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Black brane solutions of Einstein-scalar-Maxwell gravity and their holographic applications

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Abstract

Black hole solutions of Einstein (and Einstein-Maxwell) gravity coupled to scalar fields have acquired a growing interest and importance in recent years. This interest is motivated both by more "classical" issues, as the problem of the uniqueness of classical black holes solutions (and related "no-hair" theorems), and mainly by recent applications of the AdS/CFT correspondence, an holographic duality which allows to describe, starting from a gravitational theory, strongly coupled quantum field theories.

In this thesis, we treat this topic both from a pure gravitational point of view and from the holographic perspective. In particular, we propose a general method for exactly solving, in some cases, the field equations of Einstein-scalar-Maxwell gravity, and present some new analytical and numerical solutions (we mainly focus on *black brane* solutions, i.e. solutions with a planar event horizon). Moreover, we discuss hyperscaling violation, a particular scaling behavior of free energy and entropy (as functions of the temperature), typical of some phase transitions of real condensed matter systems. Hyperscaling violation can be described, via AdS/CFT, starting from a gravitational solution with a particular symmetry.

Finally, we perform some interesting results about the mass spectrum and stability of black brane solutions in a wide class of gravitational models. In particular, the thermodynamics of some solutions of these models provides important information about the possible existence of physically-relevant phase transitions in the dual field theories.

To my family

"The search for truth is more precious than its possession"

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Contents

Preface

Chapters 2, 3, 5 and 6 are mainly based, respectively, on the following papers:

- Mariano Cadoni, Salvatore Mignemi, Matteo Serra, *Exact solutions with AdS asymptotics of Einstein and Einstein-Maxwell gravity minimally coupled to a scalar field*, published in Physical Review D 84, 084086 (2011).
- Mariano Cadoni, Salvatore Mignemi, Matteo Serra, *Black brane solutions and their solitonic extremal limit in Einstein-scalar gravity*, published in Physical Review D 85, 086001 (2012).
- Mariano Cadoni, Matteo Serra, *Hyperscaling violation for black branes in arbitrary dimensions*, published in JHEP 11, 136 (2012).
- Mariano Cadoni, Paolo Pani, Matteo Serra, *Infrared behavior of scalar condensates in effective holographic theories*, published in JHEP 06, 29 (2013).

List of abbreviations:

AdS: Anti-de Sitter
BB: Black Brane
BF: Breitenlohner-Freedman
CFT: Conformal Field Theory
DW: Domain Wall
EHT: Effective Holographic Theories
EM: Electromagnetic
ESM: Einstein-Scalar-Maxwell
HV: Hyperscaling Violation
IR: Infrared
PET: Positive Energy Theorem
QFT: Quantum Field Theory
RG: Renormalization Group
RN: Reissner-Nordström
SAdS: Schwarzschild-AdS
UV: Ultraviolet

Introduction

In recent years there has been a renewed growing interest for the static black hole solutions of Einstein (and Einstein-Maxwell) gravity coupled to scalar fields.

In the past, the interest for these models, and in particular for solutions characterized by a nontrivial profile of the scalar field (scalar "hair"), was basically motivated either by the issue of the uniqueness of the Schwarzschild black hole and related no-hair theorems [1, 2], or by cosmological issues, in particular in the context of dark energy models [3–5], or by the quest for new black hole solutions in low-energy string models [6–10]. Moreover, the search for this black holes was mainly focused on asymptotically flat solutions.

More recently, the interest for these solutions was shifted from asymptotically flat to asymptotically anti-de Sitter (AdS) solutions. From a pure gravitational point of view, the shift to asymptotically AdS solutions allows to circumvent standard no-hair theorems, which relate the existence of black hole solutions with scalar hair to the violation of the positive energy theorem (PET) [11, 12]. Unlike the flat case, a scalar field in the AdS spacetime may have negative squared-mass m^2, without destabilizing the AdS vacuum, provided m^2 is above the so-called Breitenlohner-Freedman (BF) bound [13].

But static black hole solutions with non trivial scalar hair and AdS asymptotics can play a crucial role in the context of applications of the anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [14], one of the most intriguing recent discoveries in theoretical physics. When the classical approximation for the bulk gravity theory is reliable, exploiting the AdS/CFT "rules", one can deal with "holographic" strongly coupled quantum field theories (QFTs) at finite temperature in d dimensions by investigating black holes in AdS in $d+1$ dimensions. In this context, the nontrivial scalar hair of the black hole solutions can be interpreted either as a running coupling constant or as a scalar condensate in the dual QFT. In the first case the bulk scalar dynamics is very useful for holographic renormalization methods [15]. In the second case the scalar condensate can describe a symmetry breaking or phase transitions in the dual QFT, in many cases reminiscent of well-known condensed matter systems [16–25].

The best-known example is represented by holographic superconductors: below a critical temperature the bulk gravity theory, in this case Einstein-Maxwell with a covariantly coupled scalar field, allows for black hole solutions with scalar hair [18–20, 22, 24]. This corresponds to the formation of a charged condensate in the dual theory that breaks spontaneously a global $U(1)$ symmetry, a typical behavior of superconducting systems. This basic structure has been generalized to a number of cases including, among others, Yang-Mills theories [26] and nonminimal couplings between the scalar and the electromagnetic field [23, 25, 27–31].

Another interesting feature of static black holes with scalar hair and AdS asymptotics, which is common to several models, is the presence in the near-horizon regime of solutions which break the conformal symmetry of AdS vacuum, but preserve some scaling symmetries [23, 25, 28, 30, 32–40]. In particular it has been realized that these solutions belong to a general class of metrics that are not scale-invariant but scale-*covariant*, in the sense that these metrics transform, under scale transformations, with a definite weight. In the context of Einstein-scalar gravity, the global solutions typically appear as scalar solitons interpolating between an asymptotic AdS spacetime and a near-horizon scale-covariant metric.

Scale-covariant metrics are very interesting also from the holographic point of view, because are dual to quantum field theories with hyperscaling violation, namely a particular scaling behavior of the free energy and entropy as a function of the temperature, typical of some phase transitions in real condensed matter systems (e.g. Ising models) [38, 39, 41–48]. In terms of the dual QFT, the scalar soliton describes a flow from a conformal fixed point in the ultraviolet regime (UV) to an hyperscaling-violating phase in the infrared (IR).

An alternative scenario is represented by models with non-AdS asymptotics [32, 35, 49], in which the scalar soliton interpolates between a near-horizon AdS spacetime and an asymptotic scale-covariant metric. Correspondingly, in the dual QFT we have a flow between a conformal fixed point in the IR and an hyperscaling-violating phase in the UV. However, in this case the correct interpretation of the dual QFT is often more involved.

The main purpose of this thesis is twofold: firstly, from a pure gravitational point of view, we look for new gravitational solutions with scalar hair, both analytical and numerical. In particular, we will focus our attention on *black brane* solutions, i.e. black holes with planar event horizons, that in general are computationally more tractable against the spherical black holes. Secondly, we will study the possible holographic applications of some of these new gravitational solutions.

Following this line of reasoning, the thesis is structured in two parts. The Part I is mainly dedicated to the presentation of new exact black brane solutions with scalar hair. We will consider both the electromagnetic uncharged and the charged case. In Chapter 1 we will give a brief introduction about the general features of Einstein gravity theories coupled to scalar fields and the relative possible solutions (mainly in the context of asymptotically AdS spacetimes, but also considering different asymptotic behavior). In Chapter 2 we will present a general method for exactly solving the field equations of Einstein-Maxwell gravity minimally coupled to scalar fields. The striking feature of this method is that it requires an ansatz for the scalar field, instead than for the potential, as usually occurs. Using this method we will derive a broad class of exact "hairy" solutions and a new no-hair theorem about the general existence of black brane and black hole solutions of Einstein-Maxwell gravity with scalar hair. In Chapter 3 we discuss a particular model of Einstein gravity minimally coupled to a scalar field and derive an exact black brane solution with scalar hair, but with a different approach than that used in Chapter 2. As we will see, the black brane solution obtained is not asymptotically AdS, but has a *domain wall* asymptotic, namely a spacetime in which the conformal symmetry is broken, while is preserved the Poincarè relativistic symmetry.

In the Part II we focus our attention on the holographic applications. The Chapter 4 will be dedicated to a brief overview of the general formulation of the AdS/CFT correspondence and to one of its most significant phenomenological extensions, the AdS/condensed matter duality. We also present with some detail several interesting applications as holographic superconductors, metallic behavior and hyperscaling violation. In Chapter 5 we extend the model already studied in Chapter 3 to a generic number of dimensions, and also discuss the thermodynamical properties of the black brane solution. For what concerns the dual field theory, we will derive the hyperscaling violation parameter and the short-distance form of the correlators for the scalar operators. In Chapter 6 we study a very general class of models of Einstein-scalar-Maxwell gravity and show some general and interesting results about the stability of black brane solutions with scalar hair. These results will be checked deriving several numerical solutions for particular models. Moreover these results, as we will see, seem to imply very important consequences about the existence of phase transitions, typical of real condensed matter systems, in the dual field theory.

Finally, in the Conclusions we summarize the results of the thesis and point out the possibile future developments.

Part I

Gravitational solutions with scalar “hair”

Chapter 1

Generalities on Einstein-scalar gravity in AdS

In this chapter we present some general features of Einstein gravity minimally coupled to a scalar field, in particular in the context of asymptotically anti-de Sitter (AdS) spacetimes. In the first two sections we analyze in particular the general setup of these theories, with a particular attention to the constraints imposed by the AdS asymptotics, the tricky question about the general existence of black hole solutions with a non-trivial profile of the scalar field, and the symmetries of the possible near-horizon behavior of the solutions. Finally in Sect. 1.3, we consider an interesting example of Einstein-scalar gravity, the so-called "fake supergravity" models. For simplicity, in this chapter we mainly consider uncharged theories, but most of the general considerations, in particular for what concerns the general setup of the asymptotically AdS models, are still valid in presence of an electromagnetic field.

1.1 AdS Einstein-scalar gravity setup

We consider theories of gravity in $d+2$ dimensions coupled to a real scalar field $\phi(r)$ with potential $V(\phi)$:

$$S = \int d^{d+2}x\sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right). \tag{1.1}$$

The potential is taken to have a negative local maximum in $\phi = 0$, corresponding to an AdS vacuum:

$$V(\phi) = -\frac{d(d+1)}{L^2} + \frac{1}{2}m^2\phi^2 + \mathcal{O}\left(\phi^3\right),$$

where L is the AdS length and $m^2 = V''(0)$ is the scalar mass. This choice guarantees that small scalar fluctuations are tachyonic, i.e. $m^2 < 0$. It has long been known that tachyonic scalars in $d+2$ AdS spacetime do not represent an instability, provided their mass is above the so-called Breitenlohner-Freedman (BF) bound [13] $m^2_{BF} = -(d+1)^2/4L^2$.

Asymptotically AdS black hole solutions with $m^2 > 0$ are in general forbidden by usual no-hair theorems, as we will discuss in Sect. 1.2. For this reason we will in general consider negative scalar squared masses. We will only consider positive scalar masses in the case of solutions with non-AdS asymptotics. In this latter case usual no-hair theorems do not apply.

We wish to study asymptotically AdS_{d+2} solutions, where the metric approaches:

$$ds^2 = \frac{r^2}{L^2}(-dt^2 + d\Omega_d^2) + \frac{L^2}{r^2}dr^2,$$

where $d\Omega_d^2$ is the d-dimensional transverse space.

In all asymptotically AdS solutions, when $m^2 > m^2_{BF}$ the scalar ϕ decays at large radius as[1]

[1] A particular case is $m^2 = m^2_{BF}$, that is when the BF bound is saturated. In this case the asymptotic behavior of the scalar presents a logarithmic branch. In particular one finds at $r \to \infty$: $\phi = O_1 r^{-(d+1)/2}\ln r + O_2 r^{-(d+1)/2}$[50].

$$\phi = \frac{O_1}{r^{\Delta_1}} + \frac{O_2}{r^{\Delta_2}},$$

where $\Delta_{1,2} = \frac{(d+1)\mp\sqrt{(d+1)^2+4m^2L^2}}{2}$. Because the AdS spacetime is not globally hyperbolic, this asymptotic behavior must be supported by boundary conditions (BC) on O_1 and O_2. Standard choices that preserve the asymptotic isometries of the AdS spacetime are $O_1 = 0$ (Dirichlet BC) or $O_2 = 0$ (Neumann BC), but recently has been shown that if the scalar mass is in the range $m^2_{BF} < m^2_s < m^2_{BF} + 1/L^2$, also boundary conditions of the form $O_1 = f O_2^{\Delta_1/\Delta_2}$ (with f an arbitrary constant), which admit a conserved total energy and preserve all the AdS asymptotic symmetries, are allowed [50, 51].

We can also consider more general boundary conditions, that are conventionally parametrized as:

$$O_2 = \frac{\partial W}{\partial O_1}, \tag{1.2}$$

where $W(O_1)$ is an arbitrary function. For a generic choice of W, some of the asymptotic AdS symmetries will be broken. When the gravitational theory admits a field theory dual (see Part II), the choice of boundary conditions (1.2) is mapped into a multitrace deformation of the action of the boundary theory [52, 53].

The simplest static solution of the field equations stemming from the action (1.1) is the Schwarzschild-AdS black hole[2]:

$$ds^2 = -\left(\frac{r^2}{L^2} + \varepsilon - \frac{M}{2r^{d-1}}\right)dt^2 + \left(\frac{r^2}{L^2} + \varepsilon - \frac{M}{2r^{d-1}}\right)^{-1} dr^2 + r^2 d\Omega_d^2, \quad \phi = 0,$$

where M is the black hole mass and $\epsilon = 0, 1, -1$ denotes, respectively, the

[2]Obviously, in presence of a coupling also with a Maxwell field, the simplest solution will be the Reissner-Nordström-AdS black hole: $ds^2 = -\left(\frac{r^2}{L^2} + \varepsilon - \frac{M}{2r^{d-1}} + \frac{\rho^2}{4r^2}\right)dt^2 + \left(\frac{r^2}{L^2} + \varepsilon - \frac{M}{2r^{d-1}} + \frac{\rho^2}{4r^2}\right)^{-1} dr^2 + r^2 d\Omega_d^2$, where ρ is the charge density.

d-dimensional planar, spherical or hyperbolic transverse space $d\Omega_d^2$ (in particular, hereafter we will define *black branes* the black hole solutions with $\epsilon = 0$, i.e. with planar transverse space).

Any other black hole solution will be characterized by a non-trivial profile of the scalar field. For a given potential, these "hairy" solutions generally form a two-parameter family defined by the values of the horizon radius r_h and of the scalar field at the horizon $\phi(r_h) = \phi_h$. Alternatively, each solution is uniquely characterized by the (O_1, O_2) coefficients related to the asymptotic behavior of the scalar field, and from the field equations we can obtain a precise mapping $(r_h, \phi_h) \leftrightarrow (O_1, O_2)$. Then, varying ϕ_h and leaving r_h fixed, one obtains a curve $O_2(O_1)$ in the (O_1, O_2) plane, that will be generally not single-valued.

Apart from the black hole solutions, the ensemble of static and spherically symmetric solutions to (1.1) is completed by the hairy solitons, which are defined by the requirement of regularity at $r = 0$ and form a one-parameter family, defined by the value of the scalar field at the origin $\phi(r = 0) = \phi_0$. In this case, the integration of the field equations provides a single curve $O_2^{'}(O_1)$ in the (O_1, O_2) plane, which can obtained as the $r_h \to 0$ limit of $O_2(O_1)$.

The case more studied in the literature is $m^2 = -2/L^2$ in four dimensions (i.e. $d = 2$), where it is simple to find that $\Delta_1 = 1$ and $\Delta_2 = 2$. In this case it was found [54] that in the limit of small ϕ and small O_1, O_2 one finds the universal behavior $O_2^{'} = -\frac{2}{\pi}O_1 + \mathcal{O}(O_1^3)$. At large O_1 the soliton function instead scales as $O_2^{'} \simeq -s_c O_1^2$ (with $s_c > 0$), as one would expect from dimensional analysis [55]. At the non-linear level the behavior of $O_2^{'}$ obviously depends on the particular theory considered.

An interesting point is to understand for which W the theory has a stable, minimum energy ground state. Defining an "effective potential" $\mathcal{V}(O_1)$:

$$\mathcal{V}(O_1) = W(O_1) + W^{'}(O_1),$$

where $W^{'}(O_1) = -\int_0^{O_1} O_2^{'}(O_1)dO_1$, it is simple to note that the extrema of $\mathcal{V}$ are precisely the points where $O_2' = O_2$. In other words, there is a soliton which satisfies the boundary conditions. In [55], following two conjectures formulated in [56], it was shown that:

1. The total energy of the theory with boundary conditions $O_2 = \frac{\partial W}{\partial O_1}$ is bounded from below if and only if $\mathcal{V}$ has a global minimum $\mathcal{V}_{\min}$;
2. The minimal energy solution is simply the soliton associated with $\mathcal{V}_{\min}$.

Substantially, we observe that for the same action, many possible boundary conditions can be fixed. But if we change the boundary conditions, we change the properties of the theory. In particular, in some theories one can "pre-order" the number and masses of solitons, so we can say that there are boundary conditions wich yield a desired result. For this reason, often these theories are called "designer gravity" theories [56].

1.2 "Hairy" solutions and no-hair theorems

A crucial issue in this context is the question about the existence of regular, static black hole solutions of Einstein-scalar gravity with AdS asymptotics, endowed with non trivial scalar hair.

"Black holes have no hair", wrote John Wheeler in the early 1970's [57]. His conjecture was inspired by some uniqueness theorems for static and asymptotically flat black hole solutions in Einstein-Maxwell theory [1, 58–60], for which these solutions are uniquely determined by only three parameters ("hair"), defined as integrals at spatial infinity: mass, charge and spin.

However, the conjecture is valid only in the context of pure General Relativity. It was shown that black hole solutions with extra hair can exist in theories where gravity is coupled to scalar or Higgs fields, as e.g. Einstein-Maxwell-dilaton, Einstein-Yang-Mills and Einstein-Skyrme theories

[7, 61, 62], although some of these solutions are unstable [63–65]. More generally, Bekenstein [66] demonstred the non-existence of regular, asymptotically flat, hairy spherical black hole solutions when the scalar field is minimally coupled and has a convex potential. This theorem was then generalized to scalar fields minimally coupled with arbitrary positive potentials [67], scalar multiplets [2] and nonminimally coupled scalars [10].

But what happens when one considers spacetimes not asymptotically flat, but asymptotically anti-de Sitter? In general, we can say that the presence of the negative cosmological constant seems to favour the existence and stability of black hole solutions with scalar hair. In particular, several examples of stable hairy black holes in AdS were found, both analytical and numerical [11, 18, 23, 25, 68–71]. However, in 2001 Torii, Maeda and Narita [11] showed that regular black hole solutions asymptotically AdS cannot exist when the scalar field is massless or has a convex potential. More recently, Hertog [12] added two other necessary conditions for the existence of asymptotically AdS hairy black hole solutions: the violation of the Positive Energy Theorem (PET) and the breaking of the full AdS symmetry group, in theories where the scalar field has negative local maxima. The first condition implies that black hole solutions with scalar hair and positive squared mass m^2 are forbidden, while in general allows solutions with negative m^2 above the BF bound.

Another important issue is the existence of stable solitonic solutions. For spherical solutions can be found boundary conditions, breaking the AdS symmetry, which allow stable solitons. But in the case of planar solutions only AdS-symmetry preserving boundary conditions are possible, and has been shown that these boundary conditions allow for a stable ground state only if the potential has a second extremum [55].

1.2.1 Exact solutions

Despite the growing importance played by static black hole solutions with scalar hair and AdS asymptotics, in particular in the context of the applications of AdS/CFT, very few exact analytical solutions are known. In fact, most of the usual methods used for finding exact, asymptotically flat, "hairy" solutions, as for examples those used in [6–9] for deriving hairy solutions with nonminimal couplings (mainly motivated by string theory), do not work in AdS spacetime. Essentially, the few examples of exact asymptotically AdS solutions are the family of four-charge black holes in $\mathcal{N} = 8$ four-dimensional gauged supergravity [72, 73], the solution with hyperbolic horizon of Ref. [71] and a few other examples, some of them generated from asymptotically flat solutions [74–79]. Actually, most solutions of this kind, which have been exploited for holographic applications and for deriving the most recent no-hair theorems, are numerical [18, 20, 22–25]. Hovewer, to find general methods for obtaining exact solutions would be obviously a very important step in this context.

1.2.2 Black brane solutions

Asymptotically AdS black brane solutions can be classified by means their, small r, near-horizon behavior (also called *infrared* behavior, in particular if the solutions have an holographic dual).

The most general scaling behavior one can find is described by the following full class of metrics [39, 42]:

$$ds^2 = r^{-2(d-\theta)/d} \left(-r^{-2(z-1)} dt^2 + dr^2 + dx_i^2 \right) . \qquad (1.3)$$

These metrics exhibit a dynamical critical exponent z and the so-called *hyperscaling violation* exponent θ. We will return in Sect. 4.4.3 on the meaning of the exponent θ in the context of holographic applications to condensed matter systems. Here we observe that a nonzero value of θ makes the metric 1.3 not scale-invariant, but scale-*covariant*, in the sense that the metric 1.3 transforms under a definite weight:

$$ds \to \lambda^{\theta/d} ds$$

under the following scale transformation:

$$t \to \lambda^z t, \quad r \to \lambda r, \quad x_i \to \lambda x_i. \tag{1.4}$$

This class of metrics typically appears in the near-horizon limit of extremal black holes and black branes in theories where the scalar field is nonminimally coupled with the Maxwell field [23, 80, 81].

Two important particular cases of the general metric 1.3 are obtained choosing, respectively, $\theta = 0$ or $z = 1$.

In the first case the 1.3 becomes:

$$ds^2 = -r^{-2z} dt^2 + r^{-2}(dr^2 + dx_i dx^i). \tag{1.5}$$

This metric describes the so-called *Lifshitz* spacetimes [82–84], which are scale-invariant under the transformations 1.4, but not conformally invariant, because relativistic symmetry is broken. Also this class of metrics typically appears in theories with a scalar field coupled to the Maxwell field [23, 25, 27, 28, 30]. The scalar, in these theories, usually presents a logarithmic behavior.

In the case $z = 1$ the 1.3 becomes:

$$ds^2 = r^{-2(d-\theta)/d} \left(-dt^2 + dr^2 + dx_i^2\right). \tag{1.6}$$

This metric, which is not scale-invariant, but preserves the Poincarè relativistic isometry of the transverse space, is often called *domain wall* [32, 33, 35], because these solutions are often related to solitonic supergravity domain-walls, resulting from various dimensional reductions of 10 and 11-dimensional maximal supergravity theories.

Domain wall solutions as 1.6 usually arise as the near-horizon limit of black holes (branes) in uncharged Einstein-scalar theories. Typically, in these theories the potential behaves exponentially, while the scalar field has a logarithmic form.

Obviously, the most symmetric case is obtained choosing in the 1.3 both $\theta = 0$ and $z = 1$, which corresponds to relativistic conformal theories as AdS.

So far we have only considered black brane solutions with AdS asymptotics and a not conformal symmetry for the near-horizon behavior.

We have seen that recent no-hair theorems forbid asymptotically AdS black brane solutions with a positive squared mass for the scalar. However, black brane solutions with non-AdS asymptotics are allowed.

In particular, recently was shown [85] that the existence of this kind of solutions is a rather generic feature of a broad class of models, for which in the extremal limit the black brane solution reduces to a fully regular scalar soliton, which interpolates between an AdS vacuum in the near-horizon region and a scale-covariant solution in the asymptotic region.

This exchange between the near-horizon and the asymptotic behavior, compared to the usual setup, has important consequences also in the context of the holographic applications (as we will see with more details in the Part II of the thesis), because these solitonic solutions represent a flow between an infrared (IR) conformal fixed point and a hyperscaling-violating phase in the ultraviolet (UV) regime of the dual field theory. This is an alternative scenario against the role played by IR and UV regimes in applications of AdS/CFT, where usually one has a conformal fixed point in the UV.

1.3 Fake supergravity

A particularly simple and interesting example of Einstein-scalar gravity is represented by the so-called fake supergravity (SUGRA) [55, 86, 87]. In these models one can define Killing spinors using fake transformations similar to real SUGRA theories. The striking feature is that the potential $V(\phi)$ for the scalar field can be derived from a superpotential $P(\phi)$:

$$V(\phi) = 2\left(\frac{dP}{d\phi}\right)^2 - 3P^2. \tag{1.7}$$

In particular, if we choose a boost invariant parametrization for the spacetime metric of the form:

$$ds^2 = r^2(-dt^2 + dx_i dx^i) + \frac{dr^2}{h(r)},$$

the second-order equation of motion stemming from the action (1.1) can be reduced to a fake Bogomol'nyi-Prasad-Sommerfeld (BPS) first order equation:

$$\phi'(r) = -\frac{2P_{,\phi}}{rP(\phi)}, \quad h(r) = r^2 P^2(\phi), \tag{1.8}$$

whose solutions automatically satisfy the second-order field equation. Moreover, assuming the following behavior for the superpotential near $\phi = 0$ (corresponding to $r \to \infty$):

$$P(\phi) = 1 + \frac{1}{4}\phi^2 - \frac{s}{6}|\phi|^3 + O\left(\phi^4\right) \tag{1.9}$$

(s is a constant), the asymptotic behavior of ϕ and h is constrained to be:

$$\phi = \frac{\alpha}{r} - \frac{s\alpha^2}{r^2} + ..., \qquad h = r^2 + \frac{\alpha^2}{2} - \frac{4s\alpha^3}{3r} + ...$$

We note that the boundary conditions on the scalar field are fixed in the form $O_2 = -sO_1^2$. This is a consequence of the scale invariance of the equations of motion under rescaling $r \to cr$, $h \to c^2 h$.

Exploiting the Witten-Nester theorem [88–90], one can show that the energy of any singularity-free solution of the first-order equation (1.8) is bounded from below. In particular one finds for the energy [55]:

$$E \geq \left[W + \frac{s}{3} |\alpha|^3 \right],$$

where W is the "boundary condition function" defined in Sect. 1.1. It is interesting to note that it is not necessary W itself to be bounded.

A key question is the following: for what range of s the superpotential P exists globally? For most potentials $V(\phi)$, P exists only globally up to a critical value s_c. This point is hard to show analitically, but can be understood by a simple argument. From (1.7) we can derive the general form of the first derivative of P:

$$P'(\phi) = \sqrt{\frac{3P^2}{2} + \frac{V(\phi)}{2}}. \tag{1.10}$$

The superpotential will exist for all ϕ unless $P' = 0$. Since the (1.10) is a first order differential equation, its solutions are obviously monotone except a singular point. Because of it, if a solution P_1 exists globally with some value $s = s_1$, then all solutions with $s < s_1$ exist globally as well (see 1.9). So one can algorithmically find s_c by numerically solving (1.10), starting from $\phi = 0$, and increasing s until P no longer exists because $P'(\phi)$ reaches zero.

Concluding remarks and summary of subsequent chapters

Summarizing, in this first chapter we have analyzed the main features of Einstein theories of gravity coupled to scalar fields, with a particular attention to black brane solutions endowed with a non-trivial scalar field and asymptotically AdS. Although the interest on this topic was considerably increased in the last times, mainly for the applications to AdS/CFT, we have seen that there are still important open problems on which we can focus our attention. In particular, we identify the following three important points:

- General methods to generate exact asymptotically AdS black hole and black brane solutions are still lacking;
- The know no-hair theorems put loose constraints about the general existence of scalar-dressed black hole and black brane solutions, while it would be very important to be able to formulate more stringent no-hair theorems;
- An interesting scenario, still not enough studied in deep, is represented by black brane solutions with non-AdS asymptotics, also in the perspective of possible holographic applications.

In the next two chapters we will mainly address to the open problems pointed out above. In particular, in Chapter 2 we shall present a general method for exactly solving the field equations of Einstein and Einstein-Maxwell gravity minimally coupled to a scalar field, and derive several exact scalar-dressed black brane and black hole solutions with different asymptotic behavior (AdS, Lifshitz, domain wall). Moreover, we shall formulate a new no-hair theorem, which adds further constraints in particular about the existence of uncharged, asymptotically AdS, hairy black brane solutions.

In Chapter 3 we shall consider a particular fake SUGRA model of Einstein-scalar gravity, characterized by a positive squared mass for the scalar, and derive an exact scalar-dressed black brane solution with domain wall asymptotics, using a more traditional approach then that used in Chapter 2. As we will see, the extremal limit of the black brane solution is a soliton interpolating between an AdS behavior near $r = 0$ and a domain wall in the asymptotic region.

Chapter 2

Exact solutions of Einstein-scalar-Maxwell gravity

We propose a general method for solving exactly the static field equations of Einstein and Einstein-Maxwell gravity minimally coupled to a scalar field. Our method starts from an ansatz for the scalar field profile and determines, together with the metric functions, the corresponding form of the scalar self-interaction potential. Using this method we prove a new no-hair theorem about the existence of hairy black hole and black brane solutions and derive broad classes of static solutions with radial symmetry of the theory, which may play an important role in applications of the AdS/CFT correspondence to condensed matter and strongly coupled QFTs. These solutions include: 1) four - or generic $(d+2)$ - dimensional solutions with planar, spherical or hyperbolic horizon topology; 2) solutions with AdS, domain wall and Lifshitz asymptotics; 3) solutions interpolating between an AdS spacetime in the asymptotic region and a domain wall or conformal Lifshitz spacetime in the near-horizon region.

2.1 Introduction

We have pointed out in Sect. 1.2.1 that very few exact black hole solutions with scalar hair and AdS asymptotics are known. Obviously, this situation has a negative impact on further developments of the subject. This is particularly true because the known no-hair theorems (see Sect. 1.2) put loose constraints on the existence of black hole solutions with scalar hair. Therefore, they do not give stringent indications that can be used when searching for exact or numerical solutions of a given model.

Starting from these considerations, in this chapter we first propose a general method for solving the field equations of Einstein and Einstein-Maxwell gravity minimally coupled to a scalar field ϕ in the static, radially symmetric, case. Our main idea is to reverse the usual method for solving the field equations. Usually, one determines the metric functions, the scalar field and the EM field, for a *given* form of the self-interaction potential $V(\phi)$. Instead of solving the field equations for a given potential, we will assume a given profile $\phi(r)$ for the scalar field and then we will solve the system for the metric functions and the potential.

This method is particularly suitable for applications to the AdS/CFT correspondence. In this case the actual exact form of the potential $V(\phi)$ is not particularly relevant. What is more important is the behavior of the scalar field, and in particular its fall-off behavior at $r = \infty$.

We will apply our solving method to two different but related issues. First, we will apply it to find exact analytic solutions of Einstein and Einstein-Maxwell gravity with scalar hair. For the scalar field we use profiles which are very common for hairy black hole solutions in flat space, gauged supergravity and Lifshitz spacetime, namely harmonic and logarithmic functions. This allows us to find exact solutions in several situations: four or generic $d+2$ spacetime dimensions, different topologies of the transverse space (planar, spherical, hyperbolic) and different asymptotics (anti-de Sitter, domain wall, conformal to Lifshitz spacetime). In particular we will derive exact solutions interpolating between an asymptotic AdS spacetime and a near-horizon domain wall or conformal Lifshitz spacetime. The models that we find contain as particular case the truncation to the abelian sector of $\mathcal{N} = 8$, $D = 4$ gauged supergravity.

Our solving method is also effective in dealing with more general profiles of the scalar field, giving rise to different types of potentials. We will show that our method allows to find exact solutions for scalar field profiles of the r^l type. In particular, we work out explicitly the solutions for $l = -1$ and show that it corresponds to a potential which is the combination of powers and trigonometric functions.

Second, our method allows to write explicitly a formal solution of the field equations for an arbitrary potential. This fact will be used to prove a new no-hair theorem about the existence of black hole and black brane solutions of Einstein and Einstein-Maxwell gravity minimally coupled to a scalar field.

The structure of the chapter is as follows. In Sect. 2.2 we present our general method for solving the field equations of Einstein-scalar-Maxwell gravity in $d + 2$ dimension and planar, spherical or hyperbolic topology of the transverse sections. In Sect. 2.3 we apply this method to find domain wall and conformal Lifshitz black hole solutions in $d = 2$ for the planar case. In Sect. 2.4 we prove a new no-hair theorem for black hole solutions of Einstein-Maxwell gravity minimally coupled to a scalar field. In Sect. 2.5 we use our method to derive planar solutions with AdS asymptotics in $d = 2$, both in the uncharged and charged case and discuss their near-horizon behavior. In Sect. 2.6 we consider scalar field profiles of the r^l type and find explicit solutions for the $l = -1$ case. The generalization of our solutions to the $d+2$-dimensional case and to spherical or hyperbolic solutions is discussed, respectively, in Sects. 2.7 and 2.8. Finally in Sect. 2.9 we present our concluding remarks.

2.2 A general method for solving the field equations

We consider Einstein-Maxwell gravity in $d+2$ dimensions (with $d \geq 2$), minimally coupled to a scalar field ϕ, and with a generic self-interaction potential $V(\phi)$. The action is:

$$I = \int d^{d+2}x\sqrt{-g}\left(\mathcal{R} - 2(\partial\phi)^2 - F^2 - V(\phi)\right). \tag{2.1}$$

The ensuing field equations take the form:

$$\begin{aligned}
&\nabla_\mu F^{\mu\nu} = 0\,,\\
&\nabla^2\phi = \frac{1}{4}\frac{dV(\phi)}{d\phi}\,,\\
&\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 2\left(F_{\mu\rho}F^{\rho}_{\nu} - \frac{g_{\mu\nu}}{4}F^{\rho\sigma}F_{\rho\sigma}\right) + 2\left(\partial_\mu\phi\partial_\nu\phi - \frac{g_{\mu\nu}}{2}\partial^\rho\phi\partial_\rho\phi\right) - \frac{g_{\mu\nu}}{2}V(\phi)\,.
\end{aligned} \tag{2.2}$$

Throughout this chapter we will investigate static solutions of the previous field equations, exhibiting radial symmetry. Moreover, we will consider only purely electric solutions; magnetic solutions can be easily generated from the electric ones using the electro-magnetic duality. We adopt a Schwarzschild gauge to write the spacetime metric:

$$ds^2 = -U(r)dt^2 + U^{-1}(r)dr^2 + R^2(r)d\Omega^2_{(\varepsilon,d)}, \tag{2.3}$$

where $\varepsilon = 0, 1, -1$ denotes, respectively, the d-dimensional planar, spherical, or hyperbolic transverse space with metric $d\Omega^2_{(\varepsilon,d)}$. In these coordinates, the electric field satisfying (2.2) reads:

$$F_{tr} = \frac{Q}{R^d}, \tag{2.4}$$

with Q the electric charge. With the parametrization (2.3), the field equations take the form:

$$\begin{aligned}\frac{R''}{R} &= -\frac{2}{d}(\phi')^2, \quad (UR^d\phi')' = \frac{1}{4}R^d\frac{dV}{d\phi},\\ (UR^d)'' &= \varepsilon d(d-1)R^{d-2} + 2\frac{d-2}{d}\frac{Q^2}{R^d} - \frac{d+2}{d}R^dV,\\ (UR^{d-1}R')' &= \varepsilon(d-1)R^{d-2} - \frac{2}{d}\frac{Q^2}{R^d} - \frac{1}{d}R^dV. \end{aligned} \tag{2.5}$$

We are mainly interested in black hole solutions of the field equations that are asymptotically AdS. As explained in Sect. 1.1, the existence of the AdS vacuum requires $V(0) < 0$ and $V'(0) = 0$, with $V(0) = -d(d+1)/L^2$, where L is the AdS length (assuming that $\phi \to 0$ as $r \to \infty$). Under this condition the simplest static black hole solution of the field equations is given by the Schwarzschild-AdS (SAdS) solution:

$$U = \frac{r^2}{L^2} + \varepsilon - \frac{M}{2r^{d-1}}, \quad R = r, \quad \phi = 0, \quad Q = 0. \tag{2.6}$$

Apart from SAdS and the $Q \neq 0$ Reissner-Nordström AdS black hole, the other solutions of (2.5), if they exist, will be characterized by a non-constant profile of the scalar field ϕ. These solutions are very difficult to find, at least analytically. Eqs. (2.5) may be solved in closed form for some particular choice of the potential V, but for a generic potential there is no a general solving method. Moreover, it is not completely clear if and when the field equations allow for regular black hole solutions. Explicit solutions, analytical or numerical, are known in a few cases. Nonetheless, a more precise statement about the existence of black hole solutions of the field equations (2.5) is still lacking.

In this chapter we will often consider solutions with scalar hair and zero temperature (no horizon). With some abuse of terminology we will always call these solutions "extremal black hole solutions". Obviously, the use of this name is only strictly pertinent when the $T = 0$ solution can be considered as the $T \to 0$ limit of black hole solutions with a regular horizon. As we will see in detail in the next sections, we will not be able to show that this is the case for all the hairy zero temperature solutions we will find. Nonetheless, we will use the word extremal black hole in the wide sense defined above.

Usually, one solves the field equations (2.5) by determining U, R and ϕ, for a *given* form of the potential $V(\phi)$. Here we will approach the problem in a reversed way. Instead of solving equations (2.5) for a given potential V, we will assume a given profile $\phi(r)$ for the scalar field and then solve the system for $U(r)$, $R(r)$ and $V(\phi)$. Although at first sight this approach may seem rather weird, it is very useful for at least two reasons.

First, focusing on solutions with AdS asymptotics, in particular for what concerns applications to the AdS/CFT correspondence, the actual exact form of the potential $V(\phi)$ is not particularly relevant. What is often more important is the behavior of the scalar field $\phi(r)$, in particular its fall-off behavior at $r = \infty$ (see Sect. 1.1):

$$\phi \sim \frac{O_1}{r^{\Delta_1}} + \frac{O_2}{r^{\Delta_2}}, \quad \Delta_{1,2} = \frac{(d+1) \mp \sqrt{(d+1)^2 + 4m^2L^2}}{2}. \tag{2.7}$$

Moreover, in applications of the AdS/CFT correspondence to condensed matter physics, as we will see in detail in Chapter 4, a nontrivial, r-dependent, profile of ϕ has a holographic interpretation in terms of a scalar condensate in the dual QFT triggering symmetry breaking and/or phase transitions [18, 20, 22–25]. If one is interested in reproducing phenomenological properties of strongly-coupled condensed matter systems, the actual form of the potential V may be rather irrelevant. Conversely, it is the behavior of the scalar condensate that contains more physical information.

