)] + \gamma \frac{v}{c^2} m_0^3 \phi'^1[l'_B(l_A)] \quad , \\
\frac{m_0^3}{c}\phi^1(l_A) &= \gamma \frac{v}{c} m_0^3 h'[l'_B(l_A)] + \gamma \phi \frac{m_0^3}{c}\phi'^1[l'_B(l_A)] \quad , \\
\frac{m_0^3}{c}\phi^2(l_A) &= \frac{m_0^3}{c}\phi'^2[l'_B(l_A)] \quad , \quad \frac{m_0^3}{c}\phi^3(l_A) = \frac{m_0^3}{c}\phi'^3[l'_B(l_A)]
\end{aligned}
$$

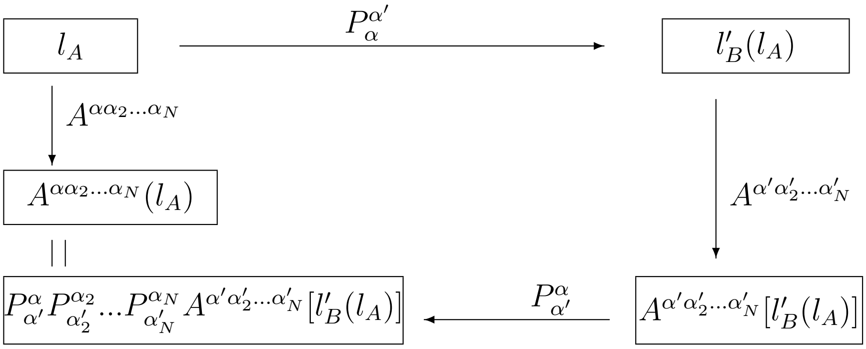
For the sake of simplicity, the particular Lorentz transformation

$$P_{\alpha}^{\alpha'} = \begin{pmatrix} \gamma & \frac{\gamma v}{c} & 0 & 0 \\ \frac{\gamma v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

has been used. If we multiply these relations by m_0^{-3}, $m_0^{-3}c$, $m_0^{-3}c$, $m_0^{-3}c$ respectively, and then take their limits as $c \to \infty$, we obtain

$$h(l_A) = h'[l'_B(l_A)], \quad \psi^k(l_A) = \psi'^k[l'_B(l_A)] \qquad \text{with } \psi^k = \phi^k - hv^k. \tag{6.20}$$

We have now to impose the Einstein's relativity principle also for the functions $A^{\alpha\alpha_2\alpha_N}$ and $B^{\alpha\alpha_2\alpha_M}$; for the first of them it imposes that both ways in the following diagram give the same result



$$\text{that is} \quad A^{\alpha\alpha_2\alpha_N}(l_A) = P_{\alpha'}^{\alpha} P_{\alpha'_2}^{\alpha_2} P_{\alpha'_N}^{\alpha_N} A^{\alpha'\alpha'_2\alpha'_N}[l'_B(l_A)]$$

or, with $\alpha\alpha_2\alpha_N = i_1 \ldots i_r 0 \ldots 0$, and using the identities

$$\begin{aligned} P_{\alpha'}^{\alpha} &= P_0^{\alpha}\delta_{\alpha'}^0 + P_{j_1}^{\alpha}\delta_{\alpha'}^{j_1}, \qquad P_0^i = \gamma\frac{v^i}{c} \\ \text{and} \quad P_{j_1}^{i_1} &= \delta_{j_1}^{i_1} + \frac{\gamma^2}{\gamma+1}\frac{v^{i_1}v_{j_1}}{c^2}, \end{aligned}$$

$$\begin{aligned} A^{i_1\ldots i_r 0\ldots 0}(l_A) = & \left[\gamma\frac{v^{i_1}}{c}\delta_{\alpha'_1}^0 + \left(\delta_{j_1}^{i_1} + \frac{\gamma^2}{\gamma+1}\frac{v^{i_1}v_{j_1}}{c^2}\right)\delta_{\alpha'_1}^{j_1}\right] \cdot \\ & \ldots \\ & \left[\gamma\frac{v^{i_r}}{c}\delta_{\alpha'_r}^0 + \left(\delta_{j_r}^{i_r} + \frac{\gamma^2}{\gamma+1}\frac{v^{i_r}v_{j_r}}{c^2}\right)\delta_{\alpha'_r}^{j_r}\right] \cdot \left[\gamma\delta_{\alpha'_{r+1}}^0 + \frac{\gamma}{c}v_{j_{r+1}}\delta_{\alpha'_{r+1}}^{j_{r+1}}\right] \cdot \\ & \ldots \\ & \left[\gamma\delta_{\alpha'_N}^0 + \frac{\gamma}{c}v_{j_N}\delta_{\alpha'_N}^{j_N}\right] A^{\alpha'_1\ldots\alpha'_r\alpha'_{r+1}\ldots\alpha'_N}[l'_B(l_A)] = \end{aligned}$$

$$\sum_{a=0}^{r}\binom{r}{a}\left(\frac{\gamma}{c}\right)^{a}v^{(i_1}...v^{i_a}\left(\delta^{i_{a+1}}_{j_{a+1}}+\frac{\gamma^2}{\gamma+1}\frac{v^{i_{a+1}}v_{j_{a+1}}}{c^2}\right)...\left(\delta^{i_r)}_{j_r}+\frac{\gamma^2}{\gamma+1}\frac{v^{i_r)}v_{j_r}}{c^2}\right)\cdot$$

$$\gamma^{N-r}\sum_{b=0}^{N-r}\binom{N-r}{b}\left(\frac{1}{c}\right)^{N-r-b}v_{j_{r+b+1}}...v_{j_N}A'^{0...0j_{a+1}...j_r0...0j_{r+b+1}...j_N}$$

from which

$$F_N^{i_1...i_r}(l_A)=\sum_{a=0}^{r}\binom{r}{a}\gamma^{a+N-r}v^{(i_1}...v^{i_a}\left(\delta^{i_{a+1}}_{j_{a+1}}+\frac{\gamma^2}{\gamma+1}\frac{v^{i_{a+1}}v_{j_{a+1}}}{c^2}\right)...$$

$$\left(\delta^{i_r)}_{j_r}+\frac{\gamma^2}{\gamma+1}\frac{v^{i_r)}v_{j_r}}{c^2}\right)\cdot\sum_{b=0}^{N-r}\binom{N-r}{b}c^{-2N+2r+2b}v_{j_{r+b+1}}...v_{j_N}F_N'^{j_{a+1}...j_rj_{r+b+1}...j_N}. \tag{6.21}$$

We can notice that the exponent of c is even and is $-2(N-r-b)\leq 0$; so that, at the limit as $c\to\infty$, only the terms with b=N-r remain, i.e.

$$F_N^{i_1...i_r}(l_A)=\sum_{a=0}^{r}\binom{r}{a}v^{(i_1}...v^{i_a}F_N'^{i_{a+1}\cdots i_r)}$$

$$\text{or, for s=r-a}\quad F_N^{i_1...i_r}(l_A)=\sum_{s=0}^{r}\binom{r}{s}F_N'^{(i_1\cdots i_s}[l'_B(l_A)]v^{i_{s+1}}...v^{i_r)}\quad .$$

This can be written also as

$$F^{i_1...i_r}(l_A)=\sum_{s=0}^{r}X^{i_1\,\cdots\,i_r}_{j_1\,\cdots\,j_s}(\vec{v})F'^{j_1\,\cdots\,j_s}[l'_B(l_A)]\quad\text{for}\quad r=0,\cdots,N-1 \tag{6.22}$$

$$\text{with}\qquad X^{i_1\,\cdots\,i_r}_{j_1\,\cdots\,j_s}=\binom{r}{s}\delta^{(i_1}_{j_1}\cdots\delta^{i_s}_{j_s}v^{i_{s+1}}...v^{i_r)}\quad .$$

