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DIRECT AND INVERSE SCATTERING OF THE MATRIX ZAKHAROV-SHABAT SYSTEM

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To Claudia

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Preface

This PhD. thesis is the fruit of three years of hard studies and I would like to seize the opportunity to acknowledge various people who have played a significant role in concluding the thesis.

Particular acknowledgments concern Claudia for the patience shown me in these years. Her smile has given me the force to continue in the periods with more difficulties and I will never forget her gentleness. I also thank various family members for having been close to me during the past few years.

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Chapter 1

Introduction

1. The Korteweg-de Vries equation. Many nonlinear evolution equations admit travelling wave solutions of the form

$$u(x,t) = \phi(x - ct), \tag{1.1}$$

where $x \in \mathbb{R}$ (or $x \in \mathbb{R}^+$) is the position variable, $t \in \mathbb{R}$ is time, and c is a parameter called the wavespeed. The first such equation formulated has been the Korteweg-de Vries (KdV) equation [30, 31, 67]

$$u_t + u_{xxx} - 6uu_x = 0 \tag{1.2}$$

to describe water waves travelling along a canal. After a long period (1895-1960) without new applications of the KdV equation, Zabusky and Kruskal [93] coined the term "soliton" for the elastically interactive solitary wave solutions of eq. (1.1) which pass each other without losing their shape and velocity. Gardner, Greene, Kruskal and Miura [47, 48] developed the so-called inverse scattering transformation (IST) to solve the initial-value problem of the KdV equation.

To understand the IST, we consider the Schrödinger equation on the line

$$-\psi''(x,\lambda) + u(x)\,\psi(x,\lambda) = \lambda\,\psi(x,\lambda), \quad x \in \mathbb{R}, \tag{1.3}$$

where $u(x)$ is the (real) potential and $\lambda = k^2$ is an eigenvalue parameter satisfying $\operatorname{Im} k \geq 0$. For u satisfying the Faddeev condition

$$\int_{-\infty}^{\infty} dx\,(1 + |x|)\,|u(x)| < \infty, \tag{1.4}$$

we introduce the Jost solutions $f_l(k,x)$ and $f_r(k,x)$ by

$$f_l(k,x) = e^{ikx}[1 + o(1)], \qquad x \to +\infty, \tag{1.5a}$$

$$f_r(k,x) = e^{-ikx}[1 + o(1)], \quad x \to -\infty, \tag{1.5b}$$

and compute the reflection coefficients $R(k)$ and $L(k)$ and the transmission coefficient $T(k)$ from the asymptotic expansions

$$f_l(k,x) = \frac{1}{T(k)}\,e^{ikx} + \frac{L(k)}{T(k)}e^{-ikx} + o(1), \quad x \to -\infty, \tag{1.6a}$$

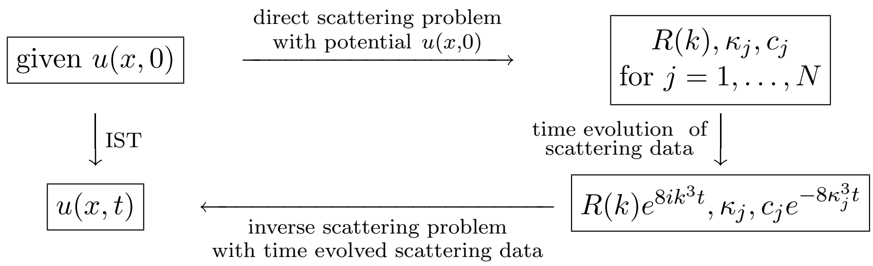
$$f_r(k,x) = \frac{1}{T(k)}\,e^{-ikx} + \frac{R(k)}{T(k)}e^{ikx} + o(1), \quad x \to +\infty, \tag{1.6b}$$

as well as the finitely many numbers k (all positive imaginary and denoted $i\kappa_1, \ldots, i\kappa_N$ where $0 < \kappa_1 < \ldots < \kappa_N$) for which eq. (1.3) has a nontrivial solution in $L^2(\mathbb{R})$, along with these so-called bound state solutions. This is the solution of the direct scattering problem. The inverse scattering problem for the Schrödinger equation on the line, first solved by Faddeev [45] and presented in [40, 43, 32, 75], consists of recovering the potential $u(x)$ of Faddeev class from one of the reflection coefficients, the bound state wave numbers $i\kappa_1, \ldots, i\kappa_N$, and N positive parameters $c_1, \ldots, c_N$ (called the *norming constants*). The method consists of converting the Riemann-Hilbert problem satisfied by the Jost solutions into a Marchenko integral equation and to compute $u(x)$ from its solution.

The inverse scattering transform (IST) now consists of three steps. First we let the initial condition $u(x, 0)$ to the KdV equation (1.1) be the potential in eq. (1.3) and solve the direct scattering problem to arrive at the scattering data

$$\left\{R(k), \{\kappa_j, c_j\}_{j=1}^N\right\}.$$

We then evolve these data in time in an elementary way. Finally we solve the inverse scattering problem starting from the time evolved data and find the potential $u(x, t)$ as the solution of eq. (1.1). The IST can be summarized by the following diagram:



2. Nonlinear Schrödinger equation. Soon after the seminal papers [47, 48], many nonlinear evolution equations were discovered that can be solved by the inverse scattering transform for some accompanying linear ordinary differential equation. In this PhD thesis we focus on the direct and inverse scattering theory for the family of systems of first order ordinary differential equations

$$-iJ\frac{\partial X(x,\lambda)}{\partial x} - V(x)X(x,\lambda) = \lambda X(x,\lambda), \quad x \in \mathbb{R}, \tag{1.7}$$

where

$$J = \begin{pmatrix} I_n & 0_{n\times m} \\ 0_{m\times n} & -I_m \end{pmatrix}, \quad V(x) = \begin{pmatrix} 0_{n\times n} & k(x) \\ l(x) & 0_{m\times m} \end{pmatrix}, \quad X(x,\lambda) = \begin{pmatrix} X^{up}(x,\lambda) \\ X^{dn}(x,\lambda) \end{pmatrix}, \tag{1.8}$$

I_n denotes the identity matrix of order n and $0_{p\times q}$ the $p \times q$ matrix with zero entries, $V(x)$ is the potential, and $\lambda \in \mathbb{C}$ is an eigenvalue parameter. The most natural such system occurs for $n = m = 1$ and $l(x) = \pm\overline{k(x)}$ and is called the Zakharov-Shabat system. It reads

$$-i\frac{\partial X^{up}(x,\lambda)}{\partial x} - k(x)X^{dn}(x,\lambda) = \lambda X^{up}(x,\lambda), \tag{1.9a}$$

$$i\frac{\partial X^{dn}(x,\lambda)}{\partial x} \mp \overline{k(x)}X^{up}(x,\lambda) = \lambda X^{dn}(x,\lambda). \tag{1.9b}$$

It was introduced in [94] to solve the nonlinear Schrödinger equation

$$iq_t = q_{xx} \pm 2|q|^2 q$$

by the inverse scattering transform. The work was generalized by Manakov [73] to solve a pair of coupled nonlinear Schrödinger equations using the inverse scattering solution of eq. (1.7) for $n = 1$ and $m = 2$, $k(x) = \begin{pmatrix} k_1(x) & k_2(x) \end{pmatrix}$, $l(x) = \begin{pmatrix} l_1(x) & l_2(x) \end{pmatrix}^T$ and $l_j(x) = \pm\overline{k_j(x)}$ $(j = 1, 2)$. In this thesis we call (1.7) the matrix Zakharov-Shabat system. In the literature one often finds the terminologies AKNS system (after the extensive study made in [2]; see also [1, 4, 3]) and canonical system (primarily within the Odessa school and followers; cf. [19] for the history and review).

In many applications such as to fiber optics we distinguish two special cases according to the symmetry properties of the potential. In the **symmetric case** we have $l(x) = k(x)^*$, where the asterisk denotes the matrix conjugate transpose; in the **antisymmetric case** we have $l(x) = -k(x)^*$. The antisymmetric case occurs in fiber optics for anomolous dispersion [61, 62], while the symmetric case occurs in the case of normal dispersion [61, 63]. In the antisymmetric case (anomolous dispersion) multisoliton solutions abound. In the symmetric case so-called dark soliton solutions (i.e., travelling wave solutions that do not vanish as $x \to \pm\infty$ for fixed t but instead oscillate) occur. In the literature the terms "defocussing" (symmetric case) and "focussing" (antisymmetric case) abound, but in this thesis we do not use these terms.

In general, it is not easy to know beforehand if a given nonlinear evolution equation can be solved by an IST relating it to the direct and inverse scattering theory of a Hamiltonian operator H. Major light was shed on this problem by Lax [72] who derived nonlinear evolution equations associated to H by means of an IST by studying so-called Lax pairs $(H, \mathcal{B})$ of linear operators H and $\mathcal{B}$ such that

$$H_t = H\mathcal{B} - \mathcal{B}H.$$

We indicate Lax pairs for the KdV equation and the matrix NLS equation in Sec. 5.3. For examples of Lax pairs we refer to the literature (e.g., [4, 1, 3]).

3. Inverse scattering. Inverse scattering has been studied for its own sake long before the inverse scattering transform supplied it with a major application. Borg [27, 28] recovered the potential in the Schrödinger equation on a finite interval from its eigenvalues under two sets of boundary conditions. Under Neumann boundary conditions one set of eigenvalues suffices [18]. In the early 1950's Gelfand and Levitan [49], Krein [68], and Marchenko [74] (also [6]) developed the inverse spectral theory (cf. [49]) and inverse scattering theory (cf. [68, 74]) for the Schrödinger equation on the half-line $\mathbb{R}^+$. Faddeev [45] constructed its inverse scattering theory on the line (also [75, 40, 43, 32]). Starting from the late 1960's, Melik-Adamjan [76] generalized the methods of [68] to develop the inverse scattering theory for eq. (1.7) on the half-line for arbitrary boundary condition at $x = 0$. The so-called Odessa group and other researchers which followed up on this work denoted eq. (1.7) by the term "canonical system." The inverse scattering theory of eq. (1.7) on the half-line and on a finite interval, has been generalized in many directions and linked to the study of reproducing kernel Hilbert spaces and specific classes of analytic operator-valued functions while using the language of linear control theory (see [19] for a review of the literature).

Using the ideas of Zakharov, Shabat and Manakov, the authors Ablowitz, Kaup, Newell and Segur [2] developed the inverse scattering theory and the inverse scattering transform for the matrix nonlinear Schrödinger (mNLS) equation which can be solved by applying the inverse scattering theory of eq. (1.7). This system of eq. (1.7) is therefore often called the AKNS system. Beals and Coifman analyzed the direct and inverse scattering theory associated with first order systems of ordinary differential equations of AKNS type with arbitrary invertible and diagonal matrix J and the corresponding nonlinear evolution equations with initial data in the Schwartz class of test functions [21, 22, 23]. Earlier results on first order systems with real diagonal J with $\mathrm{Tr}(J^{-1}) = 0$ have been given Gerdjikov and Kulish [51]. A comprehensive treatment of the direct and inverse scattering theory for higher order systems of linear differential equations and the corresponding IST can be found in [24]. More recent developments on higher order systems and their accompanying IST appeared in [95, 41]. In [3] the direct and inverse scattering of eq. (1.7) has been given in almost full generality.

4. State space methods. In linear system theory (e.g., [20, 39]), so-called state space methods are used to study various transformations of linear systems (such as conjugation, cascades, cascade decomposition, input-output reversal, etc.) by using so-called transfer functions. When the reflection coefficient is a rational matrix function, the inverse scattering problem for eq. (1.7) can be solved in closed form by state space methods. Alpay and Gohberg [12, 13, 14, 15], and Gohberg, Kaashoek and A.L. Sakhnovich [55, 56] have thus solved the inverse spectral problem for the canonical system eq. (1.7) on the half-line as well as the inverse scattering problem on the half-line in the symmetric case where $l(x) = k(x)^*$. The inverse scattering problem on the full-line with rational reflection coefficient has been solved by similar methods by Aktosun et al. [9] if $l(x) = k(x)^*$, and by van der Mee [90] if $l(x) = -k(x)^*$ and there are no bound states. Recently, Aktosun and van der Mee [11] have used the solution of the inverse scattering problem for the Schrödinger equation on the line to obtain explicit solutions of the KdV equation for initial data which are potentials having rational reflection coefficients.

So far state space methods in inverse scattering have primarily been used to solve inverse problems for the matrix Zakharov-Shabat system on the half-line in the symmetric case for $n = m$. Very few of these papers deal with the more interesting antisymmetric case. Further, if nonlinear evolution equations were solved, the issues of local vs. global in time existence and the obtainability of these solutions by the inverse scattering transform were never raised. In this thesis we deal with all of these issues. For a comprehensive account of the literature on (primarily local in time) existence of solutions of the multidimensional nonlinear Schrödinger equation in Sobolev spaces we refer to [29].

5. Contents of the thesis. In [3] the inverse scattering theory of eq. (1.7) has been developed in full generality, but some details have not been given. In [3] discrete eigenvalues are always assumed algebraically and geometrically simple and the definition of the norming constants reflects this limitation. Further, as in [2] bound state norming constants are defined as if the reflection coefficients extend analytically off the real line, which requires very strong decay assumptions on the potential. The compactness of the Marchenko integral operator has not been proved in full generality. The unique solvability of the Marchenko integral equation has not been established in sufficient generality. In this thesis we intend to fill most of these gaps, while developing inverse scattering theory in the notations of [9, 90]. Also in the symmetric and antisymmetric cases we derive explicit solutions for rational scattering data if there are bound states, as well as the most general multi-soliton solutions.

This PhD thesis is organized as follows. In Chapter 2 we specify the domain of the full

Hamiltonian $H = -iJ\frac{d}{dx} - V$ and define some basic notions of spectral theory. In Chapters 3 and 4 we develop the direct and inverse scattering theory for eq. (1.7). In particular, we study the Jordan structure of the discrete eigenvalues of the matrix Zakharov-Shabat system and derive the Marchenko equations irrespective of Jordan structure. Further, in the symmetric and antisymmetric cases we prove the unique solvability of the Marchenko equations and characterize the scattering data, the latter if there are no bound states. Then, in Chapter 5 we give a brief exposition of fiber optics transmission and study the inverse scattering transform for the nonlinear Schrödinger equation in more detail. Finally, in Chapter 6 we introduce the state space method in the antisymmetric and symmetric cases, derive multi-soliton solutions in a systematic way, and plot some of these solutions. We also give a necessary and sufficient conditions for a state space solution of the mNLS equation to be time periodic. In Appendix A we compare our notations to those used in [2, 3] and to those used in fiber optics theory. In Appendix B we discuss the symmetry properties of various functions arising in direct and inverse scattering theory.

6. Notations and definitions. Let us now introduce some notations used throughout the thesis. By $\mathbb{R}$ and $\mathbb{R}^\pm$ we denote the real line and the (closed) positive and negative half-lines. By $\mathbb{C}^+$ and $\mathbb{C}^-$ we denote the open upper half and lower half complex planes, respectively. We write $\overline{\mathbb{C}^+} = \mathbb{C}^+ \cup \mathbb{R}$ and $\overline{\mathbb{C}^-} = \mathbb{C}^- \cup \mathbb{R}$. Furthermore, let us write $\hat{f} = \mathcal{F}f$ for the Fourier transform

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} dx\, e^{i\xi x} f(x).$$

Then, according to Plancherel's theorem, $(2\pi)^{-1/2}\mathcal{F}$ is a unitary operator on $L^2(\mathbb{R})$, implying the inversion formula

$$f(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} d\xi\, e^{-i\xi x} \hat{f}(\xi).$$

The Fourier transform is a contraction from $L^1(\mathbb{R})$ into the Banach space of continuous complex-valued functions on $\mathbb{R}$ vanishing at $\pm\infty$, endowed with the supremum norm. We shall use the same symbol to denote the Fourier transform on direct sums of n copies of $L^2(\mathbb{R})$ or $L^1(\mathbb{R})$.

For $n \in \mathbb{N}$ we denote by $\mathcal{H}_n$ the direct sum of n copies of $L^2(\mathbb{R})$ endowed with the scalar product

$$\langle \{f_k\}_{k=1}^n, \{g_k\}_{k=1}^n \rangle = \sum_{k=1}^n \langle f_k, g_k \rangle_{L^2(\mathbb{R})} = \sum_{k=1}^n \int_{-\infty}^{\infty} dx\, f_k(x)\overline{g_k(x)}.$$

Letting $H^s(\mathbb{R})$ stand for the Sobolev space of those measurable functions f whose Fourier transform $\hat{f}$ satisfies

$$\|f\|_{H^s(\mathbb{R})} = \left[\int_{-\infty}^{\infty} d\xi\, (1+\xi^2)^s |\hat{f}(\xi)|^2\right]^{1/2} < \infty, \tag{1.10}$$

we denote by $\mathcal{H}_n^s$ the direct sum of n copies of $H^s(\mathbb{R})$ endowed with the scalar product

$$\langle \{f_k\}_{k=1}^n, \{g_k\}_{k=1}^n \rangle = \sum_{k=1}^n \langle f_k, g_k \rangle_{H^s(\mathbb{R})}.$$

As a result, $\mathcal{H}_n^0 = \mathcal{H}_n$ (apart from a factor $(2\pi)^{-1/2}$ in the definition of the norms) for each $n \in \mathbb{N}$. Introducing the Hilbert spaces

$$L^{2,s}(\mathbb{R}) = L^2(\mathbb{R}; (1+\xi^2)^s d\xi) \tag{1.11}$$

for any $s \in \mathbb{R}$ and letting $L^{2,s}_n(\mathbb{R})$ stand for the orthogonal direct sum of n copies of $L^{2,s}(\mathbb{R})$, we see that the Fourier transform $\mathcal{F}$ is a unitary transformation from $\mathcal{H}^s_n(\mathbb{R})$ onto $L^{2,s}_n(\mathbb{R})$.

Chapter 2

Domains of the Hamiltonian

It is well-known [64, 83] that, under sufficiently general conditions on the real potential, the spectrum of the Hamiltonian H of the Schrödinger equation with real potential V in the Faddeev class consists of the continuous spectrum $[0, \infty)$ and at most countably many isolated negative eigenvalues which can only accumulate at zero.

In this chapter we specify the domains of the free Hamiltonian $H_0 = -iJ\frac{d}{dx}$ and the (full) Hamiltonian $H = -iJ\frac{d}{dx} - V$ on the direct sum $\mathcal{H}_{n+m}$ of $n+m$ copies of $L^2(\mathbb{R})$. We distinguish between potentials having their entries in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ where H_0 and H have the same domain, and potentials having their entries in $L^1(\mathbb{R})$. In Sec. 2.3 we define the Jordan structure of analytic operator-valued functions on open subsets of the complex plane. This concept is then applied in Sec. 2.4 to prove that the nonreal spectrum of H consists of isolated eigenvalues of finite algebraic multiplicity which can only accumulate on the real line. In Sec. 2.5 we briefly discuss the effect of certain symmetries on the spectrum of H.

2.1 Domain of the free Hamiltonian

The free Hamiltonian H_0 is defined as the unbounded linear operator

$$H_0 = -iJ(d/dx)$$

on the dense domain $\mathcal{H}^1_{n+m}$ in $\mathcal{H}_{n+m}$, where

$$J = \operatorname{diag}(I_n, -I_m) = \operatorname{diag}(\underbrace{1, \ldots, 1}_{n \text{ copies}}, \underbrace{-1, \ldots, -1}_{m \text{ copies}}).$$

Denoting the Fourier transform on $\mathcal{H}_{n+m}$ by $\mathcal{F}$ we obtain

$$(\mathcal{F} H_0 \mathcal{F}^{-1} \hat{f})(\xi) = -J\xi \hat{f}(\xi) = \left(\operatorname{diag}(\underbrace{-\xi, \ldots, -\xi}_{n \text{ copies}}, \underbrace{\xi, \ldots, \xi}_{m \text{ copies}}) \right) \hat{f}(\xi).$$

Thus H_0 is selfadjoint and its spectrum is continuous and fills up the complete real line.

To compute the resolvent of H_0, we choose alternatively $\lambda \in \mathbb{C}^+$ and $\lambda \in \mathbb{C}^-$ and consider the system of differential equations

$$-iJ\frac{\partial X(x,\lambda)}{\partial x} = \lambda X(x,\lambda) - F(x),$$

where $F = \begin{pmatrix} F_{up} & F_{dn} \end{pmatrix}^T$ with $F_{up} \in \mathcal{H}_n$ and $F_{dn} \in \mathcal{H}_m$. Choosing $\lambda \in \mathbb{C}^+$ and partitioning $X = \begin{pmatrix} X_{up} & X_{dn} \end{pmatrix}^T$ as we did for F, we get

$$\begin{cases} X_{up}(x,\lambda) = \left(T_0(\lambda)F_{up}\right)(x) \stackrel{\text{def}}{=} -i\displaystyle\int_{-\infty}^{x} dy\, e^{i\lambda(x-y)}F_{up}(y), \\ X_{dn}(x,\lambda) = \left(S_0(\lambda)F_{dn}\right)(x) \stackrel{\text{def}}{=} -i\displaystyle\int_{x}^{+\infty} dy\, e^{i\lambda(y-x)}F_{dn}(y), \end{cases}$$

whereas for $\lambda \in \mathbb{C}^-$ we have

$$\begin{cases} X_{up}(x,\lambda) = \left(T_0(\lambda)F_{up}\right)(x) \stackrel{\text{def}}{=} +i\displaystyle\int_{x}^{\infty} dy\, e^{-i\lambda(y-x)}F_{up}(y), \\ X_{dn}(x,\lambda) = \left(S_0(\lambda)F_{dn}\right)(x) \stackrel{\text{def}}{=} +i\displaystyle\int_{-\infty}^{x} dy\, e^{-i\lambda(x-y)}F_{dn}(y). \end{cases}$$

For $\lambda \in \mathbb{C} \setminus \mathbb{R}$ we have therefore written the resolvent of H_0 in the form

$$\left(\lambda I_{\mathcal{H}_{n+m}} - H_0\right)^{-1} = \begin{pmatrix} T_0(\lambda) & 0_{n\times m} \\ 0_{m\times n} & S_0(\lambda) \end{pmatrix}, \tag{2.1}$$

where $T_0(\lambda)$ and $S_0(\lambda)$ are bounded operators on $\mathcal{H}_n$ and $\mathcal{H}_m$, respectively.

2.2 Domain of the Hamiltonian

Recall that $\mathcal{H}_{n+m}^{-1}$ is defined as the orthogonal direct sum of $n+m$ copies of $H^{-1}(\mathbb{R})$, where $H^{-1}(\mathbb{R})$ is given by (1.10) with $s = -1$. We have the following

Lemma 2.1 *Let W be an $(n+m)\times(n+m)$ matrix function whose elements belong to $L^2(\mathbb{R})$. Then multiplication by W is a bounded linear operator from $\mathcal{H}_{n+m}^1$ into $\mathcal{H}_{n+m}$ and from $\mathcal{H}_{n+m}$ into $\mathcal{H}_{n+m}^{-1}$.*

Proof. It suffices to prove this lemma for $W \in L^2(\mathbb{R})$ and $n = m = 1$. In other words, it suffices to prove that the operator of multiplication by $W \in L^2(\mathbb{R})$ is bounded from $H^1(\mathbb{R})$ into $L^2(\mathbb{R})$ and from $L^2(\mathbb{R})$ into $H^{-1}(\mathbb{R})$. Indeed, for $f, g \in L^2(\mathbb{R})$ we have

$$\begin{aligned} \langle \widehat{Wf}, \frac{\hat{g}}{(1+k^2)^{\frac{1}{2}}} \rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k-\hat{k})x} W(x) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\hat{k}) d\hat{k}\, dx \frac{\overline{\hat{g}(k)}}{(1+k^2)^{\frac{1}{2}}} dk \\ &= (2\pi)^{-1/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{W}(k-\hat{k})\hat{f}(\hat{k})d\hat{k} \frac{\overline{\hat{g}(k)}}{(1+k^2)^{\frac{1}{2}}} dk, \end{aligned}$$

where $\widehat{W}$ is the Fourier transform of W. Thus

$$\left|\langle\widehat{Wf},\frac{\hat{g}}{(1+k^2)^{\frac{1}{2}}}\rangle\right| \leq (2\pi)^{\frac{1}{2}}\|W\|_2\|f\|_2\int_{-\infty}^{\infty}\frac{|\hat{g}(k)|}{(1+k^2)^{\frac{1}{2}}}dk \leq$$
$$\leq \pi\sqrt{2}\,\|W\|_2\|f\|_2\|\hat{g}\|_2,$$

where we have used the Cauchy-Schwartz inequality. Thus, for any $f\in L^2(\mathbb{R})$ we have $(1+k^2)^{-\frac{1}{2}}\widehat{Wf}\in L^2(\mathbb{R})$ and therefore $Wf\in H^{-1}(\mathbb{R})$. Moreover, $\|Wf\|_{H^{-1}(\mathbb{R})}\leq\pi\sqrt{2}\,\|W\|_2\|f\|_2$, i.e., the operator of multiplication by $W\in L^2(\mathbb{R})$ is a bounded linear operator from $L^2(\mathbb{R})$ into $H^{-1}(\mathbb{R})$.

Next, for $f\in H^1(\mathbb{R})$ and $g\in L^2(\mathbb{R})$ we have

$$\langle\widehat{Wf},\hat{g}\rangle = (2\pi)^{-1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{W}(k-\hat{k})\hat{f}(\hat{k})d\hat{k}\,\overline{\hat{g}(k)}\,dk =$$
$$= (2\pi)^{-1/2}\int_{-\infty}^{\infty}\hat{f}(\hat{k})\overline{\int_{-\infty}^{\infty}\overline{\hat{W}(k-\hat{k})}\hat{g}(k)dk}.$$

Thus

$$|\langle\widehat{Wf},\hat{g}\rangle| \leq \|W\|_2\|\hat{g}\|_2\int_{-\infty}^{\infty}|\hat{f}(\tilde{k})|(1+\tilde{k}^2)^{\frac{1}{2}}\frac{d\tilde{k}}{(1+\tilde{k}^2)^{\frac{1}{2}}} \leq$$
$$\leq \sqrt{\pi}\|W\|_2\|\hat{g}\|_2\|f\|_{H^1(\mathbb{R})}.$$

Hence $Wf\in L^2(\mathbb{R})$ for $f\in H^1(\mathbb{R})$ and $\|Wf\|_2\leq 2^{-1/2}\|W\|_2\|f\|_{H^1(\mathbb{R})}$. ■

It is now easy to prove the subsequent

Corollary 2.2 *Let V be a potential whose elements belong to*

$$L^1(\mathbb{R})\cap L^2(\mathbb{R}).$$

Then the premultiplication by V is a bounded linear operator from $\mathcal{H}^1_{n+m}$ into $\mathcal{H}_{n+m}$ and from $\mathcal{H}_{n+m}$ into $\mathcal{H}^{-1}_{n+m}$. Consequently, for any $\lambda\in\mathbb{C}\backslash\mathbb{R}$, $(\lambda I_{\mathcal{H}_{n+m}}-H_0)^{-1}V$ and $V(\lambda I_{\mathcal{H}_{n+m}}-H_0)^{-1}$ are bounded operators on $\mathcal{H}_{n+m}$.

As a result, if V is a potential whose elements belong to $L^1(\mathbb{R})\cap L^2(\mathbb{R})$, then

$$H = H_0 - V = -iJ\frac{d}{dx} - V$$

is a selfadjoint operator on $\mathcal{H}_{n+m}$ with domain $\mathcal{D}(H)=\mathcal{D}(H_0)=\mathcal{H}^1_{n+m}$. A similar result exists for the Schrödinger equation ([64], Sec.5.3).

For general potentials V whose elements belong to $L^1(\mathbb{R})$, it is in general no longer possible to define the Hamiltonian H as above or even to have a Hamiltonian H defined on the same domain as the free Hamiltonian H_0. We shall define the domain of H and discuss its discrete spectrum of H in Subsection 2.4.2.

2.3 Jordan normal form

In the literature (e.g., [46, 71]) the Jordan normal form is usually defined for square matrices. Given an eigenvalue λ_0 of the $n \times n$ matrix A, we call x_0 an eigenvector of A at the eigenvalue λ_0 if

$$(\lambda_0 I_n - A)\, x_0 = 0, \quad x_0 \neq 0.$$

We call $\{x_0, x_1, \cdots, x_{q-1}\}$ a Jordan chain of A of length q at the eigenvalue λ_0 if $(\lambda_0 I_n - A)\, x_{j-1} + x_{j-2} = 0$ for $j = 2, \cdots, q$ and $(\lambda_0 I_n - A)\, x_0 = 0$, $x_0 \neq 0$. The vectors $x_1, \cdots, x_{q-1}$ are called generalized eigenvectors. It is clear that $q \leq n$ and $\dim \mathrm{Ker}\, (\lambda_0 I_n - A) \leq n$. When looking for a complete set of maximal Jordan chains of A at λ_0 for which the corresponding eigenvectors x_0 are linearly independent and span Ker $(\lambda_0 I_n - A)$, we obtain Jordan chains of A at λ_0 of lengths $q_1 \geq q_2 \geq \cdots \geq q_r$ such that

$$\alpha_m = \dim \mathrm{Ker}\, (\lambda_0 I_n - A)^m = \sum_{s=1}^{m} \sharp \{j = 1, \cdots, r : q_j \geq s\}.$$

Let us now generalize this concept. To this end, let $\mathcal{X}$ be a complex Banach space, $\lambda_0 \in \mathbb{C}$ and $\mathcal{H}(\mathcal{X}, \lambda_0)$ the linear vector space of germs of $\mathcal{X}$-valued analytic functions in a neighborhood of λ_0. This means that we identify $\mathcal{X}$-valued analytic functions whenever they have the same values in some neighborhood of λ_0. Now let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ stand for the Banach space of bounded linear operators from $\mathcal{X}$ into the complex Banach space $\mathcal{Y}$, where we adopt the notation $\mathcal{L}(\mathcal{X})$ if $\mathcal{X} = \mathcal{Y}$.

Let Ω be an open set in $\mathbb{C}$ and let $\mathcal{X}^*$ denote the adjoint of a complex Banach space $\mathcal{X}$, i.e., $\mathcal{X}^* \stackrel{\text{def}}{=} \mathcal{L}(X, \mathbb{C})$. Then $F : \Omega \to \mathcal{X}$ is said to be *analytic* if either of the two following conditions is satisfied:

- $\forall x_0 \in \mathcal{X}^*$, $\langle F(\cdot), x_0 \rangle$ is analytic (*weak analyticity*);
- $\forall \lambda \in \Omega$ we have $\lim_{z \to \lambda} \left\| \frac{F(z) - F(\lambda)}{z - \lambda} - F'(\lambda) \right\|_{\mathcal{X}} = 0$ and $\lambda \mapsto F'(\lambda)$ is continuous in $\lambda \in \Omega$ (*strong analyticity*).

It is well-known ([92], Theorem V3.1) that the above conditions are equivalent. Moreover, $F : \Omega \to \mathcal{L}(\mathcal{X})$ is said to be analytic if one of the following conditions is satisfied:

- $\forall x \in \mathcal{X}$, $\forall x_0 \in \mathcal{X}^*$, $\langle F(\cdot)x, x_0 \rangle$ is analytic;
- $\forall x \in \mathcal{X}$, $F(\cdot)x$ is strongly analytic;
- $F(\cdot)$ is strongly analytic with respect to the norm of $\mathcal{L}(\mathcal{X})$,

and, also in this case, it is easy to establish that the three preceding conditions are equivalent.

Given $F \in \mathcal{H}(\mathcal{L}(\mathcal{X}, \mathcal{Y}), \lambda_0)$, for any $p = 0, 1, 2, \cdots$ and writing

$$F(\lambda) = \sum_{j=0}^{\infty} (\lambda - \lambda_0)^j F_j, \qquad |\lambda - \lambda_0| \leq \varepsilon,$$

we define the linear operator $\Phi_p(F;\lambda_0) : \mathcal{X}^p \to \mathcal{Y}^p$ by

$$\Phi_p(F;\lambda_0) = \begin{pmatrix} F_0 & 0 & \cdots & \cdots & \cdots \\ F_1 & F_0 & 0 & \cdots & \cdots \\ \vdots & & \ddots & 0 & \\ \vdots & & & \ddots & \\ F_{p-1} & F_{p-2} & \cdots & \cdots & F_0 \end{pmatrix}. \tag{2.2}$$

Here $\mathcal{X}^p$ and $\mathcal{Y}^p$ denote the direct sums of p copies of $\mathcal{X}$ and $\mathcal{Y}$, respectively. Then we easily obtain

$$\Phi_p(F;\lambda_0)\Phi_p(G;\lambda_0) = \Phi_p(H;\lambda_0) \tag{2.3}$$

whenever $FG = H$ with $F \in \mathcal{H}(\mathcal{L}(\mathcal{Y},\mathcal{Z}),\lambda_0)$, $G \in \mathcal{H}(\mathcal{L}(\mathcal{X},\mathcal{Y}),\lambda_0)$ and $H \in \mathcal{H}(\mathcal{L}(\mathcal{X},\mathcal{Z}),\lambda_0)$. It is easily seen that $\Phi_p(F;\lambda_0)$ is boundedly invertible for some (and hence all) p if and only if $F_0 = F(\lambda_0)$ is boundedly invertible.

Let us restrict ourselves to those $F \in \mathcal{H}(\mathcal{L}(\mathcal{X},\mathcal{Y}),\lambda_0)$ that have only invertible values in a deleted neighborhood of λ_0 and for which $F(\lambda_0)$ is a Fredholm operator. Then

$$\alpha_p(F,\lambda_0) = \dim \operatorname{Ker} \Phi_p(F,\lambda_0)$$

is finite and for some $q \in \mathbb{N}$ we have

$$\alpha_0(F,\lambda_0) \leq \alpha_1(F,\lambda_0) \leq \ldots \leq \alpha_{q-1}(F,\lambda_0) < \alpha_q(F,\lambda_0) = \alpha_{q+1}(F,\lambda_0) = \ldots < +\infty.$$

The index q is called the *ascent* of F in λ_0 and coincides with the order of the pole of $F(\lambda)^{-1}$ in λ_0. The numbers $\{\alpha_p(F,\lambda_0)\}_{p=0}^{q-1}$ determine the *Jordan characteristics* of F in λ_0.

Let $F \in \mathcal{H}(\mathcal{L}(\mathcal{X},\mathcal{Y}),\lambda_0)$ and $G \in \mathcal{H}(\mathcal{L}(\mathcal{Z},\mathcal{W}),\lambda_0)$. Then F and G are called *equivalent* in λ_0 (see, e.g., [54], Chapter II) if there exist operator functions $E \in \mathcal{H}(\mathcal{L}(\mathcal{X},\mathcal{Z}),\lambda_0)$ and $\tilde{E} \in \mathcal{H}(\mathcal{L}(\mathcal{Y},\mathcal{W}),\lambda_0)$ such that $E(\lambda_0)$ and $\tilde{E}(\lambda_0)$ are boundedly invertible and

$$\tilde{E}F = GE$$

as germs of analytic functions in a neighborhood of λ_0. Then F is boundedly invertible in a deleted neighborhood of λ_0 and has a Fredholm operator as its value in λ_0 whenever G has these properties, and in this case we have

$$\alpha_p(F,\lambda_0) = \alpha_p(G,\lambda_0), \quad p = 1,2,\ldots.$$

Let $F \in \mathcal{H}(\mathcal{L}(\mathcal{X}),\lambda_0)$ and let $\mathcal{Y}$ be a complex Banach space. Then by the $\mathcal{Y}$*-extension* of F we mean the operator function $F \oplus I_{\mathcal{Y}} \in \mathcal{H}(\mathcal{L}(\mathcal{X} \oplus \mathcal{Y}),\lambda_0)$. We then have

$$\alpha_p(F \oplus I_{\mathcal{Y}},\lambda_0) = \alpha_p(F,\lambda_0), \quad p = 1,2,\ldots.$$

Considering $F \in \mathcal{H}(\mathcal{L}(\mathcal{X},\mathcal{Y}),\lambda_0)$ and $G \in \mathcal{H}(\mathcal{L}(\mathcal{Y},\mathcal{X}),\lambda_0)$, it is easily seen ([54], p. 38) that the $\mathcal{X}$-extension of $I_{\mathcal{Y}} - FG$ is equivalent in λ_0 to the $\mathcal{Y}$-extension of $I_{\mathcal{X}} - GF$.

Let $y \in \mathcal{H}(\mathcal{Y}, \lambda_0)$. Then the vector function $y(\lambda) = \sum_{n=0}^{\infty}(\lambda - \lambda_0)^n y_n$ with a power series having a positive or infinite radius of convergence, is called a *root function* of F at λ_0 if

$$F(\lambda)y(\lambda) = 0$$

in a neighborhood in λ_0 and $y(\lambda_0) \neq 0$. Then it is easily verified that

$$\Phi_p(F;\lambda_0)\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{p-1}\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0\end{pmatrix}$$

for any $p \geq 1$. Thus the root functions of F at λ_0 correspond to the nontrivial vectors in the kernel of $\Phi_p(F;\lambda_0)$ for any $p \in \mathbb{N}$. Then for $p = 1, \ldots, q$ the dimension of the vector space of root functions $y(\lambda)$ which are $\mathcal{Y}$-valued polynomials of degree at most $p-1$ coincides with $\alpha_p(F, \lambda_0)$.

2.4 Nature of the discrete spectrum

In this section we define the Hamiltonian operator H for all potentials V having their entries in $L^1(\mathbb{R})$ by specifying its domain $\mathcal{D}(H)$. We also prove that the nonreal spectrum of H consists of eigenvalues of finite algebraic multiplicity which can only accumulate on the real line.

We first prove the following two technical lemmas. For $n = m = 1$ the first lemma appears as Lemma 4.1 of [66]. In the case of the Schrödinger equation it is known as the Birman-Schwinger principle.

Lemma 2.3 *Let W_1 and W_2 be $(n+m)\times(n+m)$ matrix functions whose elements belong to $L^2(\mathbb{R})$, and let $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Then*

$$W_1(\lambda - H_0)^{-1}W_2$$

is a Hilbert-Schmidt operator on $\mathcal{H}_{n+m}$ whose Hilbert-Schmidt norm vanishes as $|Im\,\lambda| \to \infty$.

Proof. It suffices to prove that $W_1T_0(\lambda)W_2$ is a Hilbert-Schmidt operator whenever $\lambda \in \mathbb{C}^+$ and W_1 and W_2 are $n \times n$ matrix functions whose elements belong to $L^2(\mathbb{R})$. Indeed, $W_1T_0(\lambda)W_2$ is an integral operator on a space of vector functions defined on $\mathbb{R}$ whose $n \times n$ matrix integral kernel is given by

$$K(x,y) = -i\, e^{i\lambda(x-y)}W_1(x)W_2(y)\chi_{\mathbb{R}^+}(x-y),$$

where $\chi_{\mathbb{R}^+}$ stands for the characteristic function of $\mathbb{R}^+$. We have

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy|K(x,y)|^2 = \int_{-\infty}^{\infty} dx|W_1(x)|^2 \int_{-\infty}^{x} dy\, e^{-2(x-y)\mathrm{Im}\,\lambda}|W_2(y)|^2$$
$$\leq \int_{-\infty}^{\infty} dx|W_1(x)|^2 \int_{-\infty}^{x} dy\,|W_2(y)|^2 \leq \|W_1\|_2^2\|W_2\|_2^2,$$

which shows $W_1T_0(\lambda)W_2$ to be Hilbert-Schmidt. Since the integrand is bounded above by the integrable function

$$|W_1(x)|^2|W_2(y)|^2,$$

we can apply the Theorem of Dominated Convergence and prove that the Hilbert-Schmidt norm of $W_1 T_0(\lambda) W_2$ vanishes as $\operatorname{Im}\lambda \to +\infty$. ∎

Lemma 2.4 *Let W_1 and W_2 be $(n+m)\times(n+m)$ matrix functions whose elements belong to $L^2(\mathbb{R})$, and let $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Then*

$$W_1(\lambda - H_0)^{-2} W_2$$

is a Hilbert-Schmidt operator on $\mathcal{H}_{n+m}$ whose Hilbert-Schmidt norm vanishes as $|Im\,\lambda| \to \infty$.

Proof. The Hilbert-Schmidt operators on a separable Hilbert space such as $\mathcal{H}_{n+m}$ are themselves elements of a separable Hilbert space (cf. [60]) if one imposes the scalar product

$$\langle T, S\rangle_{HS} \stackrel{\text{def}}{=} \operatorname{tr}(T_1(T_2)^*),$$

where tr denotes the trace of the trace class operator $T_1(T_2)^*$. If we now follow the proof of Lemma 2.3 for the integral kernel

$$\frac{\partial K}{\partial \lambda}(x,y) = e^{i\lambda(x-y)}(x-y)W_1(x)W_2(y)\chi_{\mathbb{R}^+}(x-y), \tag{2.4}$$

we see that the $\frac{d}{d\lambda} W_1(\lambda - H_0)^{-1} W_2$ is itself the integral operator with integral kernel (2.4). Moreover, applying the Theorem of Dominated Convergence we have for any Hilbert-Schmidt operator T on $\mathcal{H}_{n+m}$

$$\frac{d}{d\lambda}\langle W_1(\lambda - H_0)^{-1}W_2, T\rangle_{HS} = -\langle W_1(\lambda - H_0)^{-2}W_2, T\rangle_{HS}, \qquad \lambda \in \mathbb{C}\setminus\mathbb{R}.$$

Thus $W_1(\lambda - H_0)^{-1}W_2$ is weakly analytic as a Hilbert-Schmidt valued vector function. But then the derivation with respect to $\lambda \in \mathbb{C}\setminus\mathbb{R}$ also holds in the Hilbert-Schmidt norm [92, Theorem V3.1]. It is also easily verified, as in the proof of Lemma 2.3, that $W_1(\lambda - H_0)^{-2}W_2$ vanishes in the Hilbert-Schmidt norm as $|\operatorname{Im}\lambda| \to \infty$ in either of $\mathbb{C}^\pm$. ∎

2.4.1 Square integrable potentials

If the potential V is an $(n+m)\times(n+m)$ matrix function whose elements belong to $L^1(\mathbb{R})\cap L^2(\mathbb{R})$, then, according to Lemma 2.3, $V(\lambda - H_0)^{-1}V$ is Hilbert-Schmidt whenever $\lambda \in \mathbb{C}\setminus\mathbb{R}$. If we now define the operator function

$$W(\lambda) = (\lambda - H)(\lambda - H_0)^{-1} = I_{\mathcal{H}_{n+m}} + V(\lambda - H_0)^{-1},$$

then for $\lambda \in \mathbb{C}\setminus\mathbb{R}$

$$[W(\lambda) - I_{\mathcal{H}_{n+m}}]^2 = \left[V(\lambda - H_0)^{-1}V\right](\lambda - H_0)^{-1}$$

is a Hilbert-Schmidt operator. Thus $W(\lambda) - I_{\mathcal{H}_{n+m}}$ has a compact square and hence is a Riesz operator [86, Sec. 9.6], implying that $W(\lambda)$ is a Fredholm operator of index zero. Since Lemma 2.3 also implies that $\|W(\lambda) - I_{\mathcal{H}_{n+m}}\|$ vanishes as $|\operatorname{Im}\lambda| \to \infty$, it follows from the analyticity of $W(\lambda)$ as a function of $\lambda \in \mathbb{C}\setminus\mathbb{R}$ that $W(\lambda)$ is boundedly invertible for every nonreal λ, except in a set of points whose only accumulation points are real. Furthermore, these exceptional points are isolated eigenvalues of W of finite algebraic multiplicity ([60], Theorem I 5.1).

Now note that W and $\lambda - H$ are obviously equivalent operator functions for $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Hence, the nonreal points of noninvertibility of $W(\lambda)$ are exactly the points of the nonreal spectrum of H. Consequently, the nonreal spectrum of H consists exclusively of isolated eigenvalues of finite algebraic multiplicity.

We have proved the following

Theorem 2.5 *Let the elements of the potential V belong to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then the nonreal spectrum of H only consists of eigenvalues of finite algebraic multiplicity which can only accumulate on the real line.*

2.4.2 General integrable potentials

In the case of the Schrödinger equation it is well-known [64, 82] that the Hamiltonian H is selfadjoint and has the same domain as the free Hamiltonian H_0 if the potential V is real and belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Under more general conditions on real potentials V one usually introduces sesquilinear forms to prove the selfadjointness of H and to specify its domain. This method can in principle be used in the case of the Hamiltonian of the matrix Zakharov-Shabat system with arbitrary L^1-potential. Since we are only interested in proving Theorem 2.5 for arbitrary L^1-potentials, we proceed along a much shorter path towards the domain of H than in [64, 82].

Let us now define $\operatorname{sgn}(V(x))$ as the partial isometry in the polar decomposition [25, 46] of the matrix $V(x)$.

Lemma 2.6 *Let k be a complex $n \times m$ matrix and l a complex $m \times n$ matrix, and let*

$$V = \begin{pmatrix} 0_{n\times n} & k \\ l & 0_{m\times m} \end{pmatrix}.$$

Then there exists a nonnegative selfadjoint $(n+m)\times(n+m)$ matrix V and a partial isometry U such that $V = U|V| = U(|V|^{1/2})^2$, where $|V| = |l| \oplus |k|$, $|V|^{1/2} = |l|^{1/2} \oplus |k|^{1/2}$, and $|l|$ and $|k|$ are nonnegative selfadjoint matrices.

Proof. Let $k = U_1|k|$ and $l = U_2|l|$ be polar decompositions of k and l, where the $n \times m$ matrix U_1 is the partial isometry defined by $U_1(k^*k)^{1/2} = k$ from the range of $|k| \stackrel{\text{def}}{=} (k^*k)^{1/2}$ onto the range of k and by zero on the kernel of k, and the $m \times n$ matrix U_2 is the partial isometry defined by $U_2(l^*l)^{1/2} = l$ from the range of $|l| \stackrel{\text{def}}{=} (l^*l)^{1/2}$ onto the range of l and by zero on the kernel of l. Putting

$$U = \begin{pmatrix} 0_{n\times n} & U_1 \\ U_2 & 0_{m\times m} \end{pmatrix}, \qquad |V| = \begin{pmatrix} |l| & 0_{n\times m} \\ 0_{m\times n} & |k| \end{pmatrix},$$

we get $V = U|V|$ with U a partial isometry, as claimed. We now define $|V|^{1/2} = |l|^{1/2} \oplus |k|^{1/2}$. ■

Now let V be an arbitrary potential whose elements belong to $L^1(\mathbb{R})$. Applying Lemma 2.6 we can write

$$V(x) = U(x)|V(x)|^{1/2}|V(x)|^{1/2},$$

where the values of the $(n+m) \times (n+m)$ matrix function $U(x)$ are partial isometries and hence have at most unit norm and the elements of $|V(x)|^{1/2}$ belong to $L^2(\mathbb{R})$. Also the elements of $U(x)|V(x)|^{1/2}$ belong to $L^2(\mathbb{R})$.

Instead of W, we now define

$$\tilde{W}(\lambda) = I_{\mathcal{H}_{n+m}} + |V|^{1/2}(\lambda - H_0)^{-1}U|V|^{1/2}, \tag{2.5}$$

where $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Then, according to Lemma 2.3, the operator $\tilde{W}(\lambda) - I$ is an integral operator of Hilbert-Schmidt type which vanishes in the Hilbert-Schmidt norm as $|\mathrm{Im}\,\lambda| \to \infty$. Thus $\tilde{W}(\lambda)$ is boundedly invertible on $\mathcal{H}_{n+m}$ for $|\mathrm{Im}\,\lambda|$ large enough. Using Theorem I 5.1 of [60] it follows that $\tilde{W}(\lambda)$ is invertible on $\mathcal{H}_{n+m}$ for all nonreal λ, with the exception of a set of points which can only accumulate on the real line. Moreover, these exceptional points of $\tilde{W}$ are isolated eigenvalues of finite algebraic multiplicity.

If we are in the antisymmetric case and $n = 1$, the above polar decomposition has been given before by Klaus [65] who employed its factors to arrive at eq. (2.5) . In fact, in the symmetric and antisymmetric cases (i.e., if $l(x) = \pm k(x)^*$) we have

$$\begin{aligned} U(x) &= \frac{1}{\rho(x)^{1/2}} \begin{pmatrix} 0_{n\times n} & k(x) \\ \pm k(x)^* & 0_{m\times m} \end{pmatrix}, \\ |V(x)| &= \frac{1}{\rho(x)^{3/2}} \begin{pmatrix} \rho(x)^2 & 0_{1\times m} \\ 0_{m\times 1} & k(x)^*k(x) \end{pmatrix}, \end{aligned}$$

where $\rho(x) = \|k(x)\|$, the Euclidean norm of $k(x)$, is assumed nonzero.

To define the domain of H (or, equivalently, the range of $(\lambda I_{\mathcal{H}_{n+m}} - H)^{-1}$ for nonreal λ outside the discrete set of points of noninvertibility of $\tilde{W}(\lambda)$), we depart from the identity

$$\begin{aligned} (\lambda I_{\mathcal{H}_{n+m}} - H)^{-1} &= (\lambda I_{\mathcal{H}_{n+m}} - H_0)^{-1} \\ &\quad - (\lambda I_{\mathcal{H}_{n+m}} - H_0)^{-1}\mathrm{sgn}(V)|V|^{1/2}\tilde{W}(\lambda)^{-1}|V|^{1/2}(\lambda I_{\mathcal{H}_{n+m}} - H_0)^{-1}, \end{aligned} \tag{2.6}$$

where all ingredients of eq. (2.6) have to be specified separately. In fact, for those $\lambda \in \mathbb{C}\setminus\mathbb{R}$ for which $\tilde{W}(\lambda)$ is invertible, the resolvent difference $D(\lambda) = (\lambda - H)^{-1} - (\lambda - H_0)^{-1}$ is bounded on $\mathcal{H}_{n+m}$, as indicated by the following diagram:

$$\begin{array}{ccccc} \mathcal{H}_{n+m} & \xrightarrow{(\lambda I_{\mathcal{H}_{n+m}} - H_0)^{-1}} & \mathcal{H}^1_{n+m} & \xrightarrow{|V|^{1/2}} & \mathcal{H}_{n+m} \\ {\scriptstyle D(\lambda)}\downarrow & & & & \downarrow{\scriptstyle \tilde{W}(\lambda)^{-1}} \\ \mathcal{H}_{n+m} & \xleftarrow[-(\lambda - H_0)^{-1}]{} & \mathcal{H}^{-1}_{n+m} & \xleftarrow[\mathrm{sgn}(V)|V|^{1/2}]{} & \mathcal{H}_{n+m} \end{array}$$

In fact, the expression (2.6) defines the resolvent of the Hamiltonian operator H, as proved in the following

Lemma 2.7 *For $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$ we have the resolvent identity*

$$(\lambda - H)^{-1} - (\mu - H)^{-1} = (\mu - \lambda)(\lambda - H)^{-1}(\mu - H)^{-1}.$$

Moreover, the expression (2.6) *defines the resolvent of an unbounded* (*but closed and densely defined*) *linear operator on $\mathcal{H}_{n+m}$.*

Proof. Indeed, if $\tilde{W}(\lambda)$ and $\tilde{W}(\mu)$ are invertible and $\lambda \neq \mu$ we simply substitute (2.6) for λ and μ into the right-hand side of $(\mu - \lambda)(\lambda - H)^{-1}(\mu - H)^{-1}$ and employ the resolvent identity

$$(\lambda - H_0)^{-1} - (\mu - H_0)^{-1} = (\mu - \lambda)(\lambda - H_0)^{-1}(\mu - H_0)^{-1}$$

to obtain $(\lambda - H)^{-1} - (\mu - H)^{-1}$. To prove that the left-hand side of (2.6) really is a resolvent, we choose $\phi \in \mathcal{H}_{n+m}$ such that $(\lambda - H)^{-1}\phi = 0$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ where $\tilde{W}(\lambda)$ is invertible. Then

$$\begin{aligned} 0 = |V|^{1/2}(\lambda - H)^{-1}\phi &= |V|^{1/2}(\lambda - H_0)^{-1} - [\tilde{W}(\lambda) - I]\tilde{W}(\lambda)^{-1}|V|^{1/2}(\lambda - H_0)^{-1}\phi \\ &= \tilde{W}(\lambda)^{-1}|V|^{1/2}(\lambda - H_0)^{-1}\phi, \end{aligned}$$

and hence

$$\begin{aligned} (\lambda - H_0)^{-1}\phi &= (\lambda - H)^{-1}\phi \\ &\quad + (\lambda - H_0)^{-1}\mathrm{sgn}(V)|V|^{1/2}\mathrm{W}(\lambda)^{-1}|V|^{1/2}(\lambda - H_0)^{-1}\phi = 0, \end{aligned}$$

which implies $\phi = 0$. Using the same argument on the adjoint of (2.6), we see that $(\lambda - H)^{-1}$ has a zero kernel and a dense range. Indeed, we should replace $\tilde{W}(\lambda)$ and $(\lambda - H)^{-1}$ by

$$\tilde{W}(\overline{\lambda})^* = I_{\mathcal{H}_{n+m}} + |V|^{1/2}U^*(\lambda - H_0)^{-1}|V|^{1/2}$$

and

$$\begin{aligned} [(\overline{\lambda} - H)^{-1}]^* &= (\lambda - H_0)^{-1} \\ &\quad - (\lambda - H_0)^{-1}|V|^{1/2}[\tilde{W}(\overline{\lambda})^{-1}]^*|V|^{1/2}[\mathrm{sgn}(V)]^*(\lambda - H_0)^{-1}. \end{aligned}$$

The resolvent identity then implies that this kernel and range do not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Therefore, we may define $\lambda - H$ as its unbounded inverse defined on the range of $(\lambda - H)^{-1}$. ■

A second diagram shows that the square of $D(\lambda) = (\lambda - H)^{-1} - (\lambda - H_0)^{-1}$ is a Hilbert-Schmidt operator on $\mathcal{H}_{n+m}$. More precisely,

$$\begin{array}{ccccccc} \mathcal{H}_{n+m} & \xrightarrow{(\lambda - H_0)^{-1}} & \mathcal{H}^1_{n+m} & \xrightarrow{|V|^{1/2}} & \mathcal{H}_{n+m} & \xrightarrow{\tilde{W}(\lambda)^{-1}} & \mathcal{H}_{n+m} \\ {\scriptstyle D(\lambda)^2}\downarrow & & & & & & \downarrow{\scriptstyle S} \\ \mathcal{H}_{n+m} & \xleftarrow[(\lambda - H_0)^{-1}]{} & \mathcal{H}^{-1}_{n+m} & \xleftarrow[\mathrm{sgn}(V)|V|^{1/2}]{} & \mathcal{H}_{n+m} & \xleftarrow[\tilde{W}(\lambda)^{-1}]{} & \mathcal{H}_{n+m} \end{array}$$

where $S = |V|^{1/2}(\lambda - H_0)^{-2}\mathrm{sgn}(V)|V|^{1/2}$. By Lemma 2.4, S is Hilbert-Schmidt.

We have shown that $\lambda - H$ is invertible whenever $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $\tilde{W}(\lambda)$ is invertible. Thus the nonreal spectrum of H is contained in $\{\lambda \in \mathbb{C} \setminus \mathbb{R} : \tilde{W}(\lambda) \text{ is not invertible}\}$. Consequently,

Theorem 2.8 *Let the elements of the potential V belong to $L^1(\mathbb{R})$. Then the nonreal spectrum of H only consists of eigenvalues of finite algebraic multiplicity which can only accumulate on the real line.*

We have not proved that $\{\lambda \in \mathbb{C} \setminus \mathbb{R} : \tilde{W}(\lambda) \text{ is not invertible}\}$ coincides with the set of nonreal eigenvalues of H. We have not related the nonreal eigenvalues of H to the points of noninvertibility of $W(\lambda) = (\lambda - H_0)^{-1}(\lambda - H)$ either. It is easily seen that $\mathcal{D}(H) = \mathcal{D}(H_0)$ if the entries of V belong to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

2.5 Symmetries of the Hamiltonian operator

In Chapter 6 we shall derive explicit solutions of the inverse problem for the matrix Zakharov-Shabat system and for the matrix nonlinear Schrödinger equation for $x \in \mathbb{R}^+$, without even bothering about the negative half-line. The rationale is that a treatment for $x \in \mathbb{R}^-$ is analogous to such an extent that a repetition of the derivations involved is unnecessary. Another approach is to introduce symmetry relations that allow one to pass from a treatment for $x \in \mathbb{R}^+$ to a treatment for $x \in \mathbb{R}^-$ and vice versa. In this section we therefore give an overview of the various symmetries of the free Hamiltonian H_0 and the full Hamiltonian H for an arbitrary L^1-potential. In Appendix B we shall discuss these symmetry relations in more detail.

1. The symmetric and antisymmetric cases. In the symmetric case ($l(x) = k(x)^*$) and the antisymmetric case ($l(x) = -k(x)^*$) the potential V and the Hamiltonian H have the following properties:

$$\begin{cases} V(x)^* = V(x) \text{ and } H^* = H, & \text{symmetric case,} \\ V(x)^* = -V(x) \text{ and } (JH)^* = JH, & \text{antisymmetric case.} \end{cases} \tag{2.7}$$

Thus in the symmetric case H is selfadjoint on $\mathcal{H}_{n+m}$, which implies that its spectrum coincides with the full real line and there do not exist any discrete eigenvalues. In the antisymmetric case H is selfadjoint with respect to the indefinite scalar product

$$[f, g]_0 = \langle Jf, g \rangle = \int_{-\infty}^{\infty} dx \left(\langle f^{\text{up}}(x), g^{\text{up}}(x) \rangle - \langle f^{\text{dn}}(x), g^{\text{dn}}(x) \rangle \right). \tag{2.8}$$

As a result [42, p. 80], the eigenvalues of H are located symmetrically with respect to the real line in the sense that if $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is an eigenvalue of H, also $\overline{\lambda}$ is an eigenvalue of H having the same Jordan structure.

2. Changing the sign of $x \in \mathbb{R}$: $V(-x)$ as a potential. For any function W of $x \in \mathbb{R}$ we define the function $W^{(\#)}$ by

$$W^{(\#)}(x) = W(-x). \tag{2.9}$$

Defining the unitary and selfadjoint operator U on $\mathcal{H}_{n+m}$ by

$$(Uf)(x) = f^{(\#)}(x) = f(-x), \qquad x \in \mathbb{R}, \tag{2.10}$$

we easily obtain the symmetry relations

$$H_0(UJ) = -(UJ)H_0, \qquad H(UJ) = -(UJ)H^{(\#)}, \tag{2.11}$$

where $H^{(\#)} = -iJ(d/dx) - V^{(\#)}$ is the Hamiltonian corresponding to the potential $V^{(\#)}$. In deriving (2.11) we have made use of the commutation relations $UJ = JU$ and $H_0 J = JH_0$ and the anticommutation relations $JV = -VJ$ (which actually means that V has zero diagonal $n \times n$ and $m \times m$ blocks) and $UH_0 = -H_0 U$. As a result of (2.11), the discrete eigenvalues of $H^{(\#)}$ are obtained from those of H (including their Jordan structure) by multiplying them by -1.

3. Changing the sign of $x \in \mathbb{R}$: $-V(-x)$ **as a potential**. For any function W of $x \in \mathbb{R}$ we define the function $W^{[\#]}$ by

$$W^{[\#]}(x) = -W(-x). \tag{2.12}$$

Using (2.10) we easily obtain the symmetry relations

$$H_0 U = -UH_0, \qquad HU = -UH^{[\#]}, \tag{2.13}$$

where $H^{[\#]} = -iJ(d/dx) - V^{[\#]}$ is the Hamiltonian corresponding to the potential $V^{[\#]}$. In deriving (2.13) we have made use of the anticommutation relations $JV = -VJ$ and $UH_0 = -H_0 U$. As a result of (2.13), the discrete eigenvalues of $H^{[\#]}$ are obtained from those of H (including their Jordan structure) by multiplying them by -1.

It is now clear that in the antisymmetric case the discrete spectrum of H is symmetric with respect to both the real and the imaginary axis if the potential is an even or an odd function of x.

Here we summarize spectral symmetries in the following table. A profound study of the effect certain symmetries of H have on its spectrum for the Zakharov-Shabat ($n = m = 1$) and Manakov ($n = 1$ and $m = 2$) systems has been made in [66, 65].

Table 2.1: For different symmetries we indicate under which transformation the discrete spectrum of H remains invariant. This invariance regards both location of the eigenvalues and Jordan structure.

antisymmetric case	$(JH)^* = JH$	complex conjugation
even potential	$H^{(\#)} = H$	sign inversion
odd potential	$H^{[\#]} = H$	sign inversion
antisymmetric case with even potential	$(UJH)^* = UJH$	reflection with respect to the imaginary axis
antisymmetric case with odd potential	$(UH)^* = UH$	reflection with respect to the imaginary axis

Chapter 3

Direct Scattering Theory

In this chapter we study the direct scattering theory of the system of differential equations

$$-iJ\frac{dX(x,\lambda)}{dx} - V(x)X(x,\lambda) = \lambda X(x,\lambda), \quad x \in \mathbb{R}, \tag{3.1}$$

where

$$J = \begin{pmatrix} I_n & 0_{n\times m} \\ 0_{m\times n} & -I_m \end{pmatrix}, \quad V(x) = \begin{pmatrix} 0_{n\times n} & k(x) \\ l(x) & 0_{m\times m} \end{pmatrix}. \tag{3.2}$$

Here the $n \times m$ matrix function k and the $m \times n$ matrix function l have complex-valued entries belonging to $L^1(\mathbb{R})$ and $\lambda \in \mathbb{R}$ is an eigenvalue parameter. We call the function V the *potential matrix* (or *potential*), k and l *potentials* and the parameter λ the *energy*. Note that $V(x)$ satisfies

$$JV(x) = -V(x)J.$$

As before we distinguish the following two special cases:

- *symmetric potentials*: $l(x) = k(x)^*$, or $V(x) = V(x)^*$, where the asterisk superscript denotes the matrix conjugate transpose;
- *antisymmetric potentials*: $l(x) = -k(x)^*$, or $JV(x) = JV(x)^*$.

For $n = m = 1$ we have the so-called Zakharov-Shabat system and for $n = 1$ and $m = 2$ the Manakov system.

In the symmetric and antisymmetric cases the direct and inverse scattering theory of (3.1) has been developed in [9] and [90], respectively, the latter only if there are no bound states. Here we extend the formalism and results of [9, 90] to the matrix Zakharov-Shabat system with general L^1 potentials with and without bound states, filling up some of the questions left unanswered in these papers. In part we cover the same material as in Chapter 4 of [3], proving many of the statements left unproved there and greatly improving the treatment of the bound state norming constants, where we rely on Section B.3 of Appendix B for some of the technical aspects involving symmetry. In Section A.1 of Appendix A we compare their formalism with ours in detail.

After introducing the Jost solutions and Faddeev functions and deriving their continuity and analyticity properties in Sections 3.1 and 3.2, we represent these functions as Fourier transforms of L^1 functions in Section 3.3. In Section 3.4 we introduce the reflection and transmission coefficients and study their properties. In Section 3.5 we relate the spectral properties of the Hamiltonian operator H to those of the inverses of the transmission coefficients, with full account of Jordan structure. Finally, in Section 3.6 we derive Wiener-Hopf factorization results for certain matrix functions built from a reflection coefficient in the symmetric and antisymmetric cases. These results are used to construct the scattering matrix from one of the reflection coefficients.

3.1 Jost solutions

In this section we define the Jost solutions of (3.1), i.e., the solutions proportional to the free solutions $e^{i\lambda Jx}$ as either $x \to +\infty$ or $x \to -\infty$. In [3] the Jost solutions are either $(n+m) \times n$ matrix functions or $(n+m) \times m$ matrix functions. We shall instead define them as square matrix functions of order $n+m$. Apart from trivial notational changes, our Jost solutions are composed of those defined in [3] by arranging them as columns in a square matrix function of order $n+m$. For details on the distinction between our notational system and that of [3] we refer to Section A.1.

For $\lambda \in \mathbb{R}$, we define the *Jost solution from the left*, $F_l(x,\lambda)$, and the *Jost solution from the right*, $F_r(x,\lambda)$, as the $(n+m) \times (n+m)$ matrix solutions of (3.1) satisfying the asymptotic conditions

$$F_l(x,\lambda) = e^{i\lambda Jx}[I_{n+m} + o(1)], \quad x \to +\infty, \tag{3.1a}$$

$$F_r(x,\lambda) = e^{i\lambda Jx}[I_{n+m} + o(1)], \quad x \to -\infty. \tag{3.1b}$$

Using (3.1), (3.1a) and (3.1b), we obtain the Volterra integral equations

$$F_l(x,\lambda) = e^{i\lambda Jx} - iJ\int_x^\infty dy e^{-i\lambda J(y-x)}V(y)F_l(y,\lambda), \tag{3.2a}$$

$$F_r(x,\lambda) = e^{i\lambda Jx} + iJ\int_{-\infty}^x dy e^{-i\lambda J(y-x)}V(y)F_r(y,\lambda). \tag{3.2b}$$

Since the entries of $k(x)$ and $l(x)$ belong to $L^1(\mathbb{R})$, for each fixed $\lambda \in \mathbb{R}$ it follows by iteration that (3.2a) and (3.2b) are uniquely solvable and hence that the Jost solutions exist uniquely. Since $F_l(x,\lambda)$ and $F_r(x,\lambda)$ are solutions of a first order linear homogeneous differential equation and hence the columns of one are linear combinations of the columns of the other, for each $\lambda \in \mathbb{R}$ there exist $(n+m) \times (n+m)$ matrices $a_l(\lambda)$ and $a_r(\lambda)$ such that

$$a_l(\lambda)F_r(x,\lambda) = F_l(x,\lambda), \tag{3.3a}$$

$$a_r(\lambda)F_l(x,\lambda) = F_r(x,\lambda). \tag{3.3b}$$

Thus, from (3.1)-(3.2) we have

$$F_l(x,\lambda) = e^{i\lambda Jx}[a_l(\lambda) + o(1)], \quad x \to -\infty, \tag{3.4a}$$

$$F_r(x,\lambda) = e^{i\lambda Jx}[a_r(\lambda) + o(1)], \quad x \to +\infty, \tag{3.4b}$$

where

$$a_l(\lambda) = I_{n+m} - iJ\int_{-\infty}^{\infty} dy\, e^{-i\lambda Jy}V(y)F_l(y,\lambda), \tag{3.5a}$$

$$a_r(\lambda) = I_{n+m} + iJ\int_{-\infty}^{\infty} dy\, e^{-i\lambda Jy}V(y)F_r(y,\lambda). \tag{3.5b}$$

We have

Proposition 3.1 *For $\lambda \in \mathbb{R}$ the matrices $a_l(\lambda)$ and $a_r(\lambda)$ appearing in* (3.4a) *and* (3.4b)*, respectively, satisfy*

$$a_l(\lambda)a_r(\lambda) = a_r(\lambda)a_l(\lambda) = I_{n+m}, \tag{3.6}$$

where

$$\det a_l(\lambda) = \det a_r(\lambda) = 1. \tag{3.7}$$

Proof. At first, we prove eq. (3.6). We have

$$F_l(x,\lambda) = \begin{cases} e^{i\lambda Jx}[a_l(\lambda) + o(1)], & x \to -\infty, \\ e^{i\lambda Jx}[I_{n+m} + o(1)], & x \to +\infty, \end{cases}$$

and

$$F_r(x,\lambda) = \begin{cases} e^{i\lambda Jx}[I_{n+m} + o(1)], & x \to -\infty, \\ e^{i\lambda Jx}[a_r(\lambda) + o(1)], & x \to +\infty. \end{cases}$$

As a result of (3.3), we obtain $F_l(x,\lambda) = a_l(\lambda)F_r(x,\lambda) = a_l(\lambda)a_r(\lambda)F_l(x,\lambda)$, which implies $a_l(\lambda)a_r(\lambda) = I_{n+m}$. Proceeding in a similar way, but starting from the identity $F_r(x,\lambda) = a_r(\lambda)F_l(x,\lambda) = a_r(\lambda)a_l(\lambda)F_r(x,\lambda)$, we find $a_r(\lambda)a_l(\lambda) = I_{n+m}$.