Second, our approach is very useful for setting up a new no-hair theorem about the existence of black hole solutions of the field equations. In fact, our method allows us to write explicitly a – albeit formal – solution of the field equations for an arbitrary potential. This result will be used in Sect. 2.4, to prove a new no-hair theorem about the existence of black hole solutions of minimally coupled Einstein-scalar-Maxwell gravity.

Our method for solving the field equations (2.5) works as follows. Assuming that the r-dependence of the scalar field $\phi(r)$ is given, and introducing the new variables F, Y and u defined as:

$$F(r) = -\frac{2}{d}(\phi')^2, \quad R = e^{\int Y}, \quad u = UR^d, \tag{2.8}$$

the field equations (2.5) become:

$$Y' + Y^2 = F, \quad (u\phi')' = \frac{1}{4}e^{d\int Y}\frac{dV}{d\phi}, \tag{2.9}$$

$$u'' - (d+2)(uY)' = -2\varepsilon(d-1)e^{(d-2)\int Y} + 4Q^2 e^{-d\int Y}, \tag{2.10}$$

$$u'' = \varepsilon d(d-1)e^{(d-2)\int Y} + 2\frac{d-2}{d}Q^2 e^{-d\int Y} - \frac{d+2}{d}e^{d\int Y}V. \tag{2.11}$$

The first equation in (2.9) is a first-order nonlinear equation for Y, known as the Riccati equation, which can be solved in a number of cases. Once the solution for Y has been found we can integrate Eq. (2.10), which is linear in u, to obtain:

$$u = R^{d+2}\left[\int\left(4Q^2\int\frac{1}{R^d} - 2\varepsilon(d-1)\int R^{d-2} - C_1\right)\frac{1}{R^{d+2}} + C_2\right], \tag{2.12}$$

where C_1 and C_2 are integration constants. Finally, we can determine the potential $V(\phi)$ by using Eq. (2.11),

$$V = \frac{d^2(d-1)}{d+2}\frac{\varepsilon}{R^2} + 2\frac{d-2}{d+2}\frac{Q^2}{R^{2d}} - \frac{d}{d+2}\frac{u''}{R^d}, \tag{2.13}$$

while the metric functions read (cfr. 2.8):

$$R = \Lambda e^{\int Y}, \qquad U = \frac{u}{R^d}, \tag{2.14}$$

where we have introduced an integration constant Λ coming from the integral of Y.

In the following sections we will use this method to find solutions of minimally coupled Einstein-scalar-Maxwell gravity in different spacetime dimensions and for planar, spherical and hyperbolic topologies of the d-dimensional transverse section of the spacetime.

2.3 Domain walls and solutions conformal to Lifshitz

In this section, we consider the case of (3+1)-dimensional spacetime, i.e. $d = 2$, and black brane solutions, i.e. $\varepsilon = 0$. This is the most useful case for applications to holography. These solutions will be generalized to $d + 2$ spacetime dimensions in Sect. 2.7 and to black holes with spherical ($\varepsilon = 1$) or hyperbolic ($\varepsilon = -1$) symmetry in Sect. 2.8.

Our method for solving the field equations (2.5) requires an ansatz for the scalar field. In this section we wish to find domain wall and Lifshitz-like solutions. Usually, these solutions appear when the scalar behaves as $\log r$ [23, 25, 27, 28, 30]. The most natural ansatz is therefore:

$$\gamma\phi = \log \frac{r}{r_-}, \tag{2.15}$$

where γ and r_- are constants. Note that r_- has no particular physical meaning, but simply sets a length-scale.

2.3.1 Uncharged (Domain wall) solutions

Let us consider the solutions (2.12), (2.14) for $d = 2$, $\varepsilon = 0$. At first we examine the simplest case in which $Q = 0$. Choosing $C_1 = 0$ and scaling the constants C_2 and Λ to 1, one gets:

$$U = R^2.$$

Hence, for this choice of the parameters, the solution takes the form of a domain wall:

$$ds^2 = U(-dt^2 + dx^2 + dy^2) + U^{-1}dr^2. \tag{2.16}$$

Notice that in the relevant cases, even when $C_1 \neq 0$ the corresponding term in Eq. (2.12) in the $r \to \infty$ limit is subleading, and therefore the solution is still asymptotical to a domain wall.

With the ansatz (2.15) the Riccati equation is solved by:

$$Y = \frac{\alpha}{r}, \quad \alpha(\alpha - 1) = -\frac{1}{\gamma^2}. \tag{2.17}$$

Parametrizing α and γ as

$$\gamma^{-1} = h\alpha = \frac{h}{h^2+1}, \tag{2.18}$$

the solution takes the form:

$$U = \left(\frac{r}{r_-}\right)^{\frac{2}{1+h^2}} - C_1 \left(\frac{r}{r_-}\right)^{-\frac{1-h^2}{1+h^2}}, \qquad R = \left(\frac{r}{r_-}\right)^{\frac{1}{1+h^2}}, \tag{2.19}$$

with potential

$$V = -\frac{2(3-h^2)}{(1+h^2)^2 r_-^2}\, e^{-2h\phi}. \tag{2.20}$$

Hence the potential has a simple exponential form. If in the theory a length-scale L is present, as happens for instance when the exponential potential arises as near-horizon approximation of an asymptotically AdS spacetime, one can trade r_- for L using the invariance of the field equations under rescaling of $V \to \lambda V, U \to \lambda U$, yielding $V = -[\frac{2(3-h^2)}{(1+h^2)L^2}]e^{-2h\phi}$.

In the extremal case, $C_1 = 0$, the solution (2.19) has the typical form of a single-scalar domain wall solution, $ds^2 = (Ar)^\delta(\eta_{\mu\nu}dx^\mu dx^\nu) + (Ar)^{-\delta}dr^2$ (see Sect. 1.2.2). Domain wall solutions are conformal to AdS spacetime and have a consistent holographic interpretation, in terms of a dual QFT with only relativistic symmetry, for $\delta \geq 1$, which in our case implies $h^2 \leq 1$.

One can easily calculate the curvature invariants for the solution (2.19):

$$\begin{aligned}
\mathcal{R} &= \frac{2}{(1+h^2)^2 r_-^2}\left[3(h^2-2)x^{\frac{-2h^2}{1+h^2}} - h^2\mu x^{\frac{-3-h^2}{1+h^2}}\right], \\
\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} &= \frac{4}{(1+h^2)^4 r_-^4}\left[(3h^4-9h^2+9)x^{\frac{-4h^2}{1+h^2}} + 3h^2(1-h^2)\mu x^{-3} + h^4\mu^2 x^{\frac{-6-2h^2}{1+h^2}}\right],
\end{aligned}$$

where $x = r/r_-$, showing that $r = 0$ is a curvature singularity.

For $h^2 \leq 3$ and $C_1 > 0$, our solution (2.19) represents a black brane with domain wall asymptotics, a singularity at $r = 0$ and a horizon at $r_h = C_1^{(1+h^2)/(3-h^2)} r_-$. The horizon is regular and has negative curvature ($\mathcal{R}(r_h) < 0$, $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}(r_h) \neq 0$). The domain wall solution for $C_1 = 0$ shows a naked singularity at the origin and may be seen as the $T = 0$ extremal limit of (2.19). Notice however that the solution at finite temperature breaks the Poincaré isometry of the extremal domain wall solution. For $h^2 > 3$, the solution is still valid, but its physical interpretation is less clear.

2.3.2 Solutions conformal to Lifshitz spacetime

Let us now consider the case of nonvanishing electric charge. In this case it is convenient to adopt the parametrization:

$$\alpha = \frac{h^2}{h^2+4}, \qquad \gamma^{-1} = \frac{2h}{h^2+4}. \tag{2.21}$$

Using again the ansatz (2.15) and Eq. (2.17), the solution (2.12) reads:

$$U = \frac{Q^2}{\Lambda^4}\frac{(4+h^2)^2 r_-^2}{(4-h^2)(2-h^2)}\left(\frac{r}{r_-}\right)^{2\frac{4-h^2}{4+h^2}}\left[1 - C_1\left(\frac{r}{r_-}\right)^{-\frac{4-h^2}{4+h^2}} + C_2\left(\frac{r}{r_-}\right)^{-4\frac{2-h^2}{4+h^2}}\right],$$

$$R = \Lambda\left(\frac{r}{r_-}\right)^{\frac{h^2}{4+h^2}}, \tag{2.22}$$

with potential:

$$V = -\frac{Q^2}{\Lambda^4}\left[\frac{4}{2-h^2}\,e^{-2h\phi} + \frac{2h^2(3h^2-4)}{(4-h^2)(2-h^2)}\,C_2\,e^{-4\phi/h}\right], \tag{2.23}$$

where the integration constants C_1 and C_2 of (2.12) have been rescaled. The solution holds for $h^2 \neq 2, 4$.

Setting $\Lambda^2 = r_- Q$, the potential becomes independent from the electric charge. If an extra length scale L is present, one can, like in the uncharged case, trade r_- for L in the potential (2.23), so that it acquires a factor $1/L^2$. The constant C_2 is a parameter of the action, that can be chosen to vanish. In such case, one is left with an exponential potential like in (2.20).

For $C_2 = 0$, the extremal $C_1 = 0$ case in Eq. (2.22) represents a solution which is conformal to the Lifshitz spacetime:

$$ds^2 = l^2\left(-\bar{r}^{2z}dt^2 + \frac{d\bar{r}^2}{\bar{r}^2} + \bar{r}^2 dx^i dx^i\right). \tag{2.24}$$

This can be easily shown by setting $\bar{r} = (r/r_-)^{-h^2/(4+h^2)}$ in Eq. (2.22). The metric (2.22) is conformal (with conformal factor $\bar{r}^{-4}$) to the Lifshitz metric (2.24) with $z = 3 - 4/h^2$ and $l = r_-\sqrt{(4-h^2)(2-h^2)}/h^2$. Obviously the anisotropic scaling transformation between space and time:

$$t \to \lambda^z t, \quad \bar{r} \to \lambda^{-1}\bar{r}, \quad x^i \to \lambda x^i, \tag{2.25}$$

which is an isometry of the Lifshitz metric (2.24), is not longer an isometry of our solution (2.22). However, its conformality with Lifshitz implies that it scales with a definite weight under the anisotropic scaling transformation (2.25): $ds^2 \to \lambda^4 ds^2$. In the remaining of this chapter we will denote solutions which are conformal to Lifshitz spacetime simply as "conformal Lifshitz" .

For $h^2 < 2$, $C_2 = 0$ and $C_1 > 0$ the solution represents a black brane with asymptotics conformal to Lifshitz, a singularity at $r = 0$ and a regular horizon at $r_h = C_1^{(4+h^2)/(4-h^2)} r_-$. The conformal Lifshitz solution with $C_1 = 0$ may be seen as the $T = 0$ extremal limit of (2.22). The solution at finite temperature does not follow the simple scaling behavior of the extremal solution.

If instead $C_2 \neq 0$, two horizons may be present, depending on the value of the parameters. Moreover, the term in C_2 becomes dominant for $r \to \infty$ if $h^2 > 2$. In any case, the solutions (2.22) constitute a two-parameter family parametrized by C_1 and Q.

2.3.3 Alternative approach

When $C_2 = 0$, the solutions of this section can be obtained also by means of a more traditional approach, introduced in [6] and developed in several papers [7–9, 91].

Parametrizing the metric and the electric field as:

$$ds^2 = -e^{2\nu} dt^2 + e^{2\nu+4\rho} d\xi^2 + e^{2\rho}(dx^2 + dy^2), \qquad F_{t\xi} = e^{2\nu} Q, \tag{2.26}$$

where $\nu = \nu(\xi)$, $\rho = \rho(\xi)$ and $\phi = \phi(\xi)$, one can in fact reduce the field equations to the form of a dynamical system, that admits a three-parameter family of regular black brane solutions. Exact solutions can be obtained in a special two-parameter case and coincide with those obtained above. The third parameter is presumably related to the scalar charge. In the next chapter we will use this approach to find an exact solution of an Einstein-scalar gravity model.

It is important to stress that only the solutions derived in this section –

and their higher-dimensional generalization –, which correspond to a purely exponential potential, can be derived using the more traditional approach. The solutions derived in the remaining sections of this chapter, corresponding either to combination of exponentials (see Sects. 2.5, 2.7, 2.8) or to combinations of powers and trigonometric functions (see Sect. 2.6), cannot be derived using standard methods.

2.4 A No-hair theorem

In Sect. 1.2 we have already pointed out the importance of the question about the general existence of black hole solutions with non-trivial scalar hair. In particular we have seen that recent "no-hair" theorems put some constraints about the existence of hairy black hole solutions asymptotically AdS, as the violation of the PET and the breaking of the full AdS symmetry group [12].

We prove here, using the reformulation of the field equations discussed in Sect. 2.2, a new no-hair theorem about the existence of regular hairy black hole solutions of Einstein-scalar-Maxwell gravity. We will consider for simplicity the $d = 2$ case, but our theorem can be trivially generalized to arbitrary $d + 2$ dimensions. We will first consider the uncharged and planar case $Q = \varepsilon = 0$ and then we will generalize our argument to the charged and $\varepsilon = \pm 1$ cases.

A key ingredient for our argument is the existence of an extremal $T = 0$ hairy black hole solution. We will prove the validity of the following three statements about black hole solutions of Einstein gravity minimally coupled to a scalar field:

1) *One-parameter families of asymptotically AdS black brane solutions with nontrivial scalar hair exist only if the field equations (2.5) admit an extremal $T = 0$, $U = R^2$ solution.*

2) *Black brane solutions that asymptotically approach the domain wall solution (2.19) exist in some range of the parameters for the case of an exponential potential* $V(\phi)$.
3) *The allowed asymptotically AdS hairy black brane solutions necessarily have a scalar hair that depends on the black brane temperature* T. *Solutions with temperature-independent scalar hair exist only for the case of domain wall spacetimes (2.19).*

In order to prove part 1) of the theorem we start from equation (2.12), set $d = 2$, $\varepsilon = 0$ and fix the physically irrelevant integration constant $C_2 = 1$. In the planar and uncharged case we are considering, the constant C_2 is physically irrelevant because the field equations (2.5) are invariant under the rescaling $R \to \lambda R$ of the metric function R. C_2 can be set to 1 by using this symmetry. Physically, C_2 parametrizes the volume of the d-dimensional transverse space, which is not fixed by the field equations.

We get in this way the general form of the solution of the field equation (2.5) for the metric function U:

$$U = R^2 \left(1 - C_1 \int \frac{1}{R^4}\right). \tag{2.27}$$

The integration constant C_1 is determined in terms of the mass M (or equivalently of the temperature T) of the solution.

Assuming the existence of a one-parameter family of black brane solutions with a regular horizon at $r = r_h$, we have to require $U(r_h) = 0$, $R(r_h) \neq 0$. This implies that the horizon is determined by the equation:

$$1 - C_1 \int \frac{1}{R^4} = 0, \tag{2.28}$$

whereas for $C_1 = 0$ we have an extremal $T = 0$ domain wall solution with $U = R^2$.

Let us now assume that the field equations do not admit the $C_1 = 0$ extremal solution. Inserting Eq. (2.27) into the third field equation in (2.5),

C_1 becomes completely determined in terms of the functions R and ϕ. This, at least in principle, can be used to eliminate the integration constant C_1 from the field equations (2.5), which can now be used to determine the solutions for R and ϕ. As a consequence, the solutions for R and ϕ will not depend on C_1, i.e. they will be *temperature independent*.

Let us now pick up a particular – albeit generic – solution of the field equations (2.5) with $C_1 = C_1^{(0)}$, denote it with (U_0, R_0, ϕ_0) and decompose the general solution of the equations (2.5) as follows:

$$(U = U_0 + \tilde{U}(C_1, r),\ R,\ \phi). \tag{2.29}$$

Because R and ϕ do not depend on C_1, we must have $\phi = \phi_0$ and $R = R_0$. Substitution of Eq. (2.29) into Eq. (2.5) gives $(\tilde{U}R^2)'' = 0$, $(\tilde{U}R^2\phi')' = 0$, which implies $\phi = c \log \frac{r}{r_0}$, with c, r_0 integration constants. According to Eqs. (2.15)-(2.20), this is only possible for an exponential potential $V(\phi)$ and gives the domain wall solution (2.19). From this, part 1) and part 2) of the no-hair theorem follow immediately.

Obviously, if the field equations allow the $C_1 = 0$ solution the previous derivation fails. We can choose in Eq. (2.29) U_0 as the $C_1 = 0$ solution, whereas ϕ and R do not need to be independent from C_1.

Statement 3) can be proved with a slight modification of the previous argument. One begins by noticing that, owing to the first equation in (2.5), a temperature-independent scalar hair implies that also the function R is temperature-independent. One then assumes the existence of a one-parameter family of black brane solutions of the field equations (2.5) $(U(C_1), R, \phi)$ with ϕ and R independent of C_1. Repeating the argument starting from Eq. (2.29) one easily finds that the one-parameter family of hairy solutions $(U(C_1), R, \phi)$ exists only in the case of an exponential potential and is given by the black brane solution (2.19).

The previous derivation can be easily extended to the charged case $Q \neq 0$ and to $\varepsilon = \pm 1$. The only new ingredient is that now the general solution of the field equations (2.5) is determined by:

$$U = R^2 \left[C_2 + \int \left(4Q^2 \int \frac{1}{R^2} - 2\varepsilon r - C_1 \right) \frac{1}{R^4} \right], \tag{2.30}$$

rather than by Eq. (2.27). Notice that for $\varepsilon = \pm 1$ and/or $Q \neq 0$ the rescaling of R is not anymore a symmetry of the equations of motion as in the $\varepsilon = 0$ case. The constant C_2 in Eq. (2.30) becomes physically relevant but enters as parameter of the potential, therefore cannot represent a new "independent hair" of the solution.

Since for $Q \neq 0$ or $\varepsilon \neq 0$ the solution (2.30) with $C_1 = 0$ is no longer given by $U = R^2$, in general, it will not necessarily be an extremal $T = 0$ solution. This is related to the fact that C_1 will now be determined not only in terms of the black hole mass M, but in terms of Q as well. As a consequence, the spacetime will in general have an inner and outer horizon. Statement 1) then holds in a much weaker form: One-parameter families of asymptotically charged AdS black hole solutions with nontrivial scalar hair exist only if the field equations (2.5) admit a black hole solution with $C_1 = 0$. Because the $C_1 = 0$ solution is not necessarily extremal, this statement is not particularly useful.

On the other hand statements 2) and 3) do not depend on the existence of an extremal solution. Their generalization to the charged and $\varepsilon = \pm 1$ case is almost trivial. Statement 2) now affirms that charged black brane/black hole solutions that asymptotically approach the conformal Lifshitz spacetime (2.22) exist, in some range of the parameters, for the case of an exponential potential $V(\phi)$. Statement 3) still remains true in the form given above also for the case of charged and $\varepsilon = \pm 1$ black holes.

Concerning statement 3) it is important to stress that this theorem does not apply to the case of a nonminimal coupling between the scalar and the gauge field. In the latter case we have an additional term depending on the derivative of the coupling function between the scalar and F^2 in the last equation of (2.5). The effect of this term is to allow for solutions with two integration constants $r_\pm$, with $Q \sim r_- r_+$, $T \sim (r_+ - r_-)$, whereas the scalar field depends on r_- only. Hence the scalar hair is independent from the black hole temperature but is related to Q. This result is perfectly consistent with the well-established existence of black holes with temperature-independent scalar charges in models with a non-minimally coupled scalar field.

We conclude this section by listing the classes of static black hole solutions of Einstein-scalar-Maxwell gravity that may exist in view of the above no-hair theorems:

- Models with an exponential potential admit, at least in some range of the parameters, a one-parameter family of domain wall ($Q = 0, \varepsilon = 0$), conformal Lifshitz ($Q \neq 0, \varepsilon = 0$), black brane or black hole ($Q = 0$ or $Q \neq 0$ and $\varepsilon = \pm 1$) solutions.
- The existence of asymptotically AdS uncharged black brane solutions with scalar hair is tightly constrained. Apart from the violation of the PET, a further necessary, but not sufficient, condition for their existence is that the field equations allow for an extremal $T = 0$ solution.
- The existence of hairy asymptotically AdS charged black branes, or charged and uncharged black holes, is very loosely constrained by the above no-hair theorem.
- For all cases (charged and uncharged, black branes and black holes) the allowed hairy AdS solutions must have a temperature-dependent scalar hair.

2.5 Asymptotically AdS solutions

In this section we will derive asymptotically AdS solutions with scalar hair of the field equations (2.5) for $d = 2$ and $\varepsilon = 0$. As explained in Sects. 1.1 and

2.2, in order to have asymptotically AdS solutions we require the potential to satisfy $V(0) < 0$, $V'(0) = 0$ and we normalize V using $V(0) = -6/L^2$.

As usual, the starting point of our solving method is an ansatz for the scalar field. Inspired by known solutions in flat spacetime and in gauged supergravity [6–9, 72], we use an ansatz in which ϕ is expressed in terms of a four-dimensional harmonic function X:

$$\gamma\phi = \log X, \qquad X = 1 - \frac{r_-}{r}, \tag{2.31}$$

where γ and r_- are constants. Notice that with this ansatz, in the asymptotical AdS region the scalar field ϕ is a tachyonic excitation with mass above the BF bound in 4D, $m^2 = -2/L^2$. Expanding Eq. (2.31) near $r = \infty$ and comparing with Eq. (2.7) one finds that the asymptotic behavior of the scalar field is characterized by $\Delta_{1,2} = 1, 2$ and by $O_2 = r_- O_1$. This tells us that we are dealing with so-called designer gravity models (see Sect. 1.1).

Given the ansatz (2.31), the Riccati equation (2.9) can be solved in terms of the harmonic function X to give:

$$R = \Lambda r X^{\beta+\frac{1}{2}}, \qquad \beta^2 - \frac{1}{4} = -\frac{1}{\gamma^2}, \tag{2.32}$$

where Λ can be set to 1 without loss of generality, if $\varepsilon = 0$ and $Q = 0$. Notice that the previous equation implies

$$-\frac{1}{2} < \beta < \frac{1}{2}. \tag{2.33}$$

As usual we will proceed by discussing separately uncharged and charged solutions.

2.5.1 Uncharged solutions

Let us set $Q = 0$ in Eq. (2.12) and first consider the $C_1 = 0$ extremal solutions. The constant C_2 essentially determines the normalization of the potential. This is fixed by choosing $C_2 = 1/L^2$. With these assumptions, Eqs. (2.12) and (2.13) give, respectively, the solution for the metric and the scalar potential:

$$U = R^2 = \frac{r^2}{L^2}\left(1 - \frac{r_-}{r}\right)^{2\beta+1}, \tag{2.34}$$

$$V_1(\gamma, \phi) = -\frac{2e^{2\gamma\beta\phi}}{L^2}\left[2 - 8\beta^2 + (1 + 8\beta^2)\cosh(\gamma\phi) - 6\beta\sinh(\gamma\phi)\right]. \tag{2.35}$$

One can easily check that these solutions represent domain walls with AdS_4 asymptotics. Calculating the periodicity of the 2D Euclidean section one can also check that the solution is an extremal $T = 0$ solution.

The potential (2.35) interpolates smoothly between the asymptotic AdS region at $\phi = 0$ and a $\phi \to \infty$ region (a near-horizon region) where the potential behaves exponentially:

$$V(\phi) = -\frac{(2\beta + 1)(4\beta + 1)}{L^2}\, e^{\gamma(2\beta-1)\phi}. \tag{2.36}$$

Moreover, it contains as a special case, $\beta = 0$, the potential resulting from truncation to the abelian sector of $\mathcal{N} = 8$, $D = 4$ gauged supergravity [51]:

$$V(\phi) = -\frac{2}{L^2}(\cosh 2\phi + 2). \tag{2.37}$$

In this case the solution (2.34) takes the particularly simple form:

$$U = R^2 = \frac{r^2}{L^2} - \frac{rr_-}{L^2},\ \phi = \frac{1}{2}\log\left(1 - \frac{r_-}{r}\right). \tag{2.38}$$

The model described by the potential (2.35) becomes very simple also for $\beta = 1/4$:

$$V(\phi) = -\frac{6}{L^2}\cosh\frac{2\phi}{\sqrt{3}}. \tag{2.39}$$

The potential (2.35) remains invariant under the duality transformation:

$$\phi \to -\phi, \quad \beta \to -\beta. \tag{2.40}$$

This symmetry of the action can be used to generate a new dual solution from Eq. (2.34):

$$U = R^2 = \frac{r^2}{L^2} X^{-2\beta+1}, \quad \gamma\phi = -\log X. \tag{2.41}$$

Notice that for the supergravity model (2.37) the symmetry transformation is simply $\phi \to -\phi$, whereas solution (2.38) becomes self-dual.

Because the model with potential (2.35) admits the $C_1 = 0$ solution, statement 1) of the no-hair theorem discussed in the previous section implies that also non extremal black brane solutions with $C_1 \neq 0$ can in principle exist. Unfortunately, our method does not allow to find such a solutions. Naively, one could think that these solutions can be derived just by using Eq. (2.32) into Eq. (2.12) with $C_1 \neq 0$. This is not the case not only because the resulting potential $\tilde{V}$ is different from (2.35) but, more importantly, because $\tilde{V}$ will depend explicitly on C_1, which instead should be a free integration constant related to the mass of the solution.

Notice that a one-parameter family of solutions can be generated from Eq. (2.12) with $C_1 \neq 0$ by using the invariance of the field equations under the rescaling $R \to \lambda R$, to let the potential depend only on the ratio C_2/C_1, whereas U depends on both C_2 and C_1. However, in this case the solution $C_1 = 0$ is not allowed. Hence the no-hair theorem of the previous section implies that this family of solutions are not black branes. Thus non-extremal black brane solutions of models with the potential (2.35), if they exist, have to be found numerically.

The hairy extremal solution (2.34) interpolates between an AdS vacuum at $r = \infty$ and a domain wall solution (2.19) near $r = r_-$. This can be easily shown by expressing solution (2.34) in the near-horizon approximation

$r \sim r_-$. Shifting $r \to r + r_-$ and expanding near $r = 0$ one finds at leading order:

$$\gamma\phi = \ln \frac{r}{r_-}, \qquad U = R^2 = A^2 \left(\frac{r}{r_-}\right)^{2\beta+1}, \qquad A = \frac{r_-}{L}. \tag{2.42}$$

As expected, this solution is easily recognized, just by setting $\beta = (1 - h^2)/(2 + 2h^2)$ and by rescaling V (in the way explained after Eq. (2.19)) as the exact solutions (2.19) with $C_1 = 0$ of a model with near-horizon exponential potential (2.36).

The near-extremal solutions with a horizon, corresponding to solutions (2.19) with $C_1 \neq 0$, are given by:

$$U = A^2 \left(\frac{r}{r_-}\right)^{2\beta+1} - \mu \left(\frac{r}{r_-}\right)^{-2\beta}, \tag{2.43}$$

whereas ϕ and R are given as in Eq. (2.42). We stress again that Eq. (2.42) and (2.43) are exact solutions of the near-horizon approximate form of the potential (2.36), but only leading-order solutions of the near-horizon approximation for the exact potential (2.35).

The near-horizon, extremal and near-extremal solutions, corresponding to the dual solution (2.41), can be easily obtained from Eq. (2.43) just by using the duality transformation (2.40).

2.5.2 Charged solutions

Following the same steps described in the previous subsection we now derive $Q \neq 0$ hairy black brane solutions of the field equations (2.5) with AdS asymptotics.

The $C_1 = C_2 = 0$ solution, corresponding to the ansatz (2.31) for the scalar, is obtained by substituting the solution (2.32) of the Riccati equation into Eq. (2.12):

$$U = \frac{r^2}{L^2}\left(1-\frac{r_-}{r}\right)^{-2\beta}\left(1-\frac{r_1}{r}\right)\left(1-\frac{r_2}{r}\right),$$
$$R = \Lambda\frac{r}{r_-}\left(1-\frac{r_-}{r}\right)^{\beta+\frac{1}{2}}, \quad \gamma\phi = \log\left(1-\frac{r_-}{r}\right), \quad \frac{1}{4}-\beta^2 = \gamma^{-2}, \tag{2.44}$$

where $\Lambda^2 = QL/(|\beta|\sqrt{\frac{3}{2}(36\beta^2-1})$, $r_{1,2} = (r_-/2)(6\beta+1\pm\sqrt{36\beta^2-1})$ and $1/6 < |\beta| < 1/2$.

Using Eq. (2.13), the corresponding potential turns out to be:

$$\begin{aligned} V_2(\gamma,\phi) &= -\frac{2}{L^2}e^{-4\gamma\beta\phi}(4\beta^2-1)\left[-(36\beta^2+1)\cosh(\gamma\phi)+27\beta^2-2-12\beta\sinh(\gamma\phi)\right] \\ &+ 3\beta^2(1+12\beta^2)\cosh(2\gamma\phi)+24\beta^3\sinh(2\gamma\phi). \end{aligned} \tag{2.45}$$

Eqs. (2.44) represent a one-parameter (the charge) family of asymptotically AdS solutions of the model (2.45) for a fixed value of the mass (or temperature). As pointed out in Sect. 2.4, in the charged case the $C_1 = 0$ solution does not necessarily correspond to extremal $T = 0$ black brane solutions. Moreover, in this case statement 1) of the no-hair theorem of the previous section is not useful for guessing about the existence of a full two-parameter (charge Q and mass M) family of hairy black brane solutions.

The potential (2.45) is invariant under the duality symmetry (2.40). The dual solutions are easily obtained using Eq. (2.40) into Eq. (2.44).

The geometrical and thermal properties of the solution (2.44) depend on the value of β. In the parameter region $-1/2 < \beta < -1/6$ we have $r_1, r_2 < r_-$. Because r_- is the origin of the radial coordinate r, there are no horizons and the solution is an extremal $T = 0$ solution. For $1/6 < \beta < 1/2$ we get $r_1 > r_-$ and we have an horizon. Because r_1 is just a simple (not double) root of U, the solution does not represent an extremal $T = 0$ solution.

Solution (2.44) for $-1/2 < \beta < -1/6$ interpolates between an AdS vacuum at $r = \infty$ and a conformal Lifshitz solution (2.22) with $C_1 = C_2 = 0$ in the near-horizon limit $r \sim r_-$. In fact, shifting $r \to r + r_-$ and expanding near $r=0$, Eq. (2.44) becomes:

$$\gamma\phi = \ln\frac{r}{r_-}, \quad U = B\left(\frac{r}{r_-}\right)^{-4\beta}, \quad R = \frac{r_-}{L}\left(\frac{r}{r_-}\right)^{\beta+\frac{1}{2}}, \tag{2.46}$$

where B is a constant depending on β and r_-. Solution (2.46) has the conformal Lifshitz form (2.22). On the other hand for $1/6 < \beta < 1/2$ we have $U \sim r$ and $R \sim$ const., which seems to indicate that in this case the solution has to interpreted as an extremal $T \neq 0$ solution.

Analogously to the $Q=0$ case, one can also write down near-extremal, approximate solutions with an horizon (black brane):

$$\gamma\phi = \ln\frac{r}{r_-}, \quad U = B\left(\frac{r}{r_-}\right)^{-4\beta} - \mu\left(\frac{r}{r_-}\right)^{-2\beta}, \quad R = \frac{r_-}{L}\left(\frac{r}{r_-}\right)^{\beta+\frac{1}{2}}. \tag{2.47}$$

Also in the charged case, both the $C_1 = 0$ solution (2.46) and the near-extremal solution (2.47) are exact solutions of an Einstein-scalar-Maxwell gravity model with an exponential potential given by the leading term in the $\phi \to \infty$ expansion of the potential (2.45).

2.5.3 Other solutions

In this subsection we present a further example of the use of our general method for generating exact solutions of (2.5) with AdS asymptotic behavior, for $Q=0$, $d=2$ and $\varepsilon = 0$.

As ansatz for the scalar field we choose a combination of harmonic functions in $n+2$ dimensions[1]:

$$\phi = \frac{\sqrt{2n-1}}{2n}\log\frac{X_+}{X_-}, \qquad X_\pm = 1 \pm \left(\frac{r_-}{r}\right)^n, \qquad n > \frac{1}{2}. \tag{2.48}$$

In an asymptotically AdS spacetime, this corresponds to a scalar excitation

[1]We do not limit ourselves to an integer n, but we take n real.

near $\phi = 0$ of mass:

$$m^2 = -\frac{n(3-n)}{L^2}. \tag{2.49}$$

The scalar excitation is a tachyon with mass above the BF bound for $1/2 < n < 3$. The PET implies the non existence of black brane solutions for $n \geq 3$.

The Riccati equation (2.9) is solved by:

$$Y = \frac{r^{2n-1}}{r^{2n} - r_-^{2n}}. \tag{2.50}$$

In the uncharged case, $Q = 0$, Eq. (2.12) with $C_1 = 0$ and $C_2 = 1/L^2$ gives the solution:

$$U = R^2 = \frac{r^2}{L^2}\left[1 - \left(\frac{r_-}{r}\right)^{2n}\right]^{\frac{1}{n}}, \quad \phi = \frac{\sqrt{2n-1}}{2n}\log\frac{r^n + r_-^n}{r^n - r_-^n}, \tag{2.51}$$

which represents an asymptotically AdS domain wall solution. As expected, also in this case the solution is a $T = 0$ extremal solution. Eq. (2.13) gives the potential:

$$V(\phi) = -\frac{2}{L^2}(\cosh\frac{a\phi}{2})^{2-\frac{2}{n}}\left[(2-n)\cosh a\phi + (n+1)\right], \quad a = \frac{2n}{\sqrt{2n-1}}. \tag{2.52}$$

Notice that this potential is invariant under the duality transformation $\phi \to -\phi$. The potential (2.52) smoothly interpolates between the asymptotical AdS region at $\phi = 0$ and the $\phi \to \infty$ near-horizon region, where the potential has the exponential behavior:

$$V(\phi) = -\frac{(2-n)}{L^2}2^{\frac{2}{n}-2}e^{2\sqrt{2n-1}\,\phi}. \tag{2.53}$$

In the special case $n = 2$ the potential takes a very simple form:

$$V(\phi) = -\frac{6}{L^2}\cosh\frac{2\phi}{\sqrt{3}}. \tag{2.54}$$

Also for the case of the potential (2.52) hold the same considerations concerning the existence of non-extremal black brane solutions as those discussed in subsection 2.5.1 for the potential (2.35).

The extremal solution (2.51) interpolates between an AdS vacuum at $r = \infty$ and a domain wall solution of the form (2.19) with $C_1 = 0$ near $r = r_-$. This can be easily seen by working in the near-horizon approximation. Shifting $r \to r + r_-$ and expanding near $r = 0$, Eq. (2.51) becomes at leading order:

$$\phi = -\frac{1}{a}\ln\frac{r}{r_-}, \quad U = R^2 = D^2\left(\frac{r}{r_-}\right)^{\frac{1}{n}}, \quad D = \frac{r_-}{L}(2n)^{1/(2n)}. \tag{2.55}$$

Near-extremal approximate solutions with a horizon have the form (2.19) with $C_1 \neq 0$ and are given by:

$$U = D^2\left(\frac{r}{r_-}\right)^{1/n} - \mu\left(\frac{r}{r_-}\right)^{1-1/n}, \tag{2.56}$$

whereas ϕ and R are given as in Eq. (2.55). As expected, solution (2.56) is an exact solution of a model with the exponential potential (2.53).

2.6 Solutions for models with nonexponential potentials

Until now we have considered the application of our method to models for which the potential turns out to behave as a combination of exponentials. This fact may have generate in the reader the wrong impression that our method works only for this class of Einstein-scalar-Maxwell (ESM) gravity models. In this section we will show that this is not the case, i.e. that our method can be used to generate solutions of models whose potential does not behave exponentially.

These solutions can be generated using a general method for solving the Riccati equation. The method works when the equation takes the form:

$$Y' + aY^2 = br^s, \tag{2.57}$$

where a, b are constants and $s = 4n/(1-2n), n = \pm 1, \pm 2 \ldots$..

To solve the Riccati equation we perform, iteratively, the following transformations until we reach $s = 0$:

$$n > 0 \quad : \quad Y = Z^{-1}r^{-2} + (ar)^{-1}, \quad r = x^{\frac{1}{s+3}}; \tag{2.58}$$

$$n < 0 \quad : \quad Y = \frac{b}{x(bxZ + s + 1)}, \quad r = x^{-\frac{1}{s+1}}. \tag{2.59}$$

In both cases, after the transformation, the Riccati equation takes the form $\frac{dZ}{dx} + \hat{a}Z^2 = \hat{b}x^{\hat{s}}$ with $\hat{a} = b/(s+3), \hat{b} = a/(s+3), \hat{s} = -(s+4)/(s+3)$ for $n > 0$ and $\hat{a} = -b/(s+1), \hat{b} = -a/(s+1), \hat{s} = -(3s+4)(s+1)$ for $n < 0$. Once we reach $s = 0$, the Riccati equation becomes separable and can be integrated using elementary methods.

As an example let us consider ESM gravity in 4D with $\varepsilon = Q = 0$ and a scalar field with an $1/r$ profile:

$$\phi = \frac{k}{r}. \tag{2.60}$$

In four-dimensional asymptotically AdS spacetimes this corresponds to a scalar field of mass $m^2 = -2/L^2$ and boundary conditions (2.7) characterized by $O_2 = 0, O_1 = k$. The Riccati Eq. (2.9) takes the form (2.57) with $s = -4$ ($n = 1$), $a = 1$ and $b = -k^2$. After a single iteration of the solving procedure, the equation is brought into the form $\frac{dZ}{dx} + k^2 Z^2 = -1$, which can be easily integrated to give, after reintroducing the initial variables, the solution:

$$Y = \frac{k}{r^2} \cot\left(c - \frac{k}{r}\right) + \frac{1}{r}, \tag{2.61}$$

where $0 < c \leq \pi/2$ is an integration constant. Using Eqs. (2.8) and (2.12) with $C_1 = 0$ and $C_2 = 1$, one finds the solutions of the field equations of ESM gravity:

$$U = R^2, \quad R = \frac{r}{L} \sin\left(c - \frac{k}{r}\right). \tag{2.62}$$

For $\frac{k}{c} < r < \infty$, the solution describes a domain wall with AdS asymptotics. The point $r = \frac{k}{c}$ is the origin of the radial coordinate, whereas asymptotically, as $r \to \infty$, we have $U = R^2 \sim r^2/L^2$. The potential can be calculated using Eq. (2.13). Fixing for simplicity the value of the constant c to $c = \pi/2$, one finds:

$$V(\phi) = -\frac{2}{L^2}\left[\frac{3}{2}(1+\cos 2\phi) + 3\phi \sin 2\phi + \phi^2(1 - 2\cos 2\phi)\right]. \tag{2.63}$$

The resulting potential is a combination of powers and trigonometric functions. The asymptotic $r = \infty$ AdS region corresponds to $\phi = 0$. As expected one has $V(0) = -6/L^2$, $V'(0) = 0$, $V''(0) = -8/L^2$. The solution (2.60), (2.62) is now defined for $0 < \phi < \pi/2$, corresponding to $2k/\pi < r < \infty$.

