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Abstract

The Cantor–Bernstein–Schröder theorem (CBS-theorem for short) of set theory was generalized by Sikorski and Tarski to σ-complete Boolean algebras. After this, several generalizations of the CBS-theorem, extending the Sikorski–Tarski version to different classes of algebras, have been established. Among these classes there are lattice ordered groups, orthomodular lattices, MV-algebras, residuated lattices, etc. This suggests to consider a common algebraic framework in which the algebraic versions of the CBS-theorem can be formulated. In this work we provide this framework establishing necessary and sufficient conditions for the validity of the theorem. We also show how this abstract framework includes the versions of the CBS-theorem already present in the literature as well as new versions of the theorem extended to other classes such as groups, modules, semigroups, rings, $*$-rings etc.

Keywords (separated by '-'): Cantor–Bernstein–Schröder theorem - Congruence lattice - Factor congruences

Mathematics Subject Classification (separated by '-'): 03G10 - 06C15

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Algebra Universalis

The Cantor–Bernstein–Schröder theorem via universal algebra

Hector Freytes

In memoriam: Roberto Cignoli (1937–2018).

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1. Introduction

The famous Cantor–Bernstein–Schröder theorem of the set theory states that

> *"if a set X can be embedded into a set Y and vice versa, then there is a one-to-one function of X onto Y".*

The history of this theorem is rather curious. The earliest record of the theorem might be a letter to Dedekind dated 5 november 1882 where Cantor conjectured the theorem. Dedekind proved it in 1887 but did not publish it. His proof was printed only in his collected works in 1932. Schröder proved the theorem in 1894 but he published it in 1898 [39,40]. However Schröder's proof was defective. Korselt wrote to Schröder about the error in 1902 and few weeks later he sent a proof of the theorem to the Mathematische Annalen. Korselt paper appeared in 1911 [32]. Bernstein, a 19 years old Cantor student, proved the theorem. His proof found its way to the public through Borel because Cantor showed the proof to Borel in the 1897 during the International Congress of Mathematicians in Zürich. The Bernstein proof was published in 1898 in the appendix of a Borel book [6] and in 1901 Bernstein's thesis appeared with his proof. Several years later, at the end of the forties, Sikorski [38] and independently Tarski [44], showed that the CBS-theorem is a particular case of a statement on σ-complete Boolean algebras. Following this idea, several authors have extended the Sikorski–Tarski version to classes of algebras more general than Boolean algebras. Among these classes there are lattice ordered groups [26], MV-algebras [12,14,24], orthomodular lattices [13], effect algebras [27], pseudo effect algebras [16], pseudo MV-algebras [25], pseudo BCK-algebras [34] and in general, algebras with an underlying lattice structure such that the central elements of this lattice determine a direct decomposition of the algebra [18]. It suggests that the CBS-theorem can be formulated in a common algebraic framework from which all the versions of the theorem mentioned above stem.

In the present work we provide this general algebraic framework for the CBS-theorem. It consists of a category $\mathcal{A}$ of algebras of the same type and a presheaf, called *congruences presheaf*, acting on the congruence lattice of each algebra of the category $\mathcal{A}$.

In this perspective each congruences presheaf determinates a CBS type theorem formulated in terms of the quotient algebras related to the congruences involving by the presheaf. Moreover, conditions for the validity of the CBS-theorem may be established in terms of properties that certain algebras in $\mathcal{A}$ should satisfy with respect to the congruence presheaf. This framework also yields new versions of the CBS-theorem, applied to several algebraic structures.

The paper is structured as follows. Section 2 contains generalities on lattice theory, universal algebra and some technical results that are used in subsequent sections. In Section 3 the crucial notion of congruences presheaf is introduced and the abstract framework for the CBS-theorem is provided. Quasi-cyclic groups are studied as an example of algebras satisfying the CBS-theorem. In Section 4 a congruences presheaf related to factor congruences is introduced and a CBS-theorem with respect to this special presheaf is established. A necessary and sufficient condition for the validity of the CBS-theorem is given. Injective modules and divisible groups are studied as examples of algebras satisfying the CBS-theorem. A useful necessary and sufficient condition for the validity of the CBS-theorem, restricted to this particular congruences presheaf, is also provided. In Section 5 our abstract version of the CBS-theorem is studied in categories of algebras having Boolean factor congruences (BFC). This particular framework allows us to consider versions of the theorem extended to algebras with an underlying lattice structure as lattice ordered groups, orthomodular lattices, residuated lattices, Łukasiewicz and Post algebras, semigroups with $0, 1$, bounded semilattices, commutative pseudo BCK-algebras, rings with unity, $*$-rings etc. Finally, we extend our abstract framework to two categories of algebras defined by partial operations.

2. Basic notions

We recall from [4,7,35] some basic notions about lattice theory and universal algebra that play an important role in what follows. Let $\langle L, \leq \rangle$ be an ordered set. An interval $[a, b]_L$ of L is defined as the set $\{x \in A : a \leq x \leq b\}$. The ordered set L is called *bounded* if it has a smallest element 0 and a greatest element 1. Let L be a bounded ordered set. A subset X of L is *orthogonal* (*dual orthogonal*) if and only if $x \wedge y = 0$ ($x \vee y = 1$) whenever x, y are distinct elements of X.