In order to prove eq. (3.7), we observe that

$$\begin{aligned} \frac{d(e^{-i\lambda Jx}F_l(x,\lambda))}{dx} &= e^{-i\lambda Jx}\frac{d(F_l(x,\lambda))}{dx} - i\lambda J e^{-i\lambda Jx}F_l(x,\lambda) \\ &= e^{-i\lambda Jx} iJ\left[V(x) + \lambda I_{n+m}\right]F_l(x,\lambda) - i\lambda J e^{-i\lambda Jx}F_l(x,\lambda) \\ &= e^{-i\lambda Jx} iJV(x)F_l(x,\lambda), \end{aligned}$$

which implies (see, for example [81])

$$\frac{d\det(e^{-i\lambda Jx}F_l(x,\lambda))}{dx} = \mathrm{tr}(e^{-i\lambda Jx}iJV(x))\det(F_l(x,\lambda)),$$

where tr denotes the matrix trace. By eq. (3.2), $e^{-i\lambda Jx}iJV(x)$ has zero trace, and hence $\det(e^{-i\lambda Jx}F_l(x,\lambda))$ is independent of x. Because of the relations $\lim_{x\to-\infty} e^{-i\lambda Jx}F_l(x,\lambda) = a_l(\lambda)$ and $\lim_{x\to+\infty} e^{-i\lambda Jx}F_l(x,\lambda) = I_{n+m}$, we have $\det(a_l(\lambda)) = \det(I_{n+m}) = 1$. Proceeding in a similar way for $e^{-i\lambda Jx}F_r(x,\lambda)$, we find $\det(a_r(\lambda)) = \det(I_{n+m}) = 1$. ■

Proposition 3.2 *Let $X(x,\lambda)$ and $Y(x,\lambda)$ be any two solutions of* (3.1)*, and let $\lambda \in \mathbb{R}$. Then for symmetric potentials the matrix $X(x,\lambda)^* J Y(x,\lambda)$ is independent of x and*

$$a_l(\lambda)^{-1} = J a_l(\lambda)^* J, \quad a_r(\lambda)^{-1} = J a_r(\lambda)^* J; \tag{3.8}$$

in particular, $a_l(\lambda)$ and $a_r(\lambda)$ are J-unitary matrices. For antisymmetric potentials the matrix $X(x,\lambda)^* Y(x,\lambda)$ is independent of x and

$$a_l(\lambda)^{-1} = a_l(\lambda)^*, \quad a_r(\lambda)^{-1} = a_r(\lambda)^*; \tag{3.9}$$

in particular, $a_l(\lambda)$ and $a_r(\lambda)$ are unitary matrices.

Proof. If we differentiate $X(x,\lambda)^* Y(x,\lambda)$ ($X(x,\lambda)^* J Y(x,\lambda)$, respectively) and use (3.1) and the selfadjointness of $JV(x)$ (in the antisymmetric case) and $V(x)$ (in the symmetric case), we obtain that the matrix $X(x,\lambda)^* Y(x,\lambda)$ ($X(x,\lambda)^* J Y(x,\lambda)$, respectively) does not depend on x.

Now we prove eq. (3.9) (the proof of (3.8) is very similar). From Proposition (3.1) we have $a_r(\lambda) F_l(x,\lambda) = F_r(x,\lambda)$. So it is not difficult to see that in the antisymmetric case $F_l(x,\lambda)^* F_l(x,\lambda) = a_l(\lambda)^* a_l(\lambda) = I_{n+m}$ as $x \to \pm\infty$ and, analogously, $F_r(x,\lambda)^* F_r(x,\lambda) = a_r(\lambda)^* a_r(\lambda) = I_{n+m}$. Then eq. (3.9) readily follows. ■

3.2 Faddeev matrices

In this section we introduce the Faddeev matrices $M_l(x,\lambda)$ and $M_r(x,\lambda)$ in terms of the Jost solutions (3.1a). Using a decomposition of these matrices in suitable blocks, we study the analyticity properties of these functions. Moreover, we study the analyticity properties of the analogous submatrices of $a_l(\lambda)$ and $a_r(\lambda)$.

Let us define the *Faddeev matrices* $M_l(x,\lambda)$ and $M_r(x,\lambda)$ as follows:

$$M_l(x,\lambda) = F_l(x,\lambda) e^{-i\lambda J x}, \quad M_r(x,\lambda) = F_r(x,\lambda) e^{-i\lambda J x}. \tag{3.10}$$

From (3.1a) and (3.1b) we get

$$M_l(x,\lambda) = I_{n+m} + o(1), \quad x \to +\infty, \tag{3.11a}$$
$$M_r(x,\lambda) = I_{n+m} + o(1), \quad x \to -\infty. \tag{3.11b}$$

Let us partition the Jost solutions and Faddeev matrices in the following way:

$$F_l(x,\lambda) = \begin{pmatrix} F_{l1}(x,\lambda) & F_{l2}(x,\lambda) \\ F_{l3}(x,\lambda) & F_{l4}(x,\lambda) \end{pmatrix}, \quad F_r(x,\lambda) = \begin{pmatrix} F_{r1}(x,\lambda) & F_{r2}(x,\lambda) \\ F_{r3}(x,\lambda) & F_{r4}(x,\lambda) \end{pmatrix}, \tag{3.12}$$

$$M_l(x,\lambda) = \begin{pmatrix} M_{l1}(x,\lambda) & M_{l2}(x,\lambda) \\ M_{l3}(x,\lambda) & M_{l4}(x,\lambda) \end{pmatrix}, M_r(x,\lambda) = \begin{pmatrix} M_{r1}(x,\lambda) & M_{r2}(x,\lambda) \\ M_{r3}(x,\lambda) & M_{r4}(x,\lambda) \end{pmatrix}, \tag{3.13}$$

where $F_{l1}, F_{r1}, M_{l1}, M_{r1}$ are $n \times n$ matrices, F_{l2}, F_{r2}, M_{l2}, and M_{r2} are $n \times m$ matrices, F_{l3}, F_{r3}, M_{l3}, and M_{r3} are $m \times n$ matrices and F_{l4}, F_{r4}, M_{l4}, and M_{r4} are $m \times m$ matrices. We shall adopt this type of partitioning of $(n+m) \times (n+m)$ matrices into blocks labeled $1, 2, 3, 4$ throughout this thesis.

Before studying the above analyticity properties of submatrices of the Faddeev matrices, we recall the subsequent (cf. e.g. [80]).

Lemma 3.3 (Gronwall) *Let $\rho \in L^1(\mathbb{R})$ be nonnegative, H a bounded positive function, and F a measurable function satisfying*

$$0 \leq F(x) \leq H(x) + \int_x^{+\infty} dy\, \rho(y) F(y).$$

Then

$$0 \leq F(x) \leq H(x) \exp\left\{ \int_x^{+\infty} dy\, H(y)\rho(y) \right\}. \tag{3.14}$$

Proof. Let us first consider the case $H(x) \equiv 1$. We observe that $G(x) = \exp\{\int_x^\infty dy \rho(y)\}$ is the exact solution of the integral equation $G(x) = 1 + \int_x^\infty dy \rho(y) G(y)$. This solution is easily obtained by iteration:

$$G_0(x) = 1, \quad G_{n+1}(x) = 1 + \int_x^\infty dy \rho(y) G_n(y).$$

Hence, any nonnegative function F such that $F(x) \leq 1 + \int_x^{+\infty} dy \rho(y) F(y)$, satisfies $0 \leq F(x) \leq G(x)$.

Let us consider the general case. Dividing eq. (3.14) by $H(x)$ and putting $\tilde{\rho}(y) = \rho(y) H(y)$, we have $0 \leq (F(x)/H(x)) \leq \exp\{\int_x^{+\infty} dy \tilde{\rho}(y)\}$, which implies the lemma. ■

We now analyze the Faddeev matrix $M_l(x, \lambda)$.

Proposition 3.4 *Assume that the entries of $k(x)$ and $l(x)$ belong to $L^1(\mathbb{R})$. Then the following statements are true*:

1. *For each fixed $x \in \mathbb{R}$, $M_{l1}(x, \lambda)$ and $M_{l3}(x, \lambda)$ can be extended to matrix functions that are continuous in $\lambda \in \overline{\mathbb{C}^+}$ and analytic in $\lambda \in \mathbb{C}^+$. Moreover, $M_{l1}(x, \lambda)$ tends to I_n as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$ and $M_{l3}(x, \lambda)$ to $0_{m\times n}$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$.*

2. *For all $\lambda \in \overline{\mathbb{C}^+}$, the matrix functions $M_{l1}(x, \lambda)$ and $M_{l3}(x, \lambda)$ are bounded in the norm by $\exp\{\int_x^{+\infty} dy\, \max(\|k(y)\|, \|l(y)\|)\}$.*

3. *For each fixed $x \in \mathbb{R}$, $M_{l2}(x, \lambda)$ and $M_{l4}(x, \lambda)$ can be extended to matrix functions that are continuous in $\lambda \in \overline{\mathbb{C}^-}$ and analytic in $\lambda \in \mathbb{C}^-$. Moreover, $M_{l4}(x, \lambda)$ tends to I_m as $\lambda \to \infty$ in $\overline{\mathbb{C}^-}$ and $M_{l4}(x, \lambda)$ to $0_{n\times m}$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^-}$.*

4. *For all $\lambda \in \overline{\mathbb{C}^-}$, the matrix functions $M_{l2}(x, \lambda)$ and $M_{l4}(x, \lambda)$ are bounded in the norm by $\exp\{\int_x^{+\infty} dy\, \max(\|k(y)\|, \|l(y)\|)\}$.*

Proof. Using eq. (3.10) in (3.2a), we obtain

$$\begin{aligned} M_l(x, \lambda) &= I_{n+m} - iJ \int_x^\infty dy\, e^{-i\lambda J(y-x)} V(y) F_l(y, \lambda) e^{-i\lambda J x} \\ &= I_{n+m} - iJ \int_x^\infty dy\, e^{-i\lambda J(y-x)} V(y) M_l(y, \lambda) e^{i\lambda J(y-x)}. \end{aligned}$$

But, using eqs. (3.2), (3.10) and because of

$$e^{i\lambda J(y-x)} = \begin{pmatrix} e^{i\lambda(y-x)}I_n & 0_{n\times m} \\ 0_{m\times n} & e^{i\lambda(x-y)}I_m \end{pmatrix},$$

we can write

$$\begin{aligned}\begin{pmatrix} M_{l1}(x,\lambda) & M_{l2}(x,\lambda) \\ M_{l3}(x,\lambda) & M_{l4}(x,\lambda) \end{pmatrix} &= \begin{pmatrix} I_n & 0_{n\times m} \\ 0_{m\times n} & I_m \end{pmatrix} - i\int_x^\infty dy \begin{pmatrix} e^{i\lambda(x-y)}I_n & 0_{n\times m} \\ 0_{m\times n} & -e^{i\lambda(y-x)}I_m \end{pmatrix}\begin{pmatrix} 0_n & k(y) \\ l(y) & 0_m \end{pmatrix}\times \\ &\times \begin{pmatrix} M_{l1}(y,\lambda) & M_{l2}(y,\lambda) \\ M_{l3}(y,\lambda) & M_{l4}(y,\lambda) \end{pmatrix}\begin{pmatrix} e^{i\lambda(y-x)}I_n & 0_{n\times m} \\ 0_{m\times n} & e^{i\lambda(x-y)}I_m \end{pmatrix} = \\ &= \begin{pmatrix} I_n & 0_{n\times m} \\ 0_{m\times n} & I_m \end{pmatrix} - i\int_x^\infty dy \begin{pmatrix} e^{i\lambda(x-y)}I_n & 0_{n\times m} \\ 0_{m\times n} & -e^{i\lambda(y-x)}I_m \end{pmatrix}\times \\ &\times \begin{pmatrix} k(y)M_{l3}(y,\lambda) & k(y)M_{l4}(y,\lambda) \\ l(y)M_{l1}(y,\lambda) & l(y)M_{l2}(y,\lambda) \end{pmatrix}\begin{pmatrix} e^{i\lambda(y-x)}I_n & 0_{n\times m} \\ 0_{m\times n} & e^{i\lambda(x-y)}I_m \end{pmatrix}.\end{aligned}$$

Thus,

$$M_{l1}(x,\lambda) = I_n - i\int_x^\infty dy\, k(y)M_{l3}(y,\lambda), \tag{3.15}$$

$$M_{l2}(x,\lambda) = -i\int_x^\infty dy\, e^{2i\lambda(x-y)}k(y)M_{l4}(y,\lambda), \tag{3.16}$$

$$M_{l3}(x,\lambda) = +i\int_x^\infty dy\, e^{2i\lambda(y-x)}l(y)M_{l1}(y,\lambda), \tag{3.17}$$

$$M_{l4}(x,\lambda) = I_m + i\int_x^\infty dy\, l(y)M_{l2}(y,\lambda). \tag{3.18}$$

Substituting (3.17) in (3.15), we have

$$\begin{aligned} M_{l1}(x,\lambda) &= I_n - i^2\int_x^\infty dy\, k(y)\int_y^\infty dz\, e^{2i\lambda(z-y)}l(z)M_{l1}(z,\lambda) \\ &= I_n + \int_x^\infty dz\, k(z)\int_z^\infty dy\, e^{2i\lambda(y-z)}l(y)M_{l1}(y,\lambda) \\ &= I_n + \int_x^\infty dy\int_x^y dz\, e^{2i\lambda(y-x)}k(z)l(y)M_{l1}(y,\lambda). \end{aligned} \tag{3.19}$$

Proceeding in a similar manner we obtain

$$\begin{aligned} M_{l2}(x,\lambda) = &-i\int_x^\infty dy\, e^{-2i\lambda(y-x)}k(y) \\ &+ \int_x^\infty dy\int_y^\infty dz\, e^{-2i\lambda(y-x)}k(y)l(z)M_{l2}(z,\lambda), \end{aligned} \tag{3.20}$$

$$M_{l3}(x,\lambda) = -i\int_x^\infty dy\, e^{2i\lambda(y-x)} l(y) + \int_x^\infty dy \int_y^\infty dz\, e^{2i\lambda(y-x)} k(z) l(y) M_{l3}(z,\lambda), \tag{3.21}$$

$$M_{l4}(x,\lambda) = I_m + \int_x^\infty dy \int_y^\infty dz\, e^{-2i\lambda(z-y)} k(z) l(y) M_{l4}(z,\lambda). \tag{3.22}$$

In order to derive the estimate in 2) we estimate (by (3.15) and (3.17)) $\|M_{l1}(x,\lambda)\| + \|M_{l3}(x,\lambda)\|$ and find

$$\|M_{l1}(x,\lambda)\| + \|M_{l3}(x,\lambda)\| \le 1 + \int_x^\infty dy\, \max(\|k(y)\|, \|l(y)\|) \times \times (\|M_{l1}(x,\lambda)\| + \|M_{l3}(x,\lambda)\|).$$

Using Gronwall's Lemma we have

$$\|M_{l1}(x,\lambda)\| + \|M_{l3}(x,\lambda)\| \le \exp\left\{\int_x^\infty dy\, \max(\|k(y)\|, \|l(y)\|)\right\},$$

i.e., the estimate in 2). Proceeding in a similar way, we get (by using (3.16) and (3.18))

$$\|M_{l2}(x,\lambda)\| + \|M_{l4}(x,\lambda)\| \le 1 + \int_x^\infty dy\, \max(\|k(y)\|, \|l(y)\|) \times \times (\|M_{l2}(x,\lambda)\| + \|M_{l4}(x,\lambda)\|),$$

and using Gronwall's Lemma, we obtain the estimate in 4).

Iterating the Volterra integral equations (3.19) and (3.21), we prove that the iterates converge absolutely and uniformly in $\lambda \in \overline{\mathbb{C}^+}$; similarly, we prove that the iterates of (3.20) and (3.22) converge absolutely and uniformly in $\lambda \in \overline{\mathbb{C}^-}$.

To prove the assertion concerning the large λ limit we first consider $M_{l2}(x,\lambda)$. Equation (3.16) implies

$$\|M_{l2}(x,\lambda)\| \le \|\omega^{(k)}(\lambda,x)\| \exp\{\int_x^\infty dy\, \max(\|k(y)\|, \|l(y)\|)\},$$

where

$$\omega^{(k)}(\lambda,x) = \int_x^\infty dy\, e^{2i\lambda(x-y)} k(y).$$

Approximating k by $k_q \in [\mathcal{D}(\mathbb{R})]^{n\times m}$, where

$$\mathcal{D}(\mathbb{R}) = \{\phi : \mathbb{R} \to \mathbb{C} : \phi \in C^\infty(\mathbb{R}),\ \phi \text{ has compact support}\},$$

and taking into account that

1. $\omega_q^{(k)}(\lambda,x) \overset{\text{def}}{=} \int_x^\infty dy\, e^{2i\lambda(x-y)} k_q(y) \to 0$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$ (as a result of integration by parts);
2. $\omega_q^{(k)}(\lambda,x)$ tends uniformly to $\omega^{(k)}(\lambda,x)$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$,

we have $\|M_{l2}(x,\lambda)\| \to 0$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$. Proceeding in the same way, we obtain the proof also for $M_{l3}(x,\lambda)$, $M_{r2}(x,\lambda)$ and $M_{r3}(x,\lambda)$. Moreover, if we define $\omega_q^{(l)}(\lambda,x) = \int_x^\infty dy\, e^{2i\lambda(x-y)} l_q(y)$ and proceed as for $M_{l2}(x,\lambda)$, we find

$$\|M_{l4}(x,\lambda) - I_m\| \leq \left(\omega^l(\lambda,x)\, \exp\left\{\int_x^\infty dy\, \max(\|k(y)\|, \|l(y)\|)\right\}\right) \to 0$$

as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$. Analogously, we obtain the proofs for $M_{l1}(x,\lambda)$ and $M_{l3}(x,\lambda)$. ■

We have a similar result for the Faddeev matrix $M_r(x,\lambda)$. We omit its proof.

Proposition 3.5 *Assume that the entries of $k(x)$ and $l(x)$ belong to $L^1(\mathbb{R})$. Then the following statements are true*:

1. *For each fixed $x \in \mathbb{R}$, $M_{r1}(x,\lambda)$ and $M_{r3}(x,\lambda)$ can be extended to matrix functions that are continuous in $\lambda \in \overline{\mathbb{C}^-}$ and analytic in $\lambda \in \mathbb{C}^-$. Moreover, $M_{r1}(x,\lambda)$ tends to I_n as $\lambda \to \infty$ in $\overline{\mathbb{C}^-}$ and $M_{r3}(x,\lambda)$ to $0_{m\times n}$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^-}$.*

2. *For all $\lambda \in \overline{\mathbb{C}^-}$, the matrix functions $M_{r1}(x,\lambda)$ and $M_{r3}(x,\lambda)$ are bounded in the norm by $\exp\{\int_{-\infty}^x dy\, \max(\|k(y)\|, \|l(y)\|)\}$.*

3. *For each fixed $x \in \mathbb{R}$, $M_{r2}(x,\lambda)$ and $M_{r4}(x,\lambda)$ can be extended to matrix functions that are continuous in $\lambda \in \overline{\mathbb{C}^+}$ and analytic in $\lambda \in \mathbb{C}^+$. Moreover, $M_{r4}(x,\lambda)$ tends to I_m as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$ and $M_{r4}(x,\lambda)$ to $0_{n\times m}$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$.*

4. *For all $\lambda \in \overline{\mathbb{C}^+}$, the matrix functions $M_{r2}(x,\lambda)$ and $M_{r4}(x,\lambda)$ are bounded in the norm by $\exp\{\int_{-\infty}^x dy\, \max(\|k(y)\|, \|l(y)\|)\}$.*

Let us now write

$$a_l(\lambda) = \begin{pmatrix} a_{l1}(\lambda) & a_{l2}(\lambda) \\ a_{l3}(\lambda) & a_{l4}(\lambda) \end{pmatrix}, \qquad a_r(\lambda) = \begin{pmatrix} a_{r1}(\lambda) & a_{r2}(\lambda) \\ a_{r3}(\lambda) & a_{r4}(\lambda) \end{pmatrix}. \tag{3.23}$$

From (3.4a) and (3.4b), we see that

$$\begin{pmatrix} a_{l1}(\lambda) & a_{l2}(\lambda) \\ a_{l3}(\lambda) & a_{l4}(\lambda) \end{pmatrix} = \lim_{x\to-\infty} \begin{pmatrix} M_{l1}(x,\lambda) & e^{-2i\lambda x} M_{l2}(x,\lambda) \\ e^{2i\lambda x} M_{l3}(x,\lambda) & M_{l4}(x,\lambda) \end{pmatrix}, \tag{3.24a}$$

$$\begin{pmatrix} a_{r1}(\lambda) & a_{r2}(\lambda) \\ a_{r3}(\lambda) & a_{r4}(\lambda) \end{pmatrix} = \lim_{x\to+\infty} \begin{pmatrix} M_{r1}(x,\lambda) & e^{-2i\lambda x} M_{r2}(x,\lambda) \\ e^{2i\lambda x} M_{r3}(x,\lambda) & M_{r4}(x,\lambda) \end{pmatrix}. \tag{3.24b}$$

Using the following expressions

$$M_l(x,\lambda) = I_{n+m} - iJ \int_x^\infty dy\, e^{-i\lambda J(y-x)} V(y) M_l(y,\lambda) e^{i\lambda J(y-x)}, \tag{3.25a}$$

$$M_r(x,\lambda) = I_{n+m} + iJ \int_{-\infty}^x dy\, e^{i\lambda J(y-x)} V(y) M_l(y,\lambda) e^{-i\lambda J(y-x)}, \tag{3.25b}$$

and eqs. (3.24a) and (3.24b) we find the integral representations

$$a_{l1}(\lambda) = I_n - i\int_{-\infty}^{\infty} dy\, k(y)M_{l3}(y,\lambda), \tag{3.26}$$

$$a_{l2}(\lambda) = -i\int_{-\infty}^{\infty} dy\, e^{-2i\lambda y}k(y)M_{l4}(y,\lambda), \tag{3.27}$$

$$a_{l3}(\lambda) = i\int_{-\infty}^{\infty} dy\, e^{2i\lambda y}l(y)M_{l1}(y,\lambda), \tag{3.28}$$

$$a_{l4}(\lambda) = I_m + i\int_{-\infty}^{\infty} dy\, l(y)M_{l2}(y,\lambda), \tag{3.29}$$

$$a_{r1}(\lambda) = I_n + i\int_{-\infty}^{\infty} dy\, k(y)M_{r3}(y,\lambda), \tag{3.30}$$

$$a_{r2}(\lambda) = i\int_{-\infty}^{\infty} dy\, e^{-2i\lambda y}k(y)M_{r4}(y,\lambda), \tag{3.31}$$

$$a_{r3}(\lambda) = -i\int_{-\infty}^{\infty} dy\, e^{2i\lambda y}l(y)M_{r1}(y,\lambda), \tag{3.32}$$

$$a_{r4}(\lambda) = I_m - i\int_{-\infty}^{\infty} dy\, l(y)M_{r2}(y,\lambda). \tag{3.33}$$

We now present the continuity and analyticity properties of the matrices of $a_{ls}(\lambda)$ and $a_{rs}(\lambda)$, where $s = 1, 2, 3, 4$.

Proposition 3.6 *Assume that the entries of $k(x)$ and $l(x)$ belong to $L^1(\mathbb{R})$. Then the following statements are true:*

1. *The matrices $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are continuous in $\lambda \in \overline{\mathbb{C}^+}$ and analytic in $\lambda \in \mathbb{C}^+$. Moreover, $a_{l1}(\lambda)$ tends to I_n and $a_{r4}(\lambda)$ to I_m as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$.*
2. *The matrices $a_{l4}(\lambda)$ and $a_{r1}(\lambda)$ are continuous in $\lambda \in \overline{\mathbb{C}^-}$ and analytic in $\lambda \in \mathbb{C}^-$. Moreover, $a_{r1}(\lambda)$ tends to I_n and $a_{l4}(\lambda)$ to I_m as $\lambda \to \infty$ in $\overline{\mathbb{C}^-}$.*
3. *The matrices $a_{l3}(\lambda)$ and $a_{r3}(\lambda)$ are continuous in $\lambda \in \mathbb{R}$ and tend to $0_{m\times n}$ as $\lambda \to \pm\infty$.*
4. *The matrices $a_{l2}(\lambda)$ and $a_{r2}(\lambda)$ are continuous in $\lambda \in \mathbb{R}$ and tend to $0_{n\times m}$ as $\lambda \to \pm\infty$.*

Proof. The results follow from (3.26)-(3.33) using Propositions 3.4 and 3.5. ∎

For symmetric and anti-symmetric potentials we have

Proposition 3.7 *For $\lambda \in \mathbb{R}$ the matrices $a_{l2}(\lambda)$, $a_{l3}(\lambda)$, $a_{r2}(\lambda)$, $a_{r4}(\lambda)$ satisfy*

$$a_{r2}(\lambda) = -a_{l3}(\lambda)^*, \quad a_{r3}(\lambda) = -a_{l2}(\lambda)^* \quad (\textit{symmetric case}),$$
$$a_{r2}(\lambda) = a_{l3}(\lambda)^*, \quad a_{r3}(\lambda) = a_{l2}(\lambda)^* \quad (\textit{anti-symmetric case}).$$

Proof. In the symmetric case we obtain the result using $a_r(\lambda)^*J = Ja_l(\lambda)$, whereas in the anti-symmetric case we prove the proposition using $a_r(\lambda)^* = a_l(\lambda)$. ∎

3.3 Representations in the Wiener algebra

In this section we introduce a particularly well-known Banach algebra: the Wiener algebra $\mathcal{W}^q$ (cf. [50, 53, 54]). In order to prove that the Faddeev functions and the coefficients $a_l(\lambda)$ and $a_r(\lambda)$ belong to $\mathcal{W}^q$ for a suitable q, we define the L^1 matrix functions $B_l(x,\lambda)$ and $B_r(x,\lambda)$ that are related to the Faddeev functions by eqs. (3.35) and (3.36). These functions are in turn related to the potentials $k(x)$ and $l(x)$ by eqs. (3.54) and (3.55) to be discussed shortly.

Let $\mathcal{W}^q$ denote the Wiener algebra of all $q \times q$ matrix functions of the form

$$Z(\lambda) = Z_\infty + \int_{-\infty}^{\infty} d\alpha\, z(\alpha) e^{i\lambda\alpha}, \tag{3.34}$$

where $z(\alpha)$ is a $q \times q$ matrix function whose entries belong to $L^1(\mathbb{R})$ and $Z_\infty = Z(\pm\infty)$. Then $\mathcal{W}^q$ is a Banach algebra with unit element endowed with the norm

$$\|Z(\lambda)\|_{\mathcal{W}^q} = \|Z_\infty\| + \int_{-\infty}^{\infty} d\alpha\, \|z(\alpha)\|.$$

By $\mathcal{W}^q_\pm$ we denote the subalgebra of those functions $Z(\lambda)$ for which $z(\alpha)$ has support in $\mathbb{R}^\pm$ and by $\mathcal{W}^q_{\pm,0}$ the subalgebra of those functions $Z(\lambda)$ for which $Z_\infty = 0$ and $z(\alpha)$ has support in $\mathbb{R}^\pm$. Then, as a consequence of the Riemann-Lebesgue lemma, $\mathcal{W}^q_\pm$ consists of those $Z \in \mathcal{W}^q$ which are continuous in $\overline{\mathbb{C}^\pm}$, are analytic in $\mathbb{C}^\pm$, and tend to Z_∞ as $\lambda \in \infty$ in $\overline{\mathbb{C}^\pm}$. Furthermore,

$$\mathcal{W}^q = \mathcal{W}^q_+ \oplus \mathcal{W}^q_{-,0} = \mathcal{W}^q_{+,0} \oplus \mathcal{W}^q_-.$$

It is important to recall the subsequent matrix generalization of a famous result by Wiener [91, 54].

Theorem 3.8 *Given $Z \in \mathcal{W}^q$ of the form* (3.34)*, let Z_∞ be invertible and $Z(\lambda)$ nonsingular for all $\lambda \in \mathbb{R}$. Then there exists w such that its elements are in $L^1(\mathbb{R})$ and*

$$Z(\lambda)^{-1} = (Z_\infty)^{-1} + \int_{-\infty}^{\infty} d\alpha\, w(\alpha) e^{i\lambda\alpha}$$

for $\lambda \in \mathbb{R}$.

In order to prove that the Faddeev functions $M_l(x,\cdot)$ and $M_r(x,\cdot)$ belong to $\mathcal{W}^{n+m}$ for any $x \in \mathbb{R}$, we proceed as follows. First we write

$$M_l(x,\lambda) = I_{n+m} + \int_0^{\infty} d\alpha\, B_l(x,\alpha) e^{i\lambda J\alpha}, \tag{3.35}$$

$$M_r(x,\lambda) = I_{n+m} + \int_0^{\infty} d\alpha\, B_r(x,\alpha) e^{-i\lambda J\alpha}, \tag{3.36}$$

where, for any $x \in \mathbb{R}$, $B_l(x,\cdot)$ and $B_r(x,\cdot)$ are L^1 matrix functions, without being concerned with justifying such a representation. We then employ (3.35) and (3.36) to convert the Volterra integral equations (3.25a)-(3.25b) into integral equations for $B_l(x,\alpha)$. By iterating the latter equations we prove the existence of the L^1 matrix functions $B_l(x,\cdot)$ and $B_r(x,\cdot)$ and show that the matrix functions defined in terms of them by (3.35) and (3.36) satisfy (3.25a)-(3.25b). In this way we avoid any appearance of circular reasoning.

Indeed, let us partition the matrix functions $B_l(x,\alpha)$ and $B_r(x,\alpha)$ in (3.35) and (3.36) into $(n+m)\times(n+m)$ blocks as follows:

$$B_l(x,\alpha) = \begin{pmatrix} B_{l1}(x,\alpha) & B_{l2}(x,\alpha) \\ B_{l3}(x,\alpha) & B_{l4}(x,\alpha) \end{pmatrix}, \quad B_r(x,\alpha) = \begin{pmatrix} B_{r1}(x,\alpha) & B_{r2}(x,\alpha) \\ B_{r3}(x,\alpha) & B_{r4}(x,\alpha) \end{pmatrix}.$$

Then eq. (3.35) can be written as

$$\begin{pmatrix} M_{l1}(x,\lambda) & M_{l2}(x,\lambda) \\ M_{l3}(x,\lambda) & M_{l4}(x,\lambda) \end{pmatrix} = \begin{pmatrix} I_n & 0_{n\times m} \\ 0_{m\times n} & I_m \end{pmatrix} + \int_0^\infty d\alpha \begin{pmatrix} B_{l1}(x,\alpha) & B_{l2}(x,\alpha) \\ B_{l3}(x,\alpha) & B_{l4}(x,\alpha) \end{pmatrix} e^{i\lambda J\alpha}.$$

So we have

$$M_{l1}(x,\lambda) = I_n + \int_0^\infty d\alpha\, B_{l1}(x,\alpha)e^{i\lambda\alpha}, \tag{3.37}$$

$$M_{l2}(x,\lambda) = \int_0^\infty d\alpha\, B_{l2}(x,\alpha)e^{-i\lambda\alpha}, \tag{3.38}$$

$$M_{l3}(x,\lambda) = \int_0^\infty d\alpha\, B_{l3}(x,\alpha)e^{i\lambda\alpha}, \tag{3.39}$$

$$M_{l4}(x,\lambda) = I_m + \int_0^\infty d\alpha\, B_{l4}(x,\alpha)e^{-i\lambda\alpha}. \tag{3.40}$$

Taking into account eqs. (3.15)-(3.18) we can proceed in the following way: First, we substitute the second member of (3.15) into the first member of (3.37) obtaining

$$-i\int_x^\infty dy\, k(y)M_{l3}(y,\lambda) = \int_0^\infty d\alpha\, B_{l1}(x,\alpha)e^{i\lambda\alpha}.$$

Now, if we apply eq. (3.39) in the first member of the previous equation we have

$$\begin{aligned} -i\int_x^\infty dy\, k(y)\int_0^\infty d\alpha\, B_{l3}(y,\alpha)e^{i\lambda\alpha} &= -i\int_0^\infty d\alpha\, e^{i\lambda\alpha}\int_x^\infty dy\, k(y)B_{l3}(y,\alpha) \\ &= \int_0^\infty d\alpha\, B_{l1}(x,\alpha)e^{i\lambda\alpha}, \end{aligned}$$

and therefore

$$B_{l1}(x,\alpha) = -i\int_x^\infty dy\, k(y)B_{l3}(y,\alpha). \tag{3.41}$$

Analogously, if we compare (3.39) with (3.17) we have

$$i\int_x^\infty dy\, e^{2i\lambda(y-x)}l(y)M_{l1}(y,\lambda) = \int_0^\infty d\alpha\, B_{l3}(x,\alpha)e^{i\lambda\alpha}.$$

Substituting eq. (3.37) in this equation we find

$$i\int_x^\infty dy\, l(y)e^{2i\lambda(y-x)} + i\int_x^\infty dy\, l(y)e^{2i\lambda(y-x)}\int_0^\infty d\hat{\alpha}\, B_{l1}(y,\hat{\alpha})e^{i\lambda\hat{\alpha}} = \int_0^\infty d\alpha\, B_{l3}(x,\alpha)e^{i\lambda\alpha}.$$

Putting $\alpha = 2(y-x)$ in the first integral in the first member and $\alpha = \hat{\alpha} + 2(y-x)$ in the second integral, we obtain

$$\int_0^\infty d\alpha\, B_{l3}(x,\alpha)e^{i\lambda\alpha} = \frac{i}{2}\int_x^\infty d\alpha\, l(x+\frac{\alpha}{2})e^{i\lambda\alpha} - i\int_0^\infty d\alpha\, e^{i\lambda\alpha}\int_x^\infty dy\, l(y)B_{l1}(y,\alpha-2(y-x)),$$

implying

$$B_{l3}(x,\alpha) = \frac{i}{2}l(x+\frac{\alpha}{2}) + i\int_x^{x+\frac{\alpha}{2}} dy\, l(y)B_{l1}(y,\alpha-2(y-x)). \tag{3.42}$$

Proceeding in a similar way we get[1]

$$B_{l2}(x,\alpha) = \frac{-i}{2}k(x+\frac{\alpha}{2}) - i\int_x^{x+\frac{\alpha}{2}} dy\, k(y)B_{l4}(y,\alpha-2(y-x)), \tag{3.43}$$

$$B_{l4}(x,\alpha) = i\int_x^\infty dy\, l(y)B_{l2}(y,\alpha), \tag{3.44}$$

$$B_{r1}(x,\alpha) = i\int_{-\infty}^x dy\, k(y)B_{r3}(y,\alpha), \tag{3.45}$$

$$B_{r3}(x,\alpha) = \frac{-i}{2}l(x-\frac{\alpha}{2}) - i\int_{x-\frac{\alpha}{2}}^x dy\, l(y)B_{r1}(y,\alpha+2(y-x)), \tag{3.46}$$

$$B_{r2}(x,\alpha) = \frac{i}{2}k(x-\frac{\alpha}{2}) + i\int_{x-\frac{\alpha}{2}}^x dy\, k(y)B_{r4}(y,\alpha+2(y-x)), \tag{3.47}$$

$$B_{r4}(x,\alpha) = -i\int_{-\infty}^x dy\, l(y)B_{r2}(y,\alpha). \tag{3.48}$$

Let us introduce the following mixed norm on the $(n+m)\times n$ or $(n+m)\times m$ matrix functions $B(x,\alpha)$ depending on $(x,\alpha) \in \mathbb{R}\times\mathbb{R}^+$:

$$\|B(\cdot,\cdot)\|_{\infty,1} = \sup_{x\in\mathbb{R}}\|B(x,\cdot)\|_1 = \sup_{x\in\mathbb{R}}\int_0^\infty d\alpha\, \|B(x,\alpha)\|. \tag{3.49}$$

Theorem 3.9 *Assume that the elements of $k(x)$ and $l(x)$ belong to $L^1(\mathbb{R})$. Then, for each $x\in\mathbb{R}$, the four pairs of equations* (3.41) *and* (3.42), (3.43) *and* (3.44), (3.45) *and* (3.46), *and* (3.47) *and* (3.48) *have unique solutions $B(x,\alpha)$ satisfying*

$$\|B(\cdot,\cdot)\|_{\infty,1} = \sup_{x\in\mathbb{R}}\|B(x,\cdot)\|_1 < \infty.$$

Consequently, $M_l(x,\cdot) \in \mathcal{W}^{n+m}$ and $M_r(x,\cdot) \in \mathcal{W}^{n+m}$, with norms bounded above by a finite constant not depending on $x\in\mathbb{R}$.

[1]Note the (same) typo in eq. (4.11) of [9, 90].

Proof. Let us give the proof only for eqs. (3.41) and (3.42), since the proof is similar for the other pairs of equations. Taking norms in eqs. (3.41) and (3.42) we have

$$\|B_{l1}(x,\alpha)\| \le \int_x^\infty dy\, \|k(y)\|\, \|B_{l3}(x,\alpha)\|,$$
$$\|B_{l3}(x,\alpha)\| \le \frac{1}{2}\|l(x+\frac{\alpha}{2})\| + \int_x^{x+\frac{\alpha}{2}} dy\, \|l(y)\|\, \|B_{l1}(y,\alpha-2(y-x))\|.$$

Integrating with respect to $\alpha \in \mathbb{R}^+$, we obtain

$$\|B_{l1}(x,\cdot)\|_1 \le \int_x^\infty dy\, \|k(y)\|\, \|B_{l3}(y,\cdot)\|_1,$$

as well as

$$\begin{aligned}\|B_{l3}(x,\cdot)\|_1 &\le \int_x^\infty dy\, \|l(y)\| + \int_0^\infty d\alpha \int_x^{x+\frac{\alpha}{2}} dy\, \|l(y)\|\, \|B_{l1}(y,\alpha-2(y-x))\| \le \\ &\le \int_x^\infty dy\, \|l(y)\| + \int_x^\infty dy\, \|l(y)\| \int_0^\infty d\hat\alpha\, \|B_{l1}(y,\hat\alpha)\| = \\ &= \int_x^\infty dy\, \|l(y)\| + \int_x^\infty dy\, \|l(y)\|\, \|B_{l1}(y,\cdot)\|_1.\end{aligned}$$

where the change of variable $\hat\alpha = \alpha + 2x - 2y$ has been applied. Summing the preceding two estimates we have

$$\begin{aligned}\|B_{l1}(x,\cdot)\|_1 + \|B_{l3}(x,\cdot)\|_1 \le& \int_x^\infty dy\, \|l(y)\| \\ &+ \int_x^\infty dy\, \max(\|k(y)\|,\|l(y)\|)\,(\|B_{l1}(y,\cdot)\|_1 + \|B_{l3}(y,\cdot)\|_1).\end{aligned}$$

Using the Gronwall Lemma 3.3 we obtain

$$\begin{aligned}\|B_{l1}(x,\cdot)\|_1 + \|B_{l3}(x,\cdot)\|_1 &\le \int_x^\infty dy\, \|l(y)\| \exp\left\{\int_x^\infty dy\, \max(\|k(y)\|,\|l(y)\|) \int_y^\infty dz\, \|l(z)\|\right\} \\ &\le \|l(\cdot)\|_1 \exp\left\{\|l(\cdot)\|_1\,(\|l(\cdot)\|_1 + \|k(\cdot)\|_1)\right\}.\end{aligned}$$

Thus eqs. (3.41)-(3.42) can be solved uniquely by iteration in the Banach space of continuous functions in $x \in \mathbb{R}$ with values in $L^1(\mathbb{R}^+;\mathbb{C}^{(n+m)\times n})$, endowed with the norm (3.49). ■

Faddeev [45] was the first to derive estimates as those given in the proof of Theorem 3.9 in his treatment of the direct scattering theory of the Schrödinger equation on the line. Tanaka [88] has generalized and corrected some mistakes in these estimates (See also [40]).

Let us now define $M_{l1}(x,\lambda)$, $M_{l2}(x,\lambda)$, $M_{l3}(x,\lambda)$ and $M_{l4}(x,\lambda)$ by (3.37)-(3.40) and $M_{r1}(x,\lambda)$,

$M_{r2}(x,\lambda)$, $M_{r3}(x,\lambda)$ and $M_{r4}(x,\lambda)$ by

$$M_{r1}(x,\lambda) = I_n + \int_0^\infty d\alpha\, B_{r1}(x,\alpha)e^{-i\lambda\alpha}, \tag{3.50}$$

$$M_{r2}(x,\lambda) = \int_0^\infty d\alpha\, B_{r2}(x,\alpha)e^{i\lambda\alpha}, \tag{3.51}$$

$$M_{r3}(x,\lambda) = \int_0^\infty d\alpha\, B_{r3}(x,\alpha)e^{-i\lambda\alpha}, \tag{3.52}$$

$$M_{r4}(x,\lambda) = I_m + \int_0^\infty d\alpha\, B_{r4}(x,\alpha)e^{i\lambda\alpha}, \tag{3.53}$$

in agreement with (3.36). Then the eight matrix functions thus constructed make up the matrix functions $M_l(x,\lambda)$ and $M_r(x,\lambda)$ as in (3.13) which belong to $\mathcal{W}^{n+m}$. Applying the Fourier transform to (3.41)-(3.48) we obtain the integral equations (3.15)-(3.18) and their analogues for $M_{r1}(x,\lambda)$, $M_{r2}(x,\lambda)$, $M_{r3}(x,\lambda)$ and $M_{r4}(x,\lambda)$. Since the latter eight equations are uniquely solvable, we have represented the Jost solutions in the form (3.35)-(3.36), where the elements of $B_l(x,\cdot)$ and $B_r(x,\cdot)$ belong to $L^1(\mathbb{R}^+)$, as claimed.

The integral equations (3.41)-(3.48) allow us to derive the following relations for the potentials $k(x)$ and $l(x)$:[2]

$$k(x) = 2iB_{l2}(x,0^+) = -2iB_{r2}(x,0^+), \tag{3.54}$$

$$l(x) = -2iB_{l3}(x,0^+) = 2iB_{r3}(x,0^+). \tag{3.55}$$

To justify (3.54)-(3.55), let us fix $\alpha > 0$ and integrate the norm of the left hand side in (3.42) with respect to $x \in \mathbb{R}$. We obtain

$$\|B_{l3}(\cdot,\alpha)\|_1 \leq \frac{1}{2}\|l\|_1 + \int_{-\infty}^\infty ds \int_{y-\frac{\alpha}{2}}^y dx\, \|l(y)\|\, \|B_{l1}(y,\alpha-2(y-x))\|,$$

and putting $z = \alpha - 2(y-x)$ in the integral in the second member of the previous formula, we have

$$\begin{aligned}\|B_{l3}(\cdot,\alpha)\|_1 &\leq \frac{1}{2}\left[\|l\|_1 + \int_{-\infty}^\infty ds \int_0^\alpha dz\, \|l(y)\|\, \|B_{l1}(y,z)\|\right] \leq \\ &\leq \frac{1}{2}\left[\|l\|_1 + \int_{-\infty}^\infty dy\, \|l(y)\|\, \|B_{l1}(y,\cdot)\|_1\right].\end{aligned}$$

Hence, for each $\alpha > 0$, $B_{\alpha 3}(\cdot,\alpha)$ is a matrix function with entries in $L^1(\mathbb{R})$. We now derive the estimate

$$\begin{aligned}\|B_{l3}(\cdot,\alpha) - \frac{i}{2}l(x+\frac{1}{2}\alpha)\|_1 \leq \frac{1}{2}\int_{-\infty}^\infty dy\, \|l(y)\| \int_0^\alpha dz\, \|l(y)\|\, \|B_{l1}(y,z)\| = \\ = o(1), \quad \alpha \to 0^+\end{aligned}$$

which justifies the identity $l(x) = 2iB_{l3}(x,0^+)$. In an analogous way one proves a similar result for $B_{l2}(\cdot,\alpha)$, $B_{r2}(\cdot,\alpha)$ and $B_{r3}(\cdot,\alpha)$.

[2] Note the sign discrepancies with eq. (4.19) in [9, 90].

Theorem 3.10 *The coefficients $a_l(\lambda)$ and $a_r(\lambda)$ are elements of $\mathcal{W}^{n+m}$.*

Proof. Using eqs. (3.5a) and (3.35) we have

$$a_l(\lambda) = I_{n+m} - iJ\int_{-\infty}^{\infty} dy e^{-i\lambda Jy}V(y)M_l(y,\lambda)e^{i\lambda Jy} =$$
$$= I_{n+m} - iJ\int_{-\infty}^{\infty} dy e^{-i\lambda Jy}V(y)\left[I_{n+m} + \int_0^{\infty} d\alpha\, B_l(x,\alpha)e^{i\lambda J\alpha}\right]e^{i\lambda Jy}.$$

On the other hand, the relation

$$\int_{-\infty}^{\infty} dy \int_0^{\infty} d\alpha\, \|e^{-i\lambda Jy}V(y)B_l(x,\alpha)e^{i\lambda J\alpha}e^{i\lambda Jy}\| =$$
$$= \int_{-\infty}^{\infty} dy \int_0^{\infty} d\alpha\, \|V(y)\|\,\|B_l(x,\alpha)\| = \|V\|_1\,\|B_l(x,\cdot)\|_1 < \infty$$

allows us to justify the use of the Dominated Convergence Theorem, which implies

$$a_l(\lambda) = I_{n+m} - iJ\int_{-\infty}^{\infty} dy\, e^{-i\lambda Jy}V(y)e^{i\lambda Jy} -$$
$$- iJ\int_0^{\infty} d\alpha \int_{-\infty}^{\infty} dy\, e^{-i\lambda Jy}V(y)B_l(x,\alpha)e^{i\lambda J\alpha}e^{i\lambda Jy}. \tag{3.56}$$

But the first two terms in the right-hand side of eq. (3.56) can be written as

$$I_{n+m} - iJ\int_{-\infty}^{\infty} dy\, e^{-i\lambda Jy}V(y)e^{i\lambda Jy} = \begin{pmatrix} I_n & -i\int_{-\infty}^{\infty} dy\, e^{-2i\lambda y}k(y) \\ i\int_{-\infty}^{\infty} dy\, e^{2i\lambda y}l(y) & I_m \end{pmatrix} =$$
$$= \begin{pmatrix} I_n & -\frac{i}{2}\int_{-\infty}^{\infty} d\alpha\, e^{-i\lambda\alpha}k(\frac{\alpha}{2}) \\ \frac{i}{2}\int_{-\infty}^{\infty} d\alpha\, e^{i\lambda\alpha}l(\frac{\alpha}{2}) & I_m, \end{pmatrix}$$

and, in this way, we see that these terms are in $\mathcal{W}^{n+m}$. Moreover, we have

$$- iJ\int_0^{\infty} d\alpha \int_{-\infty}^{\infty} dy\, e^{-i\lambda Jy}V(y)B_l(x,\alpha)e^{i\lambda J\alpha}e^{i\lambda Jy} =$$
$$= -iJ\int_0^{\infty} d\alpha \underbrace{\left[\int_{-\infty}^{\infty} dy\, e^{-i\lambda Jy}V(y)B_l(x,\alpha)e^{i\lambda Jy}\right]}_{\text{in } L^1 \text{ as a function of } \alpha} e^{i\lambda J\alpha},$$

where $\left[\int_{-\infty}^{\infty} dy\, e^{-i\lambda Jy}V(y)B_l(x,\alpha)e^{i\lambda Jy}\right]$ is an element of L^1 as a function of α because of the following estimate:

$$\int_0^{\infty} d\alpha \left\|\int_{-\infty}^{\infty} dy\, e^{-i\lambda Jy}V(y)B_l(x,\alpha)e^{i\lambda Jy}\right\| \leq \int_0^{\infty} d\alpha \int_{-\infty}^{\infty} dy\|e^{-i\lambda Jy}V(y)B_l(x,\alpha)e^{i\lambda Jy}\| =$$
$$= \int_0^{\infty} d\alpha\|B_l(x,\alpha)\| \int_{-\infty}^{\infty} dy\|V(y)\| < \infty.$$

So, we have found that also the third term in (3.56) is an element of $\mathcal{W}^{n+m}$ and consequently, $a_l(\lambda)$ is in $\mathcal{W}^{n+m}$ too.

The proof for $a_r(\lambda)$ is similar. ■

3.4 The scattering matrix

In analogy with the introduction of the reflection and transmission functions for the Schrödinger equation on the line, here we introduce the scattering matrix for the matrix Zakharov-Shabat system (3.1). First, we discuss the properties of this matrix function in the absence of symmetries of the potentials. Successively we analyze the symmetric and anti-symmetric cases.

Recall that (cf. eqs. (3.1a) and (3.4a))

$$F_l(x,\lambda) \cong \begin{cases} e^{i\lambda Jx}, & x \to +\infty \\ e^{i\lambda Jx} a_l(\lambda) = e^{i\lambda Jx} \begin{pmatrix} a_{l1}(\lambda) & a_{l2}(\lambda) \\ a_{l3}(\lambda) & a_{l4}(\lambda) \end{pmatrix}, & x \to -\infty. \end{cases} \tag{3.57}$$

Let us write

$$F_l(x,\lambda) \begin{pmatrix} T_l(\lambda) & 0_{n\times m} \\ 0_{m\times n} & \overline{T}_r(\lambda) \end{pmatrix} \cong \begin{cases} e^{i\lambda Jx} \begin{pmatrix} T_l(\lambda) & 0_{n\times m} \\ 0_{m\times n} & \overline{T}_r(\lambda) \end{pmatrix}, & x \to +\infty, \\ e^{i\lambda Jx} \begin{pmatrix} I_n & \overline{L}(\lambda) \\ L(\lambda) & I_m \end{pmatrix}, & x \to -\infty, \end{cases} \tag{3.58}$$

as well as

$$F_r(x,\lambda) \begin{pmatrix} \overline{T}_l(\lambda) & 0_{n\times m} \\ 0_{m\times n} & T_r(\lambda) \end{pmatrix} \cong \begin{cases} e^{i\lambda Jx} \begin{pmatrix} \overline{T}_l(\lambda) & 0_{n\times m} \\ 0_{m\times n} & T_r(\lambda) \end{pmatrix}, & x \to -\infty, \\ e^{i\lambda Jx} \begin{pmatrix} I_n & R(\lambda) \\ \overline{R}(\lambda) & I_m \end{pmatrix}, & x \to +\infty, \end{cases} \tag{3.59}$$

where we put

$$T_l(\lambda) = a_{l1}(\lambda)^{-1}, \qquad \overline{T}_r(\lambda) = a_{l4}(\lambda)^{-1}, \tag{3.60}$$

$$\overline{T}_l(\lambda) = a_{r1}(\lambda)^{-1}, \qquad T_r(\lambda) = a_{r4}(\lambda)^{-1}, \tag{3.61}$$

$$\overline{L}(\lambda) = a_{l2}(\lambda)a_{l4}(\lambda)^{-1}, \qquad L(\lambda) = a_{l3}(\lambda)a_{l1}(\lambda)^{-1}, \tag{3.62}$$

$$R(\lambda) = a_{r2}(\lambda)a_{r4}(\lambda)^{-1}, \qquad \overline{R}(\lambda) = a_{r3}(\lambda)a_{r1}(\lambda)^{-1}, \tag{3.63}$$

provided the inverses appearing in (3.60)-(3.63) exist. The quantities defined in eqs. (3.60) and (3.61) are called *transmission coefficients* and those in eqs. (3.62) and (3.63) *reflection coefficients*. We observe that

$$a_l(\lambda) = \begin{pmatrix} I_n & \overline{L}(\lambda) \\ L(\lambda) & I_m \end{pmatrix} \begin{pmatrix} T_l(\lambda)^{-1} & 0_{n\times m} \\ 0_{m\times n} & \overline{T}_r(\lambda)^{-1} \end{pmatrix}, \tag{3.64a}$$

$$a_r(\lambda) = \begin{pmatrix} I_n & R(\lambda) \\ \overline{R}(\lambda) & I_m \end{pmatrix} \begin{pmatrix} \overline{T}_l(\lambda)^{-1} & 0_{n\times m} \\ 0_{m\times n} & T_r(\lambda)^{-1} \end{pmatrix}. \tag{3.64b}$$

In this way it is possible to define *the scattering matrices* as

$$S(\lambda) = \begin{pmatrix} T_l(\lambda) & R(\lambda) \\ L(\lambda) & T_r(\lambda) \end{pmatrix}, \tag{3.65a}$$

$$\overline{S}(\lambda) = \begin{pmatrix} \overline{T}_l(\lambda) & \overline{L}(\lambda) \\ \overline{R}(\lambda) & \overline{T}_r(\lambda) \end{pmatrix}, \tag{3.65b}$$

In order to define the reflection and transmission coefficients as matrix functions that are continuous in $\lambda \in \mathbb{R}$, we need the inverses appearing in (3.60)-(3.63) to exist. We therefore make the following *technical hypothesis*:

The matrices $a_{l1}(\lambda)$, $a_{r1}(\lambda)$, $a_{l4}(\lambda)$ and $a_{r4}(\lambda)$ are invertible for all $\lambda \in \mathbb{R}$.

The following proposition is immediate from Theorem 3.10 with the help of Theorem 3.8.

Proposition 3.11 *Under the technical hypothesis, the reflection and transmission coefficients belong to $\mathcal{W}^{p\times q}$ for suitable p and q.*

As a result, under the technical hypothesis, the reflection and transmission matrices are continuous in $\lambda \in \mathbb{R}$, while as $\lambda \to \pm\infty$ the reflection coefficients vanish and the transmission coefficients tend to the identity. Further, $T_l(\lambda)$ and $T_r(\lambda)$ are meromorphic functions of $\lambda \in \mathbb{C}^+$ having finitely many poles. Also, $\overline{T}_l(\lambda)$ and $\overline{T}_r(\lambda)$ are meromorphic functions of $\lambda \in \mathbb{C}^-$ having finitely many poles.

Proposition 3.12 *Under the technical hypothesis, we have*

$$\det a_{l1}(\lambda) = \det a_{r4}(\lambda), \quad \lambda \in \overline{\mathbb{C}^+},$$
$$\det a_{r1}(\lambda) = \det a_{l4}(\lambda), \quad \lambda \in \overline{\mathbb{C}^-},$$

and therefore

$$\det T_l(\lambda) = \det T_r(\lambda), \tag{3.66a}$$

$$\det \overline{T}_l(\lambda) = \det \overline{T}_r(\lambda). \tag{3.66b}$$

Proof. We prove only eq. (3.66a), because the proof of (3.66b) is analogous. Put

$$f_+(x,\lambda) = \begin{pmatrix} F_{l1}(x,\lambda) & F_{r2}(x,\lambda) \\ F_{l3}(x,\lambda) & F_{r4}(x,\lambda) \end{pmatrix}.$$

Using eq. (3.1) we have

$$\frac{d}{dx}\left[f_+(x,\lambda)e^{-i\lambda Jx}\right] = \left[iJV(x)f_+(x,\lambda) + i\lambda f_+(x,\lambda)\right]e^{-i\lambda Jx} - f_+(x,\lambda)i\lambda J e^{-i\lambda Jx} = f_+(x,\lambda)iJV(x)e^{-i\lambda Jx},$$

which implies (see, for example, [81])

$$\frac{d}{dx}\left[\det(f_+(x,\lambda)e^{-i\lambda Jx})\right] = tr\left(iJV(x)e^{-i\lambda Jx}\right)\det f_+(x,\lambda) = 0,$$

because $iJV(x)e^{-i\lambda Jx}$ has zero trace. So, $\det(f_+(x,\lambda)e^{-i\lambda Jx})$ is independent of x. Taking into account the relations

$$\lim_{x\to+\infty} f_+(x,\lambda)e^{-i\lambda Jx} = \begin{pmatrix} I_n & a_{r2}(\lambda)e^{2i\lambda x} \\ 0_{m\times n} & a_{r4}(\lambda)\end{pmatrix},$$
$$\lim_{x\to-\infty} f_+(x,\lambda)e^{-i\lambda Jx} = \begin{pmatrix} a_{l1}(\lambda) & 0_{n\times m} \\ a_{l3}(\lambda)e^{-2i\lambda x} & I_m\end{pmatrix},$$

we find $\det(f_+(x,\lambda)e^{-i\lambda Jx}) = \det a_{r4}(\lambda) = \det a_{l1}(\lambda)$. As a result, using the technical hypothesis that $\det a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are nonsingular for $\lambda\in\mathbb{R}$, we have

$$\det T_l(\lambda) = \frac{1}{\det a_{l1}(\lambda)} = \frac{1}{\det a_{r4}(\lambda)} = \det T_r(\lambda),$$

that is eq. (3.66a). The extension of Proposition 3.12 to $\lambda\in\mathbb{C}^+$ proceeds by analytic continuation. ∎

Let us now derive alternative expressions for the reflection coefficients. Taking into account Proposition 3.1 we can write

$$\begin{pmatrix} a_{l1}(\lambda) & a_{l2}(\lambda) \\ a_{l3}(\lambda) & a_{l4}(\lambda)\end{pmatrix}\begin{pmatrix} a_{r1}(\lambda) & a_{r2}(\lambda) \\ a_{r3}(\lambda) & a_{r4}(\lambda)\end{pmatrix} = \begin{pmatrix} I_n & 0_{n\times m} \\ 0_{m\times n} & I_m\end{pmatrix}, \tag{3.67a}$$
$$\begin{pmatrix} a_{r1}(\lambda) & a_{r2}(\lambda) \\ a_{r3}(\lambda) & a_{r4}(\lambda)\end{pmatrix}\begin{pmatrix} a_{l1}(\lambda) & a_{l2}(\lambda) \\ a_{l3}(\lambda) & a_{l4}(\lambda)\end{pmatrix} = \begin{pmatrix} I_n & 0_{n\times m} \\ 0_{m\times n} & I_m\end{pmatrix}, \tag{3.67b}$$

and from these we obtain

$$a_{l3}(\lambda)a_{r1}(\lambda) + a_{l4}(\lambda)a_{r3}(\lambda) = 0, \tag{3.68}$$
$$a_{r1}(\lambda)a_{l2}(\lambda) + a_{r2}(\lambda)a_{l4}(\lambda) = 0, \tag{3.69}$$
$$a_{l1}(\lambda)a_{r2}(\lambda) + a_{l2}(\lambda)a_{r4}(\lambda) = 0, \tag{3.70}$$
$$a_{r3}(\lambda)a_{l1}(\lambda) + a_{r4}(\lambda)a_{l3}(\lambda) = 0. \tag{3.71}$$

Equation (3.68) implies

$$\overline{R}(\lambda) = a_{r3}(\lambda)a_{r1}(\lambda)^{-1} = -a_{l4}(\lambda)^{-1}a_{l3}(\lambda), \tag{3.72}$$

while eq. (3.69), (3.70) and (3.71) lead to the expressions

$$\overline{L}(\lambda) = a_{l2}(\lambda)a_{l4}(\lambda)^{-1} = -a_{r1}(\lambda)^{-1}a_{r2}(\lambda), \tag{3.73}$$
$$R(\lambda) = a_{r2}(\lambda)a_{r4}(\lambda)^{-1} = -a_{l1}(\lambda)^{-1}a_{l2}(\lambda), \tag{3.74}$$
$$L(\lambda) = a_{l3}(\lambda)a_{l1}(\lambda)^{-1} = -a_{r4}(\lambda)^{-1}a_{r3}(\lambda). \tag{3.75}$$

In analogy with (3.64a) and (3.64b) we observe that

$$a_l(\lambda) = \begin{pmatrix} T_l(\lambda)^{-1} & 0_{n\times m} \\ 0_{m\times n} & \overline{T}_r(\lambda)^{-1}\end{pmatrix}\begin{pmatrix} I_n & -\overline{R}(\lambda) \\ -R(\lambda) & I_m\end{pmatrix}, \tag{3.76a}$$
$$a_r(\lambda) = \begin{pmatrix} \overline{T}_l(\lambda)^{-1} & 0_{n\times m} \\ 0_{m\times n} & T_r(\lambda)^{-1}\end{pmatrix}\begin{pmatrix} I_n & -L(\lambda) \\ -\overline{L}(\lambda) & I_m\end{pmatrix}. \tag{3.76b}$$

We now easily prove that $\overline{S}(\lambda)=S(\lambda)^{-1}$ for $\lambda\in\mathbb{R}$. Indeed,

$$\begin{aligned}\overline{S}(\lambda)S(\lambda)&=\begin{pmatrix}\overline{T}_l(\lambda)&\overline{L}(\lambda)\\ \overline{R}(\lambda)&\overline{T}_r(\lambda)\end{pmatrix}\begin{pmatrix}T_l(\lambda)&R(\lambda)\\ L(\lambda)&T_r(\lambda)\end{pmatrix}=\\ &=\begin{pmatrix}a_{r1}(\lambda)^{-1}&-a_{r1}(\lambda)^{-1}a_{r2}(\lambda)\\ -a_{l4}(\lambda)^{-1}a_{l3}(\lambda)&a_{l4}(\lambda)^{-1}\end{pmatrix}\begin{pmatrix}a_{l1}(\lambda)^{-1}&a_{r2}(\lambda)a_{r4}(\lambda)^{-1}\\ a_{l3}(\lambda)a_{l1}(\lambda)^{-1}&a_{r4}(\lambda)^{-1}\end{pmatrix}=\\ &=\begin{pmatrix}a_{r1}(\lambda)^{-1}&0_{n\times m}\\ 0_{m\times n}&a_{l4}(\lambda)^{-1}\end{pmatrix}\begin{pmatrix}I_n-a_{r2}(\lambda)a_{l3}(\lambda)&0_{n\times m}\\ 0_{m\times n}&I_m-a_{l3}(\lambda)a_{r2}(\lambda)\end{pmatrix}\times\\ &\times\begin{pmatrix}a_{l1}(\lambda)^{-1}&0_{n\times m}\\ 0_{m\times n}&a_{r4}(\lambda)^{-1}\end{pmatrix}=\begin{pmatrix}a_{r1}(\lambda)^{-1}&0_{n\times m}\\ 0_{m\times n}&a_{l4}(\lambda)^{-1}\end{pmatrix}\times\\ &\times\begin{pmatrix}a_{r1}(\lambda)a_{l1}(\lambda)&0_{n\times m}\\ 0_{m\times n}&a_{l4}(\lambda)a_{r4}(\lambda)\end{pmatrix}\begin{pmatrix}a_{l1}(\lambda)^{-1}&0_{n\times m}\\ 0_{m\times n}&a_{r4}(\lambda)^{-1}\end{pmatrix}=I_{n+m},\end{aligned}$$

where we have used (3.74) and (3.75) as well as (3.6).

In the *antisymmetric* case the matrices $a_l(\lambda)$ and $a_r(\lambda)$ are unitary, so we have

$$a_{r1}(\lambda)=a_{l1}(\overline{\lambda})^*,\qquad a_{l4}(\lambda)=a_{r4}(\overline{\lambda})^*,\tag{3.77a}$$

$$a_{r2}(\lambda)=a_{l3}(\lambda)^*,\qquad a_{l2}(\lambda)=a_{r3}(\lambda)^*.\tag{3.77b}$$

Thus, under the technical hypothesis, it is easy to see that

$$\overline{T}_l(\lambda)=T_l(\overline{\lambda})^*,\qquad \overline{T}_r(\lambda)=T_r(\overline{\lambda})^*.$$

Moreover,

$$\overline{R}(\lambda)^*=-a_{l3}(\lambda)^*\left[a_{l4}(\lambda)^{-1}\right]^*=-a_{r2}(\lambda)a_{r4}(\lambda)^{-1}=-R(\lambda).$$

From eqs. (3.73) and (3.75) we obtain in the same way

$$\overline{L}(\lambda)^*=\left[a_{l4}(\lambda)^{-1}\right]^*a_{l2}(\lambda)^*=a_{r4}(\lambda)^{-1}a_{r3}(\lambda)=-L(\lambda).$$

In other words, in the antisymmetric case the scattering matrices $S(\lambda)$ and $\overline{S}(\lambda)$ are J-unitary and

$$\overline{S}(\lambda)=S(\lambda)^{-1}=JS(\lambda)^*J,\quad \lambda\in\mathbb{R}.$$

In the *symmetric* case the matrices $a_l(\lambda)$ and $a_r(\lambda)$ are J-unitary, implying that

$$\begin{aligned}a_{l1}(\lambda)^*a_{l1}(\lambda)-a_{l3}(\lambda)^*a_{l3}(\lambda)&=I_n,\\ a_{l4}(\lambda)^*a_{l4}(\lambda)-a_{l2}(\lambda)^*a_{l2}(\lambda)&=I_m,\\ a_{r1}(\lambda)^*a_{r1}(\lambda)-a_{r3}(\lambda)^*a_{r3}(\lambda)&=I_n,\\ a_{r4}(\lambda)^*a_{r4}(\lambda)-a_{r2}(\lambda)^*a_{r2}(\lambda)&=I_m.\end{aligned}$$

These equations imply that $a_{l1}(\lambda)$, $a_{l4}(\lambda)$, $a_{r1}(\lambda)$, and $a_{r4}(\lambda)$ are invertible for $\lambda\in\mathbb{R}$ and hence the technical hypothesis is always satisfied. As in the antisymmetric case, we have

$$\overline{T}_l(\lambda)=T_l(\overline{\lambda})^*,\qquad \overline{T}_r(\lambda)=T_r(\overline{\lambda})^*.$$

Moreover,

$$\overline{R}(\lambda)^* = -a_{l3}(\lambda)^* \left[a_{l4}(\lambda)^{-1}\right]^* = a_{r2}(\lambda)a_{r4}(\lambda)^{-1} = R(\lambda),$$

and

$$\overline{L}(\lambda)^* = \left[a_{l4}(\lambda)^{-1}\right]^* a_{l2}(\lambda)^* = -a_{r4}(\lambda)^{-1}a_{r3}(\lambda) = L(\lambda).$$

In other words, in the symmetric case the scattering matrices $S(\lambda)$ and $\overline{S}(\lambda)$ are unitary and

$$\overline{S}(\lambda) = S(\lambda)^{-1} = S(\lambda)^*, \quad \lambda \in \mathbb{R}.$$

In the symmetric case, we prove the following important properties.