2.7 Generalization to $d + 2$ dimensions

In this section we generalize the black brane solutions found in the $d = 2$ case to $d + 2$ dimensions.

2.7.1 Domain wall solutions

We start again from the ansatz (2.15). When $d \neq 2$ the Riccati equation (2.9) is solved by:

$$Y = \frac{\alpha}{r}, \quad \alpha(\alpha - 1) = -\frac{2}{d}\,\gamma^{-2}. \tag{2.64}$$

It is useful to parametrize α and γ as follows:

$$\alpha = \frac{2}{2d + h^2}, \qquad \gamma^{-1} = \frac{dh}{2 + dh^2}. \tag{2.65}$$

Redefining the constant C_2 in Eqs. (2.8) and (2.12), and rescaling C_1, the solution takes the form:

$$U = \left(\frac{r}{r_-}\right)^{\frac{4}{2+dh^2}} - C_1 \left(\frac{r}{r_-}\right)^{\frac{dh^2-2d+2}{2+dh^2}}, \tag{2.66}$$

$$R = \left(\frac{r}{r_-}\right)^{\frac{2}{2+dh^2}}, \tag{2.67}$$

$$V = -\frac{2d[2(d+1) - dh^2]}{(2 + dh^2)^2 r_-^2}\, e^{-2h\phi}. \tag{2.68}$$

As usual, the parameter r_- in the potential can be substituted by the AdS scale L, using the invariance of the field equations under rescaling of V and U.

As in $d = 2$, choosing $C_1 = 0$ we obtain the typical domain wall solution $U = R^2$. For $h^2 \leq 2/d$, the domain wall solution (2.66) with $C_1 = 0$ has a consistent holographic interpretation and a singularity at $r = 0$. For $C_1 \geq 0$ and $h^2 \leq 2 + 2/d$, the solution (2.66) is asymptotical to the domain wall solution, and has a horizon at $r_h = C_1^{(2+dh^2)/(2d+2-dh^2)} r_-$.

2.7.2 Charged solutions

The previous solution can be generalized to the case $Q \neq 0$. This is the only charged solution, among those found for $d = 2$, that can be computed in closed form in $d + 2$ dimensions.

The ansatz for the scalar field is still given by (2.15), whereas the solution for the Riccati equation is the same as in Eq. (2.64). In the case at hand, it is convenient to choose the following parametrization for α and γ:

$$\alpha = \frac{h^2}{2d+h^2}, \quad \gamma^{-1} = \frac{dh}{2d+h^2}.$$

Equations (2.12), (2.13) and (2.64) give (after a rescaling of the integration constants C_1 and C_2):

$$\begin{aligned} U &= \frac{2(2d+h^2)^2 r_-^2 Q^2}{[2d-(d-1)h^2](2d-dh^2)\Lambda^{2d}} \left(\frac{r}{r_-}\right)^{2\frac{2d-(d-1)h^2}{2d+h^2}} \left[1 - C_1 \left(\frac{r}{r_-}\right)^{-\frac{2d-(d-1)h^2}{2d+h^2}} \right. \\ &\left. + C_2 \left(\frac{r}{r_-}\right)^{-\frac{4d-2dh^2}{2d+h^2}} \right], \\ R &= \Lambda \left(\frac{r}{r_-}\right)^{\frac{h^2}{2d+h^2}}, \qquad (2.69) \\ V &= -\frac{2Q^2}{(2-h^2)\Lambda^{2d}} \left[2e^{-2h\phi} + \frac{h^2[-2d+(d+1)h^2]C_2}{[2d-(d-1)h^2]} e^{-4\phi/h}\right]. \qquad (2.70) \end{aligned}$$

In order to make the potential independent from the electric charge, one must choose the integration constant $\Lambda = (r_- Q)^{1/d}$. As usual, one can introduce a further length scale L in the potential, by performing a rescaling of the variables.

For $C_1 = C_2 = 0$, the solution is conformal to $(d+2)$-dimensional Lifshitz spacetime. For $h^2 < 2d/(d-1)$, $C_2 = 0$ and $C_1 > 0$ the solution represents a black brane asymptotical to the conformal Lifshitz spacetime, with a singularity at $r = 0$ and a horizon at $r_h = C_1^{\frac{2d+h^2}{2d-(d-1)h^2}} r_-$.

2.7.3 Asymptotically AdS solutions

In order to derive asymptotically AdS_{d+2} solutions of our field equations (2.5), we consider again the ansatz (2.31), which expresses the scalar field in terms of a harmonic function X given as in (2.31).

Near the AdS vacuum the scalar field is tachyonic and has mass:

$$m^2 = -d/L^2,$$

which is always above the BF bound in $d+2$ dimensions. The Riccati equation (2.9) is now solved by:

$$R = r\left(1 - \frac{r_-}{r}\right)^{\beta+\frac{1}{2}}, \qquad \frac{1}{4} - \beta^2 = \frac{2}{d}\gamma^{-2}, \qquad -\frac{1}{2} < \beta < \frac{1}{2}. \tag{2.71}$$

We search again for extremal solutions with $C_1 = 0$. Setting $C_2 = 1/L^2$ in (2.12) and (2.13), we obtain the following asymptotically AdS_{d+2} domain wall solution and the corresponding potential:

$$U = R^2 = \frac{r^2}{L^2}\left(1 - \frac{r_-}{r}\right)^{2\beta+1}, \quad \gamma\phi = \ln\left(1 - \frac{r_-}{r}\right) \tag{2.72}$$

$$V(\phi) = -\frac{d}{L^2}e^{2\gamma\beta\phi}\left\{\frac{1}{2}(d+2)(1-4\beta^2) + \frac{1}{2}\left[4\beta^2(d+2)+d\right]\cosh(\gamma\phi) - 2\beta(d+1)\sinh(\gamma\phi)\right\}. \tag{2.73}$$

One can easily check that the previous potential satisfies, as expected, $V(0) = -d(d+1)/L^2$ and $V'(0) = 0$. Notice that the metric part of the solutions for the generic case (2.72) is exactly the same as in the $d = 2$ case (see Eq. (2.34)). Only the scalar field and the potential are changed. Also in $d+2$ dimensions the potential (2.73) is invariant under the duality transformation (2.40). Dual solutions are easily obtained using (2.40) into Eq. (2.72) and (2.71).

Since the metric functions U and R do not depend on the spacetime dimension, the near-horizon and near-extremal approximate behavior of U and R is the same as in the $d = 2$ case. Thus, the hairy extremal solution (2.72) always interpolates between an AdS_{d+2} vacuum at $r = \infty$ and a domain wall solution (2.66) near $r = r_-$.

As in $d = 2$, the case $\beta = 0$ is particularly simple. The metric part of the solution is still the same as in $d=2$ and is given by Eq. (2.38), whereas the scalar field and the potential are:

$$\phi = \frac{1}{2}\sqrt{\frac{d}{2}}\log\left(1-\frac{r_-}{r}\right), \quad V(\phi) = -\frac{d}{L^2}\left[\frac{d}{2}\cosh\left(2\sqrt{\frac{2}{d}}\phi\right)+\frac{d+2}{2}\right].$$

For what concerns the existence of nonextremal $C_1 \neq 0$ solutions in $d+2$ dimensions, and the consequences of the no-hair theorem of Sect. 2.4, the same considerations as in the $d=2$ case hold.

2.7.4 Other solutions

It is also easy to work out the generalization to $d+2$ dimensions of the model described in subsection 2.5.3.

We consider the following ansatz for the scalar field:

$$a\phi = \log\frac{X_+}{X_-}, \quad X_\pm = 1 \pm \left(\frac{r_-}{r}\right)^n, \quad a = \sqrt{\frac{8n^2}{d(2n-1)}}, \quad n > \frac{1}{2}. \tag{2.74}$$

Near the AdS_{d+2} vacuum the scalar field is a tachyon with mass:

$$m^2 = -\frac{n(d+1-n)}{L^2},$$

which is always above the BF bound in $d+2$ dimensions. The PET forbids the existence of black brane solutions for $n \geqslant d+1$, when the square-mass of the scalar becomes positive.

The Riccati equation gives the same solution (2.50) as in the $d=2$ case, the metric function U (with $C_1 = 0$ and $C_2 = \frac{1}{L^2}$) is given by Eq. (2.51), while the potential becomes:

$$V(\phi) = -\frac{d}{L^2}\left(\cosh a\frac{\phi}{2}\right)^{2-\frac{2}{n}}\left[\left(\frac{d+2}{2}-n\right)\cosh a\phi + \left(n+\frac{d}{2}\right)\right]. \tag{2.75}$$

Because the metric part of the solution is the same obtained for $d = 2$, the near-horizon, near-extremal approximate solutions for U and R are identical to those obtained in four dimensions.

2.8 Spherical and hyperbolic solutions

The results of the previous sections can be easily generalized to the case in which the two-dimensional sections of the solutions are spherical or hyperbolic. Contrary to the planar case, where it is dimensionless, the metric function R, and hence the integration constant Λ in the solution (2.14), is now usually taken to have the physical dimension of a length. Therefore, when Λ is not determined by the field equations, we shall identify it with the AdS length L.

2.8.1 Uncharged black hole solutions

We first consider the case of four dimensions ($d = 2$). The field equations are given by (2.5), with $\varepsilon = \pm 1$ and the solutions by (2.9)-(2.14).

The generalization of the black brane solutions of Sect. 2.3.1 to the case of spherical (or hyperbolic) symmetry is obtained adopting the ansatz (2.15). Substituting the solutions (2.17), with parametrization (2.18), in the general solution of Sect. 2.2, after rescaling C_1 and putting $C_2 = 0$, the metric functions take the form:

$$U = \frac{(1+h^2)\varepsilon r_-^2}{(1-h^2)L^2}\left(\frac{r}{r_-}\right)^{\frac{2h^2}{1+h^2}}\left(1 - \frac{C_1 r_-}{r}\right), \qquad R = L\left(\frac{r}{r_-}\right)^{\frac{1}{1+h^2}}, \tag{2.76}$$

with potential

$$V = \frac{2h^2\varepsilon}{(h^2-1)L^2}\, e^{-2\phi/h}, \tag{2.77}$$

having a simple exponential form, as in the planar case. Notice that we have identified the integration constant Λ with the the AdS length L.

If $\varepsilon = 1$ and $h^2 < 1$, the solutions represent spherically symmetric black holes with conformal Lifshitz asymptotics, exhibiting a singularity at $r = 0$, shielded by a horizon at $r_h = C_1 r_-$. Solutions exist also for $\varepsilon = -1$ and $h^2 > 1$: they are black holes with conformal Lifshitz asymptotics and horizons with hyperbolic topology.

2.8.2 Charged black hole solutions

We now try to extend the previous solutions to the case of nonvanishing electric charge, generalizing those of Sect. 2.3.2. With the parametrization (2.21) the solution reads, after a redefinition of the constants C_1 and C_2:

$$\begin{aligned} U &= \frac{(4+h^2)r_-^2}{4-h^2}\left[-\frac{\varepsilon}{\Lambda^2}\left(\frac{r}{r_-}\right)^{\frac{8}{4+h^2}} + \frac{(4+h^2)Q^2}{(2-h^2)\Lambda^4}\left(\frac{r}{r_-}\right)^{2\frac{4-h^2}{4+h^2}} + C_2\left(\frac{r}{r_-}\right)^{\frac{2h^2}{4+h^2}} \right. \\ & \left. -C_1\left(\frac{r}{r_-}\right)^{\frac{4-h^2}{4+h^2}}\right], \qquad R = \Lambda\left(\frac{r}{r_-}\right)^{\frac{h^2}{4+h^2}}, \end{aligned} \tag{2.78}$$

with

$$V = -\frac{4Q^2}{(2-h^2)\Lambda^4}\, e^{-2h\phi} + \frac{8\,\varepsilon}{(4-h^2)\Lambda^2}\, e^{-h\phi} + \frac{2h^2(4-3h^2)C_2}{(4+h^2)(4-h^2)}\, e^{-4\phi/h}. \tag{2.79}$$

Contrary to the planar case, if $\varepsilon \neq 0$ one cannot eliminate from the potential the dependence on Q by a suitable choice of the integration constants: one ought in fact to impose $\Lambda^2 = Q = 1$. However, the solution with $Q = 0$, $C_2 \neq 0$ may still have interest. For $\varepsilon > 0$, $h^2 > 4$ and $C_1 > 0$, such solution represents a black hole with domain wall asymptotic behavior, a singularity at $r = 0$ and one or two horizons, depending on the value of C_2. The asymptotic behavior is dictated by the C_2 term. The potential is the sum of two exponential.

2.8.3 Asymptotically AdS solutions

In this section, we wish to generalize the asymptotically anti-de Sitter solutions obtained using the ansatz (2.31), in the case $\varepsilon \neq 0$. The solution (2.32) for the radial function still holds, while, after the usual rescaling of C_2, the metric function U becomes in the special case $C_1 = 0$, corresponding to an extremal black hole:

$$U = -\frac{\varepsilon r^2}{2\beta(1+4\beta)L^2} X^{-2\beta} \left[1 - (1+4\beta)\frac{r_-}{r}\right] + \frac{C_2 r^2}{L^2} X^{2\beta+1}, \qquad (2.80)$$

where we have rescaled C_2, set $L = \Lambda r_-$, and $-1/4 < \beta < 0$. The potential is then:

$$V(\phi) = -\frac{\varepsilon}{2\beta(1+4\beta)} V_1(-\gamma, \phi) + C_2 V_1(\gamma, \phi), \qquad (2.81)$$

where $V_1(\gamma, \phi)$ is given by Eq. (2.35). The metric is singular at $r = 0$, while, when $C_2 \neq 0$, in general the solution is a black hole, whose horizon structure cannot be determined analytically. As in the planar case, solutions with $C_1 \neq 0$ exist, but it is not possible to eliminate C_1 from the potential. Hence if a family of black hole solutions exist for the potential (2.81), it must be determined numerically.

An interesting property of the potential (2.81) is the symmetry between its two terms for $\phi \to -\phi$. In particular, choosing $C_2 = \varepsilon/2\beta(1+4\beta)L^2$, the potential becomes:

$$\begin{aligned} V(\phi) \;&=\; -\frac{2\varepsilon}{\beta(1+4\beta)L^2}\{(\beta+\frac{1}{2})(4\beta+1)\sinh[2(\beta-\frac{1}{2})\gamma\phi] - (8\beta^2-2)\sinh[2\beta\gamma\phi] \\ &\quad +(\beta-\frac{1}{2})(4\beta-1)\sinh[(2\beta+1)\gamma\phi]\}. \end{aligned}$$

The most interesting case is $C_2 = 0$, $\varepsilon = 1$. With this assumption, using the invariance of the field equations under the rescaling $R \to \frac{1}{\lambda}R$, $U \to \lambda^2 U$, $V \to \lambda^2 V$ and changing the sign of γ in Eq. (2.31), the potential (2.81) can be brought into the form $V = V_1(\gamma, \phi)$ where, as usual, $V_1(\gamma, \phi)$ is given by Eq. (2.35). Solution (2.80) becomes:

$$U = -\frac{r^2}{L^2} X^{-2\beta} \left[1 - (1+4\beta)\frac{r_-}{r} \right], \quad R = \frac{L\, r}{r_- \sqrt{-2\beta(1+4\beta)}} X^{\beta+\frac{1}{2}}, \quad \gamma\phi = -\log X. \tag{2.82}$$

In the range of definition, $-1/4 < \beta < 0$, the solution has no horizon. However, as we will see when we consider the near-horizon, near extremal solution, it cannot be considered an extremal black hole.

We conclude by observing that a solution can be found also in the particular case $\beta = 0$, $\gamma = 2$. In this case:

$$\phi = \frac{1}{2} \log X, \quad R = \Lambda r X^{1/2},$$

and then:

$$\begin{aligned} U &= \frac{2\varepsilon r(r-r_-)}{L^2} \left[\frac{r_-}{r-r_-} + \log\left(\frac{r-r_-}{r} \right) \right], \\ V &= -\frac{4\varepsilon}{L^2} [4\phi + 2\phi\cosh(2\phi) - 3\sinh(2\phi)]. \end{aligned}$$

2.8.4 Charged asymptotically AdS solutions

We consider now the solutions of the previous subsection with $Q \neq 0$, but $C_2 = 0$. The only change is in the function U:

$$\begin{aligned} U &= -\frac{\varepsilon r^2}{2\beta(1+4\beta)L^2} X^{-2\beta} \left[1 - (1+4\beta)\frac{r_-}{r} \right] \\ &\quad + \frac{8\mu r^2}{3(1-36\beta^2)} X^{-4\beta} \left[1 + (1+6\beta)\frac{r_-}{r} + 3\beta(1+6\beta)\frac{r_-^2}{r^2} \right], \end{aligned} \tag{2.83}$$

and in the potential:

$$V(\phi) = -\frac{\varepsilon}{2\beta(1+4\beta)} V_1(-\gamma, \phi) - \frac{2\mu L^2}{3\beta(1+4\beta)} V_2(\gamma, \phi),$$

where we have defined $L = \Lambda r_-$, $\mu = \frac{Q^2}{r_-^3\Lambda^4}$, while $V_1(\gamma,\phi)$ and $V_2(\gamma,\phi)$ are given respectively by Eqs. (2.35) and (2.45). Therefore μ, L and β are parameters of the action, Q is a free parameter and $r_- = \mu L^4/Q^2$. Multiple horizons may occur, but cannot be determined analytically for generic β.

2.8.5 Spherical solutions generated from the planar ones

For $d = 2$, spherical solutions can simply be generated from the planar ones just by exploiting the fact that the field equations (2.5) are linear in the metric function U. This method permits to find spherical solutions for a given form of the potential. This fact may be very useful when one wants to compare planar and spherical solutions of the same model or when the method described in the previous subsection gives a singular result (e.g. $\beta = 0$ in Eq. (2.80)).

Indicating with U_0, $R_0 = rX^{\beta+1/2}, \phi_0 = \gamma^{-1}\ln X$, where X is the harmonic function (2.31), a solution of the field equations (2.5) for $\varepsilon = 0$ and $d = 2$, it follows from the linearity in U of the field equations that a solution of (2.5) for $\varepsilon = \pm 1$ and $d = 2$ is given by:

$$U = U_0 + \varepsilon X^{-2\beta}, \quad R = R_0, \quad \phi = \phi_0. \tag{2.84}$$

This method can be used to generate $\varepsilon = \pm 1$ solutions for the potentials (2.35) and (2.45) from the planar solutions respectively given by (2.34) and (2.44). In the uncharged case, i.e for the potential (2.35), we have:

$$U = X^{2\beta+1}\frac{r^2}{L^2}+\varepsilon X^{-2\beta}, \quad R = rX^{\beta+\frac{1}{2}}, \quad \gamma\phi = \ln X, \quad X = 1-\frac{r_-}{r}, \quad \frac{1}{4}-\beta^2 = \gamma^{-2}. \tag{2.85}$$

Notice that, differently from Eqs. (2.82), this solution holds in the full range $-1/2 < \beta < 1/2$ of the parameter β. For $\beta = 0$ we get the spherical extremal solution of the model (2.37):

$$U = \frac{r^2}{L^2} - \frac{rr_-}{L^2} + \varepsilon, \quad R = rX^{1/2}, \quad \phi = \frac{1}{2}\ln X. \tag{2.86}$$

One can easily check that solution (2.85) is an extremal $T = 0$ solution, in fact the zeros of U are always behind the origin of the radial coordinate at $r = r_-$.

Eqs. (2.82) and (2.85) are solutions of the same model with potential $V_1(\gamma, \phi)$ given by (2.35). Because $V_1(\gamma, \phi)$ is invariant under the duality transformation (2.40), one can generate from (2.82), (2.85) two other solutions of the same model just by reversing the sign of β and ϕ.

2.8.6 Near-horizon, near-extremal spherical solutions

Let us now consider the near-horizon approximation of the solutions (2.82) and (2.85). One can obtain the near-horizon, near-extremal solution by first shifting $r \to r + r_-$ and expanding near $r = 0$. Then one solves the field equations perturbatively near $r = 0$ using the extremal, near-horizon solution as zero-th order approximation.

In the case of solution (2.82) this procedure gives the solution:

$$\begin{gathered} U = \left(\frac{r_-}{L}\right)^2 x^{-2\beta}\left[(1-4\beta(1+2\beta))x - C\right], \\ R = \frac{Lx^{\beta+\frac{1}{2}}}{\sqrt{-2\beta(1+4\beta)}}\left[1 - \frac{2}{C}\beta(2\beta-1)x\right], \quad \gamma\phi = -\ln x + \frac{4\beta}{C}x, \end{gathered} \tag{2.87}$$

where C is an integration constant and $x = r/r_-$. For $C > 0$ the solution describes black holes with a regular horizon at $x = x_h = C/(1 - 4\beta(1 + 2\beta))$ with $R(x_h) \neq 0$ and a singularity at $r = 0$. The Hawking temperature of the horizon is $T = (1/4\pi)(r_-^2/L^2){x_h}^{-2\beta}$. The solution (2.87) is singular for $C = 0$, although $T \to 0$ as $C \to 0$. For $C < 0$ we have solutions with no horizon and in particular for $C = C_0 = 4\beta$ we get the solution (2.82) in the near-horizon approximation.

Because solution (2.87) has a singularity at $r = 0$ (corresponding to the singularity of (2.82) at $r = r_-$), one should reject solutions with $C < 0$ as unphysical. This gives a strong hint about the nature of our solution (2.82): it is an isolated solution disconnected from the continuous part of the black hole spectrum at $C > 0$ by solutions with naked singularities.

In the case of solution (2.85), near-horizon, near-extremal solutions are given by:

$$\begin{aligned} U &= \frac{r_-^2}{L^2}\left(x^{2\beta+1} - Cx^{-2\beta}\right) + 2\beta x^{-2\beta+1}, \\ R &= r_- x^{\beta+\frac{1}{2}}\left(1 + \frac{L^2}{Cr_-^2}(\beta - \frac{1}{2})x\right), \quad \gamma\phi = \ln x + \frac{L^2}{Cr_-^2}x, \end{aligned} \tag{2.88}$$

where C is an integration constant. For $C > 0$ the solution has a regular horizon, which at leading order is located at $x = x_h = C^{1/(4\beta+1)}$ when C is small, $C < \left(r_-^2/(L^2(1-2\beta))\right)^{-(4\beta+1)/(4\beta)}$. This is consistent with a near-extremal approximation. The Hawking temperature of the horizon is $T = (1/4\pi)(r_-^2/L^2)(4\beta+1){x_h}^{2\beta}$. The solution (2.88) is singular for $C = 0$. For $C < 0$ we have solutions with no horizon and, in particular, for $C = C_0 = L^2/r_-^2$ we get the solution (2.85) in the near-horizon approximation.

Also solution (2.88) has a singularity at $r = 0$, which corresponds to the singularity of (2.85) at $r = r_-$. Hence, solutions with $C < 0$ have naked singularities and solution (2.85) is disconnected from the continuous part of the black hole spectrum at $C > 0$.

2.9 Conclusions

In this chapter we have presented a general method for finding static, radially symmetric, analytic solutions of Einstein and Einstein-Maxwell gravity minimally coupled to a scalar field. Rather than assuming a particular form of the scalar self-interaction potential, our method starts from an ansatz for the scalar field profile and determines, together with the metric functions, the corresponding form of the potential. For this reason it is particularly suitable for applications to the AdS/CFT correspondence.

We have investigated in detail two related applications of our method. Firstly, we have derived a new no-hair theorem about the existence of black hole solutions of Einstein gravity with scalar hair. As a second application, we have derived broad classes of exact analytic hairy solutions of Einstein and Einstein-Maxwell gravity minimally coupled to a scalar. These solutions have been derived using rather general and simple ansätze for the scalar (in terms of harmonic, logarithmic functions and r^l behaviour). They cover many different situations: four or higher dimensions; solutions with planar, spherical or hyperbolic horizon topology; solutions with AdS, domain wall and conformal Lifshitz asymptotics; solutions interpolating between an AdS spacetime in the asymptotic region and domain wall or conformally Lifshitz behavior in the near-horizon region. Also the class of potentials for the scalar field characterizing these models is broad, ranging from the simple exponential potential – known to give rise in many situations to domain wall and Lifshitz solutions – to more general forms such as combinations of exponentials – containing as a particular case $\mathcal{N} = 8$ gauged supergravity in 4D truncated to the $U(1)$ sector – or combinations of powers and trigonometric functions.

Our investigation has shown that Einstein gravity minimally coupled to a scalar field has a rich spectrum of solutions with non trivial scalar hair and AdS asymptotics, which may play an important role in applications of the AdS/CFT correspondence to condensed matter and strongly coupled QFTs.

Our approach has a main drawback. In some situations it does not allow to find a full one-parameter family of black holes, i.e. the full spectrum of solutions for different temperatures, but only "extremal" $T = 0$ solutions. Moreover, what we have called extremal solutions always present a curvature singularity at $r = 0$.

Although our method always allows to find one-parameter families of near-horizon near-extremal solutions, interpolating solutions with AdS asymptotics can be found only in the extremal case. Moreover, in many situations, it is not even clear whether or not such solutions exist. This is a particularly important question in the cases in which the exact solution interpolates between the AdS spacetime and a near-horizon domain wall or Lifshitz spacetime. In the spherical case we have found strong evidence that our exact solutions represent isolated solutions disconnected from the continuous part of the spectrum. A final answer to these questions involves numerical computations.

Chapter 3

Black brane and solitonic solutions in Einstein-scalar gravity

In this chapter we investigate static, planar solutions of Einstein-scalar gravity admitting an AdS vacuum. When the squared mass of the scalar field is positive and the scalar potential can be derived from a superpotential, minimum energy theorems indicate the existence of a scalar soliton. On the other hand, for these models, no-hair theorems forbid the existence of hairy black brane solutions with AdS asymptotics. By considering a specific example in four dimensions (an exact integrable model which has the form of a Toda molecule) and by deriving an explicit exact solution, we show that these models allow for hairy black brane solutions with non-AdS domain wall asymptotics, whose extremal limit is a scalar soliton. The soliton smoothly interpolates between a non-AdS domain wall solution at $r = \infty$ and an AdS solution near $r = 0$.

3.1 The model

Let us investigate static, radially symmetric, planar solutions of four-dimensional Einstein gravity, minimally coupled to a scalar field with self-interaction potential $V(\phi)$. The action is:

$$I = \int d^4x\sqrt{-g}\left[R - 2(\partial\phi)^2 - V(\phi)\right]. \tag{3.1}$$

Differently from the previous chapter, in which we have considered potentials having a negative maximum and corresponding negative squared mass m^2 for the scalar (but above the BF bound), here we assume that $V(\phi)$ has a negative minimum at $\phi = 0$, thus allowing an AdS_4 vacuum, corresponding to a positive squared mass m^2 for the scalar excitation. In this case, positive-energy theorems (PET) allow for a stable ground state solitonic solution, but standard no-hair theorems forbid the existence of black brane (BB) solutions asymptotically AdS (see Sect. 1.2).

However, our new no-hair theorem (see Sect. 2.4) implies that only BB solutions with AdS asymptotics are forbidden, leaving open the possibility of having BB solutions with generic domain wall asymptotics.

We also assume that $V(\phi)$ can be derived from a superpotential $P(\phi)$:

$$V(\phi) = 2\left(\frac{dP}{d\phi}\right)^2 - 6P^2. \tag{3.2}$$

This means that our theory is a fake SUGRA model (see Sect. 1.3). So, if we parametrize the spacetime metric as $ds^2 = r^2(-dt^2 + dx_i dx^i) + h^{-1}dr^2$, the second-order field equations stemming from (3.1) reduce to first order equations:

$$\phi'(r) = -\frac{P_{,\phi}}{rP(\phi)}, \quad h(r) = r^2P^2(\phi). \tag{3.3}$$

For definiteness, we will focus on a fake SUGRA model defined by (L is the AdS length):

$$V(\phi) = -\frac{6}{\gamma L^2}\left(e^{2\sqrt{3}\beta\phi} - \beta^2 e^{\frac{2\sqrt{3}}{\beta}\phi}\right), \quad P(\phi) = \frac{1}{\gamma L}\left(e^{\sqrt{3}\beta\phi} - \beta^2 e^{\frac{\sqrt{3}}{\beta}\phi}\right), \quad \gamma = 1-\beta^2. \tag{3.4}$$

The potential is defined for every $\beta \neq 0, 1$. It has always a minimum at $\phi = 0$, with $V(0) = -6/L^2$, corresponding to the AdS_4 solution and to a scalar excitation with positive squared mass $m^2 = 18/L^2$. We use standard (Dirichlet) boundary conditions for ϕ, which set to zero the dominant term in the $r \to \infty$ expansion. The fall-off behavior of the scalar field is therefore given by $\phi \sim \frac{\beta}{r^6}$.

We will look for BB solutions of (3.1) with asymptotics:

$$ds^2 = r^\eta(-dt^2 + dx_i dx^i) + r^{-\eta} dr^2, \tag{3.5}$$

with $0 \leq \eta \leq 2$. For $\eta = 0, 2$, Eq. (3.5) describes flat or AdS spacetime, respectively. When $0 < \eta < 2$ (3.5) describes a domain wall (DW).

3.2 The exact solution

The field equations of the Einstein-scalar gravity model with potential (3.4) can be exactly integrated. This can be achieved using a parametrization of the metric introduced in [6] and used in several investigations of dilatonic black holes [8, 9, 91–95]:

$$ds^2 = -e^{2\nu} dt^2 + e^{2\nu+4\rho} d\xi^2 + e^{2\rho} dx_i dx^i. \tag{3.6}$$

Using this parametrization, the field equations can be recast in the form of the $SU(2) \times SU(2)$ Toda molecule [96]. In fact, defining new variables $\Omega = \nu + 2\rho + \sqrt{3}\beta\phi$, $\Sigma = \nu + 2\rho + \frac{\sqrt{3}}{\beta}\phi$, and taking into account that the field equations imply $\rho = \nu + c\xi$, with c an integration constant, one obtains the second-order equations:

$$\ddot{\Omega} = \frac{9}{L^2} e^{2\Omega}, \qquad \ddot{\Sigma} = \frac{9}{L^2} e^{2\Sigma}, \tag{3.7}$$

subject to the constraint:

$$\dot{\Omega}^2 - \beta^2 \dot{\Sigma}^2 - \gamma c^2 = \frac{9}{L^2}(e^{2\Omega} - \beta^2 e^{2\Sigma}). \tag{3.8}$$

These equations can be solved to give the general solution:

$$\begin{aligned}
e^{2\nu} &= \left(\frac{2L}{3}\right)^{2/3} a^{\frac{2}{3\gamma}} b^{\frac{-2\beta^2}{3\gamma}} e^{\frac{2b\beta^2\xi_0}{3\gamma}} e^{2(a-\beta^2 b - 2\gamma c)\xi/3\gamma} \left[\frac{\left(1 - e^{2b(\xi-\xi_0)}\right)^{\beta^2}}{1 - e^{2a\xi}}\right]^{2/3\gamma}, \\
e^{2\rho} &= \left(\frac{2L}{3}\right)^{2/3} a^{\frac{2}{3\gamma}} b^{\frac{-2\beta^2}{3\gamma}} e^{\frac{2b\beta^2\xi_0}{3\gamma}} e^{2(a-\beta^2 b + \gamma c)\xi/3\gamma} \left[\frac{\left(1 - e^{2b(\xi-\xi_0)}\right)^{\beta^2}}{1 - e^{2a\xi}}\right]^{2/3\gamma}, \\
\phi &= \frac{\beta}{\sqrt{3}\,\gamma} \log\left[\frac{b \sinh a\xi}{a \sinh b(\xi - \xi_0)}\right],
\end{aligned} \tag{3.9}$$

where ξ_0 is an arbitrary integration constant and a, b, c must satisfy the constraint $\gamma c^2 = a^2 - \beta^2 b^2$.

We are interested in solutions with a regular horizon at $\xi = \xi_h$. Requiring $e^{2\nu}(\xi_h) = 0$ and $e^{2\rho}(\xi_h) = const.$, one easily realizes that this is only possible for $\xi_h \to -\infty$, when $\gamma c = \beta^2 b - a$. This condition, together with the constraint, implies $a = b = -c$. In the case $\xi_0 = 0$, we obtain the planar Schwarzschild-anti de Sitter solution with $\phi = 0$. As one can show by expanding (3.9) near $\xi = 0$ and $\xi = -\infty$, all the other solutions with AdS asymptotics and non-trivial scalar hair have a naked singularity at $r = 0$ with $\phi \sim \log r$. This is in complete accordance with the results of well-established no-hair theorems.

In the general case $\xi_0 \neq 0$ we have solutions with a regular horizon, but they do not approach AdS_4 asymptotically, and it is not possible to write them in a Schwarzschild form in terms of elementary functions.

Let us first consider the case $\beta^2 < 1$. In this case the asymptotic region corresponds to the limit $\xi \to 0$. Defining the new radial coordinate $\sigma r = (1 - e^{2a\xi})^{-(1+3\beta^2)/3\gamma}$ with σ constant, for $0 < \xi_0 < \infty$ the solution (3.9) becomes:

$$
\begin{aligned}
ds^2 &= \left(1 + \frac{\mu_2}{r^\delta}\right)^{2\beta^2/3\gamma} \left[- \left(1 - \frac{\mu_1}{r^\delta}\right) r^{2/(1+3\beta^2)} dt^2 \right. \\
&\quad \left. + \frac{E(1+\mu_2/r^\delta)^{4\beta^2/3\gamma}}{(1-\mu_1/r^\delta) r^{2/(1+3\beta^2)}} dr^2 + r^{2/(1+3\beta^2)} dx_i dx^i \right], \\
e^{2\phi} &= D \left(1 + \frac{\mu_2}{r^\delta}\right)^{-2\beta/\sqrt{3}\gamma} r^{-2\sqrt{3}\beta/(1+3\beta^2)}, \qquad (3.10)
\end{aligned}
$$

where $\mu_1 \geq 0, \mu_2 > 0$ are free parameters, $\delta = 3\gamma/(1+3\beta^2)$, $D = [\mu_2(\mu_1 + \mu_2)]^{\beta/\sqrt{3}\gamma}$, and $E = [\gamma L/(1+3\beta^2)]^2 D^{-\sqrt{3}\beta}$.

The asymptotic behavior of this solution for $r \to \infty$ is that of a domain wall (3.5) with $\eta = 2/(1+3\beta^2)$ and $\phi = -[(\sqrt{3}\beta)/(1+3\beta^2)] \ln r$. For $\mu_1 > 0$, the metric (3.10) exhibits a singularity at $r = 0$ shielded by a horizon at $r = \mu_1^{1/\delta}$, and therefore represents a regular black brane. Owing to the fact that the scalar ϕ depends on μ_1, the existence of this BB solution is perfectly consistent with our no-hair theorem (see Sect. 2.4). Notice that although the scalar field remains finite at $r = 0$, the scalar curvature R of spacetime diverges as $R \sim r^{-3(1+\beta^2)(1+3\beta^2)}$.

The extremal, zero temperature, solution is obtained for $\mu_1 = 0$:

$$
ds^2 = \left(1 + \frac{\mu_2}{r^\delta}\right)^{2\beta^2/3\gamma} \left[r^{2/(1+3\beta^2)} \left(-dt^2 + dx_i dx^i\right) + E r^{-2/(1+3\beta^2)} \left(1 + \frac{\mu_2}{r^\delta}\right)^{4\beta^2/3\gamma} dr^2 \right], \qquad (3.11)
$$

while the scalar field maintains the form of Eq. (3.10). The extremal solution (3.11) represents a regular soliton. In fact, not only the scalar field is finite at $r = 0$ ($e^{2\phi} = D(\mu_2)^{-(2\beta)/(\sqrt{3}\gamma)}$), but also the scalar curvature of the spacetime remains finite both ar $r = 0$ and $r = \infty$. The extremal soliton has the form of a brane, for which the metric behaves for small and large r as in Eq. (3.5) with a different power of r in the $r = \infty$ and $r = 0$ region. Whereas for $r \to \infty$ we have $\eta = 2/(1 + 3\beta^2)$ and $\phi \sim \ln r$, near the origin $\eta = 2$ and $\phi = const.$. Hence, our soliton (3.11) interpolates between a DW solution at infinity and AdS spacetime at $r = 0$. As expected the soliton (3.11) satisfies the fake BPS equations (3.3).

A similar procedure allows one to find the solution when $\beta^2 > 1$. Now the asymptotic region $r \to \infty$ corresponds to $\xi \to \xi_0$. As before, the metric can be written in terms of a new radial coordinate $\sigma r = (1 - e^{2a(\xi-\xi_0)})^{(3+\beta^2)/3\gamma}$:

$$\begin{aligned} ds^2 &= \left(1 + \frac{\mu_2}{r^\delta}\right)^{-2/3\gamma} \Bigg[-\left(1 - \frac{\mu_1}{r^\delta}\right) r^{2\beta^2/(3+\beta^2)} dt^2 \\ &\quad + \frac{E(1 + \mu_2/r^\delta)^{-4/3\gamma}}{(1 - \mu_1/r^\delta) r^{2\beta^2/(3+\beta^2)}} dr^2 + r^{2\beta^2/(3+\beta^2)} dx_i dx^i \Bigg], \\ e^{2\phi} &= D\left(1 + \frac{\mu_2}{r^\delta}\right)^{2\beta/\sqrt{3}\gamma} r^{-2\sqrt{3}\beta/(3+\beta^2)}, \end{aligned} \tag{3.12}$$

where now $\delta = -3\gamma/(3 + \beta^2) > 0$, $D = [\mu_2(\mu_1 + \mu_2)]^{\beta/\sqrt{3}\gamma}$, and $E = [\gamma L/(3+\beta^2)]^2 D^{-\sqrt{3}\beta}$. At infinity, the solution behaves as a domain wall with $\eta = 2\beta^2/(3 + \beta^2)$ and $\phi = -[(\sqrt{3}\beta)/(3 + \beta^2)] \ln r$.

As in the previous case, if $\mu_1 > 0$, the metric exhibits a singularity at $r = 0$ and a horizon at $r = \mu_1^{1/\delta}$, and therefore describes a regular black brane with non-AdS domain wall asymptotics.