Let $\langle L, \vee, \wedge \rangle$ be a lattice. If $a \leq b$ in L then $\langle [a, b]_L, \vee, \wedge, a, b \rangle$ is a bounded lattice. Given a, b, c in L, we write: $(a, b, c)D$ if and only if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ and $(a, b, c)D^*$ if and only if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$. Further, we write $(a, b, c)T$ if and only if $(a, b, c)D$ and $(a, b, c)D^*$ hold for all permutations of a, b, c. An element z of the lattice L is called a *neutral element* if and only if for all elements $a, b \in L$ we have $(a, b, z)T$. The lattice L is σ*-complete* if and only if L admits denumerable supremum and denumerable infimum. In particular, L is said to be *orthogonal* σ*-complete* (*dual orthogonal* σ*-complete*) if and only if every denumerable orthogonal (dual orthogonal) subset of L has supremum (infimum) in L.

Let $\langle L, \vee, \wedge, 0, 1 \rangle$ be a bounded lattice. A *complement* of an element $a \in L$ is an element $\neg a \in L$ such that $a \vee \neg a = 1$ and $a \wedge \neg a = 0$. The lattice L is called *complemented* when every element of L has a complement. In particular, L is a *Boolean algebra* if and only if it is a complemented distributive lattice. If L is a Boolean algebra then every element in L has a unique complement. Let $\langle L, \vee, \wedge, 0, 1 \rangle$ be a bounded lattice. An element $z \in L$ is called a *central element* if and only if z is a neutral element having a complement. The set of all central elements of L is called the *center* of L and it is denoted by $Z(L)$. The center $Z(L)$ is a Boolean sublattice of L [35, Theorem 4.15].

Proposition 2.1 [18, Proposition 3.1]. *Let L be a bounded lattice and $z \in Z(L)$. Then*

(1) $Z(L) \cap [z, 1]_L = Z([z, 1]_L)$.

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(2) *If* $x \in Z([z,1]_L)$ *and* $\neg x$ *is the complement of* x *in* $Z(L)$ *then the complement of* x *relative to* $[z,1]_L$ *is* $\neg_z x = z \vee \neg x$.
(3) $\langle Z([z,1]_L), \vee, \wedge, \neg_z, z, 1\rangle$ *is a Boolean Algebra.*

Proposition 2.2. *Let* A *be Boolean algebra. Then* A *is orthogonal (dual orthogonal)* σ*-complete if and only if* A *is* σ*-complete.*

Proof. Suppose that A is an orthogonal σ-complete Boolean algebra and let $(x_i)_{i\in\mathbb{N}}$ be a denumerable set in A. Let us consider the sequence $(t_i)_{i\in\mathbb{N}}$ such that $t_1 = x_1$, $t_2 = \neg x_1 \wedge x_2$ and, in general, $t_n = \bigwedge_{i=1}^{n-1} \neg x_i \wedge x_n$. Note that $(t_i)_{i\in\mathbb{N}}$ is an orthogonal set then, by hypothesis, there exists the supremum $t = \bigvee_{i\in\mathbb{N}} t_i$. We will show that $t = \bigvee_{i\in\mathbb{N}} x_i$.

We first prove, by induction, that for each $n \in \mathbb{N}$, $\bigvee_{i=1}^{n} x_i = \bigvee_{i=1}^{n} t_i$. If $n = 2$ then $t_1 \vee t_2 = x_1 \vee (\neg x_1 \wedge x_2) = x_1 \vee x_2$. Let us assume that $\bigvee_{i=1}^{n-1} t_i = \bigvee_{i=1}^{n-1} x_i$. Then

$$\bigvee_{i=1}^{n} t_i = \bigvee_{i=1}^{n-1} t_i \vee t_n = \bigvee_{i=1}^{n-1} x_i \vee \left(\bigwedge_{i=1}^{n-1} \neg x_i \wedge x_n\right)$$
$$= \bigvee_{i=1}^{n-1} x_i \vee \left(\left(\neg \bigvee_{i=1}^{n-1} x_i\right) \wedge x_n\right) = \bigvee_{i=1}^{n} x_i.$$

By the above result we can see that for each $n \in \mathbb{N}$,

$$x_n \leq \bigvee_{i=1}^{n} x_i = \bigvee_{i=1}^{n} t_i \leq t.$$

Therefore t is an upper bound of the set $(x_i)_{i\in\mathbb{N}}$. Let M be an upper bound of the set $(x_i)_{i\in\mathbb{N}}$. Then for each $n \in \mathbb{N}$, $\bigvee_{i=1}^{n} t_i = \bigvee_{i=1}^{n} x_i \leq M$ and then $t = \bigvee_{i\in\mathbb{N}} t_i \leq M$. It proves that $t = \bigvee_{i\in\mathbb{N}} x_i$. Hence A is a σ-complete Boolean algebra. By the dual argument we can prove that dual orthogonal σ-completeness also implies σ-completeness.