Proposition 3.13 *In the symmetric case the transmission coefficients $T_l(\lambda)$ and $T_r(\lambda)$ are continuous in $\lambda \in \overline{\mathbb{C}^+}$ and analytic in $\lambda \in \mathbb{C}^+$, while*

$$\sup_{\lambda\in\overline{\mathbb{C}^+}} \|T_l(\lambda)\| > 0, \quad \sup_{\lambda\in\overline{\mathbb{C}^+}} \|T_r(\lambda)\| > 0. \tag{3.78}$$

Moreover, the reflection coefficients $R(\lambda)$ and $L(\lambda)$ satisfy the inequalities

$$\sup_{\lambda\in\mathbb{R}} \|R(\lambda)\| < 1, \quad \sup_{\lambda\in\mathbb{R}} \|L(\lambda)\| < 1. \tag{3.79}$$

Proof. If $\lambda_0 \in \mathbb{C}^+$ is a zero of $\det a_{l1}(\lambda)$, then there exists $0 \neq \xi \in \mathbb{C}^n$ such that $a_{l1}(\lambda_0)\xi = 0$. Then the asymptotic properties of $F_l(x, \lambda_0)$ as $x \to \pm\infty$ imply that

$$F_l(x, \lambda_0) \begin{pmatrix} \xi \\ 0_{m\times 1} \end{pmatrix}$$

belongs to $L^2(\mathbb{R}; \mathbb{C}^{n\times m})$. Hence λ_0 is a nonreal eigenvalue of the Hamiltonian operator H, which contradicts its selfadjointness. Thus $\det a_{l1}(\lambda) \neq 0$ for $\lambda \in \mathbb{C}^+$. In the same way we prove that $\det a_{r4}(\lambda) \neq 0$ for $\lambda \in \mathbb{C}^+$. Hence, $T_l(\lambda)$ and $T_r(\lambda)$ are analytic in $\mathbb{C}^+$.

Using the unitarity of the scattering matrix $S(\lambda)$ we obtain the identities

$$T_l(\lambda)\, T_l(\lambda)^* + R(\lambda)\, R(\lambda)^* = I_n, \qquad T_r(\lambda)^*\, T_r(\lambda) + R(\lambda)^*\, R(\lambda) = I_m, \tag{3.80}$$
$$T_l(\lambda)^*\, T_l(\lambda) + L(\lambda)^*\, L(\lambda) = I_n, \qquad T_r(\lambda)\, T_r(\lambda)^* + L(\lambda)\, L(\lambda)^* = I_m, \tag{3.81}$$
$$T_r(\lambda)\, R(\lambda)^* + L(\lambda)\, T_l(\lambda)^* = 0_{m\times n}, \quad T_r(\lambda)^*\, L(\lambda) + R(\lambda)^*\, T_l(\lambda) = 0_{n\times m}. \tag{3.82}$$

Since $T_l(\lambda)$ and $T_r(\lambda)$ are continuous in $\lambda \in \mathbb{R}$, tend to identity as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$, and take nonsingular values only, there exist ε_l, $\varepsilon_r \in (0, 1)$ such that

$$\|T_l(\lambda)\xi\| \geq \varepsilon_l\|\xi\|, \quad \xi \in \mathbb{C}^n,$$
$$\|T_r(\lambda)\eta\| \geq \varepsilon_r\|\eta\|, \quad \eta \in \mathbb{C}^m,$$

As a result,

$$\|L(\lambda)\xi\| \leq \sqrt{1-\varepsilon_l^2}\, \|\xi\|, \quad \xi \in \mathbb{C}^n,$$
$$\|R(\lambda)\eta\| \leq \sqrt{1-\varepsilon_r^2}\, \|\eta\|, \quad \eta \in \mathbb{C}^m,$$

which completes the proof. ∎

We conclude this subsection by analyzing the cases in which the potential $V(x)$ is supported on the half-line $\mathbb{R}^+$ or $\mathbb{R}^-$ (cf. [90, 9]). We have the following:

Proposition 3.14 *If $k(x)$ and $l(x)$ are supported on $\mathbb{R}^+$, then $L(\lambda)$ and $\overline{L}(\lambda)$ belong to $\mathcal{W}_{0,+}^{m\times n}$ and have continuations that are continuous on $\overline{\mathbb{C}^+}$, are analytic on $\mathbb{C}^+$, and vanish as $\lambda\to\infty$ in $\overline{\mathbb{C}^+}$.*

Proof. If $k(x)$ has support in $\mathbb{R}^+$, then from (3.28) and Proposition 3.4 we find that a_{l3} belong to $\mathcal{W}_{0,+}^{m\times n}$, has a continuation that is analytic in $\mathbb{C}^+$, is continuous in $\overline{\mathbb{C}^+}$ and converges to $0_{m\times n}$ as $\lambda\to\infty$ in $\overline{\mathbb{C}^+}$. As a result, using eq. (3.75) we can conclude that also $L(\lambda)$ extends to a function that is continuous on $\overline{\mathbb{C}^+}$, is analytic on $\mathbb{C}^+$ and vanish as $\lambda\to\infty$ in $\mathbb{C}^+$. ■

Proceeding in a similar way, it is easy to see that also the following proposition holds.

Proposition 3.15 *If $k(x)$ and $l(x)$ are supported on $\mathbb{R}^-$, then $R(\lambda)$ and $\overline{R}(\lambda)$ belong to $\mathcal{W}_{0,+}^{n\times m}$ and have continuations that are continuous on $\overline{\mathbb{C}^+}$, are analytic on $\mathbb{C}^+$, and vanish as $\lambda\to\infty$ in $\overline{\mathbb{C}^+}$.*

3.5 Bound states and scattering coefficients

In this section we prove that $T_l(\lambda)$ and $T_r(\lambda)$ have the same pole structures in each pole $\lambda\in\mathbb{C}^+$. More precisely, we prove that $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ have the same Jordan structure in all zeros of their determinants in $\mathbb{C}^+$.

We say that $\lambda_0\in\mathbb{C}^+$ is a *bound state* if λ_0 is an eigenvalue of the Hamiltonian $H=H_0-V$ and the corresponding eigenfunction belongs to $\mathcal{H}_{n+m}$. Now we prove the following important

Theorem 3.16 *Let $\lambda_0\in\mathbb{C}^+$ be a bound state. Then the Jordan structures of $\lambda I_{\mathcal{H}_{n+m}}-H$, $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ at λ_0 coincide. Analogously, let $\lambda_0\in\mathbb{C}^-$ be a bound state. Then the Jordan structures of $\lambda I_{\mathcal{H}_{n+m}}-H$, $a_{r1}(\lambda)$ and $a_{l4}(\lambda)$ at λ_0 coincide.*

Proof. Let us first find the eigenfunctions of H corresponding to a nonreal eigenvalue. Recall that any column vector solution of (3.1) has either of the equivalent forms

$$F_l(x,\lambda)\boldsymbol{\varepsilon}=F_r(x,\lambda)\boldsymbol{\eta}$$

for certain vectors $\boldsymbol{\varepsilon},\boldsymbol{\eta}\in\mathbb{C}^{n+m}$. For $\lambda\in\mathbb{C}^+$ these solutions have the asymptotic properties

$$\begin{cases} e^{i\lambda Jx}[1+o(1)]\boldsymbol{\varepsilon}=e^{i\lambda Jx}[a_r(\lambda)\boldsymbol{\eta}+o(1)], & x\to+\infty,\\ e^{i\lambda Jx}[a_l(\lambda)\boldsymbol{\varepsilon}+o(1)]=e^{i\lambda Jx}[1+o(1)]\boldsymbol{\eta}, & x\to-\infty,\end{cases}$$

but in order for them to be eigenfunctions of H they should also belong to $L^2(\mathbb{R};\mathbb{C}^{n+m})$. Thus for $\lambda\in\mathbb{C}^+$ the vectors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\eta}$ should have the form

$$\boldsymbol{\varepsilon}=\begin{pmatrix}\varepsilon_0\\ 0_{m\times1}\end{pmatrix},\qquad \boldsymbol{\eta}=\begin{pmatrix}0_{n\times1}\\ \eta_0\end{pmatrix},$$

where $a_{l1}(\lambda)\varepsilon_0 = a_{r4}(\lambda)\eta_0 = 0$. Hence the eigenvalues of H in $\mathbb{C}^+$ are the zeros of $\det a_{l1}(\lambda) = 0$ or, equivalently [cf. Proposition 3.12], of $\det a_{r4}(\lambda) = 0$ and the eigenfunctions correspond to the nontrivial vectors $\varepsilon_0 \in \operatorname{Ker} a_{l1}(\lambda)$ and $\eta_0 \in \operatorname{Ker} a_{r4}(\lambda)$, which implies that these null spaces have the same dimension. Therefore, for $\lambda \in \mathbb{C}^+$ the eigenfunctions of H have either of the equivalent forms

$$\begin{pmatrix} F_{l1}(x,\lambda)\varepsilon_0 \\ F_{l3}(x,\lambda)\varepsilon_0 \end{pmatrix} = \begin{pmatrix} F_{r2}(x,\lambda)\eta_0 \\ F_{r4}(x,\lambda)\eta_0 \end{pmatrix}, \tag{3.83}$$

where $0 \neq \varepsilon_0 \in \mathbb{C}^n$ and $0 \neq \eta_0 \in \mathbb{C}^m$ satisfy $a_{l1}(\lambda)\varepsilon_0 = a_{r4}(\lambda)\eta_0 = 0$. Analogously, for $\lambda \in \mathbb{C}^-$ the eigenfunctions of H have either of the equivalent forms

$$\begin{pmatrix} F_{r1}(x,\lambda)\tilde{\varepsilon}_0 \\ F_{r3}(x,\lambda)\tilde{\varepsilon}_0 \end{pmatrix} = \begin{pmatrix} F_{l2}(x,\lambda)\tilde{\eta}_0 \\ F_{l4}(x,\lambda)\tilde{\eta}_0 \end{pmatrix}, \tag{3.84}$$

where $0 \neq \tilde{\varepsilon}_0 \in \mathbb{C}^n$ and $0 \neq \tilde{\eta}_0 \in \mathbb{C}^m$ satisfy $a_{r1}(\lambda)\tilde{\varepsilon}_0 = a_{l4}(\lambda)\tilde{\eta}_0 = 0$.

To obtain the Jordan chains corresponding to a eigenvalue $\lambda \in \mathbb{C}^+$ of H, we should solve the differential equations

$$-iJ\frac{dX_0(x,\lambda)}{dx} - V(x)X_0(x,\lambda) = \lambda X_0(x,\lambda), \tag{3.85a}$$

$$-iJ\frac{dX_s(x,\lambda)}{dx} - V(x)X_s(x,\lambda) = \lambda X_s(x,\lambda) + X_{s-1}(x,\lambda), \tag{3.85b}$$

where $X_0(x,\lambda)$ is any of the two vector functions in (3.83). Further, $0 \neq \varepsilon_0 \in \mathbb{C}^n$ and $0 \neq \eta_0 \in \mathbb{C}^m$ should satisfy $a_{l1}(\lambda)\varepsilon_0 = a_{r4}(\lambda)\eta_0 = 0$ and all of the vector functions $X_s(\cdot,\lambda)$ should belong to $L^2(\mathbb{R};\mathbb{C}^{n+m})$. Similarly, to find the Jordan chains corresponding to any eigenvalue $\lambda \in \mathbb{C}^-$ of H, we should solve eqs. (3.85) starting from any of the two vector functions in (3.84), where $0 \neq \tilde{\varepsilon}_0 \in \mathbb{C}^n$ and $0 \neq \tilde{\eta}_0 \in \mathbb{C}^m$ satisfy $a_{r1}(\lambda)\varepsilon_0 = a_{l4}(\lambda)\eta_0 = 0$ and all of the vector functions $X_s(\cdot,\lambda) \in L^2(\mathbb{R};\mathbb{C}^{n+m})$.

To derive equations of the type (3.85) from any solution $X(x,\lambda)$ of (3.1), we calculate its successive derivatives with respect to λ and obtain

$$\begin{aligned}
-iJ\frac{dX(x,\lambda)}{dx} - V(x)X(x,\lambda) &= \lambda X(x,\lambda), \\
-iJ\frac{\partial}{\partial x}\frac{\partial X}{\partial \lambda}(x,\lambda) - V(x)\frac{\partial X}{\partial \lambda}(x,\lambda) &= \lambda\frac{\partial X}{\partial \lambda}(x,\lambda) + X(x,\lambda), \\
-iJ\frac{\partial}{\partial x}\frac{\partial^2 X}{\partial \lambda^2}(x,\lambda) - V(x)\frac{\partial^2 X}{\partial \lambda^2}(x,\lambda) &= \lambda\frac{\partial^2 X}{\partial \lambda^2}(x,\lambda) + 2\frac{\partial X}{\partial \lambda}(x,\lambda), \\
\vdots \qquad\qquad\qquad \vdots \qquad &\qquad\quad \vdots \qquad\qquad\qquad \vdots \\
-iJ\frac{\partial}{\partial x}\frac{\partial^s X}{\partial \lambda^s}(x,\lambda) - V(x)\frac{\partial^s X}{\partial \lambda^s}(x,\lambda) &= \lambda\frac{\partial^s X}{\partial \lambda^s}(x,\lambda) + s\frac{\partial^{s-1} X}{\partial \lambda^{s-1}}(x,\lambda).
\end{aligned}$$

Thus $X_s(x,\lambda) = (s!)^{-1}(\partial/\partial\lambda)^s X(x,\lambda)$ satisfies the differential equations (3.85). Once the solutions $X_0(x,\lambda),\ldots,X_{s-1}(x,\lambda)$ are known, any two particular solutions $X_s(x,\lambda)$ of the subsequent differential equation differ by an arbitrary (λ-dependent) linear combination of the solutions $X_0(x,\lambda),\ldots,X_{s-1}(x,\lambda)$.

Putting $F_{l,s}(x,\lambda) = (s!)^{-1}(\partial/\partial\lambda)^s F_l(x,\lambda)$, we depart from $\varepsilon_0 \in \mathbb{C}^n$ satisfying $a_{l1}(\lambda)\varepsilon_0 = 0$, where $\lambda \in \mathbb{C}^+$, and define $\varepsilon_1, \ldots, \varepsilon_{s-1}, \ldots$ by

$$\begin{aligned}
\psi_0(x) &= F_{l,0}(x,\lambda)\begin{pmatrix}\varepsilon_0\\0\end{pmatrix},\\
\psi_1(x) &= F_{l,1}(x,\lambda)\begin{pmatrix}\varepsilon_0\\0\end{pmatrix} + F_{l,0}(x,\lambda)\begin{pmatrix}\varepsilon_1\\0\end{pmatrix},\\
\psi_2(x) &= F_{l,2}(x,\lambda)\begin{pmatrix}\varepsilon_0\\0\end{pmatrix} + F_{l,1}(x,\lambda)\begin{pmatrix}\varepsilon_1\\0\end{pmatrix} + F_{l,0}(x,\lambda)\begin{pmatrix}\varepsilon_2\\0\end{pmatrix},\\
\vdots \quad & \vdots \quad \vdots \quad \vdots\\
\psi_{s-1}(x) &= \sum_{\sigma=0}^{s-1} F_{l,\sigma}(x,\lambda)\begin{pmatrix}\varepsilon_{s-\sigma-1}\\0\end{pmatrix}.
\end{aligned}$$

Then $\{\psi_0, \psi_1, \ldots, \psi_{s-1}\}$ is a Jordan chain of length s if and only if $\varepsilon_0 \neq 0$ and $\psi_0(-\infty) = \psi_1(-\infty) = \ldots = \psi_{s-1}(-\infty) = 0$. Computing the asymptotic behavior of the first n components of each of the vectors $\psi_0(x), \ldots, \psi_{s-1}(x)$ we obtain for $t = 1, \ldots, s-1$

$$\psi_t^{\text{up}}(x) = \begin{cases} e^{i\lambda x}\displaystyle\sum_{\sigma=0}^{t} \frac{(ix)^\sigma}{\sigma!}\varepsilon_{t-\sigma}[1+o(1)], & x \to +\infty,\\ \displaystyle\sum_{\sigma=0}^{t} \frac{1}{\sigma!}\left(\frac{d}{d\lambda}\right)^\sigma e^{i\lambda x} a_{l1}(\lambda)\varepsilon_{t-\sigma} + o(1), & x \to -\infty. \end{cases} \tag{3.86}$$

Writing the second line of (3.86) as

$$e^{i\lambda x}\sum_{r=0}^{t}\frac{(ix)^r}{r!}\sum_{\rho=0}^{t-r}\frac{1}{\rho!}\left(\frac{d}{d\lambda}\right)^\rho a_{l1}(\lambda)\varepsilon_{t-r-\rho},$$

we obtain the identity

$$\Phi_s(a_{l1};\lambda)\begin{pmatrix}\varepsilon_0\\ \vdots\\ \varepsilon_{s-1}\end{pmatrix} = 0, \tag{3.87}$$

where $0 \neq \varepsilon_0 \in \mathbb{C}^n$ and Φ_s is defined by eq. (2.2).

Next, putting $F_{r,s}(x,\lambda) = (s!)^{-1}(\partial/\partial\lambda)^s F_r(x,\lambda)$, we depart from $\eta_0 \in \mathbb{C}^m$ and define the vectors $\eta_1, \ldots, \eta_{s-1}, \ldots$ by

$$\begin{aligned}
\tilde{\psi}_0(x) &= F_{r,0}(x,\lambda)\begin{pmatrix}0\\ \eta_0\end{pmatrix},\\
\tilde{\psi}_1(x) &= F_{r,1}(x,\lambda)\begin{pmatrix}0\\ \eta_0\end{pmatrix} + F_{r,0}(x,\lambda)\begin{pmatrix}0\\ \eta_1\end{pmatrix},
\end{aligned}$$

$$\tilde{\psi}_2(x) = F_{r,2}(x,\lambda)\begin{pmatrix}0\\ \eta_0\end{pmatrix} + F_{r,1}(x,\lambda)\begin{pmatrix}0\\ \eta_1\end{pmatrix} + F_{r,0}(x,\lambda)\begin{pmatrix}0\\ \eta_2\end{pmatrix},$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\tilde{\psi}_{s-1}(x) = \sum_{\sigma=0}^{s-1} F_{r,\sigma}(x,\lambda)\begin{pmatrix}0\\ \eta_{s-\sigma-1}\end{pmatrix}.$$

Then $\{\tilde{\psi}_0, \tilde{\psi}_1, \ldots, \tilde{\psi}_{s-1}\}$ is a Jordan chain of length s if and only if $\eta_0 \neq 0$ and $\tilde{\psi}_0(+\infty) = \tilde{\psi}_1(+\infty) = \ldots = \tilde{\psi}_{s-1}(+\infty) = 0$. Computing the asymptotic behavior of the last m components of each of the vectors $\eta_0(x), \ldots, \eta_{s-1}(x)$ we obtain for $t = 1, \ldots, s-1$

$$\eta_t^{\mathrm{dn}}(x) = \begin{cases} \displaystyle\sum_{\sigma=0}^{t} \frac{1}{\sigma!}\left(\frac{d}{d\lambda}\right)^{\sigma} e^{-i\lambda x} a_{r4}(\lambda)\eta_{t-\sigma} + o(1), & x \to +\infty, \\ \displaystyle e^{-i\lambda x}\sum_{\sigma=0}^{t} \frac{(-ix)^{\sigma}}{\sigma!}\eta_{t-\sigma}, & x \to -\infty. \end{cases} \tag{3.88}$$

Writing the second line of (3.88) as

$$e^{-i\lambda x}\sum_{r=0}^{t} \frac{(-ix)^r}{r!}\sum_{\rho=0}^{t-r}\frac{1}{\rho!}\left(\frac{d}{d\lambda}\right)^{\rho} a_{r4}(\lambda)\eta_{t-r-\rho},$$

we obtain the identity

$$\Phi_s(a_{r4};\lambda)\begin{pmatrix}\eta_0\\ \vdots \\ \eta_{s-1}\end{pmatrix} = 0, \tag{3.89}$$

where $0 \neq \eta_0 \in \mathbb{C}^m$.

We have therefore derived the Jordan structure of H at any eigenvalue $\lambda \in \mathbb{C}^+$ in two different ways, relating it first to the Jordan structure of $a_{l1}(\cdot)$ at λ and then to the Jordan structure of $a_{r4}(\cdot)$ at λ. Consequently, for $s = 1, 2, 3, \ldots$ and $\lambda \in \mathbb{C}^+$ we have

$$\dim \operatorname{Ker} \Phi_s(a_{l1};\lambda) = \dim \operatorname{Ker} \Phi_s(a_{r4};\lambda).$$

In the same way we deduce that for $s = 1, 2, 3, \ldots$ and $\lambda \in \mathbb{C}^-$

$$\dim \operatorname{Ker} \Phi_s(a_{r1};\lambda) = \dim \operatorname{Ker} \Phi_s(a_{l4};\lambda),$$

which completes the proof. ■

Corollary 3.17 *Let $\lambda_0 \in \mathbb{C}^+ \cup \mathbb{C}^-$ be a bound state. Then the geometrical multiplicity of λ_0 as an eigenvalue of H does not exceed* $\min(n, m)$.

Proof. Suppose $n \leq m$ and $\lambda_0 \in \mathbb{C}^+$. Then Theorem 3.16 implies that the geometric multiplicity of λ_0 as an eigenvalue of H coincides with the dimension of the kernel of $a_{l1}(\lambda_0)$, which cannot exceed n, the order of the square matrix $a_{l1}(\lambda_0)$. The proof is similar in the cases $n \geq m$ and/or $\lambda_0 \in \mathbb{C}^-$. ■

Corollary 3.17 implies that the discrete eigenvalues of the Zakharov-Shabat ($n = m = 1$) and Manakov ($n = 1$, $m = 2$) systems are geometrically simple, i.e., that there cannot be more than one Jordan block per eigenvalue.

3.6 From reflection coefficient to scattering matrix

In this section we review some well-known results on the canonical Wiener-Hopf factorization of matrix functions on the line and apply them to construct the scattering matrix from one reflection coefficient in the symmetric case and in the antisymmetric case.

Suppose W is a $p \times p$ matrix function defined on the extended real line which is continuous on $\mathbb{R}$ and at $\pm\infty$. Then

$$W(\lambda) = W_+(\lambda)W_-(\lambda), \qquad \lambda \in \mathbb{R} \cup \{\infty\},$$

is called a *left canonical (Wiener-Hopf) factorization* of W if

1. $W_\pm(\lambda)$ extends to a $p \times p$ matrix function that is analytic in $\lambda \in \mathbb{C}^\pm$, continuous in $\lambda \in \overline{\mathbb{C}^\pm}$, and has a limit as $\lambda \to \infty$ in $\overline{\mathbb{C}^\pm}$.
2. $\det W_\pm(\lambda) \neq 0$ for all $\lambda \in \overline{\mathbb{C}^\pm}$ and as $\lambda \to \infty$ in $\overline{\mathbb{C}^\pm}$.

A factorization of W of the form

$$W(\lambda) = W_-(\lambda)W_+(\lambda), \qquad \lambda \in \mathbb{R} \cup \{\infty\},$$

where the factors $W_\pm$ have the properties 1-2 stated above, is called a *right canonical (Wiener-Hopf) factorization* of W. If $W = W_+W_- = \tilde{W}_+\tilde{W}_-$ are two left canonical factorizations of the same matrix function, then there exists a nonsingular matrix G such that $W_+(\lambda)G = \tilde{W}_+(\lambda)$ and $W_-(\lambda) = G\tilde{W}_-(\lambda)$. The same uniqueness property is true for right canonical factorizations. If W has either a left or a right canonical factorization, then $\det W(\lambda) \neq 0$ for each $\lambda \in \mathbb{R}$ and as $\lambda \to \pm\infty$ and its winding number (with respect to $+i$) vanishes. Wiener-Hopf factorizations of matrix functions have been studied in detail in [59, 38].

Let us recall the following well-known result [59, 53, 38] on the canonical factorization of positive selfadjoint matrix functions of Wiener class.

Theorem 3.18 *Let $F \in L^1(\mathbb{R}; \mathbb{C}^{p\times p})$ be such that*

$$\hat{W}(\lambda) = I_p + \int_{-\infty}^{\infty} dt\, e^{i\lambda t} F(t) \tag{3.90}$$

is positive and selfadjoint for $\lambda \in \mathbb{R}$. Then there exist unique functions $F_+ \in L^1(\mathbb{R}^+; \mathbb{C}^{p\times p})$ and $G_+ \in L^1(\mathbb{R}^+; \mathbb{C}^{p\times p})$ such that

$$\hat{W}(\lambda) = \left[I_p + \int_0^{\infty} dt\, e^{i\lambda t} F_+(t)\right] \left[I_p + \int_0^{\infty} dt\, e^{-i\lambda t} F_+(t)\right]^*, \tag{3.91}$$

$$\hat{W}(\lambda) = \left[I_p + \int_0^{\infty} dt\, e^{-i\lambda t} G_+(t)\right]^* \left[I_p + \int_0^{\infty} dt\, e^{i\lambda t} G_+(t)\right], \tag{3.92}$$

while

$$\det\left(\left[I_p+\int_0^\infty dt\, e^{i\lambda t}F_+(t)\right]\right)\neq 0, \qquad \lambda\in\overline{\mathbb{C}^+}, \tag{3.93}$$

$$\det\left(\left[I_p+\int_0^\infty dt\, e^{i\lambda t}G_+(t)\right]\right)\neq 0, \qquad \lambda\in\overline{\mathbb{C}^+}. \tag{3.94}$$

Next, we apply this result to the antisymmetric and symmetric cases. First we consider the *antisymmetric case.* In this case, we know that the scattering matrix $S(\lambda)$ is J-unitary. So, using this information, it is possible to construct the scattering matrix $S(\lambda)$ in terms of $L(\lambda)$ or $R(\lambda)$ alone. In fact, we find the following identities

$$T_l(\lambda)\,T_l(\lambda)^*-R(\lambda)\,R(\lambda)^*=I_n, \qquad T_r(\lambda)^*\,T_r(\lambda)-R(\lambda)^*\,R(\lambda)=I_m, \tag{3.95}$$

$$T_l(\lambda)^*\,T_l(\lambda)-L(\lambda)^*\,L(\lambda)=I_n, \qquad T_r(\lambda)\,T_r(\lambda)^*-L(\lambda)\,L(\lambda)^*=I_m, \tag{3.96}$$

$$T_r(\lambda)\,R(\lambda)^*-L(\lambda)\,T_l(\lambda)^*=0_{m\times n}, \quad T_r(\lambda)^*\,L(\lambda)-R(\lambda)^*\,T_l(\lambda)=0_{n\times m}. \tag{3.97}$$

Now given $R(\lambda)$ for $\lambda\in\mathbb{R}$ and assuming it to be continuous for $\lambda\in\mathbb{R}$ and to belong to $\mathcal{W}_0^{n\times n}$ (see Proposition 3.11 for a comparison), by applying Theorem 3.18 we obtain the unique matrix function $T_{l0}(\lambda)$ which is continuous on $\overline{\mathbb{C}^+}$, is analytic on $\mathbb{C}^+$, tends to I_n as $\lambda\to\infty$ in $\overline{\mathbb{C}^+}$, and which satisfies the following relation

$$T_{l0}(\lambda)\,T_{l0}(\lambda)^*=I_n+R(\lambda)\,R(\lambda)^*, \qquad \lambda\in\mathbb{R}. \tag{3.98}$$

In a similar way, the matrix function $T_{r0}(\lambda)$ which is continuous on $\overline{\mathbb{C}^+}$, is analytic on $\mathbb{C}^+$, belongs to $\mathcal{W}_0^{n\times n}$ and tends to I_m as $\lambda\to\infty$ in $\overline{\mathbb{C}^+}$, is constructed in the following way:

$$T_{r0}(\lambda)^*\,T_{r0}(\lambda)=I_m+R(\lambda)^*\,R(\lambda), \qquad \lambda\in\mathbb{R}. \tag{3.99}$$

We then define the matrix function

$$L_0(\lambda)=T_{r0}(\lambda)\,R(\lambda)^*\,[T_{l0}(\lambda)^*]^{-1}, \qquad \lambda\in\mathbb{R}. \tag{3.100}$$

It easily follows that $T_{l0}(\lambda)$, $T_{r0}(\lambda)$, $R(\lambda)$, and $L_0(\lambda)$ are the scattering coefficients if $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are nonsingular for $\lambda\in\overline{\mathbb{C}^+}$ (i.e., in the absence of bound states).

Next consider the four factorizations given in (3.95) and (3.96), where both $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are nonsingular for $\lambda\in\mathbb{R}$ and at least one (and hence both) of $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ is singular for some $\lambda\in\mathbb{C}^+$ (i.e., in the presence of bound states). Then there exist matrices $B_l(\lambda)$ and $B_r(\lambda)$ such that

$$T_l(\lambda)=T_{l0}(\lambda)B_l(\lambda), \qquad T_r(\lambda)=B_r(\lambda)T_{r0}(\lambda). \tag{3.101}$$

Then we easily see that

$$L(\lambda)=B_r(\lambda)L_0(\lambda)B_l(\lambda). \tag{3.102}$$

The matrices $B_l(\lambda)$ and $B_r(\lambda)$ have the following properties:

- they are unitary for $\lambda\in\mathbb{R}$;
- they tend to the identity matrix as $\lambda\to\pm\infty$;

- they have no real poles;
- they have the same Jordan structure,

where the last statement follows from Theorem 3.16. Further, B_l and B_r can be extended meromorphically to $\lambda \in \mathbb{C}^-$ by putting $B_{l,r}(\lambda) = B_{l,r}(\overline{\lambda})^{-1}$. Since multiplication of the extended $B_{l,r}(\lambda)$ by a suitable scalar polynomial creates an entire matrix function of polynomial growth at infinity and hence a matrix polynomial, the matrix functions $B_l(\lambda)$ and $B_r(\lambda)$ must be rational. So we can conclude that $B_l(\lambda)$ and $B_r(\lambda)$ are rational matrix functions with the same Jordan structure.

Next we consider the *symmetric case*. Using the unitarity of the scattering matrix $S(\lambda)$, we obtain the identities

$$T_l(\lambda)\,T_l(\lambda)^* + R(\lambda)\,R(\lambda)^* = I_n, \qquad T_r(\lambda)^*\,T_r(\lambda) + R(\lambda)^*\,R(\lambda) = I_m, \tag{3.103}$$

$$T_l(\lambda)^*\,T_l(\lambda) + L(\lambda)^*\,L(\lambda) = I_n, \qquad T_r(\lambda)\,T_r(\lambda)^* + L(\lambda)\,L(\lambda)^* = I_m, \tag{3.104}$$

$$T_r(\lambda)\,R(\lambda)^* + L(\lambda)\,T_l(\lambda)^* = 0_{m\times n}, \quad T_r(\lambda)^*\,L(\lambda) + R(\lambda)^*\,T_l(\lambda) = 0_{n\times m}. \tag{3.105}$$

We must take into account that $T_l(\lambda)$ and $T_r(\lambda)$ are continuous in $\lambda \in \mathbb{R}$, tend to identity as $\lambda \to \infty$, and satisfy (3.78)-(3.79). Then, given $R(\lambda)$ for $\lambda \in \mathbb{R}$ and assuming it to be continuous for $\lambda \in \mathbb{R}$, applying Theorem 3.18 we obtain the matrix functions $T_l(\lambda)$ and $T_r(\lambda)$ which are continuous on $\overline{\mathbb{C}^+}$, are analytic on $\mathbb{C}^+$ and tend to the identity matrix as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$, which satisfy the factorizations

$$T_l(\lambda)\,T_l(\lambda)^* = I_n - R(\lambda)\,R(\lambda)^*, \qquad \lambda \in \mathbb{R}, \tag{3.106}$$

$$T_r(\lambda)^*\,T_r(\lambda) = I_m - R(\lambda)^*\,R(\lambda), \qquad \lambda \in \mathbb{R}. \tag{3.107}$$

Note that the absence of bound states implies that $T_l(\lambda)$ and $T_r(\lambda)$ are analytic in $\lambda \in \mathbb{C}^+$. We then define the reflection coefficient from the left by

$$L(\lambda) = -T_r(\lambda)\,R(\lambda)^*\,[T_l(\lambda)^*]^{-1}, \qquad \lambda \in \mathbb{R}. \tag{3.108}$$

Chapter 4

Inverse Scattering Theory

In this chapter we develop the inverse scattering theory of determining the potentials $k(x)$ and $l(x)$ from one of the reflection coefficients and bound state data. The potentials follow immediately from the values at $\alpha = 0^+$ of the solutions $B(x,\alpha)$ (where $x \in \mathbb{R}$ and $\alpha \in \mathbb{R}^+$) of the Marchenko integral equations, which are in turn derived from the Riemann-Hilbert problem satisfied by the Faddeev matrices. We first discuss the general form of the Marchenko integral equations, both in the so-called coupled and uncoupled forms, and how to retrieve the potentials from its solution. We then go on to discuss its symmetries (separately for the symmetric and antisymmetric cases), the compactness of the integral operator, and, in the symmetric and antisymmetric cases, its unique solvability. Next, starting from the Riemann-Hilbert problem for the Faddeev matrices, we derive its integral kernel explicitly, first in the absence of bound states, next if there are only algebraically simple eigenvalues, and then in general. After a discussion of the symmetries of the Marchenko integral kernels, we conclude by characterizing the scattering data in the absence of bound states.

4.1 Analysis of the Marchenko equations

In this section we analyze the Marchenko integral equations without bothering with the precise form of their integral kernels.

4.1.1 Coupled and uncoupled Marchenko equations

The Marchenko integral equations have as their solutions one of the eight matrix functions $B_{ls}(x,\alpha)$ and $B_{rs}(x,\alpha)$ $(s = 1,2,3,4)$, where $\alpha \in \mathbb{R}^+$ is the independent variable and $x \in \mathbb{R}$ is a parameter. These eight matrix functions are related to the Faddeev matrices $M_l(x,\lambda)$ and $M_r(x,\lambda)$ by (3.35) and (3.36). According to Theorem 3.9, they satisfy

$$\sup_{x\in\mathbb{R}} \int_0^\infty d\alpha \left(\|B_{ls}(x,\alpha)\| + \|B_{rs}(x,\alpha)\|\right) < \infty, \qquad s = 1,2,3,4,$$

provided the potentials $k(x)$ and $l(x)$ have their entries in $L^1(\mathbb{R})$. The relationship between the potentials and the solutions of the Marchenko equations is given by (cf. (3.54) and (3.55))

$$k(x) = 2iB_{l2}(x,0^+) = -2iB_{r2}(x,0^+), \tag{4.1}$$

$$l(x) = -2iB_{l3}(x,0^+) = 2iB_{r3}(x,0^+). \tag{4.2}$$

In Sec. 4.2 the integral kernels Ω_l, Ω_r, $\overline{\Omega}_l$, and $\overline{\Omega}_r$ will be expressed in the reflection coefficients and bound state data. Here we are primarily concerned with their properties rather than their explicit form.

The eight *coupled Marchenko equations* are given by

$$B_{r1}(x,\alpha) = -\int_0^\infty d\beta\, B_{r2}(x,\beta)\,\Omega_r(\alpha+\beta-2x), \tag{4.3a}$$

$$B_{l2}(x,\alpha) = -\Omega_l(\alpha+2x) - \int_0^\infty d\beta\, B_{l1}(x,\beta)\,\Omega_l(\alpha+\beta+2x), \tag{4.3b}$$

$$B_{r3}(x,\alpha) = -\Omega_r(\alpha-2x) - \int_0^\infty d\beta\, B_{r4}(x,\beta)\,\Omega_r(\alpha+\beta-2x), \tag{4.3c}$$

$$B_{l4}(x,\alpha) = -\int_0^\infty d\beta\, B_{l3}(x,\beta)\,\Omega_l(\alpha+\beta+2x), \tag{4.3d}$$

and

$$B_{l1}(x,\alpha) = -\int_0^\infty d\beta\, B_{l2}(x,\beta)\,\overline{\Omega}_l(\alpha+\beta+2x), \tag{4.4a}$$

$$B_{r2}(x,\alpha) = -\overline{\Omega}_r(\alpha-2x) - \int_0^\infty d\beta\, B_{r1}(x,\beta)\,\overline{\Omega}_r(\alpha+\beta-2x), \tag{4.4b}$$

$$B_{l3}(x,\alpha) = -\overline{\Omega}_l(\alpha+2x) - \int_0^\infty d\beta\, B_{l4}(x,\beta)\,\overline{\Omega}_l(\alpha+\beta+2x), \tag{4.4c}$$

$$B_{r4}(x,\alpha) = -\int_0^\infty d\beta\, B_{r3}(x,\beta)\,\overline{\Omega}_r(\alpha+\beta-2x). \tag{4.4d}$$

In other words, these eight equations consist of four pairs of two coupled integral equations.

Let us formally write (4.3a)-(4.4d) as follows:

$$\begin{pmatrix} B_{l1} & B_{l2} \\ B_{l3} & B_{l4} \end{pmatrix} + \begin{pmatrix} B_{l1} & B_{l2} \\ B_{l3} & B_{l4} \end{pmatrix} \begin{pmatrix} 0_n & \Omega_l \\ \overline{\Omega}_l & 0_m \end{pmatrix} = -\begin{pmatrix} 0_n & \Omega_l \\ \overline{\Omega}_l & 0_m \end{pmatrix} \tag{4.5a}$$

$$\begin{pmatrix} B_{r1} & B_{r2} \\ B_{r3} & B_{r4} \end{pmatrix} + \begin{pmatrix} B_{r1} & B_{r2} \\ B_{r3} & B_{r4} \end{pmatrix} \begin{pmatrix} 0_n & \overline{\Omega}_r \\ \Omega_r & 0_m \end{pmatrix} = -\begin{pmatrix} 0_n & \overline{\Omega}_r \\ \Omega_r & 0_m \end{pmatrix}, \tag{4.5b}$$

where the integral operators depend on the parameter $x \in \mathbb{R}$. Passing to the conjugate transpose we have

$$\begin{pmatrix} B_{l1}^* & B_{l3}^* \\ B_{l2}^* & B_{l4}^* \end{pmatrix} + \begin{pmatrix} 0_n & \overline{\Omega}_l^* \\ \Omega_l^* & 0_m \end{pmatrix} \begin{pmatrix} B_{l1}^* & B_{l3}^* \\ B_{l2}^* & B_{l4}^* \end{pmatrix} = -\begin{pmatrix} 0_n & \overline{\Omega}_l^* \\ \Omega_l^* & 0_m \end{pmatrix} \tag{4.6a}$$

$$\begin{pmatrix} B_{r1}^* & B_{r3}^* \\ B_{r2}^* & B_{r4}^* \end{pmatrix} + \begin{pmatrix} 0_n & \Omega_r^* \\ \overline{\Omega}_r^* & 0_m \end{pmatrix} \begin{pmatrix} B_{r1}^* & B_{r3}^* \\ B_{r2}^* & B_{r4}^* \end{pmatrix} = -\begin{pmatrix} 0_n & \Omega_r^* \\ \overline{\Omega}_r^* & 0_m \end{pmatrix}. \tag{4.6b}$$

Substituting one equation of a pair of coupled equations into the other and vice versa, we obtain eight uncoupled Marchenko equations. In fact, in view of eqs. (4.1) and(4.2) we only list the equations for $B_{l2}(x,\alpha)$, $B_{l3}(x,\alpha)$, $B_{r2}(x,\alpha)$ and $B_{r3}(x,\alpha)$, which turn out to have less complicated inhomogeneous terms than those for $B_{l1}(x,\alpha)$, $B_{l4}(x,\alpha)$, $B_{r1}(x,\alpha)$ and $B_{r4}(x,\alpha)$. For example, the uncoupled Marchenko equation for $B_{r1}(x,\alpha)$ is rather complicated and reads

$$\begin{aligned} B_{r1}(x,\alpha) = &\int_0^\infty d\gamma\, \overline{\Omega}_r(\alpha+\gamma-2x)\,\Omega_r(\gamma-2x)+ \\ &+\int_0^\infty d\beta B_{r1}(x,\beta)\int_0^\infty d\gamma\,\overline{\Omega}_r(\gamma+\beta-2x)\,\Omega_r(\alpha+\gamma-2x). \end{aligned}$$

Formally we thus obtain the *uncoupled Marchenko equations*

$$B_{l2}\left(I-\overline{\Omega}_l\Omega_l\right) = -\Omega_l, \tag{4.7a}$$
$$B_{l3}\left(I-\Omega_l\overline{\Omega}_l\right) = -\overline{\Omega}_l, \tag{4.7b}$$
$$B_{r2}\left(I-\Omega_r\overline{\Omega}_r\right) = -\overline{\Omega}_r, \tag{4.7c}$$
$$B_{r3}\left(I-\overline{\Omega}_r\Omega_r\right) = -\Omega_r, \tag{4.7d}$$

as well as their conjugate transposes

$$\left(I-\Omega_l^*\,\overline{\Omega}_l^*\right)B_{l2}^* = -\Omega_l^*, \tag{4.8a}$$
$$\left(I-\overline{\Omega}_l^*\,\Omega_l^*\right)B_{l3}^* = -\overline{\Omega}_l^*, \tag{4.8b}$$
$$\left(I-\overline{\Omega}_r^*\,\Omega_r^*\right)B_{r2}^* = -\overline{\Omega}_r^*, \tag{4.8c}$$
$$\left(I-\Omega_r^*\,\overline{\Omega}_r^*\right)B_{r3}^* = -\Omega_r^*, \tag{4.8d}$$

where the operators and right-hand sides depend on the parameter $x \in \mathbb{R}$.

In the symmetric case ($l(x) = k(x)^*$) and the antisymmetric case ($l(x) = -k(x)^*$), the integral kernels turn out to satisfy the following symmetry relations (cf. Subsection 4.2.4):

$$\begin{cases} \overline{\Omega}_l(\alpha) = \Omega_l(\alpha)^* \text{ and } \overline{\Omega}_r(\alpha) = \Omega_r(\alpha)^*, & \text{symmetric case,} \\ \overline{\Omega}_l(\alpha) = -\Omega_l(\alpha)^* \text{ and } \overline{\Omega}_r(\alpha) = -\Omega_r(\alpha)^*, & \text{antisymmetric case.} \end{cases} \tag{4.9}$$

Specifying the (adjoint) uncoupled Marchenko equations (4.5) in these special cases we obtain in the **symmetric case**

$$\left(I-\Omega_l^*\,\Omega_l\right)B_{l2}^* = -\Omega_l^*, \tag{4.10a}$$
$$\left(I-\Omega_l\,\Omega_l^*\right)B_{l3}^* = -\Omega_l, \tag{4.10b}$$
$$\left(I-\Omega_r\,\Omega_r^*\right)B_{r2}^* = -\Omega_r, \tag{4.10c}$$
$$\left(I-\Omega_r^*\,\Omega_r\right)B_{r3}^* = -\Omega_r^*, \tag{4.10d}$$

and in the **antisymmetric case**

$$(I+\Omega_l^*\,\Omega_l)\,B_{l2}^* = -\Omega_l^*, \tag{4.11a}$$
$$(I+\Omega_l\,\Omega_l^*)\,B_{l3}^* = -\Omega_l, \tag{4.11b}$$
$$(I+\Omega_r\,\Omega_r^*)\,B_{r2}^* = -\Omega_r, \tag{4.11c}$$
$$(I+\Omega_r^*\,\Omega_r)\,B_{r3}^* = -\Omega_r^*. \tag{4.11d}$$

4.1.2 Compactness of the Marchenko integral operator

In the sequel it will be apparent that, for every $x \in \mathbb{R}$, each of the eight Marchenko integral kernels is a matrix function $\Omega(\alpha+\beta)$ depending only on the sum $\alpha+\beta$ of its arguments $\alpha, \beta \in \mathbb{R}^+$, where each entry of Ω belongs to $L^1(\mathbb{R}^+)$. Such integral operators are specific cases of so-called Hankel operators and are known to be compact on a variety of function spaces [54]. Here we shall actually prove these important properties.

Let Ω belong to $L^1(\mathbb{R}^+)$. We consider the operator

$$(H_\Omega b)\,(\alpha) = \int_0^\infty d\beta\,\Omega(\alpha+\beta)b(\beta)$$

and prove the following

Proposition 4.1 *Let p be a real number such that $1 \le p \le \infty$. Then H_Ω is bounded in $L^p(\mathbb{R}^+)$ and the inequality*

$$\|H_\Omega\| \le \|\Omega\|_{L^1(\mathbb{R}^+)}$$

is satisfied.

Proof. When $p = 1$, we have

$$\int_0^\infty d\alpha\ |(H_\Omega b)\,(\alpha)| \le \int_0^\infty d\alpha \int_0^\infty d\beta\ |\Omega(\alpha+\beta)|\ |b(\beta)|$$
$$= \int_0^\infty d\beta \int_\beta^\infty d\gamma\ |\Omega(\gamma)|\ |b(\beta)| \le \|\Omega\|_{L^1(\mathbb{R}^+)}\,\|b\|_{L^1(\mathbb{R}^+)}\,.$$

For $p = \infty$, we find

$$|(H_\Omega b)\,(\alpha)| \le \|b\|_{L^\infty(\mathbb{R}^+)} \int_0^\infty d\beta\ |\Omega(\alpha+\beta)| \le \|\Omega\|_1\ \|b\|_\infty\,.$$

When $p = 2$, we consider the Fourier transform. For $b^\sharp(\beta) = b(-\beta)$ we have the identity $\hat{b^\sharp}(\omega) = \int_{-\infty}^\infty d\beta\, e^{i\omega\beta} b(-\beta) = \hat{b}(-\omega)$, and hence

$$\left(\widehat{H_\Omega b}\right)(\omega) = \hat{\Omega}(\omega)\hat{b}(-\omega).$$

It is clear that H_Ω can also be represented by the diagram

$$L^2(\mathbb{R}^+) \xrightarrow{\mathcal{F}} H^2(\mathbb{C}^+) \xrightarrow{\sharp} H^2(\mathbb{C}^-) \xrightarrow{\hat{\Omega}-\phi} L^2(\mathbb{R}) \xrightarrow{\mathcal{F}^{-1}} L^2(\mathbb{R}) \xrightarrow{\Pi^+} L^2(\mathbb{R}^+),$$

where ϕ is any bounded analytic function in $\mathbb{C}^-$ [i.e., $\phi \in H^\infty(\mathbb{C}^-)$]: As a result,

$$\|H_\Omega\|_{L^2 \to L^2} \le \sup_{\omega \in \mathbb{R}} \left| \hat{\Omega}(\omega) - \phi(\omega) \right|,$$

and therefore

$$\|H_\Omega\| \le \sup_{\omega \in \mathbb{R}} \left| \hat{\Omega}(\omega) \right| \le \|\Omega\|_{L^1(\mathbb{R}^+)}.$$

The boundedness of H_Ω on the other L^p spaces follows with the help of the Riesz-Thorin interpolation Theorem [69]. In fact,

$$\|H_\Omega\|_{L^p \to L^p} \le \|H_\Omega\|_{L^1 \to L^1}^{\frac{1}{p}} \|H_\Omega\|_{L^\infty \to L^\infty}^{1-\frac{1}{p}} \le \|\Omega\|_{L^1(\mathbb{R}^+)},$$

which completes the proof. ∎

According to Nehari's Theorem [79] we have the norm equality

$$\|H_\Omega\|_{L^2 \to L^2} = \inf_{\phi \in H^\infty(\mathbb{C}^-)} \sup_{\omega \in \mathbb{R}} \left| \hat{\Omega}(\omega) - \phi(\omega) \right|.$$

In the general case we mimic the proof given in [54] to derive

Theorem 4.2 *Let $1 \le p \le \infty$ and $\Omega \in L^1(R^+)$. Then H_Ω is a compact operator on $L^p(\mathbb{R}^+)$.*

Proof. The proof is easy for $p = 2$. In this case we know that there exists a sequence $\Omega_n \in L^1(\mathbb{R}^+)$ of functions of compact support such that $\|\Omega - \Omega_n\|_1 \to 0$. So, we obtain

$$\|H_\Omega - H_{\Omega_n}\|_{L^2(\mathbb{R}^+)} = \|H_{\Omega - \Omega_n}\|_{L^2(\mathbb{R}^+)} \le \|\Omega - \Omega_n\|_1.$$

Since obviously

$$\int_0^\infty d\alpha \int_0^\infty d\beta \, |\Omega_n(\alpha+\beta)|^2 = \int_0^\alpha d\gamma \, \gamma |\Omega_n(\gamma)|^2 \le \infty$$

and hence H_{Ω_n} is a Hilbert-Schmidt operator, the compactness of H_Ω is clear.

In the general case we proceed in the following way: For $\alpha > -1$ the Laguerre polynomials $\left((\frac{n!}{\Gamma(n+\alpha+1)})^{\frac{1}{2}} L_n^{(\alpha)} \right)_{n=0}^\infty$, where Γ denotes the Gamma function, form an orthonormal basis of $L^2(\mathbb{R}^+; x^\alpha \, e^{-x} dx)$. Thus

$$\left(\left(\frac{n!}{\Gamma(n+\alpha+1)} \right)^{\frac{1}{2}} L_n^\alpha(\cdot) x^{\frac{\alpha}{2}} \, e^{-\frac{x}{2}} \right)_{n=0}^\infty$$

is an orthonormal basis of $L^2(\mathbb{R}^+)$. So there exists a sequence of polynomials $(p_n)_{n=0}^\infty$ such that

$$\lim_{n \to \infty} \left\| \Omega - p_n x^{\frac{\alpha}{2}} \, e^{-\frac{x}{2}} \right\|_1 = 0,$$

and hence for $\alpha = 0$ we get

$$p_n(x+y) \, e^{-\frac{(x+y)}{2}} = \sum_s \gamma_s \, c_s(x) c_s(y),$$

where the summation is finite, each γ_s is a real number and $c_s(x) = x^s\, e^{\frac{-x}{2}}$. Now it is easy to check that the integral operator $H_{p_n e^{-\frac{x}{2}}}$ defined by the expression

$$\left(H_{p_n\, e^{-\frac{x}{2}}} b\right)(\alpha) = \sum_s \gamma_s\, c_s(\alpha) \int_0^\infty d\beta\, c_s(\beta) b(\beta)$$

has finite rank. Thus $H_{p_n\, e^{-\frac{x}{2}}}$ is a compact operator. Since

$$\left\| H_\Omega - H_{p_n\, e^{-\frac{x}{2}}} \right\|_{L^p(\mathbb{R}^+)} \leq \left\| \Omega - p_n e^{-\frac{x}{2}} \right\|_1 \to 0,$$

also H_Ω is a compact operator on $L^p(\mathbb{R}^+)$. ■

4.1.3 Unique solvability of the Marchenko integral equations

In this subsection we prove the unique solvability of the Marchenko integral equations in the symmetric and antisymmetric cases. The proofs will be carried out for the integral equations in adjoint form, since they can more easily be modelled as the effect of a linear operator (actually, the sum of the identity and a compact operator) acting on a vector in a Banach function space. It is almost trivial to prove these results in an L^2 setting. Compactness arguments are used to transfer them from the familiar L^2 setting to the general L^p setting.

Theorem 4.3 *Let $1 \leq p \leq \infty$. Then in the symmetric and antisymmetric cases the Marchenko equations* (4.3)-(4.4) *are uniquely solvable in $L^p(\mathbb{R}^+; \mathbb{C}^k)$ for some suitable $k \in \mathbb{N}$.*

Proof. First we assume that $p = 2$. In the symmetric case there are no bound states and Ω_l and Ω_r coincide with the Fourier inverse transforms of the reflection matrices $R(\lambda)$, $L(\lambda)$, $\overline{R}(\lambda)$ and $\overline{L}(\lambda)$ apart from the factors $e^{\pm 2i\lambda x}$. Since the euclidean norms of the reflection coefficients are strictly less than 1 for any $\lambda \in \mathbb{R}$ and these coefficients are continuous in $\lambda \in \mathbb{R}$ and vanish as $\lambda \to \pm\infty$ [cf. Proposition 3.13], the Marchenko integral operators are strict contractions on $L^2(\mathbb{R}^+; \mathbb{C}^k)$, which implies the unique solvability of eqs. (4.3)-(4.4) for $p = 2$.

In the antisymmetric case we consider the uncoupled Marchenko equations. Then the linear operators $I + \Omega\,\Omega^*$ and $I + \Omega^*\,\Omega$ governing these equations are bounded on $L^2(\mathbb{R}^+; \mathbb{C}^k)$ and satisfy

$$((I + \Omega\,\Omega^*)\, b, b) = \|b\|_2^2 + \|\Omega^*\, b\|_2^2 \geq \|b\|_2^2,$$
$$((I + \Omega^*\,\Omega)\, b, b) = \|b\|_2^2 + \|\Omega\, b\|_2^2 \geq \|b\|_2^2,$$

which implies their unique solvability for $p = 2$.

Let us consider the same uncoupled Marchenko equations for arbitrary p. Such an equation has the form

$$(I + \Gamma)\, b = b_0,$$

where Γ is a compact operator on $L^p(\mathbb{R}^+; \mathbb{C}^k)$ $(1 \leq p \leq \infty)$. Let us now consider the following diagram, where all arrows represent natural (dense and continuous) imbeddings between Banach spaces:

$$\begin{array}{ccc} L^2(\mathbb{R}^+;\mathbb{C}^k) & \longrightarrow & L^2(\mathbb{R}^+;\mathbb{C}^k)+L^p(\mathbb{R}^+;\mathbb{C}^k) \\ \uparrow & & \uparrow \\ L^2(\mathbb{R}^+;\mathbb{C}^k)\cap L^p(\mathbb{R}^+;\mathbb{C}^k) & \longrightarrow & L^p(\mathbb{R}^+;\mathbb{C}^k) \end{array}$$

Here $L^2(\mathbb{R}^+;\mathbb{C}^k)\cap L^p(\mathbb{R}^+;\mathbb{C}^k)$ is a Banach space with respect to the norm

$$\|b\| = \|b\|_2 + \|b\|_p,$$

while $L^2(\mathbb{R}^+;\mathbb{C}^k)+L^p(\mathbb{R}^+;\mathbb{C}^k)$ is a Banach space with respect to the norm

$$\|b\| = \inf_{\substack{b=b_2+b_p \\ b_2\in L^2,\, b_p\in L^p}} \max\left(\|b_2\|_2, \|b_p\|_p\right),$$

If Γ is compact on $L^2(\mathbb{R}^+;\mathbb{C}^k)$ and $L^p(\mathbb{R}^+;\mathbb{C}^k)$, it is also compact on $L^2(\mathbb{R}^+;\mathbb{C}^k)\cap L^p(\mathbb{R}^+;\mathbb{C}^k)$. Hence, $I+\Gamma$ satisfies the Fredholm alternative on all four spaces $L^2(\mathbb{R}^+;\mathbb{C}^k)\cap L^p(\mathbb{R}^+;\mathbb{C}^k)$, $L^2(\mathbb{R}^+;\mathbb{C}^k)$, $L^p(\mathbb{R}^+;\mathbb{C}^k)$ and $L^2(\mathbb{R}^+;\mathbb{C}^k)+L^p(\mathbb{R}^+;\mathbb{C}^k)$. Obviously, using the invertibility of $I+\Gamma$ on $L^2(\mathbb{R}^+;\mathbb{C}^k)$, we have

$$\text{Ker}_{\,L^2(\mathbb{R}^+;\mathbb{C}^k)\cap L^p(\mathbb{R}^+;\mathbb{C}^k)}\,(I+\Gamma) = \{0\}$$

and hence, by the Fredholm alternative, $I+\Gamma$ is invertible on $L^2(\mathbb{R}^+;\mathbb{C}^k)\cap L^p(\mathbb{R}^+;\mathbb{C}^k)$. Next,

$$L^p(\mathbb{R}^+;\mathbb{C}^k) \supseteq \text{Im}_{L^p(\mathbb{R}^+;\mathbb{C}^k)}\,(I+\Gamma) \supseteq \text{Im}_{L^2(\mathbb{R}^+;\mathbb{C}^k)\cap L^p(\mathbb{R}^+;\mathbb{C}^k)}\,(I+\Gamma) = L^2(\mathbb{R}^+;\mathbb{C}^k)\cap L^p(\mathbb{R}^+;\mathbb{C}^k),$$

while the image $\text{Im}_{L^p(\mathbb{R}^+;\mathbb{C}^k)}\,(I+\Gamma)$ is closed in $L^p(\mathbb{R}^+;\mathbb{C}^k)$ and $L^2(\mathbb{R}^+;\mathbb{C}^k)\cap L^p(\mathbb{R}^+;\mathbb{C}^k)$ is dense in $L^p(\mathbb{R}^+;\mathbb{C}^k)$. Therefore

$$\text{Im}_{L^p(\mathbb{R}^+;\mathbb{C}^k)}\,(I+\Gamma) = L^p(\mathbb{R}^+;\mathbb{C}^k).$$

Hence, by the Fredholm alternative, $I+\Gamma$ is invertible on $L^p(\mathbb{R}^+;\mathbb{C}^k)$. ■

4.1.4 The x-dependence of Marchenko operators

In this subsection we prove that the solutions of the previous Marchenko integral equations lead to potentials $k(x)$ and $\ell(x)$ having their entries in $L^1(\mathbb{R})$. This requires us to study these equations as functions of $x\in\mathbb{R}$. Therefore, we introduce the notations $\Omega_l^{(x)}$, $\Omega_r^{(x)}$, $\overline{\Omega}_l^{(x)}$ and $\overline{\Omega}_r^{(x)}$ to express their dependence on x. We show that these integral operators depend continuously on x and vanish as $x\to\pm\infty$ in the operator norm, irrespective of the L^p-space of vector functions on $\mathbb{R}^+$ in which we are working.

We have the following:

Proposition 4.4 *For $x\geq 0$, the Marchenko operators $\Omega_l^{(x)}$ and $\overline{\Omega}_l^{(x)}$ depend continuously on x in the norm, while*

$$\lim_{x\to+\infty}\|\Omega_l^{(x)}\| = 0, \qquad \lim_{x\to+\infty}\|\overline{\Omega}_l^{(x)}\| = 0. \tag{4.12}$$

For $x\leq 0$, the Marchenko operators $\Omega_r^{(x)}$ and $\overline{\Omega}_r^{(x)}$ depend continuously on x in the norm, while

$$\lim_{x\to-\infty}\|\Omega_r^{(x)}\| = 0, \qquad \lim_{x\to-\infty}\|\overline{\Omega}_r^{(x)}\| = 0. \tag{4.13}$$

Proof. We give the proof only for the first equation of (4.12), because the proof of the other equations is similar. We have for $x_2 \geq x_1 \geq 0$

$$\left[H_{\Omega_l^{(x_2)}} b - H_{\Omega_l^{(x_1)}} b\right](\alpha) = \int_0^\infty d\beta \, [\Omega_l(2x_2+\alpha+\beta) - \Omega_l(2x_1+\alpha+\beta)] \, b(\beta).$$

Then, according to Proposition 4.1, in the operator norm we obtain, as a result of the Theorem of Dominated Convergence,

$$\|H_{\Omega_l^{(x_2)}} - H_{\Omega_l^{(x_1)}}\| \leq \int_0^\infty d\alpha \|\Omega_l(2x_2+\alpha) - \Omega_l(2x_1+\alpha)\| \to 0$$

as $x_2 \to x_1$. In the same way,

$$\|H_{\Omega_l^{(x)}}\| \leq \int_0^\infty d\alpha \|\Omega_l(2x+\alpha)\| = \int_{2x}^\infty d\alpha \|\Omega_l(\alpha)\| \to 0$$

as $x \to +\infty$. ∎

Corollary 4.5 *The Marchenko integral equations* (4.7a) *and* (4.7b) *are uniquely solvable for sufficiently large x. Similarly, the integral equations* (4.7c) *and* (4.7d) *are uniquely solvable for sufficiently large $-x$.*

Let us now suppose that the Marchenko integral equations (4.7a) and (4.7b) are uniquely solvable for any $x \geq 0$ and the equations (4.7c) and (4.7d) are uniquely solvable for any $x \leq 0$. This is in particularly true in the symmetric and the antisymmetric cases (cf. Theorem 4.3). Because of Proposition 4.4, the linear operators $I - \overline{\Omega}_l\Omega_l$, $I - \Omega_l\overline{\Omega}_l$, $I - \Omega_r\overline{\Omega}_r$, and $I - \overline{\Omega}_r\Omega_r$ appearing in the respective equations (4.7) depend continuously on x and vanish as $|x| \to \infty$ in the operator norm on the half-line indicated in the table below. Thus

$$\sup_{x\geq 0} \|(I - \overline{\Omega}_l^{(x)}\Omega_l^{(x)})^{-1}\| < \infty, \qquad \sup_{x\geq 0} \|(I - \Omega_l^{(x)}\overline{\Omega}_l^{(x)})^{-1}\| < \infty, \tag{4.14a}$$

$$\sup_{x\leq 0} \|(I - \Omega_r^{(x)}\overline{\Omega}_r^{(x)})^{-1}\| < \infty, \qquad \sup_{x\leq 0} \|(I - \overline{\Omega}_r^{(x)}\Omega_r^{(x)})^{-1}\| < \infty. \tag{4.14b}$$

Theorem 4.6 *Suppose that the Marchenko integral equations* (4.7a) *and* (4.7b) *are uniquely solvable for any $x \geq 0$ and the equations* (4.7c) *and* (4.7d) *are uniquely solvable for any $x \leq 0$. Then the potentials $k(x)$ and $\ell(x)$ follow from* (4.1) *and* (4.2) *and have their entries in $L^1(\mathbb{R})$.*

Proof. The estimates (4.14a) imply that

$$\sup_{x\geq 0} \int_0^\infty d\alpha \, \|B_{l2}(x,\alpha)\| < \infty.$$

Using (4.4a) we get

$$\sup_{x\geq 0} \int_0^\infty d\alpha \, \|B_{l1}(x,\alpha)\| \leq \|\overline{\Omega}_l^{(x)}\|_1 \sup_{x\geq 0} \int_0^\infty d\alpha \, \|B_{l2}(x,\alpha)\| < \infty.$$

We now estimate (4.3b):

$$\int_0^\infty dx\,\|B_{l2}(x,0^+)\| \le \int_0^\infty dx\,\|\Omega_l(2x)\| + \left(\sup_{x\ge 0}\int_0^\infty d\alpha\,\|B_{l1}(x,\alpha)\|\right)\sup_{\beta\ge 0}\int_0^\infty dx\,\|\Omega_l(\beta+2x)\|$$
$$\le \left(1+\sup_{x\ge 0}\int_0^\infty d\alpha\,\|B_{l1}(x,\alpha)\|\right)\|\Omega_l^{(x)}\|_1 < \infty.$$

As a result of (4.1) we have

$$\int_0^\infty dx\,\|k(x)\| = 2\int_0^\infty d\alpha\,\|B_{l2}(x,0^+)\| < \infty.$$

To prove that $\int_{-\infty}^0 dx\,\|k(x)\|$, $\int_0^\infty dx\,\|\ell(x)\|$, and $\int_{-\infty}^0 dx\,\|\ell(x)\|$ are finite, we apply the same argument to B_{r2}, B_{l2}, and B_{l3}, respectively, in accordance with the table below. ■

equation	solution	potential computed
(4.7a)	$B_{l2}(x,\alpha)$	$k(x)$ for $x\ge 0$
(4.7b)	$B_{l3}(x,\alpha)$	$\ell(x)$ for $x\ge 0$
(4.7c)	$B_{r2}(x,\alpha)$	$k(x)$ for $x\le 0$
(4.7d)	$B_{r3}(x,\alpha)$	$\ell(x)$ for $x\le 0$

4.2 Deriving the Marchenko equations

As we have seen in Chapter 3, the Jost solutions and Faddeev matrices are $(n+m)\times(n+m)$ matrix functions, each partitioned into blocks of sizes $n\times n$, $n\times m$, $m\times n$, and $m\times m$, half of which are analytic in $\mathbb{C}^+$ and the other half in $\mathbb{C}^-$. To arrive at the Riemann-Hilbert problems satisfied by (blocks of) the Faddeev matrices and ultimately at Marchenko integral equations, we introduce modifications of the Faddeev matrices and Jost functions which are analytic in either $\mathbb{C}^+$ or $\mathbb{C}^-$, by rearranging the blocks into newly defined $(n+m)\times(n+m)$ matrix functions $m_+(x,\lambda)$, $m_-(x,\lambda)$, $f_+(x,\lambda)$, and $f_-(x,\lambda)$. Next, we employ the asymptotic properties of $m_+(x,\lambda)$ and $m_-(x,\lambda)$ as $x\to\pm\infty$ to derive a Riemann-Hilbert problem satisfied by these functions. This Riemann-Hilbert problem is then Fourier transformed to derive the Marchenko integral equations.

Using the notations of (3.12), let us define the following *modified Jost solutions*:

$$f_+(x,\lambda) = \begin{pmatrix} F_{l1}(x,\lambda) & F_{r2}(x,\lambda) \\ F_{l3}(x,\lambda) & F_{r4}(x,\lambda)\end{pmatrix}, \tag{4.15a}$$

$$f_-(x,\lambda) = \begin{pmatrix} F_{r1}(x,\lambda) & F_{l2}(x,\lambda) \\ F_{r3}(x,\lambda) & F_{l4}(x,\lambda)\end{pmatrix}, \tag{4.15b}$$

thus repeating the definition of $f_+(x,\lambda)$ given in the proof of Proposition 3.12. From Propositions 3.4 and 3.5 it follows that $f_+(x,\lambda)$ is a solution of (3.1) that is continuous in $\lambda\in\overline{\mathbb{C}^+}$ and analytic in $\lambda \in \mathbb{C}^+$; similarly, $f_-(x,\lambda)$ is a solution of (3.1) that is continuous in $\lambda \in \overline{\mathbb{C}^-}$ and analytic in $\lambda \in \mathbb{C}^-$. Using (3.13) and (4.15), let us also define the *modified Faddeev matrices*

$$m_+(x,\lambda) = \begin{pmatrix} M_{l1}(x,\lambda) & M_{r2}(x,\lambda) \\ M_{l3}(x,\lambda) & M_{r4}(x,\lambda) \end{pmatrix} = f_+(x,\lambda)e^{-i\lambda Jx}, \tag{4.16a}$$

$$m_-(x,\lambda) = \begin{pmatrix} M_{r1}(x,\lambda) & M_{l2}(x,\lambda) \\ M_{r3}(x,\lambda) & M_{l4}(x,\lambda) \end{pmatrix} = f_-(x,\lambda)e^{-i\lambda Jx}. \tag{4.16b}$$

Then according to (3.35) and (3.36) we have

$$m_\pm(x,\lambda) = I_{n+m} + \int_0^\infty d\alpha\, b_\pm(x,\alpha)e^{\pm i\lambda\alpha}, \tag{4.17}$$

where $b_\pm(x,\cdot) \in L^1(\mathbb{R}^+;\mathbb{C}^{n+m})$. Here

$$b_+(x,\alpha) = \begin{pmatrix} B_{l1}(x,\alpha) & B_{r2}(x,\alpha) \\ B_{l3}(x,\alpha) & B_{r4}(x,\alpha) \end{pmatrix}, \quad b_-(x,\alpha) = \begin{pmatrix} B_{r1}(x,\alpha) & B_{l2}(x,\alpha) \\ B_{r3}(x,\alpha) & B_{l4}(x,\alpha) \end{pmatrix}. \tag{4.18}$$

Let us now derive the Riemann-Hilbert problems satisfied by $m_+(x,\lambda)$ and $m_-(x,\lambda)$.