Similarly for $B^{\alpha\alpha_2...\alpha_M}$ we find (eq. (6.21) with M instead of N)

$$F_M^{i_1...i_r}(l_A)=\sum_{a=0}^{r}\binom{r}{a}\gamma^{a+M-r}v^{(i_1}...v^{i_a}\left(\delta^{i_{a+1}}_{j_{a+1}}+\frac{\gamma^2}{\gamma+1}\frac{v^{i_{a+1}}v_{j_{a+1}}}{c^2}\right)... \tag{6.23}$$

$$\left(\delta^{i_r)}_{j_r}+\frac{\gamma^2}{\gamma+1}\frac{v^{i_r)}v_{j_r}}{c^2}\right)\cdot\sum_{b=0}^{M-r}\binom{M-r}{b}c^{-2M+2r+2b}v_{j_{r+b+1}}...v_{j_M}F_M'^{j_{a+1}...j_rj_{r+b+1}...j_M}.$$

Let us now consider eqs. (6.10), into which we substitute eq. (6.21) and (6.23) finding $F^{i_1\cdots i_re_1e_1\cdots e_{\frac{N+M-1-2r}{2}}e_{\frac{N+M-1-2r}{2}}}$ in terms of $F_N'^{j_{a+1}...j_rj_{r+b+1}...j_N}$ and $F_M'^{j_{a+1}...j_rj_{r+b+1}...j_M}$.
In the resulting expression, we substitute $F_M'^{j_{a+1}...j_rj_{r+b+1}...j_M}$ firstly obtained

from eqs. (6.10) written in Σ'; in this way we find $F^{i_1\cdots i_r e_1 e_1 \cdots e_{\frac{N+M-1-2r}{2}} e_{\frac{N+M-1-2r}{2}}}$ in terms of $F_N^{'j_{a+1}\dots j_r j_{r+b+1}\dots j_N}$ and of $F^{'i_1\cdots i_r e_1 e_1 \cdots e_{\frac{N+M-1-2r}{2}} e_{\frac{N+M-1-2r}{2}}}$; finally, we do the limit as $c \to \infty$ and we find

$$F^{i_1\cdots i_r e_1 e_1 \cdots e_{\frac{N+M-1-2r}{2}} e_{\frac{N+M-1-2r}{2}}} = \sum_{h=0}^{N-1} Y^{i_1\cdots i_r}_{j_1\cdots j_h}(\vec{v}) F'^{j_1\cdots j_h} +$$

$$\sum_{\eta=r}^{M-1} Z^{i_1\cdots i_r}_{j_1\cdots j_\eta}(\vec{v}) F'^{j_1\cdots j_\eta e_1 e_1\cdots e_{\frac{N+M-1-2\eta}{2}} e_{\frac{N+M-1-2\eta}{2}}} \quad \text{for r= 0,..., M-1,} \qquad (6.24)$$

with

$$Y^{i_1\cdots i_r}_{j_1\cdots j_h} = \sum_{k=sup\{0,h-N-M+1+2r\}}^{inf\{h,r\}} \quad \sum_{p_1=sup\left\{0,h-k-\frac{N+M-1-2r}{2}\right\}}^{\left[\frac{h-k}{2}\right]}$$

$$\binom{r}{k} 2^{h-k-2p_1} \frac{\left(\frac{N+M-1-2r}{2}\right)!}{p_1!(h-k-2p_1)!\left(\frac{N+M-1-2r}{2}+p_1-h+k\right)!}$$

$$\delta^{(i_1}_{(j_1}\cdots\delta^{i_k}_{j_k} v^{i_{k+1}}\cdots v^{i_r)} v_{j_{k+1}}\cdots v_{j_{h-2p_1}} \delta_{j_{h-2p_1+1}j_{h-2p_1+2}}\cdots\delta_{j_{h-1}j_h)}$$

$$(v^2)^{\frac{N+M-1-2r}{2}+p_1-h+k} ,$$

$$Z^{i_1\cdots i_r}_{j_1\cdots j_\eta} = \sum_{k=sup\{0,2r-\eta\}}^{r} \quad \sum_{p_1=sup\left\{0,\frac{N+M-1}{2}-\eta-k+r\right\}}^{\left[\frac{N+M-1-\eta-k}{2}\right]} \binom{r}{k} 2^{M+N-1-\eta-k-2p_1}$$

$$\frac{\left(\frac{N+M-1-2r}{2}\right)!}{p_1!(M+N-1-\eta-k-2p_1)!\left(-r+p_1+k-\frac{N+M-1}{2}+\eta\right)!}$$

$$\delta^{(i_1}_{(j_1}\cdots\delta^{i_k}_{j_k} v^{i_{k+1}}\cdots v^{i_r)} v_{j_{k+1}}\cdots v_{j_{N+M-1-\eta-2p_1}}$$

$$\delta_{j_{N+M-\eta-2p_1}j_{N+M-\eta-2p_1+1}}\cdots\delta_{j_{\eta-1}j_\eta)}(v^2)^{-r+p_1+k-\frac{N+M-1}{2}+\eta} .$$

For the sake of brevity we leave the transformation of the productions which, on the other hand, is obvious. We notice that as independent variables l_A we can take $F^{i_1\dots i_s}$ and $F^{i_1\dots i_r l_1 l_1\dots l_{\frac{N+M-1-2r}{2}} l_{\frac{N+M-1-2r}{2}}}$; in this case eqs. (6.22) and (6.24) give $l_A = l_A(l'_B)$ while the same equations with s+1 and r+1 instead of s and r give conditions on the costitutive functions, besides that for h and ψ^k. Alternatively, we can take as independent variables the lagrange multipliers defined by

$$
\begin{aligned}
dh &= \sum_{s=0}^{N-1} \lambda^*_{i_1 \ldots i_s} dF^{i_1 \ldots i_s} + \sum_{r=0}^{M-1} \mu^*_{i_1 \ldots i_r} dF^{i_1 \ldots i_r e_1 e_1 \ldots e_{\frac{N+M-1-2r}{2}} e_{\frac{N+M-1-2r}{2}}} \\
&= \sum_{h=0}^{N-1} \lambda^{*'}_{j_1 \ldots j_h} dF'^{j_1 \ldots j_h} + \sum_{\eta=0}^{M-1} \mu^{*'}_{j_1 \ldots j_\eta} dF'^{j_1 \ldots j_\eta e_1 e_1 \ldots e_{\frac{N+M-1-2\eta}{2}} e_{\frac{N+M-1-2\eta}{2}}}
\end{aligned}
\tag{6.25}
$$

with

$$
\begin{aligned}
&\lambda^{*'}_{j_1 \ldots j_h} = \sum_{s=h}^{N-1} \lambda^*_{i_1 \ldots i_s} X^{i_1 \ldots i_s}_{j_1 \ldots j_h} + \sum_{r=0}^{M-1} \mu^*_{i_1 \ldots i_r} Y^{i_1 \ldots i_r}_{j_1 \ldots j_h} = \sum_{s=h}^{N-1} \binom{s}{h} \lambda_{j_1 \ldots j_h j_{h+1} \cdots j_s} \\
&v^{j_{h+1}} \ldots v^{j_s} + \sum_{r=0}^{M-1} \sum_{k=sup\{0, h-N-M+1+2r\}}^{inf\{h,r\}} \sum_{p_1 = sup\left\{0, h-k-\frac{N+M-1-2r}{2}\right\}}^{\left[\frac{h-k}{2}\right]} \binom{r}{k} 2^{h-k-2p_1} \\
&\frac{\left(\frac{N+M-1-2r}{2}\right)!}{p_1!(h-k-2p_1)!\left(\frac{N+M-1-2r}{2}+p_1-h+k\right)!} v^{p_{k+1}} \ldots v^{p_r} \mu^*_{p_{k+1} \ldots p_r (j_1 .. j_k} \\
&v_{j_{k+1}} \ldots v_{j_{h-2p_1}} \delta_{j_{h-2p_1+1} j_{h-2p_1+2}} \ldots \delta_{j_{h-1} j_h)} (v^2)^{\frac{N+M-1-2r}{2}+p_1-h+k}
\end{aligned}
\tag{6.26}
$$

and

$$
\begin{aligned}
&\mu^{*'}_{j_1 \ldots j_\eta} = \sum_{r=0}^{\eta} \mu^*_{i_1 \ldots i_r} Z^{i_1 \ldots i_r}_{j_1 \ldots j_\eta} = \sum_{r=0}^{\eta} \sum_{k=sup\{0,2r-\eta\}}^{r} \sum_{p_1 = sup\left\{0, \frac{N+M-1}{2}-\eta-k+r\right\}}^{\left[\frac{N+M-1-\eta-k}{2}\right]} \binom{r}{k} \\
&2^{M+N-1-\eta-k-2p_1} \frac{\left(\frac{N+M-1-2r}{2}\right)!}{p_1!(M+N-1-\eta-k-2p_1)!\left(-r+p_1+k-\frac{N+M-1}{2}+\eta\right)!} \\
&(v^2)^{-r+p_1+k-\frac{N+M-1}{2}+\eta} v^{p_{k+1}} \ldots v^{p_r} \mu^*_{p_{k+1} \ldots p_r (j_1 .. j_k} v_{j_{k+1}} \ldots v_{j_{N+M-1-\eta-2p_1}} \\
&\delta_{j_{N+M-\eta-2p_1} j_{N+M-\eta-2p_1+1}} \ldots \delta_{j_{\eta-1} j_\eta)}
\end{aligned}
\tag{6.27}
$$

In the following subsection we will see what happens if we take these as independent variables.