The other direction of the proof is trivial. □

Let τ be a type of algebras and X be a denumerable set of variables such that $\tau \cap X = \emptyset$. We denote by $\mathrm{Term}_\tau(X)$ the set of terms built from the set of variables X. Each element $t \in \mathrm{Term}_\tau(X)$ is referred as a τ-*term*. For a τ-term t we often write $t(x_1, x_2, \ldots, x_n)$ to indicate that the variables occurring in t are among $x_1, x_2, \ldots, x_n$. If $t \in \mathrm{Term}_\tau(X)$ and A is an algebra of type τ then we denote by t^A the interpretation of t in the algebra A. A τ-*homomorphism* is a function between algebras of type τ that preserves the τ-operations. We write $A \cong_\tau B$ to indicate that there exists a τ-isomorphism between the algebras A and B of type τ. An *equation* of type τ is an expression of the form $s = t$ such that $s, t \in \mathrm{Term}_\tau(X)$ and the symbol $=$ is interpreted as the identity. A *quasi equation* is an expression of the form $(\&_{i=1}^{n} s_i = t_i) \Longrightarrow s = t$ where $t_i, s_i, s, t \in \mathrm{Term}_\tau(X)$ and $\&_{i=1}^{n}$ denotes a logical n-conjunction.

Let $\mathcal{A}$ be a class of algebras of type τ. The *language of* $\mathcal{A}$ is the first order language with identity built from the set $\mathrm{Term}_\tau(X)$. If Φ is a sentence in the language of $\mathcal{A}$ and $A \in \mathcal{A}$ then $A \models \Phi$ means that Φ *holds in* A. The sentence

Φ *holds in the class* $\mathcal{A}$, abbreviated as $\mathcal{A} \models \Phi$, if and only if for each $A \in \mathcal{A}$, $A \models \Phi$. If Σ is a set of sentences in the language of $\mathcal{A}$ then $A \models \Sigma$ means that $A \models \Phi$ for each $\Phi \in \Sigma$. The class $\mathcal{A}$ is a *variety* (*quasivariety*) if and only if there exists a set Σ of equations (quasi equations) in the language of $\mathcal{A}$ such that $\mathcal{A} = \{A : A \models \Sigma\}$. Equivalently, $\mathcal{A}$ is a variety if and only if it is closed under homomorphic images, subalgebras and direct products. The class $\mathcal{A}$ is a quasivariety if and only if $\mathcal{A}$ contains a trivial algebra and it is closed under subalgebras, isomorphisms, direct products and ultraproducts. Let us notice that a quasivariety is not necessarily closed under homomorphic images.

Let A be an algebra of type τ. We denote by $\mathrm{Con}(A)$ the congruence lattice of A. The largest congruence on A, given by A^2, is denoted by ∇_A and the smallest one, given by the diagonal $\{(a,a) : a \in A\}$, is denoted by Δ_A. If $f\colon A \to B$ is a τ-homomorphism then the *kernel congruence of* f (i.e. the congruence $\{(x,y) \in A^2 : f(x) = f(y)\}$) is denoted by $\ker(f)$. For $a \in A$ and $\theta \in \mathrm{Con}(A)$, $a_{/\theta}$ denotes the congruence class of a modulo θ. Let $\theta_1, \theta_2 \in \mathrm{Con}(A)$. Then we say that θ_1, θ_2 are *permutable* if and only if $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$ where $\circ$ is the relational product defined as $\theta_1 \circ \theta_2 = \{(x,y) \in A^2 : \exists w \in A, \text{with } (x,w) \in \theta_1 \text{ and } (w,y) \in \theta_2\}$. In [7, Theorem 5.9] it is proved that the congruences θ_1, θ_2 are permutable if and only if $\theta_1 \vee \theta_2 = \theta_1 \circ \theta_2$. Let $\sigma \in \mathrm{Con}(A)$. If $\theta \in [\sigma, \nabla_A]_{\mathrm{Con}(A)}$ then

$$\theta/\sigma = \{(x_{/\sigma}, y_{/\sigma}) \in (A/\sigma)^2 : (x,y) \in \theta\} \tag{2.1}$$

is a congruence on A/σ. The following theorem plays an important role in the next sections:

Theorem 2.3. *Let A be an algebra of type τ and $\sigma \in \mathrm{Con}(A)$. Then*

1. *If $\sigma \subseteq \theta$ then $f\colon (A/\sigma)/(\theta/\sigma) \to (A/\theta)$ such that $f((a_{/\sigma})_{/(\theta/\sigma)}) = a_{/\theta}$ is a τ-isomorphism.*
2. *$u_\sigma\colon [\sigma, \nabla_A]_{\mathrm{Con}(A)} \to \mathrm{Con}(A/\sigma)$ such that $u_\sigma(\theta) = \theta/\sigma$ is a lattice isomorphism.*
3. *If $\sigma \subseteq \theta_1$ and $\sigma \subseteq \theta_2$ then $(a,b) \in \theta_1 \circ \theta_2$ if and only if $(a_{/\sigma}, b_{/\sigma}) \in \theta_1/\sigma \circ \theta_2/\sigma$.*

Proof. (1) See [7, Theorem 6.15]. (2) See [7, Theorem 6.20]. (3) $(a,b) \in \theta_1 \circ \theta_2$ if and only if there exists $c \in A$ such that $(a,c) \in \theta_1$ and $(c,b) \in \theta_2$ if and only if $(a_{/\sigma}, c_{/\sigma}) \in \theta_1/\sigma$ and $(c_{/\sigma}, b_{/\sigma}) \in \theta_2/\sigma$ if and only if $(a_{/\sigma}, b_{/\sigma}) \in \theta_1/\sigma \circ \theta_2/\sigma$. □

A congruence θ on A is a *factor congruence* if and only if there exists $\neg\theta \in \mathrm{Con}(A)$, called a *factor complement of* θ, such that $\theta \cap \neg\theta = \Delta_A$, $\theta \vee \neg\theta = \nabla_A$ and θ permutes with $\neg\theta$ (or equivalently, by [7, Theorem 5.9], $\theta \cap \neg\theta = \Delta_A$ and $\theta \circ \neg\theta = \nabla_A$). In this case A is τ-isomorphic to $A/\theta \times A/\neg\theta$. The pair $(\theta, \neg\theta)$ is called a *pair of factor congruences*. We denote by $\mathrm{FC}(A)$ the set of factor congruences on A.