Theorem 4.7 *Under the technical hypothesis, we have the Riemann-Hilbert problems*

$$m_-(x,\lambda) = m_+(x,\lambda)G(x,\lambda), \text{ where } G(x,\lambda) = e^{i\lambda Jx}JS(\lambda)Je^{-i\lambda Jx}, \tag{4.19}$$

$$m_+(x,\lambda) = m_-(x,\lambda)\overline{G}(x,\lambda), \text{ where } \overline{G}(x,\lambda) = e^{i\lambda Jx}J\overline{S}(\lambda)Je^{-i\lambda Jx}. \tag{4.20}$$

Proof. It suffices to prove that

$$f_-(x,\lambda) = f_+(x,\lambda)e^{-i\lambda Jx}G(x,\lambda)e^{i\lambda Jx}, \tag{4.21}$$

where $e^{-i\lambda Jx}G(x,\lambda)e^{i\lambda Jx} = JS(\lambda)J$. In fact, on either side we have an $(n+m)\times(n+m)$ matrix solution of (3.1). Using the asymptotic properties of the Jost solutions as $x \to +\infty$ we have to prove that

$$\begin{pmatrix} a_{r1}(\lambda) & 0_{n\times m} \\ a_{r3}(\lambda) & I_m \end{pmatrix} = \begin{pmatrix} I_n & a_{r2}(\lambda) \\ 0_{m\times n} & a_{r4}(\lambda) \end{pmatrix} \begin{pmatrix} T_l(\lambda) & -R(\lambda) \\ -L(\lambda) & T_r(\lambda) \end{pmatrix}, \qquad \lambda \in \mathbb{R}. \tag{4.22}$$

Using the asymptotic properties of the Jost solutions as $x \to -\infty$ we also have to prove that

$$\begin{pmatrix} I_n & a_{l2}(\lambda) \\ 0_{m\times n} & a_{l4}(\lambda) \end{pmatrix} = \begin{pmatrix} a_{l1}(\lambda) & 0_{n\times m} \\ a_{l3}(\lambda) & I_m \end{pmatrix} \begin{pmatrix} T_l(\lambda) & -R(\lambda) \\ -L(\lambda) & T_r(\lambda) \end{pmatrix}, \qquad \lambda \in \mathbb{R}. \tag{4.23}$$

Indeed, easy calculations on the right-hand side of eq. (4.22) show that

$$T_l(\lambda) - a_{r2}(\lambda)L(\lambda) = a_{l1}(\lambda)^{-1} - a_{r2}(\lambda)a_{l3}(\lambda)a_{l1}(\lambda)^{-1} = a_{r1}(\lambda)$$

where, in the second identity of the preceding equation, we have taken into account that from eq. (3.67b) we find $a_{r1}(\lambda)a_{l1}(\lambda) + a_{r2}(\lambda)a_{l3}(\lambda) = I_n$, which implies $a_{l1}(\lambda)^{-1} - a_{r2}(\lambda)a_{l3}(\lambda)a_{l1}(\lambda)^{-1} = a_{r1}(\lambda)$. Also, using (3.75)

$$-a_{r4}(\lambda)L(\lambda) = -a_{r4}(\lambda)a_{r4}(\lambda)^{-1}a_{r3}(\lambda) = a_{r3}(\lambda).$$

Finally, using (3.63) and (3.61),

$$-R(\lambda)+T_r(\lambda)a_{r2}(\lambda)=-a_{r2}(\lambda)a_{r4}(\lambda)^{-1}+a_{r2}(\lambda)a_{r4}(\lambda)^{-1}=0_{n\times m},$$
$$a_{r4}(\lambda)T_r(\lambda)=a_{r4}(\lambda)a_{r4}(\lambda)^{-1}=I_m,$$

i.e., eq. (4.22) is satisfied. Moreover, if we consider the right-hand side of eq. (4.23) we obtain using (3.60) and (3.63)

$$a_{l1}(\lambda)T_l(\lambda)=a_{l1}(\lambda)a_{l1}(\lambda)^{-1}=I_n,$$
$$-L(\lambda)+T_l(\lambda)a_{l3}(\lambda)=-a_{l3}(\lambda)a_{l1}(\lambda)^{-1}+a_{l3}(\lambda)a_{l1}(\lambda)^{-1}=0_{m\times n},$$

and, using (3.74)

$$-a_{l1}(\lambda)R(\lambda)=a_{l1}(\lambda)a_{l1}(\lambda)^{-1}a_{l2}(\lambda)=a_{l2}(\lambda),$$

Then

$$T_r(\lambda)-a_{l3}(\lambda)R(\lambda)=a_{r4}(\lambda)^{-1}-a_{l3}(\lambda)a_{r2}(\lambda)a_{r4}(\lambda)^{-1}=a_{l4}(\lambda),$$

where, in the second identity of the preceding equation, we have taken into account that from eq. (3.67a) we have $a_{l3}(\lambda)a_{r2}(\lambda)+a_{l4}(\lambda)a_{r4}(\lambda)=I_m$, which implies $a_{r4}^{-1}(\lambda)-a_{l3}(\lambda)a_{r2}(\lambda)a_{r4}^{-1}(\lambda)=a_{l4}(\lambda)$. So, we have proved that also eq. (4.22) is satisfied and all components of (4.19) belong to the Wiener algebra $\mathcal{W}_p$ for suitable p. ■

The Marchenko integral equations are now derived by

(a) projecting (4.19) onto $\mathcal{W}_{-,0}^{n+m}$ and projecting (4.20) onto $\mathcal{W}_{+,0}^{n+m}$;

(b) stripping off the Fourier transforms to arrive at integral equations for the blocks of $b_+(x,\alpha)$ and $b_-(x,\alpha)$ defined by (4.18).

This is straightforward if the transmission coefficients are analytic in $\mathbb{C}^+$, i.e., if there are no bound states. We shall therefore discuss this case first. The result consists of Marchenko integral equations of the type (4.3) or (4.4) in which $\Omega_l=\hat{R}$, $\Omega_r=\hat{L}$, $\overline{\Omega}_l=\hat{\overline{R}}$, and $\overline{\Omega}_r=\hat{\overline{L}}$. Next, we assume that the poles of the transmission coefficients in $\mathbb{C}^+$ are simple (as in [3]). This will lead to additional terms in the kernel functions Ω_l, Ω_r, $\overline{\Omega}_l$, and $\overline{\Omega}_r$, which amount to a finite rank perturbation of the integral operators. Finally, we consider the most general case, where the transmission coefficients may have higher order poles in $\mathbb{C}^+$ and the additional terms are more complicated.

Throughout the remainder of this chapter we assume that the matrices $a_{l1}(\lambda)$, $a_{r1}(\lambda)$, $a_{l4}(\lambda)$ and $a_{r4}(\lambda)$ are invertible for all $\lambda\in\mathbb{R}$, i.e., we depart from the technical hypothesis. According to Theorem 3.10, the elements of $a_l(\lambda)$ and $a_r(\lambda)$ belong to some $\mathcal{W}$. Because of the hypothesis made, we can apply Theorem 3.8 to prove that the reflection and transmission coefficients have their entries in $\mathcal{W}$. We may therefore write

$$R(\lambda)=\int_{-\infty}^{+\infty}d\alpha\,\hat{R}(\alpha)e^{-i\lambda\alpha}, \tag{4.24a}$$

$$L(\lambda)=\int_{-\infty}^{+\infty}d\alpha\,\hat{L}(\alpha)e^{-i\lambda\alpha}, \tag{4.24b}$$

$$\overline{R}(\lambda)=\int_{-\infty}^{+\infty}d\alpha\,\hat{\overline{R}}(\alpha)e^{i\lambda\alpha}, \tag{4.24c}$$

$$\overline{L}(\lambda) = \int_{-\infty}^{+\infty} d\alpha\, \hat{\overline{L}}(\alpha) e^{i\lambda\alpha}, \tag{4.24d}$$

where $\hat{R}$, $\hat{L}$, $\hat{\overline{R}}$, and $\hat{\overline{L}}$ have their entries in $L^1(\mathbb{R})$.

In Sec. 3.4 we have derived the following symmetries for the reflection coefficients:

$$\begin{cases} \overline{R}(\lambda)^* = R(\lambda) \text{ and } \overline{L}(\lambda)^* = L(\lambda), & \text{symmetric case,} \\ \overline{R}(\lambda)^* = -R(\lambda) \text{ and } \overline{L}(\lambda)^* = -L(\lambda), & \text{antisymmetric case,} \end{cases}$$

where $\lambda \in \mathbb{R}$. We thus obtain

$$\begin{cases} \hat{\overline{R}}(\alpha) = \hat{R}(\alpha)^* \text{ and } \hat{\overline{L}}(\alpha) = \hat{L}(\alpha)^*, & \text{symmetric case,} \\ \hat{\overline{R}}(\alpha) = -\hat{R}(\alpha)^* \text{ and } \hat{\overline{L}}(\alpha) = -\hat{L}(\alpha)^*, & \text{antisymmetric case,} \end{cases}$$

which agrees with (4.9) if $\Omega_l = \hat{R}$, $\Omega_r = \hat{L}$, $\overline{\Omega}_l = \hat{\overline{R}}$, and $\overline{\Omega}_r = \hat{\overline{L}}$.

4.2.1 When there are no bound state eigenvalues

It is well-known that there are no bound states in the symmetric case. Here we discuss the derivation of the Marchenko integral equations in general if there are no bound states, without specializing them in the symmetric and antisymmetric cases, thus recovering the equations derived in [9, 90].

Let us now derive the first four Marchenko integral equations.

Theorem 4.8 *For $\alpha \geq 0$ we have the Marchenko integral equations*

$$B_{r1}(x,\alpha) = -\int_0^\infty d\beta\, B_{r2}(x,\beta)\hat{L}(\alpha+\beta-2x), \tag{4.25a}$$

$$B_{l2}(x,\alpha) = -\hat{R}(\alpha+2x) - \int_0^\infty d\beta\, B_{l1}(x,\beta)\hat{R}(\alpha+\beta+2x), \tag{4.25b}$$

$$B_{r3}(x,\alpha) = -\hat{L}(\alpha-2x) - \int_0^\infty d\beta\, B_{r4}(x,\beta)\hat{L}(\alpha+\beta-2x), \tag{4.25c}$$

$$B_{l4}(x,\alpha) = -\int_0^\infty d\beta\, B_{l3}(x,\beta)\hat{R}(\alpha+\beta+2x). \tag{4.25d}$$

Proof. We note that eq. (4.19) is an identity, where $m_-(x,\cdot) - I_{n+m} \in \mathcal{W}^{n+m}_{-,0}$, $m_+(x,\cdot) - I_{n+m} \in \mathcal{W}^{n+m}_{+,0}$ and $G(x,\cdot) - I_{n+m} \in \mathcal{W}^{n+m}_0$. In the absence of bound states we also have $T_l(\cdot) - I_n \in \mathcal{W}^n_{+,0}$ and $T_r(\cdot) - I_m \in \mathcal{W}^n_{+,0}$. Projecting (4.19) onto $\mathcal{W}^{n+m}_{-,0}$, we obtain

$$M_{r1}(x,\lambda) - I_n = \Pi_{-,0}(M_{r1}(x,\lambda)) = -\Pi_{-,0}(M_{r2}(x,\lambda)L(\lambda)e^{-2i\lambda x}), \tag{4.26a}$$

$$M_{l2}(x,\lambda) = \Pi_{-,0}M_{l2}(x,\lambda) = -\Pi_{-,0}(M_{l1}(x,\lambda)R(\lambda)e^{2i\lambda x}), \tag{4.26b}$$

$$M_{r3}(x,\lambda) = \Pi_{-,0}(M_{r3}(x,\lambda)) = -\Pi_{-,0}(M_{r4}(x,\lambda)L(\lambda)e^{-2i\lambda x}), \tag{4.26c}$$

$$M_{l4}(x,\lambda) - I_m = \Pi_{-,0}(M_{l4}(x,\lambda)) = -\Pi_{-,0}(M_{l3}(x,\lambda)R(\lambda)e^{2i\lambda x}), \tag{4.26d}$$

where $\Pi_{-,0}$ is the projection of $\mathcal{W}^p$ onto $\mathcal{W}^p_{-,0}$ along $\mathcal{W}^p_+$. Moreover, we have taken into account that $M_{l1}(x,\lambda)-I_n$, $M_{r2}(x,\lambda)$, $M_{l3}(x,\lambda)$ and $M_{r4}(x,\lambda)-I_m$ are elements of $\mathcal{W}_{+,0}$, therefore the terms $M_{l1}(x,\lambda)T_l(\lambda)$, $M_{l3}(x,\lambda)T_l(\lambda)$, $M_{r2}(x,\lambda)T_r(\lambda)$ and $M_{r4}(x,\lambda)T_r(\lambda)$ belong to $\mathcal{W}_{+,0}$ and can be deleted in the projection onto $\mathcal{W}^p_{-,0}$.

Let us now simplify the right-hand sides of (4.26). Using (4.24a) and (4.24b) we can write

$$\begin{aligned} R(\lambda)e^{2i\lambda x} &= \int_{-\infty}^{+\infty} d\alpha\, \hat{R}(\alpha+2x)e^{-i\lambda\alpha}, \\ L(\lambda)e^{-2i\lambda x} &= \int_{-\infty}^{+\infty} d\alpha\, \hat{L}(\alpha-2x)e^{-i\lambda\alpha}. \end{aligned}$$

Now, easy calculations show that

$$\Pi_{-,0}(M_{r2}(x,\lambda)L(\lambda)e^{-2i\lambda x}) = \int_0^{+\infty} d\alpha\, e^{-i\lambda\alpha} \int_0^{+\infty} d\beta\, B_{r2}(x,\beta)\hat{L}(\alpha+\beta-2x), \tag{4.27a}$$

$$\begin{aligned} \Pi_{-,0}(M_{l1}(x,\lambda)R(\lambda)e^{2i\lambda x}) &= \int_0^{+\infty} d\alpha\, \hat{R}(\alpha+2x)e^{-i\lambda\alpha} \\ &+ \int_0^{+\infty} d\alpha\, e^{-i\lambda\alpha} \int_0^{+\infty} d\beta\, B_{l1}(x,\beta)\hat{R}(\alpha+\beta+2x), \end{aligned} \tag{4.27b}$$

$$\begin{aligned} \Pi_{-,0}(M_{r4}(x,\lambda)L(\lambda)e^{-2i\lambda x}) &= \int_0^{+\infty} d\alpha\, \hat{L}(\alpha-2x)e^{-i\lambda\alpha} \\ &+ \int_0^{+\infty} d\alpha\, e^{-i\lambda\alpha} \int_0^{+\infty} d\beta\, B_{r4}(x,\beta)\hat{L}(\alpha+\beta-2x), \end{aligned} \tag{4.27c}$$

$$\Pi_{-,0}(M_{l3}(x,\lambda)R(\lambda)e^{2i\lambda x}) = \int_0^{+\infty} d\alpha\, e^{-i\lambda\alpha} \int_0^{+\infty} d\beta\, B_{l3}(x,\beta)\hat{R}(\alpha+\beta+2x). \tag{4.27d}$$

Recalling eqs. (3.50), (3.38), (3.52), (3.40), (4.26) and (4.27) we obtain

$$\int_0^{\infty} d\alpha\, e^{-i\lambda\alpha} B_{r1}(x,\alpha) = -\int_0^{\infty} d\alpha\, e^{-i\lambda\alpha} \int_0^{\infty} d\beta\, B_{r2}(x,\beta)\hat{L}(\alpha+\beta-2x), \tag{4.28a}$$

$$\begin{aligned} \int_0^{\infty} d\alpha\, e^{-i\lambda\alpha} B_{l2}(x,\alpha) &= -\int_0^{\infty} d\alpha\, e^{-i\lambda\alpha} \hat{R}(\alpha+2x) \\ &- \int_0^{\infty} d\alpha\, e^{-i\lambda\alpha} \int_0^{\infty} d\beta\, B_{l1}(x,\beta)\hat{R}(\alpha+\beta+2x), \end{aligned} \tag{4.28b}$$

$$\begin{aligned} \int_0^{\infty} d\alpha\, e^{-i\lambda\alpha} B_{r3}(x,\alpha) &= -\int_0^{\infty} d\alpha\, e^{-i\lambda\alpha} \hat{L}(\alpha-2x) \\ &- \int_0^{\infty} d\alpha\, e^{-i\lambda\alpha} \int_0^{\infty} d\beta\, B_{r4}(x,\beta)\hat{L}(\alpha+\beta-2x), \end{aligned} \tag{4.28c}$$

$$\int_0^{\infty} d\alpha\, e^{-i\lambda\alpha} B_{l4}(x,\alpha) = -\int_0^{\infty} d\alpha\, e^{-i\lambda\alpha} \int_0^{\infty} d\beta\, B_{l3}(x,\beta)\hat{R}(\alpha+\beta+2x). \tag{4.28d}$$

Removing the Fourier transforms from the preceding equations, we get the Marchenko integral equations (4.25). ∎

The proof of the following Theorem 4.9 is very similar. We now project (4.20) onto $\mathcal{W}_{+,0}^{n+m}$ and use (4.24c) and (4.24d) instead. To facilitate checking some of the subsequent Marchenko equations, we give a short proof of Theorem 4.9.

Theorem 4.9 *For $\alpha \geq 0$ we have the Marchenko integral equations*

$$B_{l1}(x,\alpha) = -\int_0^\infty d\beta\, B_{l2}(x,\beta)\hat{\overline{R}}(\alpha+\beta+2x), \tag{4.29a}$$

$$B_{r2}(x,\alpha) = -\hat{\overline{L}}(\alpha+-x) - \int_0^\infty d\beta\, B_{r1}(x,\beta)\hat{\overline{L}}(\alpha+\beta-2x), \tag{4.29b}$$

$$B_{l3}(x,\alpha) = -\hat{\overline{R}}(\alpha+2x) - \int_0^\infty d\beta\, B_{l4}(x,\beta)\hat{\overline{R}}(\alpha+\beta+2x), \tag{4.29c}$$

$$B_{r4}(x,\alpha) = -\int_0^\infty d\beta\, B_{r3}(x,\beta)\hat{\overline{L}}(\alpha+\beta-2x). \tag{4.29d}$$

Thus eqs. (4.3) and (4.4) are satisfied for $\Omega_r = \hat{L}$, $\Omega_l = \hat{R}$, $\overline{\Omega}_l = \hat{\overline{R}}$, and $\overline{\Omega}_r = \hat{\overline{L}}$.

Proof. We note that $m_+(x,\cdot) - I_{n+m} \in \mathcal{W}_{+,0}^{n+m}$, $m_-(x,\cdot) - I_{n+m} \in \mathcal{W}_{-,0}^{n+m}$, and $G(x,\cdot) - I_{n+m} \in \mathcal{W}_0^{n+m}$. In the absence of bound states we also have $\overline{T}_l(\cdot) - I_n \in \mathcal{W}_{-,0}^n$ and $\overline{T}_r(\cdot) - I_m \in \mathcal{W}_{-,0}^n$. Projecting (4.20) onto $\mathcal{W}_{+,0}^{n+m}$, we obtain

$$M_{l1}(x,\lambda) - I_n = \Pi_{+,0}(M_{l1}(x,\lambda)) = -\Pi_{+,0}(M_{l2}(x,\lambda)\overline{R}(\lambda)e^{-2i\lambda x}), \tag{4.30a}$$

$$M_{r2}(x,\lambda) = \Pi_{+,0}M_{r2}(x,\lambda) = -\Pi_{+,0}(M_{r1}(x,\lambda)\overline{L}(\lambda)e^{2i\lambda x}), \tag{4.30b}$$

$$M_{l3}(x,\lambda) = \Pi_{+,0}(M_{l3}(x,\lambda)) = -\Pi_{+,0}(M_{l4}(x,\lambda)\overline{R}(\lambda)e^{-2i\lambda x}), \tag{4.30c}$$

$$M_{r4}(x,\lambda) - I_m = \Pi_{+,0}(M_{r4}(x,\lambda)) = -\Pi_{+,0}(M_{r3}(x,\lambda)\overline{L}(\lambda)e^{2i\lambda x}), \tag{4.30d}$$

where $\Pi_{+,0}$ is the projection of $\mathcal{W}^p$ onto $\mathcal{W}_{+,0}^p$ along $\mathcal{W}_-^p$.

Let us now simplify the right-hand sides of (4.30). Using (4.24c) and (4.24d) we can write

$$\overline{R}(\lambda)e^{-2i\lambda x} = \int_{-\infty}^{+\infty} d\alpha\, \hat{\overline{R}}(\alpha+2x)e^{i\lambda\alpha},$$

$$\overline{L}(\lambda)e^{2i\lambda x} = \int_{-\infty}^{+\infty} d\alpha\, \hat{\overline{L}}(\alpha-2x)e^{i\lambda\alpha}.$$

We now proceed as in the proof of (4.27) and (4.28). Removing the Fourier transforms from the preceding equations, we get the Marchenko integral equations (4.29). ■

4.2.2 When the eigenvalues are algebraically simple

By hypothesis, $a_{l1}(\lambda)$, $a_{r1}(\lambda)$, $a_{l4}(\lambda)$, and $a_{r4}(\lambda)$ are invertible for every $\lambda \in \mathbb{R}$. Therefore the transmission coefficients $T_l(\lambda)$ and $T_r(\lambda)$ have at most finitely many poles in $\mathbb{C}^+$ and these necessarily coincide. We denote these distinct poles in $\mathbb{C}^+$ by $i\kappa_1, \ldots, i\kappa_N$. Similarly, the transmission coefficients $\overline{T}_l(\lambda)$ and $\overline{T}_r(\lambda)$ have at most finitely many poles in $\mathbb{C}^-$ and these necessarily coincide. We denote these distinct poles in $\mathbb{C}^-$ by $-i\tilde{\kappa}_1, \ldots, -i\tilde{\kappa}_N$. Let us suppose that **all of these**

poles are simple. Then there exist unique $n \times n$ and $m \times m$ matrix functions $T_{l1}(\lambda)$ and $T_{r1}(\lambda)$, continuous in $\lambda \in \overline{\mathbb{C}^+}$, analytic in $\lambda \in \mathbb{C}^+$, and approaching I_n and I_m (respectively) as $\lambda \to \infty$ from within $\overline{\mathbb{C}^+}$, such that

$$T_l(\lambda) = i \sum_{j=1}^{N} \frac{\tau_{lj0}}{\lambda - i\kappa_j} + T_{l1}(\lambda), \tag{4.31}$$

$$T_r(\lambda) = i \sum_{j=1}^{N} \frac{\tau_{rj0}}{\lambda - i\kappa_j} + T_{r1}(\lambda), \tag{4.32}$$

where $\lambda \in \overline{\mathbb{C}^+}$. Analogously, there exist unique $n \times n$ and $m \times m$ matrix functions $\overline{T_{l1}}(\lambda)$ and $\overline{T_{r1}}(\lambda)$, continuous in $\lambda \in \overline{\mathbb{C}^-}$, analytic in $\lambda \in \mathbb{C}^-$, and approaching I_n and I_m (respectively) as $\lambda \to \infty$ from within $\overline{\mathbb{C}^-}$, such that

$$\overline{T_l}(\lambda) = -i \sum_{j=1}^{\tilde{N}} \frac{\tilde{\tau}_{lj0}}{\lambda + i\tilde{\kappa}_j} + \overline{T}_{l1}(\lambda), \tag{4.33}$$

$$\overline{T_r}(\lambda) = -i \sum_{j=1}^{\tilde{N}} \frac{\tilde{\tau}_{rj0}}{\lambda + i\tilde{\kappa}_j} + \overline{T}_{r1}(\lambda), \tag{4.34}$$

where $\lambda \in \overline{\mathbb{C}^-}$.

For $j = 1, \ldots, N$, put $m_+^{j0}(x) = m_+(x, i\kappa_j)$ and for $j = 1, \ldots, \tilde{N}$ put $m_-^{j0}(x) = m_-(x, -i\tilde{\kappa}_j)$. Then for $j = 1, \ldots, N$ we have the partitioning

$$m_+^{j0}(x) = \begin{pmatrix} M_{l1}^{j0}(x) & M_{r2}^{j0}(x) \\ M_{l3}^{j0}(x) & M_{r4}^{j0}(x) \end{pmatrix}, \quad m_-^{j0}(x) = \begin{pmatrix} M_{r1}^{j0}(x) & M_{l2}^{j0}(x) \\ M_{r3}^{j0}(x) & M_{l4}^{j0}(x) \end{pmatrix}. \tag{4.35}$$

We prove the following

Proposition 4.10 *For $\alpha \geq 0$ we have the following integral equations :*

$$B_{r1}(x,\alpha) = -\sum_{j=1}^{N} M_{l1}^{j0}(x)\tau_{lj0}e^{-\kappa_j\alpha} - \int_0^\infty d\beta\, B_{r2}(x,\beta)\hat{L}(\alpha+\beta-2x), \tag{4.36a}$$

$$B_{l2}(x,\alpha) = -\sum_{j=1}^{N} M_{r2}^{j0}(x)\tau_{rj0}e^{-\kappa_j\alpha} - \hat{R}(\alpha+2x) - \int_0^\infty d\beta\, B_{l1}(x,\beta)\hat{R}(\alpha+\beta+2x), \tag{4.36b}$$

$$B_{r3}(x,\alpha) = -\sum_{j=1}^{N} M_{l3}^{j0}(x)\tau_{lj0}e^{-\kappa_j\alpha} - \hat{L}(\alpha-2x) - \int_0^\infty d\beta\, B_{r4}(x,\beta)\hat{L}(\alpha+\beta-2x), \tag{4.36c}$$

$$B_{l4}(x,\alpha) = -\sum_{j=1}^{N} M_{r4}^{j0}(x)\tau_{rj0}e^{-\kappa_j\alpha} - \int_0^\infty d\beta\, B_{l3}(x,\beta)\hat{R}(\alpha+\beta+2x). \tag{4.36d}$$

Proof. Let us mimic the proof of Theorem 4.8 while concentrating on the extra terms stemming from the transmission coefficients in the Riemann-Hilbert problem (4.19). To retrieve the extra term in (4.36a) we consider the proof of (4.25a). Instead of (4.26a) we now have

$$\begin{aligned}M_{r1}(x,\lambda)-I_n &= \Pi_{-,0}(M_{l1}(x,\lambda)T_l(\lambda))-\Pi_{-,0}(M_{r2}(x,\lambda)L(\lambda)e^{-2i\lambda x})\\ &= i\sum_{j=1}^{N}\Pi_{-,0}\frac{M_{l1}(x,\lambda)\,\tau_{lj0}}{\lambda-i\kappa_j}-\Pi_{-,0}(M_{r2}(x,\lambda)L(\lambda)e^{-2i\lambda x}).\end{aligned}$$

Let us analyze the extra term separately. Using that $M_{l1}(x,i\kappa_j)=m_{l1}^{j0}(x)$ we have

$$i\Pi_{-,0}\left(\frac{M_{l1}(x,\lambda)\,\tau_{lj0}}{\lambda-i\kappa_j}\right)=i\frac{M_{l1}^{j0}(x)\,\tau_{lj0}}{\lambda-i\kappa_j}=-\int_0^\infty dy\,e^{-i\lambda y}e^{-\kappa_j y}M_{l1}^{j0}(x)\,\tau_{lj0},$$

which implies the extra term in (4.36a). The other equations are proved analogously. ■

In the same way we prove the following

Proposition 4.11 *For $\alpha\geq 0$ we have the following integral equations :*

$$B_{l1}(x,\alpha)=-\sum_{j=1}^{\tilde N}M_{r1}^{j0}(x)\tilde\tau_{lj0}e^{-\tilde\kappa_j\alpha}-\int_0^\infty d\beta\,B_{l2}(x,\beta)\hat{\bar R}(\alpha+\beta+2x), \tag{4.37a}$$

$$B_{r2}(x,\alpha)=-\sum_{j=1}^{\tilde N}M_{l2}^{j0}(x)\tilde\tau_{rj0}e^{-\tilde\kappa_j\alpha}-\hat{\bar L}(\alpha-2x)-\int_0^\infty d\beta\,B_{r1}(x,\beta)\hat{\bar L}(\alpha+\beta-2x), \tag{4.37b}$$

$$B_{l3}(x,\alpha)=-\sum_{j=1}^{\tilde N}M_{r3}^{j0}(x)\tilde\tau_{lj0}e^{-\tilde\kappa_j\alpha}-\hat{\bar R}(\alpha+2x)-\int_0^\infty d\beta\,B_{l4}(x,\beta)\hat{\bar R}(\alpha+\beta+2x), \tag{4.37c}$$

$$B_{r4}(x,\alpha)=-\sum_{j=1}^{\tilde N}M_{l4}^{j0}(x)\tilde\tau_{rj0}e^{-\tilde\kappa_j\alpha}-\int_0^\infty d\beta\,B_{r3}(x,\beta)\hat{\bar L}(\alpha+\beta-2x). \tag{4.37d}$$

In the proof of Theorem 3.16 we have made clear that all eigenfunctions of (3.1) at an eigenvalue $\lambda=i\kappa_j\in\mathbb{C}^+$ have one of the equivalent forms

$$F_l(x,i\kappa_j)\begin{pmatrix}\varepsilon_j\\ 0_{m\times1}\end{pmatrix}=F_r(x,i\kappa_j)\begin{pmatrix}0_{n\times1}\\ \eta_j\end{pmatrix}, \tag{4.38}$$

where ε_j and η_j are nontrivial vectors satisfying $a_{l1}(i\kappa_j)\varepsilon_j=0_{n\times1}$ and $a_{r4}(i\kappa_j)\eta_j=0_{m\times1}$. Since $a_{l1}(\lambda)\varepsilon_j=(\lambda-i\kappa_j)f(\lambda)$ for a vector function $f(\lambda)$ that is analytic in a neighborhood of $i\kappa_j$, we obtain $\varepsilon_j=i\tau_{lj0}f(i\kappa_j)$. On the other hand, if $\varepsilon_j=\tau_{lj0}h\in\operatorname{Im}\tau_{lj0}$, then $a_{l1}(\lambda)\varepsilon_j=-i\,(\lambda-i\kappa_j)\,[h-T_{l1}(\lambda)h]$ vanishes as $\lambda\to i\kappa_j$ and hence $a_{l1}(\lambda_0)\varepsilon_j=0_{n\times1}$. Consequently, the subspace of $\mathbb{C}^n$ generated by the vectors ε_j coincides with the range of τ_{lj0}. Similarly, the subspace of $\mathbb{C}^m$ generated by the vectors η_j coincides with the range of τ_{rj0}. Because these ranges have the same dimension, we can find an $m\times n$ matrix C_{j0} and an $n\times m$ matrix D_{j0} such that $\eta_j=C_{j0}\varepsilon_j$

and $\varepsilon_j = D_{j0}\eta_j$ whenever ε_j and η_j are related by (4.38). In other words, C_{j0} and D_{j0} are each other's inverses on the subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$ generated by ε_j and η_j. Letting them annihilate the orthogonal complements of these subspaces, we have

$$C_{j0} = C_{j0}D_{j0}C_{j0}, \qquad D_{j0} = D_{j0}C_{j0}D_{j0},$$

where $C_{j0}D_{j0}$ is the orthogonal projection onto the range of τ_{rj0} and $D_{j0}C_{j0}$ is the orthogonal projection onto the range of τ_{lj0}. In fact, D_{j0} is the Moore-Penrose generalized inverse of C_{j0} and vice versa (cf. [25]). We call the matrices C_{j0} and D_{j0} *dependency constant matrices.* Since (4.38) implies that

$$e^{-\kappa_j x}\begin{pmatrix} M_{l1}(x,i\kappa_j)\varepsilon_j \\ M_{l3}(x,i\kappa_j)\varepsilon_j \end{pmatrix} = e^{\kappa_j x}\begin{pmatrix} M_{r2}(x,i\kappa_j)\eta_j \\ M_{r4}(x,i\kappa_j)\eta_j \end{pmatrix}, \tag{4.39}$$

we get

$$M_{r2}^{j0}(x)\eta_j = e^{-2\kappa_j x}M_{l1}^{j0}(x)\varepsilon_j, \tag{4.40}$$

$$M_{l3}^{j0}(x)\varepsilon_j = e^{2\kappa_j x}M_{r4}^{j0}(x)\eta_j. \tag{4.41}$$

Consequently,

$$M_{r2}^{j0}(x)\tau_{rj0} = e^{-2\kappa_j x}M_{l1}^{j0}(x)D_{j0}\tau_{rj0} = e^{-2\kappa_j x}M_{l1}^{j0}(x)\Gamma_{lj}, \tag{4.42a}$$

$$M_{l3}^{j0}(x)\tau_{lj0} = e^{2\kappa_j x}M_{r4}^{j0}(x)C_{j0}\tau_{lj0} = e^{2\kappa_j x}M_{r4}^{j0}(x)\Gamma_{rj}, \tag{4.42b}$$

$$M_{r4}^{j0}(x)\tau_{rj0} = e^{-2\kappa_j x}M_{l3}^{j0}(x)D_{j0}\tau_{rj0} = e^{-2\kappa_j x}M_{l3}^{j0}(x)\Gamma_{lj}, \tag{4.42c}$$

$$M_{l1}^{j0}(x)\tau_{lj0} = e^{2\kappa_j x}M_{r2}^{j0}(x)C_{j0}\tau_{lj0} = e^{2\kappa_j x}M_{r2}^{j0}(x)\Gamma_{rj}, \tag{4.42d}$$

where the *norming constant matrices* are defined by

$$\Gamma_{lj} = D_{j0}\tau_{rj0}, \qquad \Gamma_{rj} = C_{j0}\tau_{lj0}. \tag{4.43}$$

Thus eqs. (4.42) can be written in the form

$$m_+(x,i\kappa_j)\begin{pmatrix} \tau_{lj0} & 0_{n\times m} \\ 0_{m\times n} & \tau_{rj0} \end{pmatrix} = m_+(x,i\kappa_j)\begin{pmatrix} 0_{n\times n} & e^{-2\kappa_j x}\Gamma_{lj} \\ e^{2\kappa_j x}\Gamma_{rj} & 0_{m\times m} \end{pmatrix}. \tag{4.44}$$

In [2] norming constant matrices are introduced for the matrix Zakharov-Shabat system with antisymmetric potential when the poles of the transmission coefficients are simple and the potential has sufficient exponential decay. These quantities are defined in terms of reflection coefficients evaluated off the real axis, which is only correct for sufficient exponential decay of the potential. In [2] appropriate decay assumptions on the potential are made to justify their definitions. Our definitions extend those given in [2] (and those given in the subsequent [1, 3]) to general L^1-potentials without symmetries. In [10] norming constant matrices are introduced to study the small energy asymptotics of the matrix Schrödinger equation with selfadjoint potential. These norming constant matrices relate two representations of the eigenfunction at a zero energy half-bound state.

Analogously, it is clear that all eigenfunctions of (3.1) at an eigenvalue $\lambda = -i\tilde{\kappa}_j \in \mathbb{C}^-$ have one of the equivalent forms

$$F_l(x, -i\tilde{\kappa}_j) \begin{pmatrix} 0_{n\times 1} \\ \overline{\eta}_j \end{pmatrix} = F_r(x, -i\tilde{\kappa}_j) \begin{pmatrix} \overline{\varepsilon}_j \\ 0_{m\times 1} \end{pmatrix}, \tag{4.45}$$

where $a_{r1}(-i\tilde{\kappa}_j)\overline{\varepsilon}_j = 0_{n\times 1}$ and $a_{l4}(-i\tilde{\kappa}_j)\overline{\eta}_j = 0_{m\times 1}$. Further, the vectors $\overline{\varepsilon}_j$ generate the range of $\tilde{\tau}_{lj0}$ and the vectors $\overline{\eta}_j$ the range of $\tilde{\tau}_{rj0}$. Because these ranges have the same dimension, we can find an $m\times n$ matrix $\overline{C}_{j0}$ and an $n\times m$ matrix $\overline{D}_{j0}$ such that $\overline{\eta}_j = \overline{C}_{j0}\overline{\varepsilon}_j$ and $\overline{\varepsilon}_j = \overline{D}_{j0}\overline{\eta}_j$ whenever $\overline{\varepsilon}_j$ and $\overline{\eta}_j$ are related by (4.45). Letting $\overline{C}_{j0}$ and $\overline{D}_{j0}$ annihilate the orthogonal complements of these subspaces, we have

$$\overline{C}_{j0} = \overline{C}_{j0}\overline{D}_{j0}\overline{C}_{j0}, \qquad \overline{D}_{j0} = \overline{D}_{j0}\overline{C}_{j0}\overline{D}_{j0},$$

where $\overline{C}_{j0}\overline{D}_{j0}$ is the orthogonal projection onto the range of $\tilde{\tau}_{rj0}$ and $\overline{D}_{j0}\overline{C}_{j0}$ is the orthogonal projection onto the range of $\tilde{\tau}_{lj0}$. In fact, $\overline{D}_{j0}$ is the Moore-Penrose generalized inverse of $\overline{C}_{j0}$ and vice versa. We also call the matrices $\overline{C}_{j0}$ and $\overline{D}_{j0}$ dependency constant matrices. Since (4.45) implies that

$$e^{\tilde{\kappa}_j x} \begin{pmatrix} M_{r1}(x, -i\tilde{\kappa}_j)\overline{\varepsilon}_j \\ M_{r3}(x, -i\tilde{\kappa}_j)\overline{\varepsilon}_j \end{pmatrix} = e^{-\tilde{\kappa}_j x} \begin{pmatrix} M_{l2}(x, -i\tilde{\kappa}_j)\overline{\eta}_j \\ M_{l4}(x, -i\tilde{\kappa}_j)\overline{\eta}_j \end{pmatrix}, \tag{4.46}$$

we get

$$M_{l2}^{j0}(x)\overline{\eta}_j = e^{2\tilde{\kappa}_j x} M_{r1}^{j0}(x)\overline{\varepsilon}_j, \tag{4.47}$$

$$M_{r3}^{j0}(x)\overline{\varepsilon}_j = e^{-2\tilde{\kappa}_j x} M_{l4}^{j0}(x)\overline{\eta_j}. \tag{4.48}$$

Consequently,

$$M_{l2}^{j0}(x)\tilde{\tau}_{rj0} = e^{2\tilde{\kappa}_j x} M_{r1}^{j0}(x)\overline{D}_{j0}\tilde{\tau}_{rj0} = e^{2\tilde{\kappa}_j x} M_{r1}^{j0}(x)\overline{\Gamma}_{rj}, \tag{4.49a}$$

$$M_{r3}^{j0}(x)\tilde{\tau}_{lj0} = e^{-2\tilde{\kappa}_j x} M_{l4}^{j0}(x)\overline{C}_{j0}\tilde{\tau}_{lj0} = e^{-2\tilde{\kappa}_j x} M_{l4}^{j0}(x)\overline{\Gamma}_{lj}, \tag{4.49b}$$

$$M_{l4}^{j0}(x)\tilde{\tau}_{rj0} = e^{2\tilde{\kappa}_j x} M_{r3}^{j0}(x)\overline{D}_{j0}\tilde{\tau}_{rj0} = e^{2\tilde{\kappa}_j x} M_{r3}^{j0}(x)\overline{\Gamma}_{rj}, \tag{4.49c}$$

$$M_{r1}^{j0}(x)\tilde{\tau}_{lj0} = e^{-2\tilde{\kappa}_j x} M_{l2}^{j0}(x)\overline{C}_{j0}\tilde{\tau}_{lj0} = e^{-2\tilde{\kappa}_j x} M_{l2}^{j0}(x)\overline{\Gamma}_{lj}, \tag{4.49d}$$

where the norming constant matrices are defined by

$$\overline{\Gamma}_{lj} = \overline{C}_{j0}\tilde{\tau}_{lj0}, \qquad \overline{\Gamma}_{rj} = \overline{D}_{j0}\tilde{\tau}_{rj0}. \tag{4.50}$$

Thus eqs. (4.49) can be written in the form

$$m_-(x, -i\tilde{\kappa}_j) \begin{pmatrix} \tilde{\tau}_{lj0} & 0_{n\times m} \\ 0_{m\times n} & \tilde{\tau}_{rj0} \end{pmatrix} = m_-(x, -i\tilde{\kappa}_j) \begin{pmatrix} 0_{n\times n} & e^{2\tilde{\kappa}_j x}\overline{\Gamma}_{rj} \\ e^{-2\tilde{\kappa}_j x}\overline{\Gamma}_{lj} & 0_{m\times m} \end{pmatrix}. \tag{4.51}$$

In [3] we find definitions of $\overline{\Gamma}_{lj}$ and $\overline{\Gamma}_{rj}$ in terms of the residues of $\overline{R}$ and $\overline{L}$ at $-i\tilde{\kappa}_j$, even though in general the reflection coefficients do not extend meromorphically off the real axis. This is correct under appropriate decay assumptions on the potential.

Substituting equations (4.42) into (4.36) and using (4.43) it is easy to prove the following

Theorem 4.12 *For $\alpha \geq 0$ we have the Marchenko integral equations*

$$B_{r1}(x,\alpha) = -\int_0^\infty d\beta\, B_{r2}(x,\beta)\,\Omega_r(\alpha+\beta-2x), \tag{4.52a}$$

$$B_{l2}(x,\alpha) = -\Omega_l(\alpha+2x) - \int_0^\infty d\beta\, B_{l1}(x,\beta)\,\Omega_l(\alpha+\beta+2x), \tag{4.52b}$$

$$B_{r3}(x,\alpha) = -\Omega_r(\alpha-2x) - \int_0^\infty d\beta\, B_{r4}(x,\beta)\,\Omega_r(\alpha+\beta-2x), \tag{4.52c}$$

$$B_{l4}(x,\alpha) = -\int_0^\infty d\beta\, B_{l3}(x,\beta)\,\Omega_l(\alpha+\beta+2x), \tag{4.52d}$$

where

$$\Omega_l(\alpha) = \hat{R}(\alpha) + \sum_{j=1}^{N} \Gamma_{lj}\, e^{-\kappa_j\alpha}, \tag{4.53a}$$

$$\Omega_r(\alpha) = \hat{L}(\alpha) + \sum_{j=1}^{N} \Gamma_{rj}\, e^{-\kappa_j\alpha}. \tag{4.53b}$$

In the same way we derive

Theorem 4.13 *For $\alpha \geq 0$ we have the Marchenko integral equations*

$$B_{l1}(x,\alpha) = -\int_0^\infty d\beta\, B_{l2}(x,\beta)\,\overline{\Omega}_l(\alpha+\beta+2x), \tag{4.54a}$$

$$B_{r2}(x,\alpha) = -\overline{\Omega}_r(\alpha-2x) - \int_0^\infty d\beta\, B_{r1}(x,\beta)\,\overline{\Omega}_r(\alpha+\beta-2x), \tag{4.54b}$$

$$B_{l3}(x,\alpha) = -\overline{\Omega}_l(\alpha+2x) - \int_0^\infty d\beta\, B_{l4}(x,\beta)\,\overline{\Omega}_l(\alpha+\beta+2x), \tag{4.54c}$$

$$B_{r4}(x,\alpha) = -\int_0^\infty d\beta\, B_{r3}(x,\beta)\,\overline{\Omega}_r(\alpha+\beta-2x). \tag{4.54d}$$

where

$$\overline{\Omega}_l(\alpha) = \hat{\overline{R}}(\alpha) + \sum_{j=1}^{\tilde{N}} \overline{\Gamma}_{lj}\, e^{-\tilde{\kappa}_j\alpha}, \tag{4.55a}$$

$$\overline{\Omega}_r(\alpha) = \hat{\overline{L}}(\alpha) + \sum_{j=1}^{\tilde{N}} \overline{\Gamma}_{rj}\, e^{-\tilde{\kappa}_j\alpha}. \tag{4.55b}$$

4.2.3 When the eigenvalues have any multiplicity

If some of the poles of $T_l(\lambda)$ or $T_r(\lambda)$ are not simple, the construction of the Marchenko integral equations is to be modified. It is comparatively simple to convert the Riemann-Hilbert problem containing the principal parts of the transmission coefficient at the poles into an integral equation of Marchenko type. However, it is not obvious how to generalize the insertion of the dependency constants to the multiple pole case. For the 1-D Schrödinger equation with energy dependent potential $ikP(x) + Q(x)$ this generalization has been accomplished in [8], but it is based on Wronskian relations relating Jost solutions of the Schrödinger equation with potential $ikP(x)+Q(x)$ to Jost solutions of the Schrödinger equation with potential $-ikP(x)+Q(x)$. Such Wronskian relations can in principle be generalized to systems of ordinary differential equations, but to have a meaningful result we need to relate Jost solutions of the original equations to those of the adjoint equations. In this subsection we shall accomplish the same result without using Wronskian relations by relating matrices of the type $\Phi_s(a_{l1};\lambda_0)$ and $\Phi_s(a_{r4};\lambda_0)$ for the poles $\lambda_0 \in \mathbb{C}^+$ of $T_l(\lambda)$ and $T_r(\lambda)$. By doing the same thing for the poles of $\overline{T}_l(\lambda)$ and $\overline{T}_r(\lambda)$ in $\mathbb{C}^-$, we avoid confining ourselves to the antisymmetric case, although Subsection 4.2.4 will be devoted to the simplifications arising from the antisymmetric case. It is not necessary to consider the symmetric case, since there is no discrete spectrum and hence no need to account for bound state information.

Let us assume that the transmission coefficients $T_l(\lambda)$ and $T_r(\lambda)$ have the form

$$T_l(\lambda) = \sum_{j=1}^{N} \sum_{s=0}^{q_j-1} \frac{i^{s+1}\,\tau_{ljs}}{(\lambda - i\kappa_j)^{s+1}} + T_{l0}(\lambda), \tag{4.56}$$

$$T_r(\lambda) = \sum_{j=1}^{N} \sum_{s=0}^{q_j-1} \frac{i^{s+1}\,\tau_{rjs}}{(\lambda - i\kappa_j)^{s+1}} + T_{r0}(\lambda), \tag{4.57}$$

where $T_{l0}(\lambda)$ and $T_{r0}(\lambda)$ are continuous in $\lambda \in \overline{\mathbb{C}^+}$, are analytic in $\lambda \in \mathbb{C}^+$, and tend to the identity matrix as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$. The usual technical hypothesis guarantees the finiteness of the set of poles, while Theorem 3.16 implies that $T_l(\lambda)$ and $T_r(\lambda)$ have the same poles $i\kappa_1, \ldots, i\kappa_N$ and the same pole orders $q_1, \ldots, q_N$. Obviously, τ_{ljq_j} and τ_{rjq_j} are nonzero matrices ($j = 1, \ldots, N$). As a result,

$$\Pi_{-,0} T_l(\lambda) = \sum_{j=1}^{N} \sum_{s=0}^{q_j-1} \frac{i^{s+1}\tau_{ljs}}{(\lambda - i\kappa_j)^{s+1}} = -\sum_{j=1}^{N} \sum_{s=0}^{q_j-1} \frac{\tau_{ljs}}{s!} \int_0^\infty d\alpha\, e^{-i\lambda\alpha} \alpha^s e^{-\kappa_j \alpha},$$

$$\Pi_{-,0} T_r(\lambda) = \sum_{j=1}^{N} \sum_{s=0}^{q_j-1} \frac{i^{s+1}\tau_{rjs}}{(\lambda - i\kappa_j)^{s+1}} = -\sum_{j=1}^{N} \sum_{s=0}^{q_j-1} \frac{\tau_{rjs}}{s!} \int_0^\infty d\alpha\, e^{-i\lambda\alpha} \alpha^s e^{-\kappa_j \alpha}.$$

Let us consider the series expansion

$$m_+(x,\lambda) = \sum_{t=0}^{\infty} (-i)^t m_+^{jt}(x) (\lambda - i\kappa_j)^t, \quad |\lambda - i\kappa_j| < \operatorname{Re}\kappa_j. \tag{4.58}$$

For $j = 1, \dots, N$ and $t = 0, 1, 2, \dots$ we introduce the partitioning

$$m_+^{jt}(x) = \begin{pmatrix} M_{l1}^{jt}(x) & M_{r2}^{jt}(x) \\ M_{l3}^{jt}(x) & M_{r4}^{jt}(x) \end{pmatrix}, \tag{4.59}$$

where $M_{l1}^{jt}(x)$, $M_{r2}^{jt}(x)$, $M_{l3}^{jt}(x)$, and $M_{r4}^{jt}(x)$ have the sizes $n \times n$, $n \times m$, $m \times n$ and $m \times m$, respectively. Letting

$$m(x,\lambda) = \int_0^\infty d\alpha\, e^{i\lambda\alpha} b(x,\alpha)$$

stand for any of two component matrices $M_{l1}(x,\lambda) - I_n$ and $M_{l3}(x,\lambda)$ of $m_+(x,\lambda) - I_{n+m}$ and $(-i)^t m^{jt}(x)$ for the coefficient of $(\lambda - i\kappa_j)^t$ in the Taylor series of $m(x,\lambda)$ at $\lambda = i\kappa_j$, we obtain after some calculation

$$\Pi_{-,0}\left(m(x,\lambda) \frac{i^{s+1}\,\tau_{ljs}}{(\lambda - i\kappa_j)^{s+1}} \right) = \sum_{t=0}^{s} \frac{i^{t+1}}{(\lambda - i\kappa_j)^{t+1}} m^{j,s-t}(x)\tau_{ljs},$$

where $s = 0, 1, \dots, q_j - 1$. Similarly, letting $m(x,\lambda)$ stand for any of two component matrices $M_{r2}(x,\lambda)$ and $M_{r4}(x,\lambda) - I_m$ of $m_+(x,\lambda) - I_{n+m}$, we obtain after some calculation

$$\Pi_{-,0}\left(m(x,\lambda) \frac{i^{s+1}\,\tau_{rjs}}{(\lambda - i\kappa_j)^{s+1}} \right) = \sum_{t=0}^{s} \frac{i^{t+1}}{(\lambda - i\kappa_j)^{t+1}} m^{j,s-t}(x)\tau_{rjs},$$

where $s = 0, 1, \dots, q_j - 1$.

By inverse Fourier transformation, we therefore obtain the following.

Proposition 4.14 *For $\alpha \geq 0$ we have the following integral equations*

$$B_{r1}(x,\alpha) = -\int_0^\infty d\beta\, B_{r2}(x,\beta)\hat{L}(\alpha+\beta-2x) - \sum_{j=1}^{N}\sum_{s=0}^{q_{lj}-1}\sum_{t=0}^{s} M_{l1}^{j,s-t}(x)\tau_{ljs}\frac{\alpha^t}{t!}e^{-\kappa_j\alpha}, \tag{4.60a}$$

$$\begin{aligned} B_{l2}(x,\alpha) = &-\hat{R}(\alpha+2x) - \int_0^\infty d\beta\, B_{l1}(x,\beta)\hat{R}(\alpha+\beta+2x) \\ &- \sum_{j=1}^{N}\sum_{s=0}^{q_{rj}-1}\sum_{t=0}^{s} M_{r2}^{j,s-t}(x)\tau_{rjs}\frac{\alpha^t}{t!}e^{-\kappa_j\alpha}, \end{aligned} \tag{4.60b}$$

$$\begin{aligned} B_{r3}(x,\alpha) = &-\hat{L}(\alpha-2x) - \int_0^\infty d\beta\, B_{r4}(x,\beta)\hat{L}(\alpha+\beta-2x) \\ &- \sum_{j=1}^{N}\sum_{s=0}^{q_{lj}-1}\sum_{t=0}^{s} M_{l3}^{j,s-t}(x)\tau_{ljs}\frac{\alpha^t}{t!}e^{-\kappa_j\alpha}, \end{aligned} \tag{4.60c}$$

$$B_{l4}(x,\alpha) = -\int_0^\infty d\beta\, B_{l3}(x,\beta)\hat{R}(\alpha+\beta+2x) - \sum_{j=1}^{N}\sum_{s=0}^{q_{rj}-1}\sum_{t=0}^{s} M_{r4}^{j,s-t}(x)\tau_{rjs}\frac{\alpha^t}{t!}e^{-\kappa_j\alpha}. \tag{4.60d}$$

Analogously, the transmission coefficients $\overline{T}_l(\lambda)$ and $\overline{T}_r(\lambda)$ have the form

$$\overline{T}_l(\lambda) = \sum_{j=1}^{\tilde{N}} \sum_{s=0}^{\tilde{q}_j-1} \frac{(-i)^{s+1}\,\tilde{\tau}_{ljs}}{(\lambda + i\tilde{\kappa}_j)^{s+1}} + \tilde{T}_{l0}(\lambda), \tag{4.61}$$

$$\overline{T}_r(\lambda) = \sum_{j=1}^{\tilde{N}} \sum_{s=0}^{\tilde{q}_j-1} \frac{(-i)^{s+1}\,\tilde{\tau}_{rjs}}{(\lambda + i\tilde{\kappa}_j)^{s+1}} + \tilde{T}_{r0}(\lambda), \tag{4.62}$$

where $\tilde{T}_{l0}(\lambda)$ and $\tilde{T}_{r0}(\lambda)$ are continuous in $\lambda \in \overline{\mathbb{C}^-}$, are analytic in $\lambda \in \mathbb{C}^-$, and tend to the identity matrix as $\lambda \to \infty$ in $\overline{\mathbb{C}^-}$. The usual technical hypothesis guarantees the finiteness of the set of poles, while Theorem 3.16 implies that $\overline{T}_l(\lambda)$ and $\overline{T}_r(\lambda)$ have the same poles $-i\tilde{\kappa}_1, \ldots, -i\tilde{\kappa}_{\tilde{N}}$ and the same pole orders $\tilde{q}_1, \ldots, \tilde{q}_{\tilde{N}}$. Obviously, $\tilde{\tau}_{lj\tilde{q}_j}$ and $\tau_{rj\tilde{q}_j}$ are nonzero matrices ($j = 1, \ldots, \tilde{N}$). As a result,

$$\Pi_{+,0}\overline{T}_l(\lambda) = \sum_{j=1}^{\tilde{N}} \sum_{s=0}^{\tilde{q}_j-1} \frac{(-i)^{s+1}\tilde{\tau}_{ljs}}{(\lambda + i\tilde{\kappa}_j)^{s+1}} = -\sum_{j=1}^{\tilde{N}} \sum_{s=0}^{\tilde{q}_j-1} \frac{\tilde{\tau}_{ljs}}{s!} \int_0^\infty d\alpha\, e^{-i\lambda\alpha} \alpha^s e^{-\tilde{\kappa}_j \alpha},$$

$$\Pi_{+,0}\overline{T}_r(\lambda) = \sum_{j=1}^{\tilde{N}} \sum_{s=0}^{\tilde{q}_j-1} \frac{(-i)^{s+1}\tilde{\tau}_{rjs}}{(\lambda + i\tilde{\kappa}_j)^{s+1}} = -\sum_{j=1}^{\tilde{N}} \sum_{s=0}^{\tilde{q}_j-1} \frac{\tilde{\tau}_{rjs}}{s!} \int_0^\infty d\alpha\, e^{-i\lambda\alpha} \alpha^s e^{-\tilde{\kappa}_j \alpha}.$$

Let us now consider the series expansion

$$m_-(x,\lambda) = \sum_{t=0}^{\infty} i^t m_-^{jt}(x)(\lambda + i\tilde{\kappa}_j)^t, \qquad |\lambda + i\tilde{\kappa}_j| < \operatorname{Re}\kappa_j. \tag{4.63}$$

For $j = 1, \ldots, \tilde{N}$ and $t = 0, 1, 2, \ldots$ we introduce the partitioning

$$m_-^{jt}(x) = \begin{pmatrix} M_{r1}^{jt}(x) & M_{l2}^{jt}(x) \\ M_{r3}^{jt}(x) & M_{l4}^{jt}(x) \end{pmatrix}, \tag{4.64}$$

where $M_{r1}^{jt}(x)$, $M_{l2}^{jt}(x)$, $M_{r3}^{jt}(x)$, and $M_{l4}^{jt}(x)$ have the sizes $n \times n$, $n \times m$, $m \times n$ and $m \times m$, respectively. Letting

$$\tilde{m}(x,\lambda) = \int_0^\infty d\alpha\, e^{-i\lambda\alpha} \tilde{b}(x,\alpha)$$

stand for any of two component matrices $M_{r1}(x,\lambda) - I_n$ and $M_{r3}(x,\lambda)$ of $m_-(x,\lambda) - I_{n+m}$ and $i^t \tilde{m}^{jt}(x)$ as the coefficient of $(\lambda + i\tilde{\kappa}_j)^t$ in the Taylor series of $\tilde{m}(x,\lambda)$ at $\lambda = -i\tilde{\kappa}_j$, we obtain after some calculation

$$\Pi_{+,0}\left(\tilde{m}(x,\lambda)\frac{(-i)^{s+1}\,\tilde{\tau}_{ljs}}{(\lambda + i\tilde{\kappa}_j)^{s+1}}\right) = \sum_{t=0}^{s} \frac{(-i)^{t+1}}{(\lambda + i\tilde{\kappa}_j)^{t+1}} \tilde{m}^{j,s-t}(x)\tilde{\tau}_{ljs},$$

where $s=0,1,\ldots,\tilde{q}_j-1$. Similarly, letting $\tilde{m}(x,\lambda)$ stand for any of two component matrices $M_{l2}(x,\lambda)$ and $M_{l4}(x,\lambda)-I_m$ of $m_-(x,\lambda)-I_{n+m}$, we obtain after some calculation

$$\Pi_{+,0}\left(\tilde{m}(x,\lambda)\frac{(-i)^{s+1}\,\tilde{\tau}_{rjs}}{(\lambda+i\tilde{\kappa}_j)^{s+1}}\right)=\sum_{t=0}^{s}\frac{(-i)^{t+1}}{(\lambda+i\tilde{\kappa}_j)^{t+1}}\tilde{m}^{j,s-t}(x)\tilde{\tau}_{rjs},$$

where $s=0,1,\ldots,\tilde{q}_j-1$.

By inverse Fourier transformation, we obtain the following

Proposition 4.15 *For $\alpha\geq 0$ we have the following integral equations*

$$B_{l1}(x,\alpha)=\int_0^\infty d\beta\,B_{l2}(x,\beta)\hat{\overline{R}}(\alpha+\beta+2x)-\sum_{j=1}^{\tilde{N}}\sum_{s=0}^{\tilde{q}_{lj}-1}\sum_{t=0}^{s}M_{r1}^{j,s-t}(x)\tilde{\tau}_{ljs}\frac{\alpha^t}{t!}e^{-\tilde{\kappa}_j\alpha},\tag{4.65a}$$

$$\begin{aligned}B_{r2}(x,\alpha)&=\hat{\overline{L}}(\alpha-2x)+\int_0^\infty d\beta\,B_{r1}(x,\beta)\hat{\overline{L}}(\alpha+\beta-2x)\\&\quad-\sum_{j=1}^{\tilde{N}}\sum_{s=0}^{\tilde{q}_{rj}-1}\sum_{t=0}^{s}M_{l2}^{j,s-t}(x)\tilde{\tau}_{rjs}\frac{\alpha^t}{t!}e^{-\tilde{\kappa}_j\alpha},\end{aligned}\tag{4.65b}$$

$$\begin{aligned}B_{l3}(x,\alpha)&=\hat{\overline{R}}(\alpha+2x)+\int_0^\infty d\beta\,B_{l4}(x,\beta)\hat{\overline{R}}(\alpha+\beta+2x)\\&\quad-\sum_{j=1}^{\tilde{N}}\sum_{s=0}^{\tilde{q}_{lj}-1}\sum_{t=0}^{s}M_{r3}^{j,s-t}(x)\tilde{\tau}_{ljs}\frac{\alpha^t}{t!}e^{-\tilde{\kappa}_j\alpha},\end{aligned}\tag{4.65c}$$

$$B_{r4}(x,\alpha)=\int_0^\infty d\beta\,B_{r3}(x,\beta)\hat{\overline{L}}(\alpha+\beta-2x)-\sum_{j=1}^{\tilde{N}}\sum_{s=0}^{\tilde{q}_{rj}-1}\sum_{t=0}^{s}M_{l4}^{j,s-t}(x)\tilde{\tau}_{rjs}\frac{\alpha^t}{t!}e^{-\tilde{\kappa}_j\alpha}.\tag{4.65d}$$

We now observe that the four equations (4.60a), (4.60b), (4.65a), and (4.65b) are coupled, as are also the four equations (4.60c), (4.60d), (4.65c), and (4.65d). When there are no bound states, we have in fact coupled pairs of two equations each. If the bound state poles are simple, dependency constants can be introduced to attain the same degree of coupling. We intend to generalize this procedure when there are higher order bound state poles.

If $\{\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_{s-1}\}$ and $\{\eta_0,\eta_1,\ldots,\eta_{s-1}\}$ are the two s-tuples of vectors in $\mathbb{C}^n$ and $\mathbb{C}^m$ satisfying (3.87) and (3.89), we can write instead of (3.88) and (3.89)

$$\begin{aligned}a_{l1}(\lambda)\sum_{\sigma=0}^{s-1}(\lambda-i\kappa_j)^\sigma\varepsilon_\sigma&=(\lambda-i\kappa_j)^s f_l(\lambda),\\a_{r4}(\lambda)\sum_{\sigma=0}^{s-1}(\lambda-i\kappa_j)^\sigma\eta_\sigma&=(\lambda-i\kappa_j)^s f_r(\lambda),\end{aligned}$$

where $f_l(\lambda)$ and $f_r(\lambda)$ are analytic vector functions in a neighborhood of $i\kappa_j$. Putting

$$T_l^{\#}(\lambda) = (\lambda - i\kappa_j)^{q_j} T_l(\lambda) = \sum_{s=0}^{q_j-1} (\lambda - i\kappa_j)^s \, i^{q_j-s} \tau_{lj,q_j-s-1} + (\lambda - i\kappa_j)^{q_j} \, T_{lj1}(\lambda),$$
$$T_r^{\#}(\lambda) = (\lambda - i\kappa_j)^{q_j} T_r(\lambda) = \sum_{s=0}^{q_j-1} (\lambda - i\kappa_j)^s \, i^{q_j-s} \tau_{rj,q_j-s-1} + (\lambda - i\kappa_j)^{q_j} \, T_{rj1}(\lambda),$$

where $T_{lj1}(\lambda)$ and $T_{rj1}(\lambda)$ are analytic in $i\kappa_j$, we obtain

$$\sum_{\sigma=q_j-s}^{q_j-1} (\lambda - i\kappa_j)^\sigma \varepsilon_{\sigma+s-q_j} = T_l^{\#}(\lambda) f_l(\lambda), \qquad \sum_{\sigma=q_j-s}^{q_j-1} (\lambda - i\kappa_j)^\sigma \eta_{\sigma+s-q_j} = T_r^{\#}(\lambda) f_r(\lambda),$$

implying that for $s = 1, \ldots, q_j$

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \varepsilon_0 \\ \vdots \\ \varepsilon_{s-1} \end{pmatrix} \in \operatorname{Im} \Phi_{q_j}(T_l^{\#}; i\kappa_j), \qquad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \eta_0 \\ \vdots \\ \eta_{s-1} \end{pmatrix} \in \operatorname{Im} \Phi_{q_j}(T_r^{\#}; i\kappa_j), \tag{4.66}$$

where either column vector contains $q_j - s$ zero entries which are by themselves zero column vectors in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Thus $\{\varepsilon_0, \ldots, \varepsilon_{s-1}\}$ induces a Jordan chain of $a_{l1}(\lambda)$ at $\lambda = i\kappa_j$ of length s or the zero string if and only if its natural extension to a chain of length q_j belongs to the range of $\Phi_{q_j}(T_l^{\#}; i\kappa_j)$, and likewise for Jordan chains of $a_{r4}(\lambda)$.