Also in this case the extremal, zero temperature solution, obtained for $\mu_1 = 0$, is a regular soliton that satisfies Eq. (3.3):

$$ds^2 = \left(1 + \frac{\mu_2}{r^\delta}\right)^{-2/3\gamma} \left[r^{2\beta^2/(3+\beta^2)} \left(-dt^2 + dx_i dx^i\right) + E r^{-2\beta^2/(3+\beta^2)} \left(1 + \frac{\mu_2}{r^\delta}\right)^{-4/3\gamma} dr^2\right]. \tag{3.13}$$

As expected, the soliton interpolates between the domain wall solution (3.5) with $\eta = 2\beta^2/(3+\beta^2)$ at infinity and an AdS solution with constant ϕ near $r = 0$.

It may be interesting to notice that the Schwarzschild-anti de Sitter solution is recovered in the singular limit $\mu_2 \to \infty$ of (3.10) or (3.12).

Let us now compare our results with those obtained in the previous chapter, when the potential had a negative maximum with $m^2_{BF} \leq m^2 < m^2_{BF}+1$. If the potential $V(\phi)$ behaves exponentially at large ϕ, one has solutions with AdS_4 asymptotics at large r and singular DW behavior near $r = 0$, with $\phi \sim \ln r$. The only known case that does not present a small-r singularity is when V has a second extremum. Apart from this case, the solutions always have opposite behavior with respect to the soliton that we get in the $m^2 > 0$ case: the solution interpolates between an AdS_4 spacetime at $r = \infty$ and a DW solution near $r = 0$.

In this context, it is also interesting to notice that also a pure exponential potential $V = -2\lambda e^{-2h\phi}$ for $h^2 < 3$ is a fake SUGRA model [55]. In fact, V can be derived from the superpotential $P = \sqrt{\lambda/(3-h^2)}\, e^{-h\phi}$. Also in this case, the field equations can be exactly integrated using the Toda molecule parametrization (3.6) for the metric. BB solutions with DW asymptotics can be found using the procedure described above. Defining a new variable $\eta = \nu+2\rho-h\phi$, the field equations can be recast in the form $\ddot{\eta} = (3-h^2)\lambda e^{2\eta}$, together with a constraint involving the integration constants. Solving these equations, one can show that the solutions with a regular horizon can be written in the form $ds^2 = -U(r)dt^2 + U(r)^{-1}dr^2 + R(r)^2 dx_i dx^i$, with:

$$U = \left(1 - \mu r^{(h^2-3)/(1+h^2)}\right) r^{2/(1+h^2)}, \quad R(r) = r^{1/(1+h^2)}, \quad \mathrm{e}^{2\phi} = C r^{2h/(1+h^2)},$$

where μ is an integration constant and $C = [(\lambda(1+h^2)^2/(2(3-h^2))]^{1/h}$. For $\mu = 0$ we get a DW solution, which is singular at $r = 0$. We can immediately note that this form of the solution is the same already derived in Sect. 2.3.1, using a different method.

3.3 Concluding remarks

In this chapter we have derived explicit exact black brane solutions of Einstein-scalar gravity with positive squared mass for the scalar field, whose extremal limit is a regular scalar soliton. We have circumvented standard no-hair theorems by allowing for solutions with non-AdS domain wall asymptotics. We have derived the solutions for 4D Einstein-scalar gravity but our derivation could be easily extended to arbitrary spacetime dimensions. The scalar soliton interpolates between AdS_4 for small r and non-AdS brane at large r. As we will see in the next Part of the thesis, this soliton has an holographic interpretation in terms of a flow of a dual 3D QFT between an IR fixed point at $r = 0$ and an UV Poincaré invariant vacuum at $r = \infty$.

Part II

Holographic applications

Chapter 4

AdS/CFT and AdS/condensed matter correspondence

Black hole and black brane solutions with scalar hair, like those presented in the previous chapters, can play a crucial role in the context of applications of the AdS/CFT correspondence. In this chapter we will briefly review the original formulation of the AdS/CFT duality and its most significant extensions and applications. In particular we will focus our attention on the so-called AdS/condensed matter duality and its most important examples, as holographic superconductors, metallic behavior and hyperscaling violation.

4.1 From the holographic principle to the AdS/CFT correspondence

One of the most intriguing features of black holes is their entropy. In the early 70's Bekenstein [97, 98] proposed that a black hole has an entropy which is proportional to the event horizon area rather than, as one would expect, to the volume:

$$S \propto \frac{\mathcal{A}}{l_P^2},$$

where l_P^2 is the Planck area. This conjecture was then proved by Hawking, Bardeen and Carter on thermodynamical basis [99].

About twenty-five years later, this concept was used by 't Hooft and Susskind [100, 101] to formulate the so-called *holographic principle*, i.e. the conjecture that the number of local degrees of freedom in a gravity theory depends on the area enclosing a volume, and not on the volume itself. In other words, the principle requires that the three dimensional world is an image of data stored on a two dimensional projection, just like an hologram.

The main realization of this suggestive idea is surely the anti-de Sitter/Conformal Field Theory correspondence (AdS/CFT), formulated by Maldacena in 1997 [14] and developed by Witten [102].

In its original formulation, AdS/CFT is an equivalence between the full type IIB string theory defined on the background $AdS_5 \times S^5$ (AdS space in five-dimensions times a five-sphere) and the $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang-Mills gauge theory, defined in the flat (3+1)-dimensional spacetime (i.e. in one dimension less than the string theory).

The $\mathcal{N} = 4$ super Yang-Mills theory in $3+1$ dimensions contains a gauge field A together with multiple scalar fields Φ and Weyl fermions Ψ, transforming in the adjoint representation of $SU(N)$. The Lagrangian is:

$$\mathcal{L}_{QFT} \sim Tr\left(F^2 + (\partial\Phi)^2 + i\overline{\Psi}\Gamma\cdot\partial\Psi + g^2\left[\Phi,\Phi\right]^2 + ig_{YM}\overline{\Psi}\left[\Phi,\Psi\right],\right. \qquad (4.1)$$

where $F = dA + g_{YM}A \wedge A$ is the nonabelian field strenght. The Yang-Mills coupling g_{YM} is exactly marginal and the theory is conformal at all couplings.

Actually, the full quantum string theory on $AdS_5 \times S^5$ is still poorly understood. However, the string theory can be well approximate by a (semi)classical gravitational theory in the so-called "t'Hooft large N limit", where N represents the rank of the gauge group of the quantum field theory (QFT) with

Lagrangian (4.1).

This limit can be better understood if we represent the duality by a mapping between the parameters of the two theories, through the following relation:

$$L \sim l_s g_{YM} N^{1/4}, \tag{4.2}$$

where L is the AdS radius and l_s is the string length.

The large N limit implies $L >> l_s$, i.e. a small curvature (in Planck units) of the spacetime, that guarantees the validity of the semiclassical approximation.

For making the correspondence more precise, in the large N limit it is natural to define an action for the bulk theory. Actually, in general the full action will be rather complicated, depending on many fields, buth often is possible to truncate it to a relatively small number of fields, that capture the physics of interest.

The simplest theory from which we can start is the universal sector, i.e. the Einstein-Hilbert action, coupled to a negative cosmological constant:

$$S = \frac{1}{2k^2} \int d^{d+1}x \sqrt{-g} \left(R + \frac{d(d-1)}{L^2} \right), \tag{4.3}$$

where R is the Ricci scalar and k the gravitational constant. The most symmetric solution of this theory is the anti-de Sitter space (AdS):

$$ds^2 = \frac{L^2}{r^2} \left(-dt^2 + dr^2 + dx^i dx^i \right), \tag{4.4}$$

whose full isometry group is the conformal group in d dimensions $SO(d, 2)$. As a consequence, the symmetries of the bulk action act on the boundary field theory as conformal transformations, implying that the dual QFT is conformally invariant.

The extra radial dimension of the bulk has a precise physical meaning: it is the renormalization group (RG) scale in the boundary QFT. In few words, some components of the gravitational field equations, determining the evolution of the bulk spacetime along the extra dimension, correspond precisely to the RG equations of the boundary field theory. In particular, the AdS spacetime corresponds to a CFT, i.e. to an UV fixed point without RG flow. In general the bulk geometry will be asymptotically AdS, so the dual QFT will approach a fixed point in the UV.

It's very interesting to observe that, also in the large N limit, the dual field theory remains strongly coupled. This makes the AdS/CFT really powerful: it puts in relation very different regimes of two completely different theories, which in turn are defined in a different number of dimensions!

For this reason, we can say that AdS/CFT is one of the most significant results found in the last years in theoretical physics, although it still must be considered a conjecture, because a definitive proof of the duality is still lacking. However, during the two last decades AdS/CFT received numerous confirmations and survived several tests of validity, so it would be very surprising to find that the conjecture is not valid.

4.2 AdS/CFT dictionary

Starting from the general definition of the AdS/CFT duality, we can define a precise *dictionary* of the correspondence, which puts in relation each gravitational object with a corresponding dual object in the boundary QFT.

The most basic entry in the dictionary is that for each dynamical field ϕ in the bulk gravitational theory, supported with appropriated boundary conditions, there is an operator $\mathcal{O}$ in the conformal dual field theory, in a one-to-one correspondence. This can also be represented by an equivalence between the gravity partition function Z_{bulk} and the gauge theory correlation functions:

$$Z_{bulk}[\phi_0(\vec{x})] = \left\langle e^{\int d^4x \phi_0(\vec{x})\mathcal{O}(\vec{x})} \right\rangle_{QFT},$$

where ϕ_0 is the boundary value of the bulk field ϕ.

In particular, the metric $g_{\mu\nu}$ will be dual to the energy-momentum tensor $T^{\mu\nu}$ of the dual field theory, but the bulk action can contain more fields than just the metric, as the Maxwell field (dual to the global current J^μ of the QFT), scalars and fermionic fields (duals to appropriate operators in the QFT). In the Table below we summarize the main entries of the dictionary.

Gravitational bulk	$\Longleftrightarrow$	**Boundary field theory**
metric tensor $g_{\mu\nu}$	$\Longleftrightarrow$	conserved energy tensor $T^{\mu\nu}$
Maxwell field A_μ	$\Longleftrightarrow$	global current J^μ
scalar field ϕ	$\Longleftrightarrow$	scalar operator $\mathcal{O}_S$
fermionic field ψ	$\Longleftrightarrow$	fermionic operator $\mathcal{O}_F$
AdS black hole	$\Longleftrightarrow$	finite-temperature quantum field theory

4.3 Extensions

The gauge theory described in the original formulation of AdS/CFT is a quantum field theory at zero temperature. However in several cases we wish to describe quantum field theories at finite temperature. In general, the temperature introduces an energy scale T which breaks the conformal invariance, and this could make difficult the computation of quantum field theories at finite temperature, even at weak coupling. However, a remarkable feature of the holographic correspondence is that finite temperature computations of strongly coupled gauge theories are essentially no harder than computations at zero temperature, as involve only classical bulk fields in a curved spacetime.

The natural extension of the bulk theory is to consider, rather than an AdS spacetime (4.4), an asymptotically AdS black hole, and the Hawking temperature of the black hole will be associated to the temperature of the boundary field theory at equilibrium.

Always starting from the action (4.3) and relaxing the conformal symmetry, the only regular static, radially symmetric solution of the theory is the Schwarzschild-AdS black hole:

$$ds^2 = \frac{L^2}{r^2}\left(-f(r)dt^2 + \frac{dr^2}{r^2} + dx^i dx^i\right), \qquad f(r) = 1 - \left(\frac{r}{r_h}\right)^d, \tag{4.5}$$

where r_h is the event horizon.

The Hawking temperature of the black hole (and of the dual field theory) is:

$$T = \frac{d}{4\pi r_h}.$$

Starting from the definition of temperature, the simplest finite temperature quantity one can compute is the free energy of the theory:

$$F = -T \log Z = TS_E[g_{BH}] = -\frac{(4\pi)^d}{2d^d} N^2 V_{d-1} T^d,$$

where in the second equality we exploited the large N relation $Z = e^{-S_E[g]}$, whith the partition function expressed as the classical Euclidean action evaluated on the black hole saddle point (4.5).

From the free energy we can derive the black hole entropy:

$$S_{BH} = -\frac{\partial F}{\partial T} = -\frac{(4\pi)^d}{d^{d-1}} N^2 V_{d-1} T^{d-1}.$$

Remarkably, it is simple to check that S_{BH}, computed in a purely gravitational background, agrees with the entropy S_{YM} of the dual Yang-Mills field theory at weak coupling.

More generally, when we introduce a central charge $c \sim N^2$, we get the entropy of a free-field CFT in d dimensions with central charge c.

Another very important extension of the basic setup of the correspondence is to introduce in the gravity bulk other fields, as e.g. the Maxwell field and scalar fields.

In the first case, we need to supplement the basic action (4.3) with the term:

$$S[A] = -\frac{1}{4g^2}\int d^{d+1}x\sqrt{-g}F^2,$$

where $F = dA$ is the field strenght. The effect of adding a Maxwell field in the bulk is to add an electric charge to the black hole and, in turn, to place the dual theory at finite chemical potential and hence induce a charge density ρ. In particular, the nonzero Maxwell potential will be:

$$A_t = \mu\left[1 - \left(\frac{r}{r_h}\right)^{d-2}\right],$$

where μ is the chemical potential and r_h the horizon.

Adding a scalar field ϕ, the additional term in the action will be:

$$S[\phi] = \int d^{d+1}x\sqrt{-g}\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right),$$

where $V(\phi)$ is a self-interaction potential. In this case, the bulk scalar field is dual to a scalar operator which represents an order parameter.

The next step is to couple the scalar field to the Maxwell field. This can be done in various ways. Depending on this coupling we will have a different behavior (e.g different transport features) in the dual QFT. If the scalar field is $U(1)$- charged (i.e complex), one can have a covariant coupling between the scalar and the Maxwell field. As we will see in the next section, this leads in the dual QFT to superconducting behavior. On the other hand, if the scalar field does not have charge (i.e. it is real) one can consider a nonminimal coupling between the two fields. As we will see in Sect. 4.4.2, this leads to metallic transport features in the dual QFT .

4.4 AdS/condensed matter duality

The original Maldacena conjecture can be considered a starting point, which immediately stimulated theoretical physicists to look for other *gauge/gravity* dual pairs, also when some of the stringent hypotesis of the AdS/CFT correspondence were not met. Actually, in the last fifteen years, many other dual field theories have been discovered or conjecturated, including non-supersymmetric and non-conformal theories. Among the several dualities conjecturated, there is no doubt that one of the most interesting and fascinating is the so-called anti-de Sitter/Condensed Matter (AdS/CM) duality: starting from effective and phenomenological models of AdS gravity (not necessarily arising from string theories), it is possible to give a dual description of strongly coupled real condensed matter systems.

But why it is so interesting and important to investigate this kind of duality? After all, we are talking about phenomena in which, owing to its weakness, the gravitational interaction is almost completely irrelevant. There are at least two good reasons. Firstly, the AdS/CFT approach provides a unique chance to get new insight into some aspects of strongly coupled condensed matter systems, because these systems are holographically related to semiclassical gravity models, which are computationally tractable and conceptually more transparent. Secondly, condensed matter systems can be investigated in detail in laboratories, thus the AdS/CFT can offer the fascinating possibility to realize experimental tests of gravitational theories, exploiting their condensed matter duals.

The AdS/CM duality allows to have dual descriptions of phenomena as e.g. the Hall effect [103] and Nernst effect [104], but the best-known example is represented by the holographic superconductors, which will be the subject of the next subsection. In the following subsections we will see how is possible to describe via AdS/CFT also typical metallic behavior and, in one of the most recent applications, hyperscaling violation.

4.4.1 Holographic superconductors

In the early part of the 20^{th} century it was discovered that the electrical resistivity of most metals drops suddenly to zero when the temperature is lowered below a critical value T_c. Moreover, in these materials a magnetic field is expelled when $T < T_c$, the so-called Meissner effect. This property was called superconductivity. A first phenomenological description of superconductivity was given by the London brothers in 1935 [105], while in 1950 Landau e Ginzburg [106] described superconductors in terms of a second order phase transition, whose order parameter is a scalar field. They also showed that the superconductive phase transition is associated to a breaking of a $U(1)$ symmetry. A more complete theory of superconductivity was finally given by Bardeen, Cooper and Schrieffer in 1957 (BCS theory [107]). They showed that interactions with phonons can cause pairs of electrons with opposite spin to bind and form a particular charged boson called *Cooper pair*. Below a critical temperature T_c, there is a second order phase transition and these bosons condense, while the DC conductivity becomes infinite producing a superconductor.

Until the mid-80s, it was thought that the highest T_c for a BCS superconductor was around $30° \ K$, but from 1986 a new class of high T_c superconductors was discovered. Today, the highest T_c known (at atmospheric pressure) is $T_c = 134° \ K$ for a mercury, barium, copper, oxide compound. There is evidence that electron pairs still form in these high T_c materials, but the pairing mechanism is not well understood because, unlike the BCS theory, it involves strong coupling.

This last feature, however, makes high-temperature superconductors the ideal arena for gauge/gravity duality. In the wake of this motivation, few years ago a gravitational dual to a strongly coupled superconductor has been formulated [18, 19, 24].

But what are the minimal ingredients of an holographic dual for a superconductor? Firstly, in a superconductor we need a temperature: in the gravity side, as already discussed in the previous section, that role can be played by a black hole with a Hawking temperature. Secondly, a superconductor needs a charged condensate, i.e. a charged operator which acquires a non-vanishing expectation value below a critical temperature T_c. In the bulk, this can be described by a nonzero field outside a black hole (i.e. a black hole "hair").

So, in few words, to describe a superconductor we need to find a black hole with a charged scalar hair at low temperatures, but no hair at high temperatures. Because the scalar has to be charged, the most natural coupling to the EM field is the usual covariant gauge coupling. We are therefore lead to consider the following gravitational action in the bulk:

$$S = \int d^4x\sqrt{-g}\left(R + \frac{6}{L^2} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |\nabla\Psi - iqA\Psi|^2 - m^2\,|\Psi|^2\right),$$

which is just Einstein gravity in anti-de Sitter, covariantly coupled to a Maxwell field and a charged scalar field Ψ with charge q and mass m.

A solution of the equations of motion stemming from this theory is the Reissner-Nordström-AdS (RN-AdS) black hole with $\Psi = 0$, i.e. with no scalar hair, but if q is large enough and for sufficiently low temperatures, the RN-AdS black hole becomes unstable against scalar pertubations [108]. As a consequence, at low temperatures the RN-AdS black hole develops a scalar hair, which breaks the $U(1)$ symmetry of the theory. Moreover, if one computes the free energy of the two configurations (RN-AdS black hole with scalar hair and without scalar hair), it turns out that for $T < T_c$ the free energy is always lower for the hairy configurations and becomes equal as $T \to T_c$. This behavior guarantees that for $T < T_c$ the "scalar-dressed" black hole is always energetically favoured against the undressed configuration. Furthermore, the difference of free energies scales like $(T_c - T)^2$ near the transition, showing a second order phase transition.

But the fundamental ingredient which characterizes a superconductor is the infinity DC conductivity. This can be obtained, following the gauge/gravity dictionary, by studying electromagnetic pertubations and considering in particular their asymptotic behavior (for details see [18, 19]). Remarkably, the DC conductivity turns out to be infinite (see Fig. 4.1). Moreover, the AC conductivity shows the frequency gap typical of real superconductors (see Fig. 4.2).

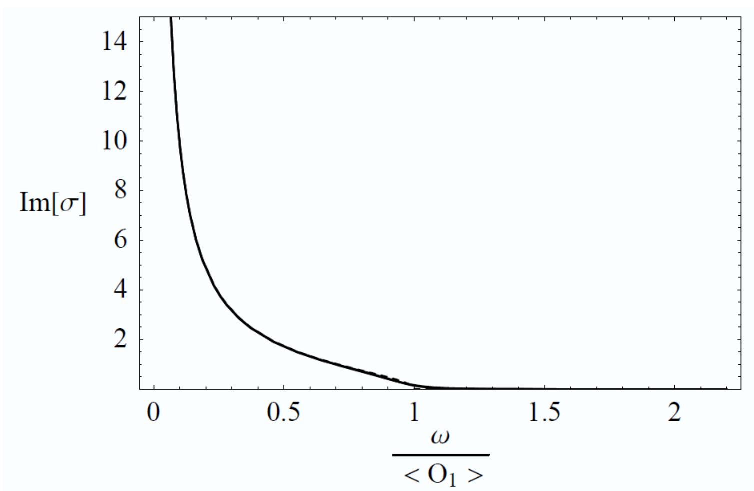

Figure 4.1: DC conductivity as a function of the frequency (normalised in terms of the condensate), at small T/T_c.

4.4.2 Metallic behavior

More recently it was shown [23] that it is possible to generate phase transitions between charged black holes with scalar uncharged hair and the RN-AdS solution, to obtain holographic dual QFTs with interesting charge transport features, starting from the following models:

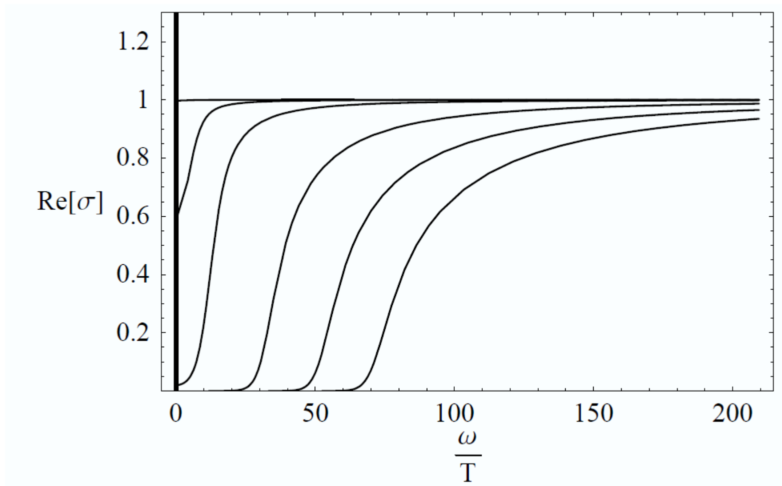

Figure 4.2: The formation of a frequency gap in the AC conductivity, as a function of ω/T, at small T/T_c.

$$S = \int d^4x\sqrt{-g}\left(R - \frac{f(\phi)}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)\right).$$

The main differences with the model of the holographic superconductors are the presence of a nonminimal coupling $f(\phi)$ between the scalar field ϕ and the Maxwell field and the choice of a neutral scalar field, instead of a charged one. However, similar results was found. In particular, also these models present a second-order phase transition between the RN-AdS black hole solution and the "scalar-dressed" charged black hole solutions, which exist only below a critical value T_c of the black hole temperature. An interesting feature of the scalar-dressed solutions is the near-horizon behavior of the extremal zero-temperature solutions. In this limit the solution has the scale-covariant form which we have described in Sect. 1.2.2 and we will also discuss in the next subsection.

From the holographic point of view, in this case we have the condensation of a neutral scalar condensate, which is related to very interesting electric transport properties (probably caused by the interaction of the charge carriers with the scalar condensate), reminiscent of electron motions in real materials. In particular, when the temperature is not too close to zero, the AC optical conductivity shows a minimum at low frequencies (the so-called "Drude peak"), reaching a constant value at $\omega = 0$ (which can be much larger than its constant value at high frequencies). Moreover, the resistivity does not increase monotonically with the temperature but displays a minimum, in a reminiscence of the well-known Kondo effect, caused in real metals with magnetic impurities by the interaction of the magnetic moment of the conduction electrons with the magnetic moment of the impurity.

In [25] these results were generalized to the case of dyonic black holes, i.e. solutions endowed with both an electrical and a magnetic charge. Also in this case, the dual field theory at finite temperature presents a rich phenomenology, reminiscent of electron motion in metals: phase transitions triggered by nonvanishing expectation values of scalar operators, non-monotonic behavior of the electric conductivities, Hall effect and sharp synchrotron resonances of the conductivity in presence of a magnetic field. Furthermore, in the zero temperature limit the AC optical conductivity for these models shows an interesting universal power-law behavior, whereas the DC conductivity in general scales as T^2 and is suppressed at small temperatures.

4.4.3 Hyperscaling violation

While the initial interest in the context of the holographic applications was focused mainly on AdS gravity theories and their conformal dual field theories, in recent years the class of metrics of interest in gauge/gravity duality has been considerably enlarged. The natural generalization is to consider metrics dual to field theories which are not conformally invariant, but scale-*covariant*. We have already defined in Sect. 1.2.2 the general form of scale-covariant metrics:

$$ds^2 = r^{-2(d-\theta)/d} \left(-r^{-2(z-1)} dt^2 + dr^2 + dx_i^2\right). \tag{4.6}$$

The hyperscaling violation exponent θ, which makes the metric scale-covariant, has a precise meaning in terms of the boundary theory [41]. Hyperscaling is a property of real critical systems for which free energy and entropy scale (as functions of the temperature T) by their naive dimensions:

$$F \sim T^{\frac{d+z}{z}}, \quad S \sim T^{d/z}. \tag{4.7}$$

When hyperscaling is violated, free energy and entropy scale in a different way:

$$F \sim T^{\frac{(d-\theta)+z}{z}}, \quad S \sim T^{(d-\theta)/z}. \tag{4.8}$$

Comparing the (4.7) and (4.8) we can observe that, very roughly speaking, in a theory with hyperscaling violation the thermodynamical behavior is as if the theory enjoyed dynamical exponent z but lived in $d-\theta$ dimensions.

In presence of hyperscaling violation, also the typical hyperscaling relation between the specific heat exponent $\hat{\alpha}$ and the critical exponent $\hat{\nu}$ [41]:

$$2 - \hat{\alpha} = d\,\hat{\nu}$$

is modified by "lowering" the dimensionality of the system from d to $d-\theta$, namely:

$$2 - \hat{\alpha} = (d-\theta)\,\hat{\nu}.$$

Obvioulsy, there are some constraints we must impose on (4.6), in order to get a physically sensible dual field theory. From the gravity side, a minimum constraint is that the null energy condition (NEC) is satisfied:

$$T_{\mu\nu} N^{\mu} N^{\nu} \geq 0,$$

where $N^{\mu} N_{\mu} = 0$. Taking into account that $G_{\mu\nu} = T_{\mu\nu}$ on shell, from the Ricci and Einstein tensors for the metric (4.6) the NEC becomes:

$$\begin{cases} (d-\theta)[d(z-1)-\theta] \geq 0 \\ (z-1)(d+z-\theta) \geq 0. \end{cases}$$

These conditions imply some consequences for the allowed values of the exponents z and θ that admit a consistent gravity dual. For example, in a Lorentz invariant theory (i.e. with $z = 1$) the first inequality implies $\theta \leq 0$ or $\theta \geq d$, while in the case of a scale invariant theory ($\theta = 0$), we recover the known result, for Lifshitz theories, $z \geq 1$.

Moreover, it was found [42] that the "area law" of the entanglement entropy requires the further constraint:

$$\theta \leq d-1.$$

Concluding remarks and summary of subsequent chapters

In this chapter we have reviewed the general formulation of the AdS/CFT correspondence and some of its extensions and applications. We have seen that one of the most intriguing applications of the AdS/CFT is the possibility to describe, starting from a gravitational theory coupled to scalar fields and an electromagnetic field, strongly coupled condensed matter systems, as e.g. superconductors. One of the most recent applications involves hyperscaling violation (HV), a particular scaling of free energy and entropy (as functions of the temperature), observed near critical points in real condensed matter phase transitions. Hyperscaling violation can be described by *scale-covariant* gravitational theories. Typically, in real systems HV is observed in the infrared (IR) regime of the field theory, while in the ultraviolet (UV) one has a conformal fixed point. This behavior can be described, in the gravitational bulk, by a "scalar-dressed" solution interpolating between a scale-covariant metric in the near-horizon behavior and AdS at infinity.

However, we have showed (see Chapter 3) that in some gravitational solutions (in particular those characterized by a scalar field with a positive squared mass) this behavior is reversed, with AdS in the near-horizon region and a scale-covariant behavior at infinity. This corresponds, in the dual field theory, to a flow between a fixed point in the IR regime and an hyperscaling-violating phase in the UV.

In the next two chapters we will attempt to move a step in order to achieve a better understanding of both these configurations, both from the point of view of the gravitational solutions and in the perspective of the dual field theory. In particular, in Chapter 5 we shall generalize to a generic number of dimensions the black brane solution derived (in four dimensions) in Chapter 3, also focusing on the thermodynamical properties of the solutions, and discuss some features of the dual field theory, in which we observe the more unusual configuration with hyperscaling violation in the UV regime and a conformal fixed point in the IR.

In Chapter 6 we shall focus our attention on scalar-dressed black brane solutions asymptotically AdS and with different near-horizon behaviors (AdS, scale covariant). We will make a detailed analysis of the infrared features of the spectrum of the black brane solutions, with a particular attention to the solutions with a scale-covariant near-horizon behavior, corresponding to an hyperscaling-violating phase in the IR regime of the dual field theory. Moreover, we will present some interesting general results about the stability of "hairy" black brane solutions in a wide class of gravitational models. These results, as we will see, can be used for understanding quantum critical points and phase transitions in the corresponding dual field theories.

Chapter 5

Hyperscaling violation for scalar black branes

We extend to black branes (BB) in arbitrary dimensions the results of Chapter 3 and of Ref. [85] obtained for scalar black 2-branes. We derive the analytic form of the $(d+1)$-dimensional scalar soliton interpolating between a conformal invariant AdS_{d+2} vacuum in the infrared and a scale-covariant metric in the ultraviolet. We show that the thermodynamical system undergoes a phase transition between Schwarzschild-AdS_{d+2} and a scalar-dressed BB. We calculate the critical exponent z and the hyperscaling violation parameter θ in the two phases. We show that our scalar BB solutions generically emerge as compactifications of $p-$brane solutions of supergravity theories. We also derive the short distance form of the correlators for the scalar operators corresponding to an UV exponential potential supporting our black brane solution. We show that also for negative θ these correlators have a short distance power-law behavior.

5.1 Introduction

Recent investigations on the application of the AdS/CFT correspondence to strongly interacting quantum field theories (QFT) have emphasized the importance played by non-AdS gravitational backgrounds and the related dual nonconformal QFTs [23, 25, 27, 28, 30, 34, 37–40, 80, 109, 110].

The standard setup for this kind of holographic applications is a black brane in a AdS background endowed with non trivial scalar field configurations and finite electromagnetic charge density. It has been shown that this produces a rich phenomenology in the dual QFT, such as spontaneous symmetry breaking, phase transitions and non-trivial transport properties [18–23, 25, 27, 28, 30, 34, 37–40, 80, 109–117]. In the case of nonminimal, exponential coupling between the scalar field and the Maxwell tensor, the bulk gravity allows for extremal, near-horizon solutions which break the conformal symmetry of the AdS vacuum [23, 25, 28, 30, 34, 36–40]. We have seen in Sect. 1.2.2 and in the previous chapter that these IR metrics belong to a general class of metrics that are not scale-invariant but only scale-covariant and lead to hyperscaling violation in the dual field theory [38, 39, 42–48]. They are characterized by two parameters, the critical exponent z and the hyperscaling violation parameter θ, which characterize both the transformation weight of the infinitesimal length ds under scale transformations and the scaling behavior of free energy and entropy as functions of the temperature.

The standard framework for obtaining, dynamically, scale-covariant metrics in the IR is given by Einstein-scalar gravity, possibly coupled – minimally or non-minimally – to a $U(1)$ field. The self-interaction potential $V(\phi)$ for the scalar field ϕ must have a negative local maximum at $\phi = 0$, with a corresponding scalar tachyonic excitation whose mass is slightly above the Breitenlohner-Freedman (BF) bound. Under suitable conditions, usually an exponential behavior of the potential and/or scalar-Maxwell tensor coupling functions, the theory admits black brane solutions with scalar hair that in the near-extremal regime approach the scale-covariant metrics.

In Chapter 3 we have shown that this framework is not the only possible way to produce scale-covariant BBs. These solutions can be also obtained from Einstein-scalar gravity with a positive squared mass for the scalar, when the potential behaves exponentially in the asymptotic region of the spacetime.

Although BB solutions with scale-covariant asymptotics have been explicitly derived for particular four-dimensional (4D) Einsten-scalar gravity models, as our exact solution described in Chapter 3, their existence is a rather generic feature of a broad class of 4D models [85]. Moreover, in the extremal limit the BB solution reduces to a fully regular scalar soliton, which interpolates between an AdS_4 vacuum in the near-horizon region and a scale-covariant solution in the asymptotic region.

These results allow to realize an alternative scenario, which exchanges IR and UV regions. In the dual QFT we have an infrared fixed point, corresponding to the AdS vacuum, whereas in the UV regime we have hyperscaling violation.

Detailed investigations of the symmetries and thermodynamics of these BB solutions revealed rather interesting and intriguing features [85]. The thermodynamical phase diagram of the system is characterized by the presence of different phases. Above a critical temperature T_c the scalar-dressed BB becomes energetically preferred with respect to the Schwarzschild-AdS_4 (SAdS) solution and the thermodynamical system undergoes a first-order phase transition. Moreover, for some values of the parameters characterizing the model, at low temperatures different phases may coexist. In the dual QFT the scalar-dressed, stable, BB corresponds to a phase with a negative hyperscaling violation parameter θ. Although negative values of θ do not have analogous in condensed matter system, they are consistent with the null energy conditions for the bulk stress-energy tensor. Moreover they also arise in string theory and supergravity constructions [35, 39, 114, 118, 119].

In this chapter we will generalize the results obtained in Chapter 3 and in Ref. [85] concerning 2-branes to branes of arbitrary dimensions. We will show that basically all the results of Ref. [85] can be generalized to arbitrary dimensions and therefore generically hold for d-branes. Moreover, we will also show that our scalar BB solutions can be obtained in several ways as compactifications of p-brane solutions of supergravity (SUGRA) theories. Finally, we will be concerned with some holographic features of QFTs with negative hyperscaling violation parameter θ. Extending the results of Ref. [39], which hold for positive θ and for a massive scalar field, we will derive the short distance form of the correlators for scalar operators corresponding to an UV exponential potential supporting our black brane solution. We show that for negative θ these correlators have a short distance power-law behavior.

The chapter is organized as follows. In Sect. 5.2 we present our Einstein-scalar gravity model, derive the BB solutions with scale-covariant asymptotics and discuss their solitonic extremal limit. In Sect. 5.3 we show how our BB solutions can be obtained as compactifications of p-brane solutions of SUGRA theories. The thermodynamics of our solutions is investigated in Sect. 5.4. The symmetries of the BB are discussed in Sect. 5.5, where also critical exponent and hyperscaling violation parameters are calculated. In Sect. 5.6 we extend our investigation to general models whose potential behaves exponentially in the asymptotic region. In Sect. 5.7 we study holographic properties of our BB solution and in particular the two-point function of scalar operators in the dual QFT. Finally, in Sect. 5.8 we state our conclusions.

5.2 Einstein-scalar gravity in $d+2$ dimensions

We consider $d+2$-dimensional (with $d \geq 2$) Einstein gravity minimally coupled to a scalar field ϕ:

$$I = \int d^{d+2}x\sqrt{-g}\left[R - 2(\partial\phi)^2 - V(\phi)\right]. \tag{5.1}$$

We will focus on models for which the scalar self-interaction potential $V(\phi)$ is given by:

$$V(\phi) = -\frac{d(d+1)}{\gamma L^2}\left(e^{2s\beta\phi} - \beta^2 e^{2\frac{s}{\beta}\phi}\right), \quad \gamma = 1-\beta^2, \quad s = \sqrt{\frac{2(d+1)}{d}}, \tag{5.2}$$

where β is a (real) parameter characterizing the model and L is the AdS length. The action (5.1) is the $d+2$-dimensional generalization of the four-dimensional Einstein-scalar gravity model investigated in Chapter 3 and in Ref. [85]. It shares with the 4D model several features. The potential (5.2) has a minimum at $\phi = 0$ with $V(0) = -d(d+1)/L^2$, corresponding to an AdS_{d+2} vacuum and a local scalar excitation of positive squared-mass $m^2 = 2(d+1)^2/L^2$.

The model is a fake SUGRA model. In fact the potential (5.2) can be derived from the superpotential:

$$P(\phi) = \sqrt{\frac{d}{2}}\gamma^{-1}L^{-1}\left(e^{s\beta\phi} - \beta^2 e^{\frac{s}{\beta}\phi}\right). \tag{5.3}$$

Moreover, the action (5.1) is invariant under the duality transformation:

$$\beta \to \frac{1}{\beta}. \tag{5.4}$$

5.2.1 Black brane solutions

We now look for static, radially symmetric, planar solutions of the field equations stemming from the action (5.1).

The presence of the $\phi = 0$ minimum of the potential (5.2) for every value of β implies the existence of the Schwarzschild-AdS (SAdS) solution with $\phi = 0$:

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 dx_i dx_i \,, \qquad f(r) = \frac{r^2}{L^2} - \frac{M}{2r^{d-1}}, \tag{5.5}$$

where M is the black brane mass and $i = 1, 2 \dots d$.

Solutions with a non-trivial scalar field can be found using the same approach used in Chapter 3. Choosing for the metric the parametrization:

$$ds^2 = -e^{2\nu}dt^2 + e^{2\nu+2d\rho}d\xi^2 + e^{2\rho}dx_i dx_i, \tag{5.6}$$

the field equations can be recast in the form of the $SU(2) \times SU(2)$ Toda molecule [96]. Solutions with a regular horizon and nontrivial scalar field exist only if they do not approach asymptotically to AdS_{d+2}. For $\beta^2 < 1$ we get the two-parameter family of solutions:

$$\begin{aligned}
ds^2 &= \left(\frac{r_0}{r}\right)^{\frac{2}{\omega}} \left\{ \Delta(r)^{\frac{2\beta^2}{(d+1)\gamma}} \left[-\Gamma(r)\, dt^2 + dx_i dx_i\right] + E\Delta(r)^{\frac{2\beta^2}{\gamma}} \Gamma(r)^{-1} dr^2 \right\}, \\
e^{2\phi} &= \left[\frac{A}{\Delta(r)}\right]^{\frac{2\beta}{s\gamma}} \left(\frac{r}{r_0}\right)^{\frac{ds\beta}{\omega}}, \quad \Gamma(r) = 1 - \mu_1 \left(\frac{r_0}{r}\right)^{\delta}, \quad \Delta(r) = 1 + \mu_2 \left(\frac{r_0}{r}\right)^{\delta}, \\
\omega &= 1 - (d+1)\beta^2, \quad \delta = -\frac{(d+1)\gamma}{\omega}, \\
A &= \sqrt{\mu_2(\mu_1 + \mu_2)}, \quad E = \left(\frac{\gamma L}{r_0 \omega}\right)^2 A^{-\frac{2\beta^2}{\gamma}},
\end{aligned} \tag{5.7}$$

where $\mu_{1,2}$ are dimensionless free parameters and r_0 is a length scale that must be introduced in order to get the correct physical dimensions. After a trivial change of parametrization for the radial coordinate r, it is simple to verify that solution (5.7) reduces to (3.10) for $d = 2$.