Proposition 2.4. *Let A be an algebra of type τ, $\sigma \in \mathrm{FC}(A)$ and a congruence $\theta \in [\sigma, \nabla_A]_{\mathrm{Con}(A)}$ such that $\theta/\sigma \in \mathrm{FC}(A/\sigma)$. Then $\theta \in \mathrm{FC}(A)$.*

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Proof. Let us suppose that $(\sigma, \neg\sigma)$ is a pair of factor congruences in $\mathrm{FC}(A)$ and $(\theta/\sigma, \neg(\theta/\sigma))$ is a pair of factor congruences in $\mathrm{FC}(A/\sigma)$. Then, by Theorem 2.3(1), we have that

$$\begin{aligned} A &\cong_\tau A/\sigma \times A/\neg\sigma \cong_\tau ((A/\sigma)/(\theta/\sigma) \times (A/\sigma)/\neg(\theta/\sigma)) \times A/\neg\sigma \\ &\cong_\tau A/\theta \times B \end{aligned}$$

where $B = (A/\sigma)/\neg(\theta/\sigma) \times A/\neg\sigma$. Consider the diagram $A \xrightarrow{f} A/\theta \times B \xrightarrow{\pi_B} B$ where f is a τ-isomorphism. Then $(\theta, \ker(\pi_B f))$ is a pair of factor congruences on A proving that $\theta \in \mathrm{FC}(A)$. □

Proposition 2.5. *Let A be an algebra of type τ and let us consider the denumerable direct product $B = \prod_{\mathbb{N}} A$. Then there exists $\sigma \in \mathrm{FC}(B)$ such that $B \cong_\tau B/\sigma$.*

Proof. If we consider $B = A \times A_{_{\mathbb{N}-\{1\}}}$ where $A_{_{\mathbb{N}-\{1\}}} = \prod_{i \in \mathbb{N}-\{1\}} A$ then $f \colon B \to A_{_{\mathbb{N}-\{1\}}}$, defined by $B \ni (b_i)_{i\in\mathbb{N}} \mapsto f((b_i)_{i\in\mathbb{N}}) = (a_i)_{i\geq 2}$ where $a_2 = b_1$; $a_3 = b_2$; $\ldots$ $a_{n+1} = b_n$; $\ldots$, is a τ-isomorphism. Thus, by considering $\sigma = \ker(\pi_{A_{_{\mathbb{N}-\{1\}}}})$, we have that $\sigma \in \mathrm{FC}(B)$ and $B \cong_\tau B/\sigma$. □

Definition 2.6. A *category of algebras* is a category $\mathcal{A}$ whose objects are algebras of type τ and whose arrows are the τ-homomorphisms (also called $\mathcal{A}$-homomorphisms) $f \colon A \to B$ such that A, B are objects of $\mathcal{A}$.

Let $\mathcal{A}$ be a category of algebras. We denote by $Ob(\mathcal{A})$ the class of objects of $\mathcal{A}$ and by $Hom_{\mathcal{A}}$ the set of all $\mathcal{A}$-homomorphisms. For the sake of simplicity if A is an object of $\mathcal{A}$ then we write $A \in \mathcal{A}$ when there is no confusion. If two objects $A, B \in \mathcal{A}$ are τ-isomorphic, i.e. there exists a bijective map between A and B that preserves τ-operations, then we denote this fact by $A \cong_{_{\mathcal{A}}} B$. Note that if $\mathcal{A}$ is a class of algebras of type τ then we can identify $\mathcal{A}$ with a category of algebras by considering the τ-homomorphisms between algebras of $\mathcal{A}$ as arrows of $\mathcal{A}$. In this sense varieties and quasivarieties can be seen as categories of algebras. A *presheaf* on a category $\mathcal{C}$ is a functor $\mathcal{F} \colon \mathcal{C}^{op} \to Set$ where $\mathcal{C}^{op}$ is the dual category of $\mathcal{C}$ and Set is the category of all sets.

3. Presheaf approach to the CBS-theorem

In this section we provide an abstract formulation of the CBS-theorem that captures the numerous algebraic versions of the theorem present in the literature. With this aim, we first analyze the Sikorski–Tarski version of the theorem focusing our attention on the congruence lattice of a Boolean algebra.

Let A be a Boolean algebra and $z \in A$. Then, by Proposition 2.1, we have that $\langle [z, 1]_A, \vee, \wedge, \neg_z, z, 1\rangle$ is a Boolean algebra. In this way, the Sikorski–Tarski version of the CBS-theorem reads as follows:

Theorem 3.1. *Let A and B be σ-complete Boolean algebras, $a \in A$, and $b \in B$. If A is Boolean-isomorphic to $[b, 1]_B$ and B is Boolean-isomorphic to $[a, 1]_A$, then A is Boolean-isomorphic to B.*

Clearly, to obtain the classical CBS-theorem it is sufficient to assume that A and B are the power sets of two sets endowed with the natural set-theoretic Boolean operations. Let us notice that the Boolean algebras $[a, 1]_A$ and $[b, 1]_B$ are isomorphic to the quotient algebras A/θ_a and B/θ_b respectively, where $\theta_a = \{(x, y) \in A^2 : x \vee a = y \vee a\} \in \mathrm{FC}(A)$ and $\theta_b = \{(x, y) \in B^2 : x \vee b = y \vee b\} \in \mathrm{FC}(B)$. Consequently, the hypothesis of σ-completeness in A and B can be equivalently expressed as σ-completeness conditions in $\mathrm{FC}(A)$ and $\mathrm{FC}(B)$ respectively. In this context we can also notice that the conditions for the validity for CBS-theorem, extended to different classes of algebras [12, 13, 16, 18, 24, 25, 26, 27, 34], can be expressed in terms of σ-completeness type conditions related to the set of factor congruences of the algebras.