In the proof of Theorem 3.16 it has been explained that the Jordan chains of H of length s at the eigenvalue $\lambda = i\kappa_j \in \mathbb{C}^+$ are given by any of the two equivalent expressions

$$\left\{ \begin{pmatrix} F_{l1,0}(x,\lambda)\varepsilon_0 \\ F_{l3,0}(x,\lambda)\varepsilon_0 \end{pmatrix}, \begin{pmatrix} \sum_{\sigma=0}^{1} F_{l1,\sigma}(x,\lambda)\varepsilon_{1-\sigma} \\ \sum_{\sigma=0}^{1} F_{l3,\sigma}(x,\lambda)\varepsilon_{1-\sigma} \end{pmatrix}, \ldots, \begin{pmatrix} \sum_{\sigma=0}^{s-1} F_{l1,\sigma}(x,\lambda)\varepsilon_{s-\sigma-1} \\ \sum_{\sigma=0}^{s-1} F_{l1,\sigma}(x,\lambda)\varepsilon_{s-\sigma-1} \end{pmatrix} \right\}$$

and

$$\left\{ \begin{pmatrix} F_{r2,0}(x,\lambda)\eta_0 \\ F_{r4,0}(x,\lambda)\eta_0 \end{pmatrix}, \begin{pmatrix} \sum_{\sigma=0}^{1} F_{r2,\sigma}(x,\lambda)\eta_{1-\sigma} \\ \sum_{\sigma=0}^{1} F_{r4,\sigma}(x,\lambda)\eta_{1-\sigma} \end{pmatrix}, \ldots, \begin{pmatrix} \sum_{\sigma=0}^{s-1} F_{r2,\sigma}(x,\lambda)\eta_{s-\sigma-1} \\ \sum_{\sigma=0}^{s-1} F_{r2,\sigma}(x,\lambda)\eta_{s-\sigma-1} \end{pmatrix} \right\},$$

where

$$\Phi_s(a_{l1}; i\kappa_j) \begin{pmatrix} \varepsilon_0 \\ \vdots \\ \varepsilon_{s-1} \end{pmatrix} = \Phi_s(a_{r4}; i\kappa_j) \begin{pmatrix} \eta_0 \\ \vdots \\ \eta_{s-1} \end{pmatrix} = 0. \tag{4.67}$$

When writing the entries in the Jordan chains pertaining to F_{l1} and F_{l3} as a column of vectors in $\mathbb{C}^n$ and $\mathbb{C}^m$, we obtain

$$\left\{ \Phi_s(F_{l1}(x,\cdot); i\kappa_j) \begin{pmatrix} \varepsilon_0 \\ \vdots \\ \varepsilon_{s-1} \end{pmatrix}, \Phi_s(F_{l3}(x,\cdot); i\kappa_j) \begin{pmatrix} \varepsilon_0 \\ \vdots \\ \varepsilon_{s-1} \end{pmatrix} \right\}. \tag{4.68}$$

Similarly we get for the Jordan chains pertaining to F_{r2} and F_{r4}

$$\left\{ \Phi_s(F_{r2}(x,\cdot); i\kappa_j) \begin{pmatrix} \eta_0 \\ \vdots \\ \eta_{s-1} \end{pmatrix}, \Phi_s(F_{r4}(x,\cdot); i\kappa_j) \begin{pmatrix} \eta_0 \\ \vdots \\ \eta_{s-1} \end{pmatrix} \right\}. \tag{4.69}$$

The $1,1$-correspondence between the Jordan chains of $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ at $\lambda = i\kappa_j$ implies the equivalence, in the sense of Sec. 2.3, of the extensions $a_{l1} \oplus I_m$ and $I_n \oplus a_{r4}$ in $\mathbb{C}^+$. In other words, there exist analytic $(n+m) \times (n+m)$ matrix functions E and $\tilde{E}$ having only nonsingular values such that

$$E(\lambda)\,[a_{l1}(\lambda) \oplus I_m] = [I_n \oplus a_{r4}(\lambda)]\,\tilde{E}(\lambda), \qquad \lambda \in \mathbb{C}^+.$$

Consequently, for $j = 1, \ldots, N$ and $s = 0, 1, \ldots, q_j - 1$ we have

$$\Phi_s(E; i\kappa_j)\Phi_s(a_{l1} \oplus I_m; i\kappa_j) = \Phi_s(I_n \oplus a_{r4}; i\kappa_j)\Phi_s(\tilde{E}; i\kappa_j),$$

where the matrices $\Phi_s(E; i\kappa_j)$ and $\Phi_s(\tilde{E}; i\kappa_j)$ are nonsingular. Letting

$$E^3(\lambda) = \begin{pmatrix} 0_{m\times n} & I_m \end{pmatrix} E(\lambda) \begin{pmatrix} I_n \\ 0_{m\times n} \end{pmatrix},$$
$$\tilde{E}^3(\lambda) = \begin{pmatrix} 0_{m\times n} & I_m \end{pmatrix} \tilde{E}(\lambda) \begin{pmatrix} I_n \\ 0_{m\times n} \end{pmatrix},$$

we obtain for $j = 1, \ldots, N$ and $s = 0, 1, \ldots, q_j - 1$

$$\Phi_s(E^3; i\kappa_j)\Phi_s(a_{l1}; i\kappa_j) = \Phi_s(a_{r4}; i\kappa_j)\Phi_s(\tilde{E}^3; i\kappa_j).$$

Starting instead from the equivalence relation

$$[a_{l1}(\lambda) \oplus I_m]\,\tilde{E}(\lambda)^{-1} = E(\lambda)^{-1}\,[I_n \oplus a_{r4}(\lambda)], \qquad \lambda \in \mathbb{C}^+,$$

we derive for $j = 1, \ldots, N$ and $s = 0, 1, \ldots, q_j - 1$

$$\Phi_s(a_{l1}; i\kappa_j)\Phi_s((\tilde{E}^{-1})^3; i\kappa_j) = \Phi_s((E^{-1})^3; i\kappa_j)\Phi_s(a_{r4}; i\kappa_j),$$

where

$$(E^{-1})^3(\lambda) = \begin{pmatrix} 0_{m\times n} & I_m \end{pmatrix} E(\lambda)^{-1} \begin{pmatrix} I_n \\ 0_{m\times n} \end{pmatrix},$$
$$(\tilde{E}^{-1})^3(\lambda) = \begin{pmatrix} 0_{m\times n} & I_m \end{pmatrix} \tilde{E}(\lambda)^{-1} \begin{pmatrix} I_n \\ 0_{m\times n} \end{pmatrix}.$$

Thus there exist matrices block Toeplitz matrices

$$\Phi_s(C_j)=\begin{pmatrix} C_{j0} & 0 & \dots & 0\\ C_{j1} & C_{j0} & \dots & 0\\ \vdots & & \ddots & \vdots\\ C_{j,s-1} & \dots & C_{j1} & C_{j0}\end{pmatrix},\ \Phi_s(D_j)=\begin{pmatrix} D_{j0} & 0 & \dots & 0\\ D_{j1} & D_{j0} & \dots & 0\\ \vdots & & \ddots & \vdots\\ D_{j,s-1} & \dots & D_{j1} & D_{j0}\end{pmatrix},$$

satisfying

$$\Phi_s(C_j)=\Phi_s(C_j)\Phi_s(D_j)\Phi_s(C_j), \qquad \Phi_s(D_j)=\Phi_s(D_j)\Phi_s(C_j)\Phi_s(D_j),$$

such that the range of $\Phi_s(D_j)\Phi_s(C_j)$ coincides with the kernel of $\Phi_s(a_{l1};i\kappa_j)$ and the range of $\Phi_s(C_j)\Phi_s(D_j)$ coincides with the kernel of $\Phi_s(a_{r4};i\kappa_j)$. In particular,

$$\Phi_s(C_j)[\operatorname{Ker}\Phi_s(a_{l1};i\kappa_j)]=\operatorname{Ker}\Phi_s(a_{r4};i\kappa_j),$$
$$\Phi_s(D_j)[\operatorname{Ker}\Phi_s(a_{r4};i\kappa_j)]=\operatorname{Ker}\Phi_s(a_{l1};i\kappa_j),$$

while $\Phi_s(C_j)$ and $\Phi_s(D_j)$ act as each other's inverses between these two subspaces. We shall call the matrices $C_{j,\sigma}$ and $D_{j,\sigma}$ ($j=1,\dots,N$, $\sigma=0,1,\dots,q_j-1$) *dependency constant matrices.*

In analogy with (4.43) we introduce the *norming constant matrices* $\Gamma_{lj,\sigma}$ and $\Gamma_{rj,\sigma}$ ($j=1,\dots,N$, $\sigma=0,1,\dots,q_j-1$) by

$$\Phi_s(\Gamma_{lj})=-i\,\Phi_s(D_j)\Phi_s(T_r^{\#};i\kappa_j), \quad \Phi_s(\Gamma_{rj})=-i\,\Phi_s(C_j)\Phi_s(T_l^{\#};i\kappa_j), \tag{4.70}$$

where

$$\Phi_s(\Gamma_{lj})=\begin{pmatrix} \Gamma_{lj0} & 0 & \dots & 0\\ \Gamma_{lj1} & \Gamma_{j0} & \dots & 0\\ \vdots & & \ddots & \vdots\\ \Gamma_{lj,s-1} & \dots & \Gamma_{lj1} & \Gamma_{lj0}\end{pmatrix}, \qquad \Phi_s(\Gamma_{rj})=\begin{pmatrix} \Gamma_{rj0} & 0 & \dots & 0\\ \Gamma_{rj1} & \Gamma_{rj0} & \dots & 0\\ \vdots & & \ddots & \vdots\\ \Gamma_{rj,s-1} & \dots & \Gamma_{rj1} & \Gamma_{rj0}\end{pmatrix}.$$

Therefore, deleting the argument $i\kappa_j$ and any x-dependence from all Φ_s matrices to avoid clutter we compute for $s=1,\dots,q_j$

$$\begin{aligned}\Phi_s(M_{r2})\Phi_s(T_r^{\#}) &= \Phi_s(F_{r2})\Phi_s^m(e^{i\lambda x})\Phi_s(T_r^{\#}) \overset{?}{=} \Phi_s^n(e^{i\lambda x})\Phi_s(F_{r2})\Phi_s(T_r^{\#})\\ &\overset{?}{=} \Phi_s^n(e^{i\lambda x})\Phi_s(F_{l1})\Phi_s(D_j)\Phi_s(T_r^{\#})\\ &\overset{(4.70)}{=} i\Phi_s^n(e^{i\lambda x})\Phi_s(F_{l1})\Phi_s(\Gamma_{lj}) \overset{?}{=} i\Phi_s(F_{l1})\Phi_s^n(e^{i\lambda x})\Phi_s(\Gamma_{lj})\\ &= i\Phi_s(M_{l1})\Phi_s^n(e^{i\lambda x})\Phi_s^n(e^{i\lambda x})\Phi_s(\Gamma_{lj})\\ &\overset{?}{=} i\Phi_s(M_{l1})\Phi_s^n(e^{2i\lambda x})\Phi_s(\Gamma_{lj}) \overset{?}{=} i\Phi_s^n(e^{2i\lambda x})\Phi_s(M_{l1})\Phi_s(\Gamma_{lj}),\end{aligned}$$

where we give the following clarifications:

a. At the equality signs carrying the first, third, and fifth question marks, we use that $\Phi_s^p(e^{i\lambda x})$ and either of $\Phi_s(F_{r2})$ or $\Phi_s(F_{l1})$ intertwine for suitable p, where

$$\Phi_s^p(e^{i\lambda x}) = \Phi_s(e^{i(\cdot)x}I_p; i\kappa_j) = e^{-\kappa_j x}\begin{pmatrix} I_p & 0 & \ldots & 0 \\ (ix)I_p & I_p & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ \frac{(ix)^{s-1}}{(s-1)!}I_p & \ldots & (ix)I_p & I_p \end{pmatrix}.$$

This intertwining relation follows from

$$F_{l1}(x,\lambda) = e^{i\lambda x}M_{l1}(x,\lambda) = M_{l1}(x,\lambda)e^{i\lambda x}$$

and likewise for F_{r2} and M_{r2}.

b. At the equality sign carrying the second question mark, we use that $\Phi_s(D_j)$ maps the η-vectors pertaining to the Jordan chains of length s into the corresponding ε-vectors, in combination with the second of (4.66). This means assuming $s = q_j$. However, this is OK for any $s = 1, \ldots, q_j$, because $\Phi_q(A)\Phi_q(B) = \Phi_q(C)$ implies $\Phi_s(A)\Phi_s(B) = \Phi_s(C)$ for any $s \in 1, \ldots, q$.

c. At the equality sign carrying the fourth question mark we apply the identity $\Phi_s^n(e^{i\lambda x})^2 = \Phi_s^n(e^{2i\lambda x})$, which is in turn based on the rather obvious relation $(e^{i\lambda x}I_n)^2 = e^{2i\lambda x}I_n$.

In this way we derive the following analogs of (4.42):

$$\begin{aligned}
\Phi_s(M_{r2}(x,\cdot);\lambda_j)\Phi_s(T_r^{\#};\lambda_j) &= i\,\Phi_s(e^{2i(\cdot)x}I_n;\lambda_j)\Phi_s(M_{l1}(x,\cdot);\lambda_j)\Phi_s(\Gamma_{lj}) \\
&= i\,\Phi_s(M_{l1}(x,\cdot);\lambda_j)\Phi_s(e^{2i(\cdot)x}I_n;\lambda_j)\Phi_s(\Gamma_{lj}), && \text{(4.71a)} \\
\Phi_s(M_{l3}(x,\cdot);\lambda_j)\Phi_s(T_l^{\#};\lambda_j) &= i\,\Phi_s(e^{-2i(\cdot)x}I_m;\lambda_j)\Phi_s(M_{r4}(x,\cdot);\lambda_j)\Phi_s(\Gamma_{rj}) \\
&= i\,\Phi_s(M_{r4}(x,\cdot);\lambda_j)\Phi_s(e^{-2i(\cdot)x}I_m;\lambda_j)\Phi_s(\Gamma_{rj}), && \text{(4.71b)} \\
\Phi_s(M_{r4}(x,\cdot);\lambda_j)\Phi_s(T_r^{\#};\lambda_j) &= i\,\Phi_s(e^{2i(\cdot)x}I_m;\lambda_j)\Phi_s(M_{l3}(x,\cdot);\lambda_j)\Phi_s(\Gamma_{lj}) \\
&= i\,\Phi_s(M_{l3}(x,\cdot);\lambda_j)\Phi_s(e^{2i(\cdot)x}I_n;\lambda_j)\Phi_s(\Gamma_{lj}), && \text{(4.71c)} \\
\Phi_s(M_{l1}(x,\cdot);\lambda_j)\Phi_s(T_l^{\#};\lambda_j) &= i\,\Phi_s(e^{-2i(\cdot)x}I_n;\lambda_j)\Phi_s(M_{r2}(x,\cdot);\lambda_j)\Phi_s(\Gamma_{rj}) \\
&= i\,\Phi_s(M_{r2}(x,\cdot);\lambda_j)\Phi_s(e^{-2i(\cdot)x}I_m;\lambda_j)\Phi_s(\Gamma_{rj}), && \text{(4.71d)}
\end{aligned}$$

where $\lambda_j = i\kappa_j$. The four equalities between the first and third member can be written in the concise form

$$\begin{aligned}
&\Phi_s(m_+(x,\cdot);i\kappa_j)\Phi_s\left(\begin{pmatrix} T_l^{\#} & 0_{n\times m} \\ 0_{m\times n} & T_r^{\#}\end{pmatrix}; i\kappa_j\right) \\
&= i\Phi_s(m_+(x,\cdot);i\kappa_j)\Phi_s\left(\begin{pmatrix} 0_{n\times n} & e^{2i(\cdot)x}\gamma_{lj} \\ e^{-2i(\cdot)x}\gamma_{rj} & 0_{m\times m}\end{pmatrix}; i\kappa_j\right), \qquad \text{(4.72)}
\end{aligned}$$

where $\gamma_{lj}(\lambda) = \sum_{\sigma=0}^{q_j-1}(\lambda - i\kappa_j)^\sigma\Gamma_{lj,\sigma}$ and $\gamma_{rj}(\lambda) = \sum_{\sigma=0}^{q_j-1}(\lambda - i\kappa_j)^\sigma\Gamma_{rj,\sigma}$.

Let us now write the entries in the Jordan chains pertaining to F_{r1} and F_{r3} corresponding to the eigenvalue $-i\tilde{\kappa}_j \in \mathbb{C}^-$ as a column of vectors, obtaining

$$\left\{ \Phi_s(F_{r1}(x,\cdot); -i\tilde{\kappa}_j) \begin{pmatrix} \bar{\varepsilon}_0 \\ \vdots \\ \bar{\varepsilon}_{s-1} \end{pmatrix}, \Phi_s(F_{r3}(x,\cdot); -i\tilde{\kappa}_j) \begin{pmatrix} \bar{\varepsilon}_0 \\ \vdots \\ \bar{\varepsilon}_{s-1} \end{pmatrix} \right\}. \tag{4.73}$$

Similarly we get for the Jordan chains pertaining to F_{l2} and F_{l4}

$$\left\{ \Phi_s(F_{l2}(x,\cdot); -i\tilde{\kappa}_j) \begin{pmatrix} \bar{\eta}_0 \\ \vdots \\ \bar{\eta}_{s-1} \end{pmatrix}, \Phi_s(F_{l4}(x,\cdot); -i\tilde{\kappa}_j) \begin{pmatrix} \bar{\eta}_0 \\ \vdots \\ \bar{\eta}_{s-1} \end{pmatrix} \right\}, \tag{4.74}$$

where

$$\Phi_s(a_{r1}; -i\tilde{\kappa}_j) \begin{pmatrix} \bar{\varepsilon}_0 \\ \vdots \\ \bar{\varepsilon}_{s-1} \end{pmatrix} = \Phi_s(a_{l4}; -i\tilde{\kappa}_j) \begin{pmatrix} \bar{\eta}_0 \\ \vdots \\ \bar{\eta}_{s-1} \end{pmatrix} = 0. \tag{4.75}$$

Putting

$$\overline{T}_l^{\#}(\lambda) = (\lambda + i\tilde{\kappa}_j)^{\tilde{q}_j} \overline{T}_l(\lambda) = \sum_{s=0}^{\tilde{q}_j-1} (\lambda + i\tilde{\kappa}_j)^s (-i)^{\tilde{q}_j - s} \tau_{lj,\tilde{q}_j-s-1} + (i + \tilde{\kappa}_j) \tilde{T}_{lj1}(\lambda),$$
$$\overline{T}_r^{\#}(\lambda) = (\lambda + i\tilde{\kappa}_j)^{\tilde{q}_j} \overline{T}_r(\lambda) = \sum_{s=0}^{\tilde{q}_j-1} (\lambda + i\tilde{\kappa}_j)^s (-i)^{\tilde{q}_j - s} \tau_{rj,\tilde{q}_j-s-1} + (i + \tilde{\kappa}_j) \tilde{T}_{rj1}(\lambda),$$

we have in analogy with (4.66)

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \bar{\varepsilon}_0 \\ \vdots \\ \bar{\varepsilon}_{s-1} \end{pmatrix} \in \operatorname{Im} \Phi_{\tilde{q}_j}(\overline{T}_l^{\#}; -i\tilde{\kappa}_j), \qquad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \bar{\eta}_0 \\ \vdots \\ \bar{\eta}_{s-1} \end{pmatrix} \in \operatorname{Im} \Phi_{\tilde{q}_j}(\overline{T}_r^{\#}; -i\tilde{\kappa}_j), \tag{4.76}$$

where either column vector contains $\tilde{q}_j - s$ zero entries which are by themselves zero column vectors in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Moreover, there exist block Toeplitz matrices

$$\Phi_s(\tilde{C}_j) = \begin{pmatrix} \tilde{C}_{j0} & 0 & \dots & 0 \\ \tilde{C}_{j1} & \tilde{C}_{j0} & \dots & 0 \\ \vdots & & \ddots & \vdots \\ \tilde{C}_{j,s-1} & \dots & \tilde{C}_{j1} & \tilde{C}_{j0} \end{pmatrix}, \qquad \Phi_s(\tilde{D}_j) = \begin{pmatrix} \tilde{D}_{j0} & 0 & \dots & 0 \\ \tilde{D}_{j1} & \tilde{D}_{j0} & \dots & 0 \\ \vdots & & \ddots & \vdots \\ \tilde{D}_{j,s-1} & \dots & \tilde{D}_{j1} & \tilde{D}_{j0} \end{pmatrix},$$

satisfying

$$\Phi_s(\tilde{C}_j) = \Phi_s(\tilde{C}_j)\Phi_s(\tilde{D}_j)\Phi_s(\tilde{C}_j), \qquad \Phi_s(\tilde{D}_j) = \Phi_s(\tilde{D}_j)\Phi_s(\tilde{C}_j)\Phi_s(\tilde{D}_j),$$

such that the range of $\Phi_s(\tilde{D}_j)\Phi_s(\tilde{C}_j)$ coincides with the kernel of $\Phi_s(a_{r1};-i\tilde{\kappa}_j)$ and the range of $\Phi_s(\tilde{C}_j)\Phi_s(\tilde{D}_j)$ equals the kernel of $\Phi_s(a_{l4};-i\tilde{\kappa}_j)$. In particular,

$$\Phi_s(\tilde{C}_j)[\operatorname{Ker}\Phi_s(a_{r1};-i\tilde{\kappa}_j)] = \operatorname{Ker}\Phi_s(a_{l4};-i\tilde{\kappa}_j),$$
$$\Phi_s(\tilde{D}_j)[\operatorname{Ker}\Phi_s(a_{l4};-i\tilde{\kappa}_j)] = \operatorname{Ker}\Phi_s(a_{r1};-i\tilde{\kappa}_j),$$

while $\Phi_s(\tilde{C}_j)$ and $\Phi_s(\tilde{D}_j)$ act as each other's inverses between these two subspaces. We shall call the matrices $\tilde{C}_{j,\sigma}$ and $\tilde{D}_{j,\sigma}$ ($j=1,\ldots,\tilde{N}$, $\sigma=0,1,\ldots,\tilde{q}_j-1$) dependency constant matrices.

In analogy with (4.50) we introduce the norming constant matrices $\overline{\Gamma}_{lj,\sigma}$ and $\overline{\Gamma}_{rj,\sigma}$ ($j=1,\ldots,\tilde{N}$, $\sigma=0,1,\ldots,\tilde{q}_j-1$) by

$$\Phi_s(\overline{\Gamma}_{lj}) = i\,\Phi_s(\tilde{C}_j)\Phi_s(\overline{T}_l^{\#};-i\tilde{\kappa}_j), \qquad \Phi_s(\overline{\Gamma}_{rj}) = i\,\Phi_s(\tilde{D}_j)\Phi_s(\overline{T}_r^{\#};-i\tilde{\kappa}_j), \tag{4.77}$$

where

$$\Phi_s(\overline{\Gamma}_{lj}) = \begin{pmatrix} \overline{\Gamma}_{lj0} & 0 & \ldots & 0 \\ \overline{\Gamma}_{lj1} & \overline{\Gamma}_{j0} & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ \overline{\Gamma}_{lj,s-1} & \ldots & \overline{\Gamma}_{lj1} & \overline{\Gamma}_{lj0} \end{pmatrix}, \qquad \Phi_s(\overline{\Gamma}_{rj}) = \begin{pmatrix} \overline{\Gamma}_{rj0} & 0 & \ldots & 0 \\ \overline{\Gamma}_{rj1} & \overline{\Gamma}_{rj0} & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ \overline{\Gamma}_{rj,s-1} & \ldots & \overline{\Gamma}_{rj1} & \overline{\Gamma}_{rj0} \end{pmatrix}.$$

We then derive the following analogs of (4.49):

$$\begin{aligned}
\Phi_s(M_{l2}(x,\cdot);\lambda_j)\Phi_s(\overline{T}_r^{\#};\lambda_j) &= -i\,\Phi_s(e^{2i(\cdot)x}I_n;\lambda_j)\Phi_s(M_{r1}(x,\cdot);\lambda_j)\Phi_s(\overline{\Gamma}_{rj}) \\
&= -i\,\Phi_s(M_{r1}(x,\cdot);\lambda_j)\Phi_s(e^{2i(\cdot)x}I_n;\lambda_j)\Phi_s(\overline{\Gamma}_{rj}), && \text{(4.78a)}\\
\Phi_s(M_{r3}(x,\cdot);\lambda_j)\Phi_s(\overline{T}_l^{\#};\lambda_j) &= -i\,\Phi_s(e^{-2i(\cdot)x}I_m;\lambda_j)\Phi_s(M_{l4}(x,\cdot);\lambda_j)\Phi_s(\overline{\Gamma}_{lj}) \\
&= -i\,\Phi_s(M_{l4}(x,\cdot);\lambda_j)\Phi_s(e^{-2i(\cdot)x}I_m;\lambda_j)\Phi_s(\overline{\Gamma}_{lj}), && \text{(4.78b)}\\
\Phi_s(M_{l4}(x,\cdot);\lambda_j)\Phi_s(\overline{T}_r^{\#};\lambda_j) &= -i\,\Phi_s(e^{2i(\cdot)x}I_m;\lambda_j)\Phi_s(M_{r3}(x,\cdot);\lambda_j)\Phi_s(\overline{\Gamma}_{rj}) \\
&= -i\,\Phi_s(M_{r3}(x,\cdot);\lambda_j)\Phi_s(e^{2i(\cdot)x}I_n;\lambda_j)\Phi_s(\overline{\Gamma}_{rj}), && \text{(4.78c)}\\
\Phi_s(M_{r1}(x,\cdot);\lambda_j)\Phi_s(\overline{T}_l^{\#};\lambda_j) &= -i\,\Phi_s(e^{-2i(\cdot)x}I_n;\lambda_j)\Phi_s(M_{l2}(x,\cdot);\lambda_j)\Phi_s(\overline{\Gamma}_{lj}) \\
&= -i\,\Phi_s(M_{l2}(x,\cdot);\lambda_j)\Phi_s(e^{-2i(\cdot)x}I_m;\lambda_j)\Phi_s(\overline{\Gamma}_{lj}), && \text{(4.78d)}
\end{aligned}$$

where $\lambda_j = i\kappa_j$. The four equalities between the first and third member can be written in the concise form

$$\begin{aligned}
&\Phi_s(m_-(x,\cdot);-i\tilde{\kappa}_j)\Phi_s\left(\begin{pmatrix}\overline{T}_l^{\#} & 0_{n\times m}\\ 0_{m\times n} & \overline{T}_r^{\#}\end{pmatrix};-i\tilde{\kappa}_j\right)\\
&= -i\Phi_s(m_-(x,\cdot);-i\tilde{\kappa}_j)\Phi_s\left(\begin{pmatrix}0_{n\times n} & e^{2i(\cdot)x}\overline{\gamma}_{rj}\\ e^{-2i(\cdot)x}\overline{\gamma}_{lj} & 0_{m\times m}\end{pmatrix};-i\tilde{\kappa}_j\right),
\end{aligned} \tag{4.79}$$

where $\overline{\gamma}_{lj}(\lambda) = \sum_{\sigma=0}^{q_j-1}(\lambda+i\tilde{\kappa}_j)^\sigma\overline{\Gamma}_{lj,\sigma}$ and $\overline{\gamma}_{rj}(\lambda) = \sum_{\sigma=0}^{q_j-1}(\lambda+i\tilde{\kappa}_j)^\sigma\overline{\Gamma}_{rj,\sigma}$.

Let us employ (4.71d) to write (4.60a) in a more convenient way. Indeed, the second term in the right-hand side of (4.60a) can be written as follows:

$$i\sum_{s=0}^{q_j-1}\sum_{t=0}^{s}\underbrace{\frac{(-i\alpha)^t}{t!}e^{-\kappa_j\alpha}}_{[\Phi_{s+1}(e^{-i(\cdot)\alpha};i\kappa_j)]_t}\underbrace{(-i)^{s-t}M_{l1}^{j,s-t}(x)}_{[\Phi_{s+1}(M_{l1}(x,\cdot);i\kappa_j)]_{s-t}}\underbrace{i^{s+1}\tau_{ljs},}_{[\Phi_{s+1}(T_l^{\#};i\kappa_j)]_{q_j-s-1}}$$

where $[\Phi_{s+1}(C)]_p=C_p$. This expression can in turn be written as

$$\begin{aligned}&+i\left[\Phi_{s+1}(e^{-i(\cdot)\alpha}M_{l1}(x,\cdot)T_l^{\#};i\kappa_j)\right]_{q_j-1}\\ &=-\left[\Phi_{s+1}(e^{-i(\cdot)\alpha}e^{-2i(\cdot)\alpha}M_{r2}(x,\cdot)\Gamma_{rj};i\kappa_j)\right]_{q_j-1}\\ &=-\sum_{s=0}^{q_j-1}\sum_{t=0}^{s}\frac{(-i[\alpha+2x])^t}{t!}e^{-\kappa_j(\alpha+2x)}(-i)^{s-t}M_{r2}^{j,s-t}(x)\Gamma_{rj,q_j-s-1}.\end{aligned}$$

Similar considerations apply to eqs. (4.60b)-(4.60d).

We now easily derive the following theorem.

Theorem 4.16 *For $\alpha\geq 0$ we have the Marchenko integral equations*

$$B_{r1}(x,\alpha)=-\int_0^\infty d\beta\,B_{r2}(x,\beta)\,\Omega_r(\alpha+\beta-2x),\tag{4.80a}$$

$$B_{l2}(x,\alpha)=-\Omega_l(\alpha+2x)-\int_0^\infty d\beta\,B_{l1}(x,\beta)\,\Omega_l(\alpha+\beta+2x),\tag{4.80b}$$

$$B_{r3}(x,\alpha)=-\Omega_r(\alpha-2x)-\int_0^\infty d\beta\,B_{r4}(x,\beta)\,\Omega_r(\alpha+\beta-2x),\tag{4.80c}$$

$$B_{l4}(x,\alpha)=-\int_0^\infty d\beta\,B_{l3}(x,\beta)\,\Omega_l(\alpha+\beta+2x),\tag{4.80d}$$

where

$$\Omega_l(\alpha)=\hat{R}(\alpha)+\sum_{j=1}^{N}\sum_{s=0}^{q_j-1}\frac{\alpha^s}{s!}\,e^{-\kappa_j\alpha}\Gamma_{lj,q_j-1-s},\tag{4.81a}$$

$$\Omega_r(\alpha)=\hat{L}(\alpha)+\sum_{j=1}^{N}\sum_{s=0}^{q_j-1}\frac{\alpha^s}{s!}\,e^{-\kappa_j\alpha}\Gamma_{rj,q_j-1-s}.\tag{4.81b}$$

In the same way we prove

Theorem 4.17 *For $\alpha \geq 0$ we have the Marchenko integral equations*

$$B_{l1}(x,\alpha) = -\int_0^\infty d\beta\, B_{l2}(x,\beta)\,\overline{\Omega}_l(\alpha+\beta+2x), \tag{4.82a}$$

$$B_{r2}(x,\alpha) = -\overline{\Omega}_r(\alpha-2x) - \int_0^\infty d\beta\, B_{r1}(x,\beta)\,\overline{\Omega}_r(\alpha+\beta-2x), \tag{4.82b}$$

$$B_{l3}(x,\alpha) = -\overline{\Omega}_l(\alpha+2x) - \int_0^\infty d\beta\, B_{l4}(x,\beta)\,\overline{\Omega}_l(\alpha+\beta+2x), \tag{4.82c}$$

$$B_{r4}(x,\alpha) = -\int_0^\infty d\beta\, B_{r3}(x,\beta)\,\overline{\Omega}_r(\alpha+\beta-2x), \tag{4.82d}$$

where

$$\overline{\Omega}_l(\alpha) = \hat{\overline{R}}(\alpha) + \sum_{j=1}^{\tilde{N}} \sum_{s=0}^{\tilde{q}_j-1} \frac{\alpha^s}{s!}\, e^{-\tilde{\kappa}_j\alpha}\overline{\Gamma}_{lj,\tilde{q}_j-s-1}, \tag{4.83a}$$

$$\overline{\Omega}_r(\alpha) = \hat{\overline{L}}(\alpha) + \sum_{j=1}^{\tilde{N}} \sum_{s=0}^{\tilde{q}_j-1} \frac{\alpha^s}{s!}\, e^{-\tilde{\kappa}_j\alpha}\overline{\Gamma}_{rj,\tilde{q}_j-s-1}. \tag{4.83b}$$

4.2.4 Symmetry relations in the antisymmetric case

Let us now discuss the simplifications in deriving the Marchenko integral equations in the antisymmetric case. As we have seen in Sec. 2.5, in this case the spectrum of the Hamiltonian H is symmetric with respect to the real line, i.e., we have $\tilde{N} = N$ (same number of poles in $C^{\pm}$), $\tilde{\kappa}_j = \overline{\kappa_j}$, and $\tilde{q}_j = q_j$ $(j = 1, \ldots, N)$. Moreover [cf. Sec. 3.4],

$$\overline{T}_l(\lambda) = T_l(\overline{\lambda})^*, \qquad \overline{T}_r(\lambda) = T_r(\overline{\lambda})^*, \tag{4.84a}$$

$$\overline{R}(\lambda) = -R(\lambda)^*, \qquad \overline{L}(\lambda) = -L(\lambda)^*, \tag{4.84b}$$

$$\hat{\overline{R}}(\alpha) = -\hat{R}(\alpha)^*, \qquad \hat{\overline{L}}(\alpha) = -\hat{L}(\alpha)^*. \tag{4.84c}$$

Thus in the absence of bound states we have the symmetry relations

$$\overline{\Omega}_l(\alpha) = -\Omega_l(\alpha)^*, \qquad \overline{\Omega}_r(\alpha) = -\Omega_r(\alpha)^*, \tag{4.85}$$

for the Marchenko integral kernels. In this subsection we show, with the help Sec. B.3 of Appendix B, that these symmetry relations remain true if there are bound states [cf. eqs. (B.28)]. In the case of simple poles of the transmission coefficients these symmetry relations have been derived before in [2, 3] on the basis of their treatment of the norming constant matrices. In their proof residues of reflection coefficients are employed, which greatly restricts the class of potentials that can be treated. Our treatment of the norming constant matrices necessitates a profound study of the symmetry relations of various quantities in Appendix B.

Using (4.56), (4.57), (4.61), and (4.62) we easily derive the following symmetry relations for the Laurent expansion coefficients of $T_l(\lambda)$ and $T_r(\lambda)$ in $i\kappa_j$:

$$\tilde{\tau}_{ljs} = (\tau_{ljs})^*, \qquad \tilde{\tau}_{rjs} = (\tau_{rjs})^*, \tag{4.86}$$

where $j = 1, \ldots, N$ and $s = 0, 1, \ldots, q_j - 1$. According to (4.85) and (4.86), eqs. (4.85) are equivalent to the equalities

$$\overline{\Gamma}_{lj,\sigma} = -(\Gamma_{lj,\sigma})^*, \quad \overline{\Gamma}_{rj,\sigma} = -(\Gamma_{rj,\sigma})^*, \tag{4.87}$$

where $j = 1, \ldots, N$ and $s = 0, 1, \ldots, q_j - 1$. In Sec. B.3 we shall prove (4.87) if the poles of the transmission coefficients are simple.

4.3 The characterization problem

The characterization problem can be described as follow: *Give necessary and sufficient conditions for a matrix*

$$\begin{pmatrix} a(k) & b(k) \\ c(k) & d(k) \end{pmatrix}$$

to be the scattering matrix of a potential $V(x) \in L^1(\mathbb{R})$. More precisely, if we specify when the diagram below is such that the correspondence between the data within the boxes is 1,1, we arrive at the so-called characterization of the scattering data.

$k(x),\ \ell(x) \in L^1$	direct scattering problem $\longrightarrow$ $\longleftarrow$ inverse scattering problem	$R(k)$ or $L(k)$ plus bound state data

In the case of the Schrödinger equation the characterization problem for the so-called Faddeev class potentials (i.e., the real potentials $V(x)$ such that $\int_{-\infty}^{\infty} dx\,(1 + |x|)|V(x)| < \infty$) has been resolved in [75, 77] (also [7]). As far as we know, no solution of the characterization problem for the matrix Zakharov-Shabat system has been published. In the symmetric case on the half-line Melik-Adamjan [76] has given a complete characterization of the Jost solution as scattering data to retrieve an L^1-potential, but his proof only implies that the tail of the potential is in L^1. In this section we present the most general characterization result in the symmetric case (where there are no bound states) and in the antisymmetric case without bound states.

4.3.1 Symmetric case

In the symmetric case the characterization problem can be solved in a particularly elementary way. In fact, in this case the potentials are linked by $\ell(x) = k(x)^*$ and there are no bound states. We then have

$$\Omega_l(\alpha) = \hat{R}(\alpha), \qquad \Omega_r(\alpha) = \hat{L}(\alpha),$$

which are related to the reflection coefficients by eqs. (4.24a)-(4.24b):

$$R(\lambda) = \int_{-\infty}^{+\infty} d\alpha\, \hat{R}(\alpha) e^{-i\lambda\alpha}, \tag{4.88a}$$

$$L(\lambda) = \int_{-\infty}^{+\infty} d\alpha\, \hat{L}(\alpha) e^{-i\lambda\alpha}. \tag{4.88b}$$

In the following diagram we indicate the 1, 1-correspondence between selfadjoint potentials with entries in $L^1(\mathbb{R})$ and the scattering data.

$$\boxed{\begin{matrix}\ell(x)=k(x)^* \\ \text{in } L^1\end{matrix}} \quad \begin{matrix}\text{direct problem} \\ \xrightarrow{\hspace{3cm}} \\ \xleftarrow{\hspace{3cm}} \\ \text{inverse problem}\end{matrix} \quad \boxed{\begin{matrix} \begin{cases} R(\lambda)=\int_{-\infty}^{+\infty} d\alpha\,\hat{R}(\alpha)e^{-i\lambda\alpha} \\ \sup_{\lambda\in\mathbb{R}}\|R(\lambda)\|<1 \\ \int_{-\infty}^{+\infty} d\alpha\,\|\hat{R}(\alpha)\|<\infty, \end{cases} \\ \text{OR}: \begin{cases} L(\lambda)=\int_{-\infty}^{+\infty} d\alpha\,\hat{L}(\alpha)e^{-i\lambda\alpha} \\ \sup_{\lambda\in\mathbb{R}}\|L(\lambda)\|<1 \\ \int_{-\infty}^{+\infty} d\alpha\,\|\hat{L}(\alpha)\|<\infty. \end{cases}\end{matrix}}$$

Indeed, let the potentials satisfy $k(x) = \ell(x)^*$ and have their entries in $L^1(\mathbb{R})$. Then $R(\lambda)$ and $L(\lambda)$ have their entries in the Wiener algebra $\mathcal{W}$ (cf. Theorem 3.10 and eqs. (3.60) and (3.63)) and hence can be represented by (4.88), where the entries of $\hat{R}(\alpha)$ and $\hat{L}(\alpha)$ belong to $L^1(\mathbb{R})$. Moreover, since the scattering matrix $\boldsymbol{S}(\lambda)$ is unitary and the transmission coefficients are nonsingular and depend continuously on $\lambda \in \mathbb{R}$, we have as a result of (3.79)

$$\sup_{\lambda\in\mathbb{R}} \|R(\lambda)\| < 1, \qquad \sup_{\lambda\in\mathbb{R}} \|L(\lambda)\| < 1. \tag{4.89}$$

On the other hand, if the reflection coefficients satisfy (4.88) with

$$\int_{-\infty}^{\infty} d\alpha \left(\|\hat{R}(\alpha)\| + \|\hat{L}(\alpha)\| \right) < \infty, \tag{4.90}$$

as well as (4.89), then the Marchenko integral equations (4.10a) and (4.10b) are uniquely solvable for any $x \geq 0$, while the equations (4.10c) and (4.10d) are uniquely solvable for any $x \leq 0$ (see Theorems 4.3). Further, the potentials defined in terms of their solutions by (4.1) and (4.2) have their entries in $L^1(\mathbb{R})$ (see Theorem 4.6).

The scattering data consist of just one reflection coefficient, while the other reflection coefficient is to be computed in the process. In fact, given $R \in \mathcal{W}^{n\times m}$ satisfying the first of (4.89), $L(\lambda)$ is to be evaluated from the unitarity of the scattering matrix

$$\boldsymbol{S}(\lambda) = \begin{pmatrix} T_l(\lambda) & R(\lambda) \\ L(\lambda) & T_r(\lambda) \end{pmatrix}, \qquad \lambda \in \mathbb{R},$$

where $L \in \mathcal{W}^{m\times n}$, T_l is an invertible element of $\mathcal{W}_+^{n\times n}$ with $T_l(\pm\infty) = I_n$, and T_r is an invertible element of $\mathcal{W}_+^{m\times m}$ with $T_r(\pm\infty) = I_m$. Indeed, using unitarity we first determine the unique matrix functions $T_l(\lambda)$ and $T_r(\lambda)$ such that T_l is an invertible element of $\mathcal{W}_+^{n\times n}$ with $T_l(\pm\infty) = I_n$, T_r is an invertible element of $\mathcal{W}_+^{m\times m}$ with $T_r(\pm\infty) = I_m$, and (3.106) and (3.107) are satisfied. These factorization problems have a unique solution, as a result of Theorem 3.18. We then define $L(\lambda)$ by (3.108). On the other hand, given $L \in \mathcal{W}^{m\times n}$ satisfying the second of (4.89), we first determine the unique matrix functions $T_l(\lambda)$ and $T_r(\lambda)$ such that T_l is an invertible element of $\mathcal{W}_+^{n\times n}$ with $T_l(\pm\infty) = I_n$, T_r is an invertible element of $\mathcal{W}_+^{m\times m}$ with $T_r(\pm\infty) = I_m$, and the two equations (3.104) are satisfied. These factorization problems again have a unique solution. We then define [cf. (3.105)]

$$R(\lambda) = -T_l(\lambda)L(\lambda)^*[T_r(\lambda)^*]^{-1}, \qquad \lambda \in \mathbb{R}.$$

We then derive a uniquely solvable Marchenko integral equation and hence a unique potential.

4.3.2 Antisymmetric case

If bound states are present, the characterization problem in the antisymmetric case is more complicated than in the symmetric case and hence we do not study this general situation. On the other hand, when *there are no bound states*, the characterization problem can be solved as in the symmetric case. In this situation the potentials are linked by $\ell(x) = -k(x)^*$. We then have

$$\Omega_l(\alpha) = \hat{R}(\alpha), \qquad \Omega_r(\alpha) = \hat{L}(\alpha),$$

which are related to the reflection coefficients by eqs. (4.24a)-(4.24b):

$$R(\lambda) = \int_{-\infty}^{+\infty} d\alpha\, \hat{R}(\alpha) e^{-i\lambda\alpha}, \tag{4.91a}$$

$$L(\lambda) = \int_{-\infty}^{+\infty} d\alpha\, \hat{L}(\alpha) e^{-i\lambda\alpha}. \tag{4.91b}$$

In the following diagram we indicate the $1,1$-correspondence between selfadjoint potentials with entries in $L^1(\mathbb{R})$ and the scattering data.

$\ell(x) = -k(x)^*$ in L^1 without bound states	direct problem $\longrightarrow$ $\longleftarrow$ inverse problem	$\begin{cases} R(\lambda) = \int_{-\infty}^{+\infty} d\alpha\, \hat{R}(\alpha) e^{-i\lambda\alpha} \\ \int_{-\infty}^{+\infty} d\alpha\, \lVert\hat{R}(\alpha)\rVert < \infty, \end{cases}$ OR : $\begin{cases} L(\lambda) = \int_{-\infty}^{+\infty} d\alpha\, \hat{L}(\alpha) e^{-i\lambda\alpha} \\ \int_{-\infty}^{+\infty} d\alpha\, \lVert\hat{L}(\alpha)\rVert < \infty. \end{cases}$

Indeed, let the potentials satisfy $k(x) = -\ell(x)^*$ and have their entries in $L^1(\mathbb{R})$. Then $R(\lambda)$ and $L(\lambda)$ have their entries in the Wiener algebra $\mathcal{W}$ (cf. Theorem 3.10 and eqs. (3.60) and (3.63)) and hence can be represented by (4.88), where the entries of $\hat{R}(\alpha)$ and $\hat{L}(\alpha)$ belong to $L^1(\mathbb{R})$. On the other hand, if the reflection coefficients satisfy (4.88) with

$$\int_{-\infty}^{\infty} d\alpha \left(\lVert\hat{R}(\alpha)\rVert + \lVert\hat{L}(\alpha)\rVert \right) < \infty, \tag{4.92}$$

then the Marchenko integral equations (4.11a) and (4.11b) are uniquely solvable for any $x \geq 0$, while the equations (4.11c) and (4.11d) are uniquely solvable for any $x \leq 0$ (see Theorems 4.3). Further, the potentials defined in terms of their solutions by (4.1) and (4.2) have their entries in $L^1(\mathbb{R})$ (see Theorem 4.6).

The scattering data consist of just one reflection coefficient, while the other reflection coefficient is to be computed in the process. In fact, given $R \in \mathcal{W}^{n\times m}$, $L(\lambda)$ is to be evaluated from the J-unitarity of the scattering matrix

$$\boldsymbol{S}(\lambda) = \begin{pmatrix} T_l(\lambda) & R(\lambda) \\ L(\lambda) & T_r(\lambda) \end{pmatrix}, \qquad \lambda \in \mathbb{R},$$

where $L \in \mathcal{W}^{m\times n}$, T_l is an invertible element of $\mathcal{W}_+^{n\times n}$ with $T_l(\pm\infty) = I_n$, and T_r is an invertible element of $\mathcal{W}_+^{m\times m}$ with $T_r(\pm\infty) = I_m$. Indeed, using J-unitarity we first determine the unique matrix functions $T_l(\lambda)$ and $T_r(\lambda)$ such that T_l is an invertible element of $\mathcal{W}_+^{n\times n}$ with $T_l(\pm\infty) = I_n$, T_r is an invertible element of $\mathcal{W}_+^{m\times m}$ with $T_r(\pm\infty) = I_m$, and the two equations (3.95) are valid. These factorization problems have a unique solution (cf. Theorem 3.18). We then define $L(\lambda)$ by (3.100). On the other hand, given $L \in \mathcal{W}^{m\times n}$, we first determine the unique matrix functions $T_l(\lambda)$ and $T_r(\lambda)$ such that T_l is an invertible element of $\mathcal{W}_+^{n\times n}$ with $T_l(\pm\infty) = I_n$, T_r is an invertible element of $\mathcal{W}_+^{m\times m}$ with $T_r(\pm\infty) = I_m$, and the two equations (3.96) are satisfied. These factorization problems again have a unique solution. We then define [cf. (3.97)]

$$R(\lambda) = T_l(\lambda)L(\lambda)^*[T_r(\lambda)^*]^{-1}, \qquad \lambda \in \mathbb{R}.$$

We then derive a uniquely solvable Marchenko integral equation and hence a unique potential.

Chapter 5

Nonlinear Schrödinger Equations and Applications to Fiber Optics

In this chapter we give a brief exposition of fiber optics transmission and discuss the inverse scattering transform method for solving the matrix nonlinear Schrödinger equation. In Sec. 5.1, after a discussion of group velocity dispersion and nonlinearity effects, we introduce the nonlinear Schrödinger equation. We then go on to discuss a generalization to account for polarization effects. In Sec. 5.2 we discuss (bright and dark) solitons and their interactions. In Sec. 5.3 we find the Lax pair of operators determining the inverse scattering transform method for solving the matrix nonlinear Schrödinger equation. In Sec. 5.5 we derive the time evolution of the integral kernels of the Marchenko integral equations.

5.1 Basic facts on fiber optics transmission

Compared to the information carriers of radio transmissions, lightwaves have a much shorter wavelength and can therefore in principle carry information much more efficiently than radio waves. In fact, lightwaves can carry hundreds of thousands of times more information in unit time than radio waves. Furthermore, in suitable transparent materials a lightwave can propagate over tens or even hundreds of kilometers before facing a serious problem of energy loss. Since a strictly monochromatic lightwave has no information content (except for its frequency, amplitude, and phase), a lightwave has to be modulated to carry information efficiently. A modulated lightwave may be expressed in the following way

$$E(z,t) = \mathrm{Re}\left[\overline{E}(z,t)e^{i(k_0 z - \omega_0 t)}\right], \tag{5.1}$$

where t is time, $\overline{E}(z,t)$ is the modulation amplitude, or modulation, of the electric field $E(z,t)$, and ω_0 and k_0 are, respectively, the frequency and wave number of the unmodulated lightwave. Contrary to the case of strictly monochromatic light, the modulation depends on the spatial coordinate z along the fiber.

At $z = 0$ where the fiber begins, the information content of the modulated lightwave depends on the width of its Fourier spectrum, i.e., on the size of the support of the function

$$\overline{E}(z,\Omega) = \int_{-\infty}^{\infty} \overline{E}(z,t)e^{i\Omega t}\,dt \tag{5.2}$$

at $z = 0$. Even if we succeed in eliminating signal loss on transmission along the fiber (for example, by using optical amplifiers), there are nevertheless two unavoidable limitations that do not allow one to have perfect transmission, namely group velocity dispersion and nonlinearity of the fiber. We shall discuss these two phenomena separately.

1. Group velocity and group velocity dispersion. In a dispersive medium the angular frequency ω depends on the wave number k in a nonlinear way, by means of the so-called **dispersion relation**. The group velocity of a wavepacket is the velocity with which the variations in the shape of the wavepacket's amplitude propagate through space. The **group velocity** v_g is defined by the equation

$$v_g \equiv \frac{\partial\omega}{\partial k} = \frac{1}{k'}, \tag{5.3}$$

where ω is the angular frequency and k is the wave number. Group velocity is to be distinguished from the phase velocities

$$v_f(k) = \frac{\omega(k)}{k} \tag{5.4}$$

of the single components of the wavepacket. Electromagnetic waves in a material such as an optical fiber and waves along the surface of a fluid are major examples of wave motion in a dispersive medium.

Group velocity dispersion originates from the slightly different velocities with which the various frequency components $\overline{E}(z,\Omega)$ of $\overline{E}(z,t)$ propagate along the fiber. In a dispersive medium such as an optical fiber the dielectric constant ε depends on the frequency ω. As a result, different components of a wavepacket travel with different speeds and tend to change phase with respect to one another. Thus, in a dispersive medium the energy flows at a speed that may greatly differ from the phase velocity $(\omega/k) = (c/n)$, where k is the wavenumber, c the speed of light in vacuum, and n the index of refraction of the medium. Considering two frequencies, ω_1 and ω_2, in a wavepacket, the relative delay Δt_D of arrival time at the distance z is given by the so-called **group delay**

$$\Delta t_D \approx k''(\omega_1 - \omega_2)z, \tag{5.5}$$

where k'' is the derivative of the reciprocal group velocity k' with respect to the frequency ω, i.e.,

$$k'' \overset{\text{def}}{=} \frac{\partial k'}{\partial\omega} = -\frac{1}{v_g^2}\frac{\partial v_g}{\partial\omega} = -(k')^2\frac{\partial(1/k')}{\partial\omega} = \frac{-1}{(\partial\omega/\partial k)^2}\frac{\partial^2\omega}{\partial k^2}. \tag{5.6}$$

Group velocity dispersion becomes a more serious problem as the frequency difference $\omega_1 - \omega_2$ increases.

We recall that, *anomolous dispersion* occurs if $k'' < 0$, and *normal dispersion* if $k'' > 0$. Here we observe that k'' and $\frac{\partial^2\omega}{\partial k^2}$ have opposite signs.

2. Nonlinearity effects. The refractive index of a material such as an optical fiber may change under the influence of an electric field, which can be either externally applied or be generated by a lightbeam travelling along the fiber. This phenomenon is known as the **optical Kerr effect**. Especially the dependence of the refractive index on the electric field generated by a lightbeam travelling along the fiber is a major contributor to information loss in the propagation along the fiber.

When an electric field is applied, the dielectric material polarizes and a polarization current is induced. The effect of polarization is expressed in the following way

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P},$$

where ε_0 is the dielectric constant of the vacuum and $\vec{P}$ is the polarization vector given by

$$\vec{P} = -en\,\vec{\varepsilon}(\vec{E}),$$

e being the absolute value of the electron charge, n the density of the electrons which contribute to the polarization and $\vec{\varepsilon}$ represents the displacement of electron position in a dielectric molecule induced by the electric field $\vec{E}$ defined in (5.1). The index of refraction n is given by the phenomenological relation

$$n = n_0 + n_2 \frac{|\overline{E}|^2}{2}, \tag{5.7}$$

where n_0 is the refractive index in the absence of an electric field and n_2 is the so-called Kerr coefficient. Thus the Kerr effect induces the nonlinear phase shift (or self phase modulation)

$$\Delta\phi_N = \frac{\omega}{2c} n_2 |\overline{E}|^2 z = \frac{\pi z n_2 |\overline{E}|^2}{\lambda}. \tag{5.8}$$

For most materials used in fiber optics the Kerr coefficient n_2 is comparatively small (about $10^{-20}\, m^2 W^{-1}$ for typical glasses).

3. Nonlinear Schrödinger equation. If we take into account group velocity dispersion and the Kerr effect to describe the evolution of $\overline{E}$ in the direction z of propagation of information (i.e., along the fiber), we arrive at a so-called master equation. This equation, first derived by Hasegawa and Tappert in 1973 in [62], has the form

$$i\frac{\partial \overline{E}}{\partial z} - \frac{k''}{2}\frac{\partial^2 \overline{E}}{\partial \tau^2} + \frac{\omega_0 n_2}{2c}|\overline{E}|^2\overline{E} = 0. \tag{5.9}$$

Here $\tau = t - k'z$ is a time variable in which the effect of group velocity dispersion has been subtracted. Further, k'' is defined by (5.6), ω_0 is the carrier angular frequency, n_2 is the Kerr coefficient, and c is the light speed in vacuum. In applications, where anomolous dispersion prevails, k'' is almost always negative.

By rescaling the physical quantities $\overline{E}$, z, and T in the master equation (5.9) we arrive at the **nonlinear Schrödinger equation**

$$\frac{\partial q}{\partial Z} = \frac{i}{2}\frac{\partial^2 q}{\partial T^2} + i|q|^2 q. \tag{5.10}$$

It is well-known that eq. (5.9) can also be derived starting from the Maxwell equations (see, for example, Hasegawa in Chapter 1 of [89]).

The nonlinear Schrödinger equation (5.10) pertains to an idealized situation, where there are no amplifiers, there is no loss beyond group velocity dispersion, and the (negative) fiber dispersion k'' does not depend on Z. Letting $G(Z)$ be the amplifier gain, Γ the loss rate over the distance z_0, and letting k'' depend on Z, we have the more general equation

$$\frac{\partial q}{\partial Z} = \frac{i}{2} d(Z) \frac{\partial^2 q}{\partial T^2} + i|q|^2 q + [G(Z) - \Gamma] q. \tag{5.11}$$

4. Generalizations of the nonlinear Schrödinger equation. The nonlinear change in the index of refraction of a medium in which an electromagnetic wave is propagating, may cancel the diffraction divergence and lead to waveguide propagation of the radiation. The theory of such phenomena in two dimensional geometry has been developed by Zakharov and Shabat [94] for waves having the same polarization everywhere. Manakov [73] generalized the method used in [94] to the case of waves of arbitrary polarization

Following Chandrasekhar [33], we introduce a plane of reference passing through the direction of propagation of the electromagnetic wave and two unit vectors, $\vec{c}_l$ paralle**l** and $\vec{c}_r$ perpendicula**r** to this plane, such that $\vec{c}_r \times \vec{c}_l$ points in the direction of propagation. Let us write the electric field vector $\vec{E}$ as a sum of right- and left- hand polarized wave:

$$\vec{E} = E_1 \vec{c}_r + E_2 \vec{c}_l.$$

Using the orthogonality of $\vec{c}_r$ and $\vec{c}_l$, we obtain the following system of equations for E_1 and E_2:

$$i\frac{\partial E_1}{\partial t} + \frac{\partial^2 E_1}{\partial x^2} + k\left(|E_1|^2 + |E_2|^2\right) E_1 = 0, \tag{5.12}$$

$$i\frac{\partial E_2}{\partial t} + \frac{\partial^2 E_2}{\partial x^2} + k\left(|E_1|^2 + |E_2|^2\right) E_2 = 0, \tag{5.13}$$

which represent a generalization of (5.10). In fact, equation (5.10) is obtainable from eqs. (5.12) and (5.13) by substituting $E_2 = 0$.

5.2 Soliton solutions

Hasegawa and Tappert [62] were the first to show theoretically that an optical pulse in a dielectric fiber forms a solitary wave based on the fact that the wave envelope satisfies the nonlinear Schrödinger equation (5.10). However, at the time, the technology was not ready to support this important intuition. In fact, neither a dielectric fiber with small signal loss existed nor the dispersion properties of the fiber were known. Consequently, it was necessary to consider the case of normal dispersion where the group dispersion, k'', is positive, i.e, when the coefficient of the first term on the right-hand side in eq. (5.10) is negative, because, in this case, a solitary wave appears as the absence of a light wave (a so-called dark soliton) [63]. In the preceding sections of this chapter, we have already discussed the solution of the nonlinear Schrödinger equation by the inverse scattering method. According to this theory, the properties of the envelope soliton of the NLS equation can be described by the complex eigenvalues of the Hamiltonian $H = -iJ\frac{d}{dx} - V(x)$, where the potential $V(x)$ specifies the initial envelope wave form. Because of this, the solitary wave solution proposed by Hasegawa and Tappert is called a *soliton*. Seven years after the prediction by Hasegawa and Tappert, Mollenauer et al. [78] succeeded for the first time in the generation and transmission of optical solitons in a fiber, thus confirming experimentally the Hasegawa-Tappert predication.

In this section we introduce the concept of soliton from the physical point of view. We divide the treatment in two subsections: the first of these is devoted to the so-called *bright solitons* which appear as solutions of eq. (5.10) when k'' is negative (anomolous dispersion), the second to the *dark solitons* which occur when $k'' > 0$ (normal dispersion).

5.2.1 Bright solitons

Let us recall the nonlinear Schrödinger equation

$$\frac{\partial q}{\partial t} = \frac{i}{2}\frac{\partial^2 q}{\partial x^2} + i|q|^2 q,$$

where, with respect to eq. (5.10), we have made the change of variables $t = Z$ and $x = T$. This equation can be solved by the inverse scattering transform, where the corresponding eigenvalue equation $H\psi = \lambda\psi$ is the Zakharov-Shabat system. For the sake of simplicity, we consider the antisymmetric case (with $n = m = 1$), where the eigenvalue equation is given by

$$-i\frac{\partial \psi_1}{\partial x} - k(x)\psi_2 = \lambda\psi_1, \tag{5.14a}$$

$$i\frac{\partial \psi_2}{\partial x} + k(x)^*\psi_1 = \lambda\psi_2. \tag{5.14b}$$

If we write the eigenvalue of this equation as

$$\lambda = \frac{B + iA}{2}, \tag{5.15}$$

the one-soliton solutions have the following form (cf. (6.68))

$$k(x,t) = \pm A\frac{e^{iBx}e^{-i(A^2-B^2)t}e^{iC_1}}{\cosh(A[x - 2Bt] + C_2)}. \tag{5.16}$$

Equation (5.16) implies that the absolute value of $k(x,t)$ has the form $\phi(x - 2Bt)$ and hence represents a travelling wave with velocity $2B$. In particular, if $B = 0$, the absolute value of $k(x,t)$ no longer depends on t. On the other hand, the argument of $k(x,t)$ is a linear function of x and t (in fact, $\arg k(x,t) = Bx - (A^2 - B^2)t + C_1$ if $\pm A > 0$, and $\arg k(x,t) = Bx - (A^2 - B^2)t + C_1 + \pi$ if $\pm A < 0$), which only depends on t if $B = 0$ and only depends on x if $A = \pm B$. For $B \neq 0$ we have $\arg k(x,t) = \psi(x - ((A^2 - B^2)/B)t)$, which represents a travelling wave with velocity $(A^2 - B^2)/B$. It is remarkable that, contrary to the case of the Korteweg-de Vries equation [47, 48], the speed of the soliton is not proportional to its amplitude.

We have plotted $|k(x,t)|$ and $\arg k(x,t)$ as functions of x for four different values of t using MatLab Version 6.5, thus illustrating the travelling wave behaviour of these functions (with possibly different, zero or nonzero, wavespeeds). In these plots we have represented $|k(x,t)|$ and $\arg k(x,t)$ for $t=0$ (red solid curve), $t=1$ (green dashed curve), $t=2$ (red dashed curve), and $t=3$ (green solid curve). When either $|k(x,t)|$ or $\arg k(x,t)$ does not depend on t, the plot shows just one green solid curve. In Fig. 5.1 we have plotted these functions for a case where $B=0$, i.e., where $|k(x,t)|$ only depends on x and $\arg k(x,t)$ only depends on t. Next, in Fig. 5.2 we have

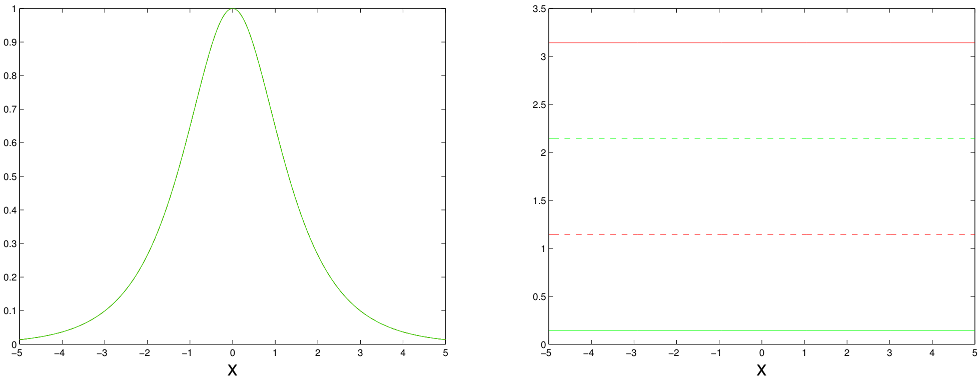


Figure 5.1: $|k(x,t)|$ as a function of x for any t and $\arg k(x,t)$ as a function of $x \in [-5,5]$ for $t=0,1,2,3$ for $A=1$, $B=0$, $C_1=\pi$, and $C_2=0$.

considered a case where $A=B>0$, i.e., where $\arg k(x,t)$ only depends on x. Finally, in Fig. 5.3

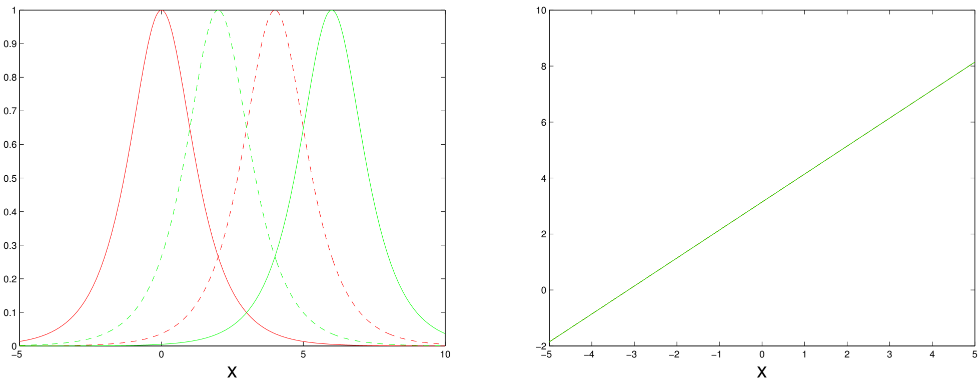


Figure 5.2: $|k(x,t)|$ as a function of $x \in [-5,10]$ and $\arg k(x,t)$ as a function of $x \in [-5,5]$ for $t=0,1,2,3$ for $A=B=1$, $C_1=\pi$, and $C_2=0$.

we have dealt with the most general case where neither $B=0$ nor $A=\pm B$. It is important to note that the amplitude A and speed $2B$ of the soliton are characterized by the imaginary and real parts of the eigenvalues λ in (5.15).

5.2.2 Dark solitons

In the wavelength range shorter than the zero group dispersion point where $k''=0$, soliton solutions no longer exist. In fact, there exist so-called dark soliton solutions which do not decay

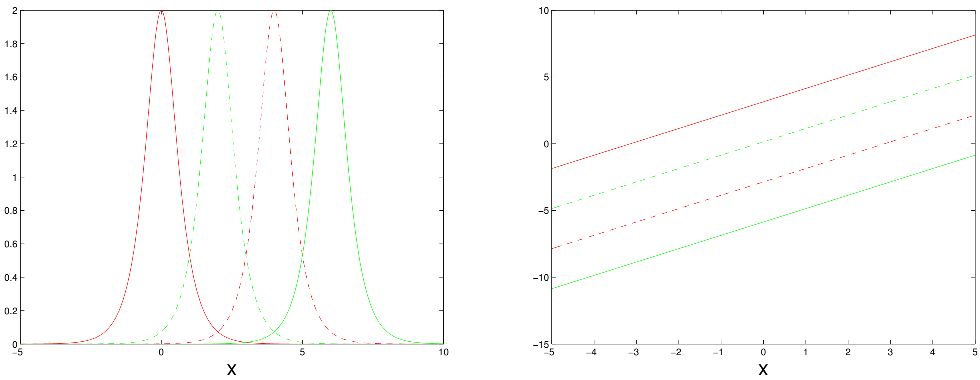


Figure 5.3: $|k(x,t)|$ as a function of $x \in [-5, 10]$ and $\arg k(x,t)$ as a function of $x \in [-5, 5]$ for $t = 0, 1, 2, 3$ for $A = 2$, $B = 1$, $C_1 = \pi$, and $C_2 = 0$.

as $x \to \pm\infty$ for fixed t.

For $k'' > 0$ eq. (5.10) can be rewritten by using a new distance normalization, $z_0 = t_0^2/k''$ as

$$\frac{\partial q}{\partial Z} = -\frac{i}{2}\frac{\partial^2 q}{\partial T^2} + i|q|^2 q. \tag{5.17}$$

In the preceding equation, q is the same rescaled quantity which appear in (5.10), while Z is the distance normalized by z_0. Now using variables x and t instead of T and Z, respectively, we have

$$\frac{\partial q}{\partial t} = -\frac{i}{2}\frac{\partial^2 q}{\partial x^2} + i|q|^2 q. \tag{5.18}$$

Dark solitons are solutions of the preceding equation which, as the bright solitons, have the property that they propagate without change of shape. Contrary to bright solitons, dark solutions do not vanish exponentially as $x \to \pm\infty$ for fixed t, but rather oscillate as $x \to \pm\infty$ for fixed t. In order to obtain the expression of the dark soliton solution, we follow the treatment of [61, Sections 4.7 and 5.4] and [89, Chapter 1]. Since we are looking for a localized solution q of eq. (5.18) that it is stationary in t (i.e., is a stationary shape of the packet), we ensure that the solution will be single-humped by imposing the following conditions:

1. $|q|^2$ is bounded below by $\rho_s > -\infty$ and above by $\rho_D < +\infty$;

2. $|q|^2$ has a global minimum at ρ_s where $\frac{\partial^2 |q|^2}{\partial x^2} > 0$;

3. ρ_D is the limit of $|q|^2$ as $x \to \pm\infty$.

In order to evaluate this solution, we introduce two real variables, $\rho = |q(x,t)|^2$ and $\sigma = \arg(q(x,t))$, i.e.,

$$q(x,t) = \rho(x,t)^{\frac{1}{2}}\, e^{i\sigma(x,t)}.$$

Substituting this expression in eq. (5.18) (and dividing the real part to the imaginary part), we find after straightforward but long calculations

$$\rho = \rho_0 \left[1 - a^2 \operatorname{sech}^2(\sqrt{\rho_0}\, ax)\right], \tag{5.19a}$$

$$\sigma = \left[\rho_0(1-a^2)\right]^{\frac{1}{2}} x + \arctan\left[\frac{a}{(1-a^2)^{\frac{1}{2}}} \tanh(\sqrt{\rho_0}\, ax)\right] - \frac{\rho_0(3-a^2)}{2} t, \tag{5.19b}$$

where ρ_0 denotes the asymptotic value of ρ and $a^2 = \frac{\rho_s - \rho_0}{\rho_0} \leq 1$. Unlike a bright soliton, a dark soliton has an additional new parameter, a, which designates the depth of modulation. We also note that at $x \to \pm\infty$, the phase of q changes. The expression for a dark soliton simplifies when $a = 1$: in this case the solution does not depend on t and reads

$$q(x,t) = \sqrt{\rho}\, \tanh(\sqrt{\rho}\, x).$$

Dark solitons were observed for the first time experimentally in fibers by Emplit et al. [44] and Krökel et al. [70] by transmitting a lightwave along a fiber under the normal dispersion.

5.2.3 Soliton interactions

In a communication system, it is desirable to launch the pulses close to each other as to increase the information carrying capacity of the fiber. But the overlap of the closely spaced solitons can lead to mutual interactions and therefore to serious performance degradation of the soliton transmission system.

By numerical investigation Chu and Desem [34] have shown that soliton interaction can lead to a significant reduction in the transmission rate by as much as a factor of ten. On the other hand, Blow and Doran [26] found that the inclusion of fiber loss results in a drastic increase in soliton interactions. In order to reduce soliton interactions, various ideas has been proposed. For example, Chu and Desem [35] suggest using Gaussian shaped pulses. In this case, the interactions are reduced because of its steep slope but this is achieved at the expense of creating larger oscillatory tails. A more realistic way of reducing the interactions is to launch adjacent pulses with unequal amplitudes [36, 37], thus effectively maintaining their initial pulse separation.

Soliton interactions are described by two-soliton solutions of the NLS equation. We have already seen that the eigenvalues of eq. (5.14) are related to the amplitudes (through the imaginary part of λ) and the velocities (through the real part of λ) of the solitons. If the real parts, A, of the eigenvalues are equal, the solitons are said to form a *bound state*. This means that the solitons undergo periodic oscillations in shape, which are determined by the imaginary part of the eigenvalues (B). However, if the values of A are different, they no longer form a bound state and the two-soliton solutions break up into diverging solitons as $t \to +\infty$.

5.3 IST for the nonlinear Schrödinger equation

In this section we change the terminology used in this chapter till now. In fact, we choose the variables in such a way that the variable that until now is denoted by Z will be indicated by t and the variable T by x.

The inverse scattering transform has been developed to solve the Korteweg-de Vries equation using the direct and inverse scattering theory of the Schrödinger equation in a series of seminal papers by Gardner, Greene, Kruskal, and Miura [47, 48]. The method was soon employed to solve the nonlinear Schrödinger equation using the direct and inverse scattering theory of the Zakharov-Shabat system [94]. Within a few years various nonlinear evolution equations came to the fore which could be solved using the direct and inverse scattering theory of a linear partial differential equation [2, 4]. Lax [72] realized that the inverse scattering transform is based on the interplay of two, generally unbounded but closed and densely defined, linear operators H and $\mathcal{B}$ on the same complex Hilbert space, forming the so-called *Lax pair* $(H, \mathcal{B})$, where H is a time dependent Hamiltonian operator and $\mathcal{B}$ is a time independent operator generating a unitary group $\{U(t)\}_{t\in\mathbb{R}}$, satisfying the time evolution equation

$$H_t = \mathcal{B}H - H\mathcal{B}. \tag{5.20}$$

Using that $U_t = \mathcal{B}U = U\mathcal{B}$ and $U(t = 0)$ is the identity operator, we see that

$$\frac{d}{dt}\left(U(-t)HU(t)\right) = -U(-t)\mathcal{B}HU(t) + U(-t)[\mathcal{B}H - H\mathcal{B}]U(t) + U(-t)H\mathcal{B}U(t) = 0.$$

Consequently,

$$H(t) = U(t)H(t = 0)U(-t), \tag{5.21}$$

which implies that the spectrum of $H(t)$ does not depend on t, which is generally known as the *isospectrality property*. In the case of the Korteweg-de Vries equation we have $H = -(d^2/dx^2) + u$ and $\mathcal{B} = -4(d^3/dx^3) + 3u(d/dx) + 3(d/dx)u$, both defined on the Hilbert space $L^2(\mathbb{R})$.