The asymptotic region of the spacetime (5.7) is given by $r \to 0$ for $\beta^2 < 1/(d+1)$, whereas it is given by $r = \infty$ when $\beta^2 \geq 1/(d+1)$. In both cases the asymptotic behavior of the solution (5.7) is given by:

$$ds^2 = \left(\frac{r_0}{r}\right)^{\frac{2}{\omega}} (-dt^2 + dx_i dx_i + dr^2), \quad \phi = \frac{sd\beta}{2\omega} \log(r/r_0). \tag{5.8}$$

This metric represents a domain wall. It is not invariant under scale transformations, but still transforms with definite weight, so it is scale-covariant. The solution (5.7) becomes singular for $\beta^2 = 1/(d+1)$. This is related to the fact that this value of β corresponds to a divergent hyperscaling parameter θ. Nevertheless, a fully regular solution can be written using a different parametrization for the radial coordinate r:

$$\begin{aligned} ds^2 &= \left(\frac{r_0}{r}\right)^2 \left\{ \Delta(r)^{\frac{2}{d(d+1)}} \left[-\Gamma(r)\, dt^2 + dx_i dx_i\right] + E\Delta(r)^{\frac{2}{d}} \Gamma(r)^{-1} \left(\frac{r_0}{r}\right)^2 dr^2 \right\}, \\ e^{2\phi} &= \left[\frac{A}{\Delta(r)}\right]^{\sqrt{\frac{2}{d}}} \left(\frac{r}{r_0}\right)^{\sqrt{2d}}, \quad \delta = -d, \quad E = \left(\frac{d^2\ L}{r_0(d+1)(d+2)}\right)^2 A^{-\frac{2}{d}}, \end{aligned} \tag{5.9}$$

whereas Δ, Γ, A are given as in Eq. (5.7).

The radial coordinate in the metric (5.8) gives the information about the various energy scales in the dual QFT. A proper energy $\mathcal{E}_0$ is redshifted according to the law:

$$\mathcal{E}(r) = r^{-\frac{1}{\omega}} \mathcal{E}_0. \tag{5.10}$$

This equation tells us that for $\omega > 0$ ($\beta^2 < 1/(d+1)$), $r \to 0$ ($\to \infty$) corresponds to the UV (IR) region of the dual QFT, whereas for $\omega < 0$ ($\beta^2 > 1/(d+1)$) the UV (IR) corresponds to $r = \infty$ ($r \to 0$).

For $\mu_1, \mu_2 \geq 0$, the metric (5.7) exhibits a singularity at $r = \infty$ ($r = 0$) for $\beta^2 < 1/(d+1)$ ($\beta^2 \geq 1/(d+1)$) shielded by a horizon at $r/r_0 = \mu_1^{1/\delta}$, and therefore represents a regular black brane.

Until now we have considered only the case $\beta^2 < 1$. The form of the solutions for $\beta^2 > 1$ can be simply found using the duality (5.4) into Eq. (5.7). All the considerations of this section can be trivially extended to the case $\beta^2 > 1$.

5.2.2 Extremal limit and scalar soliton

The extremal limit of the solution (5.7) is obtained setting $\mu_1 = 0$. When $\mu_2 = 0$ this extremal limit is singular, with a naked singularity at $r = \infty$ for $\beta^2 < 1/(d+1)$ (at $r = 0$ for $\beta^2 > 1/(d+1)$) with $\phi \sim \ln r$. On the other hand, for $\mu_2 > 0$ the extremal BB is a regular scalar soliton that interpolates between a scale-covariant solution in the UV and AdS_{d+1} in the IR:

$$\begin{aligned} ds^2 &= \left(\frac{r_0}{r}\right)^{\frac{2}{\omega}} \left\{ \Delta(r)^{\frac{2\beta^2}{(d+1)\gamma}} \left[-dt^2 + dx_i dx_i\right] + E\Delta(r)^{\frac{2\beta^2}{\gamma}} dr^2 \right\}, \\ e^{2\phi} &= \left[\frac{\mu_2}{\Delta(r)}\right]^{\frac{2\beta}{s\gamma}} \left(\frac{r_0}{r}\right)^{-\frac{ds\beta}{\omega}}. \end{aligned} \tag{5.11}$$

Let us now consider the UV (asymptotic) and IR (near-horizon) limit of the scalar soliton (5.11). For $\beta^2 < 1/(d+1)$ this corresponds to take, respectively, $r \to 0$ and $r \to \infty$. For $\beta^2 \geq 1/(d+1)$ these limits are reversed (the UV corresponds to $r \to \infty$ and the IR to $r \to 0$).

In the IR limit, the scalar field ϕ vanishes, the length scale r_0 decouples and the metric (5.11) becomes that of AdS_{d+2}. The length-scale r_0 is an UV scale, which decouples in the IR, where conformal invariance is restored. On the other hand, in the UV limit it is the AdS length L that decouples: the metric (5.11) can be written in terms of r_0 only and takes the scale-covariant form given by Eq. (5.8).

5.3 Compactifications of p-brane solutions of SUGRA theories

In this section we will look for string theory realizations that produce, after compactification, an Einstein-scalar model (5.1) with potential of the form (5.2). This means that we are considering our models just as an effective description, which breaks down in the far UV. The short-distance physics will be therefore described by the UV completion of our effective model.

We will show in the following that BB solutions (5.7) arise as compactifications of black p-brane solutions of SUGRA theories. We will see that they emerge from the p-brane both as simple spherical compactification or also as a more general Kaluza-Klein compactification parametrized by a parameter.

Black p-branes are classical Ramond-Ramond charged solutions of D-dimensional SUGRA theories supported by a $(p+2)$-form field strength G_{p+2} [120, 121]. In the Einstein frame the bosonic part of the action is:

$$I = \int d^D x \sqrt{-g} \left(R - \frac{1}{2}(\partial\Phi)^2 - e^{a\Phi} \frac{1}{2(p+2)!} G_{(p+2)} \right), \tag{5.12}$$

where Φ is the dilaton field and a is constant, which is zero for non-dilatonic p-branes, whereas

$$a^2 = 4 - [(p+1)(D-p-3)]/(D-2), \tag{5.13}$$

for dilatonic branes. The metric part of the p-brane solution is given in terms of two integration constants h_0, g_0 by:

$$
\begin{aligned}
ds_D^2 &= H(r)^{-\frac{2\tilde{d}}{\rho}}\left(-g(r)dt^2+\sum_{i=1}^{p}dx_idx_i\right)+H(r)^{\frac{2d}{\rho}}\left(g(r)^{-1}dr^2+r^2d\Omega_q^2\right),\\
H(r) &= 1+\left(\frac{h_0}{r}\right)^{\tilde{d}},\ g(r)=1-\left(\frac{g_0}{r}\right)^{\tilde{d}},\\
\rho &= (p+1)\tilde{d}+a^2\frac{D-2}{2},\ \tilde{d}=D-p-3,
\end{aligned}
\tag{5.14}
$$

where $d\Omega_q^2$ is the line element of a compact space $\mathcal{K}^q$ with $q = D-p-2$ dimensions.

Let us first consider nondilatonic p-branes. The simplest diagonal ansatz for the D-dimensional metric, which gives the $p+2$-dimensional theory in the Einstein frame, is obtained by setting $\mathcal{K}^q = S^q$ and

$$
ds_D^2 = e^{-\frac{2q}{p}\psi}ds_{p+2}^2+e^{2\psi}d\Omega_q^2. \tag{5.15}
$$

Taking into account that for nondilatonic branes $e^{\psi} = rH^{1/\tilde{d}}$ one finds after compactification the BB metric:

$$
\begin{aligned}
ds_{p+2}^2 &= r^{-\frac{2}{p}(p+2-D)}\left[H(r)^{\frac{2(D-2)}{p(p+1)(D-p-3)}}\left(-g(r)dt^2+\sum_{i=1}^{p}dx_idx_i\right)\right.\\
&\left.\quad+H(r)^{\frac{2(D-2)}{p(D-p-3)}}g(r)^{-1}dr^2\right],
\end{aligned}
\tag{5.16}
$$

one can easily see that the metric (5.16) matches exactly, after some trivial identification of the parameters, the metric (5.7) if we take $d = p$ and

$$
\beta^2 = \frac{D-2}{(p+1)(D-p-2)}. \tag{5.17}
$$

It is important to notice that this value of β always satisfies the inequality $1/(p+1) < \beta^2 < 1$. Particularly interesting cases are represented by the 2 and 5-brane in $D = 11$ corresponding, respectively, to $\beta^2 = 3/7$ and $\beta^2 = 3/8$.

Compactification of p-branes with the diagonal ansatz (5.15) produces BB solutions of the form (5.7) only for the particular values of the parameter β given in Eq. (5.17). This limitation can be removed by considering the more general diagonal ansatz of Ref. [80] for the D-dimensional metric.

Let us now briefly consider compactification of dilatonic p-branes. In this case we must use in (5.14) the value (5.13) for a giving $\rho = 2(D-2)$. The diagonal ansatz (5.15) produces now the BB solution:

$$ds^2_{p+2} = r^{-\frac{2}{p}(p+2-D)} \left[H(r)^{\frac{1}{p}} \left(-g(r)dt^2 + \sum_{i=1}^{p} dx_i dx_i \right) H(r)^{\frac{p+1}{p}} g(r)^{-1}\, dr^2 \right]. \tag{5.18}$$

Matching this BB solution with Eq. (5.7) requires:

$$D = \frac{3p+1}{p-1} + p + 2, \quad \beta^2 = \frac{p+1}{3p+1}. \tag{5.19}$$

These are very stringent constraints which however are satisfied by a very interesting case, the 3-brane in $D = 10$ which gives $\beta^2 = 2/5$. It is likely that also for dilatonic branes the use of the more general diagonal ansatz of Ref. [80] would allow to circumvent the constraints (5.19).

5.4 Thermodynamics and phase transitions

In this section we will consider the BB solution (5.7) as a thermodynamical system, using the Euclidean action formulation of Martinez et al. [71]. As it has been already noted in Ref. [85] for the 4D case, the two-parameter family of solutions (5.7) is not suitable for setting up a consistent BB thermodynamics. The problem is the explicit dependence of the scalar field on the parameter μ_1, which causes divergences in the boundary action, that determines the mass of the solution. This explicit dependence can be eliminated by constraining the possible values of $\mu_{1,2}$ in Eq. (5.7) with $\mu_2(\mu_2 + \mu_1) = 1$.

We end up with the one-parameter family of solutions:

$$\begin{aligned} ds^2 &= r^{-\frac{2}{\omega}}\left\{\left(\frac{\gamma L}{\omega}\right)^2\left[-\Delta(r)^{\frac{2\beta^2}{(d+1)\gamma}}\Gamma(r)dt^2+\Delta(r)^{\frac{2\beta^2}{\gamma}}\Gamma(r)^{-1}dr^2\right]+\Delta(r)^{\frac{2\beta^2}{(d+1)\gamma}}dx_i dx_i\right\}, \\ e^{2\phi} &= \Delta(r)^{-\frac{2\beta}{s\gamma}}r^{\frac{sd\beta}{\omega}}, \quad \Gamma(r)=1-\frac{\nu_1}{r^\delta}, \quad \Delta(r)=1+\frac{\nu_2}{r^\delta}, \end{aligned} \tag{5.20}$$

where the parameters ν_{12} are constrained by:

$$\nu_1=\frac{1}{\nu_2}-\nu_2, \quad 0<\nu_2\le 1, \quad 0\le\nu_1<\infty. \tag{5.21}$$

Notice that in writing Eq. (5.20) we have introduced dimensionless coordinates t, r and parameters ν_{12}. This is necessary because (5.11) is a global solution interpolating between the IR and the UV regimes that are characterized by two different length scales r_0 and L.

Starting from Eq. (5.20) one can now calculate, using standard formulas, the temperature T and entropy S of the BB. One has:

$$T=\frac{1}{4\pi}\frac{d(d+1)\gamma}{d+2(d+1)\beta^2}\nu_2^{-\frac{\omega}{(d+1)\gamma}}(1-\nu_2^2)^{1/(d+1)}, \quad S=4\pi V\nu_2^{-\frac{d}{(d+1)\gamma}}(1-\nu_2^2)^{\frac{d}{d+1}}, \tag{5.22}$$

where V is the volume of the transverse d-dimensional space.

We construct the thermodynamics of our BB solutions using the Euclidean action formalism. Thermodynamical potentials are given by boundary terms of the action. We use the parametrization of the metric of Ref. [71]:

$$ds^2=N^2f^2dt^2+f^{-2}dr^2+R^2dx_idx_i.$$

The variations of the boundary terms of the action evaluated for the solution

(5.20) are:

$$
\begin{aligned}
\delta I_G^\infty &= -\frac{Vd^2}{T\left[(d+2(d+1)\beta^2)\right]}\left[\delta\nu_1 + \frac{2\beta^2}{\gamma}(2-\beta^2)\delta\nu_2\right],\\
\delta I_\phi^\infty &= \frac{2\beta^2 V d^2}{\gamma T\left[(d+2(d+1)\beta^2)\right]}\delta\nu_2,\\
\delta I_G\big|_{r_h} &= -\frac{Vd^2}{T\left[(d+2(d+1)\beta^2)\right]}\left(\nu_1+\nu_2\right)^{-1}\left[(\nu_1+\gamma\nu_2)\delta\nu_1 + \beta^2\nu_1\delta\nu_2\right],\\
\delta I_\phi\big|_{r_h} &= 0, \qquad (5.23)
\end{aligned}
$$

where I_G and I_ϕ are, respectively, the gravitational and scalar field part of the boundary action.

One can easily show that the BB entropy S is correctly given by $S = I_G|_{r_h}$. The mass of the BB is given in terms of the free energy F and the entropy S by $M = F + TS = -T(I_G^\infty + I_\phi^\infty)$. Using Eqs. (5.23) one finds:

$$
M = \frac{Vd^2}{d+2(d+1)\beta^2}\left(\nu_1 + 2\beta^2\nu_2\right) = \frac{Vd^2}{d+2(d+1)\beta^2}\left\{\frac{1}{\nu_2} + \left(2\beta^2-1\right)\nu_2\right\}. \qquad (5.24)
$$

Using Eqs. (5.22), (5.24) and the constraint (5.21) one can now check that the first principle $dM = TdS$ is satisfied. As usual the results can be trivially extended to the parameter region $\beta^2 > 1$ just by using the duality $\beta \to 1/\beta$ in Eqs. (5.20), (5.22) and (5.24).

5.4.1 Phase transition

The global stability of our BB solution, considered as a thermodynamical system, can be investigated by computing the free energy and the specific heat. In particular, comparison of the free energies of different configurations at fixed temperature allows us to single out the energetically preferred phase, whereas a positive (negative) specific heat indicates local stability (instability) of a given phase. We start with the case $\beta^2 < 1/(d+1)$, where, as we will see, we observe a phase transition.

Free energy

In the case under consideration, the two competitive phases are represented by the black brane with scalar hair (SB) (5.20) and the SAdS BB (5.5). The free energy of the scalar black brane is:

$$F_{SB}(T) = M - TS = \frac{Vd}{d+2(d+1)\beta^2}\left\{-\frac{\omega}{\nu_2(T)} + \left[1+(d-1)\beta^2\right]\nu_2(T)\right\}, \tag{5.25}$$

where $\nu_2(T)$ is defined implicitly by the first equation in (5.22). For the free energy of the SAdS black brane we have instead:

$$F_{SAdS}(T) = -V\left(\frac{4\pi}{d+1}\right)^{d+1} T^{d+1}.$$

The relevant quantity $\Delta F(T) = F_{SB}(T) - F_{SAdS}(T)$ cannot be computed explicitly in closed form because $\nu_2(T)$ is only implicitly defined. Nevertheless, one can show that for $\beta^2 < 1/(d+1)$, $\Delta F(T)$ is positive for small T, vanishes at finite value of the temperature and becomes negative at large T.

A qualitative way to see this change of sign of $\Delta F(T)$ is to consider the small-T ($\nu_2 \sim 1$) and the large-T ($\nu_2 \sim 0$) behavior of F_{SB}. At small temperatures we have:

$$F_{SB}(T) = V\left\{\frac{2\beta^2 d^2}{d+2(d+1)\beta^2} - \frac{(4\pi)^{d+1}}{(d+1)^{d+1}}\left[\frac{d+2(d+1)\beta^2}{d\gamma}\right]^d T^{d+1}\right\}. \tag{5.26}$$

The small-T behavior is determined by the $T=0$, AdS_{d+2} extremal limit and is pertinent to a holographically dual $(d+1)$-dimensional CFT. Conversely for the large-T ($\nu_2 \sim 0$) behavior we have:

$$F_{SB} = -\frac{\omega V d}{d+2(d+1)\beta^2}\left\{\frac{4\pi\left[d+2(d+1)\beta^2\right]}{d(d+1)\gamma}\right\}^{\frac{(d+1)\gamma}{\omega}} T^{\frac{(d+1)\gamma}{\omega}}. \tag{5.27}$$

The free energy for the hairy BB is positive at small T implying $\Delta F > 0$.

For $\beta^2 < 1/(d+1)$, ΔF becomes negative at large T. This shows the existence of a critical temperature T_c such that $\Delta F(T_c) = 0$.

In general the critical temperature cannot be determined analytically. However, we can show the existence of T_c graphically. By setting $y = \nu_2^2$, the equation $F_{SB} = F_{SAdS}$ gives:

$$\begin{aligned} g(y) &= \frac{1-(d+1)\beta^2 - [1+(d-1)\beta^2]\,y}{1-y} = f(y) = \left[\frac{d}{d+2(d+1)\beta^2}\right]^d \gamma^{d+1} y^{\frac{d\beta^2}{2\gamma}}, \\ 0 &\le y \le 1. \end{aligned}$$

While the curve $f(y)$ is always positive, the behavior of $g(y)$ depends on the value of β. For $\beta^2 > \frac{1}{d+1}$, the curve starts from a negative point and is always negative; for $\beta^2 = \frac{1}{d+1}$, the curve starts from $y = 0$ and is always negative; for $\beta^2 < \frac{1}{d+1}$, the curve starts from a positive point and decreases monotonically to $-\infty$. Then the two curves $f(y)$ and $g(y)$ do not intersect for $\beta^2 \geqslant \frac{1}{d+1}$, while they intersect at a finite critical value of the temperature for $\beta^2 < \frac{1}{d+1}$.

In Fig. 5.1 we show the behavior of the free energy density for $d = 3$ and for $\beta^2 = 1/8$, a value in the range $0 \le \beta^2 < 1/(d+1)$. The critical temperature can also be determined numerically. For the case described in Fig. 5.1 we have $T_c = 0.20917$.

We have therefore discovered, in the $\beta^2 < 1/(d+1)$ case, a cross-over behavior for the free energies of the SAdS and scalar black brane solutions. The relevant question is now the following: can we interpret this behavior as a phase transition between two different configurations of the same bulk gravity theory? This question can be answered only if one clarifies the role played by boundary conditions in the definition of canonical thermodynamical ensembles. In fact, the two classical configurations - the SAdS and the scalar brane - actually are two different solutions of the same bulk theory defined by the action (5.1). On the other hand, these solutions correspond to different asymptotic values of the scalar field in the UV ($\phi = 0$ and $\phi = -\infty$

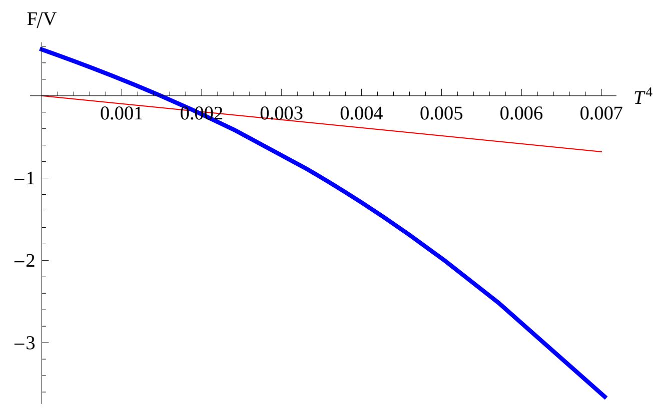

Figure 5.1: The free energy density F/V, as function of T^4, of the scalar black brane for $d = 3$ and $\beta^2 = 1/8$ (blue, thick line) and of the SAdS black brane for $d = 3$ (red, thin line).

respectively for the SAdS solution (5.5) and the hairy black brane solution (5.7)) and to different asymptotic geometries. However, there is no obstruction in considering solutions of the same bulk theory with different boundary conditions as belonging to the same canonical ensemble. Although this is not an usual situation in the AdS/CFT correspondence, where one refers to fixed boundary conditions, one can define a canonical partition function just by evaluating the Euclidean action on the particular bulk solution, without any reference to the asymptotics of the solutions. This is exactly the way we have calculated the free energy using Eqs. (5.23). This is a strong argument supporting the interpretation of the free energy cross-over described in this section as a truly first-order phase transition (the phase transition is first-order because at $T = T_c$, $dF_{SB}/dT \neq dF_{SAdS}/dT$).

A definitive answer about the existence of the phase transition could be obtained by showing that for $T > T_c$ the SAdS solution decays with finite half-life in the scalar brane solution.

Because of the change in asymptotics of the two competitive bulk solutions, the holographic interpretation of the phase transition is rather involved. In the usual gravity/gauge theory correspondence dictionary, the sources J in the dual QFT are related to small perturbations of the UV boundary conditions. The gravity/gauge theory correspondence rules allow then to compute the n-point functions for dual operators differentiating the bulk partition function with respect to J. The dramatic change in the boundary conditions for the scalar field we have in our case seems to suggest that the two different phases we have on the gravity side correspond to different sources in the dual QFT. Because different sources generally lead to different Lagrangians, we are led to the conclusion that the two phases of the gravity theory - the SAdS and the scalar brane phase - correspond to two distinct dual QFTs, not to two distinct phases of a single QFT.

An other argument supporting this interpretation is the analogy with what happens in bulk theories allowing for a flow between an AdS in the IR and an other AdS in the UV. Such solutions are known in the literature. Analogously to the case discussed in this chapter, we have also here three different boundary QFTs. In particular we have two CFTs with no flow, corresponding to fixed IR or UV fixed values for the scalar field and a QFT describing the flow between the IR and the UV fixed point, corresponding to a r-dependent scalar field.

Specific heat

It is easy to check, using Eqs. (5.22) and (5.24), that for $\beta^2 < 1/(d+1)$ the function $M(T)$ is a monotonic increasing function of the temperature T. Then the specific heat $c = \partial M/\partial T$ is positive for all values of T. Similarly, the specific heat of the SAdS black brane is: $c_{SAdS}(T) = \frac{(4\pi)^{d+1} V d}{(d+1)^d} T^d > 0$.

5.4.2 $1/(d+1) \leq \beta^2 < 1$

For $\beta^2 = 1/(d+1)$, the scalar black brane solution exists only for temperatures below the critical value $T = T_c = \frac{d^2}{4\pi(d+2)}$, while for $T > T_c$ only the

SAdS solution (5.5) exists. The free energy is:

$$F_{SB} = \frac{2Vd^2}{(d+1)(d+2)}\sqrt{1-\left(\frac{T}{T_c}\right)^{d+1}}.$$

The free energy is positive definite and vanishes for $T = T_c$, while F_{SAdS} is always negative. Then we have $F_{SAdS} < F_{SB}$ in the whole range $T \leq T_c$, that is the SAdS solution is always energetically favored. The specific heat of the black brane solution is always positive and diverges at the critical temperature.

For $\beta^2 > 1/(d+1)$, the function $T(\nu_2)$ is not monotonic. It has a maximum at $\nu_2 = \nu_0 = \sqrt{[(d+1)\beta^2 - 1] / [(d-1)\beta^2 + 1]}$. Also in this case the black brane solution exists only below a maximum, critical temperature $T = T_c$.

For what concerns the free energy, from Eq. (5.25) it is easy to realize that, for $\beta^2 > 1/(d+1)$, F_{SB} is always positive. Hence also in this case $F_{SAdS} < F_{SB}$ and the SAdS solution is energetically preferred with respect to the scalar-dressed black brane. However, the non-monotonicity of the function $T(\nu_2)$ implies the existence of two different branches of the SB phase for $T \leq T_c$, as it has been already observed in Ref. [85] for the 4D case. The first branch (obtained for $\nu_0 \leq \nu_2 \leq 1$) is the analogue of the AdS_{d+2} phase obtained for $\beta^2 < 1/(d+1)$ at small T, while the second branch (obtained for $0 < \nu_2 \leq \nu_0$) has no analogue for $\beta^2 < 1/(d+1)$. In this case the free energy scales at small temperatures as $F \sim T^\alpha$, with $\alpha = (d+1)\gamma/\omega$. But α is negative, hence F_{SB} has a singularity at $T = 0$.

For what concerns the specific heat we have an interesting peculiarity: in the first branch the specific heat is positive and hence it corresponds to a locally stable phase (although this phase is not energetically preferred with respect to the SAdS solution), while in the second branch $c(T)$ is always negative, corresponding to an unstable phase.

5.4.3 Dual solution

As already observed, using the duality (5.4) into the (5.20) we obtain the solution for $\beta^2 > 1$. The thermodynamical properties of these solutions follow easily from the case $\beta^2 < 1$ by duality. We note that in this case the phase transition between the scalar-dressed black brane solution and the SAdS solution is present for $\beta^2 > (d+1)$, while for $\beta^2 \leq (d+1)$ the SAdS solution is always energetically favored respect to the SB solution. The behavior of the free energy and the specific heat in the three cases are qualitatively analogous to those discussed for $\beta^2 < 1$.

5.5 Hyperscaling violation

The thermodynamical behavior of our scalar BB described in the previous sections is strongly related to the symmetries of the solutions in the UV and IR regimes.

The UV regime, where the solution takes the form (5.8), is characterized by violation of the scale symmetry, whereas in the IR regime we have the conformal invariant AdS_{d+2} extremal solution. For the dual QFT this corresponds to an hyperscaling-violating phase in the UV and to a scaling-preserving phase in the IR.

To describe holographic hyperscaling violation in $d+2$ dimensions we consider the parametrization of the scale-covariant metric defined in Sect. 1.2.2:

$$ds^2 = r^{-2(d-\theta)/d}\left(-r^{-2(z-1)}dt^2 + dx_i dx_i + dr^2\right), \tag{5.28}$$

where θ is the hyperscaling violation parameter and z is the dynamic critical exponent.

Comparing Eq. (5.28) with Eq. (5.8) one can easily read off the parameters θ, z for our BB solution:

$$z = 1, \quad \theta = \frac{d(d+1)\beta^2}{(d+1)\beta^2 - 1}. \tag{5.29}$$

As usual the case $\beta^2 > 1$ is covered just by using the duality (5.4). We have:

$$z = 1, \quad \theta = \frac{d(d+1)}{(d+1) - \beta^2}. \tag{5.30}$$

As expected, we have $z = 1, \theta \neq 0$ in the scalar black brane phase, whereas we get $z = 1, \theta = 0$ in the SAdS phase. This gives the deviation from the conformal scaling of the free energy of a $d+1$ conformal field theory.

One can easily check from Eq. (5.29) that $\theta < 0$ for $\beta^2 < \frac{1}{d+1}$ and $\theta > d$ for $\frac{1}{d+1} < \beta^2 < 1$, while θ diverges for $\beta^2 = \frac{1}{d+1}$ (for the dual case (5.30) we have $\theta < 0$ for $\beta^2 > d+1$ and $\theta > d$ for $\beta^2 < d+1$). The null energy conditions for the bulk stress-energy tensor are satisfied: in fact for $z = 1$ these conditions require either $\theta \leq 0$ or $\theta \geq d$ (see Sect. 4.4.3) .

A negative value of θ is not common in condensed matter critical systems, for which θ is positive. However in our solutions the case $\theta < 0$ is physically more interesting (in particular for the possible holographic applications) because in this case we observe a phase transition between the scalar black brane solution and the SAdS solution, and the specific heat of the BB solution is always positive.

5.6 General models

In the previous sections we have investigated the Einstein-scalar gravity model defined by the potential (5.2). However, the main features of our models are dictated not by the full form of the potential but only by the behavior of the potential at $\phi = 0$ and $\phi = -\infty$. We will show that the two main features of the model (hyperscaling violation and the SAdS$\rightarrow$ scalar

BB phase transition) are pertinent to all models satisfying the conditions: 1) $V(\phi)$ has a local minimum for $\phi = 0$ with $V(0) < 0$; 2) The potential approaches zero exponentially as $\phi \to -\infty$. The previous conditions ensure the existence of an AdS_{d+1} vacuum and of a Schwarzschild-AdS (SAdS) black brane solution with $\phi = 0$.

In Chapter 2 we have derived the general BB solution of a model with an exponential potential in $d+2$ dimensions. In particular, for the metric parametrization we are using in this chapter, the asymptotic behavior of the solution for the exponential potential $V = -\lambda^2 e^{2h\phi}$ is given by:

$$\phi = -\frac{dh}{dh^2-2}\,\log r + \frac{1}{2h}\ln C_1, \quad ds^2 = r^{\frac{4}{dh^2-2}}\left(-dt^2 + dx_i dx_i + dr^2\right), \tag{5.31}$$

where $h > 0$ and $C_1 = \{2d[2(d+1) - dh^2]\}/[\lambda^2(dh^2-2)^2]$.

The case $\beta^2 < 1$ described in the previous section for the model (5.1) is covered by setting $h^2 < 2(d+1)/d$, whereas the two cases $\beta^2 < 1/(d+1)$ and $\beta^2 > 1/(d+1)$ correspond, respectively, to $h^2 < 2/d$ and $h^2 > 2/d$.

For a generic model, the existence of a global scalar black brane solution interpolating between the AdS_{d+2} vacuum and the asymptotic scale-covariant solution has to be shown numerically. If we can prove that such a solution exists, the thermodynamical system for $h^2 < 2/d$ must have a scalar black brane $\to$ SAdS phase transition.

The derivation follows closely that used in Sect. 5.4. At small T the free energy of the scalar black brane must have a behavior similar to that of Eq. (5.26), i.e. $F_{SB} = C_2 - C_3 T^{d+1}$, with $C_{2,3}$ positive constants. This implies that at small T, $F_{SB} - F_{SAdS} > 0$. On the other hand, at large T, the free energy scales as $F_{SB} \sim -T^{(2+2d-dh^2)/(2-dh^2)}$. For $h^2 < 2/d$ we have $T^{(2+2d-dh^2)/(2-dh^2)} > T^{d+1}$, from which follows that at large T, $F_{SB} - F_{SAdS} < 0$.

Comparing Eq. (5.31) with Eq. (5.28), one can read off the hyperscaling violation parameter and the dynamic critical exponent:

$$\theta = \frac{d^2 h^2}{dh^2 - 2}, \quad z = 1. \tag{5.32}$$

Notice that θ is negative for $h^2 < 2/d$, whereas $\theta > d$ for $h^2 > 2/d$.

5.7 Holographic properties and two-point functions for scalar operators

Holographic features of theories with hyperscaling violation have been discussed in Ref. [39]. Most of the results derived in [39] for general scale-covariant metrics apply directly to the model discussed in this chapter. Imposing on the gravity side the null energy conditions on the stress-energy tensor constrains the range of the possible values of the parameters z, θ. In our case, being $z = 1$, the conditions become simply $\theta \leq 0$ or $\theta > d$. Taking into account Eq. (5.29) one can easily see that these conditions are always satisfied for every value of β, being $\theta < 0$ for $\{0 < \beta^2 < 1/(d+1)\} \cup \{d+1 < \beta^2 < \infty\}$ and $\theta > d$ for $\{1/(d+1) < \beta^2 < d+1,\ \beta^2 \neq 1\}$.

In Ref. [39] it has been also calculated the short distance form of the two-point function of a scalar operator $\mathcal{O}$ dual to a scalar field with a potential $2m^2\phi^2$. It has been shown that it has a power-law form and for $z = 1$, $0 < \theta < d$ is given by:

$$\langle \mathcal{O}(x)\mathcal{O}(x')\rangle = \frac{1}{|x - x'|^{2(d+1)-\theta}}. \tag{5.33}$$

The problem is that the derivation of [39] does not hold for $\theta < 0$, which is the most interesting case for the models under consideration in this chapter. Moreover, for $\theta > d$, $r \to 0$ corresponds to the IR regime of the dual QFT. This means that for $\theta > d$, Eq. (5.33) gives the large distance behavior of the two-point function instead of the short distance behavior.

Let us now first observe that for $\theta < 0$ Eq. (5.33) gives the IR behavior of the two-point function. This means that for $\theta < 0$ the mass term is irrelevant in the IR and dominates in the UV. Conversely, for $\theta > 0$ we have the opposite behavior: the mass term is irrelevant in the UV and becomes relevant in the IR. It is exactly this feature that allows one to use scaling arguments to determine the form (5.33) for the two-point function.

Obviously, if the theory whose solution is given by the metric (5.28) has an UV (or IR) completion with an UV (or IR) fixed point, the far short (far large) behavior of the two-point function (5.33) will be modified accordingly. This is for instance the case of the models discussed in this chapter, which have an IR fixed point.

We are therefore left with the problem of finding a short distance form for two-point functions of scalar operators in the case $\theta < 0$. A strong hint for tackling the problem can be obtained by looking at the gravitational dynamics that produces solution (5.28). One can easily realize that, at least in the context of Einstein-scalar gravity, what is needed is an exponential potential and a $\ln r$ short distance behavior for the scalar (see Eq. (5.31)). We will therefore look for the UV behavior of two-point functions of a scalar operator $\mathcal{O}$ dual to a scalar field that supports our black brane solution and therefore has near the UV a potential $-\lambda^2 e^{2h\phi}$. The equation of motion for ϕ in the background (5.31) is:

$$\left(\partial_r^2 - \frac{d-\theta}{r}\partial_r + \partial_i^2 - \partial_t^2\right)\phi + \frac{h\lambda^2}{2}e^{2h\phi}r^{-2+\frac{2\theta}{d}} = 0, \tag{5.34}$$

where h has to be expressed as a function of θ using Eq. (5.32). Eq. (5.34) can be solved perturbatively for $\theta < 0$ ($h^2 < 2/d$) by expanding ϕ around the background solution ϕ_0 given by Eq. (5.31): $\phi = \phi_0 + \delta\phi$. Using Eqs. (5.31) and (5.32) one gets, for the perturbation $\delta\phi$, the equation of motion satisfied by a massive scalar field in AdS in $d+2-\theta$ "bulk dimensions":

$$\left(\partial_r^2 - \frac{d-\theta}{r}\partial_r + \partial_i^2 - \partial_t^2\right)\delta\phi - \frac{m^2}{r^2}\delta\phi = 0, \tag{5.35}$$

with $m^2 = -C_1 h^2 \lambda^2 = [2\theta(d+1-\theta)]/d$. Eq. (5.35) can be solved with the usual power-law ansatz $\delta\phi \propto r^\alpha(1+\mathcal{O}(r^2))$, with α given by the standard AdS formula in $d+1-\theta$ dimensions:

$$\alpha_{12} = \frac{1}{2}\left(d+1-\theta \pm \sqrt{(d+1-\theta)^2 + 4m^2}\right) = \frac{1}{2}(d+1-\theta)\left(1 \pm \sqrt{1+\frac{8\theta}{d(d+1-\theta)}}\right). \tag{5.36}$$

The two solutions for α, corresponding to a faster and slower falloff mode of the scalar for $r \to 0$, always exist for $d \geq 8$, whereas for $d < 8$ we must require $\theta \geq -d(d+1)/(8-d)$.

The general solution to Eq. (5.35) is given by a superposition of the slower and faster falloff modes:

$$\delta\phi = a(kr)^{\alpha_2}(1+\mathcal{O}(r^2)) + b(kr)^{\alpha_1}(1+\mathcal{O}(r^2)), \tag{5.37}$$

where a, b are $\mathcal{O}(1)$ constants determined by the boundary conditions, we have taken the (t, x_i)-Fourier transform and $k^2 = -k_0^2 + k_i k_i$. The retarded Green's function $G^R(k)$ for the scalar operator dual to the bulk scalar field is given by the ratio of the coefficients of the r^{α_1} and r^{α_2} terms in Eq. (5.37) (see for instance [122]):

$$G^R(k) \sim k^{\alpha_1-\alpha_2}\ , \tag{5.38}$$

where α_{12} are given by Eq. (5.36). Taking the Fourier transform, in the coordinate space we get the power-law form for the two-point function for the scalar operator dual to a bulk scalar field with exponential potential:

$$\langle \mathcal{O}(x)\mathcal{O}(x')\rangle = \frac{1}{|x-x'|^{d+1+\alpha_1-\alpha_2}}. \tag{5.39}$$

It is also of interest to compute the two-point function (5.39) for small negative values of θ:

$$\langle \mathcal{O}(x)\mathcal{O}(x')\rangle = \frac{1}{|x-x'|^{2(d+1)-(d-4)\theta/d}}. \tag{5.40}$$

5.8 Conclusions

In this chapter we have analyzed the thermodynamics and the scaling symmetries of BB solutions of Einstein-scalar gravity in arbitrary dimensions for models with positive scalar squared mass and a potential that has an exponential asymptotic behavior. We have generalized the results of Chapter 3 and Ref. [85], which hold for two-dimensional scalar branes, to branes of arbitrary spacetime dimensions.

We have been mainly concerned with an integrable model, which also arises as compactification of black p-brane solutions of SUGRA theories. However, the relevant features of this model can easily be extended to a broad class of Einstein-scalar gravity models.

The striking features of these d-dimensional scalar BB solutions are an unexpected phase diagram and non-trivial behavior in the ultraviolet regime of the holographically dual QFT, which is characterized by hyperscaling violation. This generates an UV length scale which decouples in the IR, where conformal invariance is restored. At high temperatures, when $\beta^2 < 1/(d+1)$ or $\beta^2 > d+1$, the scalar-dressed BB solution, with scale-covariant asymptotical behavior, becomes energetically preferred.

The hyperscaling-violating phase is characterized by the two parameters normally used for critical systems with hyperscaling violation, namely the dynamical critical exponent z and the hyperscaling violation parameter θ.