6.3.2 The Galilean relativity principle in terms of the Lagrange multipliers

From eqs. (6.26) and (6.27) we have that

$$\begin{aligned}
\frac{\partial \lambda^{*'}_{j_1\cdots j_h}}{\partial v_{j_{h+1}}} &= (h+1)\lambda^{*'}_{j_1\cdots j_{h+1}} \quad \text{for } h \le N-2 \\
\frac{\partial \lambda^{*'}_{j_1\cdots j_{N-1}}}{\partial v_j} &= (M-1)\mu^{*'}_{j(j_1\ldots j_{M-2}}\delta_{j_{M-1}j_M}\cdots\delta_{j_{N-2}j_{N-1})} \\
&+ (N-M+1)\mu^{*'}_{(j_1\ldots j_{M-1}}\delta_{j_M j_{M+1}}\cdots\delta_{j_{N-1})j}, \\
\frac{\partial \mu^{*'}_{j_1\cdots j_n}}{\partial v_{j_1}} &= (n-1)\mu^{*'}_{j(j_1\ldots j_{n-2}}\delta_{j_{n-1}j_n)} \\
&+ (N+M+1-2n)\mu^{*'}_{(j_1\ldots j_{n-1}}\delta_{j_n)j} \quad \text{for } 1 \le n \le M-1. \\
\text{while} \quad \frac{\partial \mu^{*'}}{\partial v_j} &= 0 \quad .
\end{aligned}$$

By defining

$$\begin{aligned}
\tilde{h} &= \lambda^*_{i_1\ldots i_s}F^{i_1\ldots i_s} + \mu^*_{i_1\ldots i_r}F^{i_1\ldots i_r e_1 e_1\ldots} - h \\
\tilde{\psi}^k &= \lambda^*_{i_1\ldots i_s}F^{i_1\ldots i_s k} + \mu^*_{i_1\ldots i_r}F^{k i_1\ldots i_r e_1 e_1\ldots} - \psi^k
\end{aligned}$$

we obtain that (6.20) holds also if h and ψ^k are substituted by $\tilde{h}$ and $\tilde{\psi}^k$ respectively, i.e.,

$$\tilde{h}(l_A) = \tilde{h}'[l'_B(l_A)] \quad , \quad \tilde{\psi}^k(l_A) = \tilde{\psi}'^k[l'_B(l_A)].$$

These become identities if calculated for $\underline{v} = 0$ so that they are equivalent to their derivatives with respect to v_j i.e.,

$$\sum_{h=0}^{N-2}(h+1)\frac{\partial \tilde{h}'}{\partial \lambda^{*'}_{j_1\ldots j_h}}\lambda^*_{j_1\ldots j_h j} + \frac{\partial \tilde{h}'}{\partial \lambda^{*'}_{j_1\ldots j_{N-1}}}\Big[(M-1)\mu^{*'}_{j j_1\ldots j_{M-2}}\delta_{j_{M-1}j_M}\cdots$$
$$\delta_{j_{N-2}j_{N-1}} + (N-M+1)\mu^*_{j_1\ldots j_{M-1}}\delta_{j_M j_{M+1}}\cdots\delta_{j_{N-1}}j\Big] + \sum_{r=1}^{M-1}\frac{\partial \tilde{h}'}{\partial \mu^{*'}_{i_1\ldots i_r}}\cdot$$
$$\Big[(r-1)\cdot\mu^{*'}_{j i_1\ldots i_{r-2}}\delta_{i_{r-1}i_r} + (N+M+1-2r)\mu^*_{i_1\ldots i_{r-1}}\delta_{i_r j}\Big] = 0, \qquad (6.28)$$

$$\sum_{h=0}^{N-2}(h+1)\frac{\partial\tilde{\psi}'^k}{\partial\lambda^*_{j_1\cdots j_h}}\lambda^*_{j_1\ldots j_h j}+\frac{\partial\tilde{\psi}'^k}{\partial\lambda^*_{j_1\cdots j_{N-1}}}\Big[(M-1)\mu^{*'}_{jj_1\ldots j_{M-2}}\delta_{j_{M-1}k_M}\cdots$$

$$\delta_{j_{N-2}j_{N-1}}+(N-M+1)\mu^{*'}_{j_1\ldots j_{M-1}}\delta_{j_M k_{M+1}}\cdots\delta_{j_{N-1}j}\Big]+\sum_{r=1}^{M-1}\frac{\partial\tilde{\psi}'^k}{\partial\mu^*_{i_1\ldots i_r}}\cdot$$

$$\Big[(r-1)\cdot\mu^{*'}_{ji_1\ldots i_{r-2}}\delta_{i_{r-1}i_r}+(N+M+1-2r)\mu^*_{i_1\ldots i_{r-1}}\delta_{i_r j}\Big]+h'\delta^k_j=0\,. \quad (6.29)$$

But from eq. (6.25) we also have

$$F^{i_1\ldots i_s}=\frac{\partial\tilde{h}}{\partial\lambda^*_{i_1\ldots i_s}}\quad,\quad F^{i_1\ldots i_r e_1 e_1\ldots e_{\frac{N+M-1-2r}{2}}e_{\frac{N+M-1-2r}{2}}}=\frac{\partial\tilde{h}}{\partial\mu^*_{i_1\ldots i_r}}$$

so that eq. (6.22) and (6.24) follow as a consequence of (6.28). Similarly, if the entropy principle holds, we also have

$$d\phi^k=\lambda^*_{i_1\ldots i_s}dF^{i_1\ldots i_s k}+\mu^*_{i_1\ldots i_r}dF^{ki_1\ldots i_r e_1 e_1\ldots e_{\frac{N+M-1-2r}{2}}e_{\frac{N+M-1-2r}{2}}}$$

from which it follows

$$F^{i_1\ldots i_s k}=\frac{\partial\tilde{\phi}^k}{\partial\lambda^*_{i_1\ldots i_s}}\quad,\quad F^{i_1\ldots i_r e_1 e_1\ldots e_{\frac{N+M-1-2r}{2}}e_{\frac{N+M-1-2r}{2}}}=\frac{\partial\tilde{\phi}^k}{\partial\mu^*_{i_1\ldots i_r}}$$

so that eqs (6.22) and (6.24) with s+1 and r+1 instead of s and r respectively, follow as consequence of eq (6.29). Consequently, only conditions (6.28) and (6.29) have to be imposed.