Let us now find a Lax pair for the matrix Zakharov-Shabat system

$$-iJ\frac{\partial}{\partial x}X(x,\lambda) - V(x)X(x,\lambda) = \lambda X(x,\lambda), \qquad x \in \mathbb{R}, \tag{5.22}$$

where λ is a spectral parameter, $X(x, \lambda)$ is a square matrix function of order $n + m$, $J = \mathrm{diag}(I_n, -I_m)$, and

$$V(x) = \begin{pmatrix} 0_{n\times n} & k(x) \\ \ell(x) & 0_{m\times m} \end{pmatrix} \tag{5.23}$$

has its entries in $L^1(\mathbb{R})$. We now seek a pair $(H, \mathcal{B})$ consisting of the *Hamiltonian operator*

$$H = -iJ\frac{d}{dx} - V(x,t)$$

and the *spectral evolution operator*

$$\mathcal{B} = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix},$$

where β_1 is an $n \times n$ differential operator in x, β_2 is an $n \times m$ differential operator in x, β_3 is an $m \times n$ differential operator in x, and β_4 is an $m \times m$ differential operator in x. The Lax pair must satisfy the operator differential equation

$$H_t = \mathcal{B}H - H\mathcal{B}, \tag{5.24}$$

where the t-dependence of H is confined to the potential term.

Applying (5.24) we obtain

$$-i\left(\beta_1\frac{d}{dx}-\frac{d}{dx}\beta_1\right)+k\beta_3-\beta_2\ell=0, \tag{5.25}$$

$$i\left(\beta_2\frac{d}{dx}+\frac{d}{dx}\beta_2\right)+k\beta_4-\beta_1 k=-k_t, \tag{5.26}$$

$$-i\left(\beta_3\frac{d}{dx}+\frac{d}{dx}\beta_3\right)+\ell\beta_1-\beta_4\ell=-\ell_t, \tag{5.27}$$

$$i\left(\beta_4\frac{d}{dx}-\frac{d}{dx}\beta_4\right)+\ell\beta_2-\beta_3 k=0. \tag{5.28}$$

Now assume that $\beta_1, \beta_2, \beta_3, \beta_4$ are second order linear differential operators in x whose matrix coefficients β_{pj} ($p = 1, 2, 3, 4,\ j = 0, 1, 2$) depend on x:

$$\beta_p=\beta_{p0}+\beta_{p1}\frac{d}{dx}+\beta_{p2}\frac{d^2}{dx^2}. \tag{5.29}$$

Then, following [43, Sec. 6.1], we obtain the four operator identities

$$\begin{aligned}
&[i(\beta_{10})_x+k\beta_{30}-\beta_{20}\ell-\beta_{21}\ell_x-\beta_{22}\ell_{xx}]\\
&+[i(\beta_{11})_x+k\beta_{31}-\beta_{21}\ell-2\beta_{22}\ell_x]\frac{d}{dx}\\
&+[i(\beta_{12})_x+k\beta_{32}-\beta_{22}\ell]\frac{d^2}{dx^2}=0,
\end{aligned} \tag{5.30}$$

$$\begin{aligned}
&[i(\beta_{20})_x+k\beta_{40}-\beta_{10}k-\beta_{11}k_x-\beta_{12}k_{xx}]\\
&+[i(\beta_{21})_x+k\beta_{41}-\beta_{11}k-2\beta_{12}k_x+2i\beta_{20}]\frac{d}{dx}\\
&+[i(\beta_{22})_x+k\beta_{42}-\beta_{12}k+2i\beta_{21}]\frac{d^2}{dx^2}+2i\beta_{22}\frac{d^3}{dx^3}=-k_t,
\end{aligned} \tag{5.31}$$

$$\begin{aligned}
&[-i(\beta_{30})_x+\ell\beta_{10}-\beta_{40}\ell-\beta_{41}\ell_x-\beta_{42}\ell_{xx}]\\
&+[-i(\beta_{31})_x+\ell\beta_{11}-\beta_{41}\ell-2\beta_{42}\ell_x-2i\beta_{30}]\frac{d}{dx}\\
&+[-i(\beta_{32})_x+\ell\beta_{12}-\beta_{42}\ell-2i\beta_{31}]\frac{d^2}{dx^2}-2i\beta_{32}\frac{d^3}{dx^3}=-\ell_t,
\end{aligned} \tag{5.32}$$

$$\begin{aligned}
&[-i(\beta_{40})_x+\ell\beta_{20}-\beta_{30}k-\beta_{31}k_x-\beta_{32}k_{xx}]\\
&+[-i(\beta_{41})_x+\ell\beta_{21}-\beta_{31}k-2\beta_{32}k_x]\frac{d}{dx}\\
&+[-i(\beta_{42})_x+\ell\beta_{22}-\beta_{32}k]\frac{d^2}{dx^2}=0.
\end{aligned} \tag{5.33}$$

Clearly, the coefficients of d/dx, d^2/dx^2 and d^3/dx^3 in (5.30)-(5.33) should vanish. If the coefficients of d^3/dx^3 vanish, we get

$$\beta_{22}=\beta_{32}=0,$$

which we substitute in (5.30)-(5.33). If the coefficients of d^2/dx^2 vanish, we get

$$\begin{aligned}(\beta_{12})_x &= 0,\\ k\beta_{42} - \beta_{12}k + 2i\beta_{21} &= 0,\\ \ell\beta_{12} - \beta_{42}\ell - 2i\beta_{31} &= 0,\\ (\beta_{42})_x &= 0,\end{aligned}$$

which means that β_{12} and β_{42} are constant matrices satisfying the relations

$$\beta_{21} = \frac{1}{2i}[\beta_{12}k - k\beta_{42}], \tag{5.34}$$

$$\beta_{31} = \frac{1}{2i}[\ell\beta_{12} - \beta_{42}\ell]. \tag{5.35}$$

If the coefficients of d/dx vanish, we get

$$i(\beta_{11})_x + k\beta_{31} - \beta_{21}\ell = 0, \tag{5.36}$$

$$i(\beta_{21})_x + k\beta_{41} - \beta_{11}k - 2\beta_{12}k_x + 2i\beta_{20} = 0, \tag{5.37}$$

$$-i(\beta_{31})_x + \ell\beta_{11} - \beta_{41}\ell - 2\beta_{42}\ell_x - 2i\beta_{30} = 0, \tag{5.38}$$

$$-i(\beta_{41})_x + \ell\beta_{21} - \beta_{31}k = 0. \tag{5.39}$$

Substituting (5.34) and (5.35) into (5.36) and (5.37) we get

$$(\beta_{11})_x = \frac{1}{2}[k\ell\beta_{12} - \beta_{12}k\ell], \tag{5.40}$$

$$(\beta_{41})_x = \frac{1}{2}[\ell k\beta_{42} - \beta_{42}\ell k]. \tag{5.41}$$

In the case $n = m = 1$ (Zakharov-Shabat system) kl and lk are scalar functions and hence β_{11} and β_{41} are scalar constants. In the case $n = 1$ and $m = 2$ (Manakov system) we can only conclude that β_{11} is a scalar constant.

Finally, equating the coefficients of the terms not containing a factor d/dx or a factor d^2/dx^2 we get

$$i(\beta_{10})_x + k\beta_{30} - \beta_{20}\ell - \beta_{21}\ell_x = 0, \tag{5.42}$$

$$i(\beta_{20})_x + k\beta_{40} - \beta_{10}k - \beta_{11}k_x - \beta_{12}k_{xx} = -k_t, \tag{5.43}$$

$$-i(\beta_{30})_x + \ell\beta_{10} - \beta_{40}\ell - \beta_{41}\ell_x - \beta_{42}\ell_{xx} = -\ell_t, \tag{5.44}$$

$$-i(\beta_{40})_x + \ell\beta_{20} - \beta_{30}k - \beta_{31}k_x = 0. \tag{5.45}$$

For general n, m we now **assume** that β_{11} and β_{41} are constant matrices. We already know that $\beta_{22} = 0_{n\times m}$ and $\beta_{33} = 0_{m\times n}$ and that β_{12} and β_{42} are constant matrices. Thus β_{12} is a constant $n \times n$ matrix commuting with $k\ell$ [See (5.40)] and β_{42} is a constant $m \times m$ matrix commuting with ℓk [See (5.41)]. Using (5.34) in (5.37) and (5.35) in (5.38) we obtain

$$2i\beta_{20} = \frac{1}{2}(3\beta_{12}k_x + k_x\beta_{42}) + (\beta_{11}k - k\beta_{41}), \tag{5.46}$$

$$2i\beta_{30} = -\frac{1}{2}(\ell_x\beta_{12} + 3\beta_{42}\ell_x) + (\ell\beta_{11} - \beta_{41}\ell), \tag{5.47}$$

respectively. With the help of (5.22)-(5.35) and (5.42)-(5.43) we get from (5.47)

$$2(\beta_{10})_x = -\frac{1}{2}(k\beta_{42}\ell)_x - \frac{3}{2}\beta_{12}(k\ell)_x, \tag{5.48}$$

where we **assume** that β_{11} commutes with $k\ell$ and β_{12} commutes with $k\ell_x$, and

$$2(\beta_{40})_x = -\frac{1}{2}(\ell\beta_{12}k)_x - \frac{3}{2}\beta_{42}(\ell k)_x, \tag{5.49}$$

where we **assume** that β_{41} commutes with ℓk and β_{42} commutes with ℓk_x. For $n = m = 1$ these assumptions are always satisfied. Thus there exist constant matrices γ_1 and γ_4 such that

$$2\beta_{10} + \frac{1}{2}k\beta_{42}\ell + \frac{3}{2}\beta_{12}k\ell = 2\gamma_1, \tag{5.50}$$

$$2\beta_{40} + \frac{1}{2}\ell\beta_{12}k + \frac{3}{2}\beta_{42}\ell k = 2\gamma_4, \tag{5.51}$$

where we recall that β_{12} commutes with $k\ell$ and β_{42} commutes with ℓk. Substituting (5.50)-(5.51) into (5.44) and using (5.46) we obtain

$$k_t = \frac{1}{4}(\beta_{12}[k_{xx} - 2k\ell k] - [k_{xx} - 2k\ell k]\beta_{42}) + \frac{1}{2}(\beta_{11}k_x + k_x\beta_{41}) + (\gamma_1 k - k\gamma_4). \tag{5.52}$$

On the other hand, substituting (5.50)-(5.51) into (5.43) and using (5.46) we obtain

$$\ell_t = -\frac{1}{4}([\ell_{xx} - 2\ell k\ell]\beta_{12} - \beta_{42}[\ell_{xx} - 2\ell k\ell]) + \frac{1}{2}(\ell_x\beta_{11} + \beta_{41}\ell_x) - (\ell\gamma_1 - \gamma_4\ell). \tag{5.53}$$

We have thus derived the coupled system (5.52)-(5.53) of nonlinear evolution equations, where the elements of the constant matrices β_{12}, β_{42}, β_{11}, β_{41}, γ_1, and γ_4 are parameters.

Let us assume $\ell(x) = k(x)^*$ (symmetric case) or $\ell(x) = -k(x)^*$ (antisymmetric case). In order to convert the coupled system (5.52)-(5.53) in two uncoupled equations, one the adjoint of the other, we must **assume** $\beta_{12}^* = -\beta_{12}$, $\beta_{42}^* = -\beta_{42}$, $\beta_{11}^* = \beta_{11}$, $\beta_{41}^* = \beta_{41}$, $\gamma_1^* = \gamma_1$, and $\gamma_4^* = -\gamma_4$ to obtain (5.53) as the adjoint of (5.52). Writing $\beta_{12} = i\tilde{\beta}_{12}$, $\beta_{42} = i\tilde{\beta}_{42}$, $\gamma_1 = i\tilde{\gamma}_1$, and $\gamma_4 = i\tilde{\gamma}_4$, where $\tilde{\beta}_{12}$, $\tilde{\beta}_{42}$, $\tilde{\gamma}_1$ and $\tilde{\gamma}_4$ are constant hermitian matrices, we obtain

$$\begin{aligned} ik_t = &\frac{1}{4}(-\tilde{\beta}_{12}[k_{xx} \mp 2kk^*k] + [k_{xx} \mp 2kk^*k]\tilde{\beta}_{42}) \\ &+ \frac{1}{2}i(\beta_{11}k_x + k_x\beta_{41}) + (-\tilde{\gamma}_1 k + k\tilde{\gamma}_4). \end{aligned} \tag{5.54}$$

Here we should take notice of the anciliary constraints that β_{11} commutes with kk^*, β_{41} commutes with k^*k, $\tilde{\beta}_{12}$ commutes with $k(k_x)^*$, and $\tilde{\beta}_{42}$ commutes with k^*k_x. Taking $\beta_{11} = \tilde{\gamma}_1 = 0_{n\times n}$, $\beta_{41} = \tilde{\gamma}_4 = 0_{m\times m}$, $\tilde{\beta}_{12} = -2I_n$, and $\tilde{\beta}_{42} = 2I_m$, we obtain from (5.54) the usual mNLS equation

$$\boxed{i\, k_t = k_{xx} \mp 2kk^*k,} \tag{5.55}$$

where the plus sign refers to the antisymmetric case and the minus sign to the symmetric case. In the literature the symmetric case is often called *defocussing* and the antisymmetric case *focussing.* Summarizing (and without using any of the symmetry relations $\ell = \pm k^*$) we now make the following special choice of β_{pq}:

β_{pq}	$p=1$	$p=2$	$p=3$	$p=4$
$q=0$	$ik\ell$	$-k_x$	$-\ell_x$	$-i\ell k$
$q=1$	$0_{n\times n}$	$-2k$	-2ℓ	$0_{m\times m}$
$q=2$	$-2iI_n$	$0_{n\times m}$	$0_{m\times n}$	$2iI_m$

For this special choice we have for the isospectrality generator

$$\mathcal{B} = \begin{pmatrix} ik\ell - 2i\dfrac{d^2}{dx^2} & -k_x - 2k\dfrac{d}{dx} \\ -\ell_x - 2\ell\dfrac{d}{dx} & -i\ell k + 2i\dfrac{d^2}{dx^2} \end{pmatrix}.$$

5.4 Time evolution of the scattering data

In this section we discuss the time evolution of the scattering data. To do so rigorously, we should in principle employ wave operators $W^{\pm}$ (e.g., [64]) defined by

$$W_{\pm}\phi = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}\phi, \qquad \phi \in \mathcal{H}_{n+m},$$

prove the existence of the limit appearing in their definition, and derive their main properties. Instead, we follow a semi-rigorous approach (as, for instance, in [4, 43, 1, 3]) to describe the time evolution of the scattering data.

Let us now derive the evolution of the Jost solutions and scattering coefficients if the initial potential $V(x,0)$ is replaced by $V(x,t)$, where

$$V(x,t) = \begin{pmatrix} 0_{n\times n} & k(x,t) \\ \ell(x,t) & 0_{m\times m} \end{pmatrix}$$

and k and ℓ satisfy the coupled system of nonlinear evolution equations (5.52) and (5.53). Let us first discuss the asymptotics of the isospectrality operator $\mathcal{B}$ as $x \to \pm\infty$. Taking into account that k and ℓ tend to 0 as $x\to\infty$ we find

$$\mathcal{B}_\infty = \begin{pmatrix} -2i\frac{d^2}{dx^2} & 0_{n\times m} \\ 0_{m\times n} & 2i\frac{d^2}{dx^2} \end{pmatrix} = -2iJ\frac{d^2}{dx^2}.$$

Put $U_\infty(t) = e^{t\mathcal{B}_\infty}$. Then for each Sobolev index $s\in\mathbb{R}$ we have the commutative diagram

$$\begin{array}{ccc} \mathcal{H}^s_{n+m} & \xrightarrow{U_\infty(t)} & \mathcal{H}^s_{n+m} \\ {\scriptstyle\mathcal{F}}\big\downarrow & & \big\downarrow{\scriptstyle\mathcal{F}} \\ L^{2,s}_{n+m}(\mathbb{R}) & \xrightarrow[e^{2i\xi^2 tJ}]{} & L^{2,s}_{n+m}(\mathbb{R}), \end{array}$$

so that $\{U_\infty(t)\}_{t\in\mathbb{R}}$ is a strongly continuous group of unitary operators on each of the Sobolev spaces $\mathcal{H}^s_{n+m}$. Formally applying $U_\infty(t)$ to $e^{i\lambda Jx}\vec{\eta}$ for some vector $\vec{\eta}$ and using the linear operators in the above diagram we get

$$U_\infty(t)e^{i\lambda Jx}\vec{\eta} \xrightarrow{(2\pi)^{-1/2}\mathcal{F}} \begin{pmatrix} e^{2it\xi^2}\delta(\xi+\lambda)\vec{\eta}^{\,\mathrm{up}} \\ e^{-2it\xi^2}\delta(\xi-\lambda)\vec{\eta}^{\,\mathrm{dn}} \end{pmatrix} \xrightarrow{(2\pi)^{1/2}\mathcal{F}^{-1}} e^{2it\lambda^2 J}e^{i\lambda Jx}\vec{\eta}. \tag{5.56}$$

Using $HF_l(\cdot,\lambda)=\lambda F_l(\cdot,\lambda)$, $HF_r(\cdot,\lambda)=\lambda F_r(\cdot,\lambda)$, and $U(t)H=H(t)U(t)$ we obtain

$$H(t)(U(t)F_l)(x,\lambda)=\lambda(U(t)F_l)(x,\lambda),$$
$$H(t)(U(t)F_r)(x,\lambda)=\lambda(U(t)F_r)(x,\lambda).$$

Hence, apart from a right factor $e^{2i\lambda^2 tJ}$, $U(t)F_l(\cdot,\lambda)$ and $U(t)F_r(\cdot,\lambda)$ are the left and right Jost solutions of the matrix Zakharov-Shabat system with potential $V(x,t)$. In other words,

$$(U(t)F_l)(x,\lambda)\simeq e^{2i\lambda^2 tJ}F_l(x,\lambda),\qquad x\to+\infty, \tag{5.57}$$
$$(U(t)F_r)(x,\lambda)\simeq e^{2i\lambda^2 tJ}F_r(x,\lambda),\qquad x\to-\infty, \tag{5.58}$$

where we compare a solution of the matrix Zakharov-Shabat system with potential $V(x,t)$ to one with potential $V(x)$ as $x\to+\infty$ (in (5.57)) and as $x\to-\infty$ (in (5.58)). Recalling eqs. (3.1a), (3.1b), (3.4a), and (3.4b) we easily obtain

$$(U(t)F_l)(x,\lambda)\simeq\begin{cases}e^{i\lambda Jx}e^{2i\lambda^2 tJ}, & x\to+\infty,\\ e^{i\lambda Jx}a_l(\lambda;t)e^{2i\lambda^2 tJ}, & x\to-\infty,\end{cases}$$

and

$$e^{2i\lambda^2 tJ}F_l(x,\lambda)\simeq\begin{cases}e^{2i\lambda^2 tJ}e^{i\lambda Jx}, & x\to+\infty,\\ e^{2i\lambda^2 tJ}e^{i\lambda Jx}a_l(\lambda), & x\to-\infty.\end{cases}$$

Now, using eq. (5.57), we find

$$e^{2i\lambda^2 tJ}a_l(\lambda)=a_l(\lambda;t)e^{2i\lambda^2 tJ}. \tag{5.59}$$

Proceeding in a similar way, but starting from the relation (5.58), we get

$$e^{2i\lambda^2 tJ}a_r(\lambda)=a_r(\lambda;t)e^{2i\lambda^2 tJ}. \tag{5.60}$$

Using eqs. (5.59) and (5.60), we can immediately write down the following set of relations

$$a_{l1}(\lambda)=a_{l1}(\lambda;t),\quad a_{l2}(\lambda;t)=e^{4it\lambda^2}a_{l2}(\lambda), \tag{5.61a}$$
$$a_{r1}(\lambda)=a_{r1}(\lambda;t),\quad a_{r2}(\lambda;t)=e^{4it\lambda^2}a_{r2}(\lambda), \tag{5.61b}$$
$$a_{l4}(\lambda)=a_{l4}(\lambda;t),\quad a_{l3}(\lambda;t)=e^{-4it\lambda^2}a_{l3}(\lambda), \tag{5.61c}$$
$$a_{r4}(\lambda)=a_{r4}(\lambda;t),\quad a_{r3}(\lambda;t)=e^{-4it\lambda^2}a_{r3}(\lambda), \tag{5.61d}$$

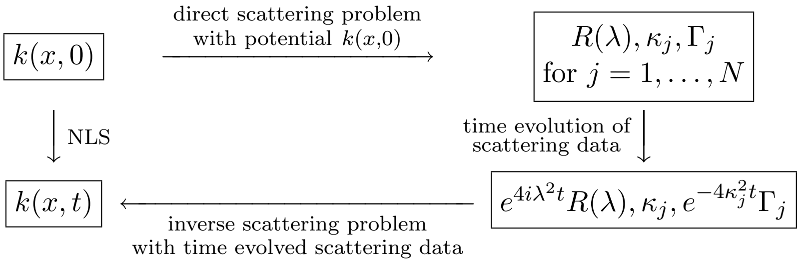
which imply

$$\begin{aligned}T_l(\lambda;t)=T_l(\lambda),&\quad \overline{T}_l(\lambda;t)=\overline{T}_l(\lambda),\\ T_r(\lambda;t)=T_r(\lambda),&\quad \overline{T}_r(\lambda;t)=\overline{T}_r(\lambda),\end{aligned} \tag{5.62}$$

and

$$\begin{aligned}R(\lambda;t)=e^{4it\lambda^2}R(\lambda),&\quad \overline{R}(\lambda;t)=e^{-4it\lambda^2}\overline{R}(\lambda),\\ L(\lambda;t)=e^{-4it\lambda^2}L(\lambda),&\quad \overline{L}(\lambda;t)=e^{4it\lambda^2}\overline{L}(\lambda).\end{aligned} \tag{5.63}$$

The IST to solve the matrix nonlinear Schrödinger equation is now described by the following diagram:



5.5 Time evolution of Marchenko integral kernels

In this section we derive a linear PDE for the Marchenko integral kernel Ω and explicitly determine the convolution integral operator mapping $\Omega(\alpha;0)$ into $\Omega(\alpha;t)$.

In the absence of bound states the time evolution of the scattering data amounts to

$$R(\lambda) \to e^{4i\lambda^2 t} R(\lambda),$$

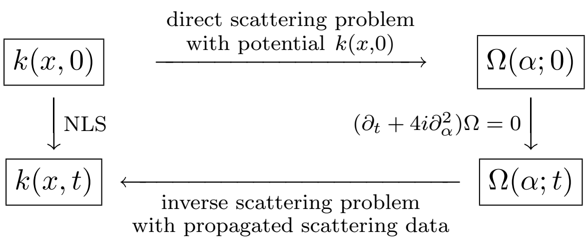
where the reflection coefficient $R(\lambda)$ can be written as the Fourier integral (4.24a) and hence

$$\hat{R}(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\lambda \, e^{i\lambda\alpha} R(\lambda).$$

Using the Theorem of Dominated Convergence it is clear that $\hat{R}$ has the time evolution

$$\partial_t \hat{R} + 4i\partial_\alpha^2 \hat{R} = 0, \tag{5.64}$$

provided $\int_{-\infty}^{\infty} d\lambda \, (1+\lambda^2) \, |R(\lambda)| < \infty$. Since $\hat{R} = \Omega$ is the Marchenko integral kernel, we can use Ω as scattering data and arrive at the following diagram:



Equation (5.64) can be interpreted in a more general way by letting the Fourier transform act from $L^{2,s}(\mathbb{R})$ onto $\mathcal{H}^s$ for any $s \in \mathbb{R}$. Since R is continuous and vanishes as $\lambda \to \pm\infty$, every entry of R belongs to $L^{2,s}(\mathbb{R})$ for $s > -\frac{1}{2}$. Thus every entry of $\hat{R}$ belongs to $\mathcal{H}_2^s$ for $s > -\frac{1}{2}$ and (5.64) remains true.

Let us now derive the linear PDE for the Marchenko integral kernel if there are bound states.

Theorem 5.1 *Suppose all of the poles of the transmission coefficients are simple. Then every entry of* Ω *belongs to* $\mathcal{H}^s$ *for* $s < -\frac{1}{2}$ *and*

$$\partial_t\Omega + 4i\partial_\alpha^2\Omega = 0. \tag{5.65}$$

Proof. Let us now study the time dependence of dependency constant matrices if the poles of the transmission coefficients are simple. Then (5.62) implies that τ_{lj0} and τ_{rj0} do not depend on t. From (5.61a) and (5.61b) it is clear that the vectors ε_j and η_j describing the eigenfunctions according to (4.38) are to be replaced by proportional vectors, where the proportionality constant depends on t. Since

$$(U(t)F_{l,r})(x,\lambda)e^{-2i\lambda^2 tJ} \simeq F_{l,r}(x,\lambda;t), \qquad x \to \pm\infty,$$

by taking $\lambda = i\kappa_j$ we obtain

$$F_l(x,i\kappa_j;t)e^{-2i\kappa_j^2 tJ}\begin{pmatrix}\varepsilon_j\\0\end{pmatrix} \simeq F_r(x,i\kappa_j;t)e^{-2i\kappa_j^2 tJ}\begin{pmatrix}0\\\eta_j\end{pmatrix}, \qquad x \to \pm\infty,$$

which leads to

$$e^{-2i\kappa_j^2 t}\varepsilon_j = e^{2i\kappa_j^2 t}D_{j0}(t)\eta_j, \qquad e^{-2i\kappa_j^2 t}C_{j0}(t)\varepsilon_j = e^{2i\kappa_j^2 t}\eta_j,$$

where $C_{j0}(t)$ and $D_{j0}(t)$ are the time dependent dependency constant matrices. Therefore,

$$C_{j0}(t) = e^{4i\kappa_j^2 t}C_{j0}, \qquad D_{j0}(t) = e^{-4i\kappa_j^2 t}D_{j0}.$$

Using (4.43) we obtain

$$\Gamma_{lj}(t) = e^{-4i\kappa_j^2 t}\Gamma_{lj}, \qquad \Gamma_{rj}(t) = e^{4i\kappa_j^2 t}\Gamma_{rj}.$$

Equation (5.65) then follows from (5.64) and (4.53a). ∎

We are interested in finding the solutions of eq. (5.65). To this end we observe that the following relations hold

$$\widehat{f'}(\xi) = \int_{-\infty}^{\infty} dy\, f'(y)e^{i\xi y} = -i\xi\widehat{f}(\xi), \qquad f \in H^1(\mathbb{R}),$$
$$\widehat{f''}(\xi) = -i\xi\widehat{f'}(\xi) = (-i\xi)^2\widehat{f}(\xi) = -\xi^2\widehat{f}(\xi), \quad f \in H^2(\mathbb{R}).$$

Then, if every entry of Ω belongs to $\mathcal{H}^2$, applying the Fourier transformation to eq. (5.65) we obtain

$$\partial_t\widehat{\Omega} = -4i\widehat{\partial_\alpha^2\Omega} = 4i\xi^2\widehat{\Omega}, \tag{5.66}$$

which admit as a unique solution the function

$$\widehat{\Omega}(\xi,t) = e^{4i\xi^2 t}\widehat{\Omega}(\xi,0). \tag{5.67}$$

Taking into account that

$$\Omega(y,t) = \lim_{N\to+\infty} \frac{1}{2\pi}\int_N^N d\xi\, e^{-iy\xi}\widehat{\Omega}(\xi,t) = \lim_{N\to+\infty}\frac{1}{2\pi}\int_N^N d\xi\, e^{-iy\xi}e^{4i\xi^2 t}\widehat{\Omega}(\xi,0)$$
$$= \lim_{N\to+\infty}\frac{1}{2\pi}\int_N^N d\xi\, e^{-iy\xi}e^{4i\xi^2 t}\int_{-\infty}^{\infty} dz\, e^{i\xi z}\Omega(z,0),$$

if we suppose that every entry of $\Omega(\cdot,0)$ belongs to $L^1(\mathbb{R})\cap L^2(\mathbb{R})$, we can apply Fubini's Theorem to obtain

$$\Omega(y,t) = \lim_{N\to+\infty}\int_{-\infty}^{\infty} dz\,\left(\frac{1}{2\pi}\int_N^N d\xi\, e^{-i\xi(y-z)}e^{4i\xi^2 t}\right)\Omega(z,0),$$

which can be written as

$$\Omega(y,t) = \lim_{N\to+\infty}\int_{-\infty}^{\infty} dz\,\left(\frac{1}{2\pi}\int_N^N d\xi\, e^{4it[\xi-\frac{1}{8t}(y-z)]^2}\right)e^{-\frac{1}{16t}i(y-z)^2}\Omega(z,0).$$

Making the following change of variable $\eta = 2\sqrt{|t|}[\xi - \frac{1}{8t}(y-z)]$, the preceding equation becomes

$$\Omega(y,t) = \lim_{N\to+\infty}\int_{-\infty}^{\infty} dz\,\left(\frac{1}{4\pi\sqrt{|t|}}\int_{2\sqrt{|t|}[-N-\frac{1}{8t}(y-z)]}^{2\sqrt{|t|}[N-\frac{1}{8t}(y-z)]} d\eta\, e^{i\eta^2 t}\right)\cdot$$
$$e^{-\frac{1}{16t}i(y-z)^2}\Omega(z,0). \tag{5.68}$$

Introducing the Fresnel integrals [5, 7.3.1-7.3.2]

$$C(z) = \int_0^z dt\,\cos\left(\frac{\pi}{2}t^2\right), \qquad S(z) = \int_0^z dt\,\sin\left(\frac{\pi}{2}t^2\right),$$

which have the property that $C(+\infty) = S(+\infty) = \frac{1}{2}$ (cf. [5, 7.3.0]), we easily calculate

$$\lim_{N\to+\infty}\frac{1}{4\pi\sqrt{|t|}}\int_{2\sqrt{|t|}[-N-\frac{1}{8t}(y-z)]}^{2\sqrt{|t|}[N-\frac{1}{8t}(y-z)]} d\eta\, e^{i\eta^2 t} = \frac{1}{2\pi\sqrt{|t|}}\int_0^{+\infty} d\eta\, e^{i\eta^2 t} =$$
$$\frac{1}{2\pi\sqrt{|t|}}\left[\int_0^{+\infty} d\eta\,\cos(\eta^2) + it\int_0^{+\infty} d\eta\,\sin(\eta^2)\right] = \frac{1+it}{4\sqrt{2\pi|t|}}.$$

As a result, using the Theorem of Dominated Convergence, eq. (5.68) can be written as

$$\Omega(y,t) = \int_{-\infty}^{\infty} dz\,\lim_{N\to+\infty}\left(\frac{1}{4\pi\sqrt{|t|}}\int_{2\sqrt{|t|}[-N-\frac{1}{8t}(y-z)]}^{2\sqrt{|t|}[N-\frac{1}{8t}(y-z)]} d\eta\, e^{i\eta^2 t}\right)\times$$
$$\times e^{-\frac{1}{16t}i(y-z)^2}\Omega(z,0) = \frac{1+it}{4\sqrt{2\pi|t|}}\int_{-\infty}^{+\infty} dz\, e^{-\frac{1}{16t}i(y-z)^2}\Omega(z,0), \tag{5.69}$$

where every entry of $\Omega(\cdot,0)$ belongs to $L^1(\mathbb{R})\cap L^2(\mathbb{R})$. If every entry of $\Omega(\cdot,0)$ belongs to $L^2(\mathbb{R})$ we have

$$\Omega(y,t) = \lim_{N\to+\infty}\frac{1+it}{4\sqrt{2\pi|t|}}\int_{-N}^{N} dz\, e^{-\frac{1}{16t}i(y-z)^2}\Omega(z,0). \tag{5.70}$$

In other words, we have proved the following

Theorem 5.2 *Let every entry of* $\Omega(\cdot,0)$ *be an element of* $L^1(\mathbb{R})\cap L^2(\mathbb{R})$*. Then the solution of eq.* (5.65) *is given by*

$$\Omega(y,t)=\frac{1+it}{4\sqrt{2\pi|t|}}\int_{-\infty}^{+\infty}dz\,e^{-\frac{1}{16t}i(y-z)^2}\Omega(z,0).$$

If every entry of $\Omega(\cdot,0)$ *belongs to* $L^2(\mathbb{R})$*, the solution of* (5.65) *is given by*

$$\Omega(y,t)=\lim_{N\to+\infty}\frac{1+it}{4\sqrt{2\pi|t|}}\int_{-N}^{N}dz\,e^{-\frac{1}{16t}i(y-z)^2}\Omega(z,0).$$

Theorem 5.2 describes the strongly continuous group of unitary transformations on the Sobolev spaces $\mathcal{H}^s_{n+m}$ generated by the skew-selfadjoint differential operator $-4i(d/d\alpha)^2$. A similar unitary group has been introduced to describe the solutions of the initial-value problem to the linearized Korteweg-de-Vries equation [4, Example 1.5.1] and the time evolution of the Marchenko integral kernel [4, Subsection 7.3.1] (also [2]). It is also known that, for $t>0$, the group action $\Omega(\cdot,0)\mapsto\Omega(\cdot,t)$ is a bounded linear operator from (a direct sum of copies of) $L^1(\mathbb{R})$ into (a direct sum of copies of) $L^\infty(\mathbb{R})$ (cf. [29]), a result which is also known to be true for the Schrödinger equation [52].

Chapter 6

State Space Solutions of the Marchenko equations

In Sec. 6.1, we introduce the so-called *state space methods* to write the Marchenko integral equations as integral equations with separated variables, thus leading to their explicit solution. By varying the Marchenko integral kernel in time while preserving its state space form, in Sec. 6.2 we also derive explicit solutions of the matrix nonlinear Schrödinger equation which encompass all known multi-soliton solutions. In the symmetric case these solutions may be local in time, but in the antisymmetric case most relevant to fiber optics they are global in time on each half-line. We shall discuss the extent to which these explicit mNLS solutions are those obtained by the inverse scattering transform. In Sec. 6.3 we derive all multi-soliton solutions of the matrix nonlinear Schrödinger equation. In Sec. 6.4 we give necessary and sufficient conditions for a multi-soliton solution to be time periodic. We then go on to plot the modulus, argument, real part, and imaginary part of the solution to the NLS equation ($n = m = 1$) in the antisymmetric case, where we illustrate various interesting special cases.

State space solutions of 1-D inverse spectral and inverse scattering problems for so-called canonical systems (in fact, matrix Zakharov-Shabat systems in the symmetric case, where $n = m$) have been studied extensively since Alpay and Gohberg [12] derived the solution of the inverse spectral problem on the half-line in state space form. The corresponding solution for the inverse scattering problem soon followed [13]. Subsequent research in this area led to a plethora of papers on these inverse spectral and inverse scattering problems [14, 15, 17, 55, 56, 58, 57, 16, 84, 85]. The inverse scattering problem for the matrix Zakharov-Shabat system (with $n = m$) on the line with rational scattering data was solved in the symmetric case [9] and in the antisymmetric case without bound states [90]. A review on the literature relating the direct and inverse spectral theory of canonical systems on finite intervals and on the half-line to the theory of certain classes of analytic operator-valued functions was given in [19].

In [55, 56] the state space formulas were modified to arrive at local in time solutions of certain nonlinear evolution equations on the half-line. Recently solutions of the Korteweg-de Vries equation on the half-line in state space form were derived in [11] in a particularly simple way.

So far state space methods in inverse scattering have primarily been used to solve inverse problems for the matrix Zakharov-Shabat system on the half-line in the symmetric case. Very few of these papers deal with the more interesting antisymmetric case. Further, if nonlinear evolution equations were solved, the issues of local vs. global in time existence and the obtainability of these solutions by the inverse scattering transform were never raised. In this chapter we apply state space methods to both the antisymmetric and symmetric cases, derive multi-soliton solutions in a systematic way, and relate these results to the literature.

6.1 State space method: symmetric and antisymmetric cases

In this section we frequently employ representations of rational $n \times m$ matrix functions $W(\lambda)$ of the form

$$W(\lambda) = \mathcal{D} - i\mathcal{C}(\lambda - i\mathcal{A})^{-1}\mathcal{B}, \tag{6.1}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are $p \times p$, $p \times m$, $n \times p$, and $n \times m$ matrices. Apart from the imaginary unit factors i, the representations (6.1) occur as transfer functions of linear continuous time systems [20, 39] and are often called *realizations.* The realization of W is called *minimal* if the matrix order p of $\mathcal{A}$ (the so-called *McMillan degree* of W) is minimal. In fact, a realization is minimal if and only if for sufficiently large integer r the $rn \times p$ matrix $\mathrm{col}_r(\mathcal{C}, \mathcal{A})$ and the $p \times mr$ matrix $\mathrm{row}_r(\mathcal{A}, \mathcal{B})$ defined by

$$\mathrm{col}_r(\mathcal{C}, \mathcal{A}) \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{C} \\ \mathcal{C}\mathcal{A} \\ \mathcal{C}\mathcal{A}^2 \\ \vdots \\ \mathcal{C}\mathcal{A}^{r-1} \end{pmatrix}, \qquad \mathrm{row}_r(\mathcal{A}, \mathcal{B}) \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{B} & \mathcal{A}\mathcal{B} & \mathcal{A}^2\mathcal{B} & \dots & \mathcal{A}^{r-1}\mathcal{B} \end{pmatrix},$$

have full rank. Thus minimality is equivalent to the pair of statements

$$\bigcap_{r=1}^{\infty} \mathrm{Ker}\,(\mathcal{C}\mathcal{A}^{r-1}) = \bigcap_{r=1}^{\infty} \mathrm{Ker}\,(\mathcal{B}^*(\mathcal{A}^*)^{r-1}) = \{0\}. \tag{6.2}$$

Minimal realizations have the property that $W(\lambda)$ and $(\lambda - i\mathcal{A})^{-1}$ have the same poles with the same pole order.

For minimal realizations the following uniqueness result is well-known (cf. [20, 39] and other books on linear systems theory).

Proposition 6.1 *Suppose*

$$W(\lambda) = \mathcal{D}_1 - i\mathcal{C}_1(\lambda - i\mathcal{A}_1)^{-1}\mathcal{B}_1 = \mathcal{D}_2 - i\mathcal{C}_2(\lambda - i\mathcal{A}_2)^{-1}\mathcal{B}_2$$

are two minimal realizations of W. Then there exists a unique nonsingular matrix $\mathcal{S}$ such that

$$\mathcal{A}_1\mathcal{S} = \mathcal{S}\mathcal{A}_2, \qquad \mathcal{B}_1 = \mathcal{S}\mathcal{B}_2, \qquad \mathcal{C}_1\mathcal{S} = \mathcal{C}_2, \qquad \mathcal{D}_1 = \mathcal{D}_2.$$

We first consider the **antisymmetric case**. If $R(\lambda)$ is a rational scattering matrix (of size $n \times m$) without real poles and vanishing as $\lambda \to \infty$, then there exist complex matrices $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ of respective sizes $p \times p$, $p \times m$, and $n \times p$ such that

$$R(\lambda) = -i\mathcal{C}(\lambda - i\mathcal{A})^{-1}\mathcal{B}, \quad \lambda \in \mathbb{C}, \tag{6.3}$$

where $\mathcal{A}$ has minimal matrix order and hence does not have purely imaginary eigenvalues. A similar minimal realization exists for $L(\lambda)$. Since $R(\lambda)$ does not have any obvious symmetry properties, there is not much that we know about $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ apriori.

We now recall that

$$R(\lambda) = \int_{-\infty}^{+\infty} d\alpha\, \hat{R}(\alpha)e^{-i\lambda\alpha}, \qquad L(\lambda) = \int_{-\infty}^{+\infty} d\alpha\, \hat{L}(\alpha)e^{-i\lambda\alpha},$$

where $\hat{R}(\alpha)$ and $\hat{L}(\alpha)$ are suitable matrix functions. Since the integral kernels of the Marchenko integral equations based on $R(\lambda)$ for $x \in \mathbb{R}^+$ only depend on $\hat{R}(\alpha)$ for $\alpha > 0$, it is sufficient to consider only the sum $\mathbf{\Pi}_+R(\lambda)$ of the principal parts of $R(\lambda)$ at the poles in $\mathbb{C}^+$. Similarly, since the integral kernels of the Marchenko integral equations based on $L(\lambda)$ for $x \in \mathbb{R}^-$ only depend on $\hat{L}(\alpha)$ for $\alpha > 0$, it is sufficient to consider only the sum $\mathbf{\Pi}_+L(\lambda)$ of the principal parts of $L(\lambda)$ at the poles in $\mathbb{C}^+$. In fact,

$$\mathbf{\Pi}_+R(\lambda) = \int_0^{+\infty} d\alpha\, \hat{R}(\alpha)e^{-i\lambda\alpha}, \qquad \mathbf{\Pi}_+L(\lambda) = \int_0^{+\infty} d\alpha\, \hat{L}(\alpha)e^{-i\lambda\alpha}.$$

We then have the following minimal realization of $\mathbf{\Pi}_+R(\lambda)$:

$$\mathbf{\Pi}_+R(\lambda) = -i\mathcal{C}(\lambda - i\mathcal{A})^{-1}\mathcal{B}, \qquad \lambda \in \mathbb{C}, \tag{6.4}$$

where the matrix order of $\mathcal{A}$ does not exceed the McMillan degree of $R(\lambda)$. Further, $\mathcal{A}$ has all of its eigenvalues in the right half-plane. We now easily get

$$\hat{R}(\alpha) = \mathcal{C}e^{-\alpha\mathcal{A}}\mathcal{B}, \qquad \alpha > 0, \tag{6.5}$$

where

$$e^{-\alpha\mathcal{A}} = \sum_{r=0}^{\infty} \frac{(-\alpha)^r}{r!}\mathcal{A}^r$$

stands for the matrix exponential.

Taking the adjoint in (6.3) and (6.4) we get

$$R(\lambda^*)^* = i\mathcal{B}^*(\lambda + i\mathcal{A}^*)^{-1}\mathcal{C}^*, \qquad \mathbf{\Pi}_-R(\lambda^*)^* = \int_0^{\infty} d\alpha\, \hat{R}(\alpha)^*e^{i\lambda\alpha},$$

where we have singled out the sum $\mathbf{\Pi}_-R(\lambda^*)^*$ of the principal parts of $R(\lambda^*)^*$ for $\lambda \in \mathbb{C}^-$. So we should in fact replace $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ by $(-\mathcal{A}^*, \mathcal{C}^*, -\mathcal{B}^*)$ to get $R(\lambda^*)^*$ instead of $R(\lambda)$. Then

$$\hat{R}(\alpha)^* = \mathcal{B}^*e^{-\alpha\mathcal{A}^*}\mathcal{C}^*, \qquad \alpha > 0.$$

We now easily compute that for $\alpha, \beta > 0$ and $x \in \mathbb{R}^+$

$$\int_0^\infty d\beta \, \hat{R}(\beta + \gamma + 2x)^* \hat{R}(\alpha + \beta + 2x) = \mathcal{B}^* e^{-(\gamma+2x)\mathcal{A}^*} \mathcal{X}_C e^{-(\alpha+2x)\mathcal{A}} \mathcal{B}, \tag{6.6}$$

$$\int_0^\infty d\beta \, \hat{R}(\beta + \gamma + 2x) \hat{R}(\alpha + \beta + 2x)^* = \mathcal{C} e^{-(\gamma+2x)\mathcal{A}} \mathcal{X}_B e^{-(\alpha+2x)\mathcal{A}^*} \mathcal{C}^*, \tag{6.7}$$

where

$$\mathcal{X}_C = \int_0^\infty d\beta \, e^{-\beta\mathcal{A}^*} \mathcal{C}^* \mathcal{C} e^{-\beta\mathcal{A}}, \tag{6.8}$$

$$\mathcal{X}_B = \int_0^\infty d\beta \, e^{-\beta\mathcal{A}} \mathcal{B} \mathcal{B}^* e^{-\beta\mathcal{A}^*}, \tag{6.9}$$

are the unique solutions of the Lyapunov equations

$$\mathcal{A}^* \mathcal{X}_C + \mathcal{X}_C \mathcal{A} = \mathcal{C}^* \mathcal{C}, \qquad \mathcal{A} \mathcal{X}_B + \mathcal{X}_B \mathcal{A}^* = \mathcal{B} \mathcal{B}^*. \tag{6.10}$$

These solutions are easily seen to be nonnegative selfadjoint. Moreover, $\mathcal{X}_B$ and $\mathcal{X}_C$ are nonsingular whenever the realization in (6.4) is minimal. Indeed, if $\mathcal{X}_C y = 0$, then

$$0 = \langle \mathcal{X}_C y, y \rangle = \int_0^\infty d\beta \, \|\mathcal{C} e^{-\beta\mathcal{A}} y\|^2,$$

and hence $\mathcal{C} e^{-\beta\mathcal{A}} y = 0$ for $\beta \geq 0$. Taking the Laplace transform we obtain $\mathcal{C}(\lambda + \mathcal{A})^{-1} y = 0$ for λ in the right half-plane and hence for λ in a neighborhood of infinity. As a result, we obtain from the Neumann series expansion $\mathcal{C}\mathcal{A}^{r-1} y = 0$ for $r = 1, 2, \ldots$, which implies $y = 0$ because of the minimality of the realization in (6.4). Consequently, $\mathcal{X}_C$ is nonsingular. In a similar way we prove the nonsingularity of $\mathcal{X}_B$. If we do not assume minimality of the realization (6.4), then the same reasoning implies that

$$\operatorname{Ker} \mathcal{X}_C = \bigcap_{r=1}^\infty \operatorname{Ker} (\mathcal{C}\mathcal{A}^{r-1}), \qquad \operatorname{Ker} \mathcal{X}_B = \bigcap_{r=1}^\infty \operatorname{Ker} (\mathcal{B}^* (\mathcal{A}^*)^{r-1}).$$

Thus minimality of (6.4) is equivalent to having both of $\mathcal{X}_C$ and $\mathcal{X}_B$ nonsingular.

Theorem 6.2 *Suppose that there are no bound states. Then for $x \in \mathbb{R}^+$ the solutions of the Marchenko integral equations* (4.11a) *and* (4.11b) (*where $\Omega(\lambda) = \hat{R}(\lambda)$*) *are given by*

$$B_{l1}(x, \alpha) = -\mathcal{C} \left[I_p + e^{-2x\mathcal{A}} \mathcal{X}_B e^{-2x\mathcal{A}^*} \mathcal{X}_C \right]^{-1} e^{-2x\mathcal{A}} \mathcal{X}_B e^{-(\alpha+2x)\mathcal{A}^*} \mathcal{C}^*; \tag{6.11}$$

$$B_{l2}(x, \alpha) = -\mathcal{C} \left[I_p + e^{-2x\mathcal{A}} \mathcal{X}_B e^{-2x\mathcal{A}^*} \mathcal{X}_C \right]^{-1} e^{-(\alpha+2x)\mathcal{A}} \mathcal{B}; \tag{6.12}$$

$$B_{l3}(x, \alpha) = +\mathcal{B}^* \left[I_p + e^{-2x\mathcal{A}^*} \mathcal{X}_C e^{-2x\mathcal{A}} \mathcal{X}_B \right]^{-1} e^{-(\alpha+2x)\mathcal{A}^*} \mathcal{C}^*; \tag{6.13}$$

$$B_{l4}(x, \alpha) = -\mathcal{B}^* \left[I_p + e^{-2x\mathcal{A}^*} \mathcal{X}_C e^{-2x\mathcal{A}} \mathcal{X}_B \right]^{-1} e^{-2x\mathcal{A}^*} \mathcal{X}_C e^{-(\alpha+2x)\mathcal{A}} \mathcal{B}. \tag{6.14}$$

Consequently, for $x \in \mathbb{R}^+$ we have

$$\boxed{k(x) = -2i\mathcal{C} \left[I_p + e^{-2x\mathcal{A}} \mathcal{X}_B e^{-2x\mathcal{A}^*} \mathcal{X}_C \right]^{-1} e^{-2x\mathcal{A}} \mathcal{B}.} \tag{6.15}$$

It is easy to see that $I_p + e^{-2x\mathcal{A}}\mathcal{X}_B e^{-2x\mathcal{A}^*}\mathcal{X}_C$ is invertible whenever $I_p + e^{-2x\mathcal{A}^*}\mathcal{X}_C e^{-2x\mathcal{A}}\mathcal{X}_B$ is, since they have the form $I_p + TS$ and $I_p + ST$ for certain matrices S and T. More precisely, let us consider the expression $I_p + e^{-2x\mathcal{A}}\mathcal{X}_B e^{-2x\mathcal{A}^*}\mathcal{X}_C$ and write it in the form $I_p + TS$ where $T = e^{-2x\mathcal{A}}\mathcal{X}_B e^{-2x\mathcal{A}^*}\mathcal{X}_C^{1/2}$ and $S = \mathcal{X}_C^{1/2}$ and $\mathcal{X}_B^{\frac{1}{2}}$ and $\mathcal{X}_C^{\frac{1}{2}}$ are the positive selfadjoint matrices having $\mathcal{X}_B$ and $\mathcal{X}_C$ as their squares. Then

$$I_p + ST = I_p + \mathcal{X}_C^{\frac{1}{2}} e^{-2x\mathcal{A}}\mathcal{X}_B^{\frac{1}{2}}\mathcal{X}_B^{\frac{1}{2}} e^{-2x\mathcal{A}^*}\mathcal{X}_C^{\frac{1}{2}} = I_p + \left(\mathcal{X}_B^{\frac{1}{2}} e^{-2x\mathcal{A}}\mathcal{X}_C^{\frac{1}{2}}\right)^* \left(\mathcal{X}_B^{\frac{1}{2}} e^{-2x\mathcal{A}^*}\mathcal{X}_C^{\frac{1}{2}}\right).$$

So, we find

$$I_p + e^{-2x\mathcal{A}}\mathcal{X}_B e^{-2x\mathcal{A}^*}\mathcal{X}_C = \left(\mathcal{X}_B^{\frac{1}{2}}\right)^{-1} (I_p + \theta(x)\theta(x)^*) \left(\mathcal{X}_B^{\frac{1}{2}}\right),$$

where

$$\theta(x) = \mathcal{X}_C^{\frac{1}{2}} e^{-2x\mathcal{A}}\mathcal{X}_B^{\frac{1}{2}}.$$

Proof. We first give the proof for eq. (6.12) and for eq. (6.15), because eq. (6.13) is proved likewise. Now, we consider the following Marchenko equation (obtained from eqs. (4.3b) and (4.4a))

$$B_{l2}(x,\alpha) = -\Omega_l(\alpha+2x) - \int_0^\infty d\beta\, B_{l2}(x,\beta) \int_0^\infty d\gamma\, \Omega_l(\gamma+\beta+2x)^*\Omega_l(\alpha+\gamma+2x).$$

Substituting (6.5) and (6.6) into the preceding equation we obtain

$$B_{l2}(x,\alpha) = -\int_0^\infty d\beta\, B_{l2}(x,\beta)\mathcal{B}^* e^{-(\beta+2x)\mathcal{A}^*}\mathcal{X}_C e^{-(\alpha+2x)\mathcal{A}}\mathcal{B} - \mathcal{C}e^{-(\alpha+2x)\mathcal{A}}\mathcal{B},$$

which can be written as

$$B_{l2}(x,\alpha) = -\left[\mathcal{C} + \left(\int_0^\infty d\beta\, B_{l2}(x,\beta)\mathcal{B}^* e^{-\beta\mathcal{A}^*}\right) e^{-2x\mathcal{A}^*}\mathcal{X}_C\right] e^{-(\alpha+2x)\mathcal{A}}\mathcal{B},$$

where

$$\begin{aligned}&\int_0^\infty d\alpha\, B_{l2}(x,\alpha)\mathcal{B}^* e^{-\alpha\mathcal{A}^*}\left[I + \int_0^\infty d\beta\, e^{-2x\mathcal{A}^*}\mathcal{X}_C e^{-2x\mathcal{A}} e^{-\beta\mathcal{A}}\mathcal{B}\mathcal{B}^* e^{-\beta\mathcal{A}^*}\right]\\ &= -\mathcal{C}\int_0^\infty d\alpha\, e^{-(\alpha+2x)\mathcal{A}}\mathcal{B}\mathcal{B}^* e^{-\alpha\mathcal{A}^*}.\end{aligned}$$

Taking into account eqs. (6.8) and (6.9), we have

$$\int_0^\infty d\alpha\, B_{l2}(x,\alpha)\mathcal{B}^* e^{-\alpha\mathcal{A}^*} = -\mathcal{C}e^{-2x\mathcal{A}}\mathcal{X}_B\left[I + e^{-2x\mathcal{A}^*}\mathcal{X}_C e^{-2x\mathcal{A}}\mathcal{X}_B\right]^{-1},$$

implying

$$B_{l2}(x,\alpha) = -\left[\mathcal{C} - \mathcal{C}e^{-2x\mathcal{A}}\mathcal{X}_B\left[I + e^{-2x\mathcal{A}^*}\mathcal{X}_C e^{-2x\mathcal{A}}\mathcal{X}_B\right]^{-1} e^{-2x\mathcal{A}^*}\mathcal{X}_C\right] e^{-(\alpha+2x)\mathcal{A}}\mathcal{B}.$$

Equation (6.12) then follows as a consequence. Equation (6.15) follows with the help of (3.54). Equations (6.11) and (6.14) follow from eqs. (6.12) and (6.13) with the help of eqs. (4.4a) and (4.3d) using (4.9). ∎

In the general situation, i.e., if bound states are present, we have

$$\Omega_l(\alpha) = \hat{R}(\alpha) + \sum_{j=1}^{N} \sum_{s=0}^{q_{l_j}-1} (-1)^s \Gamma_{lj} \frac{\alpha^s}{s!} e^{-\kappa_j \alpha}. \tag{6.16}$$

Again, from the theory of the transfer functions, since the second term on the right hand of eq. (6.16) tends to 0 as $\alpha \to \pm\infty$, it follows that $\Omega_l(\alpha)$ can be represented in the form

$$\Omega_l(\alpha) = \mathcal{C}e^{-\alpha\mathcal{A}}\mathcal{B} + \tilde{\mathcal{C}}e^{-\alpha\tilde{\mathcal{A}}}\tilde{\mathcal{B}} = \begin{pmatrix}\mathcal{C} & \tilde{\mathcal{C}}\end{pmatrix} e^{-\alpha\begin{pmatrix}\mathcal{A} & 0\\ 0 & \tilde{\mathcal{A}}\end{pmatrix}} \begin{pmatrix}\mathcal{B}\\ \tilde{\mathcal{B}}\end{pmatrix}. \tag{6.17}$$

Now, if we consider the following correspondences

$$\mathcal{A} \to \begin{pmatrix}\mathcal{A} & 0\\ 0 & \tilde{\mathcal{A}}\end{pmatrix}, \quad \mathcal{B} \to \begin{pmatrix}\mathcal{B}\\ \tilde{\mathcal{B}}\end{pmatrix}, \quad \mathcal{B} \to \begin{pmatrix}\mathcal{C} & \tilde{\mathcal{C}}\end{pmatrix},$$

we reduce the above derivations to the case already analyzed in the absence of bound states. In general, the third member of (6.17) does not lead to a minimal realization and hence the corresponding Lyapunov solutions are not nonsingular. However, the matrix $\mathcal{A} \oplus \tilde{\mathcal{A}}$ can always be replaced by a matrix of reduced order as to lead to a minimal realization. Nevertheless, eqs. (6.11)-(6.15) based on the matrices appearing in the third member of (6.17) are correct (albeit potentially cumbersome).

Let us return to (6.15) and write instead of (6.15)

$$\begin{aligned}
k(x) &= -2i\mathcal{C}\mathcal{X}_C^{-1}\left[\mathcal{X}_C^{-1} + e^{-2x\mathcal{A}}\mathcal{X}_B e^{-2x\mathcal{A}^*}\right]^{-1} e^{-2x\mathcal{A}}\mathcal{B}\\
&= -2i\mathcal{C}\mathcal{X}_C^{-1}e^{2x\mathcal{A}^*}\left[e^{2x\mathcal{A}}\mathcal{X}_C^{-1}e^{2x\mathcal{A}^*} + \mathcal{X}_B\right]^{-1}\mathcal{B}\\
&= -2i\mathcal{C}\mathcal{X}_C^{-1}e^{2x\mathcal{A}^*}\left[\mathcal{X}_B^{-1}e^{2x\mathcal{A}}\mathcal{X}_C^{-1}e^{2x\mathcal{A}^*} + I_p\right]^{-1}\mathcal{X}_B^{-1}\mathcal{B}\\
&= -2i\mathcal{C}\mathcal{X}_C^{-1}\left[e^{2x\mathcal{A}^*}\mathcal{X}_B^{-1}e^{2x\mathcal{A}}\mathcal{X}_C^{-1} + I_p\right]^{-1} e^{2x\mathcal{A}^*}\mathcal{X}_B^{-1}\mathcal{B}\\
&= -2i\mathcal{C}\mathcal{X}_C^{-\frac{1}{2}}\left[\mathcal{Z}^*\mathcal{Z} + I_p\right]^{-1}\mathcal{X}_C^{-\frac{1}{2}}e^{2x\mathcal{A}^*}\mathcal{X}_B^{-1}\mathcal{B},
\end{aligned}$$

where $\mathcal{Z} = \mathcal{X}_B^{-\frac{1}{2}}e^{2x\mathcal{A}}\mathcal{X}_C^{-\frac{1}{2}}$ and the realization (6.4) is minimal. Then $\mathcal{X}_\mathcal{B}$ and $\mathcal{X}_\mathcal{C}$ are nonsingular and

$$\|k(x)\| \le 2\|\mathcal{C}\|\|\mathcal{X}_C^{-1}\|\,\|e^{2x\mathcal{A}}\|\,\|\mathcal{X}_B^{-1}\mathcal{B}\|, \tag{6.18}$$

which is exponentially decreasing as $x \to -\infty$. Thus $k(x)$ is a C^∞-function which decays exponentially as $x \to \pm\infty$.

In the **symmetric case** we do not have to take into account the bound states, while we employ the symmetry relation $\overline{\Omega}_l(\alpha) = \Omega_l(\alpha)^*$ instead of $\overline{\Omega}_l(\alpha) = -\Omega_l(\alpha)^*$. The above derivation can then be repeated in full, except for occasional sign changes. As a result, we arrive at the following theorem.

Theorem 6.3 *For $x \in \mathbb{R}^+$ the solutions of the Marchenko integral equations* (4.10a) *and* (4.10b) (*where* $\Omega(\lambda) = \hat{R}(\lambda)$) *are given by*

$$B_{l1}(x,\alpha) = +\mathcal{C}\left[I_p - e^{-2x\mathcal{A}}\mathcal{X}_B\, e^{-2x\mathcal{A}^*}\mathcal{X}_C\right]^{-1} e^{-2x\mathcal{A}}\mathcal{X}_B e^{-(\alpha+2x)\mathcal{A}^*}\mathcal{C}^*; \tag{6.19}$$

$$B_{l2}(x,\alpha) = -\mathcal{C}\left[I_p - e^{-2x\mathcal{A}}\mathcal{X}_B\, e^{-2x\mathcal{A}^*}\mathcal{X}_C\right]^{-1} e^{-(\alpha+2x)\mathcal{A}}\mathcal{B}; \tag{6.20}$$

$$B_{l3}(x,\alpha) = -\mathcal{B}^*\left[I_p - e^{-2x\mathcal{A}^*}\mathcal{X}_C\, e^{-2x\mathcal{A}}\mathcal{X}_B\right]^{-1} e^{-(\alpha+2x)\mathcal{A}^*}\mathcal{C}^*; \tag{6.21}$$

$$B_{l4}(x,\alpha) = +\mathcal{B}^*\left[I_p - e^{-2x\mathcal{A}^*}\mathcal{X}_C\, e^{-2x\mathcal{A}}\mathcal{X}_B\right]^{-1} e^{-2x\mathcal{A}^*}\mathcal{X}_C e^{-(\alpha+2x)\mathcal{A}}\mathcal{B}. \tag{6.22}$$

Consequently, for $x \in \mathbb{R}^+$ we have

$$k(x) = -2i\mathcal{C}\left[I_p - e^{-2x\mathcal{A}}\mathcal{X}_B e^{-2x\mathcal{A}^*}\mathcal{X}_C\right]^{-1} e^{-2x\mathcal{A}}\mathcal{B}. \tag{6.23}$$

The only remaining issue is the existence of the inverses appearing in (6.20)-(6.23). However, when repeating the above calculations these inverses must exist in order to make the Marchenko equation uniquely solvable. If we were to replace $\mathcal{C}$ by $\varepsilon\mathcal{C}$ for an arbitrary $\varepsilon \in \mathbb{C}$ with $|\varepsilon| \leq 1$, we would replace $\Omega_l(\alpha)$ by $\varepsilon\Omega_l(\alpha)$ without compromising the unique solvability of the Marchenko equation. This is due to the fact that the spectral radius of the integral operator is strictly less than 1. In that case $\mathcal{X}_C$ is replaced by $|\varepsilon|^2\mathcal{X}_C$ and hence the inverted matrix in (6.20) by $I_p - |\varepsilon|^2 e^{-2x\mathcal{A}}\mathcal{X}_B e^{-2x\mathcal{A}^*}\mathcal{X}_C$, implying the invertibility of

$$I_p - |\varepsilon|^2\theta(x)\theta(x)^* = I_p - |\varepsilon|^2\mathcal{X}_C^{\frac{1}{2}} e^{-2x\mathcal{A}}\mathcal{X}_B e^{-2x\mathcal{A}^*}\mathcal{X}_C^{\frac{1}{2}}.$$

As a result, for $x \geq 0$ we have for the spectral norm

$$\|\theta(x)\| = \left\|\mathcal{X}_C^{\frac{1}{2}} e^{-2x\mathcal{A}}\mathcal{X}_B^{\frac{1}{2}}\right\| < 1. \tag{6.24}$$

6.2 Solving matrix NLS equations by state space methods

We know (see eq. (5.65)) that the Marchenko integral kernel satisfies the following equation

$$\partial_t\Omega + 4i\partial_\alpha^2\Omega = 0.$$

It is easy to see that a solution of the preceding equation is given by

$$\Omega(\alpha;t) = \mathcal{C}e^{-\alpha\mathcal{A}}e^{-4it\mathcal{A}^2}\mathcal{B}. \tag{6.25}$$

In fact, if we calculate the derivatives with respect to t and α, we obtain

$$\partial_t\Omega = -4i\mathcal{C}e^{-\alpha\mathcal{A}}\mathcal{A}^2e^{-4it\mathcal{A}^2}\mathcal{B}, \qquad \partial_\alpha^2\Omega = \mathcal{C}\mathcal{A}^2e^{-\alpha\mathcal{A}}e^{-4it\mathcal{A}^2}\mathcal{B},$$

which imply eq. (5.65). Note that we can write (6.25) in the following way

$$\Omega(\alpha;t) = \mathcal{C}e^{-2it\mathcal{A}^2}e^{-\alpha\mathcal{A}}e^{-2it\mathcal{A}^2}\mathcal{B}. \tag{6.26}$$

We therefore expect to derive solutions of the matrix nonlinear Schrödinger equation by solving the inverse scattering problem as in Sec. 6.1, but replacing $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ by $\mathcal{A}$, $e^{-2it\mathcal{A}^2}\mathcal{B}$ and $\mathcal{C}e^{-2it\mathcal{A}^2}$, respectively. The only potential limitation is the possible nonexistence of the inverse matrix constructed during the inversion procedure.

Exactly as in the preceding section we first discuss the **antisymmetric case**. Here, eqs. (6.8) and (6.9) become

$$\mathcal{X}_{C(t)} = \int_0^\infty d\beta\, e^{-\beta\mathcal{A}^*} e^{2it(\mathcal{A}^*)^2}\mathcal{C}^*\mathcal{C}e^{-2it\mathcal{A}^2}e^{-\beta\mathcal{A}} = e^{2it(\mathcal{A}^*)^2}\mathcal{X}_C e^{-2it\mathcal{A}^2}, \tag{6.27}$$

$$\mathcal{X}_{B(t)} = \int_0^\infty d\beta\, e^{-\beta\mathcal{A}} e^{-2it\mathcal{A}^2}\mathcal{B}\mathcal{B}^* e^{2it(\mathcal{A}^*)^2}e^{-\beta\mathcal{A}^*} = e^{-2it\mathcal{A}^2}\mathcal{X}_B e^{2it(\mathcal{A}^*)^2}, \tag{6.28}$$

so we can easily compute for $\alpha, \beta > 0$ and $x \in \mathbb{R}^+$

$$\int_0^\infty d\beta\, \Omega_l(\beta+\gamma+2x;t)^*\Omega_l(\alpha+\beta+2x;t) = \mathcal{B}^* e^{2it(\mathcal{A}^*)^2} e^{-(\gamma+2x)\mathcal{A}^*}\mathcal{X}_{C(t)}e^{-(\alpha+2x)\mathcal{A}}e^{-2it\mathcal{A}^2}\mathcal{B}, \tag{6.29}$$

$$\int_0^\infty d\beta\, \Omega_l(\beta+\gamma+2x;t)\Omega_l(\alpha+\beta+2x;t)^* = \mathcal{C}e^{-2it\mathcal{A}^2}e^{-(\gamma+2x)\mathcal{A}}\mathcal{X}_{B(t)}e^{-(\alpha+2x)\mathcal{A}^*}e^{2it(\mathcal{A}^*)^2}\mathcal{C}^*. \tag{6.30}$$

We have the following

Theorem 6.4 *For $x \in \mathbb{R}^+$ and $t \geq 0$ (where $\Omega(\lambda;t) = \hat{R}(\lambda;t)$ in the absence of bound states) the matrix functions*

$$\begin{aligned} B_{l1}(x,\alpha;t) = -\mathcal{C}e^{-2it\mathcal{A}^2}&\left[I_p + e^{-2x\mathcal{A}}e^{-2it\mathcal{A}^2}\mathcal{X}_B e^{4it(\mathcal{A}^*)^2}e^{-2x\mathcal{A}^*}\mathcal{X}_C e^{-2it\mathcal{A}^2}\right]^{-1}\times \\ &\times e^{-2x\mathcal{A}}e^{-2it\mathcal{A}^2}\mathcal{X}_B e^{4it(\mathcal{A}^*)^2}e^{-(\alpha+2x)\mathcal{A}^*}\mathcal{C}^*; \end{aligned} \tag{6.31}$$

$$\begin{aligned} B_{l2}(x,\alpha;t) = -\mathcal{C}e^{-2it\mathcal{A}^2}&\left[I_p + e^{-2x\mathcal{A}}e^{-2it\mathcal{A}^2}\mathcal{X}_B e^{4it(\mathcal{A}^*)^2}e^{-2x\mathcal{A}^*}\mathcal{X}_C e^{-2it\mathcal{A}^2}\right]^{-1}\times \\ &\times e^{-(\alpha+2x)\mathcal{A}}e^{-2it\mathcal{A}^2}\mathcal{B}; \end{aligned} \tag{6.32}$$

$$\begin{aligned} B_{l3}(x,\alpha;t) = +\mathcal{B}^* e^{2it(\mathcal{A}^*)^2}&\left[I_p + e^{-2x\mathcal{A}^*}e^{2it(\mathcal{A}^*)^2}\mathcal{X}_C e^{-4it\mathcal{A}^2}e^{-2x\mathcal{A}}\mathcal{X}_B e^{2it(\mathcal{A}^*)^2}\right]^{-1}\times \\ &\times e^{-(\alpha+2x)\mathcal{A}^*}e^{2it(\mathcal{A}^*)^2}\mathcal{C}^*; \end{aligned} \tag{6.33}$$

$$\begin{aligned} B_{l4}(x,\alpha;t) = -\mathcal{B}^* e^{2it(\mathcal{A}^*)^2}&\left[I_p + e^{-2x\mathcal{A}^*}e^{2it(\mathcal{A}^*)^2}\mathcal{X}_C e^{-4it\mathcal{A}^2}e^{-2x\mathcal{A}}\mathcal{X}_B e^{2it(\mathcal{A}^*)^2}\right]^{-1}\times \\ &\times e^{-2x\mathcal{A}^*}e^{2it(\mathcal{A}^*)^2}\mathcal{X}_C e^{-4it\mathcal{A}^2}e^{-(\alpha+2x)\mathcal{A}}\mathcal{B}; \end{aligned} \tag{6.34}$$

are the solutions of the Marchenko integral equations (4.11a) *and* (4.11b). *Consequently, for $x \in \mathbb{R}^+$ and $t \geq 0$ we have the following solution of the matrix nonlinear Schrödinger equation:*

$$\boxed{k(x;t) = -2i\mathcal{C}e^{-2it\mathcal{A}^2}\left[I_p + e^{-2x\mathcal{A}}e^{-2it\mathcal{A}^2}\mathcal{X}_B e^{4it(\mathcal{A}^*)^2}e^{-2x\mathcal{A}^*}\mathcal{X}_C e^{-2it\mathcal{A}^2}\right]^{-1} e^{-2x\mathcal{A}}e^{-2it\mathcal{A}^2}\mathcal{B}.} \tag{6.35}$$

We omit the proof, because it is very similar to the proof of the Theorem 6.2. The existence of the inverses appearing in (6.32)-(6.35) can be proved as in the paragraph following the statement of Theorem 6.2.