The most important peculiarity of our models is that for scalar black branes that are stable at high temperatures, the hyperscaling parameter θ is always *negative*. In QFTs with hyperscaling violation the scaling law for the free energy is that pertinent to a CFT in $d - \theta$ dimensions. For positive θ we have therefore a lowering of the effective dimensions. This is an important feature of the small temperature behavior of traditional hyperscaling-violating critical systems [41]. On the other hand, the scalar BB brane solutions investigated by us are characterized by a negative hyperscaling violation parameter θ, producing a raising of the "effective dimensions".

It is important to notice in this context that the most general compactification of p-brane solutions of SUGRA theories produces hyperscaling violation in the dual QFT with both $\theta < 0$ or $\theta > d$. Both cases are consistent with the null energy condition for the bulk stress energy tensor, but for $\theta > d$ the SAdS phase is always energetically preferred (see Sect. 5.4). On the other hand the simplest diagonal ansatz (5.15) for the D-dimensional metric leads to BB solutions with $\theta > d$.

We have also determined, for the case of negative θ, the short distance behavior of two-point functions for scalar operators of the QFT dual to a bulk scalar field with an exponential potential. We have shown that it has a power-law behavior. Our calculation completes the derivation of Ref. [39]. In that paper the short distance, power-law, form of the two-point functions for scalar operators dual to a scalar field with a mass term potential was determined only for positive θ.

A puzzling point which still remains to be clarified is the holographic interpretation of the phase transition between the two bulk phases - the SAdS and the scalar brane phase. The cross-over of the free energies for SAdS and scalar branes observed in Sect. 5.4 seems to have a very different interpretation than a conventional phase transition in the gravity/gauge theory correspondence, such as for instance the Hawking-Page phase transition [123].

Usually, in the gravity/gauge theory correspondence, we fix the boundary conditions for the fields and consider two distinct extensions into the bulk.

The corresponding dual solutions contribute to the same canonical ensemble of the QFT. In the large N limit the solution with lower free energy is energetically preferred. On the other hand the two competing phases of the QFT holographically dual to the SAdS-scalar brane phases seem to correspond to different boundary QFTs. Therefore they do not contribute to the same canonical ensemble.

This is obviously related to the unusual feature that our scalar black brane solutions exhibit hyperscaling violation in the UV and conformal symmetry in the IR. In the conventional setting where the solution has a UV fixed point and an emergent nonzero θ in the IR, the holographic interpretation of the phase transition is not problematic. In this latter case the SAdS and the hyperscaling-violating phase contribute to the same canonical ensemble.

Chapter 6

Infrared behavior in effective holographic theories

In this chapter we investigate the infrared behavior of the spectrum of scalar-dressed, asymptotically anti-de Sitter (AdS) black brane (BB) solutions of effective holographic models. These solutions describe scalar condensates in the dual field theories. We show that for zero charge density the ground state of these BBs must be degenerate with the AdS vacuum, must satisfy conformal boundary conditions for the scalar field and it is isolated from the continuous part of the spectrum. When a finite charge density is switched on, the ground state is not anymore isolated and the degeneracy is removed. Depending on the coupling functions, the new ground state may possibly be energetically preferred with respect to the extremal Reissner-Nordström AdS BB. We derive several properties of BBs near extremality and at finite temperature. As a check and illustration of our results we derive and discuss several analytic and numerical BB solutions of Einstein-scalar-Maxwell AdS gravity with different coupling functions and different potentials. We also discuss how our results can be used for understanding holographic quantum critical points, in particular their stability and the associated quantum phase transitions leading to superconductivity or hyperscaling violation.

6.1 Introduction

We have seen in the previous chapters that holographic models have been widely used as a powerful tool to describe the strongly coupled regime of a quantum field theory (QFT). In particular, these effective holographic theories (EHTs) can be very useful to give a holographic description of many interesting quantum phase transitions, such as those leading to critical superconductivity and hyperscaling violation.

Most of the EHTs with hyperscaling violation in the infrared (IR) investigated in the literature are low-energy effective models in which the ultraviolet (UV) behavior is not specified. An UV completion of these models is not strictly necessary. In fact all the thermodynamical parameters and properties are well defined also for models with hyperscaling violation in the UV (see the previous chapter). However, there are several reasons to consider models that have hyperscaling violation in the IR but flow to an AdS spacetime in the UV. First of all many EHTs are low-energy approximations of string theory and the AdS/CFT correspondence is the cornerstone of all holographic applications. Moreover, the AdS background preserves the Poincaré symmetries of the dual QFT and models with near-horizon hyperscaling-violating geometries and an asymptotic AdS spacetime will describe a flow between an UV conformal fixed point and hyperscaling violation in the IR. Last but not least the existence of an AdS solution will allow for interesting, highly nontrivial phase transitions.

Thus, if we assume that the asymptotic geometry is the AdS spacetime, the dual QFT shows a universal conformal fixed point in UV. The nontrivial dynamics therefore occurs in the IR region, at the corresponding critical points. In a Wilsonian approach, EHTs should be first classified in terms of flows, driven by relevant operators, between critical points corresponding to scale-invariant (more generally scale-covariant) QFTs. Two other relevant characterizations of the critical points are: a) the distinction between *fractionalized* phases (sourced by non-zero electric flux in the IR) and *cohesive* phases (sourced by zero electric flux in the IR); *b*) phases with broken and unbroken $U(1)$ symmetry [81, 124].

Progress in the classification and understanding of IR critical points have been achieved following various directions. In particular, it has been shown that in the case of hyperscaling-preserving and hyperscaling-violating solutions, quantum critical theories may appear as fixed lines rather than fixed points [81]. Hyperscaling-preserving solutions appear indeed as fixed points and correspond to AdS_4, $\mathrm{AdS}_2 \times R^2$ and Lifshitz bulk geometries. However, hyperscaling-violating solutions are characterized by an explicit scale and therefore appear rather as critical lines generated by changing that scale or equivalently the charge density [39, 81].

A crucial point for understanding these quantum critical points is the presence of a scalar condensate. Indeed nontrivial configurations of (generically charged or uncharged) scalar fields play several crucial roles: (i) nontrivial scalar fields are dual to relevant operators that drive the renormalization group (RG) flow from the UV fixed point to the IR critical point (or line); (ii) scalar fields are the sources that support the IR hyperscaling-violating geometry allowing for both fractionalized and cohesive phases [81, 124]; (iii) charged scalar condensates break the $U(1)$ symmetry and generate a superconducting phase [81, 124].

Despite the recognized relevance of scalar condensates for describing holographic critical points, we are far from having a complete understanding of the physics behind them, in particular we have very few information about their stability. For instance, one would like to understand why at zero (and small) temperature the hyperscaling-violating phase is energetically preferred with respect to the hyperscaling-preserving phase. In this chapter we will move a step forward in this direction by asking ourselves a simple, but relevant question: what is the energy of the ground state of a neutral asymptotically AdS BB sourced by a generic nontrivial scalar field? We show that for zero charge density the BB ground state must be degenerate with the AdS vacuum. This degeneracy is the result of an exact cancellation between a positive gravitational contribution to the energy and a negative contribution due to the scalar condensate.

Moreover, we also show that for the BB ground state the symmetries of the field equations force conformal boundary conditions for the scalar field, i.e. boundary conditions preserving the asymptotic symmetry group of the AdS spacetime. The conformal boundary conditions correspond to dual multitrace scalar operators driving the dynamics from the UV conformal fixed point to the IR critical point. In the case of an IR hyperscaling-violating geometry sourced by a pure scalar field with a potential behaving exponentially, a scale is generated in the IR. On the other hand we will show that, in the case of pure Einstein-scalar gravity at finite temperature T, the boundary conditions for the scalar are determined by the dynamics and are, therefore, generically nonconformal. This means that the ground state for scalar BBs is *isolated*, i.e. it cannot be obtained as the $T \to 0$ limit of finite-T BBs with conformal boundary conditions for the scalar field.

When a finite charge density ρ is switched on, the degeneracy of the ground state is removed. Because an additional degree of freedom (the EM potential) is present, the boundary conditions for the scalar field are not anymore determined by the dynamics. The freedom in choosing the boundary conditions arbitrarily can be used to impose conformal boundary conditions also for BBs at finite temperature. The ground state for scalar BBs is therefore not anymore isolated from the continuous part of the spectrum. The coupling between the bulk scalar and the EM field determines if it is energetically preferred with respect to the extremal Reissner-Nordström (RN) AdS BB.

We also derive several properties of the scalar BBs near extremality and at finite temperature. For instance, we show that scalar-dressed, neutral (charged), BB solutions of radius r_h (and charge density ρ) only exist for a temperature T bigger than the temperature of the Schwarzschild-AdS

(Reissner-Nordström AdS) BB with the same r_h (and with the same ρ).

As a check and illustration of our results we give and discuss – both analytically and numerically – several (un)charged, scalar-dressed BB solutions of Einstein-scalar-Maxwell AdS (ESM-AdS) gravity with minimal, nonminimal and covariant coupling functions and different potentials (quadratic, quartic, exponential).

Finally, we also discuss the relevance of our results for understanding holographic quantum critical points, in particular their stability and the associated quantum phase transitions.

The structure of the chapter is the following. In Sect. 6.2 we present the general form of the EHTs we consider. In Sect. 6.3 we investigate the spectrum of this class of theories in the IR region. In Sect. 6.4 we derive extremal, near-extremal and finite-temperature BB solutions of pure Einstein-scalar gravity theories in the case of a quadratic, quartic and exponential potential. We also derive their thermodynamical behavior and their critical exponents. In Sect. 6.5 we derive and discuss charged solutions with the scalar minimally, nonminimally and covariantly coupled to the EM field. Finally in Sect. 6.6 we end the chapter with some concluding remarks about the relevance that our results have for understanding the dual QFT, holographic quantum critical points, and in particular the stability of the latter and the associated quantum phase transitions leading to superconductivity or hyperscaling violation. In Appendix 6.7 we discuss perturbative solutions in the small scalar field limit.

6.2 Effective holographic theories

We consider Einstein gravity coupled to a real scalar field and to an EM field in four dimensions:

$$I = \int d^4x\sqrt{-g}\left[R - \frac{1}{2}\left(\partial\phi\right)^2 - \frac{Z(\phi)}{4}F^2 - V(\phi) - Y(\phi)A^2\right], \tag{6.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell field-strength. The model is parametrized by the gauge coupling function $Z(\phi)$, by the self-interaction potential $V(\phi)$ for the scalar field and by the coupling function $Y(\phi)$ giving mass to the Maxwell field.

The action (6.1) defines Einstein-scalar Maxwell (ESM) theories of gravity, which are also called Effective Holographic Theories (EHTs) because are relevant for holographic applications. Moreover, models like (6.1) generically appear, after dimensional reduction, as low-energy effective string theories. The action (6.1) can be also interpreted as an EHT for a complex scalar field that enjoys a $U(1)$ symmetry [81]. In this context the real scalar ϕ describes the modulus of the charged scalar and the phase with broken (unbroken) $U(1)$ symmetry is obtained by $Y \neq 0$ ($Y = 0$).

Although our considerations can be easily extended to the case $Y \neq 0$, we will focus for simplicity on the case of unbroken $U(1)$ symmetry, $Y = 0$. We will briefly comment on the case $Y \neq 0$ in Sect. 6.5.3.

We are interested in electrically charged BB solutions of the theory. Using the following parametrization for the metric:

$$ds^2 = -\lambda(r)dt^2 + \frac{dr^2}{\lambda(r)} + H^2(r)(dx^2 + dy^2), \tag{6.2}$$

the Einstein and scalar equations read:

$$\frac{H''}{H} = -\frac{(\phi')^2}{4}, \quad (\lambda H^2)'' = -2H^2 V, \tag{6.3}$$

$$(\lambda H H')' = -H^2 \left[\frac{V}{2} + \frac{Z A_0'^2}{4}\right], \tag{6.4}$$

$$(\lambda H^2 \phi')' = H^2 \left(\frac{dV}{d\phi} - \frac{A_0'^2}{2}\frac{dZ}{d\phi}\right). \tag{6.5}$$

The ansatz (6.2) is very convenient, as in these coordinates Maxwell's equations can be directly solved for A_0':

$$A_0' = \frac{\rho}{Z H^2}, \tag{6.6}$$

where ρ is the charge density of the solution. Note that only Eqs. (6.4) and (6.5) depend on the EM field and only through A_0'. Therefore, substituting the solution above into the remaining field equations, we can completely eliminate the EM field and solve Eqs. (6.3)–(6.5) for λ, H and ϕ.

As we are looking for asymptotically AdS solutions, we will consider models for which the potential $V(\phi)$ has a maximum at $\phi = 0$ and $Z'(\phi = 0) = 0$, with the local mass of the scalar $m_s^2 = V''(0)$ satisfying the condition $m_{BF}^2 < m_s^2 \leq -2/L^2$ and with $V(0) = -6/L^2$, where $m_{BF}^2 = -9/(4L^2)$ is the BF bound in four dimensions[1]. The presence of an extremum of $V(\phi)$ and $Z(\phi)$ at $\phi = 0$ implies the existence of a Reissner-Nordström-AdS (RN-AdS) BB solution:

$$\lambda = \frac{r^2}{L^2} - \frac{M}{2r} + \frac{\rho^2}{4r^2}, \quad H = r, \quad \phi = 0, \tag{6.7}$$

which is characterized by a trivial scalar field configuration. Each other solution of the field equations will be characterized by a non-trivial profile for the scalar field.

[1]The results of this chapter can be easily extended to the scalar-mass range $m_{BF}^2 < m_s^2 < m_{BF}^2 + 1/L^2$, where the dual CFT is known to be unitary.

The AdS, $r = \infty$, asymptotic behavior requires the following leading behavior of the metric and the scalar field (see Sect. 1.1):

$$
\begin{aligned}
ds^2 &= -\frac{r^2}{L^2}dt^2 + \frac{L^2}{r^2}dr^2 + r^2(dx^2 + dy^2) \\
\phi &= \frac{O_1}{r^{\Delta_1}} + \frac{O_2}{r^{\Delta_2}},
\end{aligned}
\tag{6.8}
$$

with $\Delta_{1,2} = \frac{3 \mp \sqrt{9 + 4m_s^2 L^2}}{2}$. As we know from Sect. 1.1, boundary conditions that preserve the asymptotic isometries of the AdS spacetime can be $O_1 = 0$, $O_2 = 0$ or $O_1 = f O_2^{\Delta_1/\Delta_2}$ (the last in the range of scalar masses $m_{BF}^2 < m_s^2 < m_{BF}^2 + 1/L^2$). More in general, boundary conditions of the form:

$$
O_1 = W(O_2) \tag{6.9}
$$

can be used. For a generic form of the function W the asymptotic AdS isometries are broken, yet an asymptotic time-like Killing vector exists and the gravitational theory admits a dual description in terms of multitrace deformations of CFTs [52, 53, 56, 125–128].

Apart from their UV AdS behavior, the scalar-dressed solutions of EHTs are also characterized by their, small r, IR behavior. This IR behavior is of crucial relevance for holographic applications, in particular in the context of the AdS/condensed matter correspondence [22]. Generically, we expect the IR regime not to be universal, but rather determined by the infrared behavior of the potential $V(\phi)$ and of the gauge coupling functions $Z(\phi)$, $Y(\phi)$. Nevertheless, we will discover in the next sections some features of the IR spectrum of EHTs, which are model-independent and related to the scaling symmetries of the UV AdS vacuum.

Although we will be concerned with general features of EHTs, for the sake of definiteness we will mainly focus on three classes of models with different IR behavior of the potential $V(\phi)$:

a) The potential has a quadratic form:

$$V(\phi) = -\frac{6}{L^2} + \frac{m_s^2\phi^2}{2}. \tag{6.10}$$

This corresponds to the simplest choice for the potential, which has been widely used in holographic models. The IR regime is dominated by the quadratic term and at $T = 0$ the scalar field diverges logarithmically in the $r = 0$, near-horizon region.

b) The potential behaves exponentially for small values of the radial coordinate r. Assuming that $r = 0$ corresponds to $\phi \to \infty$, we have in this case:

$$V(\phi) \sim e^{b\phi}, \tag{6.11}$$

where b is a positive constant. As we shall discuss later in Sect. 6.4.3, this case produces a scale-covariant solution in the IR, corresponding to hyperscaling violation in the dual QFT.

c) The origin $r = 0$ corresponds to an other extrema (a minimum) at $\phi = \phi_1$ of the potential $V(\phi)$. In this case the theory flows to a second AdS_4 vacuum in the infrared.

The IR regime of the EHT (6.1) is also characterized by the IR behavior of the gauge coupling function Z. In particular, Z is crucial for determining the contribution of bulk degrees of freedom inside or outside the event horizon to the boundary charge density. This distinction is captured by the behavior of the electric flux in the IR:

$$\Phi = \left(\int_{R^2} Z(\phi)\tilde{F} \right)_{\mathrm{IR}}, \tag{6.12}$$

where $\tilde{F}$ is the dual Maxwell tensor. Using a terminology borrowed from condensed matter physics, the phase with $\Phi = 0$ has been called *cohesive* and describes dual confined gauge invariant operators. The phase $\Phi \neq 0$ has been named *fractionalized* and describes a dual deconfined phase [81, 124]. In this chapter we will consider two choices for the gauge coupling function $Z(\phi)$: (1) a minimal coupling, $Z(\phi) = 1$; (2) a coupling which behaves exponentially in the IR, $Z \sim e^{a\phi}$.

Since in the following we shall make often use of the thermodynamical properties of the BB solutions, we find it convenient to summarize them here. The temperature T, entropy S and free energy F of the solutions (6.2) are given by:

$$T = \frac{\lambda'(r_h)}{4\pi}, \qquad S = 4\pi\mathcal{V}H^2(r_h), \qquad F = M - TS, \tag{6.13}$$

where M is the *total* mass of the solution, $\mathcal{V}$ is the volume of the $2D$ sections of the spacetime and r_h is the location of the outer event horizon.

6.3 Spectrum of Einstein-scalar-Maxwell AdS gravity in the Infrared region

In this section we investigate general features of the mass spectrum of ESM-AdS gravity in the IR region. Assuming the existence of scalar-dressed BBs with AdS asymptotic behavior, the two basic questions in this context are about the existence and features of the $T = 0$ extremal state and of the states near-extremality. We will treat separately the EM charged and uncharged cases. We will first consider the theory with zero charge density ($Z = Y = 0$ in the action (6.1)), i.e. a vanishing Maxwell field (Einstein-scalar AdS gravity). Later, we will extend our considerations to the case of finite charge density.

6.3.1 Einstein-scalar AdS gravity

A nontrivial point is the determination of the total mass M (i.e. the energy) of the BB solutions. As discussed in Ref. [56], the usual definition of energy in AdS diverges whenever $O_1 \neq 0$ (with a divergent term proportional to r). This is because the backreaction of the scalar field causes certain metric components to fall off slower than usual. However, this divergent term is exactly canceled out if one considers that for $O_1 \neq 0$ there is an additional scalar contribution to the surface terms which determine the mass.

Using the Euclidean action formalism, in the case $m_s^2 = -2/L^2$ the total mass turns out to be [56]:

$$M = M_G + \frac{\mathcal{V}}{L^4}\left[O_1 O_2 + P(O_1)\right], \tag{6.14}$$

where M_G is the gravitational contribution to the mass, we have chosen the following boundary conditions for the scalar: $O_2 = O_2(O_1)$, and $P(O_1) = \int O_2(O_1) dO_1$.

In the following we will need an expression for the mass when m_s is in the range of values considered in this chapter, $-9/4 < m_s^2 L^2 \leq -2$. Furthermore, working with the parametrization of the metric given by Eq. (6.2), it is useful to express the total mass M in terms of the coefficient of the $1/r$ term in the $r = \infty$ expansion of the metric functions. To derive such an expression, as in the previous chapter, we use the Euclidean action formalism of Martinez et al. [71]. Using the parametrization of the metric (6.2), the gravitational and scalar part of the variation of the boundary terms are given respectively by [71]:

$$\begin{aligned} \delta I_G &= \frac{2\mathcal{V}}{T}\left[(HH'\delta\lambda - \lambda' H\delta H) + 2\lambda H(\delta H')\right]\Big|_{r_h}^{\infty}, \\ \delta I_\phi &= \frac{\mathcal{V}}{T} H^2 \lambda \phi' \delta\phi\Big|_{r_h}^{\infty}. \end{aligned} \tag{6.15}$$

From the definition of the free energy $F = M - TS$, taking into account that

$I_\phi|_{r_h} = 0$, $S = I_G|_{r_h}$ and from $F = -IT$, it follows (see also Sect. 5.4):

$$M = TS - TI = -T(I_G^\infty + I_\phi^\infty). \tag{6.16}$$

To calculate the mass M (6.16) we need the subleading terms in the $r = \infty$ expansion of the metric (6.8). By means of a translation of the radial coordinate r, the asymptotic expansion of the solution can be put in the general form:

$$\begin{aligned} \lambda &= \frac{r^2}{L^2} + pr^\beta - \frac{m_0}{2r} + \mathcal{O}(r^{\beta-1}), \\ H^2 &= \frac{r^2}{L^2} + qr^\alpha + \frac{s}{r} + \mathcal{O}(r^{\alpha-1}), \\ \phi &= \frac{O_1}{r^{\Delta_1}} + \frac{O_2}{r^{\Delta_2}} + \mathcal{O}(r^{-\Delta_1-1}), \end{aligned} \tag{6.17}$$

where p, q, α, β, s are constants. Inserting this expansion in the field equations one gets (at the first and second subleading order) the following relations between the constants:

$$\beta = \alpha = 2(1-\Delta_1), \quad p = q = \frac{\Delta_1 O_1^2}{4L^2(1-2\Delta_1)}, \quad s = -\frac{\Delta_1\Delta_2 O_1 O_2}{6L^2}. \tag{6.18}$$

Substituting Eq. (6.17) into (6.15) and using $p = q$, we obtain:

$$\delta I_G^\infty = -\frac{\mathcal{V}}{TL^2}\left(\delta m_0 + 6\delta s - 2\delta p(\beta-1)r^{\beta+1}\right), \tag{6.19}$$

$$\delta I_\phi^\infty = -\frac{\mathcal{V}}{TL^4}\left(\Delta_1 O_1 \delta O_1 r^{\beta+1} + \Delta_1 O_1 \delta O_2 + \Delta_2 O_2 \delta O_1\right). \tag{6.20}$$

Notice that both the gravitational and the scalar contribution to the mass contain a term which diverges as $r^{\beta+1}$. Using Eq. (6.18) one easily finds that the two divergent terms cancel out in $\delta I^\infty = \delta I_G^\infty + \delta I_\phi^\infty$. Finally, we obtain the total mass of the solution:

$$M = -TI^{\infty} = \frac{\mathcal{V}}{L^2}\left(m_0 + \frac{(\Delta_1 - \Delta_2)}{L^2}\int dO_2 W(O_2) + \frac{\Delta_2(1-\Delta_1)}{L^2}O_2 O_1\right), \tag{6.21}$$

where we have parametrized the boundary conditions for the scalar in terms of the function $O_1 = W(O_2)$. It is also useful to split the total mass into the gravitational and scalar contributions M_G and M_ϕ, arising separately from the two terms in Eq. (6.21):

$$M_G = \frac{\mathcal{V}}{L^2}\left(m_0 - \frac{\Delta_1\Delta_2}{L^2}O_1 O_2\right), \quad M_\phi = \frac{\mathcal{V}}{L^4}\left[\Delta_1 O_1 O_2 + (\Delta_2 - \Delta_1)P(O_1)\right], \tag{6.22}$$

where $P(O_1)$ is defined as in Eq. (6.14). One can easily check that the previous equations reproduce correctly Eq. (6.14) in the case $m_s^2 = -2/L^2$, i.e. $\Delta_1 = 1, \Delta_2 = 2$.

Let us now investigate general features of the mass spectrum of ES-AdS gravity in the IR region. In particular, assuming the existence of scalar-dressed BBs with AdS asymptotic behavior, we wish to characterize the features of the $T = 0$ extremal state and of the near-extremal states.

In the uncharged case a general, albeit implicit, form of the solution for the metric function λ in a generic ES-AdS gravity theory has been derived in Chapter 2 (see in particular Sect. 2.4):

$$\lambda = H^2 - C_1 H^2 \int \frac{dr}{H^4}, \tag{6.23}$$

where C_1 is an integration constant. The equation above implies that if an extremal $T = 0$ hairy BB solution exists, this must have $C_1 = 0$, i.e. $\lambda = H^2$. We can prove this statement by the following argument. Differentiating the equation above and using Eqs. (6.13), we find the following relation between the temperature and the entropy density $\mathcal{S}$ of the solution:

$$T = \frac{\lambda'(r_h)}{4\pi} = \frac{(2\lambda H H' - C_1)_{r_h}}{\mathcal{S}}. \tag{6.24}$$

Therefore we get $C_1 = [2\lambda H H']_{r_h} - \mathcal{S}T$. An extremal solution satisfies $T = 0$ and $\lambda(r_h) = 0$. Assuming that H and H' are finite at the horizon (to avoid curvature singularities), the existence of an extremal solution imposes $C_1 = 0$, i.e.:

$$\lambda = H^2. \tag{6.25}$$

Note that this argument applies both when the entropy of the extremal solution is vanishing or when it is finite.

Obviously, an extremal uncharged solution with AdS asymptotics (besides the trivial AdS vacuum) may not exist. For the moment, we assume such a solution exists and derive a general and very important result. We shall prove that if such a solution exists it must have zero energy, i.e. *must be degenerate with the AdS vacuum*.

To prove this statement, let us first show that every scalar-dressed solution with AdS asymptotics, which is characterized by $\lambda = H^2$, requires necessarily conformal boundary conditions:

$$O_1 = f O_2^{\Delta_1/\Delta_2} \tag{6.26}$$

for the scalar field. The field equations (6.3)–(6.6) with $\rho = 0$ and the metric (6.2) are invariant under the scale transformation $r \to \mu r$, $\lambda \to \mu^2 \lambda$, $t \to \mu^{-1} t$. In the extremal case, the asymptotic expansion (6.17) implies that the full solution ($\lambda = H^2$ and ϕ) is invariant under this scale transformation if $O_{1,2}$ scale as follows: $O_1 \to \mu^{\Delta_1} O_1$, $O_2 \to \mu^{\Delta_2} O_2$, which in turn implies the conformal boundary condition (6.26).

We can now calculate the mass (6.21) of the extremal solution, which has

$\lambda = H^2$, hence $m_0 = -2s$. We get:

$$M = \frac{\mathcal{V}}{L^4}\left[(\Delta_2 - \Delta_1)P(O_1) + \Delta_1(1 - \frac{2}{3}\Delta_2)O_2 O_1\right], \tag{6.27}$$

where $P(O_1)$ is defined as in Eq. (6.14). For the conformal boundary conditions (6.26) we have $O_2 = \hat{f} O_1^{\frac{\Delta_2}{\Delta_1}}$ and $P(O_1) = \frac{\Delta_1}{3}\hat{f} O_1^{\frac{3}{\Delta_1}}$, where $\hat{f}$ is a constant and we have used the equation $\Delta_1 + \Delta_2 = 3$. Substituting the previous equations into Eq. (6.27) it follows immediately that for conformal boundary conditions (6.26) the mass M vanishes. This is an important and extremely nontrivial result. It means that in ES-AdS gravity with no EM field, if an extremal scalar-dressed BB solution exists, the AdS_4 vacuum of the theory must necessarily be degenerate. Physically, this degeneration is a consequence of the fact that the scalar condensate gives a negative contribution to the energy. Therefore we may have configurations in which the positive gravitational energy is exactly canceled by the negative energy of the scalar condensate. This cancellation is a consequence of the conformal symmetry of the extremal solution; it necessarily occurs because the extremality condition $\lambda = H^2$ forces the conformal boundary conditions (6.26).

It is also important to notice that the argument leading to the degeneracy of the $T = 0$ ground state holds true also when a condition much weaker than Eq. (6.25) is satisfied:

$$\lambda = H^2 + \mathcal{O}(r^{-2}). \tag{6.28}$$

In fact the mass (6.21) and the scaling arguments leading to the conformal boundary conditions for the scalar field depend only on terms up to $\mathcal{O}(r^{-1})$ and are completely insensitive to higher order terms in $1/r$.

Let us now consider near-extremal solutions. We assume that the theory allows for scalar-dressed BBs at finite temperature with AdS asymptotics. In the next section, we shall prove the existence of finite temperature solutions, by constructing AdS-BBs, numerically, for three classes of ES-AdS gravity models.

The BB spectrum near-extremality can be investigated by considering the $T \to 0$ limit of the finite T solutions. However, one can show that this $T \to 0$ limit is singular. In order to prove the statement we expand the fields in the near-horizon region:

$$\lambda = \sum_{n=1}^{\infty} a_n (r - r_h)^n, \quad H = \sum_{n=0}^{\infty} b_n (r - r_h)^n, \quad \phi = \sum_{n=0}^{\infty} c_n (r - r_h)^n . \tag{6.29}$$

At first order we get for $b_0 \neq 0$:

$$b_2 = -\frac{b_0}{4} c_1^2, \quad b_1 a_1 = -\frac{b_0}{2} V(c_0), \quad a_1 c_1 = \left(\frac{dV}{d\phi}\right)_{c_0}, \tag{6.30}$$

whereas the temperature of the dressed solutions, from Eqs. (6.13), becomes:

$$T = \frac{a_1}{4\pi} = -\frac{b_0 V(c_0)}{8\pi b_1} . \tag{6.31}$$

Because in the case under consideration (V has a maximum) the potential V is limited from above ($V(\phi) \leq -6/L^2$), the $T \to 0$ limit can only reached by letting $b_0 \to 0$. But on the other hand from the third equation in (6.30) it follows immediately that $a_1 = 0$ is a singular point of the perturbative expansion (6.29) unless $(dV/d\phi)_{c_0} = 0$ (corresponding to the Schwarzschild-anti-de Sitter (SAdS) BB). Thus the $T \to 0$ limit is a singular point of the perturbation theory. It should be stressed that this result has been derived by first considering the near-horizon limit, then taking $T \to 0$. In Sect. 6.4.4 we will see what happens if the two limits are taken in the reversed order.

Note that the above results are strictly true only if one considers AdS solutions with negative squared mass for the scalar field. If the scalar potential has a local minimum at $\phi = 0$, then our argument above does not apply. This is for instance the case with the class of models studied in Chapters 3 and 5 which, however, turn out not to have BB solutions with AdS asymptotics.

The singularity of the $T \to 0$ limit in the near-horizon perturbation theory indicates that the ground state (6.25) is isolated, i.e. it cannot be reached as the $T \to 0$ limit of finite-T scalar BB solutions. This conclusion can be also inferred by reasoning on the $r = \infty$ boundary conditions for the scalar field. We have previously shown that the symmetries of the field equations together with Eq. (6.25) force conformal boundary conditions (6.26) for the scalar field. On the other hand, one can easily show that in the case of T finite, the field equations together with the conditions for the existence on an event horizon imply boundary conditions of the form (6.9), hence in general nonconformal boundary conditions. In fact, the field equations (6.3)–(6.6) have the following symmetries:

$$\begin{aligned}
&r \to kr, \quad t \to kt, \quad L \to kL, \quad H \to kH, \\
&r \to kr, \quad \lambda \to k^2\lambda, \quad t \to k^{-1}t, \quad A_0 \to kA_0, \\
&H \to kH, \quad (x, y) \to k^{-1}(x, y), \\
&\lambda \to k\lambda, \quad t \to k^{-1}t, \quad H^2 \to k^{-1}H^2, \quad L \to kL\,, \quad A_0^2 \to k^{-1}A_0^2.
\end{aligned} \tag{6.32}$$

These symmetries can be used to fix all but one parameter in the perturbative expansion (6.29). The solutions become in this way a one-parameter family of solutions. The near-horizon expansion (6.29) depends on a single free parameter, which can be chosen to be r_h. For each value of r_h, we can extract the two functions $O_1(r_h)$ and $O_2(r_h)$, which define implicitly the boundary condition $O_1 = W(O_2)$.

It follows that in general the finite-T solutions require boundary conditions for the scalar, which are different from the conformal ones required for the ground state (6.25). Therefore, the solution (6.25) is generically isolated, i.e. it cannot be reached as the $T \to 0$ limit of finite-T scalar BB solutions.

It should be stressed that the above feature is a key general feature of the BB solutions of AdS Einstein-scalar gravity which holds true also for the numerical solutions discussed in the next sections. If one assumes an analytic expansion close to the horizon, an asymptotically AdS behavior at infinity and if one requires the existence of hairy BB solutions, then the boundary conditions at infinity cannot be arbitrarily imposed but are determined by the field equations. These boundary conditions will have the form (6.9), with the function W determined by the dynamics. In the dual QFT the function W characterizes the scalar condensate. Thus, the particular form of the condensate is determined by the gravitational dynamics.

It is obvious that this is true only in the case of pure Einstein-scalar gravity. For instance it does not hold for electrically charged solutions[2]. In this latter case the near-horizon solution has always more than one free parameter, that can be fixed by prescribing some boundary conditions for the scalar field.

We can also compare the temperature of the dressed solution of radius r_h with the temperature T_0 of the SAdS BB with the same radius. We can use Eqs. (6.32) to set $r_h = L,\ b_1 = c_1 = L^{-1}, b_0 = 1$, so that the only free parameter is $c_0 = \phi(r_h)$ and the temperature becomes $-8\pi T = LV(c_0)$. We have therefore:

$$T - T_0 = 8\pi L(V(0) - V(c_0)\,). \tag{6.33}$$

In the case under consideration, $V(\phi)$ has a local maximum at $\phi = 0$, so that $V(0) \geq V(c_0)$. Therefore, we obtain $T > T_0$ for *any* finite temperature solution. That is, there exists a critical temperature given by the temperature of the SAdS BB: $T_0 = \frac{3r_h}{4\pi L^2}$ such that scalar-dressed solutions of the same radius r_h only exist when $T > T_0$.

6.3.2 Einstein-scalar-Maxwell AdS gravity

Let us now consider the EM charged case, i.e. a finite charge density in the dual QFT. In general, one expects that the finite charge density will remove

[2] It does not hold also for black hole solutions of ES-AdS gravity, i.e. for solutions which spherical horizons. This is because in this case the field equations are not anymore invariant under the full set of transformations (6.32).

the degeneracy of the $T = 0$ extremal state we have found in the uncharged case. This can be shown explicitly. Indeed, when $\rho \neq 0$, the field equations imply:

$$\frac{\rho^2}{ZH^2} + 2\lambda H'^2 + 2\lambda H H'' = H^2 \lambda'' , \tag{6.34}$$

which is solved by Eq. (6.23) only when the charge is vanishing. In particular, $\lambda = H^2$ is not a solution of the equation above when $\rho \neq 0$. Moreover in the charged case Eq. (6.23) becomes (see Sect. 2.4):

$$\lambda = H^2 \left[1 - C_1 \int \frac{dr}{H^4} + \rho^2 \int dr \left(\frac{1}{H^4} \int \frac{dr}{ZH^2} \right) \right] . \tag{6.35}$$

By using the same procedure leading to Eq.(6.25), we get that the extremal solution in the EM charged case is attained for:

$$C_1 = \rho^2 \left(\int \frac{dr}{ZH^2} \right)_{r_h} . \tag{6.36}$$

Because $C_1 \neq 0$, not even the weaker condition (6.28) is satisfied in the charged case. This implies that Eq. (6.27) does not hold if $\rho \neq 0$. In general, the mass of the extremal scalar-dressed solution will be different from the mass of the extremal RN-AdS solution, so that the degeneration of the $T = 0$ ground state in the EM charged case is removed.

Notice that in the charged case the $T = 0$ solution is not necessarily forced to have conformal boundary conditions for the scalar field. In fact, the argument used for the uncharged case is based both on the relation $\lambda = H^2$ and on the scale symmetries of the field equations. Both do not hold anymore at finite charge density. Nevertheless, in this case the presence of an additional field (the EM potential A_μ) allows to choose arbitrary boundary conditions for the scalar. As discussed in the previous section, the boundary conditions are not anymore imposed by the dynamics of the system as in the uncharged case. It follows that in the charged case the $T = 0$ ground state is not anymore isolated but can be reached continuously as the $T = 0$ limit of finite-temperature scalar-dressed BB solutions.

For what concerns the BB spectrum near extremality, the results we have found in the uncharged case still hold in the charged case. In fact the first and the third equation in (6.30) are not modified by the nonvanishing charge, whereas the second becomes: $a_1 b_1 = -(Z^{-1}(c_0)\rho^2 + 2b_0^4 V(c_0))/4b_0^3$.

The temperature is:

$$T = \frac{a_1}{4\pi} = -\frac{Z^{-1}(c_0)\rho^2 + 2b_0^4 V(c_0)}{16\pi b_1 b_0^3}\,. \tag{6.37}$$

Also here, we can compare the temperature of a scalar-dressed BB with that of the RN-AdS BB with the same charge ρ and radius r_h. One easily finds that Eq. (6.33) still holds for the charged case and that $T > T_0^{RN}$ for *any* finite-temperature solution, where the critical temperature T_0^{RN} is given by:

$$T_0^{RN} = \frac{12 r_h^4 - L^2 Z^{-1}(c_0)\rho^2}{16\pi r_h^3 L^2}\,. \tag{6.38}$$

Scalar-dressed EM charged solutions of the same radius r_h and charge of the RN-AdS solution exist only for $T > T_0^{RN}$.

An important issue when dealing with finite EM charge density is the characterization of the phase as fractionalized or cohesive [81, 124]. For the generic theory (6.1) with $Y \neq 0$, this characterization will depend on the IR behavior of both $Z(\phi)$ and $Y(\phi)$. However, one can easily show that in the case of unbroken $U(1)$ symmetry, $Y = 0$, only the fractionalized phase may exist. In fact using Eq. (6.6) into Eq. (6.12) one easily finds $\Phi \sim \rho$.

To summarize, the following interesting picture emerges for the IR spectrum of scalar-dressed BB solutions of ESM-AdS gravity with $m_{BF}^2 < m_s^2 < m_{BF}^2 + 1/L^2$. If a scalar-dressed, neutral, extremal solution exists at $\rho = 0$, it must necessarily be degenerate with the AdS vacuum. This is due to a precise cancellation of the contributions to the total energy from the gravitational and scalar part and, in turn, it is due to the conformal symmetries of the boundary theory. Moreover, the $T \to 0$ limit of finite-T BB solutions is singular and the ground state is isolated from the continuous part of the spectrum.

When an EM charge is switched on, the degeneracy of the ground state is removed and the ground state can be reached continuously as the $T \to 0$ limit of finite-T solutions. Scalar-dressed uncharged (EM charged) solutions of the same radius r_h (and charge) of SAdS (RN-AdS) solution exist only for $T > T_0$ ($T > T_0^{RN}$). Cohesive phases may exist only when the $U(1)$ symmetry is broken. In the $U(1)$ symmetry-preserving case only the fractionalized phases are allowed.