6.4 The system of balance equations

It has been shown that the non-relativistic limit of Relativistic Extended Thermodynamics suggests to consider the balance equations (6.1), that we write now in the following simpler way

$$\begin{cases}\partial_t F^{i_1\cdots i_n}+\partial_k F^{ki_1\cdots i_n}=P^{i_1\cdots i_n} & \text{for n=0, ..., N,}\\ \partial_t F_*^{i_1\cdots i_r}+\partial_k G^{ki_1\cdots i_r}=Q^{i_1\cdots i_r} & \text{for r=0, ..., M.}\end{cases} \quad (6.30)$$

where N and M are two integers such that $N>M$ and $N+M$ is an odd number. They are evolution equations for moments of order n=0,..., N and suitable traces of some equations for moments of higher order. We want now exploit the Galilean relativity principle for this new system and extend to it the new methodology found by Pennisi and Ruggeri in [15] for a less general

case.
The kinetic counterpart of the variables appearing in this system is

$$
\begin{aligned}
F^{i_1\cdots i_n} &= \int f c^{i_1}\cdots c^{i_n} d\underline{c} \\
F^{ki_1\cdots i_n} &= \int f c^k c^{i_1}\cdots c^{i_n} d\underline{c} \\
F_*^{i_1\cdots i_r} &= \int f c^{i_1}\cdots c^{i_r} (c^2)^{\frac{N+M+1-2r}{2}} d\underline{c} \\
G^{ki_1\cdots i_r} &= \int f c^k c^{i_1}\cdots c^{i_r} (c^2)^{\frac{N+M+1-2r}{2}} d\underline{c}\,, \qquad (6.31)
\end{aligned}
$$

from which we see that

• All the tensors are symmetric,
• $F^{ki_1\cdots i_n}$ for n=0,..., N-1 is differentiated with respect to time in the subsequent equation,
• $F_*^{i_1\cdots i_M} = F^{ki_1\cdots i_N}\delta_{ki_{M+1}}\delta_{i_{M+2}i_{M+3}}\cdots\delta_{i_{N-1}i_N}$,
• $F_*^{i_1\cdots i_r} = G^{ki_1\cdots i_{r+1}}\delta_{ki_{r+1}}$ for r=0,...,M-1, (6.32)
• G^k is completely free.

The last 4 of the above conditions are called compatibility conditions.
From eqs. (6.31) we see also that if we consider two galileanly equivalent frames we have that the transformations for the various tensors are

$$
\begin{aligned}
F^{i_1\cdots i_n} &= \sum_{h=0}^{n} X^{i_1\cdots i_n}_{j_1\cdots j_h}(\underline{v})\, F'^{j_1\cdots j_h} \\
F_*^{i_1\cdots i_r} &= \sum_{h=0}^{N} Y^{i_1\cdots i_r}_{j_1\cdots j_h}(\underline{v})\, F'^{j_1\cdots j_h} + \sum_{p=r}^{M} Z^{i_1\cdots i_r}_{j_1\cdots j_p}(\underline{v})\, F'^{j_1\cdots j_p}_* \\
G^{i_1\cdots i_{r+1}} &= \sum_{h=0}^{N+1} P^{i_1\cdots i_{r+1}}_{j_1\cdots j_h}(\underline{v})\, F'^{j_1\cdots j_h} + \sum_{p=r}^{M} Q^{i_1\cdots i_{r+1}}_{j_1\cdots j_{p+1}}(\underline{v})\, G'^{j_1\cdots j_{p+1}} \quad . \qquad (6.33)
\end{aligned}
$$

with

$$
X^{i_1\cdots i_n}_{j_1\cdots j_h} = \binom{n}{h} \delta^{(i_1}_{j_1}\cdots\delta^{i_h}_{j_h} v^{i_{h+1}}\cdots v^{i_n)} \qquad (6.34)
$$

$$Y^{i_1\cdots i_r}_{j_1\cdots j_h} = \sum_{q_2=sup\left\{h-\frac{N+M+1}{2},0\right\}}^{inf\left\{\left[\frac{h}{2}\right],\frac{N+M+1-2r}{2}\right\}} \sum_{q_1=sup\left\{0,h-\frac{N+M+1-2r}{2}-q_2\right\}}^{inf\{r,h-2q_2\}} \binom{r}{q_1}$$

$$\frac{\left(\frac{N+M+1-2r}{2}\right)!}{q_2!(h-q_1-2q_2)!(q_1+q_2-h+\frac{N+M+1-2r}{2})!}$$

$$(v^2)^{q_1+q_2-h+\frac{N+M+1-2r}{2}} 2^{h-q_1-2q_2} v_{(j_1}\cdots v_{j_{h-q_1-2q_2}}$$

$$\delta_{j_{h-q_1-2q_2+1}j_{h-q_1-2q_2+2}}\cdots\delta_{j_{h-q_1-1}j_{h-q_1}}\delta^{(i_1}_{j_{h-q_1+1}}\cdots\delta^{i_{q_1}}_{j_h)}v^{i_{q_1+1}}\cdots v^{i_r)}$$

$$P^{i_1\cdots i_{r+1}}_{j_1\cdots j_h} = \sum_{q_2=sup\left\{h-\frac{N+M+1}{2}-1,0\right\}}^{inf\left\{\left[\frac{h}{2}\right],\frac{N+M+1-2r}{2}\right\}} \sum_{q_1=sup\left\{0,h-\frac{N+M+1-2r}{2}-q_2\right\}}^{inf\{r+1,h-2q_2\}} \binom{r+1}{q_1}$$

$$\frac{\left(\frac{N+M+1-2r}{2}\right)!}{q_2!(h-q_1-2q_2)!(q_1+q_2-h+\frac{N+M+1-2r}{2})!}$$

$$(v^2)^{q_1+q_2-h+\frac{N+M+1-2r}{2}} 2^{h-q_1-2q_2} v_{(j_1}\cdots v_{j_{h-q_1-2q_2}}$$

$$\delta_{j_{h-q_1-2q_2+1}j_{h-q_1-2q_2+2}}\cdots\delta_{j_{h-q_1-1}j_{h-q_1}}\delta^{(i_1}_{j_{h-q_1+1}}\cdots\delta^{i_{q_1}}_{j_h)}v^{i_{q_1+1}}\cdots v^{i_{r+1})}$$

$$Q^{i_1\cdots i_{r+1}}_{j_1\cdots j_{p+1}} = \sum_{q_2=\frac{N+M+1}{2}-p}^{inf\left\{\left[\frac{N+M+2-p}{2}\right],\frac{N+M+1-2r}{2}\right\}} \sum_{q_1=sup\left\{0,\frac{N+M+1}{2}-p-q_2+1+r\right\}}^{inf\{r+1,N+M+2-p-2q_2\}} \binom{r+1}{q_1}$$

$$\frac{\left(\frac{N+M+1-2r}{2}\right)!}{q_2!(N+M+2-p-q_1-2q_2)!(q_1+q_2+p-\frac{N+M+3+2r}{2})!}$$

$$(v^2)^{q_1+q_2+p-\frac{N+M+3+2r}{2}} 2^{N+M+2-p-q_1-2q_2} v_{(j_1}\cdots v_{j_{N+M+2-p-q_1-2q_2}}$$

$$\delta_{j_{N+M+3-p-q_1-2q_2}j_{N+M+4-p-q_1-2q_2}}\cdots\delta_{j_{p-q_1}j_{p-q_1+1}}$$

$$\delta^{(i_1}_{j_{p-q_1+2}}\cdots\delta^{i_{q_1}}_{j_{p+1})}v^{i_{q_1+1}}\cdots v^{i_{r+1})}$$

$$Z^{i_1\cdots i_r}_{j_1\cdots j_p} = \sum_{q_2=\frac{N+M+1}{2}-p}^{inf\left\{\left[\frac{N+M+1-p}{2}\right],\frac{N+M+1-2r}{2}\right\}} \sum_{q_1=sup\left\{0,\frac{N+M+1}{2}-p-q_2+r\right\}}^{inf\{r,N+M+1-p-2q_2\}} \binom{r}{q_1}$$

$$\frac{\left(\frac{N+M+1-2r}{2}\right)!}{q_2!(N+M+1-p-q_1-2q_2)!(q_1+q_2+p-\frac{N+M+1+2r}{2})!}$$

$$(v^2)^{q_1+q_2+p-\frac{N+M+1+2r}{2}} 2^{N+M+1-p-q_1-2q_2} v_{(j_1}\cdots v_{j_{N+M+1-p-q_1-2q_2}}$$

$$\delta_{j_{N+M+2-p-q_1-2q_2}j_{N+M+3-p-q_1-2q_2}}\cdots\delta_{j_{p-q_1-1}j_{p-q_1}}$$

$$\delta^{(i_1}_{j_{p-q_1+1}}\cdots\delta^{i_{q_1}}_{j_p)}v^{i_{q_1+1}}\cdots v^{i_r)}.$$