Following this idea and in order to establish a general algebraic version of CBS-theorem, our abstract framework for the CBS-theorem will consist on a category of algebras $\mathcal{A}$ where for each $A \in \mathcal{A}$, instead of the set of factor congruences, a subset $\mathcal{K}(A) \subseteq \mathrm{Con}(A)$ is considered. The set $\mathcal{K}(A)$ will be uniformly determined in each algebra $A \in \mathcal{A}$ through a presheaf. In this perspective, in Sections 4 and 5 where the particular case $\mathcal{K} = \mathrm{FC}(A)$ is studied, we will show how order-theoretic properties imposed on the set $\mathcal{K}(A)$ allow us to establish conditions for the validity of the CBS-theorem formulated in this abstract framework. In this way our abstract framework captures the already known algebraic versions of the CBS-theorem.

The use of a presheaf defining the set $\mathcal{K}(A) \subseteq \mathrm{Con}(A)$ in each $A \in \mathcal{A}$ is very useful due to its contravariant character. Indeed, since our abstract formulation of the CBS-theorem will be established in terms of properties related to a set of congruences of an algebra then it will be necessary to express properties about homomorphic images of an algebra $A \in \mathcal{A}$ in terms of properties related to congruences that define the mentioned homomorphic images. This task is performed by the presheaf $\mathcal{K}$ introduced in Definition 3.6. In particular, for each $\mathcal{A}$-homomorphism $f \colon A \to B$, the application $\mathcal{K}(f) \colon \mathcal{K}(B) \to \mathcal{K}(A)$ will be an order preserving map defined in terms of the function f^* introduced below.

Let A, B two algebras of type τ and $f \colon A \to B$ be a τ-homomorphism. Then we define the following sets:

$$f^*(\theta) = \{(a, b) \in A^2 : (f(a), f(b)) \in \theta\}, \text{ for each } \theta \in \mathrm{Con}(B). \quad (3.1)$$

$$f_*(\theta) = \{(f(a), f(b)) \in B^2 : (a, b) \in \theta\}, \text{ for each } \theta \in \mathrm{Con}(A). \quad (3.2)$$

Proposition 3.2. *Let A, B be two algebras of type τ and $f \colon A \to B$ be a τ-homomorphism. Then we have:*

1. *The assignment $\mathrm{Con}(B) \ni \theta \mapsto f^*(\theta)$ defines an order homomorphism $f^* \colon \mathrm{Con}(B) \to \mathrm{Con}(A)$.*
2. *$(gf)^* = f^* g^*$ whenever the composition of τ-homomorphisms gf is defined.*
3. *$1^*_A = 1_{\mathrm{Con}(A)}$.*
4. *If f is a τ-isomorphism then the assignment $\mathrm{Con}(A) \ni \theta \mapsto f_*(\theta)$ defines an order isomorphism $f_* \colon \mathrm{Con}(A) \to \mathrm{Con}(B)$ and $f_* = (f^*)^{-1} = (f^{-1})^*$.*

Moreover, $f'\colon A/\theta \to B/f_*(\theta)$ *such that* $f'(x_{/\theta}) = f(x)_{/f_*(\theta)}$ *is a* τ*-isomorphism.*

(5) *If* f *is a* τ*-isomorphism and* $\theta_1, \theta_2 \in \mathrm{Con}(A)$ *are permutable then* $f_*(\theta_1)$, $f_*(\theta_2)$ *are permutable in* $\mathrm{Con}(B)$.

Proof. (1) Straightforward calculation.

(2) Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a composition of τ-homomorphisms. Consider the diagram $\mathrm{Con}(A) \xleftarrow{f^*} \mathrm{Con}(B) \xleftarrow{g^*} \mathrm{Con}(C)$. If $\theta \in \mathrm{Con}(C)$ then $f^*g^*(\theta) = \{(x,y) \in A^2 : (f(a), f(b)) \in g^*(\theta)\} = \{(x,y) \in A^2 : (gf(a), gf(b)) \in \theta\} = (gf)^*(\theta)$. Hence $(gf)^* = f^*g^*$.

(3) Immediate.