Our results are fairly general and only assume the existence of scalar-dressed solutions, which has to be investigated numerically. In the next two sections we will show that the picture above is realized for three wide classes of models with quadratic, quartic and exponential potentials $V(\phi)$ and for two classes of gauge couplings ($Z = 1$ and $Z \sim e^{a\phi}$). Numerical computations confirm the degeneracy of the ground state in the uncharged case and the peculiarity of the $T \to 0$ limit of finite-temperature scalar-dressed BB solutions. We will discuss separately the EM neutral and charged solutions.

6.4 Neutral solutions

6.4.1 Quadratic potential

In this section we will construct numerical solutions of Einstein-scalar AdS (ES-AdS) gravity models with the quadratic potential (6.10) and we shall check the validity of the general results of Sect. 6.3. The case of a quadratic potential is the simplest possible choice and it is therefore our first example. Moreover, this is the usual choice for models describing holographic superconductors. We will come back to this point later in Sect. 6.5.3.

Extremal solutions

The near-horizon behavior of the extremal solution of the model (6.10) with an EM field covariantly coupled to a charged scalar field has been derived in Ref. [129]. The near-horizon, extremal solution of a pure Einstein-scalar gravity model (both the EM and the charge of the scalar field are zero) can be obtained as a particular case of the solution given in Ref. [129]. In the gravitational gauge used in [129], we have:

$$ds^2 = -g(\hat{r})e^{-\chi}dt^2 + \frac{d\hat{r}^2}{g(\hat{r})} + \hat{r}^2(dx^2 + dy^2), \tag{6.39}$$

and with our normalization for the kinetic term of the scalar field, the solution reads:

$$ds^2 = \frac{d\hat{r}^2}{g_0\hat{r}^2(-\ln\hat{r})} + \hat{r}^2(-dt^2+dx^2+dy^2), \quad \phi = 2\sqrt{2}(-\ln\hat{r})^{1/2}, \quad g_0 = -\frac{2m_s^2}{3}. \tag{6.40}$$

The near-horizon, extremal solution (6.40) can be written in the gauge (6.2) by a suitable reparametrization of the radial coordinate. We get:

$$\lambda = H^2 = e^{-\frac{g_0 X^2(r)}{2}}, \quad \phi = -\sqrt{2g_0}\,X(r), \quad r = \sqrt{\frac{\pi}{g_0}} + \int^X dt e^{-\frac{g_0 t^2}{4}}, \tag{6.41}$$

where the last equation defines implicitly the function $X(r)$. We note that also in these coordinates the horizon is located at $r = 0$.

The global, extremal, solution interpolating between the near-horizon behavior (6.41) and the asymptotic AdS behavior (6.8) has to be found numerically. We have integrated the field equations numerically for several values of m_s^2. In all cases we have found $\lambda = H^2$, which implies the conformal boundary condition (6.26). Indeed the total mass M of the scalar-dressed solution is zero. In Fig. 6.1 we show the profiles of the metric functions and the scalar field for $m_s^2 = -2/L^2$.

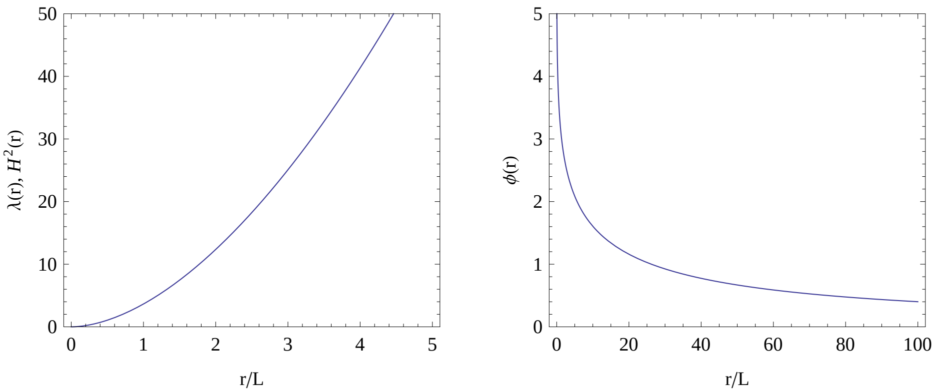

Figure 6.1: The metric functions $\lambda = H^2$ (left) and the scalar field ϕ (right) as functions of r/L, in the extremal case, for a quadratic potential with $m_s^2 L^2 = -2$.

Finite-temperature solutions

Let us now consider BB solutions of Einstein-scalar theory at finite temperature. Again we have to construct global solutions, which interpolate between the asymptotic AdS expansion given by Eq. (6.8) and a near-horizon expansion as in Eq. (6.29).

We have constructed these solutions numerically, starting from the near-horizon solution above and integrating outwards to infinity, where the asymptotic behavior of the solution is AdS_4. In Fig. 6.2[3] we show an example of the metric and scalar profiles and of the function $O_2(O_1)$ in the case $m_s^2 = -2/L^2$. In the large O_1 limit, our data are well fitted by $O_2 \sim -0.57 O_1^2$, which is consistent with the conformal boundary condition (6.26). However, for small values of O_1 the behavior reads $O_2 \sim -0.36 O_1$ and the global behavior interpolates between these two asymptotic regimes. Therefore, the function $O_2(O_1)$ does not generically satisfy the conformal boundary condition (6.26).

[3]In Fig. 6.2 and in all the figures we show in this chapter, all the dimensional quantities $(O_{1,2}, F, \mathcal{F}, c, T)$ are normalized with appropriate powers of the AdS length L.

This is a general statement that we have verified also for different choices of the parameters and different models. This fact confirms that extremal solutions are isolated from finite-temperature solutions.

As expected, solutions dressed with scalar hair only exist *above* a certain critical temperature $T_0 = 3/(4\pi)$ and a critical mass M_0 which correspond to the temperature and mass of the Schwarzschild-AdS BB, after a rescaling that sets $r_h = L = 1$. This is shown in Fig. 6.3, where we present the *total* mass M of the solutions as a function of the temperature T. The absence of dressed solutions (irrespectively of the boundary conditions $O_2(O_1)$) for $T < T_0$ confirms numerically the existence of the critical temperature T_0.

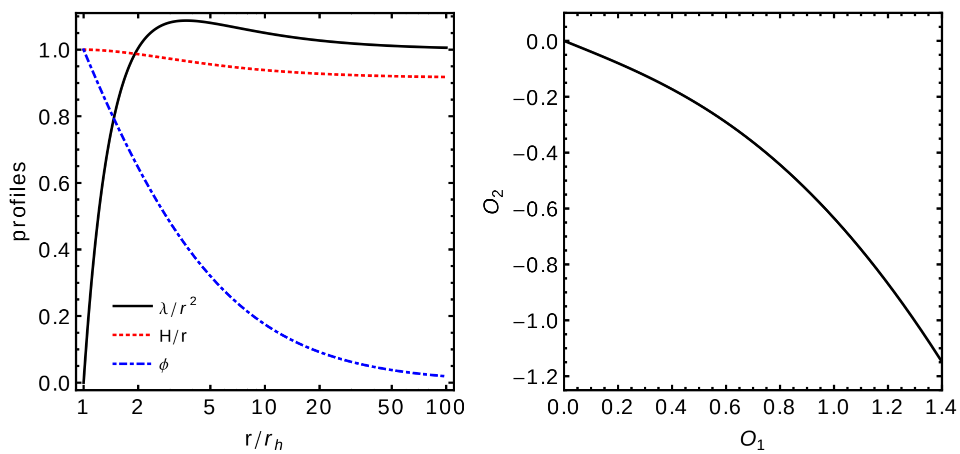

Figure 6.2: Left panel: metric and scalar profiles as functions of the (nonrescaled) coordinate r for a quadratic potential with $m_s^2 L^2 = -2$. Right panel: the function $O_2 = O_2(O_1)$ for the same model.

6.4.2 Quartic potentials

In this section we will check numerically the results of Sect. 6.3, and the validity of the picture that has emerged from our results, in the case of a theory with a potential $V(\phi)$ having the behavior described as type c) in Sect. 6.2, i.e. a theory with a IR fixed point. As an example of such a theory

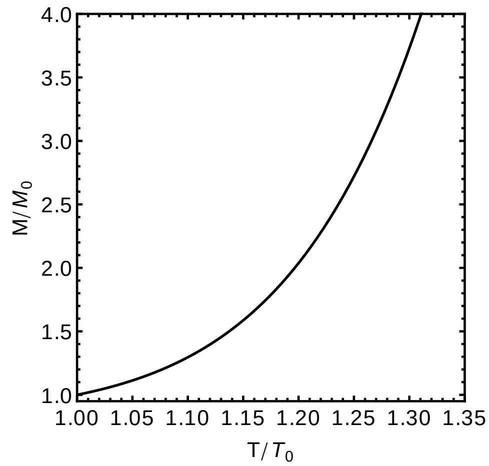

Figure 6.3: Total mass M as a function of the temperature T for a quadratic potential with $m_s^2 L^2 = -2$.

we take the quartic potential:

$$V(\phi) = \Lambda^4 \phi^4 - \frac{\hat{m}^2}{2}\phi^2 - \frac{6}{L^2}. \tag{6.42}$$

This potential has the typical mexican hat form with a maximum at $\phi = 0$ with $V(0) = -6/L^2$, $V''(0) = -\hat{m}^2$ and a minimum at $\phi_{12} = \pm\frac{\hat{m}}{2\Lambda^2}$ with $V(\phi_1) = -6/l^2 = -\hat{m}^4/(16\Lambda^4) - 6/L^2$, $V''(\phi_1) = 2\hat{m}^2$. The potential is invariant under the discrete transformation $\phi \to -\phi$, so that we will just consider $\phi \geq 0$. The theory allows for two AdS_4 vacua: an UV AdS_4 at $\phi = 0$ (corresponding to $r = \infty$), with AdS length L and with squared mass of the scalar given by $-\hat{m}^2$, and an IR AdS_4 at $\phi = \phi_1$ (corresponding to $r = 0$) with AdS length l and with squared mass of the scalar given by $2\hat{m}^2$. Again, we focus on $-9/4 < -\hat{m}^2 L^2 \leq -2$.

Extremal solutions

A scalar-dressed, extremal solution of the kind discussed in the previous section would represent a flow between an UV AdS_4 and an IR AdS_4. Let us first investigate numerically the existence of such a solution. If it exists we know from the results of the previous section that it must have zero mass, i.e. it must be degenerate with the (UV) AdS vacuum. In order to construct such solution numerically we need its perturbative expansion in the UV (near $r = \infty$) and in the IR (near horizon, $r = 0$). The UV expansion is given by Eq. (6.8). For what concerns the near-horizon $r = 0$ expansion, the field equations (6.3)-(6.5) give instead:

$$\lambda = \frac{r^2}{l^2} - \frac{\gamma^2}{12l^4}r^4 + \mathcal{O}(r^6),\ H = \frac{r}{l} - \frac{\gamma^2}{24l^3}r^3 + \mathcal{O}(r^5),\ \phi = \phi_1 + \frac{\gamma}{l}r + \mathcal{O}(r^2), \tag{6.43}$$

where γ is an arbitrary constant. Moreover, Eq. (6.5) constrains the possible values of the parameter $\hat{m}$ in Eq. (6.42) to $\hat{m}^2 = 2/l^2$. Introducing a dimensionless parametrization for Λ in Eq. (6.42), $\Lambda^{-4} = kl^2$, one finds that the restriction on $\hat{m}^2$ implies:

$$0 < k < \frac{8}{3}, \quad \frac{l^2}{L^2} = 1 - \frac{k}{24}. \tag{6.44}$$

We have integrated the field equations numerically, starting from $r \sim 0$ outwards to infinity. When $\phi \geq 0$, regular solutions only exist for $\gamma < 0$. These solutions interpolate between the $r = \infty$ AdS behavior (6.8) and the near-horizon solution (6.43).

In Fig. 6.4 we show the profiles of the metric functions and of the scalar for $k = 1$, and the function $O_2(O_1)$ (obtained by varying the free parameter γ) for selected values of k. Again we have found that $\lambda = H^2$, which implies the conformal boundary condition (6.26) and that the total mass M of the scalar-dressed solution is vanishing.

Finite-temperature solutions

Using the same method described in Sect. 6.4.1 we have constructed, numerically, dressed BB solutions at finite temperature for models with the potential (6.42). We have generated global BB solutions for $m_s^2 L^2 = -2$ and

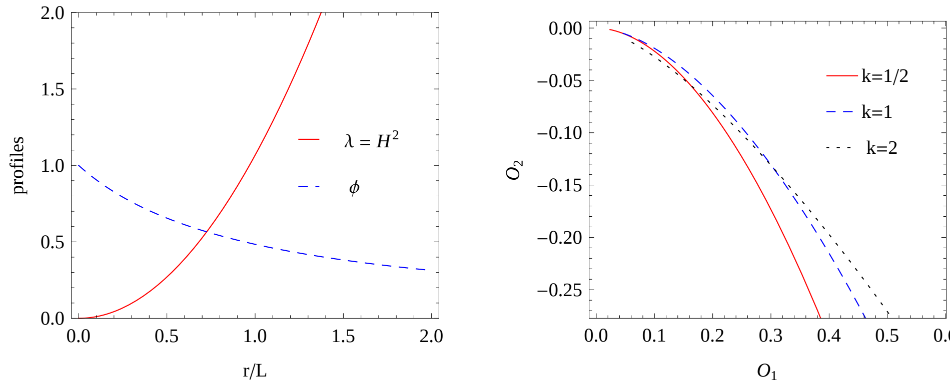

Figure 6.4: Left panel: profiles of the metric functions and the scalar field, in the extremal case, for a quartic potential with $k = 1$ and $\gamma = -1$. Right panel: the function $O_2(O_1)$ for three different values of k.

for several values of the parameter Λ. These solutions interpolate between the near-horizon expansion (6.29) and the asymptotic AdS_4 form. In Fig. 6.5 we show an example of the metric and scalar profiles and the function $O_2(O_1)$ for the case $m_s^2 L^2 = -2$ and for some selected values of Λ. As it is clear from Fig. 6.5, the function $O_2(O_1)$ displays a universal linear behavior at small O_1, which already confirms that the boundary conditions are not conformal for any value of Λ. In addition, for larger values of O_1 the slope of $O_2(O_1)$ depends on the quartic coupling.

In Fig. 6.6 we show the total mass of the solution as a function of the temperature for fixed horizon radius $r_h = 1$ and $L = 1$. As expected the dressed solutions exist only for $T > T_0$, confirming numerically the existence of the critical temperature T_0. It should be noticed that we have generated the numerical finite-temperature solutions for values of the parameters m_s^2 and Λ, which are different from those used to generate the extremal solutions. The reason for this choice is a numerical instability of the solutions for positive values of m_s^2.

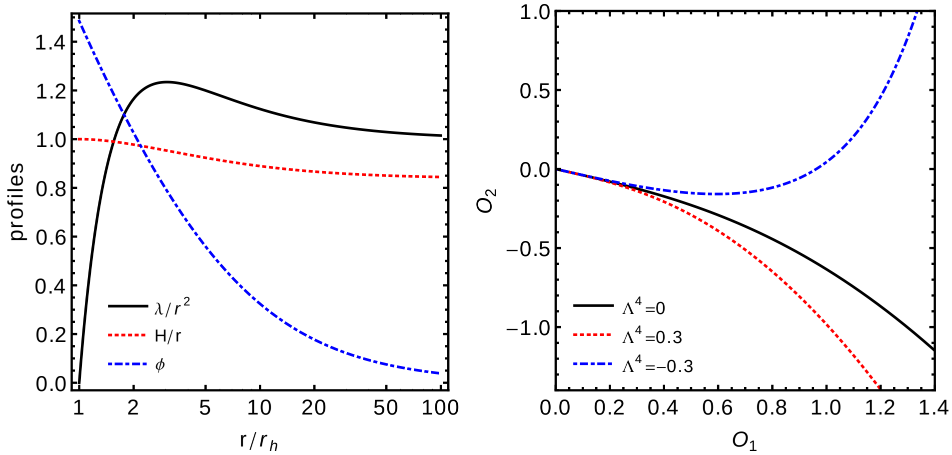

Figure 6.5: Left panel: metric and scalar profiles as functions of the (nonrescaled) coordinate r for a quartic potential with $m_s^2 L^2 = -2$ and $\lambda^4 = 0.3$. Right panel: the functions $O_2 = O_2(O_1)$ for different values of Λ.

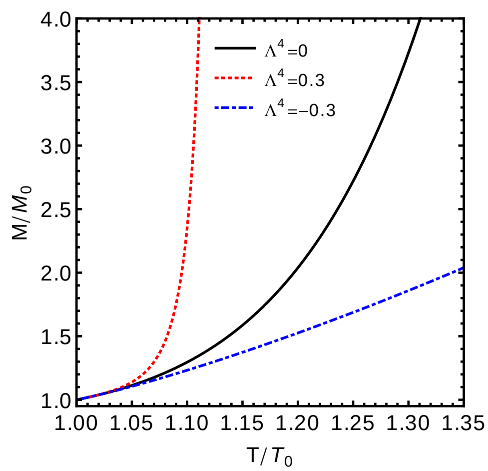

Figure 6.6: Total mass M as a function of the temperature T for a quartic potential and for selected values of the parameter Λ.

6.4.3 Exponential potentials

In this section we will investigate the case of a theory with a potential $V(\phi)$ having the behavior described as type b) in Sect. 6.2, i.e. the potential behaves exponentially $\sim e^{b\phi}$ for $\phi \to \infty$ (corresponding to $r = 0$).

We search for scalar-dressed BB solutions that smoothly interpolate between an asymptotic AdS spacetime and a near-horizon scale-covariant metric. In the dual QFT they correspond to a flow between an UV fixed point and hyperscaling violation in the IR. In general these interpolating solutions cannot be found analytically but have to be computed numerically. To be more concrete, in the following we will focus on a class of models defined by the potential:

$$V(\phi) = -\frac{2}{b^2 L^2}[\cosh(b\phi) + 3b^2 - 1]. \tag{6.45}$$

This potential is such that the mass of the scalar is independent from the parameter b, $m_s^2 = -2/L^2$. Moreover, it contains as particular cases $b = 1/\sqrt{3}$, $b = 1$ models emerging from string theory compactifications, for which analytical solutions are known [51] (see also Chapter 2).

Extremal solutions

The leading near-horizon behavior of the extremal solutions can be captured by approximating the potential in the $\phi \to \infty$ region with the exponential form $V(\phi) = -(1/b^2)e^{b\phi}$. In this case the field equations (6.3)-(6.5) give:

$$\begin{aligned} \lambda &= \alpha_0 \left(\frac{r}{r_-}\right)^w, \quad H = \left(\frac{r}{r_-}\right)^{w/2}, \quad \phi = \phi_0 - bw \ln\left(\frac{r}{r_-}\right), \\ \alpha_0 &= \frac{e^{b\phi_0} r_-^2}{b^2 w(2w-1)}, \quad w = \frac{2}{1+b^2}. \end{aligned} \tag{6.46}$$

Notice that $\alpha_0 > 0$ requires $w > 1/2$. This restricts the parameter range to $1/2 < w < 2$ ($0 < b^2 < 3$). This ansatz provides an exact solution to the equations of motion with an exponential potential $-(1/b^2)e^{b\phi}$ but only the leading near-horizon, extremal, behavior of the solutions with $V(\phi)$ generic. Solution (6.46) is scale-covariant, and the metric transforms under rescaling in the following way:

$$r \to kr, \quad (t,x,y) \to k^{1-w}(t,x,y), \quad ds^2 \to k^{2-w}ds^2. \tag{6.47}$$

The extremal solution (6.46) contains an IR length-scale r_-. However, in the case of neutral BBs the scaling transformations (6.47) may change this scale. The metric part of the solution is scale-covariant whereas the leading $\log r$ term of the scalar is left invariant. The only parameter that flows when IR length-scale r_- is changed, is the constant mode ϕ_0 of the scalar.

To reduce the number of independent parameters, we can exploit the symmetries of the field equations previously discussed [cf. Eqs. (6.32)] to fix $L = 1$ and $\phi_0 = 0$ in Eq. (6.46). So we can start from the more simple ansatz containing only one free parameter r_-.

Starting from this scaling behavior near the horizon and imposing an AdS behavior (6.8) for the metric and the scalar field at infinity, we have integrated numerically the field equations with a potential given by Eq. (6.45), with different values of the parameter $0 < b < \sqrt{3}$. We have found BB solutions with scalar hair, that interpolate between the near-horizon (6.46) and asymptotic (6.8) behavior.

In Fig. 6.7 we show the metric functions and the scalar field of these extremal BBs for $b = 1/2$ and the function $P(O_1)$ (obtained by varying the free parameter r_-) for different values of the parameter b. Also in this case we have checked numerically that $\lambda = H^2$ and that the conformal boundary conditions $P(O_1) \sim O_1^2$ are satisfied. We have also explicitly checked that the mass of the extremal solutions vanishes.

For the two cases $b = 1/\sqrt{3}$ and $b = 1$ the extremal solutions are known analytically (see Sect. 2.5.1). They are respectively given by:

$$\begin{aligned}\lambda &= H^2 = \frac{(r+r_-)^{\frac{1}{2}}}{L^2} r^{\frac{3}{2}}, \quad \phi = -\frac{\sqrt{3}}{2}\log\left(\frac{r}{r+r_-}\right), \\ \lambda &= H^2 = \frac{r+r_-}{L^2} r, \quad \phi = -\log\left(\frac{r}{r+r_-}\right), \end{aligned} \tag{6.48}$$

where r_- is a constant. From solutions (6.48) one can easily derive the function $P(O_1)$ defining the asymptotic boundary conditions for the scalar field. We have $P = (2/\sqrt{3})O_1^2$ for $b = 1/\sqrt{3}$ and $P = O_1^2$ for $b = 1$.

In order to compare these analytical solutions with those obtained numerically, we need to eliminate a linear term in the asymptotic behavior of $\lambda(r)$. Taking into account this translation, we have checked explicitly that our numerical solutions with $b = 1/\sqrt{3}$ and $b = 1$ and the numerical calculated functions P exactly reproduce the analytical results. In general, the proportionality factor f depends on the value of b. We observe that for $b < 1$ f is negative, for $b = 1$ $f = 0$, while for $b > 1$ f becomes positive, as shown in Fig. 6.7.

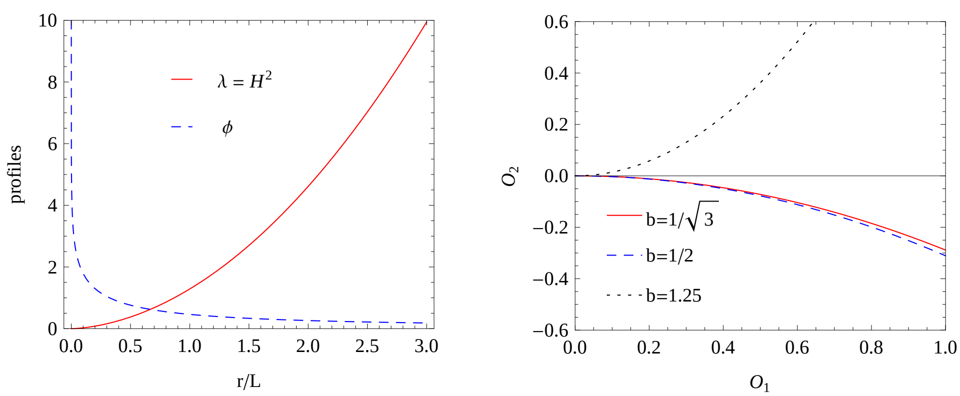

Figure 6.7: Left panel: profiles of the metric functions and the scalar field, in the extremal case, for the potential (6.45) with $b = 1/2$ and $r_- = 1$. Right panel: the function $O_2(O_1)$ for three different values of b.

Solutions at finite temperature

Following the same method as the one used in the previous subsections, one can generate generic hairy BB solutions with AdS asymptotics at finite temperature, i.e. solutions interpolating between the near-horizon (6.29) and the AdS (6.8) behavior. We have generated numerically these BB solutions and found, as in the case of a quartic potential discussed above, that for every value of the parameter b in the allowed range, they exist only for $r_h \geq 1$. This implies the existence of a critical temperature T_0 below which only the SAdS BB exists.

A summary of our results is presented in Fig. 6.8, which is qualitatively similar to Fig. 6.5 for the case of a quartic scalar potential. Again we have verified that the function $O_2(O_1)$ does not define conformal boundary conditions (6.26) for the scalar, i.e. the extremal solutions are isolated from those at finite temperature.

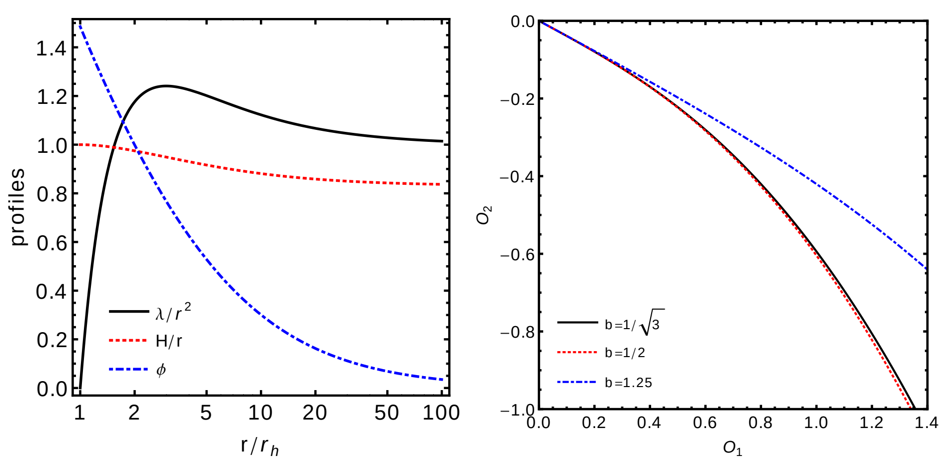

Figure 6.8: Left panel: metric and scalar profiles as functions of the (nonrescaled) coordinate r for the potential (6.45) with $b = 1/2$. Right panel: the functions $O_2(O_1)$ for different values of b.

6.4.4 Perturbative solutions near-extremality

In Sect. 6.3.1 we have seen that the $T \to 0$ limit of finite-temperature BB solutions is singular and that the ground state (6.25) is isolated from the continuous part of the spectrum. A way to gain information about the behavior near-extremality is to consider separately the near-horizon and near-extremal expansions. In general the two limits do not commute. In this section we will perform this perturbative analysis for the potential (6.45). Similar results can be obtained for other classes of potentials.

We look for perturbative solutions of the field equations (6.3)-(6.5) in the near-extremal, near-horizon regime. The near-extremal regime is obtained by expanding the metric functions λ, H and ϕ in power series of an extremality parameter m, with $m \to 0$ when the temperature $T \to 0$ (or the BB radius $r_h \to 0$). On the other hand the near-horizon regime is obtained by expanding the metric functions and the scalar field in power series of $r - r_h$. Because in general the two limits $m \to 0$ and $r \to r_h$ do not commute, we have to consider separately the two cases.

$\mathbf{r} \to \mathbf{r}_h$, $\mathbf{r}_h \to \mathbf{0}$

We first expand λ, H and ϕ in powers of m:

$$\lambda(r) = \sum_{n=0}^{\infty} \lambda_n(r) m^n, \quad H(r) = \sum_{n=0}^{\infty} H_n(r) m^n, \quad \phi(r) = \sum_{n=0}^{\infty} \phi_n(r) m^n. \quad (6.49)$$

For small BB radius $r_h << L$ (or equivalently small T, i.e. $T << 1/L$) we can truncate in the perturbative expansion (6.49) to first order in m. At leading order we find that λ_0, H_0 and ϕ_0 must satisfy the same field equations

(6.3)-(6.5). At subleading order we find instead:

$$H_1'' = -\frac{1}{4}\left(2H_0\phi_0'\phi_1 + (\phi_0')^2 H_1\right), \tag{6.50}$$

$$\left(2\lambda_0 H_1 + H_0^2\lambda_1\right)'' = 4\left[\lambda_0\left(H_0H_1\right)' + \lambda_1 H_0 H_0'\right], \tag{6.51}$$

$$\left(2\lambda_0 H_1 + H_0^2\lambda_1\right)'' = -2\phi_1 H_0^2\frac{dV(\phi_0)}{d\phi} - 4H_0H_1V(\phi_0), \tag{6.52}$$

$$\left(\lambda_0 H_0^2\phi_1' + 2\lambda_0 H_1\phi_0' + \lambda_1 H_0^2\phi_0'\right)' = 2H_0H_1\frac{dV(\phi_0)}{d\phi} + H_0^2\phi_1\frac{d^2V(\phi_0)}{d\phi^2}. \tag{6.53}$$

A solution of Eqs. (6.50)-(6.53) can be obtained by setting $\phi_1 = H_1 = 0$, so that they reduce to:

$$\left(H_0^2\lambda_1\right)'' = 0, \quad \left(H_0'H_0\lambda_1\right)' = 0, \quad \left(H_0^2\phi_0'\lambda_1\right)' = 0. \tag{6.54}$$

Equations (6.3)-(6.5) for the near-extremal leading order functions λ_0, H_0, ϕ_0 can be now solved as a near-horizon expansion in powers of r, the leading term in this expansion being obviously given by Eq. (6.46):

$$\begin{aligned} \lambda_0(r) &= \left(\frac{r}{r_-}\right)^w \sum_{n=0}^{\infty}\alpha_n\left(\frac{r}{r_-}\right)^n, \quad H_0(r) = \left(\frac{r}{r_-}\right)^{\frac{w}{2}} \sum_{n=0}^{\infty}\beta_n\left(\frac{r}{r_-}\right)^n, \\ \phi_0(r) &= -bw\ln\frac{r}{r_-} + \sum_{n=0}^{\infty}\gamma_n\left(\frac{r}{r_-}\right)^n. \end{aligned} \tag{6.55}$$

For each order in the r-expansion we can then determine the corresponding term $\lambda_1^{(n)}$ for λ_1 by solving Eqs. (6.54). One could worry about compatibility of the three equations (6.54). However, one can easily realize that for $H_0^2 = c_1 r^l$, the system (6.54) is always solved by $\lambda_1 = c_2 r^{-l+1}$ with $c_{1,2}$ constants. This follows from the first equation in (6.3), which implies $H_0' \propto 1/r$. The leading order in the near-horizon expansion involves $w, \alpha_0, \beta_0, \gamma_0$. The symmetry of the field equations (6.3)-(6.5) under a rescaling of H allows to fix $\beta_0 = 1$, whereas as expected w and α_0 turn out to be given as in Eq.

(6.46). At this order Eqs. (6.54) give in turn:

$$\lambda_1^{(0)} \propto r^{-w+1}. \tag{6.56}$$

At the $n-$th order in the near-horizon expansion we find $\lambda_1^{(n)} \propto r^{-w-n+1}$. The form of the near-extremal solution is therefore given by:

$$\lambda = \lambda_0 + \frac{m}{r^{w-1}} \left(\sum_{n=0}^{\infty} \frac{\epsilon_n}{r^n} \right) + \mathcal{O}(m^2), \quad H = H_0, \quad \phi = \phi_0, \tag{6.57}$$

where λ_0, H_0, ϕ_0 are given by Eqs. (6.55). Assuming $m < 0$ in the previous equation, we find that at leading order the relation between m and r_h is $m \propto r_h^{2w-1}$. Notice that this is an expansion in $1/r$. This means that terms with higher n give smaller contributions for $r \to \infty$.

$\mathbf{r}_h \to \mathbf{0}$, $\mathbf{r} \to \mathbf{r}_h$

This limit has been already discussed in Sect. 6.3.1. The expansion in powers of $(r - r_h)$ is given by Eq. (6.29) and at leading order the field equations (6.3)-(6.5) give the relations (6.30) involving the parameters $a_{1,2}, b_{0,1,2}, c_{0,1}$. At the next to leading order we have three more parameters a_3, b_3, c_2 and three more relations. We are therefore left with 4 independent parameters b_0, c_0, a_1, r_h. As previously discussed, the field equations have the symmetries (6.32), so that r_h is the only independent parameter. In principle, one can now expand $a_n(r_h), b_n(r_h), c_n(r_h)$ in powers of r_h, substitute in Eq. (6.29) and reorganize it as the power expansion in m given by Eq. (6.49). By retaining only the linear terms in m one could then compare the result with Eq. (6.57). Unfortunately, this is a very cumbersome task. Indeed, terms of order $\mathcal{O}(r_h)$ are generated at any order in the near-horizon expansion (6.29). The problem has to be solved numerically. Numerically, one can look for global solutions interpolating between the near-horizon behavior (6.29) with a given r_h and the AdS asymptotic solution (6.8). There is no guarantee that the solutions obtained in this way match Eq. (6.57). This is because the two limits $r \to r_h$ and $r_h \to 0$ do not commute.

Near-extremal numerical solutions

We have generated numerically, for the case of the potential (6.45), the solutions interpolating between the AdS asymptotic behavior (6.8) and the near-extremal regime given by Eq. (6.57). In Fig. 6.9 (left panel) we can see the profiles of the metric functions and the scalar field for $b = 1/\sqrt{3}$ and $r_h = 10^{-2}$ (corresponding to $m = -10^{-4}$).

We see that although the $T \to 0$ limit of the near-horizon perturbation theory is singular and isolated, global solutions obtained interpolating the near-horizon, near-extremal behavior (6.57) with AdS_4 exist also for $T > 0$. This is a manifestation of the noncommutativity of the near-horizon and near-extremal limit. From the point of view of perturbation theory, the $T = 0$ singularity means that the perturbative series in m (6.49) do not converge and that solutions (6.57) are only perturbative solutions valid for $r_h << L$.

The hairy near-extremal solutions shown in Fig. 6.9 describe small thermal perturbations of the extremal solution, but they do not describe the small-T limit of finite-temperature solutions. These results confirm that the ground state solutions (6.25) are not smoothly connected to the finite-T solutions, because of the existence of the discontinuity.

Perturbative solutions in the small scalar field limit can be also constructed. These kind of solutions are described in the Appendix.

6.4.5 Hyperscaling violation and critical exponents

The extremal $T = 0$ hairy solutions found in Sect. 6.4.3 for the case of the potential (6.45) describe a flow between the near-horizon (IR) scale-covariant regime and an asymptotic AdS fixed point. From a QFT point of view, this translates into a hyperscaling-violating phase in the IR and a scaling-

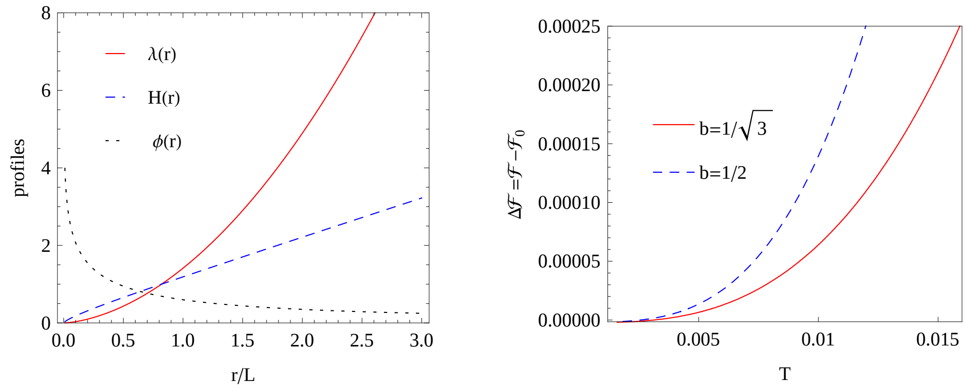

Figure 6.9: Left panel: interpolating solutions between AdS at infinity and the near-extreme regime given by Eq. (6.57) with $b = 1/\sqrt{3}$ and $r_h = 10^{-2}$. Right panel: difference between the free energy density of the near-extremal BB solution and the free energy density of the SAdS BB, for two values of b.

preserving phase in the UV. We can characterize the holographic features of this flow by giving the scaling exponents in the conformal (AdS) phase and nonconformal (hyperscaling-violating) phase. The IR behavior is dictated by Eqs. (6.46). The UV metric is instead that of AdS_4.

To describe hyperscaling violation in four dimensions we reconsider the parametrization (1.3) of the metric, with $d = 2$:

$$ds^2 = r^{\theta-2}(-r^{-2(z-1)}dt^2 + dx_i^2 + dr^2). \tag{6.58}$$

By a simple redefinition of the radial coordinate and a rescaling of the coordinates, we can write the metric (6.46) in the form (6.58). We obtain:

$$ds^2 = r^{\frac{w}{1-w}}(-dt^2 + dx_i^2 + dr^2). \tag{6.59}$$

Comparing Eq. (6.59) with Eq. (6.58), we can easily extract the parameters θ and z of our solution:

$$z = 1, \quad \theta = \frac{2-w}{1-w}. \tag{6.60}$$

While the value $z = 1$ of the dynamic critical exponent is largely expected for uncharged solutions, we see that $\theta \leq 0$ for $1 < w \leq 2$ and $\theta > 2$ for $1/2 < w < 1$, while θ diverges for $w = 1$ (recall that in our case $1/2 < w \leq 2$). This is in agreement with the null energy conditions for the stress-energy tensor, which require, for $z = 1$ and in the general case of $d + 2$ dimensions, either $\theta \leq 0$ or $\theta \geq d$.

Trivially, the parameters of the UV AdS conformally invariant solution are $z = 1,\ \theta = 0$.

The general behavior of the free energy in four dimensions (see Eq. (4.8) with $d = 2$) is:

$$F \sim T^{\frac{2-\theta+z}{z}}. \tag{6.61}$$

From Eq. (6.61), substituting the (6.60), we get that the free energy scales as:

$$F \sim T^{\frac{1-2w}{1-w}}. \tag{6.62}$$

We see from Eq. (6.62) that the exponent of T is negative for $0 < w < 1$ or, equivalently, when $\theta > 2$. So in this case the free energy diverges for $T \to 0$ and the corresponding phase is always unstable.

6.4.6 Thermodynamics of the near-extremal solutions

The hairy near-extremal solutions discussed above can be interpreted as small thermal fluctuations of extremal $T = 0$ hairy BBs. The thermodynamical features of these BB solutions – in particular the free energy and the specific heat – will provide important information about the stability of the ground state. Properties such as the scaling exponents are determined by the behavior of the system at the quantum critical point, namely by the $T = 0$ scale-covariant extremal near-horizon solution (6.46). On the other hand the stability properties are global features and they must be investigated using the global $T \neq 0$ solutions.