In order to find the tensors (6.34) let us firstly consider the tensor

$$
\begin{aligned}
&H^{i_1\cdots i_r} = \int f c^{i_1}\cdots c^{i_r}(c^2)^\alpha d\underline{c} = \\
&= \int f(c'^{i_1}+v^{i_1})\cdots(c'^{i_r}+v^{i_r})(c'^2+2c'^i v_i+v^2)^\alpha d\underline{c}' = \\
&\sum_{q_1=0}^{r}\sum_{q_2+q_3\le\alpha}\binom{r}{q_1}\int f c'^{(i_1}\cdots c'^{i_{q_1}}v^{i_{q_1+1}}\cdots v^{i_r)}\frac{\alpha!}{q_2!q_3!\,(\alpha-q_2-q_3)!} \\
&(c'^2)^{q_2}2^{q_3}c'^{j_1}\cdots c'^{j_{q_3}}v_{j_1}\cdots v_{j_{q_3}}(v^2)^{\alpha-q_2-q_3}d\underline{c}' \\
&= \sum_{(q_1,q_2,q_3)\in\Delta}\binom{r}{q_1}\frac{\alpha!}{q_2!q_3!\,(\alpha-q_2-q_3)!}2^{q_3} \\
&(v^2)^{\alpha-q_2-q_3}H'^{e_1e_1\cdots e_{q_2}e_{q_2}j_1\cdots j_{q_3}(i_1\cdots i_{q_1}}v^{i_{q_1+1}}\cdots v^{i_r)}v_{j_1}\cdots v_{j_{q_3}}
\end{aligned}
\tag{6.35}
$$

where we have used the relation $c^i = c'^i + v^i$ between galileanly equivalent frames, also the binomial and trinomial rules for powers; moreover, $\sum_{(q_1,q_2,q_3)\in\Delta}$ means that the summation have to be done with respect to every tern of indexes (q_1,q_2,q_3) belonging to the set

$$\Delta = \left\{(q_1,q_2,q_3)\,:\,0\le q_1\le r\,,\,0\le q_2\,,\,0\le q_3\,,\,q_2+q_3\le\ \alpha\right\}.$$

With the following change of index from q_3 to h, defined by $q_3 = h-q_1-2q_2$, the set Δ converts into

$$\Delta' = \left\{(q_1,q_2,h)\,:\,0\le q_1\le r\,,\,0\le q_2\,,\,q_1+2q_2\le h\,,\,h-q_1-q_2\le\alpha\right\}.$$

Let us now transform suitably Δ'. The 1^{th}, 3^{th} and 4^{th} inequalities defining it are:

$$0\le q_1\le r,\qquad q_1\le h-2q_2,\qquad h-q_2-\alpha\le q_1. \tag{6.36}$$

The compatibilities between 1^{th} and 2^{th}, 1^{th} and 3^{th}, 2^{th} and 3^{th} of these are

$$0\le h-2q_2,\quad h-q_2-\alpha\le r,\quad h-q_2-\alpha\le h-2q_2,$$

or, by adding also the remaining 2^{th} inequality which defines Δ',

$$q_2\le\left[\frac{h}{2}\right],\qquad h-\alpha-r\le q_2,\qquad q_2\le\alpha,\qquad 0\le q_2. \tag{6.37}$$

After that, eqs. (6.36) become

$$sup\,\{0,h-q_2-\alpha\}\le q_1\le inf\,\{r,h-2q_2\}\,.$$

The compatibility conditions between eqs. (6.37) are

$$h-\alpha-r\leq\left[\frac{h}{2}\right],\qquad 0\leq h,\qquad h-\alpha-r\leq\alpha\,.$$

The first of these is a consequence of the others; in fact, from the $3^{\rm th}$ one we have $h\leq 2\alpha+r\leq 2\alpha+2r$ from which the $1^{\rm th}$ one follows when h is even. In the other case, h odd, we have still $h\leq 2\alpha+2r$, as above, but it implies $h\leq 2\alpha+2r-1$ because h is odd; therefore the $1^{\rm th}$ follows also in this case. Therefore, of the above relations it remains $0\leq h\leq 2\alpha+r$.
After that eqs. (6.37) become

$$sup\,\{0,h-\alpha-r\}\leq q_2\leq inf\left\{\left[\frac{h}{2}\right],\alpha\right\}.$$

Consequently, $\displaystyle\sum_{(q_1,q_2,q_3)\in\Delta}$ becomes $\displaystyle\sum_{h=0}^{2\alpha+r}\ \sum_{q_2=sup\{0,h-\alpha-r\}}^{inf\left\{\left[\frac{h}{2}\right],\alpha\right\}}\ \sum_{q_1=sup\{0,h-q_2-\alpha\}}^{inf\{r,h-2q_2\}}$.

Let now β and M be arbitrary integers such that $0\leq\beta\leq 2\alpha+r$. We can split $\sum_{h=0}^{2\alpha+r}$ in $\sum_{h=0}^{\beta}$ and in $\sum_{h=\beta+1}^{2\alpha+r}$. In the first of these we use the change of index (from h to p) defined by $h=\beta+M+1-p$ and it becomes

$$\sum_{p=\beta+M+1-2\alpha-r}^{M}\ \sum_{q_2=sup\{0,\beta+M+1-p-\alpha-r\}}^{inf\left\{\left[\frac{\beta+M+1-p}{2}\right],\alpha\right\}}\ \sum_{q_1=sup\{0,\beta+M+1-p-\alpha-q_2\}}^{inf\{r,\beta+M+1-p-2q_2\}}$$

Consequently, eq. (6.35) becomes

$$\begin{aligned}
H^{i_1\cdots i_r} &= \sum_{h=0}^{\beta}\ \sum_{q_2=sup\{0,h-\alpha-r\}}^{inf\left\{\left[\frac{h}{2}\right],\alpha\right\}}\ \sum_{q_1=sup\{0,h-q_2-\alpha\}}^{inf\{r,h-2q_2\}}\binom{r}{q_1}\\
&\frac{\alpha!}{q_2!(h-q_1-2q_2)!\,(\alpha+q_2-h+q_1)!}2^{h-q_1-2q_2}(v^2)^{\alpha+q_2-h+q_1}\\
&H'^{e_1e_1\cdots e_{q_2}e_{q_2}j_1\cdots j_{h-q_1-2q_2}(i_1\cdots i_{q_1}}v^{i_{q_1+1}}\cdots v^{i_r)}v_{j_1}\cdots v_{j_{h-q_1-2q_2}}\\
&+\sum_{p=\beta+M+1-2\alpha-r}^{M}\ \sum_{q_2=sup\{0,\beta+M+1-p-\alpha-r\}}^{inf\left\{\left[\frac{\beta+M+1-p}{2}\right],\alpha\right\}}\ \sum_{q_1=sup\{0,\beta+M+1-p-\alpha-q_2\}}^{inf\{r,\beta+M+1-p-2q_2\}}\binom{r}{q_1}\\
&\frac{\alpha!}{q_2!(\beta+M+1-p-q_1-2q_2)!\,(\alpha+q_2+q_1-\beta-M-1+p)!}\\
&2^{\beta+M+1-p-q_1-2q_2}\,H'^{e_1e_1\cdots e_{q_2}e_{q_2}j_1\cdots j_{\beta+M+1-p-q_1-2q_2}(i_1\cdots i_{q_1}}v^{i_{q_1+1}}\cdots v^{i_r)}\\
&v_{j_1}\cdots v_{j_{\beta+M+1-p-q_1-2q_2}}(v^2)^{\alpha+q_2+q_1+p-\beta-M-1}\,. \qquad (6.38)
\end{aligned}$$

From the definition (6.35) of $H^{i_1\cdots i_r}$ and from (6.38),

- For $\alpha = 0$, $r = n \in [0, N]$, $\beta = n$, we obtain $(6.33)_1$ with $(6.34)_1$.
- For $\alpha = 0$, $r = N+1$, $\beta = N+1$, we obtain $(6.33)_1$ for $n = N+1$, with $(6.34)_1$.
- For $\alpha = \frac{N+M+1-2r}{2}$, $r \in [0, M]$, $\beta = N$, we obtain $(6.33)_2$ with $(6.34)_{2,5}$.
- Let us firstly write (6.35) and (6.38) with $r+1$ instead of r; after that, substitute $\alpha = \frac{N+M+1-2r}{2}$, $r \in [0, M]$, $\beta = N+1$, so obtaining $(6.33)_3$ with $(6.34)_{3,4}$.