(4) Let us assume that f is a τ-isomorphism. Then f_* defines a bijective function $f_*\colon \mathrm{Con}(A) \to \mathrm{Con}(B)$. We first prove that $f^*f_* = 1_{\mathrm{Con}(A)}$. Let $\theta \in \mathrm{Con}(A)$. Then, $(x,y) \in f^*f_*(\theta)$ if and only if $(f(x), f(y)) \in f_*(\theta)$ if and only if $(x,y) \in \theta$. Therefore $f^*f_* = 1_{\mathrm{Con}(A)}$. Now we prove that $f_*f^* = 1_{\mathrm{Con}(B)}$. Let $\theta \in \mathrm{Con}(B)$. Then $(x,y) \in f_*f^*(\theta)$ if and only if there exists $(x_0, y_0) \in f^*(\theta)$ such that $f(x_0) = x$ and $f(y_0) = y$. Since $(x_0, y_0) \in f^*(\theta)$ if and only if $(x,y) = (f(x_0), f(y_0)) \in \theta$ then we have that $f_*f^* = 1_{\mathrm{Con}(B)}$. Thus $f_* = (f^*)^{-1}$.

Let f^{-1} be the inverse of f and $\theta \in \mathrm{Con}(A)$. Then, $(x,y) \in (f^{-1})^*(\theta) \subseteq B^2$ if and only if $(f^{-1}(x), f^{-1}(y)) \in \theta$ if and only if $(ff^{-1}(x), ff^{-1}(y)) \in f_*(\theta)$ if and only if $(x,y) \in f_*(\theta)$. It proves that $f_* = (f^*)^{-1} = (f^{-1})^*$.

Now we prove that f_* is an order preserving function. Suppose that $\theta_1 \subseteq \theta_2$ in $\mathrm{Con}(A)$. Let $(c,d) \in f_*(\theta_1)$. Then $(f^{-1}(c), f^{-1}(d)) \in \theta_1 \subseteq \theta_2$ and $(c,d) \in f_*(\theta_2)$. Hence $f_*(\theta_1) \subseteq f_*(\theta_2)$ and f_* is an order isomorphism from $\mathrm{Con}(A)$ onto $\mathrm{Con}(B)$.

We first prove that f' is well defined. If $x_{/\theta} = y_{/\theta}$ then we have that $(x,y) \in \theta$, $(f(x), f(y)) \in f_*(\theta)$ and $f'(x_{/\theta}) = f(x)_{/f_*(\theta)} = f(y)_{/f_*(\theta)} = f'(y_{/\theta})$. Thus, f' is well defined. If $f'(x_{/\theta}) = f'(y_{/\theta})$ then $(f(x), f(y)) \in f_*(\theta)$ and $(x,y) \in \theta$. Thus, $x_{/\theta} = y_{/\theta}$ and f' is injective. Now we prove that f' is surjective. Let $y_{/f_*(\theta)} \in B/f_*(\theta)$. Since f is surjective then there exists $x \in A$ such that $f(x) = y$. Thus, $y_{/f_*(\theta)} = f(x)_{/f_*(\theta)} = f'(x_{/\theta})$ and f' is surjective.

Let $t(x_1 \ldots x_n) \in \mathrm{Term}_\tau(X)$. Then for $a_1 \ldots a_n \in A$ we have that:

$$\begin{aligned}
f'(t^{A/\theta}(a_{1/\theta}, \ldots, a_{n/\theta})) &= f'(t^A(a_1, \ldots, a_n)_{/\theta}) \\
&= f(t^A(a_1, \ldots, a_n))_{/f_*(\theta)} \\
&= t^B(f(a_1), \ldots, f(a_n))_{/f_*(\theta)} \\
&= t^{B/f_*(\theta)}(f(a_1)_{/f_*(\theta)}, \ldots, f(a_n)_{/f_*(\theta)}) \\
&= t^{B/f_*(\theta)}(f'(a_{1/\theta}), \ldots, f'(a_{n/\theta})).
\end{aligned}$$

It proves that f' preserves τ-operations. Hence, f' is a τ-isomorphism.

(5) Let us assume that $\theta_1, \theta_2 \in \mathrm{Con}(A)$ are permutable. Since f is a τ-isomorphism, each pair in $f_*(\theta_1) \circ f_*(\theta_2)$ has the form $(f(x), f(y))$ where $x, y \in A$. Suppose that $(f(x), f(y)) \in f_*(\theta_1) \circ f_*(\theta_2)$. Then, by definition of relational product, there exists $w \in A$ such that $(f(x), f(w)) \in f(\theta_1)$ and

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$(f(w), f(y)) \in f(\theta_2)$. Thus $(x, w) \in \theta_1$, $(w, y) \in \theta_2$ and $(x, y) \in \theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$. It implies that there exists $v \in A$ such that $(x, v) \in \theta_2$ and $(v, x) \in \theta_1$; consequently $(f(x), f(v)) \in f_*(\theta_2)$ and $(f(v), f(x)) \in f_*(\theta_1)$. Therefore, $(f(x), f(y)) \in f_*(\theta_2) \circ f_*(\theta_1)$ and then $f_*(\theta_1)$, $f_*(\theta_2)$ are permutable. □

Proposition 3.3. *Let A be an algebra, $\sigma \in \mathrm{Con}(A)$ and the order isomorphism $u_\sigma \colon [\sigma, \nabla_A]_{\mathrm{Con}(A)} \to \mathrm{Con}(A/\sigma)$ given by $u(\theta) = \theta/\sigma$. If $p \colon A \to A/\sigma$ is the natural homomorphism then $p^* = u_\sigma^{-1}$.*

Proof. Let $\theta \in [\sigma, \nabla_A]_{\mathrm{Con}(A)}$. Then, by Eq. (2.1), we have that

$$\begin{aligned} p^*(\theta/\sigma) &= \{(x, y) \in A^2 : (p(x), p(y)) \in \theta/\sigma\} \\ &= \{(x, y) \in A^2 : (x_{/\sigma}, y_{/\sigma}) \in \theta/\sigma\} \\ &= \{(x, y) \in A^2 : (x, y) \in \theta\} = \theta = u^{-1}(\theta/\sigma). \end{aligned}$$