By Eqs. (6.13), the temperature and the entropy density of the near-horizon, near-extremal solution (6.57) are given at leading order by:

$$T = \frac{2w-1}{4\pi}\alpha_0 r_h^{w-1}, \quad \mathcal{S} = \frac{(4\pi)^{\frac{2w-1}{w-1}}}{[\alpha_0(2w-1)]^{\frac{w}{w-1}}} T^{\frac{w}{w-1}}. \tag{6.63}$$

Notice that in these subsections we are using dimensionless coordinates, so that the IR length-scale r_- drops out from our formulae, as in Eq. (6.59), and we set $L = 1$. Temperature and entropy density are therefore also dimensionless.

The scaling exponent of the entropy becomes negative when $1/2 < w < 1$ (corresponding to $1 < b^2 < 3$), implying a negative specific heat, and the corresponding solutions are therefore unstable. This is in agreement with the results of the previous subsection concerning the scaling of the free energy for $w < 1$. Moreover, in this case small values of the temperature correspond to high values of the horizon r_h and of the parameter m, so that we cannot obtain near-extremal solutions (in the sense of small temperature solutions) with $r_h << 1$, which is the range of validity of the perturbative solutions (6.57).

For what concerns the entropy density and free energy density $\mathcal{F}_0$ of the SAdS BB, we have:

$$\mathcal{S}_0 = \frac{(4\pi)^3}{9}T^2, \quad \mathcal{F}_0 = -\left(\frac{4\pi}{3}\right)^3 T^3.$$

We have derived numerically the free energy of the numerical near-extremal solutions as a function of the temperature, for $T << 1$. In Fig. 6.9 (right panel) we show the behavior of the free energy density of the hairy BB solution compared with the free energy density of the SAdS BB for two selected values of the parameter b (both such that $1 < w < 2$), and for small values $T << 1$ of the temperature. We observe that the scalar-dressed solutions are energetically disfavored against the SAdS BB. This result can also be verified analytically by comparing $\mathcal{F}_0$ with the free energy density $\mathcal{F}$ of the hairy near-extremal solution, which can be expressed as a function of the temperature using Eq. (6.13).

6.4.7 Thermodynamics of the finite-temperature solutions

We have also computed the free energy F and the specific heat c of the finite-temperature numerical scalar-dressed solutions derived in the previous sections for the case of the quartic (6.42) and exponential (6.45) potential. The results are shown in Fig. 6.10 where we plot $\Delta F/F_0$ and c as a function of the temperature, with $\Delta F = F - F_0$. The free energy F is always larger than that of the corresponding Schwarzschild-AdS BB at the same temperature and the specific heat is negative. Hence, these solutions are energetically disfavored against the undressed ones.

6.5 Charged solutions

In this section, we will extend the numerical results previously obtained for neutral BBs in ES-AdS gravity to the case of finite charge density, i.e to the case in which an EM field is present in the bulk. We will focus our attention on models with exponential (6.45) or quadratic (6.10) potential. We will discuss separately the cases of: i) minimal gauge coupling $Z = 1$; ii) exponential gauge coupling in the $U(1)$-symmetry preserving phase; iii) Minimal gauge coupling in the $U(1)$-symmetry breaking phase ($Z = 1$, $Y \neq 0$).

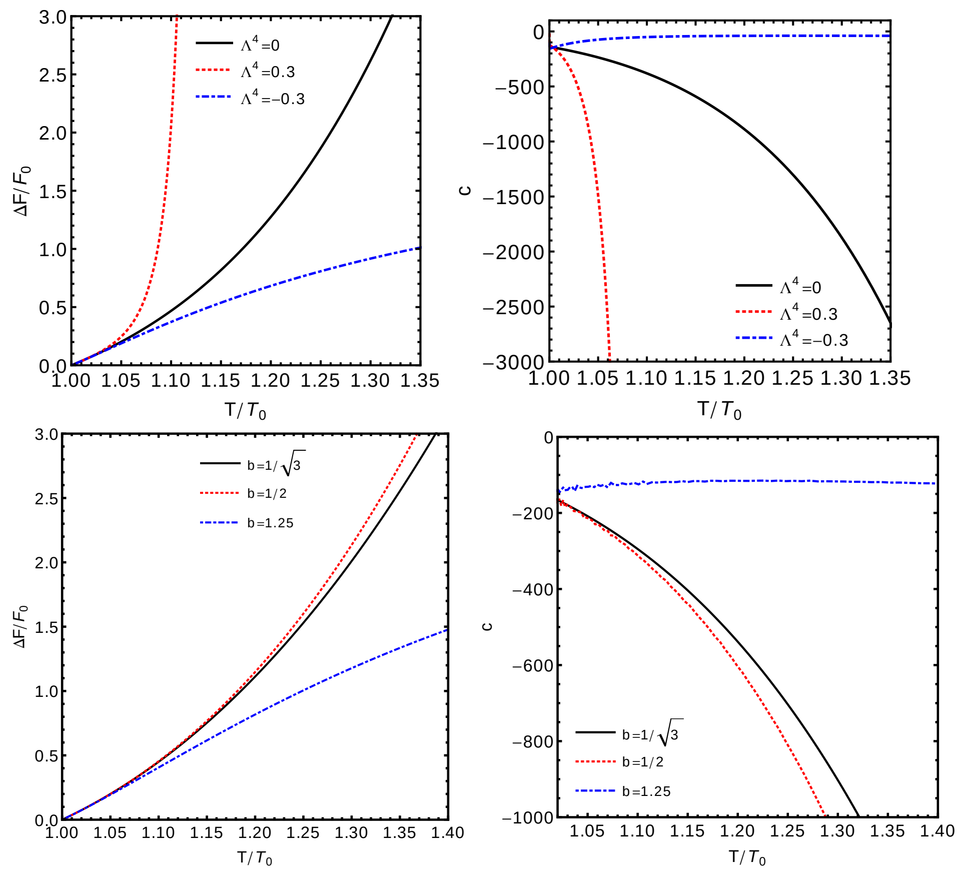

Figure 6.10: Free energy (left panels) and specific heat (right panels) of the BBs as a function of the temperature. Top and bottom panels refer to theories with a quartic potential with $m_s^2 = -2$ and with the potential (6.45), respectively.

6.5.1 Minimal gauge coupling

In this subsection we will construct numerical BB solutions for the model (6.1) with $Z = 1$, $Y = 0$ and the potential (6.45). As usual we discuss separately extremal and finite-temperature solutions.

Extremal solutions

Following the same approach as the one used for the case of electrically neutral solutions, we look for numerical scalar-dressed BB solutions interpolating between an asymptotic AdS spacetime and a near-horizon scale-covariant metric. Also in this case, the near-horizon behavior can be captured by approximating the potential (6.45) in the $\phi \to \infty$ region with the exponential form $V(\phi) = -(1/b^2)e^{b\phi}$. The field equations (6.3)–(6.5) give the scale-covariant solution, which in the dual QFT corresponds to hyperscaling violation:

$$\begin{aligned}
\lambda &= \alpha_0 \left(\frac{r}{r_-}\right)^w, \quad H = \left(\frac{r}{r_-}\right)^h, \quad \phi = \phi_0 - \frac{b}{4}(w+2)\ln\left(\frac{r}{r_-}\right), \\
w &= 2 - 4h = \frac{8-2b^2}{4+b^2}, \\
\alpha_0 &= \frac{8e^{b\phi_0}r_-^2}{b^2 w(w+2)}, \quad \rho^2 = \frac{2e^{b\phi_0}(3w-2)}{b^2(w+2)},
\end{aligned} \tag{6.64}$$

where ρ is the charge density of the solution. The solution above, together with the condition $\alpha_0 > 0$, restricts the parameter range to $2/3 < w < 2$ (corresponding to $0 < b^2 < 2$). We can exploit the symmetries of the field equations to fix $L = 1$ and $\phi_0 = 0$, leaving r_- the only free parameter. We immediately note an important feature of this solution: in the limit $\rho \to 0$ it does not reduce to the near-horizon solution (6.46) obtained in the electrically neutral case. This means that the uncharged solution (6.46) and the electrically charged solutions (6.64) represent two disjoint classes of solutions.

As usual, starting from this near-horizon scaling and imposing an AdS behavior (6.8) at infinity, we have integrated numerically the field equations for different values of the parameter b, finding numerical solutions only for $b > 1/2$. In Fig. 6.11 we show the fields for $b = 1$. As expected, here we find in general $\lambda \neq H^2$, hence the mass of the solution is nonvanishing and the degeneracy with the AdS vacuum is removed.

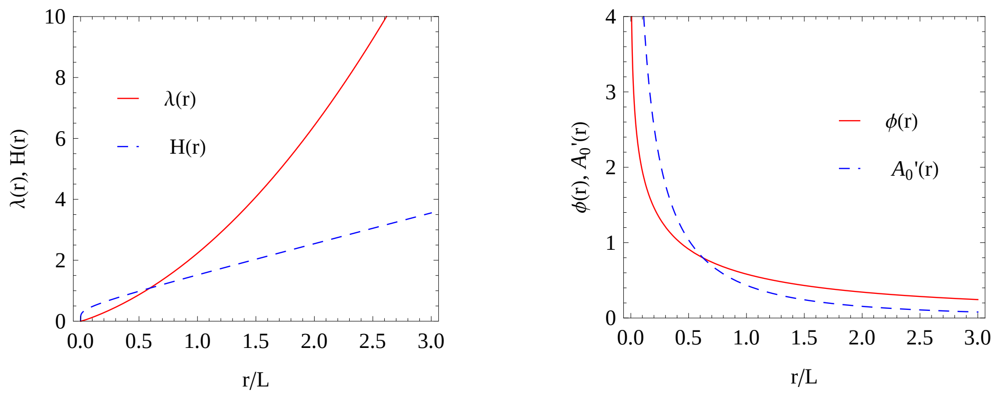

Figure 6.11: Metric functions (left panel) and scalar and Maxwell field (right), in the extremal case, for the potential (6.45) with $b = 1$ and $r_- = 1$.

Finite-temperature solutions

At variance with the extremal case, the charge of finite-temperature solutions is a free parameter and the uncharged case is obtained setting $\rho = 0$. Using a straightforward extension of the numerical integration previously discussed, we can construct finite-temperature solutions at fixed charge density ρ. Some examples are shown in Fig. 6.12 for the potential (6.45) with $b = 1$. In the left panel we show the radial profiles of the fields, in the central panel we show the function $O_2 = O_2(O_1)$ for different values of ρ, and in the right panel we show the difference between the free energy of the dressed solution and that of a RN-AdS BB with same radius and same charge. Notice that, as already stressed, in the charged case the boundary conditions can be arbitrarily chosen. In particular, one can also choose conformal boundary conditions of the form $O_1 = 0$. However, in the case at hand we have found that such conditions do not allow for scalar-dressed BBs.

Similarly to the uncharged case, these dressed solutions are always energetically disfavored with respect to the undressed ones.

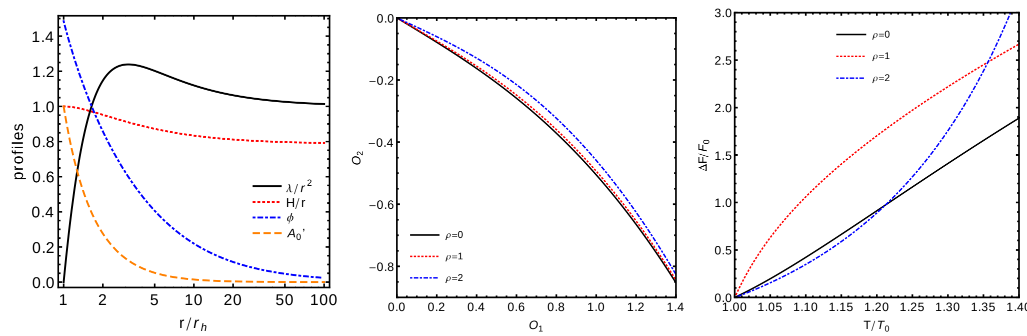

Figure 6.12: Left panel: profiles for the metric coefficients, scalar field and potential as functions of the (nonrescaled) coordinate r for the (6.45) with $b = 1$ and different values of ρ. Central panel: the functions $O_2 = O_2(O_1)$ for different values of ρ and $b = 1$. Right panel: difference between the free energy of the solution and that of a RN-AdS BB with same radius and same charge.

6.5.2 Nonminimal gauge coupling

As an example of a model with nonminimal gauge coupling we consider here the model presented in Ref. [23] and already discussed in Sect. 4.4.2. The gauge coupling Z and the potential V are [23]:

$$Z(\phi) = 2Z_0 \cosh a\phi, \qquad V(\phi) = -2V_0 \cosh b\phi. \tag{6.65}$$

In the IR ($\phi \to \infty$) both the gauge coupling Z and the potential V behave exponentially. The model therefore belongs to the wide class of EHTs that flow to a hyperscaling-violating phase in the IR [23, 27, 28, 39, 42, 81, 130]. The extremal solution of the field equations in the near-horizon approximation is the scale-covariant metric [23]:

$$\begin{aligned}
\lambda &= \alpha_0 \left(\frac{r}{r_-}\right)^w, \quad H = \left(\frac{r}{r_-}\right)^h, \quad \phi = \phi_0 - \xi \ln\left(\frac{r}{r_-}\right), \\
\xi &= \frac{4(a+b)}{4+(a+b)^2}, \quad w = 2 - 4ch, \qquad c = \frac{b}{a+b}, \\
\alpha_0 &= \frac{2V_0 e^{b\phi_0} r_-^2}{(w+2h)(w+2h-1)}, \quad \frac{\rho^2}{Z_0} e^{-a\phi_0} = \frac{2V_0 e^{b\phi_0}(2-2h-b\xi)}{(w+2h)}.
\end{aligned} \tag{6.66}$$

In Ref. [23] $T = 0$ global solutions interpolating between the near-horizon hyperscaling-violating metric (6.66) and the asymptotic AdS_4 geometry have been constructed numerically. Furthermore, numerical finite-temperature solutions have been found and their properties have been discussed in detail. In particular, it has been shown that below a critical temperature T_c the system undergoes a phase transition: the scalar-dressed BB solution becomes energetically preferred with respect to the RN-AdS BB (see also Sect. 4.4.2).

The results of Ref. [23] fully confirm the general results of Sect. 6.3. The finite charge density removes the degeneracy of the $T = 0$ solution we have in the uncharged case. Comparing the charged solution (6.66) with the neutral solution (6.46), one easily realizes that although the IR behavior of the two solutions belongs to the same class (hyperscaling-violating), the critical exponents change. Moreover, the nonminimal coupling between the scalar field and the Maxwell field is such that the energy of the extremal scalar-dressed solution is smaller than that of the RN-AdS solution. This determines an IR quantum phase transition between the $\mathrm{AdS}_2 \times R^2$ near-horizon geometry of the RN-AdS BB and the near-horizon scale-covariant geometry (6.66). In the dual QFT this corresponds to a phase transition between a conformal and a hyperscaling-violating fractionalized phase.

Because the thermodynamical properties of the system at small temperatures are essentially determined by the $T = 0$ quantum phase transition, this also explains why the hyperscaling-violating phase is stable at small temperatures, below T_c.

The near-horizon solutions for the charged BBs (6.64) and (6.66) depend on the same IR length-scale r_- as the neutral BB solution (6.46). However, in the charged case the scaling transformations under which the metric part of the solutions is scale-covariant, change not only the constant mode of the scalar ϕ_0 but also the charge density ρ. Thus, changing the IR scale r_- corresponds to a flow of the charge density ρ. As noticed already in Ref. [81], this is an irrelevant deformation along the hyperscaling-violating critical line.

It is also interesting to notice the different role played in the quantum phase transition by the finite charge density and the nonminimal gauge coupling. The finite charge density lifts the degeneracy of the $T = 0$ vacuum and changes the values of the critical exponents of the hyperscaling-violating solution, but it is by itself not enough to make the hyperscaling-violating phase energetically competitive with respect to the conformal $\mathrm{AdS}_2 \times R^2$ phase. Indeed, in the case of minimal gauge couplings discussed in the previous subsection, the energy of the extremal RN-AdS BB is lesser than the energy of the scalar-dressed $T = 0$ solution. It is the nonminimal coupling between the gauge and the scalar field that makes the extremal RN-AdS solution energetically disfavored with respect to the extremal scalar-dressed solution.

Hyperscaling violation and critical exponents

In the case of a potential behaving exponentially in the IR, the near-horizon, extremal solutions are scale-covariant for both zero or finite charge density and for both minimal or nonminimal gauge couplings. On the other hand, the critical exponents are affected by switching on a finite charge density. In particular in the case of charged solutions we will always have $z \neq 1$.

In the minimal case, after a redefinition of the radial coordinate and a rescaling of the coordinates, the metric (6.64) reads:

$$ds^2 = r^2\left(-r^{\frac{2(3w-2)}{2-w}}dt^2 + dr^2 + dx_i^2\right),$$

from which we can easily extract the critical parameters:

$$\theta = 4, \quad z = \frac{2(2-2w)}{2-w}.$$

We note immediately that the hyperscaling violation exponent θ is a (positive) constant, independent from the parameters of the potential. The range of w implies $z < 1$, which is in agreement with the NEC conditions. Indeed the latter impose, for these values of θ and z, the conditions $z > 2$ or $z < 1$. Moreover we note that for $1 < w < 2$, z is negative.

On the other hand, for $2/3 < w < 1$ (corresponding to $0 < z < 1$), the free energy scales with a negative exponent:

$$F \sim T^{\frac{2-\theta+z}{z}} = T^{\frac{w}{2w-2}},$$

which implies an instability of the corresponding phase and a negative specific heat.

Finally, we consider the case of a nonminimal gauge coupling given by Eq. (6.65). The critical exponents can be read off from Eq. (6.66), after an appropriate reparametrization of the radial coordinate. We have:

$$\theta = \frac{4c}{2c-1} \quad , \quad z = \frac{2c(2-2w)}{(2c-1)(2-w)},$$

while the free energy scales as:

$$F \sim T^{\frac{2-\theta+z}{z}} = T^{\frac{(2c-1)w+2-2c}{c(2w-2)}}. \tag{6.67}$$

6.5.3 Symmetry-breaking phase

As an example of a model having a $U(1)$-symmetry-breaking phase we consider here the model discussed in Ref. [18–20, 129] and in Sect. 4.4.1, which gives the simplest realization of holographic superconductors. The gauge coupling is minimal, while the potential V and the function $Y(\phi)$ in the action (6.1) are quadratic [129]:

$$Z(\phi) = 1, \qquad V(\phi) = \frac{m_s^2}{2}\phi^2, \qquad Y(\phi) = q^2\phi^2,$$

where q is the electric charge of the complex scalar field whose modulus is ϕ.

The metric and scalar field associated to the $T = 0$ solution of the field equations in the near-horizon approximation are given as in the neutral case discussed in Sect. 6.4.1, i.e. by Eq. (6.40), whereas the EM potential is $A_0 = \phi_0 \hat{r}^\beta (-\log \hat{r})^{1/2}$, with $2\beta = -1 \pm (1 - 48q^2/m_s^2)^{1/2}$. Numerical, extremal solutions interpolating between the near-horizon solution (6.40) and AdS_4 have been constructed for $q^2 > |m_s^2|/6$ in Ref. [129]. Numerical finite-temperature solutions have been also considered [18–20]. In particular, it is well known that below a critical temperature the superconducting phase (corresponding in the bulk to the scalar-dressed BB solution) becomes energetically preferred.

The results of Ref. [18–20, 129] for the holographic superconductors fully confirm our general results of Sect. 6.3. The finite charge density removes the degeneracy of the $T = 0$ solution in the uncharged case. Moreover, the nonvanishing coupling function Y gives a mass to the $U(1)$ gauge field and makes the extremal scalar-dressed solution energetically competitive with respect to the RN-AdS solution. The system represents an IR quantum phase transition between the $AdS_2 \times R^2$ near-horizon geometry of the RN-AdS BB and the near-horizon geometry (6.40). In the dual QFT this corresponds to the superconducting phase transition [18–20, 129], which occurs below the critical temperature.

Similarly to the nonminimal case, also here the finite charge density and the nonvanishing function Y play a very different role. The finite charge density simply lifts the degeneracy of the $T = 0$ vacuum we have in the uncharged case. But it is the coupling between the scalar field and the EM potential A_0 that causes the superconducting phase transition to occur at the critical temperature. It is also interesting to notice that in this case the finite charge density does not change the metric (and scalar) part of the IR solution, which is determined by the near-horizon solution and it is described as in the EM neutral case by Eq. (6.40).

6.6 Concluding remarks

In this chapter we have discovered several interesting features of scalar condensates in EHTs, which may be relevant for understanding holographic quantum phase transitions. In particular, we have shown that for zero charge density the ground state for scalar-dressed, asymptotically AdS, BB solutions must be degenerate with the AdS vacuum, must be isolated from the finite-temperature branch of the spectrum and must satisfy conformal boundary conditions for the scalar field. This degeneracy is the consequence of a cancellation between a gravitational positive contribution to the energy and a negative contribution due to the scalar condensate. When the scalar BB is sourced by a pure scalar field with a potential behaving exponentially in the IR, a scale is generated in the IR.

Thus, we see that fixing the UV behavior of IR hyperscaling-violating geometries to be AdS provides new crucial insights on the IR scaling geometries. In particular, the UV conformal symmetries of the AdS spacetime are relevant also in the IR and determine the degeneracy of the ground state.

Switching on a finite charge density ρ for the scalar BB, the degeneracy of the ground state is removed, the ground state is not anymore isolated from the continuous part of the spectrum and the flow of the IR scale typical of hyperscaling-violating geometries determines a flow of ρ. Depending on the gauge coupling between the bulk scalar and EM fields, the new ground state may be or may not be energetically preferred with respect to the extremal RN-AdS BB. We have also explicitly checked these features in the case of several charged and uncharged scalar BB solutions in theories with minimal, nonminimal and covariant gauge couplings. In the following subsections we will briefly discuss the consequences our results have for the dual QFT and for quantum phase transitions.

6.6.1 Dual QFT

One striking feature of the uncharged scalar BB solutions we discussed is that the boundary conditions for the scalar field are either determined by the symmetries (for the ground state) or by the dynamics (for finite-temperature solutions). Because the only free function in the model is the scalar potential $V(\phi)$, this means that the information about boundary conditions for the scalar field is entirely encoded in the symmetries of the field equations and in V. Since the scalar field drives the holographic renormalization group flow, this fact has some interesting consequences for the dual QFT.

We have seen in Sect. 6.3 that in the case of zero charge density the ground state for the scalar BB must be characterized by conformal boundary conditions. From the point of view of the dual QFT this corresponds to a multi-trace deformation of the Lagrangian of the CFT. This is a relevant deformation, associated to a relevant operator, which will produce a renormalization group flow from an UV CFT to an IR QFT. The nature of the IR QFT is entirely determined by the self-interaction potential $V(\phi)$. In the case of the quartic potential (6.42) – which is characterized by two extrema – the IR QFT has the form of a further CFT. In the case of the exponential potential (6.11), the IR QFT is characterized by hyperscaling violation. In the case of the quadratic potential (6.10), the characterization of the IR QFT is much less clear because of the absence of scaling symmetries.

The characterization of the dual QFT at finite temperature is much more involved. In this case we have generically nonconformal boundary conditions for the scalar field and the asymptotic AdS isometries are broken. Nonetheless, an asymptotic time-like killing vector exists and both the UV and the IR QFT should admit a description in terms of multi-trace deformations of a CFT.

On the other hand we have shown that the ground state and finite-T states are not continuously connected. This means that we are dealing here with two different disjoint sets of theories.

This picture changes completely when one adds a finite charge density. Now the boundary conditions for the scalar field can be arbitrarily chosen, for instance in the form of the usual conformal Neumann or Dirichlet boundary conditions. Thus, in the case of finite charge, we have the usual description borrowed from the AdS/CFT correspondence with single trace operators dual to the scalar field.

6.6.2 Scalar condensates and quantum criticality

The results of this chapter improve our understanding of quantum critical points in EHTs. In particular they shed light on the phase structure of these critical points proposed in Ref. [81] and on their stability.

The degeneracy of the ground state for uncharged BBs simply means that at zero charge density the hyperscaling-violating critical point (or line) and the hyperscaling-preserving critical point have the same energy. The potential V for the scalar field determines completely the scaling symmetry and the critical exponents of the hyperscaling-violating critical point. The renormalization group flow from the UV conformal fixed point into the IR introduces an emergent IR scale. Changing this IR scale produces a flow of the constant mode of the scalar field. As already noted in Ref. [81], the presence of this arbitrary scale implies that hyperscaling-violating critical points appear as critical lines rather than critical points. On the other hand, for scalar-dressed BBs the ground state is isolated from the finite-T part of the spectrum and the states at finite temperature are always energetically disfavored with respect to the SAdS BB. Thus, at zero charge density there is no phase transition between the hyperscaling-preserving phase and the hyperscaling-violating phase.

Considering charged scalar BBs, i.e. introducing a finite charge density ρ in the dual QFT, generates several effects. First of all the degeneracy of the ground state is lifted and the ground state is not anymore isolated from the $T > 0$ continuous branch of the spectrum. The change of the IR scale typical of hyperscaling-violating critical lines now also produces a flow of the charge density ρ. Although the critical exponents are modified by the presence of a finite charge density (for instance the dynamical critical exponent z becomes $\neq 1$), the scaling symmetries characterizing the critical point are very similar to those we have in the case of $\rho = 0$. The similarity between the ground state geometries in the $\rho \neq 0$ and $\rho = 0$ case is even more striking in the case of a covariant gauge coupling (the case of holographic superconductors). In this latter case the metric and scalar part of the near-horizon solution is exactly the same for $\rho \neq 0$ and $\rho = 0$.

The stability of the hyperscaling-violating critical line is a far more involved question. It turns out that it depends crucially on the coupling between the scalar condensate and the EM field, i.e. on the two coupling functions $Z(\phi)$ and $Y(\phi)$ in the action (6.1). In all cases that we have considered with a minimal gauge coupling $Z = 1$, and in absence of $U(1)$-symmetry breaking ($Y = 0$), the hyperscaling-preserving phase is always energetically preferred with respect to the hyperscaling-violating one. In this case, an IR phase transition between the hyperscaling-preserving phase and the hyperscaling-violating phase does not occur.

Conversely, in the two cases of a nonminimal gauge coupling behaving exponentially in the IR ($Z \sim e^{a\phi}$, $Y = 0$) and covariant gauge coupling ($Z = 1, Y \sim \phi^2$), the hyperscaling-violating phase is energetically preferred. This gives, respectively, the IR phase transitions between the hyperscaling-preserving phase and the hyperscaling-violating phase found in Ref. [23] and the well-known superconducting phase transition of Ref. [18–20, 129]. On the other hand, considering charged BBs at finite temperature, the critical temperature of the phase transition between the hyperscaling-preserving/hyperscaling-violating phases is settled by the charge density ρ [23], i.e by the IR emergent scale typical of the hyperscaling-violating critical line.

Summarizing, our results strongly indicate that for EHTs described by (6.1), the three coupling functions $V(\phi)$, $Z(\phi)$, $Y(\phi)$ determine different features of holographic quantum critical points. The self-interaction potential $V(\phi)$ determines the scaling symmetries but not the stability of hyperscaling-violating phases. Conversely Z and Y are crucial in determining the stability, the breaking of the $U(1)$ symmetry and the characterization as fractionalized or cohesive of the hyperscaling-violating phase.

6.7 Appendix: Uncharged perturbative solutions in the small scalar field limit

In the neutral case, it is possible to construct analytical BB solutions in the small scalar field limit perturbatively, i.e. expanding the solution as follows:

$$\lambda(r) = \frac{r^2}{L^2} - \frac{M}{2r} + \epsilon^2 \lambda_2(r)\,, \qquad (6.68)$$

$$H(r) = r + \epsilon^2 H_2(r)\,, \qquad (6.69)$$

$$\phi(r) = \epsilon \phi_1(r)\,, \qquad (6.70)$$

where ϵ is a book-keeping parameter of the expansion. The solution for the scalar field can be obtained by solving the scalar equation at first order. The regular solution can then be inserted into the Einstein equations that, to second order, can be solved for λ_2 and H_2.

Let us start with the $T = 0$ AdS_4 vacuum, i.e. we set $M = 0$ in the equations above. To second order in the scalar field, the solution reads:

$$\lambda(r) = \frac{r^2}{L^2} + \left(-\frac{O_1^2}{4L^2} - \frac{O_2^2}{6L^2 r^2} + \frac{2rC_1}{L^2} + \frac{C_2}{r} \right) \epsilon^2 + \mathcal{O}(\epsilon^4)\,, \qquad (6.71)$$

$$H(r) = r + \left(-\frac{O_2^2}{12r^3} - \frac{O_1 O_2}{6r^2} - \frac{O_1^2}{8r} + C_1 + rC_2 \right) \epsilon^2 + \mathcal{O}(\epsilon^4)\,, \qquad (6.72)$$

$$\phi(r) = \epsilon \left(\frac{O_1}{r} + \frac{O_2}{r^2} \right) + \mathcal{O}(\epsilon^3)\,, \qquad (6.73)$$

where C_i are integration constants. This is a solution for the classes of potentials presented in the main text. Although not presented, the solutions can be obtained in closed form at least to fourth order. The constant C_1 can be set to zero by performing a coordinate translation such that the asymptotic form of the metric reads as in Eq. (6.17) with $C_2 = -m_0/2$, being related to the metric contribution to the gravitational mass and after a rescaling $H \to H/(1 + \epsilon^2 C_2)$, which can be performed by rescaling the transverse coordinates. Interestingly, there exists an event horizon, so the solution represents a BB endowed with a scalar field. Let us consider two cases separately: $O_2 = 0$ and $O_2 = O_2(O_1)$ (without loss of generality, we assume $O_2 \geq 0$). For the latter case, the horizon is located at:

$$r_h = \frac{\sqrt{O_2}\epsilon}{6^{1/4}} + \frac{\sqrt{3}m_0}{4\sqrt{2}O_2}\epsilon + \frac{3^{1/4}\left(4O_1^2O_2^2 - 3m_0^2\right)}{2^{3/4}32O_2^{5/2}}\epsilon^{3/2} + \mathcal{O}(\epsilon^{5/2})\,, \tag{6.74}$$

and, to first order, the temperature of the solution is:

$$T = \frac{\sqrt{O_2}\sqrt{\epsilon}}{6^{1/4}\pi}\,. \tag{6.75}$$

On the other hand, if $O_2 = 0$, the horizon and the temperature read:

$$r_h = \frac{m_0^{1/3}\epsilon^{2/3}}{2^{1/3}} + \frac{O_1^2\epsilon^{4/3}}{2^{2/3}6m_0^{1/3}}\,, \tag{6.76}$$

$$T = \frac{3m_0^{1/3}\epsilon^{2/3}}{2^{1/3}4\pi} + \frac{O_1^4\epsilon^2}{96m_0\pi}. \tag{6.77}$$

In general, these solutions describe a BB whose horizon shrinks to zero in the $\mathcal{O}_i \to 0$ limit. The total mass of the BB is given by Eq. (6.14) and it coincides with m_0 when $O_2 = 0$. It is interesting to compare the free energy of this solution with that of a SAdS BB at the same temperature. When $O_2 \neq 0$, we obtain:

$$F - F_0 = \frac{37(\epsilon O_2)^{3/2}}{6^{3/4}27} + \mathcal{O}(\epsilon^2)\,, \tag{6.78}$$

so that $F > F_0$ for any $O_2 \neq 0$ and the dressed solution is always energetically disfavored. Note that this result is valid for any boundary condition $O_2 = O_2(O_1) \neq 0$ and for any scalar potential whose expansion reads $V \sim -6/L^2 - \phi^2/L^2$. On the other hand, if $O_2 = 0$, $F = F_0$ to second order in O_1, so that the two solutions are degenerate.

Finally, we can adopt the same technique to construct perturbative solutions of the SAdS BB at finite temperature. At first order, the general solution of the scalar field equation reads:

$$\phi_1 = \alpha P_{-1/3}\left[r^3/(L^2M) - 1\right] + \beta Q_{-1/3}\left[r^3/(L^2M) - 1\right] , \tag{6.79}$$

where P_n and Q_n are Legendre functions of order n and α and β are integration constants. Imposing regularity at the horizon $r_h = (2L^2M)^{1/3}$ requires $\beta = 0$. In principle, this solution can be inserted in the Einstein equations in order to obtain two equations for $H_2(r)$ and $\lambda_2(r)$. Unfortunately, these equations do not appear to be solved in closed form.

Conclusions

In this thesis we have investigated several aspects of Einstein-scalar gravity models, and related "hairy" black brane solutions, both from a pure gravitational point of view and from the perspective of possible holographic applications, motivated by recent applications of the AdS/CFT correspondence. We found some interesting original results that we summarize now.

In the Part I we have focused our attention on gravitational aspects, presenting several new black brane solutions with scalar hair, and studying their main features. In particular, in Chapter 2 we proposed a new general method for obtaining exact solutions of Einstein and Einstein-Maxwell gravity minimally coupled to a scalar field. The particularity of this method is that it imposes to fix *a priori* the general form of the scalar field, for determining the metric functions and the potential. Usually one starts from an opposite approach (with the potential as an input and the scalar field an output), but in this way it is possible, starting from several forms for the scalar field, to find exact solutions. Moreover, the method is suitable for applications of AdS/CFT, because in these cases what is most important is not the form of the potential, but the behavior (especially asymptotic) of the scalar field, which can be interpreted, in the dual field theory, as a running coupling constant or as a scalar condensate. We applied the method for deriving broad classes of new exact black brane and black hole solutions with scalar hair. In particular, we get solutions both in four and in a generic number d of dimensions, and with different asymptotic behavior (AdS, domain wall, Lifshitz-like).

Furthermore, our method allowed us to formulate a new important no-hair theorem about the general existence of black hole and black brane solutions, that puts some constraints in particular about the existence of uncharged asymptotically AdS black brane solutions. This result is particularly important because it represents an useful guideline in the search for new black brane solutions with scalar hair, which represent scalar condensates in the dual QFT.

In Chapter 3 we presented an exactly integrable fake SUGRA model of Einstein-scalar gravity and derived a black brane solution, using a more traditional approach. In this approach, the field equations for static, spherically symmetric solutions are reduced to an integrable dynamical system, namely a Toda lattice system. The most interesting feature of the solution is that its extremal limit is a regular scalar soliton interpolating between a non-AdS domain wall behavior at $r = \infty$ and an AdS solution at $r = 0$. In terms of the dual QFT this means that we have an IR conformal fixed point and an UV hyperscaling-violating phase.

In the Part II we studied other black brane solutions with scalar hair, but focusing more on the holographic applications. In Chapter 5 we considered an extension to a generic number of dimensions of the model presented in the Chapter 3. Studying the thermodynamics of the exact black brane solutions we showed that, in a certain range of the parameter of the potential, one observes a phase transition between the background Schwarzschild-AdS (SAdS) black brane solution (without scalar hair) and the "scalar-dressed" black brane solution, with the latter becoming energetically favoured at high temperatures. Actually, the correct physical interpretation of this phase transition remains quite involved, because the two phases (the SAdS and the hairy black branes) are characterized by two different asymptotic behavior.

From the holographic point of view, the black brane solution (when is stable) describes a field theory with hyperscaling violation in the ultraviolet regime, while the hyperscaling violation critical exponent θ is negative. Also here, the interpretation of this result is problematic, because a negative value of θ corresponds to an anomalous "raising" of the effective dimensions of the theory and has no analogue in real condensed matter systems.

Finally, in Chapter 6 we investigated a very general class of models with black brane solutions asymptotically AdS, focusing on the IR behavior of the black branes spectrum. Firstly, we studied the spectrum of energy of these solutions. We found that for zero charge density the extremal $T = 0$ black brane solutions have always zero mass, so are degenerate with the AdS vacuum, and are isolated from the continuous part of the spectrum. In presence of a finite charge density, the degeneracy is removed and the $T = 0$ ground state is not anymore isolated. We have checked these general results performing some numerical solutions with different potentials. In particular we found complete agreement with the no-hair theorem shown in Chapter 2.

Our results have improved our understanding of quantum critical points in effective holographic theories. The study of the stability of these solutions, in particular the charged ones, compared with the results of some recent works [23, 129], seem to indicate that the scalar-dressed black brane solutions can be energetically favoured against the Reissner-Nordström-AdS BB only in presence of a covariant or nonminmal coupling between the scalar field and the EM field. As the stability of the scalar-dressed phase against the AdS undressed background phase is a necessary condition for having a physically-relevant phase transition in the dual field theory, these results are very important for selecting the bulk gravity models which can produce in the dual QFT physically interesting phase transitions.

Open problems

We close the thesis with a list of open problems, in part already mentioned in the previous chapters, both concerning the gravitational solutions and about the holographic applications.

- The general method presented in Chapter 2, in some cases, does not allow to find full finite-temperature families of black hole and black brane solutions, but only extremal solutions with $T = 0$ (see for example the solutions described in Sects. 2.5.1, 2.5.2 and 2.5.3). The existence of the full spectrum of solutions for these models should be investigated numerically. Actually, in Chapter 6 we have seen that the difficulty of generating analitically the full spectrum of solutions is related to its IR behavior and we have derived finite-temperature numerical solutions for a model studied in Sect. 2.5.1. However, could be interesting to verify the existence of these finite-temperature solutions also for other models.

- A very puzzling point, as we have pointed out more times in this thesis, is to find the correct holographic interpretation of the black brane solutions described in Chapters 3 and 5 (respectively in four and in generic d dimensions). In particular, what is the physical interpretation of a negative value for the hyperscaling violation critical exponent θ? And, mostly, there exists a real condensed matter system which could describe, holographically, a gravitational solution of this kind?

  Actually, it is likely that this class of solutions could be more interesting for understanding some peculiarities of the gravitational interaction (e.g. its holographic nature) rather than for condensed matter applications. Support to this point of view has been given in a recent paper [131], where it has been argued that after analytic continuation the black brane solutions of these models produce FLRW cosmological solutions, which could be used to model inflation.

- We have seen from Chapter 6, on the base of analytical and numerical solutions, that the problem of the stability of the various phases (AdS, superconducting, hyperscaling-violating) is a rather involved issue to which we could not give a definite answer. We have argued that the relevant information about stability is encoded in the coupling functions between the Maxwell and the scalar field. Our results strongly indicate that in the case of $U(1)$-symmetry breaking and minimal coupling, at low temperature, the superconducting phase is preferred, whereas in the case of nonminimal coupling and real scalar the hyperscaling-violating phase is preferred.

  An important question is: are these results completely general? Can we prove them in an exact way by imposing on the coupling functions appropriate constraints?

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