This results are very interesting: if we call I the variables occurring in eq. (6.30), and I' their counterparts in the other frame, we have found that I are expressed in terms of I' and no other moment has slipped in their relation! This fact confirms that our equations are the physically correct ones.
From eqs. (6.33) the following properties hold

$$
\begin{aligned}
F^{ki_1\cdots i_n} - v^k F^{i_1\cdots i_n} &= \sum_{h=0}^{n} X^{i_1\cdots i_n}_{j_1\cdots j_h}(\underline{v}) F'^{kj_1\cdots j_h} \\
G^{ki_1\cdots i_r} - v^k F_*^{i_1\cdots i_r} &= \sum_{h=0}^{N} Y^{i_1\cdots i_r}_{j_1\cdots j_h}(\underline{v}) F'^{kj_1\cdots j_h} + \sum_{p=r}^{M} Z^{i_1\cdots i_r}_{j_1\cdots j_p}(\underline{v}) G'^{kj_1\cdots j_p}
\end{aligned} \tag{6.39}
$$

$$
\frac{\partial}{\partial v^j} X^{i_1\cdots i_n}_{j_1\cdots j_h} = \begin{cases} 0 & \text{for n=h,} \\ (h+1) X^{i_1\cdots i_n}_{j_1\cdots j_h j} & \text{for n=h+1,...,N.} \end{cases}
$$

$$
\frac{\partial}{\partial v^j} Y^{i_1\cdots i_r}_{j_1\cdots j_h} = \begin{cases} (h+1) Y^{i_1\cdots i_r}_{j_1\cdots j_h j} & \text{for } h = 0, ..., N-1, \\ (N+1) Z^{i_1\cdots i_r}_{(j_1\cdots j_M} \delta_{j_{M+1} j_{M+2}} \cdots \delta_{j_{N-2} j_{N-1}} \delta_{j_N j)} & \text{for } h = N. \end{cases}
$$

$$
\frac{\partial}{\partial v^j} Z^{i_1\cdots i_r}_{j_1\cdots j_p} = \begin{cases} 0 & \text{for r=p,} \\ (p-1)\delta_{(j_1 j_2} Z^{i_1\cdots i_r}_{j_3\cdots j_p)j} + (N+M+3-2p) & Z^{i_1\cdots i_r}_{(j_1\cdots j_{p-1}} \delta_{j_p)j} \\ & \text{for r=0, ... , p-1} \end{cases}
$$

The entropy principle for our system (6.30), with usual passages, is equivalent to assume the existence of Lagrange Multipliers $\lambda_{i_1\cdots i_n}$ and $\mu_{i_1\cdots i_r}$ such that

$$
\begin{aligned}
dh &= \lambda_{i_1\cdots i_n} dF^{i_1\cdots i_n} + \mu_{i_1\cdots i_r} dF_*^{i_1\cdots i_r} \\
d\phi^k &= \lambda_{i_1\cdots i_n} dF^{ki_1\cdots i_n} + \mu_{i_1\cdots i_r} dG^{ki_1\cdots i_r}
\end{aligned}
$$

plus a residual inequality.

6.4.1 The first method

Let's impose now the galilean relativity principle to our system by using the classical methodology. We have to decompose the quantities into their convective and non-convective parts. In order to do it we define

$$v^i = \frac{F^i}{F} \tag{6.40}$$

and we use eq. (6.33), where the quantity $\underline{v}$ is defined in (6.40) and isn't more the relative velocity between two frames. Moreover $F^{'\cdots}$, $F_*^{'\cdots}$ and $G^{'\cdots}$ are all non-convective quantities.
From eqs. $(6.33)_1$ and (6.40) we have that

$$F^{'j_1} = 0. \tag{6.41}$$

The decomposition of h and ϕ^k is

$$h = h' \qquad\qquad \phi^k = \phi^{'k} + hv^k\,, \tag{6.42}$$

where h' and $\phi^{'k}$ are non-convective quantities, i.e. they don't depend on velocity.
By substituting eqs. $(6.31)_{1,2}$ into eq. $(6.39)_1$, we obtain

$$dh = \sum_{n=0}^{N} \lambda_{i_1\cdots i_n} \sum_{h=0}^{n} X^{i_1\cdots i_n}_{j_1\cdots j_h} dF^{'j_1\cdots j_h} + dv^j \left\{ \sum_{n=0}^{N} \lambda_{i_1\cdots i_n} \sum_{h=0}^{n} \left(\frac{\partial}{\partial v_j} X^{i_1\cdots i_n}_{j_1\cdots j_h} \right) \right.$$
$$F^{'j_1\cdots j_h} + \sum_{r=0}^{M} \mu_{i_1\cdots i_r} \left[\sum_{h=0}^{N} \left(\frac{\partial}{\partial v_j} Y^{i_1\cdots i_r}_{j_1\cdots j_h} \right) F^{'j_1\cdots j_h} + \sum_{p=r}^{M} \left(\frac{\partial}{\partial v_j} Z^{i_1\cdots i_r}_{j_1\cdots j_p} \right) \right.$$
$$\left.\left. F_*^{'j_1\cdots j_p} \right] \right\} + \sum_{r=0}^{M} \mu_{i_1\cdots i_r} \left\{ \sum_{h=0}^{N} Y^{i_1\cdots i_r}_{j_1\cdots j_h} dF^{'j_1\cdots j_h} + \sum_{p=r}^{M} Z^{i_1\cdots i_r}_{j_1\cdots j_p} dF_*^{'j_1\cdots j_p} \right\} \tag{6.43}$$

The galilean relativity principle imply that the coefficient of dv^j must be equal to zero; so, by exchanging the order of summations, it remains

$$dh = \sum_{h=0}^{N} \left[\sum_{n=h}^{N} \lambda_{i_1\cdots i_n} X^{i_1\cdots i_n}_{j_1\cdots j_h} + \sum_{r=0}^{M} \mu_{i_1\cdots i_r} Y^{i_1\cdots i_r}_{j_1\cdots j_h} \right] dF^{'j_1\cdots j_h} +$$
$$\sum_{p=0}^{M} \sum_{r=0}^{p} \mu_{i_1\cdots i_r} Z^{i_1\cdots i_r}_{j_1\cdots j_p} dF_*^{'j_1\cdots j_p},$$

Similarly, by substituting eqs. $(6.32)_{1,2}$ in $(6.39)_2$ and by using also eq. $(6.42)_2$, eq. $(6.39)_2$ becomes

$$d\phi'^k + \underline{v^k dh} + h dv^k = \sum_{n=0}^{N} \lambda_{i_1\cdots i_n} \left[\underline{v^k dF^{i_1\cdots i_n}} + F^{i_1\cdots i_n} dv^k + \right.$$
$$\left. \sum_{h=0}^{n} X^{i_1\cdots i_n}_{j_1\cdots j_h} dF'^{kj_1\cdots j_h} \right] + \sum_{r=0}^{M} \mu_{i_1\cdots i_r} \left[\underline{v^k dF_*^{i_1\cdots i_r}} + F_*^{i_1\cdots i_r} dv^k + \right.$$
$$\left. \sum_{h=0}^{N} Y^{i_1\cdots i_r}_{j_1\cdots j_h} dF'^{kj_1\cdots j_h} + \sum_{p=r}^{M} Z^{i_1\cdots i_r}_{j_1\cdots j_p} dG'^{kj_1\cdots j_p} \right] +$$
$$dv^j \left\{ \sum_{n=0}^{N} \lambda_{i_1\cdots i_n} \sum_{h=0}^{n} \frac{\partial X^{i_1\cdots i_n}_{j_1\cdots j_h}}{\partial v_j} F'^{kj_1\cdots j_h} + \sum_{r=0}^{M} \mu_{i_1\cdots i_r} \cdot \right.$$
$$\left. \left[\sum_{h=0}^{N} \frac{\partial Y^{i_1\cdots i_r}_{j_1\cdots j_h}}{\partial v_j} F'^{kj_1\cdots j_h} + \sum_{p=r}^{M} \frac{\partial Z^{i_1\cdots i_r}_{j_1\cdots j_p}}{\partial v_j} G'^{kj_1\cdots j_p} \right] \right\}.$$