In the **symmetric case** we have employed the unique solvability of the Marchenko integral equations to prove the existence of the inverse matrices appearing in (6.20)-(6.23). When introducing time dependence, we can no longer be sure that the (modified) Marchenko integral equations with $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ replaced by the respective matrices $\mathcal{A}$, $e^{-2it\mathcal{A}^2}\mathcal{B}$ and $\mathcal{C}e^{-2it\mathcal{A}^2}$ remain uniquely solvable. Therefore Theorem 6.3 should be generalized as if the potential $k(x,t)$ only exists for small t.

Theorem 6.5 *There exists $\tau_1 > 0$ (possibly $\tau_1 = +\infty$) such that for $x \in \mathbb{R}^+$ and $0 \le t < \tau_1$ (in the symmetric case $\Omega(\lambda;t) = \hat{R}(\lambda;t)$ because there are not bound states) the matrix functions*

$$B_{l1}(x,\alpha;t) = +\mathcal{C}e^{-2it\mathcal{A}^2}\left[I_p - e^{-2x\mathcal{A}}e^{-2it\mathcal{A}^2}\mathcal{X}_B e^{4it(\mathcal{A}^*)^2}e^{-2x\mathcal{A}^*}\mathcal{X}_C e^{-2it\mathcal{A}^2}\right]^{-1}\times$$
$$\times\, e^{-2x\mathcal{A}}e^{-2it\mathcal{A}^2}\mathcal{X}_B e^{4it(\mathcal{A}^*)^2}e^{-(\alpha+2x)\mathcal{A}^*}\mathcal{C}^*; \tag{6.36}$$

$$B_{l2}(x,\alpha;t) = -\mathcal{C}e^{-2it\mathcal{A}^2}\left[I_p - e^{-2x\mathcal{A}}e^{-2it\mathcal{A}^2}\mathcal{X}_B e^{4it(\mathcal{A}^*)^2}e^{-2x\mathcal{A}^*}\mathcal{X}_C e^{-2it\mathcal{A}^2}\right]^{-1}\times$$
$$\times\, e^{-(\alpha+2x)\mathcal{A}}e^{-2it\mathcal{A}^2}\mathcal{B}; \tag{6.37}$$

$$B_{l3}(x,\alpha;t) = -\mathcal{B}^* e^{2it(\mathcal{A}^*)^2}\left[I_p - e^{-2x\mathcal{A}^*}e^{2it(\mathcal{A}^*)^2}\mathcal{X}_C e^{-4it\mathcal{A}^2}e^{-2x\mathcal{A}}\mathcal{X}_B e^{2it(\mathcal{A}^*)^2}\right]^{-1}\times$$
$$\times\, e^{-(\alpha+2x)\mathcal{A}^*}e^{2it(\mathcal{A}^*)^2}\mathcal{C}^*; \tag{6.38}$$

$$B_{l4}(x,\alpha;t) = +\mathcal{B}^* e^{2it(\mathcal{A}^*)^2}\left[I_p - e^{-2x\mathcal{A}^*}e^{2it(\mathcal{A}^*)^2}\mathcal{X}_C e^{-4it\mathcal{A}^2}e^{-2x\mathcal{A}}\mathcal{X}_B e^{2it(\mathcal{A}^*)^2}\right]^{-1}\times$$
$$\times\, e^{-2x\mathcal{A}^*}e^{2it(\mathcal{A}^*)^2}\mathcal{X}_C e^{-4it\mathcal{A}^2}e^{-(\alpha+2x)\mathcal{A}}\mathcal{B}; \tag{6.39}$$

satisfy the Marchenko integral equations (4.10a) *and* (4.10b). *Consequently, for $x \in \mathbb{R}^+$ and $0 \le t < \tau_1$ we have the following solution of the matrix nonlinear Schrödinger equation:*

$$k(x;t) = -2i\mathcal{C}e^{-2it\mathcal{A}^2}\left[I_p - e^{-2x\mathcal{A}}e^{-2it\mathcal{A}^2}\mathcal{X}_B e^{4it(\mathcal{A}^*)^2}e^{-2x\mathcal{A}^*}\mathcal{X}_C e^{-2it\mathcal{A}^2}\right]^{-1}e^{-2x\mathcal{A}}e^{-2it\mathcal{A}^2}\mathcal{B}. \tag{6.40}$$

Proof. Equations (6.37)-(6.40) are immediate from (6.20)-(6.23), on changing $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ in $\mathcal{A}$, $e^{-2it\mathcal{A}^2}\mathcal{B}$ and $\mathcal{C}e^{-2it\mathcal{A}^2}$, provided the inverses occurring in them exist. However, these inverses exist, provided

$$\|\theta(x,t)\| < 1, \tag{6.41}$$

where

$$\theta(x,t) = \mathcal{X}_{C(t)}^{1/2}e^{-2x\mathcal{A}}\mathcal{X}_{B(t)}^{1/2}.$$

To prove the existence of the inverse of

$$\Gamma(x,t) = I_p - e^{-2x\mathcal{A}}\mathcal{X}_{B(t)}e^{-2x\mathcal{A}^*}\mathcal{X}_{C(t)},$$

we observe that the inverse matrix $\Gamma(x,0)^{-1}$ exists for every $x \geq 0$ and tends to I_p as $x \to +\infty$. Thus $\sup_{x\geq 0} \|\Gamma(x,0)^{-1}\|$ is finite. We now write

$$\begin{aligned}\Gamma(x,t) - \Gamma(x,0) &= e^{-2x\mathcal{A}}[\mathcal{X}_B - \mathcal{X}_{B(t)}]e^{-2x\mathcal{A}^*}\mathcal{X}_C \\ &\quad + e^{-2x\mathcal{A}}[\mathcal{X}_B - (\mathcal{X}_B - \mathcal{X}_{B(t)})]e^{-2x\mathcal{A}^*}[\mathcal{X}_C - \mathcal{X}_{C(t)}].\end{aligned}$$

Put $f(t) = \|I_p - e^{-2it\mathcal{A}^2}\| = \|I_p - e^{2it(\mathcal{A}^*)^2}\|$. Then $f(t)$ depends continuously on $t \in \mathbb{R}^+$ and vanishes as $t \to 0^+$. Hence,

$$\begin{aligned}\|\mathcal{X}_B - \mathcal{X}_{B(t)}\| &\leq f(t)[1 + f(t)]\|\mathcal{X}_B\|, \\ \|\mathcal{X}_C - \mathcal{X}_{C(t)}\| &\leq f(t)[1 + f(t)]\|\mathcal{X}_C\|.\end{aligned}$$

Moreover, $\|e^{-2x\mathcal{A}}\| = \|e^{-2x\mathcal{A}^*}\| \leq Me^{-\alpha x}$ for certain $M, \alpha > 0$. Consequently,

$$\|\Gamma(x,t) - \Gamma(x,0)\| \leq M^2 e^{-2\alpha x}\|\mathcal{X}_B\|\|\mathcal{X}_C\| f(t)[1 + f(t)][1 + f(t)(1 + f(t))].$$

Thus there exists $\tau_1 > 0$ such that the right-hand side is strictly less than $\inf_{x\geq 0} \|\Gamma(x,0)^{-1}\|^{-1}$ for $0 \leq t < \tau_1$. For these t the matrix $\Gamma(x,t)$ is invertible for every $x \geq 0$. Since $\det \Gamma(x,t) = \det(I_p - \theta(x,t)\theta(x,t)^*)$, we finally obtain (6.41) for $0 \leq t < \tau_1$ and $x \geq 0$, as claimed. ■

Similar state space methods can be used to derive explicit solutions of the Korteweg-de Vries equation where the initial condition is a real Schrödinger equation potential corresponding to a rational reflection coefficient. In this case the explicit solution has the form

$$u(x,t) = -2\frac{\partial^2}{\partial x^2} \log \det \Gamma(x,t),$$

where $\det \Gamma(x,t) > 0$ for $0 \leq t < \tau_1$ and $x \geq 0$ (cf. [11]). Here τ_1 can be finite as well as infinite. In fact, the matrix trace can be used to substantially simplify the expression for $u(x,t)$ compared to the analogous expression for the matrix KdV equation. No such simplification occurs for the matrix nonlinear Schrödinger equation, not even in the case $n = m = 1$.

The exact solutions (6.35) and (6.40) of the matrix nonlinear Schrödinger equation are based on the application of the inverse scattering transform **as if** changing the triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ into the triple $(\mathcal{A}, e^{-2it\mathcal{A}^2}\mathcal{B}, \mathcal{C}e^{-2it\mathcal{A}^2}\mathcal{B})$ to produce a solution of the differential equation (5.65) is a correct way to implement the time evolution of the scattering data. When dealing with the multi-soliton solutions in Sec. 6.3, this is indeed the correct way to go. Unfortunately, in general we have derived state space solutions on the positive and negative half-lines by using separate realizations of the form (6.4).[1] This leads to the correct result for $t = 0$, i.e., we have solved the inverse scattering problem in a correct way. For $0 \neq t \in \mathbb{R}$, however, it is by no means clear (and, in fact, it is in general not true) that the solutions obtained belong to the same Sobolev space of matrix functions of x as their initial data. Similar considerations have induced the authors of [11] to give explicit solution of the KdV equation on the positive half-line only.

[1]To do so on the negative half-line, we should in principle repeat the calculations of Sections 6.1 and 6.2 or apply the symmetry relations of Sec. B.2 of Appendix B to convert the results on the positive half-line to those on the negative half-line.

6.3 Multi-soliton solutions

In this section we find all multi-soliton solutions of the matrix nonlinear Schrödinger equation on the line in the **antisymmetric case**. This means solving the inverse scattering problem for the matrix Zakharov-Shabat system in the reflectionless case where $R(\lambda) \equiv 0_{n\times m}$ and $L(\lambda) \equiv 0_{m\times n}$.

Let us assume that

$$\Omega_l(\alpha) = \mathcal{C}e^{-\alpha\mathcal{A}}\mathcal{B}, \qquad \alpha > 0,$$

where $\mathcal{A}$ is a $p \times p$ matrix with all of its eigenvalues in the open right half-plane, $\mathcal{B}$ is a $p \times m$ matrix and $\mathcal{C}$ is an $n \times p$ matrix such that

$$\bigcap_{j=0}^{\infty} \operatorname{Ker}(\mathcal{C}\mathcal{A}^j) = \bigcap_{j=0}^{\infty} \operatorname{Ker}(\mathcal{B}^*(\mathcal{A}^*)^j) = \{0\}.$$

Then the solutions of the Marchenko equations (4.3a)-(4.3d) and (4.4a)-(4.4d) are given by (6.31)-(6.34). Using (3.35) we obtain

$$M_{l1}(x,\lambda) = I_n + \int_0^\infty d\alpha\, B_{l1}(x,\alpha)e^{i\lambda\alpha}, \tag{6.42}$$

$$M_{l2}(x,\lambda) = \int_0^\infty d\alpha\, B_{l2}(x,\alpha)e^{-i\lambda\alpha}, \tag{6.43}$$

$$M_{l3}(x,\lambda) = \int_0^\infty d\alpha\, B_{l3}(x,\alpha)e^{i\lambda\alpha}, \tag{6.44}$$

$$M_{l4}(x,\lambda) = I_m + \int_0^\infty d\alpha\, B_{l4}(x,\alpha)e^{-i\lambda\alpha}. \tag{6.45}$$

where $x \in \mathbb{R}^+$. Further,

$$k(x) = -2iB_{l2}(x,0^+) = 2iB_{r2}(x,0^+).$$

We have seen that $k(x)$ as given by (6.15) makes sense for $x \in \mathbb{R}$ and is in fact an $n \times m$ matrix function with entries in $\mathbb{C}^\infty(\mathbb{R})$ which are exponentially decaying as $x \to \pm\infty$. Let us now solve the *direct* scattering problem for this *anti*symmetric potential. According to (3.26)-(3.29) we have

$$a_{l1}(\lambda) = I_n - i\int_{-\infty}^\infty dy\, k(y)M_{l3}(y,\lambda),$$

$$a_{l2}(\lambda) = -i\int_{-\infty}^\infty dy\, e^{-2i\lambda y}k(y)M_{l4}(y,\lambda),$$

$$a_{l3}(\lambda) = -i\int_{-\infty}^\infty dy\, e^{2i\lambda y}k(y)^*M_{l1}(y,\lambda),$$

$$a_{l4}(\lambda) = I_m - i\int_{-\infty}^\infty dy\, k(y)^*M_{l2}(y,\lambda).$$

It is easy to prove the following

Theorem 6.6 *In the antisymmetric case, the following equations hold*

$$a_{l1}(\lambda) = I_n - i\mathcal{C}\,\mathcal{X}_C^{-1}\,(\lambda + i\mathcal{A}^*)^{-1}\,\mathcal{C}^*, \tag{6.46}$$

$$a_{l2}(\lambda) = 0_{n\times m}, \tag{6.47}$$

$$a_{l3}(\lambda) = 0_{m\times n}, \tag{6.48}$$

$$a_{l4}(\lambda) = I_m + i\mathcal{B}^*\,\mathcal{X}_B^{-1}\,(\lambda - i\mathcal{A})^{-1}\,\mathcal{B}. \tag{6.49}$$

Consequently,

$$T_l(\lambda) = a_{l1}(\lambda)^{-1} = I_n + i\mathcal{C}\,\mathcal{X}_C^{-1}\,\left(\lambda + i\left[\mathcal{A}^* - \mathcal{C}^*\mathcal{C}\,\mathcal{X}_C^{-1}\right]\right)^{-1}\,\mathcal{C}^*, \tag{6.50}$$

$$T_r(\lambda) = \left[a_{l4}(\overline{\lambda})^*\right]^{-1} = I_m + i\mathcal{B}^*\,\left(\lambda + i\left[\mathcal{A}^* - \mathcal{X}_B^{-1}\,\mathcal{B}\mathcal{B}^*\right]\right)^{-1}\,\mathcal{X}_B^{-1}\,\mathcal{B}. \tag{6.51}$$

Moreover, the bound states are given by the poles of $T_l(\lambda)$ (or $T_r(\lambda)$) in $\mathbb{C}^+$, which we observe to coincide and to have the same order.

Proof. In the antisymmetric case, using eqs. (4.11a)-(4.11d) and Theorem 6.2 we immediately obtain

$$M_{l1}(x,\lambda) = I_n - i\mathcal{C}\left[I_p + e^{-2x\mathcal{A}}\mathcal{X}_B e^{-2x\mathcal{A}^*}\mathcal{X}_C\right]^{-1} e^{-2x\mathcal{A}}\mathcal{X}_B e^{-2x\mathcal{A}^*}\,(\lambda + i\mathcal{A}^*)^{-1}\,\mathcal{C}^*, \tag{6.52}$$

$$M_{l2}(x,\lambda) = i\mathcal{C}\left[I_p + e^{-2x\mathcal{A}}\mathcal{X}_B e^{-2x\mathcal{A}^*}\mathcal{X}_C\right]^{-1} e^{-2x\mathcal{A}}\,(\lambda - i\mathcal{A})^{-1}\,\mathcal{B}, \tag{6.53}$$

$$M_{l3}(x,\lambda) = i\mathcal{B}^*\left[I_p + e^{-2x\mathcal{A}^*}\mathcal{X}_C e^{-2x\mathcal{A}}\mathcal{X}_B\right]^{-1} e^{-2x\mathcal{A}^*}\,(\lambda + i\mathcal{A}^*)^{-1}\,\mathcal{C}^*, \tag{6.54}$$

$$M_{l4}(x,\lambda) = I_m + i\mathcal{B}^*\left[I_p + e^{-2x\mathcal{A}^*}\mathcal{X}_C e^{-2x\mathcal{A}}\mathcal{X}_B\right]^{-1} e^{-2x\mathcal{A}^*}\mathcal{X}_C e^{-2x\mathcal{A}}\,(\lambda - i\mathcal{A})^{-1}\,\mathcal{B}, \tag{6.55}$$

where we have taken into account the following relations

$$\int_0^{+\infty} d\alpha\, e^{i\lambda\alpha} e^{-\alpha\mathcal{A}^*} = i\,(\lambda + i\mathcal{A}^*)^{-1},$$

$$\int_0^{+\infty} d\alpha\, e^{-i\lambda\alpha} e^{-\alpha\mathcal{A}} = -i\,(\lambda - i\mathcal{A})^{-1}.$$

In the sequel of the proof we also use the well-known formula

$$\frac{d}{dx}F(x)^{-1} = -F(x)^{-1}\left(\frac{d}{dx}F(x)\right)F(x)^{-1}. \tag{6.56}$$

Applying formula (6.56) we find

$$\begin{aligned}
&\frac{d}{dx}\left[I_p + e^{-2x\mathcal{A}}\,\mathcal{X}_B\,e^{-2x\mathcal{A}^*}\,\mathcal{X}_C\right]^{-1} = 2\left[I_p + e^{-2x\mathcal{A}}\,\mathcal{X}_B\,e^{-2x\mathcal{A}^*}\,\mathcal{X}_C\right]^{-1}\times\\
&\times e^{-2x\mathcal{A}}\left\{\mathcal{A}\mathcal{X}_B + \mathcal{X}_B\mathcal{A}^*\right\}\,e^{-2x\mathcal{A}^*}\mathcal{X}_C\,\left[I_p + e^{-2x\mathcal{A}}\,\mathcal{X}_B\,e^{-2x\mathcal{A}^*}\,\mathcal{X}_C\right]^{-1}\\
&= 2\left[I_p + e^{-2x\mathcal{A}}\,\mathcal{X}_B\,e^{-2x\mathcal{A}^*}\,\mathcal{X}_C\right]^{-1}\times\\
&\times e^{-2x\mathcal{A}}\mathcal{B}\mathcal{B}^*\,e^{-2x\mathcal{A}^*}\,\left[I_p + \mathcal{X}_C\,e^{-2x\mathcal{A}}\,\mathcal{X}_B\,e^{-2x\mathcal{A}^*}\right]^{-1}\,\mathcal{X}_C,
\end{aligned} \tag{6.57}$$

where, in the second identity, we have used eq. (6.10). Now, in order to prove eq. (6.46), we substitute in eq. (3.26) the expressions of the potential $k(x)$ and $M_{l3}(x,\lambda)$ given by (6.15) and (6.54), respectively, getting

$$\begin{aligned} a_{l1}(\lambda) = I_n - 2i \int_{-\infty}^{+\infty} dx\, \mathcal{C} \left[I_p + e^{-2x\mathcal{A}}\, \mathcal{X}_B\, e^{-2x\mathcal{A}^*}\, \mathcal{X}_C \right]^{-1} \times \\ \times\, e^{-2x\mathcal{A}} \mathcal{B}\mathcal{B}^* \left[I_p + e^{-2x\mathcal{A}^*}\, \mathcal{X}_C\, e^{-2x\mathcal{A}} \mathcal{X}_B \right]^{-1} e^{-2x\mathcal{A}^*} \left(\lambda + i\mathcal{A}^*\right)^{-1} \mathcal{C}^*. \end{aligned}$$

Then, because the following identity holds

$$\left[I_p + e^{-2x\mathcal{A}^*}\, \mathcal{X}_C\, e^{-2x\mathcal{A}} \mathcal{X}_B \right]^{-1} = e^{-2x\mathcal{A}^*} \left[I_p + \mathcal{X}_C\, e^{-2x\mathcal{A}} \mathcal{X}_B\, e^{-2x\mathcal{A}^*} \right]^{-1},$$

we can write (taking into account eq. (6.57))

$$a_{l1}(\lambda) = I_n - i \int_{-\infty}^{+\infty} dx\, \frac{d}{dx} \mathcal{C} \left[I_p + e^{-2x\mathcal{A}}\, \mathcal{X}_B\, e^{-2x\mathcal{A}^*}\, \mathcal{X}_C \right]^{-1} \mathcal{X}_C^{-1} \left(\lambda + i\mathcal{A}^*\right)^{-1} \mathcal{C}^*. \tag{6.58}$$

If we now consider that

$$\left[I_p + e^{-2x\mathcal{A}}\, \mathcal{X}_B\, e^{-2x\mathcal{A}^*}\, \mathcal{X}_C \right]^{-1} \longrightarrow \begin{cases} I_p, & \text{as } x \to +\infty \\ 0_{p\times p}, & \text{as } x \to -\infty \end{cases}$$

where

$$\begin{aligned} \left[I_p + e^{-2x\mathcal{A}}\, \mathcal{X}_B\, e^{-2x\mathcal{A}^*}\, \mathcal{X}_C \right]^{-1} = \mathcal{X}_C^{-1} \left[\mathcal{X}_C^{-1} + e^{-2x\mathcal{A}}\, \mathcal{X}_B\, e^{-2x\mathcal{A}^*} \right]^{-1} \\ = \mathcal{X}_C^{-1}\, e^{-2x\mathcal{A}^*} \left[e^{2x\mathcal{A}}\, \mathcal{X}_C\, e^{2x\mathcal{A}^*} + X_B \right] e^{2x\mathcal{A}}, \end{aligned}$$

we obtain

$$a_{l1}(\lambda) = I_n - i\mathcal{C}\, \mathcal{X}_C^{-1} \left(\lambda + i\mathcal{A}^*\right)^{-1} \mathcal{C}^*,$$

which completes the proof of eq. (6.46). Moreover, if we use the well-known formula

$$(I + TR^{-1}S)^{-1} = I - T(R + ST)^{-1}S \tag{6.59}$$

for $T = i\mathcal{C}$, $R = (\lambda + i\mathcal{A}^*)$, and $S = \mathcal{C}^*$, we find that eq. (6.50) is satisfied.

Now, using the same scheme we prove eq. (6.49). In fact, substituting in eq. (3.29) the expressions of $-k(x)^*$ (obtainable by (6.15)) and $M_{l2}(x,\lambda)$ (given by (6.43)) we get

$$\begin{aligned} a_{l4}(\lambda) = I_m + 2i \int_{-\infty}^{+\infty} dx\, \mathcal{B}^* e^{-2x\mathcal{A}^*} \left[I_p + \mathcal{X}_C\, e^{-2x\mathcal{A}}\, \mathcal{X}_B\, e^{-2x\mathcal{A}^*} \right]^{-1} \times \\ \times\, \mathcal{C}^*\mathcal{C} \left[I_p + e^{-2x\mathcal{A}}\, \mathcal{X}_B\, e^{-2x\mathcal{A}^*} \mathcal{X}_C \right]^{-1} e^{-2x\mathcal{A}} \left(\lambda - i\mathcal{A}\right)^{-1} \mathcal{B} \\ = I_m + 2i \int_{-\infty}^{+\infty} dx\, \mathcal{B}^* \left[I_p + e^{-2x\mathcal{A}^*}\, \mathcal{X}_C\, e^{-2x\mathcal{A}}\, \mathcal{X}_B \right]^{-1} e^{-2x\mathcal{A}^*} \times \\ \times\, \mathcal{C}^*\mathcal{C}\, e^{-2x\mathcal{A}} \left[I_p + \mathcal{X}_B\, e^{-2x\mathcal{A}^*} \mathcal{X}_C\, e^{-2x\mathcal{A}} \right]^{-1} \left(\lambda - i\mathcal{A}\right)^{-1} \mathcal{B}. \end{aligned}$$

Exactly as above, we calculate

$$\frac{d}{dx}\left[I_p+e^{-2x\mathcal{A}^*}\mathcal{X}_C\,e^{-2x\mathcal{A}}\,\mathcal{X}_B\right]^{-1}=2\left[I_p+e^{-2x\mathcal{A}^*}\mathcal{X}_C\,e^{-2x\mathcal{A}}\,\mathcal{X}_B\right]^{-1}\times$$
$$\times\, e^{-2x\mathcal{A}^*}\{\mathcal{A}^*\mathcal{X}_C+\mathcal{X}_C\mathcal{A}\}\,e^{-2x\mathcal{A}}\mathcal{X}_B\left[I_p+e^{-2x\mathcal{A}^*}\mathcal{X}_C\,e^{-2x\mathcal{A}}\,\mathcal{X}_B\right]^{-1}$$
$$=2\left[I_p+e^{-2x\mathcal{A}^*}\mathcal{X}_C\,e^{-2x\mathcal{A}}\,\mathcal{X}_B\right]^{-1}\times$$
$$\times\, e^{-2x\mathcal{A}^*}\mathcal{C}^*\mathcal{C}\,e^{-2x\mathcal{A}}\left[I_p+\mathcal{X}_B\,e^{-2x\mathcal{A}^*}\mathcal{X}_C\,e^{-2x\mathcal{A}}\right]^{-1}\mathcal{X}_B, \tag{6.60}$$

where eq. (6.10) has been used in the second identity. Consequently, considering eq. (6.60), we have

$$a_{l4}(\lambda)=I_m+i\int_{-\infty}^{+\infty}dx\,\frac{d}{dx}\mathcal{B}^*\left[I_p+e^{-2x\mathcal{A}^*}\mathcal{X}_C\,e^{-2x\mathcal{A}}\,\mathcal{X}_B\right]^{-1}\mathcal{X}_B^{-1}\,(\lambda-i\mathcal{A})^{-1}\,\mathcal{B}. \tag{6.61}$$

Therefore,

$$a_{l4}(\lambda)=I_m+i\mathcal{B}^*\,\mathcal{X}_B^{-1}\,(\lambda-i\mathcal{A})^{-1}\,\mathcal{B}, \tag{6.62}$$

which implies

$$a_{r4}(\lambda)=a_{l4}(\overline{\lambda})^*=I_m-i\mathcal{B}^*\,(\lambda+i\mathcal{A}^*)^{-1}\,\mathcal{X}_B^{-1}\mathcal{B}, \tag{6.63}$$

and eq. (6.49) is proved. Moreover, if we use (6.59) with $T=i\mathcal{B}^*$, $R=(\lambda+i\mathcal{A}^*)$ and $S=\mathcal{X}_B^{-1}\mathcal{B}$ we find that eq. (6.51) is satisfied.

Next, we prove eq. (6.47). Substituting in eq. (3.27) the expressions of the potential $k(x)$ and $M_{l4}(x,\lambda)$ given by (6.15) and (6.55), respectively, we have

$$a_{l2}(\lambda)=-2\int_{-\infty}^{+\infty}dx\,e^{-2i\lambda x}\mathcal{C}\left[I_p+\,e^{-2x\mathcal{A}}\,\mathcal{X}_B\,e^{-2x\mathcal{A}^*}\,\mathcal{X}_C\right]^{-1}\times$$
$$\times\, e^{-2x\mathcal{A}}\,\mathcal{B}\left\{I_m+i\,\mathcal{B}^*\left[I_p\,+e^{-2x\mathcal{A}^*}\mathcal{X}_C\,e^{-2x\mathcal{A}}\mathcal{X}_B\right]^{-1}e^{-2x\mathcal{A}^*}\mathcal{X}_C\,e^{-2x\mathcal{A}}\,(\lambda-i\mathcal{A})^{-1}\,\mathcal{B}\right\}$$
$$=-2\int_{-\infty}^{+\infty}dx\,e^{-2i\lambda x}\mathcal{C}\left[I_p+\,e^{-2x\mathcal{A}}\,\mathcal{X}_B\,e^{-2x\mathcal{A}^*}\,\mathcal{X}_C\right]^{-1}e^{-2x\mathcal{A}}\,\mathcal{B}$$
$$-\,2i\int_{-\infty}^{+\infty}dx\,e^{-2i\lambda x}\left(\frac{d}{dx}\mathcal{C}\left[I_p+\,e^{-2x\mathcal{A}}\,\mathcal{X}_B\,e^{-2x\mathcal{A}^*}\,\mathcal{X}_C\right]^{-1}\right)e^{-2x\mathcal{A}}\,(\lambda-i\mathcal{A})^{-1}\,\mathcal{B}.$$

where we have used (6.56) to write the second identity. Integrating by parts the second integral in the second identity of the preceding equation, we get

$$a_{l2}(\lambda)=-2\int_{-\infty}^{+\infty}dx\,e^{-2i\lambda x}\mathcal{C}\left[I_p+\,e^{-2x\mathcal{A}}\,\mathcal{X}_B\,e^{-2x\mathcal{A}^*}\,\mathcal{X}_C\right]^{-1}e^{-2x\mathcal{A}}\,\mathcal{B}$$
$$-\,i\left[\mathcal{C}\left[I_p+\,e^{-2x\mathcal{A}}\,\mathcal{X}_B\,e^{-2x\mathcal{A}^*}\,\mathcal{X}_C\right]^{-1}e^{-2ix(\lambda-i\mathcal{A})}\,(\lambda-i\mathcal{A})^{-1}\,\mathcal{B}\right]_{-\infty}^{+\infty}$$
$$+\,i\int_{-\infty}^{+\infty}dx\,\mathcal{C}\left[I_p+\,e^{-2x\mathcal{A}}\,\mathcal{X}_B\,e^{-2x\mathcal{A}^*}\,\mathcal{X}_C\right]^{-1}(-2i)\,(\lambda-i\mathcal{A})\,e^{-2ix(\lambda-i\mathcal{A})}\,(\lambda-i\mathcal{A})^{-1}\,\mathcal{B}$$
$$=-i\left[\mathcal{C}\left[I_p+\,e^{-2x\mathcal{A}}\,\mathcal{X}_B\,e^{-2x\mathcal{A}^*}\,\mathcal{X}_C\right]^{-1}e^{-2ix(\lambda-i\mathcal{A})}\,(\lambda-i\mathcal{A})^{-1}\,\mathcal{B}\right]_{-\infty}^{+\infty}=0_{n\times m}.$$

Since the scattering matrix is J-unitary, we also obtain

$$a_{l3}(\lambda) = 0_{m\times n},$$

which completes the proof. ■

We need the following lemma to prove that the transmission coefficients $T_l(\lambda)$ and $T_r(\lambda)$ are unitary matrices for $\lambda \in \mathbb{R}$.

Lemma 6.7 *The following relations hold :*

$$\mathcal{X}_C^{-1}\left[\mathcal{A}^* - \mathcal{C}^*\mathcal{C}\,\mathcal{X}_C^{-1}\right]\mathcal{X}_C = -\mathcal{A}, \qquad \mathcal{X}_B\left[\mathcal{A}^* - \mathcal{X}_B^{-1}\mathcal{B}\mathcal{B}^*\right]\mathcal{X}_B^{-1} = -\mathcal{A}. \tag{6.64}$$

Proof. We have (cf. with (6.10))

$$\mathcal{A}^*\mathcal{X}_C + \mathcal{X}_C\mathcal{A} = \mathcal{C}^*\mathcal{C}, \qquad \mathcal{A}\mathcal{X}_B + \mathcal{X}_B\mathcal{A}^* = \mathcal{B}\mathcal{B}^*.$$

Thus

$$\mathcal{X}_C^{-1}\left[\mathcal{A}^* - \mathcal{C}^*\mathcal{C}\,\mathcal{X}_C^{-1}\right]\mathcal{X}_C = \mathcal{X}_C^{-1}\mathcal{A}^*\mathcal{X}_C - \mathcal{X}_C^{-1}\mathcal{C}^*\mathcal{C} = \mathcal{X}_C^{-1}\mathcal{A}^*\mathcal{X}_C - \mathcal{X}_C^{-1}\left(\mathcal{A}^*\mathcal{X}_C + \mathcal{X}_C\mathcal{A}\right) = -\mathcal{A},$$

and, in a similar way

$$\mathcal{X}_B\left[\mathcal{A}^* - \mathcal{X}_B^{-1}\mathcal{B}\mathcal{B}^*\right]\mathcal{X}_B^{-1} = \mathcal{X}_B\mathcal{A}^*\mathcal{X}_B^{-1} - \mathcal{B}\mathcal{B}^*\mathcal{X}_B^{-1} = \mathcal{X}_B\mathcal{A}^*\mathcal{X}_B^{-1} - \left(\mathcal{A}\mathcal{X}_B + \mathcal{X}_B\mathcal{A}^*\right)\mathcal{X}_B^{-1} = -\mathcal{A},$$

which completes the proof. ■

Lemma 6.7 implies that

$$\mathcal{A}^* - \mathcal{X}_B^{-1}\,\mathcal{B}\mathcal{B}^* = \mathcal{X}_B^{-1}(-\mathcal{A})\mathcal{X}_B, \qquad \mathcal{A}^* - \mathcal{C}^*\mathcal{C}\,\mathcal{X}_C^{-1} = \mathcal{X}_C(-\mathcal{A})\mathcal{X}_C^{-1}. \tag{6.65}$$

As a result, eqs. (6.50)-(6.51) can be written as

$$T_l(\lambda) = I_n + i\mathcal{C}\,(\lambda - i\mathcal{A})^{-1}\,\mathcal{X}_C^{-1}\,\mathcal{C}^*, \tag{6.66}$$

$$T_r(\lambda) = I_m + i\mathcal{B}^*\,\mathcal{X}_B^{-1}\,(\lambda - i\mathcal{A})^{-1}\,\mathcal{B}. \tag{6.67}$$

Hence,

$$T_l(\overline{\lambda})^* = I_n - i\mathcal{C}\,\mathcal{X}_C^{-1}\,(\lambda + i\mathcal{A}^*)^{-1}\,\mathcal{C}^* = a_{l1}(\lambda) = T_l(\lambda)^{-1},$$

$$T_r(\overline{\lambda})^* = I_m - i\mathcal{B}^*\,(\lambda + i\mathcal{A}^*)^{-1}\,\mathcal{X}_B^{-1}\mathcal{B} = a_{l4}(\lambda) = T_r(\lambda)^{-1},$$

and, consequently, $T_l(\lambda)$ and $T_r(\lambda)$ are unitary matrices for $\lambda \in \mathbb{R}$.

We have seen (cf. with Sec. 5.5-6.2) that the time evolution of the scattering data is governed by eq. (5.65). Thus if we are interested in finding the evolution of the transmission coefficients, we have to make the following change of data in eq. (6.66)-(6.67):

$$\mathcal{A} \mapsto \mathcal{A}, \quad \mathcal{B} \mapsto e^{-2it\mathcal{A}^2}\mathcal{B}, \quad \mathcal{C} \mapsto \mathcal{C}e^{-2it\mathcal{A}^2}, \quad \mathcal{X}_B \mapsto e^{-2it\mathcal{A}^2}\mathcal{X}_B e^{2it\mathcal{A}^2}, \quad \mathcal{X}_C \mapsto e^{2it\mathcal{A}^2}\mathcal{X}_C e^{-2it\mathcal{A}^2}.$$

With the above changes we get

$$\begin{aligned} T_l(\lambda;t) &= I_n + i\mathcal{C}\, e^{-2it\mathcal{A}^2}\,(\lambda - i\mathcal{A})^{-1}\, e^{2it\mathcal{A}^2}\,\mathcal{X}_C^{-1}\, e^{2it(\mathcal{A}^*)^2} e^{-2it(\mathcal{A}^*)^2}\mathcal{C}^* = \\ &= I_n + i\mathcal{C}\,(\lambda - i\mathcal{A})^{-1}\,\mathcal{X}_C^{-1}\,\mathcal{C}^* = T_l(\lambda), \\ T_r(\lambda;t) &= I_m + i\mathcal{B}^*\, e^{-2it(\mathcal{A}^*)^2} e^{2it(\mathcal{A}^*)^2}\,\mathcal{X}_B^{-1} e^{-2it\mathcal{A}^2}\,(\lambda - i\mathcal{A})^{-1}\, e^{2it\mathcal{A}^2}\,\mathcal{B} = \\ &= I_m + i\mathcal{B}^*\,\mathcal{X}_B^{-1}\,(\lambda - i\mathcal{A})^{-1}\,\mathcal{B} = T_r(\lambda), \end{aligned}$$

which shows that the transmission coefficients are invariant while the scattering data evolve.

Equation (6.35) represents a multi-soliton solution of the matrix nonlinear Schrödinger equation for $x \in \mathbb{R}$ and $t \in \mathbb{R}$. When interpreting $k(x,t)$ as an antisymmetric potential in the matrix Zakharov-Shabat system, we have zero reflection coefficients. The matrix $\mathcal{A}$ in the minimal representation of the Marchenko kernel

$$\Omega_l(\alpha) = \mathcal{C}\, e^{-\alpha\mathcal{A}}\,\mathcal{B}, \quad \alpha > 0,$$

corresponds to the poles of the transmission coefficients $T_l(\lambda)$ and $T_r(\lambda)$.

Applying the same method in the symmetric case leads to a guaranteed breakdown for some $x < 0$, because

$$\det\left[I_p - e^{-2x\mathcal{A}}\mathcal{X}_B e^{-2x\mathcal{A}^*}\mathcal{X}_C\right] = \det\left[I_p - \left(\mathcal{X}_C^{\frac{1}{2}} e^{-x\mathcal{A}}\mathcal{X}_B^{\frac{1}{2}}\right)\left(\mathcal{X}_C^{\frac{1}{2}} e^{-x\mathcal{A}}\mathcal{X}_B^{\frac{1}{2}}\right)^*\right]$$

is the determinant of a selfadjoint matrix whose eigenvalues tend to $-\infty$ as $x \to -\infty$. Hence this determinant must vanish for certain negative x. Thus in the symmetric case there do not exist multi-soliton solutions. This is understandable, because there are no bound states in the symmetric case and multi-soliton solutions are believed to correspond to bound state poles of the transmission coefficients. Instead, in the symmetric case there exist so-called dark soliton solutions, but it is by no means clear how to get them by the state space method.

The literature abounds with multi-soliton solutions, in most cases for $n = m = 1$. For example, in [96] the following expression

$$k(x,t) = \pm A\,\frac{e^{iBx - i(A^2 - B^2)t + iC_1}}{\cosh(Ax - 2ABt + C_2)} \tag{6.68}$$

where A, B, C_1, C_2 are arbitrary real constants, represents a soliton solution of the nonlinear Schrödinger equation (5.55).[2] Putting

$$\mathcal{A} = (a)\,, \quad \mathcal{B} = (1)\,, \quad \mathcal{C} = (c)\,,$$

where $a = p + iq$ with $p > 0$ and $0 \neq c \in \mathbb{C}$ and taking into account that

$$\begin{aligned} \mathcal{X}_B &= \int_0^\infty d\beta\, e^{-\beta(p+iq)} e^{-\beta(p-iq)} = \frac{1}{2p}, \\ \mathcal{X}_C &= \int_0^\infty d\beta\, e^{-\beta(p+iq)}\,|c|^2\, e^{-\beta(p-iq)} = \frac{|c|^2}{2p}, \end{aligned}$$

[2] We have substituted $k = 2$ and made the substitution $t \mapsto -t$ to account for the different NLS equation.

eq. (6.35) becomes

$$k(x,t) = \frac{-2ice^{-2x(p+iq)}e^{-4it(p^2-q^2)}e^{8tqp}}{1+\frac{|c|^2}{4p^2}e^{-4xp}e^{16tpq}} \times \frac{\frac{2p}{|c|}e^{2p(x-4qt)}}{\frac{2p}{|c|}e^{2p(x-4qt)}},$$

where $2px_0 = \ln\left(\frac{|c|}{2p}\right)$. Thus we have

$$k(x,t) = \frac{-2ip\frac{c}{|c|}e^{-4it(p^2-q^2)-2ixq}}{\cosh(2p(x-x_0-4qt))}. \tag{6.69}$$

Equation (6.69) coincides with eq. (6.68) if we choose

$$A = 2p, \qquad B = -2q, \qquad \pm e^{iC_1} = -i\frac{c}{|c|}, \qquad C_2 = -2px_0 = \ln\left(\frac{2p}{|c|}\right).$$

The transmission coefficients are easily seen to be given by

$$T_l(\lambda) = T_r(\lambda) = \frac{\lambda + i\overline{a}}{\lambda - ia}.$$

In the same way we can make the second soliton solution in [96] correspond to ours by taking $\mathcal{A} = (a)$ with $a > 0$, $\mathcal{B} = (1)$, and $C = (c)$ with $0 \neq c \in \mathbb{C}$. The multi-soliton solution in [96] corresponds to $\mathcal{A}$ being a diagonal matrix with distinct entries in the right half-plane, $\mathcal{B}$ being a column vector with entries 1, and $\mathcal{C}$ a row vector with nonzero complex entries.

6.4 Graphical representation of multi-soliton solutions

In this section we give the graphical representation of the solutions of the matrix nonlinear Schrödinger equation obtained by the state space method in the antisymmetric case. We know (see eq. (6.35)) that these solutions exist globally in $t \in \mathbb{R}$. When we have constructed the plots corresponding to these solutions, we have observed that sometimes the figure obtained displays time periodicity. This fact is very interesting because the solutions obtained with the state space method of the KdV equation are almost never periodic. Thus, we have studied this question in detail.

Before giving the main result we prove the following lemma.

Lemma 6.8 *Let A be an $n \times n$ matrix. Then e^{itA} is periodic if and only if A is diagonalizable and its eigenvalues are integer multiples of the same nonzero real number.*

Proof. (⇒) Let e^{itA} be periodic. Then there exists $\tau > 0$ such that

$$e^{i(t+\tau)A} = e^{itA}, \quad t \in \mathbb{R}.$$

From the preceding equation we get

$$e^{i\tau A} = I_n.$$

Moreover, every eigenvalue λ of A satisfies the following equation

$$e^{i\tau\lambda} = 1,$$

which implies

$$\lambda = \frac{2\pi k}{\tau}, \qquad k \in \mathbb{Z}.$$

Thus there exist an invertible matrix S and integers $k_1 < \ldots < k_r$ such that

$$S^{-1}AS = \bigoplus_{s=1}^{r} \left(\frac{2\pi k_s}{\tau} I_{m_s} + N_s \right), \tag{6.70}$$

where $m_1 + \ldots + m_r = n$ and $N_1, \ldots, N_r$ are nilpotent matrices, that is $N_s^{p_s} = 0_{m_s}$. Then we can write

$$I_n = S^{-1}e^{i\tau A}S = \bigoplus_{s=1}^{r} \underbrace{e^{\frac{2\pi i k_s}{\tau}}}_{=1} e^{i\tau N_s} = \bigoplus_{s=1}^{r} \sum_{j=0}^{p_s-1} \frac{(i\tau)^j}{j!} N_s^j,$$

from which we obtain

$$I_{m_s} = \sum_{j=0}^{p_s-1} \frac{(i\tau)^j}{j!} N_s^j,$$

and consequently, we get

$$i\tau N_s \sum_{j=1}^{p_s-1} \frac{(i\tau)^{j-1}}{j!} N_s^{j-1} = 0.$$

Now, taking into account that $\sum_{j=1}^{p_s-1} \frac{(i\tau)^j}{j!} N_s^{j-1}$ is invertible because it has the form $I_{m_s} + M$ where M is a nilpotent matrix, we find

$$N_s = 0_{m_s}, \quad \text{for} \quad s = 1, \ldots, r.$$

From eq. (6.70) we obtain

$$S^{-1}AS = \bigoplus_{s=1}^{r} \left(\frac{2\pi k_s}{\tau} I_{m_s} \right),$$

which implies that A is diagonalizable and $\lambda_s = \frac{2\pi k_s}{\tau}$, for $s = 1, \ldots, r$ are its eigenvalues.

($\Leftarrow$) Let A be diagonalizable with distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ with respective multiplicities $m_1, \ldots, m_r$. Then there exists an invertible matrix S such that

$$S^{-1}AS = \bigoplus_{s=1}^{r} \left(\lambda_s I_{m_s} \right).$$

Thus

$$e^{i\tau A} = S \bigoplus_{s=1}^{r} \left(e^{i\tau\lambda_s} I_{m_s} \right) S^{-1}.$$

As a consequence, $e^{i\tau A} = I_n$ iff there exist integers $k_1, \ldots, k_r$ such that $\tau\lambda_s = 2\pi k_s$ ($s = 1, \ldots, r$). In other words, $\lambda_1, \ldots, \lambda_r$ are integer multiples of $2\pi/\tau$. ■

We now have the following main result regarding periodicity of solutions of the matrix NLS equation in the antisymmetric case.

Theorem 6.9 *Let $\mathcal{A}$ have minimal square matrix order in the representation* (6.35) *of the solution of the matrix nonlinear Schrödinger equation. Then this solution is periodic if and only if $e^{4it\mathcal{A}^2}$ is periodic. The latter is satisfied if and only if $\mathcal{A}$ is diagonalizable with positive eigenvalues and the squares of these eigenvalues have rational ratios.*

Proof. Let us represent the Marchenko integral kernel Ω_l corresponding to the antisymmetric potential $k(x,0)$ as

$$\Omega_l(\alpha) = \mathcal{C}e^{-\alpha\mathcal{A}}\mathcal{B}, \qquad \alpha \in \mathbb{R}^+,$$

where $\mathcal{A}$ has minimal square matrix order and has its eigenvalues in the open right half-plane. Then the realization $\mathcal{C}(k - i\mathcal{A})^{-1}\mathcal{B}$ is minimal and the corresponding solution of the matrix NLS equation is given by (6.35). Letting $\Omega_l(\alpha;t)$ stand for the Marchenko kernel corresponding to the antisymmetric potential $k(x,t)$, we have

$$\Omega_l(\alpha;t) = \mathcal{C}e^{-\alpha\mathcal{A}}e^{4i\mathcal{A}^2 t}\mathcal{B}, \qquad \alpha \in \mathbb{R}^+,\ t \in \mathbb{R}. \tag{6.71}$$

Now let $k(x,t)$ be periodic in t. Then eqs. (3.43) and (3.44) imply that the solution $B_{l2}(x,\alpha;t)$ of the Marchenko integral equation with kernel $\Omega_l(\alpha;t)$ is periodic. We now apply Theorem B.1 given in Appendix B and the analyticity of $\Omega_l(\alpha;t)$ in α to prove that there exists $\tau > 0$ such that

$$\Omega_l(\alpha;t+\tau) = \Omega_l(\alpha;t), \qquad \alpha \in \mathbb{R}^+,\ t \in \mathbb{R}.$$

Substituting (6.71) and expanding the resulting equation in a power series in α we obtain for $k = 0,1,2,\ldots$ and $t \in \mathbb{R}$

$$\mathcal{C}\mathcal{A}^k e^{4i\mathcal{A}^2 t}\mathcal{B} = \mathcal{C}\mathcal{A}^k e^{4i\mathcal{A}^2(t+\tau)}\mathcal{B}.$$

By the minimality of the realization we get for $t \in \mathbb{R}$

$$e^{4i\mathcal{A}^2 t}\mathcal{B} = e^{4i\mathcal{A}^2(t+\tau)}\mathcal{B},$$

which implies

$$\mathcal{B}^*(\mathcal{A}^*)^l \left[e^{4i\mathcal{A}^2 t}\right]^* = \mathcal{B}^*(\mathcal{A}^*)^l \left[e^{4i\mathcal{A}^2(t+\tau)}\right]^*$$

for $l = 0,1,2,\ldots$ and $t \in \mathbb{R}$. Using minimality again we get

$$e^{4i\mathcal{A}^2 t} = e^{4i\mathcal{A}^2(t+\tau)}, \qquad t \in \mathbb{R}.$$

We now apply Lemma 6.8 plus the fact that all eigenvalues of $\mathcal{A}$ have a positive real part. Then $\mathcal{A}^2$ is diagonalizable, its eigenvalues are real (and hence positive), and the ratios are rational. Since a nonsingular matrix with diagonalizable square is itself diagonalizable, we conclude that $\mathcal{A}$ is diagonalizable, has only positive eigenvalues, and has rational ratios for the squares of these eigenvalues, as claimed. ■

We have applied the Symbolic Toolbox of Matlab Version 6.5 to plot the real and imaginary parts as well as the absolute values and arguments of the NLS solution $k(x,t)$ given by eq. (6.35) in various cases. The different types of plots are given. On one hand, we have plotted $\mathrm{Re}\, k(x,t)$, $\mathrm{Im}\, k(x,t)$, $|k(x,t)|$, and $\arg k(x,t)$ as functions of (x,t), which results in a surface in $\mathbb{R}^3$. On the other hand, we have plotted these quantities as functions of x for different values of t, leading to curves in the same figure corresponding to different t-values.

In Figures 6.1 and 6.2 the solution (6.35) of the NLS equation is shown for $p = n = m = 1$, $\mathcal{A} = (3)$, $\mathcal{B} = (1)$, and $\mathcal{C} = (2)$, which represents a one soliton solution that is periodic in time [cf. Theorem 6.9]. In Figure 6.1 we have plotted $\mathrm{Re}\, k(x,t)$ and $\mathrm{Im}\, k(x,t)$ as a function of $x \in [-3,3]$

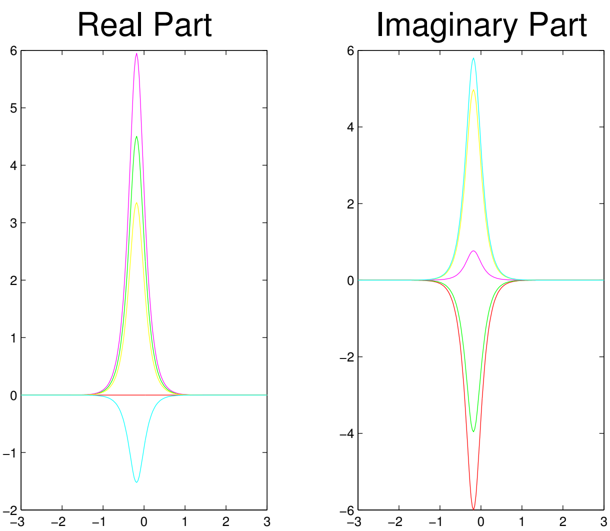


Figure 6.1: The real and imaginary parts of $k(x,t)$ as a function of $x \in [-3,3]$ for $t = 0, 0.5, 1, 1.5, 2$ for the model specified by the realization with matrices $\mathcal{A} = (3)$, $\mathcal{B} = (1)$, and $\mathcal{C} = (2)$.

for five different values of t. It is interesting to observe that these curves assume their maximal and minimal values for the same x, which suggests that $\arg k(x,t)$ is constant for a fixed t. In Figure 6.2 we have therefore plotted $|k(x,t)|$ and $\arg k(x,t)$ as a function of $(x,t) \in [-3,3] \times [0,2]$, which confirms $|k(x,t)|$ only depends on x and $\arg k(x,t)$ is time periodic.

In Figures 6.3 and 6.4 the solution (6.35) of the NLS equation is shown for $p = n = m = 1$, $\mathcal{A} = (3+i)$, $\mathcal{B} = (1)$, and $\mathcal{C} = (2)$, which represents a soliton solution that is not periodic [cf. Theorem 6.9]. In Figure 6.3 we have plotted $\mathrm{Re}\, k(x,t)$ and $\mathrm{Im}\, k(x,t)$ as a function of $x \in [-3,3]$ for five different values of t, with virtually the same graph for certain different t-values. The surfaces which represent $|k(x,t)|$ and $\arg k(x,t)$ and appear in Figure 6.4, are more complicated than those drawn in Figure 6.2.

In Figures 6.5 and 6.6 the solution (6.35) of the NLS equation is shown for $p = 2$, $n = 1$, $m = 1$, $\mathcal{A} = \mathrm{diag}\,(1,2)$, $\mathcal{B} = (1,1)^T$, and $\mathcal{C} = (3,2)$, which represents a time periodic two-soliton solution [cf. Theorem 6.9]. In Figure 6.5 we have plotted $\mathrm{Re}\, k(x,t)$ and $\mathrm{Im}\, k(x,t)$ as a function of $x \in [-3,3]$ for five different values of t, displaying time periodicity (with period π). The surfaces which represent $|k(x,t)|$ and $\arg k(x,t)$ and appear in Figure 6.6, are very complicated, but the

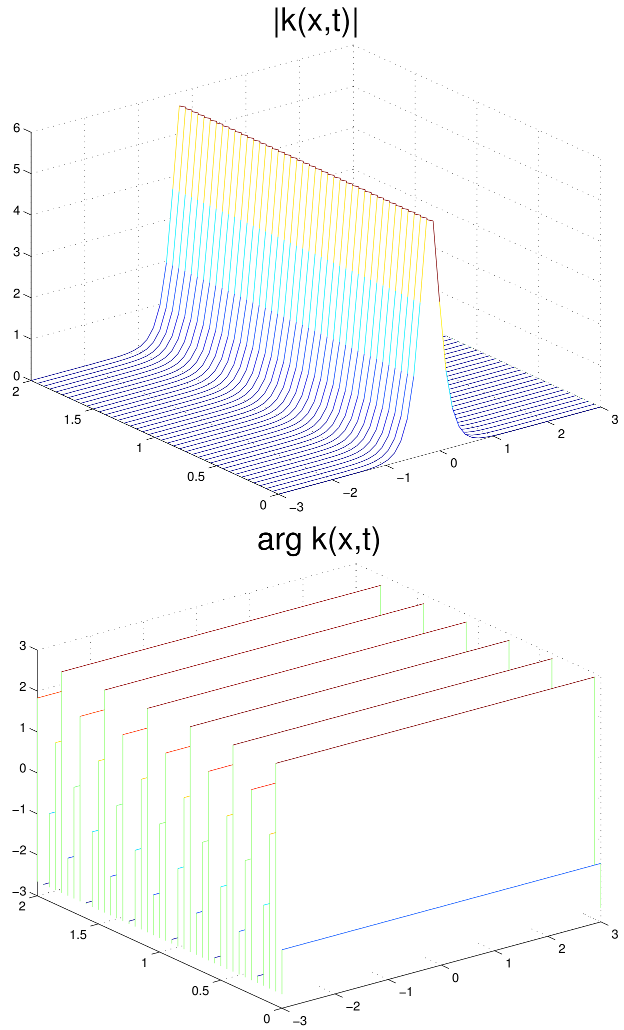


Figure 6.2: $|k(x,t)|$ and $\arg k(x,t)$ as a function of $(x,t) \in [-3,3] \times [0,2]$ for the same model as in Figure 6.1.

time periodicity of $|k(x,t)|$ is apparent.

In Figures 6.7 and 6.8 the solution (6.35) of the NLS equation is shown for $p = 2$, $n = 1$, $m = 1$, $\mathcal{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $\mathcal{B} = (1,1)^T$, and $\mathcal{C} = (3,2)$, which represents a two-soliton solution that is not periodic [cf. Theorem 6.9]. In Figure 6.7 we have plotted $\operatorname{Re} k(x,t)$ and $\operatorname{Im} k(x,t)$ as a function of $x \in [-3,3]$ for five different values of t. The surfaces which represent $|k(x,t)|$ and $\arg k(x,t)$ appear in Figure 6.8.

Aktosun has produced graphical representations of the solutions of the KdV equations on the half-line using Mathematica. This has led, among other things, to an animation of the solution of the KdV equation as a function of time [97]. Although we have not displayed them in this thesis, we have also produced similar animations of the modulus of $k(x,t)$ using MatLab.

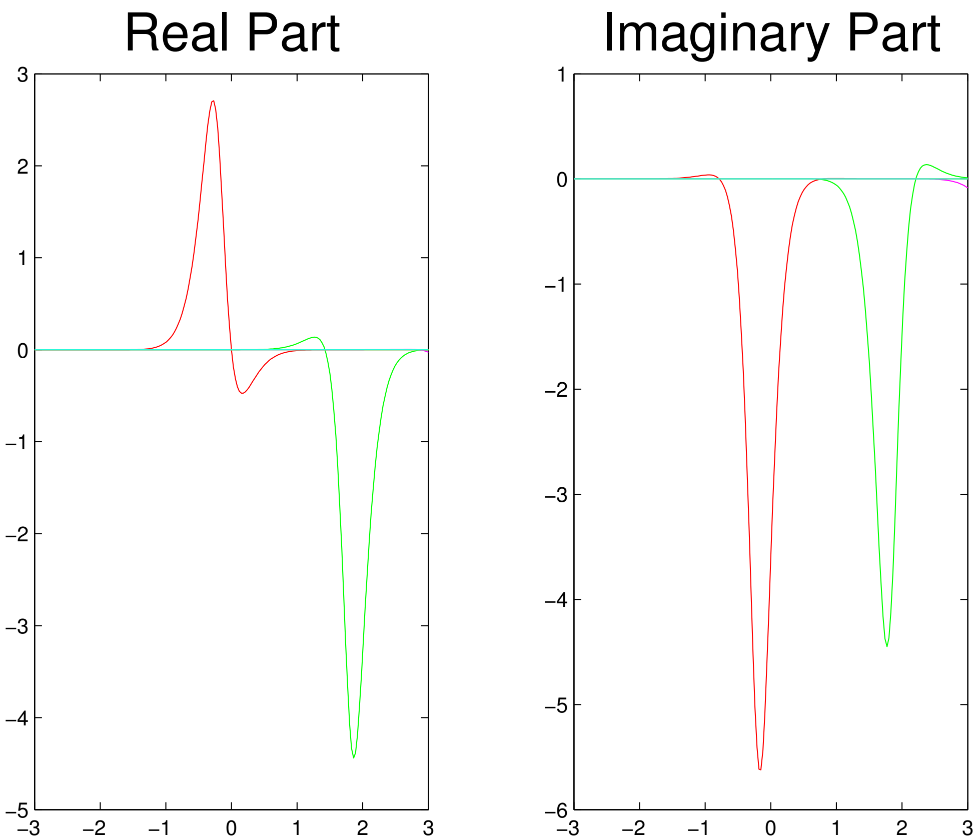


Figure 6.3: The real and imaginary parts of $k(x,t)$ as a function of $x \in [-3,3]$ for $t = 0, 0.5, 1, 1.5, 2$ for the model specified by the realization with matrices $\mathcal{A} = (3+i)$, $\mathcal{B} = (1)$, and $\mathcal{C} = (2)$.

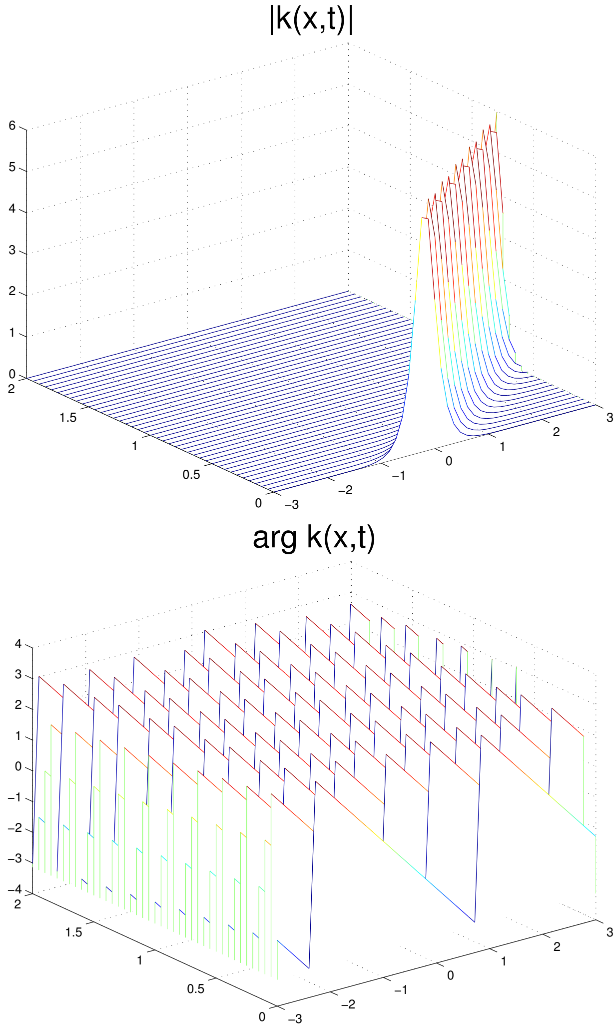


Figure 6.4: $|k(x,t)|$ and $\arg k(x,t)$ as a function of $(x,t) \in [-3,3] \times [0,2]$ for the same model as in Figure 6.3.

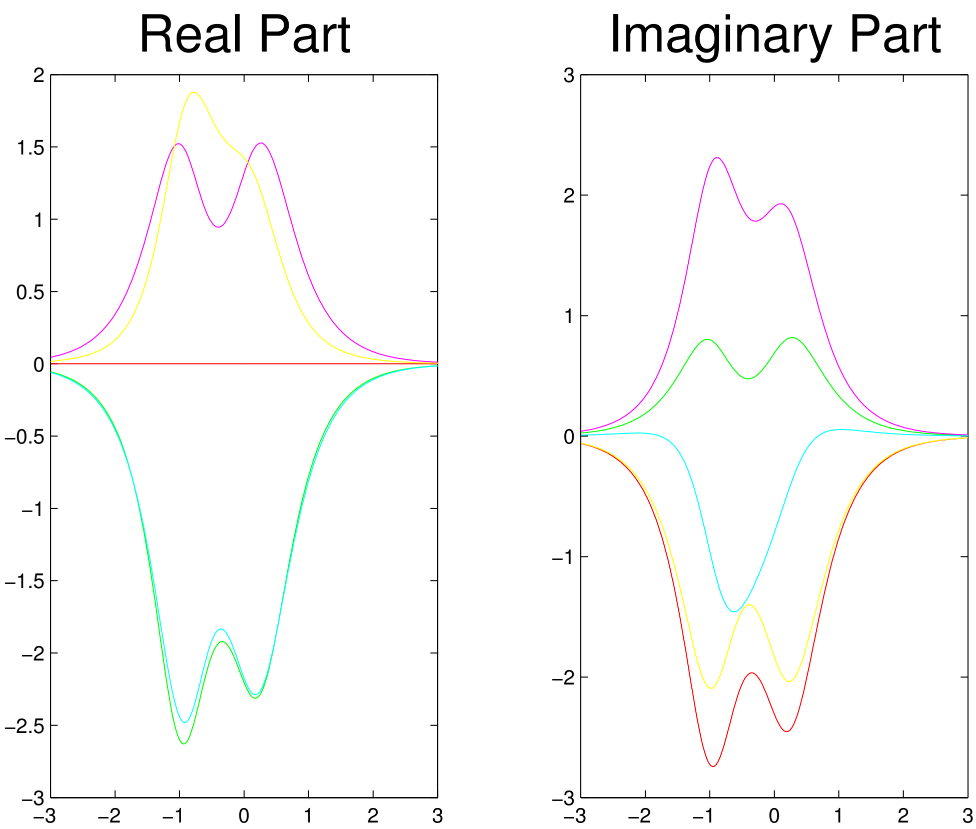


Figure 6.5: The real and imaginary parts of $k(x,t)$ as a function of $x \in [-3,3]$ for $t = 0, 0.5, 1, 1.5, 2$ for the model specified by the realization with matrices $\mathcal{A} = \text{diag}\,(1,2)$, $\mathcal{B} = (1,1)^T$, and $\mathcal{C} = (3,2)$.

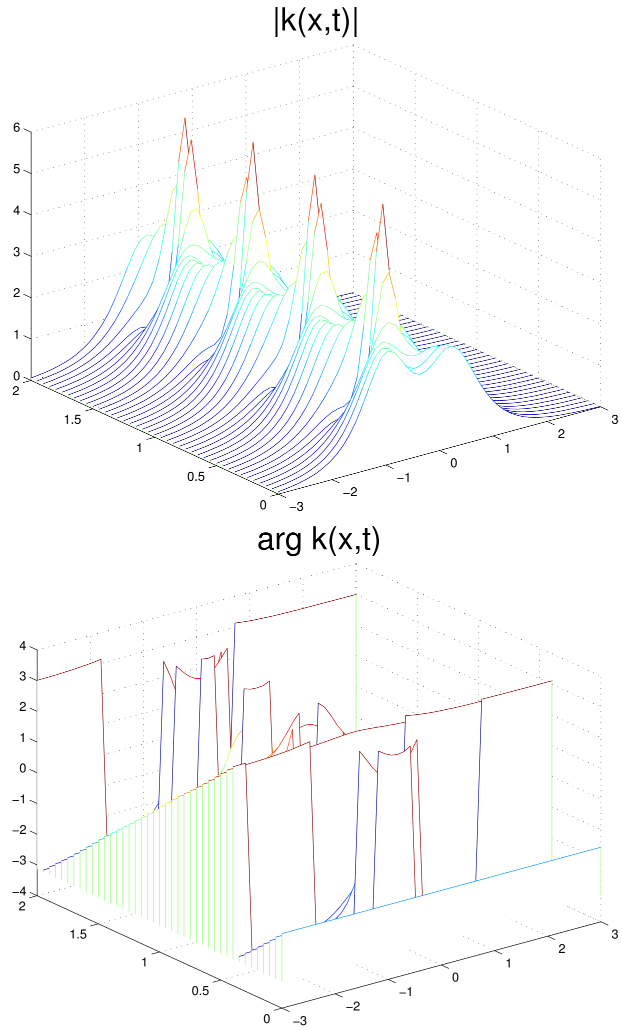


Figure 6.6: $|k(x,t)|$ and $\arg k(x,t)$ as a function of $(x,t) \in [-3,3] \times [0,2]$ for the same model as in Figure 6.5.