Hence our claim. □

Definition 3.4. Let $\mathcal{A}$ be a category of algebras. A *congruences operator* over $\mathcal{A}$ is a class operator of the form $\mathcal{A} \ni A \mapsto \mathcal{K}(A) \subseteq \mathrm{Con}(A)$ such that,

(1) $\Delta_A \in \mathcal{K}(A)$.
(2) For each $\sigma \in \mathcal{K}(A)$, $A/\sigma \in \mathcal{A}$.
(3) If $f \colon A \to B$ is a $\mathcal{A}$-isomorphism then the restriction $f^*\restriction_{\mathcal{K}(B)} \colon \mathcal{K}(B) \to \mathcal{K}(A)$ is an order isomorphism.

Proposition 3.5. *Let $\mathcal{A}$ be a category of algebras and $\mathcal{K}$ be a congruences operator over $\mathcal{A}$. Let us define the class*

$$Hom_{\mathcal{A}_{\mathcal{K}}} = \{A \xrightarrow{f} B \in Hom_{\mathcal{A}} : f \text{ is surjective and } \ker(f) \in \mathcal{K}(A)\}. \tag{3.3}$$

Then the following statements are equivalent:

(1) *$\mathcal{A}_{\mathcal{K}} = \langle Ob(\mathcal{A}), Hom_{\mathcal{A}_{\mathcal{K}}}\rangle$ is a category and, by defining $\mathcal{K}(f) = f^*\restriction_{\mathcal{K}(B)}$ for each $A \xrightarrow{f} B \in Hom_{\mathcal{A}_{\mathcal{K}}}$, $\mathcal{K} \colon \mathcal{A}_{\mathcal{K}} \to Set$ is a presheaf.*
(2) *For each $A \in \mathcal{A}$ and $\sigma \in \mathcal{K}(A)$, if $p \colon A \to A/\sigma$ is the natural $\mathcal{A}$-homomorphism then the restriction $p^*\restriction_{\mathcal{K}(A/\sigma)}$ is an order isomorphism from $\mathcal{K}(A/\sigma)$ onto $\mathcal{K}(A) \cap [\sigma, \nabla_A]_{\mathrm{Con}(A)}$.*
(3) *$\theta \in \mathcal{K}(A) \cap [\sigma, \nabla_A]_{\mathrm{Con}(A)}$ if and only if $\theta/\sigma \in \mathcal{K}(A/\sigma)$, for all $A \in \mathcal{A}$ and $\sigma \in \mathcal{K}(A)$.*

Proof. $1 \Longrightarrow 2$. Let us suppose that $\mathcal{A}_{\mathcal{K}}$ is a category and $\mathcal{K} \colon \mathcal{A}_{\mathcal{K}} \to Set$ is a presheaf. Let $A \in \mathcal{A}$, $\sigma \in \mathrm{Con}(A)$ and $p \colon A \to A/\sigma$ be the natural $\mathcal{A}$-homomorphism. Note that $\mathrm{Imag}(p^*\restriction_{\mathcal{K}(A/\sigma)}) = \mathrm{Imag}(\mathcal{K}(p)) \subseteq \mathcal{K}(A)$ because $\mathcal{K}$ is a presheaf. Then, by Proposition 3.3, $p^*\restriction_{\mathcal{K}(A/\sigma)}$ is an injective order homomorphism of the form $p^*\restriction_{\mathcal{K}(A/\sigma)} \colon \mathcal{K}(A/\sigma) \to \mathcal{K}(A) \cap [\sigma, \nabla_A]_{\mathrm{Con}(A)}$. We want to prove that $p^*\restriction_{\mathcal{K}(A/\sigma)}$ is a surjective map. With this aim we need to show that if $\theta \in \mathcal{K}(A) \cap [\sigma, \nabla_A]_{\mathrm{Con}(A)}$ then $\theta/\sigma \in \mathcal{K}(A/\sigma)$. Indeed, by Theorem 2.3(1), $A/\theta \cong_{\mathcal{A}} (A/\sigma)/(\theta/\sigma)$ and therefore the natural $\mathcal{A}$-homomorphism $A/\sigma \to (A/\sigma)/(\theta/\sigma)$ can be identified with the $\mathcal{A}_{\mathcal{K}}$-homomorphism $g \colon A/\sigma \to$

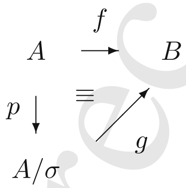
A/θ such that $g(x_{/\sigma}) = x_{/\theta}$. By hypothesis we have that $\mathcal{K}(g) = g^*\restriction_{\mathcal{K}(A/\theta)} \colon \mathcal{K}(A/\theta) \to \mathcal{K}(A/\sigma)$ and $\Delta_{A/\theta} \in \mathcal{K}(A/\theta)$. Then