It follows

$$\begin{aligned} dh &= \lambda'_{j_1\cdots j_h} dF'^{j_1\cdots j_h} + \mu'_{j_1\cdots j_p} dF_*'^{j_1\cdots j_p} \\ d\phi'^k &= \lambda'_{j_1\cdots j_h} dF'^{kj_1\cdots j_h} + \mu'_{j_1\cdots j_p} dG'^{kj_1\cdots j_p}, \end{aligned} \tag{6.44}$$

$$\sum_{h=0}^{N-1} (h+1)\lambda'_{j_1\cdots j_h j} F'^{j_1\cdots j_h} + (N+1)\mu'_{(j_1\cdots j_M} \delta_{j_{M+1}j_{M+2}} \cdots \delta_{j_N j_{N+1})} F'^{j_1\cdots j_N} \delta_j^{j_{N+1}} \cdot$$
$$+ \sum_{p=1}^{M} \left[(p-1)\mu'_{jj_1\cdots j_{p-2}} \delta_{j_{p-1}j_p} + (N+M+3-2p)\mu'_{j_1\cdots j_{p-1}} \delta_{j_p j} \right] F_*'^{j_1\cdots j_p} = 0$$

$$\sum_{h=0}^{N-1} (h+1)\lambda'_{j_1\cdots j_h j} F'^{j_1\cdots j_h k} + (N+1)\mu'_{(j_1\cdots j_M} \delta_{j_{M+1}j_{M+2}} \cdots \delta_{j_N j_{N+1})} \cdot$$
$$F'^{j_1\cdots j_N k} \delta_j^{j_{N+1}} + \sum_{p=1}^{M} \left[(p-1)\mu'_{jj_1\cdots j_{p-2}} \delta_{j_{p-1}j_p} + (N+M+3-2p)\mu'_{j_1\cdots j_{p-1}} \delta_{j_p j} \right] \cdot$$
$$G'^{j_1\cdots j_p k} + \left[\sum_{n=0}^{N} \lambda'_{i_1\cdots i_n} F'^{i_1\cdots i_n} + \sum_{r=0}^{M} \mu'_{i_1\cdots i_r} F_*'^{i_1\cdots i_r} - h \right] \delta^{kj} = 0, \tag{6.45}$$

$$\begin{aligned} \text{with} \quad \lambda'_{j_1\cdots j_h} &= \sum_{n=h}^{N} \lambda_{i_1\cdots i_n} X^{i_1\cdots i_n}_{j_1\cdots j_h} + \sum_{r=0}^{M} \mu_{i_1\cdots i_r} Y^{i_1\cdots i_r}_{j_1\cdots j_h}, \\ \mu'_{j_1\cdots j_p} &= \sum_{r=0}^{p} \mu_{i_1\cdots i_r} Z^{i_1\cdots i_r}_{j_1\cdots j_p}. \end{aligned} \tag{6.46}$$

Eqs. (6.45) express the condition that h and ϕ'^k don' t depend on the velocity v^j. From eq. (6.44), in the independent variables $F'^{j_1\cdots j_h}$ and $F_*'^{j_1\cdots j_p}$, we find

$$\lambda'_{j_1\cdots j_h} = \frac{\partial h}{\partial F'^{j_1\cdots j_h}} \quad \text{for } h \neq 1, \qquad \mu'_{j_1\cdots j_p} = \frac{\partial h}{\partial F_*'^{j_1\cdots j_p}}$$

which imply that $\lambda'_{j_1\cdots j_h}$, for $h \neq 1$, and $\mu'_{j_1\cdots j_p}$ are non-convective quantities. Eq. $(6.46)_1$ for h=1 defines λ'_{j_1}. But we see in eq. $(6.45)_1$ that $\lambda'_j F'$ appears for $h = 0$ and can be obtained from this equation in terms of quantities which are already proved to be non-convective, so that also λ'_j is non-convective too. Let's define now $\widetilde{h}$ and $\widetilde{\phi}^k$ from

$$\begin{aligned} h &= -\widetilde{h} + \lambda'_{j_1\cdots j_h} F'^{j_1\cdots i_h} + \mu'_{j_1\cdots j_p} F_*'^{j_1\cdots j_p} \\ \phi'^k &= -\widetilde{\phi}^k + \lambda'_{j_1\cdots j_h} F'^{j_1\cdots i_h k} + \mu'_{j_1\cdots j_p} G'^{j_1\cdots j_p k}. \end{aligned}$$

Eqs. (6.44), with v_j, λ', $\lambda'_{j_1 j_2}$, ..., $\lambda'_{j_1\cdots j_N}$, and $\mu'_{j_1\cdots j_p}$ as independent variables

$$\begin{aligned} \text{becomes} \quad d\widetilde{h} &= F'^{j_1\cdots i_h} d\lambda'_{j_1\cdots j_h} + F_*'^{j_1\cdots j_p} d\mu'_{j_1\cdots j_p} \\ d\widetilde{\phi}^k &= F'^{j_1\cdots i_h k} d\lambda'_{j_1\cdots j_h} + G'^{j_1\cdots j_p k} d\mu'_{j_1\cdots j_p}, \end{aligned}$$

which, taking into account also eq. (6.41), are equivalent to

$$F' = \frac{\partial \widetilde{h}}{\partial \lambda'} \qquad \frac{\partial \widetilde{\phi}^k}{\partial \lambda'} = F'^{jk} \frac{\partial \lambda'_j}{\partial \lambda'} \tag{6.47}$$

$$F'^{j_1\cdots j_n} = \frac{\partial \widetilde{h}}{\partial \lambda'_{j_1\cdots j_n}} \qquad \frac{\partial \widetilde{\phi}^k}{\partial \lambda'_{j_1\cdots j_n}} = F'^{jk} \frac{\partial \lambda'_j}{\partial \lambda'_{j_1\cdots j_n}} + F'^{j_1\cdots j_n k} \qquad \text{for } n = 2, \cdots, N$$

$$F_*'^{j_1\cdots j_p} = \frac{\partial \widetilde{h}}{\partial \mu'_{j_1\cdots j_p}} \qquad \frac{\partial \widetilde{\phi}^k}{\partial \mu'_{j_1\cdots j_p}} = F'^{jk} \frac{\partial \lambda'_j}{\partial \mu'_{j_1\cdots j_p}} + G'^{j_1\cdots j_p k} \qquad \text{for p=0,...,M.}$$