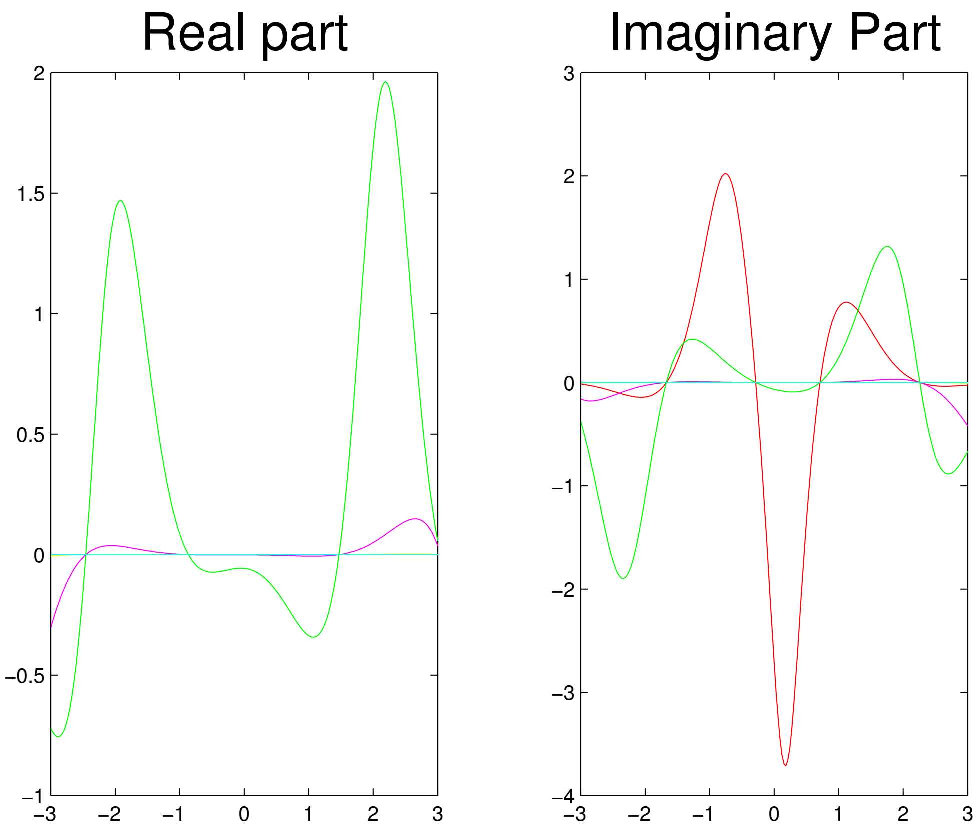


Figure 6.7: The real and imaginary parts of $k(x,t)$ as a function of $x \in [-3,3]$ for $t = 0, 0.5, 1, 1.5, 2$ for the model specified by the realization with matrices $\mathcal{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $\mathcal{B} = (1,1)^T$, and $\mathcal{C} = (3,2)$.

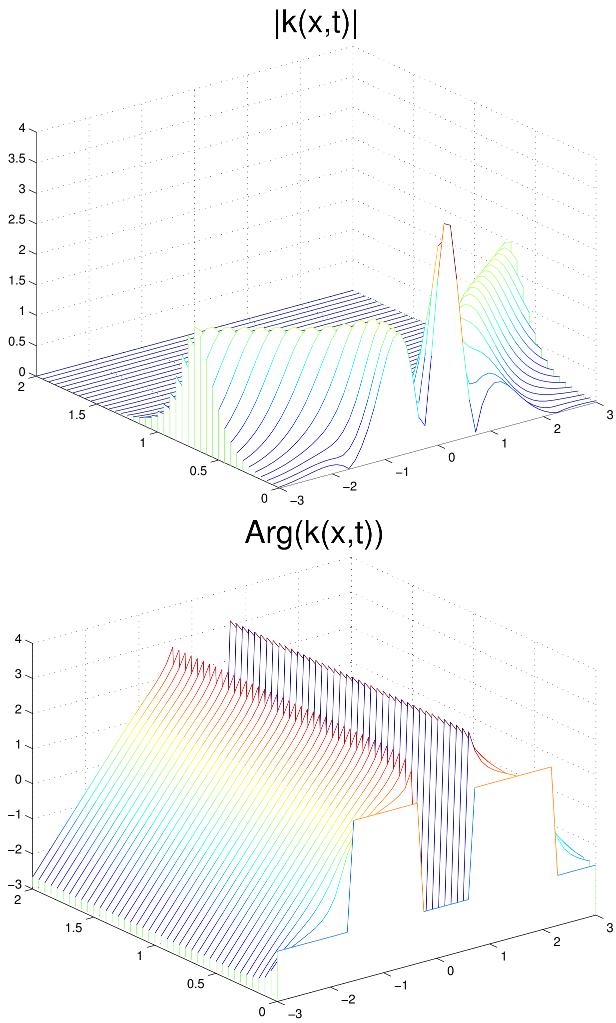


Figure 6.8: $|k(x,t)|$ and $\arg k(x,t)$ as a function of $(x,t) \in [-3,3] \times [0,2]$ for the same model as in Figure 6.7.

Appendix A

Comparison of Notations

In this thesis three different notational systems have been used, namely our own system based on [9, 90], that of Ablowitz and co-workers (see [3]), and that adopted in fiber optics [61]. Here we specify how to move from one system to the other.

A.1 Comparison with notations of Ablowitz et al.

In [3] the differential equation studied is as follows:

$$\frac{\partial \boldsymbol{v}}{\partial x} = \begin{pmatrix} -ikI_N & \boldsymbol{Q} \\ \boldsymbol{R} & ikI_M \end{pmatrix} \boldsymbol{v}, \tag{A.1}$$

where $\boldsymbol{Q}$ is an $N \times M$ matrix function with entries in $L^1(\mathbb{R})$, $\boldsymbol{R}$ is an $M \times N$ matrix function with entries in $L^1(\mathbb{R})$, $\boldsymbol{v}$ is an $(N+M)$-component vector, and k is a spectral parameter. Thus by putting

$$k = -\lambda, \qquad \boldsymbol{Q}(x) = ik(x), \qquad \boldsymbol{R}(x) = -i\ell(x),$$

we convert (A.1) into (3.1). These authors use $\boldsymbol{J} = (-I_N) \oplus I_M$, which corresponds with our $-J$ if $N = n$ and $M = m$. In [3] the Jost solutions $\phi(x,k)$, $\overline{\phi}(x,k)$, $\psi(x,k)$, and $\overline{\psi}(x,k)$ are defined as those $(N+M) \times N$, $(N+M) \times M$, $(N+M) \times M$, and $(N+M) \times N$ solutions that satisfy

$$\phi(x,k) \sim \begin{pmatrix} I_N \\ 0_{M\times N} \end{pmatrix} e^{-ikx}, \quad \overline{\phi}(x,k) \sim \begin{pmatrix} 0_{N\times M} \\ I_M \end{pmatrix} e^{ikx}, \quad x \to -\infty, \tag{A.2a}$$

$$\psi(x,k) \sim \begin{pmatrix} 0_{N\times M} \\ I_N \end{pmatrix} e^{ikx}, \quad \overline{\psi}(x,k) \sim \begin{pmatrix} I_N \\ 0_{N\times M} \end{pmatrix} e^{-ikx}, \quad x \to +\infty. \tag{A.2b}$$

Thus in our notations

$$\begin{pmatrix} \overline{\psi}(x,k) & \psi(x,k) \end{pmatrix} = F_l(-k,x), \qquad \begin{pmatrix} \phi(x,k) & \overline{\phi}(x,k) \end{pmatrix} = F_r(-k,x). \tag{A.3}$$

These authors then introduce[1]

$$\boldsymbol{M}(x,k) = e^{ikx}\phi(x,k), \quad \overline{\boldsymbol{M}}(x,k) = e^{-ikx}\overline{\phi}(x,k), \tag{A.4a}$$

[1] What the authors of [3] call Jost solutions, we call Faddeev functions.

$$\boldsymbol{N}(x,k) = e^{-ikx}\psi(x,k), \quad \overline{\boldsymbol{N}}(x,k) = e^{ikx}\overline{\psi}(x,k). \tag{A.4b}$$

We have in our notations

$$\begin{pmatrix}\overline{\boldsymbol{N}}(x,k) & \boldsymbol{N}(x,k)\end{pmatrix} = M_l(-k,x), \ \begin{pmatrix}\boldsymbol{M}(x,k) & \overline{\boldsymbol{M}}(x,k)\end{pmatrix} = M_r(-k,x). \tag{A.5}$$

For any $(N+M)\times J$ matrix $\boldsymbol{A}$ the authors of [3] introduce the notation

$$\boldsymbol{A} = \begin{pmatrix}\boldsymbol{A}^{(up)} \\ \boldsymbol{A}^{(dn)}\end{pmatrix},$$

where $\boldsymbol{A}^{(up)}$ is $N\times J$ and $\boldsymbol{A}^{(dn)}$ is $M\times J$.

In [3] the so-called scattering matrix[2]

$$\boldsymbol{S}(\xi) = \begin{pmatrix}\boldsymbol{a}(\xi) & \overline{\boldsymbol{b}}(\xi) \\ \boldsymbol{b}(\xi) & \overline{\boldsymbol{a}}(\xi)\end{pmatrix}, \qquad \xi\in\mathbb{R}, \tag{A.6}$$

is introduced such that

$$\begin{aligned}\phi(x,k) &= \psi(x,k)\boldsymbol{b}(k) + \overline{\psi}(x,k)\boldsymbol{a}(k),\\ \overline{\phi}(x,k) &= \psi(x,k)\overline{\boldsymbol{a}}(k) + \overline{\psi}(x,k)\overline{\boldsymbol{b}}(k),\end{aligned}$$

where $\boldsymbol{a}(k)$ is $N\times N$, $\overline{\boldsymbol{a}}(k)$ is $M\times M$, $\boldsymbol{b}(k)$ is $M\times N$, and $\overline{\boldsymbol{b}}(k)$ is $N\times M$. Thus in our notations

$$\begin{pmatrix}\boldsymbol{a}(k) & \overline{\boldsymbol{b}}(k) \\ \boldsymbol{b}(k) & \overline{\boldsymbol{a}}(k)\end{pmatrix} = a_r(-k) = \begin{pmatrix}a_{r1}(-k) & a_{r2}(-k) \\ a_{r3}(-k) & a_{r4}(-k)\end{pmatrix}. \tag{A.7}$$

On the other hand, in [3] we have

$$\begin{aligned}\psi(x,k) &= \phi(x,k)\boldsymbol{d}(k) + \overline{\phi}(x,k)\boldsymbol{c}(k),\\ \overline{\psi}(x,k) &= \phi(x,k)\overline{\boldsymbol{c}}(k) + \overline{\phi}(x,k)\overline{\boldsymbol{d}}(k),\end{aligned}$$

where $\boldsymbol{c}(k)$ is $N\times N$, $\overline{\boldsymbol{c}}(k)$ is $M\times M$, $\boldsymbol{d}(k)$ is $M\times N$, and $\overline{\boldsymbol{d}}(k)$ is $N\times M$. Thus in our notations

$$\begin{pmatrix}\overline{\boldsymbol{c}}(k) & \boldsymbol{d}(k) \\ \overline{\boldsymbol{d}}(k) & \boldsymbol{c}(k)\end{pmatrix} = a_l(-k) = \begin{pmatrix}a_{l1}(-k) & a_{l2}(-k) \\ a_{l3}(-k) & a_{l4}(-k)\end{pmatrix}. \tag{A.8}$$

The authors of [3] observe that the matrices in (A.7) and (A.8) are each others inverses and have determinant 1, and that $\det\boldsymbol{a}(k) = \det\boldsymbol{c}(k)$ and $\det\overline{\boldsymbol{a}}(k) = \det\overline{\boldsymbol{c}}(k)$.

[2]We use this terminology for a different matrix.

In [43], for $n = m = 1$, the authors introduce the four 2-column vector functions

$$\begin{cases} \psi_r = Re^{-i\zeta x}, & R \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{as } x \to -\infty \\ \tilde{\psi}_r = \tilde{R}e^{-i\zeta x}, & \tilde{R} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{as } x \to -\infty \\ \psi_l = Le^{-i\zeta x}, & L \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{as } x \to +\infty \\ \tilde{\psi}_l = \tilde{L}e^{-i\zeta x}, & \tilde{L} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{as } x \to +\infty \end{cases}$$

Here $\psi_r,\ \tilde{\psi}_r,\ \psi_l,\ \tilde{\psi}_l$ are the Jost solutions of the Zakharov-Shabat system

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} -i\zeta & q \\ r & i\zeta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Because ψ_r and $\tilde{\psi}_r$ (so as ψ_l and $\tilde{\psi}_l$) are linearly independent, for ζ real, we have

$$\begin{cases} \psi_r = r_+\psi_l + r_-\tilde{\psi}_l \\ \tilde{\psi}_r = \tilde{r}_+\psi_l + \tilde{r}_-\tilde{\psi}_l \end{cases} \quad \text{and} \quad \begin{cases} \psi_l = l_-\psi_r + l_+\tilde{\psi}_r \\ \tilde{\psi}_l = \tilde{l}_-\psi_r + \tilde{l}_+\tilde{\psi}_r \end{cases}$$

where the scattering coefficients $r_-, \tilde{r}_-$ only depend on $\zeta \in \mathbb{R}$, and with $r_-\tilde{r}_+ - \tilde{r}_-r_+ = 1$. Putting $\lambda = \zeta$, we have in our notations

$$F_l = \left(\psi_l|\tilde{\psi}_l\right), \quad F_r = \left(\psi_r|\tilde{\psi}_r\right), \quad a_r = \begin{pmatrix} r_- & \tilde{r}_- \\ r_- & \tilde{r}_+ \end{pmatrix}, \quad a_l = \begin{pmatrix} \tilde{l}_- & l_- \\ \tilde{l}_+ & l_+ \end{pmatrix}.$$

A.2 Notations used for the NLS equation

In the literature there exist various conventions regarding the way to write the NLS equation. The aim of this section is to discuss the main notations used and to make a comparison between their notations and ours.

We recall that in this PhD thesis we have essentially followed the notations used by Hasegawa, Tappert and Matsumoto in [62, 63, 61] when discussing fiber optics applications. Following these authors, in order to derive the NLS equation we consider the equation

$$i\frac{\partial \overline{E}}{\partial z} - \frac{k''}{2}\frac{\partial^2 \overline{E}}{\partial \tau^2} + \frac{\omega_0 n_2}{2c}|\overline{E}|^2\overline{E} = 0. \tag{A.9}$$

where $\overline{E}$ represents the modulation of the electric field and the quantities k'', z, ω_0 and τ are already introduced in Chapter 5. Now if t_0 is the scale size of the wavepacket, putting $z_0 = \frac{t_0}{-k''}$ and applying the following rescaling

$$Z = z/z_0, \quad T = \tau/t_0, \quad q = \sqrt{\frac{\omega_0 g\, \overline{n}_2 z_0}{2c}}\overline{E},$$

we obtain the NLS equation

$$\boxed{\frac{\partial q}{\partial Z} = \frac{i}{2}\frac{\partial^2 q}{\partial T^2} + i|q|^2 q.} \tag{A.10}$$

Our notation is different. In fact, we obtain our NLS equation

$$\boxed{\frac{\partial q}{\partial t} = \frac{i}{2}\frac{\partial^2 q}{\partial x^2} + i|q|^2 q.} \tag{A.11}$$

from theirs by making the change of variables $t \to Z$ and $x \to T$. Now in [3, 1, 2] we see that the NLS equation is written as

$$\boxed{\frac{\partial q}{\partial t} = -i\frac{\partial^2 q}{\partial x^2} \mp 2i|q|^2 q.} \tag{A.12}$$

Also this equation can be found starting from eq. (A.9), in fact if we put $z_0 = \frac{t_0}{k''}$ and consider the normalized variables (always in eq. (A.9))

$$Z = z/(2 * z_0), \quad T = \tau/t_0, \quad q = \sqrt{\frac{\omega_0 g\,\overline{n}_2 z_0}{2c}}\overline{E},$$

we get eq. (A.12).

A more complete discussion of the normalization of eq. (A.9) can be found in [87]. In this book the NLS equation has the following form

$$\boxed{i\frac{\partial u}{\partial \zeta} = \frac{\text{sign}(\beta_2)}{2}\frac{\partial^2 u}{\partial \tau^2} - |u|^2 u.} \tag{A.13}$$

Taking $q = u$, $t \to \zeta$ and $x \to \tau$, we arrive at

$$i\frac{\partial q}{\partial t} = \frac{\text{sign}(\beta_2)}{2}\frac{\partial^2 q}{\partial x^2} - |q|^2 q,$$

which implies

$$\boxed{\frac{\partial q}{\partial t} = -i\frac{\text{sign}(\beta_2)}{2}\frac{\partial^2 q}{\partial x^2} - |q|^2 q.} \tag{A.14}$$

When $\text{sign}(\beta_2) < 0$ eq. (A.14) is very similar to eq. (A.11) and can be reduced to eq. (A.11). On the other hand, when $\text{sign}(\beta_2) > 0$ we find a copy of eq. (A.12). We have already observed in Chapter 5 that in the optical communications literature the case $\text{sign}(\beta_2) < 0$ corresponds to anomalous dispersion, while $\text{sign}(\beta_2) > 0$ corresponds to normal dispersion.

Appendix B

Symmetry Relations

In this appendix we discuss four different transformations of the potential, namely conjugation ($V(x) \mapsto V(x)^*$), anticonjugation ($V(x) \mapsto -V(x)^*$), inversion ($V(x) \mapsto V(-x)$), and antiinversion ($V(x) \mapsto -V(x)$), and discuss their impact on the Jost solutions, the Faddeev matrices, the scattering coefficients, the reflection and transmission coefficients, the matrices $B_l(x, \alpha)$ and $B_r(x, \alpha)$, and the Marchenko integral kernels. We also discuss specific relations for these quantities in the symmetric case ($V(x)^* = V(x)$), in the antisymmetric case ($V(x)^* = -V(x)$), for even potentials ($V(-x) = V(x)$), and for odd potentials ($V(-x) = -V(x)$). In this way we elaborate on the symmetry theory expounded in Sec. 2.5. We start with the detour of constructing the Marchenko integral kernels from the functions $B_l(x, \alpha)$ and $B_r(x, \alpha)$.

B.1 Integral equations for the Marchenko integral kernels

In this section we study the extent to which the Marchenko integral kernel can be evaluated from the corresponding solution $B(x, \alpha)$ by solving the Marchenko integral equation with the roles of known and unknown function reversed. It appears that $\Omega_l(\alpha)$ ($\alpha \geq 2x$) can be computed uniquely from $B_{l2}(x, \alpha)$ ($\alpha \geq 0$) for sufficiently large x. Similarly, it appears that $\Omega_r(\alpha)$ ($\alpha \geq -2x$) can be computed uniquely from $B_{r2}(x, \alpha)$ ($\alpha \geq 0$) for sufficiently large $-x$.

Let us write the Marchenko integral equations (4.3b), (4.3c), (4.4b), and (4.4c) as follows:

$$\Omega_l(\alpha + 2x) + \int_\alpha^\infty d\beta\, B_{l1}(x, \beta - \alpha)\Omega_l(\beta + 2x) = -B_{l2}(x, \alpha), \tag{B.1a}$$

$$\Omega_r(\alpha - 2x) + \int_\alpha^\infty d\beta\, B_{r4}(x, \beta - \alpha)\Omega_r(\beta - 2x) = -B_{r3}(x, \alpha), \tag{B.1b}$$

$$\overline{\Omega}_r(\alpha - 2x) + \int_\alpha^\infty d\beta\, B_{r1}(x, \beta - \alpha)\overline{\Omega}_r(\beta - 2x) = -B_{r2}(x, \alpha), \tag{B.1c}$$

$$\overline{\Omega}_l(\alpha + 2x) + \int_\alpha^\infty d\beta\, B_{l4}(x, \beta - \alpha)\overline{\Omega}_l(\beta + 2x) = -B_{l3}(x, \alpha). \tag{B.1d}$$

Letting $B_{ls}(x, \alpha)$ and $B_{rs}(x, \alpha)$ be known and have their entries in $L^1(\mathbb{R}^+)$ for any $x \in \mathbb{R}$ and $s = 1, 2, 3, 4$, we thus obtain Volterra integral equations of convolution type.

Proposition B.1 *Given the matrix functions $B_{ls}(x,\alpha)$ or $B_{rs}(x,\alpha)$ in each of eqs.* (B.1) *and fixing $x \in \mathbb{R}$, the following statements are true:*

a. *Equation* (B.1a) *is uniquely solvable if and only if* $\det M_{l1}(x,\lambda) \neq 0$ *for every* $\lambda \in \overline{\mathbb{C}^+}$. *Moreover, there exists $x_0 \in \mathbb{R}$ such that* (B.1a) *is uniquely solvable for $x \geq x_0$.*

b. *Equation* (B.1b) *is uniquely solvable if and only if* $\det M_{r4}(x,\lambda) \neq 0$ *for every* $\lambda \in \overline{\mathbb{C}^+}$. *Moreover, there exists $x_0 \in \mathbb{R}$ such that* (B.1b) *is uniquely solvable for $x \geq x_0$.*

c. *Equation* (B.1c) *is uniquely solvable if and only if* $\det M_{r1}(x,\lambda) \neq 0$ *for every* $\lambda \in \overline{\mathbb{C}^-}$. *Moreover, there exists $x_0 \in \mathbb{R}$ such that* (B.1c) *is uniquely solvable for $x \leq x_0$.*

d. *Equation* (B.1d) *is uniquely solvable if and only if* $\det M_{l4}(x,\lambda) \neq 0$ *for every* $\lambda \in \overline{\mathbb{C}^-}$. *Moreover, there exists $x_0 \in \mathbb{R}$ such that* (B.1d) *is uniquely solvable for $x \leq x_0$.*

Proof. Each of eqs. (B.1) has the form

$$\Omega(\alpha) + \int_\alpha^\infty d\beta\, B(\beta-\alpha)\Omega(\beta) = F(\alpha), \qquad \alpha \in \mathbb{R}^+,$$

where the entries of B and F belong to $L^1(\mathbb{R}^+)$ and a solution Ω is sought with entries in $L^1(\mathbb{R}^+)$. Applying the standard theory of systems of convolution equations on the half-line [59, 53], we have unique solvability if and only if the so-called symbol

$$W(\lambda) \stackrel{\text{def}}{=} I_p + \int_0^\infty d\gamma\, e^{i\lambda\gamma} B(\gamma)$$

is invertible for each $\lambda \in \overline{\mathbb{C}^+}$, where $p = n, m, n, m$ respectively. In fact, by Fourier transformation we obtain

$$W(\lambda) \int_0^\infty d\alpha\, e^{i\lambda\alpha}\Omega(\alpha) = \int_0^\infty d\alpha\, e^{i\lambda\alpha} F(\alpha).$$

Thus we obtain for the Fourier transformed solution

$$\int_0^\infty d\alpha\, e^{i\lambda\alpha}\Omega(\alpha) = W(\lambda)^{-1} \int_0^\infty d\alpha\, e^{i\lambda\alpha} F(\alpha).$$

Since Theorem 3.8 implies the existence of $\mathcal{Z}$ with entries in $L^1(\mathbb{R}^+)$ such that

$$W(\lambda)^{-1} = I_p + \int_0^\infty d\alpha\, e^{i\lambda\alpha} \mathcal{Z}(\alpha),$$

we obtain the unique solution

$$\Omega(\alpha) = F(\alpha) + \int_\alpha^\infty d\beta\, \mathcal{Z}(\beta-\alpha) F(\beta).$$

The second statement in part a follows from the fact that $M_{l1}(x,\lambda) \to I_n$ as $x \to +\infty$, uniformly in $\lambda \in \overline{\mathbb{C}^+}$. Parts b-d are proved in the same way. ■

B.2 Inversion and antiinversion

As in Sec. 2.5, for any function W of $x \in \mathbb{R}$ we define the functions $W^{(\#)}$ and $W^{[\#]}$ by

$$W^{(\#)}(x) = W(-x), \qquad W^{[\#]}(x) = -W(-x).$$

Defining the unitary and selfadjoint operator U on $\mathcal{H}_{n+m}$ by

$$(Uf)(x) = f(-x), \qquad x \in \mathbb{R},$$

we then have the following symmetry relations [cf. (2.11) and (2.13)]:

inversion	$H^{(\#)} = -iJ(d/dx) - V^{(\#)}$	$H_0(UJ) = -(UJ)H_0$ $H(UJ) = -(UJ)H$
antiinversion	$H^{[\#]} = -iJ(d/dx) - V^{[\#]}$	$H_0U = -UH_0$ $HU = -UH$

Replacing (x, λ) by $(-x, -\lambda)$ in (3.2b) and making the change of variable $y \mapsto -y$ we obtain the Volterra integral equation

$$\begin{aligned} F_r(-x, -\lambda) &= e^{i\lambda Jx} + iJ \int_x^\infty dy\, e^{-i\lambda J(y-x)} V(-y) F_r(-y, -\lambda) \\ &= e^{i\lambda Jx} - iJ \int_x^\infty dy\, e^{-i\lambda J(y-x)} V^{[\#]}(y) F_r(-y, -\lambda). \end{aligned} \tag{B.2}$$

Multiplying the first line of (B.2) from the left and the right by J and using $JV(x)J = -V(x)$, we obtain

$$\begin{aligned} JF_r(-x, -\lambda)J &= e^{i\lambda Jx} - iJ \int_x^\infty dy\, e^{-i\lambda J(y-x)} V(-y) JF_r(-y, -\lambda)J \\ &= e^{i\lambda Jx} - iJ \int_x^\infty dy\, e^{-i\lambda J(y-x)} V^{(\#)}(y) JF_r(-y, -\lambda)J. \end{aligned} \tag{B.3}$$

In a similar way we can manipulate (3.2a). Because (3.2a) and (3.2b) are uniquely solvable for any L^1-potential V, we obtain for the Jost solutions corresponding to $H^{(\#)}$ and $H^{[\#]}$

$$F_l^{(\#)}(x, \lambda) = JF_r(-x, -\lambda)J, \qquad F_r^{(\#)}(x, \lambda) = JF_l(-x, -\lambda)J, \tag{B.4a}$$

$$F_l^{[\#]}(x, \lambda) = F_r(-x, -\lambda), \qquad F_r^{[\#]}(x, \lambda) = F_l(-x, -\lambda). \tag{B.4b}$$

Using (3.10) we get for the Faddeev matrices

$$M_l^{(\#)}(x, \lambda) = JM_r(-x, -\lambda)J, \qquad M_r^{(\#)}(x, \lambda) = JM_l(-x, -\lambda)J, \tag{B.5a}$$

$$M_l^{[\#]}(x, \lambda) = M_r(-x, -\lambda), \qquad M_r^{[\#]}(x, \lambda) = M_l(-x, -\lambda). \tag{B.5b}$$

Taking the appropriate limits of (B.4) as $x \to \pm\infty$, we obtain for the scattering matrices

$$a_l^{(\#)}(\lambda) = Ja_r(-\lambda)J, \qquad a_r^{(\#)}(\lambda) = Ja_l(-\lambda)J, \tag{B.6a}$$

$$a_l^{[\#]}(\lambda) = a_r(-\lambda), \qquad a_r^{[\#]}(\lambda) = a_l(-\lambda). \tag{B.6b}$$

Equations (B.6) lead to identities of the following type:

$$\begin{aligned} a_{l1}^{(\#)}(\lambda) &= a_{r1}(-\lambda), & a_{l4}^{(\#)}(\lambda) &= a_{r4}(-\lambda), \\ a_{l2}^{(\#)}(\lambda) &= -a_{r2}(-\lambda), & a_{l3}^{(\#)}(\lambda) &= -a_{r3}(-\lambda), \\ a_{l1}^{[\#]}(\lambda) &= a_{r1}(-\lambda), & a_{l4}^{[\#]}(\lambda) &= a_{r4}(-\lambda), \\ a_{l2}^{[\#]}(\lambda) &= a_{r2}(-\lambda), & a_{l3}^{[\#]}(\lambda) &= a_{r3}(-\lambda). \end{aligned}$$

Using (3.35) and (3.36) in (B.5) we obtain

$$B_l^{(\#)}(x,\alpha) = JB_r(-x,\alpha)J, \qquad B_r^{(\#)}(x,\alpha) = JB_l(-x,\alpha)J, \tag{B.7a}$$

$$B_l^{[\#]}(x,\alpha) = B_r(-x,\alpha), \qquad B_r^{[\#]}(x,\alpha) = B_l(-x,\alpha). \tag{B.7b}$$

It is now easily understood that the technical hypothesis applies to the inverted and antiinverted Hamiltonians $H^{(\#)}$ and $H^{[\#]}$ whenever it applies to H. Under the technical hypothesis we then easily prove the following identities:

$T_l^{(\#)}(\lambda) = \overline{T}_l(-\lambda)$	$T_r^{(\#)}(\lambda) = \overline{T}_r(-\lambda)$	$T_l^{[\#]}(\lambda) = \overline{T}_l(-\lambda)$	$T_r^{[\#]}(\lambda) = \overline{T}_r(-\lambda)$
$\overline{T}_l^{(\#)}(\lambda) = T_l(-\lambda)$	$\overline{T}_r^{(\#)}(\lambda) = T_r(-\lambda)$	$\overline{T}_l^{[\#]}(\lambda) = T_l(-\lambda)$	$\overline{T}_r^{[\#]}(\lambda) = T_r(-\lambda)$
$R^{(\#)}(\lambda) = -\overline{L}(-\lambda)$	$L^{(\#)}(\lambda) = -\overline{R}(-\lambda)$	$R^{[\#]}(\lambda) = \overline{L}(-\lambda)$	$L^{[\#]}(\lambda) = \overline{R}(-\lambda)$
$\overline{R}^{(\#)}(\lambda) = -L(-\lambda)$	$\overline{L}^{(\#)}(\lambda) = -R(-\lambda)$	$\overline{R}^{[\#]}(\lambda) = L(-\lambda)$	$\overline{L}^{[\#]}(\lambda) = R(-\lambda)$

These identities can be written as four identities for the scattering matrices

$$S^{(\#)}(\lambda) = J\overline{S}(-\lambda)J = JS(-\lambda)^{-1}J, \tag{B.8a}$$

$$S^{[\#]}(\lambda) = \overline{S}(-\lambda) = S(-\lambda)^{-1}. \tag{B.8b}$$

It is now immediate from the above table and eqs. (4.56), (4.57), (4.61), and (4.62) that we have the following correspondences for the poles of the transmission coefficients, their orders, and their Laurent series expansion coefficients ($j = 1, \ldots, \tilde{N}$, $s = 1, \ldots, \tilde{q}_j$):

$$\begin{cases} \kappa_j^{(\#)} = \kappa_j^{[\#]} = \tilde{\kappa}_j, & q_j^{(\#)} = q_j^{[\#]} = \tilde{q}_j, \\ \tau_{ljs}^{(\#)} = \tau_{ljs}^{[\#]} = \tilde{\tau}_{ljs}, & \tau_{rjs}^{(\#)} = \tau_{rjs}^{[\#]} = \tilde{\tau}_{rjs}, \\ \tilde{\kappa}_j^{(\#)} = \tilde{\kappa}_j^{[\#]} = \kappa_j, & \tilde{q}_j^{(\#)} = \tilde{q}_j^{[\#]} = q_j, \\ \tilde{\tau}_{ljs}^{(\#)} = \tilde{\tau}_{ljs}^{[\#]} = \tau_{ljs}, & \tilde{\tau}_{rjs}^{(\#)} = \tilde{\tau}_{rjs}^{[\#]} = \tau_{rjs}. \end{cases} \tag{B.9}$$

Let us now study the effect of inversion and antiinversion on eigenfunctions. Applying (4.38) to the inverted potential $V^{(\#)}$ we have

$$F_l^{(\#)}(x,i\kappa_j)\begin{pmatrix}\varepsilon_j^{(\#)} \\ 0_{m\times1}\end{pmatrix} = F_r^{(\#)}(x,i\kappa_j)\begin{pmatrix}0_{n\times1} \\ \eta_j^{(\#)}\end{pmatrix}.$$

Using (B.4a) and making the change $x \mapsto -x$ we get

$$F_r(x,-i\kappa_j)\begin{pmatrix}\varepsilon_j^{(\#)} \\ 0_{m\times1}\end{pmatrix} = -F_l(x,-i\kappa_j)\begin{pmatrix}0_{n\times1} \\ \eta_j^{(\#)}\end{pmatrix}.$$

A comparison with (4.45) leads to $\overline{\varepsilon}_j = \varepsilon_j^{(\#)}$ and $\overline{\eta}_j = -\eta_j^{(\#)}$. Therefore,

$$\overline{C}_{j0} = -C_{j0}^{(\#)}, \quad \overline{D}_{j0} = -D_{j0}^{(\#)}, \quad C_{j0} = -\overline{C}_{j0}^{(\#)}, \quad D_{j0} = -\overline{D}_{j0}^{(\#)}. \tag{B.10a}$$

In the same way we obtain with the help of (B.4b)

$$\overline{C}_{j0} = C_{j0}^{[\#]}, \qquad \overline{D}_{j0} = D_{j0}^{[\#]}, \qquad C_{j0} = \overline{C}_{j0}^{[\#]}, \qquad D_{j0} = \overline{D}_{j0}^{[\#]}. \tag{B.10b}$$

With the help of (B.9) we arrive at the identities

$$\overline{\Gamma}_{lj} = -\Gamma_{rj}^{(\#)}, \quad \overline{\Gamma}_{rj} = -\Gamma_{lj}^{(\#)}, \quad \Gamma_{lj} = -\overline{\Gamma}_{rj}^{(\#)}, \quad \Gamma_{rj} = -\overline{\Gamma}_{lj}^{(\#)}. \tag{B.11a}$$

In the same way we obtain with the help of (B.4b)

$$\overline{\Gamma}_{lj} = \Gamma_{rj}^{[\#]}, \qquad \overline{\Gamma}_{rj} = \Gamma_{lj}^{[\#]}, \qquad \Gamma_{lj} = \overline{\Gamma}_{rj}^{[\#]}, \qquad \Gamma_{rj} = \overline{\Gamma}_{lj}^{[\#]}. \tag{B.11b}$$

The previous table and, for algebraically simple eigenvalues only, the previous table plus eqs. (B.10) lead to the following symmetry relations for the Marchenko integral kernels:

$$\Omega_l^{(\#)}(\alpha) = -\overline{\Omega}_r(\alpha), \qquad \Omega_r^{(\#)}(\alpha) = -\overline{\Omega}_l(\alpha), \tag{B.12a}$$
$$\overline{\Omega}_l^{(\#)}(\alpha) = -\Omega_r(\alpha), \qquad \overline{\Omega}_r^{(\#)}(\alpha) = -\Omega_l(\alpha), \tag{B.12b}$$
$$\Omega_l^{[\#]}(\alpha) = \overline{\Omega}_r(\alpha), \qquad \Omega_r^{[\#]}(\alpha) = \overline{\Omega}_l(\alpha), \tag{B.12c}$$
$$\overline{\Omega}_l^{[\#]}(\alpha) = \Omega_r(\alpha), \qquad \overline{\Omega}_r^{[\#]}(\alpha) = \Omega_l(\alpha). \tag{B.12d}$$

For **even** potentials ($V = V^{(\#)}$) we have $\Omega_l = \Omega_l^{(\#)}$ and $\Omega_r = \Omega_r^{(\#)}$ and therefore we obtain the symmetry relations

$$\Omega_l(\alpha) = -\overline{\Omega}_r(\alpha), \qquad \Omega_r(\alpha) = -\overline{\Omega}_l(\alpha),$$
$$\overline{\Omega}_l(\alpha) = -\Omega_r(\alpha), \qquad \overline{\Omega}_r(\alpha) = -\Omega_l(\alpha).$$

Instead, for **odd** potentials ($V = -V^{(\#)} = V^{[\#]}$) we get the symmetry relations

$$\Omega_l(\alpha) = \overline{\Omega}_r(\alpha), \qquad \Omega_r(\alpha) = \overline{\Omega}_l(\alpha),$$
$$\overline{\Omega}_l(\alpha) = \Omega_r(\alpha), \qquad \overline{\Omega}_r(\alpha) = \Omega_l(\alpha).$$

B.3 Conjugation and anticonjugation

In this section we study the impact of conjugation and anticonjugation on various quantities, assuming there are no bound states.

1. Dualizing the matrix Zakharov-Shabat system. Let us begin our discussion with the following variation on the matrix Zakharov-Shabat system for $\lambda \in \mathbb{R}$:

$$\begin{aligned} -i\frac{\partial}{\partial x}\left(F(x,\lambda)^{-1}\right)J &= i\,F(x,\lambda)^{-1}\frac{\partial F}{\partial x}(x,\lambda)F(x,\lambda)^{-1}J \\ &= -F(x,\lambda)^{-1}J\left\{-iJ\frac{\partial F}{\partial x}(x,\lambda)\right\}F(x,\lambda)^{-1}J \\ &= -F(x,\lambda)^{-1}J\left\{\lambda I_{n+m}+V(x)\right\}F(x,\lambda)F(x,\lambda)^{-1}J \\ &= -F(x,\lambda)^{-1}J\left\{\lambda I_{n+m}+V(x)\right\}J \\ &= -F(x,\lambda)^{-1}\left\{\lambda I_{n+m}-V(x)\right\}, \end{aligned} \tag{B.13}$$

where F stands for either the left or the right Jost solution. Here we note that Jost solutions are invertible matrices of order $n+m$. Let us now consider the matrix Zakharov-Shabat system itself satisfied by a Jost solution:

$$-iJ\frac{\partial F}{\partial x}(x,\lambda) = \left\{\lambda I_{n+m}+V(x)\right\}F(x,\lambda).$$

Let us now take the complex conjugate of this equation and write it in the following two equivalent forms:

$$\begin{aligned} -i\frac{\partial}{\partial x}\left(JF(x,\lambda)^*J\right)J &= -JF(x,\lambda)^*J\left\{\lambda I_{n+m}-V(x)^*\right\}, \\ -i\frac{\partial}{\partial x}\left(F(x,\lambda)^*\right)J &= -F(x,\lambda)^*\left\{\lambda I_{n+m}+V(x)^*\right\}. \end{aligned}$$

Then using the superscript $(*)$ for quantities pertaining to the conjugate potential $V^{(*)}(x) \stackrel{\text{def}}{=} V(x)^*$ and the superscript $[*]$ for quantities pertaining to the anticonjugate potential $V^{[*]}(x) = -V(x)^*$, we obtain

$$F^{(*)}(x,\lambda)^{-1} = JF(x,\lambda)^*J, \qquad F^{[*]}(x,\lambda)^{-1} = F(x,\lambda)^*. \tag{B.14}$$

2. Symmetries for Jost solutions, Faddeev matrices, and scattering coefficients. Let us now employ (3.1a), (3.1b), (3.4a), and (3.4b) in (B.14). We get

$$F_l^{(*)}(x,\lambda) = J[F_l(x,\lambda)^*]^{-1}J, \qquad F_r^{(*)}(x,\lambda) = J[F_r(x,\lambda)^*]^{-1}J, \tag{B.15a}$$

$$F_l^{[*]}(x,\lambda) = [F_l(x,\lambda)^*]^{-1}, \qquad F_r^{[*]}(x,\lambda) = [F_r(x,\lambda)^*]^{-1}. \tag{B.15b}$$

For the scattering coefficients we get

$$a_l^{(*)}(\lambda) = J[a_l(\lambda)^*]^{-1}J = Ja_r(\lambda)^*J, \tag{B.16a}$$

$$a_r^{(*)}(\lambda) = J[a_r(\lambda)^*]^{-1}J = Ja_l(\lambda)^*J, \tag{B.16b}$$

$$a_l^{[*]}(\lambda) = [a_l(\lambda)^*]^{-1} = a_r(\lambda)^*, \tag{B.16c}$$

$$a_r^{[*]}(\lambda) = [a_r(\lambda)^*]^{-1} = a_l(\lambda)^*. \tag{B.16d}$$

From (B.15) and (3.10) we get the following symmetry relations for the Faddeev matrices:

$$M_l^{(*)}(x,\lambda) = J[M_l(x,\lambda)^*]^{-1}J, \qquad M_r^{(*)}(x,\lambda) = J[M_r(x,\lambda)^*]^{-1}J, \tag{B.17a}$$

$$M_l^{[*]}(x,\lambda) = [M_l(x,\lambda)^*]^{-1}, \qquad M_r^{[*]}(x,\lambda) = [M_r(x,\lambda)^*]^{-1}. \tag{B.17b}$$

Consequently, in the symmetric case ($V^{(*)} = V$) the Jost solutions and scattering coefficients are J-unitary matrices, while in the antisymmetric case ($V^{[*]} = V$) these matrices are unitary. This is in agreement with Proposition 3.2.

It is now easily understood that the technical hypothesis applies to the conjugate and anticonjugate Hamiltonians $H_0 - V^{(*)}$ and $H_0 - V^{[*]}$ whenever it applies to H. Under the technical hypothesis we then easily prove the following identities:

$T_l^{(*)}(\lambda) = \overline{T}_l(\overline{\lambda})^*$	$T_r^{(*)}(\lambda) = \overline{T}_r(\overline{\lambda})^*$	$T_l^{[*]}(\lambda) = \overline{T}_l(\overline{\lambda})^*$	$T_r^{[*]}(\lambda) = \overline{T}_r(\overline{\lambda})^*$
$\overline{T}_l^{(*)}(\lambda) = T_l(\overline{\lambda})^*$	$\overline{T}_r^{(*)}(\lambda) = T_r(\overline{\lambda})^*$	$\overline{T}_l^{[*]}(\lambda) = T_l(\overline{\lambda})^*$	$\overline{T}_r^{[*]}(\lambda) = T_r(\overline{\lambda})^*$
$R^{(*)}(\lambda) = \overline{L}(\lambda)^*$	$L^{(*)}(\lambda) = \overline{R}(\lambda)^*$	$R^{[*]}(\lambda) = -\overline{L}(\lambda)^*$	$L^{[*]}(\lambda) = -\overline{R}(\lambda)^*$
$\overline{R}^{(*)}(\lambda) = L(\lambda)^*$	$\overline{L}^{(*)}(\lambda) = R(\lambda)^*$	$\overline{R}^{[*]}(\lambda) = -L(\lambda)^*$	$\overline{L}^{[*]}(\lambda) = -R(\lambda)^*$

These identities can be written as four identities for the scattering matrices

$$S^{(*)}(\lambda) = \overline{S}(\lambda)^* = [S(\lambda)^*]^{-1}, \tag{B.18a}$$

$$S^{[*]}(\lambda) = J\overline{S}(\lambda)^*J = J[S(\lambda)^*]^{-1}J, \tag{B.18b}$$

Thus $S(\lambda)$ is unitary in the symmetric case and J-unitary in the antisymmetric case. It is now immediate from the above table and eqs. (4.56), (4.57), (4.61), and (4.62) that we have the following correspondences for the poles of the transmission coefficients, their orders, and their Laurent series expansion coefficients ($j = 1, \ldots, \tilde{N}$, $s = 1, \ldots, \tilde{q}_j$):

$$\begin{cases} \kappa_j^{(*)} = \kappa_j^{[*]} = \overline{\tilde{\kappa}_j}, & q_j^{(*)} = q_j^{[*]} = \tilde{q}_j, \\ \tau_{ljs}^{(*)} = \tau_{ljs}^{[*]} = (\tilde{\tau}_{ljs})^*, & \tau_{rjs}^{(*)} = \tau_{rjs}^{[*]} = (\tilde{\tau}_{rjs})^*, \\ \tilde{\kappa}_j^{(*)} = \tilde{\kappa}_j^{[*]} = \overline{\kappa_j}, & \tilde{q}_j^{(*)} = \tilde{q}_j^{[*]} = q_j, \\ \tilde{\tau}_{ljs}^{(*)} = \tilde{\tau}_{ljs}^{[*]} = (\tau_{ljs})^*, & \tilde{\tau}_{rjs}^{(*)} = \tilde{\tau}_{rjs}^{[*]} = (\tau_{rjs})^*. \end{cases}$$

3. Symmetries for the functions $B_{l,r}(x,\alpha)$. It is in general not easy to derive symmetry relations for $B_l(x,\alpha)$ and $B_r(x,\alpha)$, and eventually for the Marchenko integral kernels in the presence of bound states, under conjugation or anticonjugation. This requires introducing "adjoint" Jost solutions and Faddeev matrices based on the adjoint matrix Zakharov-Shabat system (B.13). From (3.10) and the proof of Proposition 3.1 it is clear that

$$\det M_l(x,\lambda) = \det M_r(x,\lambda) = 1. \tag{B.19}$$

Also, $M_l^{[*]}(x,\lambda)$ and $M_r^{[*]}(x,\lambda)$ can be divided into blocks labeled $1,2,3,4$ in such a way that $M_{l1}^{[*]}(x,\lambda)$, $M_{l3}^{[*]}(x,\lambda)$, $M_{r2}^{[*]}(x,\lambda)$, are $M_{r4}^{[*]}(x,\lambda)$ are analytic in $\mathbb{C}^+$ and $M_{r1}^{[*]}(x,\lambda)$, $M_{r3}^{[*]}(x,\lambda)$, $M_{l2}^{[*]}(x,\lambda)$, and $M_{l4}^{[*]}(x,\lambda)$ are analytic in $\mathbb{C}^-$. From (B.17b) it then follows that the blocks $1,2$ of

$M_r(x,\lambda)^{-1}$ and the blocks 3,4 of $M_l(x,\lambda)^{-1}$ are analytic in $\mathbb{C}^+$ and the blocks 3,4 of $M_r(x,\lambda)^{-1}$ and the blocks 1,2 of $M_l(x,\lambda)^{-1}$ are analytic in $\mathbb{C}^-$. In fact, these blocks of the inverse Faddeev matrices belong to the corresponding Wiener algebras. We therefore write

$$M_l(x,\lambda)^{-1} = I_{n+m} + \int_0^\infty d\alpha\, e^{-i\lambda J\alpha} C_l(x,\alpha), \tag{B.20a}$$

$$M_r(x,\lambda)^{-1} = I_{n+m} + \int_0^\infty d\alpha\, e^{i\lambda J\alpha} C_r(x,\alpha), \tag{B.20b}$$

where $\|C_l(x,\cdot)\|_1$ and $\|C_r(x,\cdot)\|_1$ are finite for $x \in \mathbb{R}$. The same result can be derived using Theorem 3.8. Equations (B.17) and (B.20) imply

$$B_l^{(*)}(x,\alpha) = JC_l(x,\alpha)^* J, \qquad B_r^{(*)}(x,\alpha) = JC_r(x,\alpha)^* J, \tag{B.21a}$$

$$B_l^{[*]}(x,\alpha) = C_l(x,\alpha)^*, \qquad B_r^{[*]}(x,\alpha) = C_r(x,\alpha)^*. \tag{B.21b}$$

We now define $n_+(x,\lambda)$ and $n_-(x,\lambda)$ as follows:

$$n_+(x,\lambda) = \begin{pmatrix} I_n & 0_{n\times m} \\ 0_{m\times n} & 0_{m\times} \end{pmatrix} M_r(x,\lambda)^{-1} + \begin{pmatrix} 0_{n\times n} & 0_{n\times m} \\ 0_{m\times n} & I_m \end{pmatrix} M_l(x,\lambda)^{-1}, \tag{B.22a}$$

$$n_-(x,\lambda) = \begin{pmatrix} I_n & 0_{n\times m} \\ 0_{m\times n} & 0_{m\times} \end{pmatrix} M_l(x,\lambda)^{-1} + \begin{pmatrix} 0_{n\times n} & 0_{n\times m} \\ 0_{m\times n} & I_m \end{pmatrix} M_r(x,\lambda)^{-1}. \tag{B.22b}$$

Then

$$n_+(x,\lambda) = I_{n+m} + \int_0^\infty d\alpha\, e^{i\lambda\alpha} c_+(x,\alpha), \tag{B.23a}$$

$$n_-(x,\lambda) = I_{n+m} + \int_0^\infty d\alpha\, e^{-i\lambda\alpha} c_-(x,\alpha). \tag{B.23b}$$

We then have the following analog of Theorem 4.7.

Theorem B.2 *Under the technical hypothesis, the Riemann-Hilbert problems*

$$n_-(x,\lambda) = H(x,\lambda) n_+(x,\lambda), \tag{B.24a}$$

$$n_+(x,\lambda) = \overline{H}(x,\lambda) n_-(x,\lambda), \tag{B.24b}$$

where $H(x,\lambda) = e^{i\lambda Jx} S(\lambda) e^{-i\lambda Jx}$ *and* $\overline{H}(x,\lambda) = e^{i\lambda Jx}\overline{S}(\lambda) e^{-i\lambda Jx}$*, hold true.*

Proof. In analogy with (B.22) we define

$$g_+(x,\lambda) = \begin{pmatrix} I_n & 0_{n\times m} \\ 0_{m\times n} & 0_{m\times} \end{pmatrix} F_l(x,\lambda)^{-1} + \begin{pmatrix} 0_{n\times n} & 0_{n\times m} \\ 0_{m\times n} & I_m \end{pmatrix} F_r(x,\lambda)^{-1},$$

$$g_-(x,\lambda) = \begin{pmatrix} I_n & 0_{n\times m} \\ 0_{m\times n} & 0_{m\times} \end{pmatrix} F_r(x,\lambda)^{-1} + \begin{pmatrix} 0_{n\times n} & 0_{n\times m} \\ 0_{m\times n} & I_m \end{pmatrix} F_l(x,\lambda)^{-1}.$$

Then (as in $F_{l,r}(x,\lambda)^{-1} = e^{-i\lambda Jx} M_{l,r}(x,\lambda)^{-1}$)

$$g_+(x,\lambda) = e^{-i\lambda Jx} n_+(x,\lambda), \qquad g_-(x,\lambda) = e^{-i\lambda Jx} n_-(x,\lambda).$$

Then it is easy to verify from the asymptotics as $x \to \pm\infty$ that

$$g_-(x,\lambda) = S(\lambda) g_+(x,\lambda), \qquad g_+(x,\lambda) = \overline{S}(\lambda) g_-(x,\lambda).$$

These two identities can then rewritten as (B.24). ■

For later use we write

$$N_l(x,\lambda) = M_l(x,\lambda)^{-1}, \qquad N_r(x,\lambda) = M_r(x,\lambda)^{-1}.$$

Then $N_{ls}(x,\lambda)$ and $N_{rs}(x,\lambda)$ ($s = 1,2,3,4$) are the blocks in the usual partitioning of $N_l(x,\lambda)$ and $N_r(x,\lambda)$. From (B.17) and (B.22) we easily derive the symmetry relations

$$n_+^{(*)}(x,\lambda) = J m_-(x,\overline{\lambda})^* J, \qquad n_+^{[*]}(x,\lambda) = m_-(x,\overline{\lambda})^*, \quad \lambda \in \overline{\mathbb{C}^+}, \tag{B.25a}$$

$$n_-^{(*)}(x,\lambda) = J m_+(x,\overline{\lambda})^* J, \qquad n_-^{[*]}(x,\lambda) = m_+(x,\overline{\lambda})^*, \quad \lambda \in \overline{\mathbb{C}^-}. \tag{B.25b}$$

Because of (B.19), the blocks of $N_l(x,\lambda)$ and $N_r(x,\lambda)$ coincide with the corresponding blocks of the cofactor matrices of $M_l(x,\lambda)$ and $M_r(x,\lambda)$. For $n = m = 1$ we find the obvious relations

$$\begin{aligned} N_{l1}(x,\lambda) &= M_{l4}(x,\lambda), & N_{l2}(x,\lambda) &= -M_{l2}(x,\lambda), \\ N_{l3}(x,\lambda) &= -M_{l3}(x,\lambda), & N_{l4}(x,\lambda) &= M_{l1}(x,\lambda), \\ N_{r1}(x,\lambda) &= M_{r4}(x,\lambda), & N_{r2}(x,\lambda) &= -M_{r2}(x,\lambda), \\ N_{r3}(x,\lambda) &= -M_{r3}(x,\lambda), & N_{r4}(x,\lambda) &= M_{r1}(x,\lambda). \end{aligned}$$

For $n = 1$ and $m \geq 2$ the only remaining elementary relations are as follows:

$$N_{l1}(x,\lambda) = \det M_{l4}(x,\lambda), \qquad N_{r1}(x,\lambda) = \det N_{r4}(x,\lambda).$$

4. Dual Marchenko integral equations. Mimicking the proof of Theorems 4.8 and 4.9, we now easily derive the "adjoint" Marchenko integral equations

$$C_{l1}(x,\alpha) = \int_0^\infty d\beta\, \Omega_l(\alpha+\beta+2x) C_{l3}(x,\beta), \tag{B.26a}$$

$$C_{l2}(x,\alpha) = \Omega_l(\alpha+2x) + \int_0^\infty d\beta\, \Omega_l(\alpha+\beta+2x) C_{l4}(x,\beta), \tag{B.26b}$$

$$C_{l3}(x,\alpha) = \overline{\Omega}_l(\alpha+2x) + \int_0^\infty d\beta\, \overline{\Omega}_l(\alpha+\beta+2x) C_{l1}(x,\beta), \tag{B.26c}$$

$$C_{l4}(x,\alpha) = \int_0^\infty d\beta\, \overline{\Omega}_l(\alpha+\beta+2x) C_{l2}(x,\beta), \tag{B.26d}$$

and

$$C_{r1}(x,\alpha) = \int_0^\infty d\beta\, \overline{\Omega}_r(\alpha+\beta-2x) C_{r3}(x,\beta), \tag{B.27a}$$

$$C_{r2}(x,\alpha) = \overline{\Omega}_r(\alpha-2x) + \int_0^\infty d\beta\, \overline{\Omega}_r(\alpha+\beta-2x) C_{r4}(x,\beta), \tag{B.27b}$$

$$C_{r3}(x,\alpha) = \Omega_r(\alpha-2x) + \int_0^\infty d\beta\, \Omega_r(\alpha+\beta-2x) C_{r1}(x,\beta), \tag{B.27c}$$

$$C_{r4}(x,\alpha) = \int_0^\infty d\beta\, \Omega_r(\alpha+\beta-2x) C_{r2}(x,\beta), \tag{B.27d}$$

in the absence of bound states.

5. Symmetries for the Marchenko integral kernels in the absence of bound states. The symmetry relations for the reflection coefficients as presented in the previous table lead to the following symmetry relations for the Marchenko integral kernels:

$$\Omega_l^{(*)}(x,\alpha) = \overline{\Omega}_l(x,\alpha)^*, \qquad \overline{\Omega}_l^{(*)}(x,\alpha) = \Omega_l(x,\alpha)^*, \tag{B.28a}$$

$$\Omega_r^{(*)}(x,\alpha) = \overline{\Omega}_r(x,\alpha)^*, \qquad \overline{\Omega}_r^{(*)}(x,\alpha) = \Omega_r(x,\alpha)^*, \tag{B.28b}$$

$$\Omega_l^{[*]}(x,\alpha) = -\overline{\Omega}_l(x,\alpha)^*, \qquad \overline{\Omega}_l^{[*]}(x,\alpha) = -\Omega_l(x,\alpha)^*, \tag{B.28c}$$

$$\Omega_r^{[*]}(x,\alpha) = -\overline{\Omega}_r(x,\alpha)^*, \qquad \overline{\Omega}_r^{[*]}(x,\alpha) = -\Omega_r(x,\alpha)^*. \tag{B.28d}$$

Here we make use of eqs. (4.24a)-(4.24d) to convert symmetry relations for the reflection coefficients into symmetry relations for their Fourier transforms. Thus in the **symmetric case** we have the symmetry relations

$$\Omega_l^{(*)}(x,\alpha) = \overline{\Omega}_l(x,\alpha)^*, \qquad \overline{\Omega}_l^{(*)}(x,\alpha) = \Omega_l(x,\alpha)^*, \tag{B.29a}$$

$$\Omega_r^{(*)}(x,\alpha) = \overline{\Omega}_r(x,\alpha)^*, \qquad \overline{\Omega}_r^{(*)}(x,\alpha) = \Omega_r(x,\alpha)^*, \tag{B.29b}$$

while in the **antisymmetric case** we have

$$\Omega_l^{[*]}(x,\alpha) = -\overline{\Omega}_l(x,\alpha)^*, \qquad \overline{\Omega}_l^{[*]}(x,\alpha) = -\Omega_l(x,\alpha)^*, \tag{B.30a}$$

$$\Omega_r^{[*]}(x,\alpha) = -\overline{\Omega}_r(x,\alpha)^*, \qquad \overline{\Omega}_r^{[*]}(x,\alpha) = -\Omega_r(x,\alpha)^*. \tag{B.30b}$$

B.4 Conjugation symmetries and bound states

So far we have derived the symmetry relations (B.28) only if there are no bound states, where we could just as well have applied the symmetry relations for the reflection coefficients. If there are bound states, the situation is far more complicated. We restrict ourselves to the case where all of the eigenvalues are algebraically simple.

Let us derive eqs. (B.26) and (B.27) if the poles of the transmission coefficients are simple. In analogy with (4.36) and (4.37) we first derive the identities

$$C_{l1}(x,\alpha) = -\sum_{j=1}^{N} e^{-\kappa_j\alpha}\tau_{lj0}N_{r1}^{j0}(x) + \int_0^\infty d\beta\, \hat{R}(\alpha+\beta+2x)C_{l3}(x,\beta),$$

$$C_{l2}(x,\alpha) = -\sum_{j=1}^{N} e^{-\kappa_j\alpha}\tau_{lj0}N_{r2}^{j0}(x) + \hat{R}(\alpha+2x) + \int_0^\infty d\beta\, \hat{R}(\alpha+\beta+2x)C_{l4}(x,\beta),$$

$$C_{r3}(x,\alpha) = -\sum_{j=1}^{N} e^{-\kappa_j\alpha}\tau_{rj0}N_{l3}^{j0}(x) + \hat{L}(\alpha-2x) + \int_0^\infty d\beta\, \hat{L}(\alpha+\beta-2x)C_{r1}(x,\beta),$$

$$C_{r4}(x,\alpha) = -\sum_{j=1}^{N} e^{-\kappa_j\alpha}\tau_{rj0}N_{l4}^{j0}(x) + \int_0^\infty d\beta\, \hat{L}(\alpha+\beta-2x)C_{r2}(x,\beta),$$

where $N_{ls}^{j0}(x) = N_{ls}(x,i\kappa_j)$ and $N_{rs}^{j0}(x) = N_{rs}(x,i\kappa_j)$ $(s = 1,2,3,4)$. We now replace eigensolutions of the matrix Zakharov-Shabat system by "left" eigensolutions which are row vector functions with entries in $L^2(\mathbb{R})$ satisfying (B.13). These left eigensolutions have either of the equivalent forms

$$\boldsymbol{\varepsilon}_j \begin{pmatrix} I_n & 0_{n\times m}\end{pmatrix} F_r(x,i\kappa_j)^{-1} = \boldsymbol{\eta}_j \begin{pmatrix} 0_{m\times n} & I_m\end{pmatrix} F_l(x,i\kappa_j)^{-1},$$

where $0 \neq \boldsymbol{\varepsilon}_j \in \mathbb{C}^{1\times n}$ and $0 \neq \boldsymbol{\eta}_j \in \mathbb{C}^{1\times m}$ satisfy

$$\boldsymbol{\varepsilon}_j a_{l1}(i\kappa_j) = 0_{1\times n}, \qquad \boldsymbol{\eta}_j a_{r4}(i\kappa_j) = 0_{1\times m}.$$

It is then clear that $\boldsymbol{\varepsilon}_j$ and $\boldsymbol{\eta}_j$ are exactly the row vectors $\boldsymbol{\xi}\tau_{lj0}$ and $\boldsymbol{\zeta}\tau_{rj0}$ for arbitrary nontrivial $\boldsymbol{\xi} \in \mathbb{C}^{1\times n}$ and $\boldsymbol{\zeta} \in \mathbb{C}^{1\times m}$, respectively.

In analogy with (4.44) we obtain

$$\begin{pmatrix} \tau_{lj0} & 0_{n\times m} \\ 0_{m\times n} & \tau_{rj0}\end{pmatrix} n_+(x,i\kappa_j) = \begin{pmatrix} 0_{n\times n} & e^{-2\kappa_j x}\Delta_{lj} \\ e^{2\kappa_j x}\Delta_{rj} & 0_{m\times m}\end{pmatrix} n_+(x,i\kappa_j), \tag{B.31}$$

where $n_+(x,i\kappa_j)$ is defined in the proof of Theorem B.2 and

$$\Delta_{lj} = \tau_{lj0}\boldsymbol{C}_{j0}, \qquad \Delta_{rj} = \tau_{rj0}\boldsymbol{D}_{j0},$$

for dependency constant matrices satisfying

$$\boldsymbol{\eta}_j = \boldsymbol{\varepsilon}_j\boldsymbol{C}_{j0}, \qquad \boldsymbol{\varepsilon}_j = \boldsymbol{\eta}_j\boldsymbol{D}_{j0},$$

which are each other's Moore-Penrose generalized inverses. Using (B.31) we then obtain the Marchenko integral equations (B.26a), (B.26b), (B.27c), and (B.27d), where

$$\Omega_l(\alpha) = \hat{R}(\alpha) + \sum_{j=1}^{N} \Delta_{lj}e^{-\kappa_j\alpha}, \tag{B.32a}$$

$$\Omega_r(\alpha) = \hat{L}(\alpha) + \sum_{j=1}^{N} \Delta_{rj}e^{-\kappa_j\alpha}. \tag{B.32b}$$

It remains to prove that we have found the same integral kernels as in (4.53).

In the same way we can depart from the left eigensolutions

$$\overline{\boldsymbol{\varepsilon}}_j \begin{pmatrix} I_n & 0_{n\times m} \end{pmatrix} F_l(x, -i\tilde{\kappa}_j)^{-1} = \overline{\boldsymbol{\eta}}_j \begin{pmatrix} 0_{m\times n} & I_m \end{pmatrix} F_r(x, -i\tilde{\kappa}_j)^{-1},$$

where $0 \neq \overline{\boldsymbol{\varepsilon}}_j \in \mathbb{C}^{1\times n}$ and $0 \neq \overline{\boldsymbol{\eta}}_j \in \mathbb{C}^{1\times m}$ satisfy

$$\overline{\boldsymbol{\varepsilon}}_j a_{r1}(-i\tilde{\kappa}_j) = 0_{1\times n}, \qquad \overline{\boldsymbol{\eta}}_j a_{l4}(-i\tilde{\kappa}_j) = 0_{1\times m}.$$

We then obtain

$$\begin{pmatrix} \tilde{\tau}_{lj0} & 0_{n\times m} \\ 0_{m\times n} & \tilde{\tau}_{rj0} \end{pmatrix} n_-(x, -i\tilde{\kappa}_j) = \begin{pmatrix} 0_{n\times n} & e^{-2\tilde{\kappa}_j x}\overline{\Delta}_{lj} \\ e^{2\tilde{\kappa}_j x}\overline{\Delta}_{rj} & 0_{m\times m} \end{pmatrix} n_-(x, -i\tilde{\kappa}_j), \tag{B.33}$$

where

$$\overline{\Delta}_{lj} = \tilde{\tau}_{lj0}\overline{\boldsymbol{C}}_{j0}, \qquad \overline{\Delta}_{rj} = \tilde{\tau}_{rj0}\overline{\boldsymbol{D}}_{j0},$$

for dependency constant matrices satisfying

$$\overline{\boldsymbol{\eta}}_j = \overline{\boldsymbol{\varepsilon}}_j\overline{\boldsymbol{C}}_{j0}, \qquad \overline{\boldsymbol{\varepsilon}}_j = \overline{\boldsymbol{\eta}}_j\overline{\boldsymbol{D}}_{j0},$$

which are each other's Moore-Penrose generalized inverses. Using (B.33) we obtain the Marchenko integral equations (B.27a), (B.27b), (B.26c), and (B.26d), where

$$\overline{\Omega}_l(\alpha) = \hat{\overline{R}}(\alpha) + \sum_{j=1}^{N} \overline{\Delta}_{lj} e^{-\kappa_j \alpha}, \tag{B.34a}$$

$$\overline{\Omega}_r(\alpha) = \hat{\overline{L}}(\alpha) + \sum_{j=1}^{N} \overline{\Delta}_{rj} e^{-\kappa_j \alpha}. \tag{B.34b}$$

It remains to prove that we have found the same integral kernels as in (4.55).

Proposition B.3 *Let the poles of the transmission coefficients be simple. Then*

$$\Delta_{lj} = \Gamma_{lj}, \qquad \Delta_{rj} = \Gamma_{rj}, \qquad \overline{\Delta}_{lj} = \overline{\Gamma}_{lj}, \qquad \overline{\Delta}_{rj} = \overline{\Gamma}_{rj}. \tag{B.35}$$

Moreover, under conjugation and anticonjugation we have the following symmetries:

$$\Gamma_{lj}^{[*]} = -\Gamma_{lj}^{(*)} = -\left(\overline{\Gamma}_{lj}\right)^*, \qquad \Gamma_{rj}^{[*]} = -\Gamma_{rj}^{(*)} = -\left(\overline{\Gamma}_{rj}\right)^*, \tag{B.36a}$$

$$\overline{\Gamma}_{lj}^{[*]} = -\overline{\Gamma}_{lj}^{(*)} = -\left(\Gamma_{lj}\right)^*, \qquad \overline{\Gamma}_{rj}^{[*]} = -\overline{\Gamma}_{rj}^{(*)} = -\left(\Gamma_{rj}\right)^*. \tag{B.36b}$$

Equation (B.35) implies the Marchenko equations (B.26) and (B.27), where the integral kernels are given by (B.32) and (B.34) and coincide with those defined by (4.53) and (4.55). As explained in Subsection 4.2.4, in the antisymmetric case these integral kernels satisfy the symmetry relations (B.30).

Proof. From linear algebra it is known that $\boldsymbol{\alpha}$ is a left eigenvector of the $p\times p$ matrix A at the eigenvalue λ (i.e., $\boldsymbol{\alpha}A = \lambda\boldsymbol{\alpha}$) if and only if $\boldsymbol{\alpha}^*$ is a right eigenvector of A^* at the eigenvalue $\overline{\lambda}$. As a result, $\boldsymbol{C}_{j0}^*$ and $\boldsymbol{D}_{j0}^*$ are each other's inverses between the ranges of τ_{lj0}^* and τ_{rj0}^*, $\overline{\boldsymbol{C}}_{j0}^*$ and $\overline{\boldsymbol{D}}_{j0}^*$ are each other's inverses between the ranges of $\tilde{\tau}_{lj0}^*$ and $\tilde{\tau}_{rj0}^*$, and the matrices extend pairwise by means of zero matrices to pairs of matrices that are each other's Moore-Penrose generalized inverses. A similar set of four matrices can be constructed by defining them by means of the intertwining relations

$$\tau_{lj0}\boldsymbol{C}_{j0} = D_{j0}\tau_{rj0}, \qquad \tau_{rj0}\boldsymbol{D}_{j0} = C_{j0}\tau_{lj0}, \tag{B.37a}$$

$$\tilde{\tau}_{lj0}\overline{\boldsymbol{C}}_{j0} = \overline{C}_{j0}\tilde{\tau}_{lj0}, \qquad \tilde{\tau}_{rj0}\overline{\boldsymbol{D}}_{j0} = \overline{D}_{j0}\tilde{\tau}_{rj0}, \tag{B.37b}$$

and extending them by zero to form pairs of matrices that are each other's Moore-Penrose generalized inverses. The uniqueness of such matrices then implies (B.35). Using (B.37) together with (B.35) we obtain (B.36), as claimed. ■

We now derive the main symmetry result.

Theorem B.4 *Let the poles of the transmission coefficients be simple. Then the Marchenko integral kernels satisfy the symmetry relations* (B.28)*.*

Proof. Equation (B.35) implies the Marchenko equations (B.26) and (B.27), where the integral kernels are given by (B.32) and (B.34) and coincide with those defined by (4.53) and (4.55). Equations (B.36) and the symmetry relations for the reflection coefficients then imply (B.28). ■

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