By using the above equations, the conditions (6.32) and eqs. (6.45) become

$$\frac{\partial \widetilde{\phi}^k}{\partial \lambda'} = \frac{\partial \widetilde{h}}{\partial \lambda'_{jk}} \frac{\partial \lambda'_j}{\partial \lambda'}$$

$$\frac{\partial \widetilde{\phi}^k}{\partial \lambda'_{j_1\cdots j_h}} = \frac{\partial \widetilde{h}}{\partial \lambda'_{jk}} \frac{\partial \lambda'_j}{\partial \lambda'_{j_1\cdots j_h}} + \frac{\partial \widetilde{h}}{\partial \lambda'_{i_1\cdots i_h k}} \quad \text{for h=2,...,N-1}$$

$$\frac{\partial \widetilde{h}}{\partial \mu'_{j_1\cdots j_M}} = \left(\frac{\partial \widetilde{\phi}^k}{\partial \lambda'_{j_1\cdots j_N}} - \frac{\partial \widetilde{h}}{\partial \lambda'_{jk}} \frac{\partial \lambda'_j}{\partial \lambda'_{j_1\cdots j_N}} \right) \delta_{k j_{M+1}} \delta_{j_{M+2} j_{M+3}} \cdots \delta_{j_{N-1} j_N}$$

$$\frac{\partial \widetilde{h}}{\partial \mu'_{j_1\cdots j_r}} = \left(\frac{\partial \widetilde{\phi}^k}{\partial \mu'_{j_1\cdots j_{r+1}}} - \frac{\partial \widetilde{h}}{\partial \lambda'_{jk}} \frac{\partial \lambda'_j}{\partial \mu'_{j_1\cdots j_{r+1}}} \right) \delta_{k j_{r+1}} \quad \text{for r=0,...,M-1}$$

$$\frac{\partial \widetilde{\phi}^{[k}}{\partial \lambda'_{j_1]\cdots j_N}} = \frac{\partial \widetilde{h}}{\partial \lambda'_{j[k}} \frac{\partial \lambda'_j}{\partial \lambda'_{j_1]\cdots j_N}}$$

$$\frac{\partial \widetilde{\phi}^{[k}}{\partial \mu'_{j_1]\cdots j_p}} = \frac{\partial \widetilde{h}}{\partial \lambda'_{j[k}} \frac{\partial \lambda'_j}{\partial \mu'_{j_1]\cdots j_p}} \quad \text{for p=0,...,M}$$

$$\sum_{h=2}^{N-1}(h+1)\lambda'_{j_1\cdots j_h j}\frac{\partial \widetilde{h}}{\partial \lambda'_{j_1\cdots j_h}} + \lambda'_j \frac{\partial \widetilde{h}}{\partial \lambda'} +$$

$$+(N+1)\mu'_{(j_1\cdots j_M}\delta_{j_{M+1}j_{M+2}}\cdots\delta_{j_N j_{N+1})}\frac{\partial \widetilde{h}}{\partial \lambda'_{j_1\cdots j_N}}\delta_j^{j_{N+1}} +$$

$$+\sum_{p=1}^{M}\left[(p-1)\mu'_{jj_1\cdots j_{p-2}}\delta_{j_{p-1}j_p} + (N+M+3-2p)\mu'_{j_1\cdots j_{p-1}}\delta_{j_p j}\right]\frac{\partial \widetilde{h}}{\partial \mu'_{j_1\cdots j_p}} = 0$$

$$\sum_{h=2}^{N-1}(h+1)\lambda'_{j_1\cdots j_h j}\left(\frac{\partial \widetilde{\phi}^k}{\partial \lambda'_{j_1\cdots j_h}} - \frac{\partial \widetilde{h}}{\partial \lambda'_{jk}}\frac{\partial \lambda'_j}{\partial \lambda'_{j_1\cdots j_h}}\right) + 2\lambda'_{j_1 j}\frac{\partial \widetilde{h}}{\partial \lambda'_{j_1 k}} +$$

$$+(N+1)\mu'_{(j_1\cdots j_M}\delta_{j_{M+1}j_{M+2}}\cdots\delta_{j_N j_{N+1})}\left(\frac{\partial \widetilde{\phi}^k}{\partial \lambda'_{j_1\cdots j_N}} - \frac{\partial \widetilde{h}}{\partial \lambda'_{jk}}\frac{\partial \lambda'_j}{\partial \lambda'_{j_1\cdots j_N}}\right)\delta_j^{j_{N+1}} +$$

$$+\sum_{p=1}^{M}\left[(p-1)\mu'_{jj_1\cdots j_{p-2}}\delta_{j_{p-1}j_p} + (N+M+3-2p)\mu'_{j_1\cdots j_{p-1}}\delta_{j_p j}\right]\cdot$$

$$\cdot\left(\frac{\partial \widetilde{\phi}^k}{\partial \mu'_{j_1\cdots j_p}} - \frac{\partial \widetilde{h}}{\partial \lambda'_{jk}}\frac{\partial \lambda'_j}{\partial \mu'_{j_1\cdots j_p}}\right) + \widetilde{h}\delta^{kj} = 0 \qquad (6.48)$$

So we have to find the functions $\widetilde{h}$, $\widetilde{\phi}^k$ and λ'_j depending on λ', $\lambda'_{j_1 j_2}$, ... , $\lambda'_{j_1\cdots j_N}$, $\mu'_{j_1\cdots j_p}$ subject to the above restrictions. After that the costitutive functions are given by $(6.47)_4$ for $n = N$ and by $(6.47)_6$.

6.4.2 The second method

We want now to simplify eqs. (6.48), by extending to our balance equations the new method already showed in the previous chapters for less general cases. To this end, let us consider the following mathematical problem: Find the functions H and H^k of the variables $\lambda'_{j_1\cdots j_h}$ with h=0,...,N and $\mu_{j_1\cdots j_p}$

with p=0,...,M subject to the following restrictions:

$$\frac{\partial H^k}{\partial \lambda'_{j_1\cdots j_h}} = \frac{\partial H}{\partial \lambda'_{j_1\cdots j_h k}} \quad \text{for h=0,...,N-1}$$

$$\frac{\partial H}{\partial \mu'_{j_1\cdots j_M}} = \frac{\partial H^k}{\partial \lambda'_{j_1\cdots j_N}} \delta_{k j_{M+1}} \delta_{j_{M+2} j_{M+3}} \cdots \delta_{j_{N-1} j_N}$$

$$\frac{\partial H}{\partial \mu'_{j_1\cdots j_r}} = \frac{\partial H^k}{\partial \mu'_{j_1\cdots j_{r+1}}} \delta_{k j_{r+1}} \quad \text{for r=0,...,M-1}$$

$$\frac{\partial H^{[k}}{\partial \lambda'_{j_1]\cdots j_N}} = 0$$

$$\frac{\partial H^{[k}}{\partial \mu'_{j_1]\cdots j_p}} = 0 \tag{6.49}$$

$$\sum_{h=0}^{N-1} (h+1)\lambda'_{j_1\cdots j_h j} \frac{\partial H}{\partial \lambda'_{j_1\cdots j_h}} + (N+1)\mu'_{(j_1\cdots j_M} \delta_{j_{M+1} j_{M+2}} \cdots \delta_{j_N j_{N+1})} \frac{\partial H}{\partial \lambda'_{j_1\cdots j_N}} \delta_j^{j_{N+1}}$$
$$+ \sum_{p=1}^{M} \left[(p-1)\mu'_{j j_1 \cdots j_{p-2}} \delta_{j_{p-1} j_p} + (N+M+3-2p)\mu'_{j_1\cdots j_{p-1}} \delta_{j_p j} \right] \frac{\partial H}{\partial \mu'_{j_1\cdots j_p}} = 0$$

$$\sum_{h=0}^{N-1} (h+1)\lambda'_{j_1\cdots j_h j} \frac{\partial H^k}{\partial \lambda'_{j_1\cdots j_h}} + (N+1)\mu'_{(j_1\cdots j_M} \delta_{j_{M+1} j_{M+2}} \cdots \delta_{j_N j_{N+1})} \frac{\partial H^k}{\partial \lambda'_{j_1\cdots j_N}} \delta_j^{j_{N+1}}$$
$$+ \sum_{p=1}^{M} \left[(p-1)\mu'_{j j_1 \cdots j_{p-2}} \delta_{j_{p-1} j_p} + (N+M+3-2p)\mu'_{j_1\cdots j_{p-1}} \delta_{j_p j} \right] \frac{\partial H^k}{\partial \mu'_{j_1\cdots j_p}}$$
$$+ H\delta^{kj} = 0$$

After that we define

$$\lambda'_j = \lambda'_j(\lambda', \lambda_{j_1 j_2}, \cdots, \lambda_{j_1\cdots j_N}, \mu'_{j_1\cdots j_p}),$$

implicitly defined by

$$\frac{\partial H}{\partial \lambda'_j} = 0. \tag{6.50}$$

If we call $\widetilde{h}$ and $\widetilde{\phi}^k$ the functions H and H^k calculated for such value of λ'_j it is easy to prove that, as consequence, they satisfy eqs. (6.48) and, consequently, they are the same functions of the first method. Uniqueness of the solution can also be proved. In such a way we have proved the equivalence of the two methods also for this new kind of system, as done in chapter 2 for the 13 and 14 moments case.

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