$$\begin{aligned}
\mathcal{K}(A/\sigma) \ni g^*(\Delta_{A/\theta}) &= g^*(\theta/\theta)\\
&= \{(x_{/\sigma}, y_{/\sigma}) \in (A/\sigma)^2 : (g(x_{/\sigma}), g(y_{/\sigma})) \in \theta/\theta\}\\
&= \{(x_\sigma, y_\sigma) \in (A/\sigma)^2 : (x_{/\theta}, y_{/\theta}) \in \theta/\theta\}\\
&= \{(x_{/\sigma}, y_{/\sigma}) \in (A/\sigma)^2 : (x, y) \in \theta\}\\
&= \theta/\sigma
\end{aligned}$$

i.e., $\theta/\sigma \in \mathcal{K}(A/\sigma)$. Thus, by Proposition 3.3, if $\theta \in \mathcal{K}(A) \cap [\sigma, \nabla_A]_{\mathrm{Con}(A)}$ then $\theta/\sigma \in \mathcal{K}(A/\theta)$. Therefore, $[\mathcal{K}(p)](\theta/\sigma) = p^*(\theta/\sigma) = \theta$ and consequently $p^*\restriction_{\mathcal{K}(A/\sigma)}$ is surjective. Hence our claim.

$2 \Longrightarrow 3$. Immediate form Proposition 3.3.

$3 \Longrightarrow 1$. We first note that for each $A \in \mathcal{A}$, $1_A \in Hom_{\mathcal{A}_\mathcal{K}}$ because $\Delta_A \in \mathcal{K}(A)$. Now we prove that the class $Hom_{\mathcal{A}_\mathcal{K}}$ is closed under compositions. Let $A \in \mathcal{K}$, $\sigma \in \mathcal{K}(A)$, $\theta/\sigma \in \mathcal{K}(A/\sigma)$ and let us consider the following diagram $A \overset{p_1}{\to} A/\sigma \overset{p_2}{\to} (A/\sigma)/(\theta/\sigma)$ in $Hom_{\mathcal{A}_\mathcal{K}}$ where p_1 and p_2 are two natural $\mathcal{A}$-homomorphisms. By Theorem 2.3(1) we have $(A/\sigma)/(\theta/\sigma) \cong_\mathcal{A} A/\theta$ and, by hypothesis, $\theta \in \mathcal{K}(A)$. Then the composition $p_2 p_1 \in Hom_{\mathcal{A}_\mathcal{K}}$ and it proves that $Hom_{\mathcal{A}_\mathcal{K}}$ is closed under compositions. Hence $\mathcal{A}_\mathcal{K}$ defines a category. Now we show that $\mathcal{K} \colon \mathcal{A}_\mathcal{K} \to Set$ is a presheaf. Let $f \colon A \to B \in Hom_{\mathcal{A}_\mathcal{K}}$. We first show that $\mathcal{K}(f) = f^*\restriction_{\mathcal{K}(B)}$ is a function of the form $\mathcal{K}(f) = \mathcal{K}(B) \to \mathcal{K}(A)$. Let us notice that f admits the following factorization in $\mathcal{A}$



where $\sigma = \ker(f) \in \mathcal{K}(A)$, p is the natural $\mathcal{A}$-homomorphism and g is a $\mathcal{A}$-isomorphism. By hypothesis and by Theorem 2.3, $p^* \colon \mathcal{K}(A/\sigma) \to \mathcal{K}(A) \cap [\sigma, \nabla_A]_{\mathrm{Con}(A)}$ is an order isomorphism and $g^*\restriction_{\mathcal{K}(B)} \colon \mathcal{K}(B) \to \mathcal{K}(A/\sigma)$ is an order isomorphism because g is a $\mathcal{A}$-isomorphism. Thus, by Proposition 3.2(2), $f^* = (gp)^* = p^* g^*$ and then $f^*\restriction_{\mathcal{K}(B)}$ is an order homomorphism from $\mathcal{K}(B)$ onto $\mathcal{K}(A)$. By Proposition 3.2 we also note that $\mathcal{K}$ is a contravariant functor. Hence $\mathcal{K} \colon \mathcal{A}_\mathcal{K} \to Set$ is a presheaf. □

Definition 3.6. Let $\mathcal{A}$ be a category of algebras. A congruences operator $\mathcal{K}$ over $\mathcal{A}$ satisfying the equivalent conditions listed in Proposition 3.5 is called a *congruences presheaf*.

If we focus our attention on the item 3 of Proposition 3.5 we can notice that the condition for a congruences operator to be a congruences presheaf is a generalization of the fact that $Z(L) \cap [z, 1]_L = Z([z, 1]_L)$ where L is a bounded lattice and $z \in Z(L)$. This result (or the equivalent dual version), introduced in Proposition 2.1, turns out to be crucial in the proof of several algebraic versions of the CBS-theorem (see for example [16, Proposition 2.8, Proposition 6.2], [18, Proposition 3.4, Theorem 3.7], [34, Lemma 3.2, Lema 4.2] etc.).

Example 3.7 [*Presheaf* Con]. Let $\mathcal{A}$ be a category of algebras closed under homomorphic images. Let us define the class operator $\mathcal{A} \ni A \mapsto \mathrm{Con}(A)$. It is not difficult to show that Con is a congruences operator and that $Hom_{\mathcal{A}_{\mathrm{Con}}}$ is the class of surjective $\mathcal{A}$-homomorphisms. Thus $\mathcal{A}_{\mathrm{Con}}$ is a category. If we define $\mathrm{Con}(f) = f^*$ then, by Proposition 3.2, Con is a congruences presheaf. In particular Con is a congruences presheaf over varieties of algebras.

Example 3.8. Let $\mathcal{A}$ be a quasivariety. For each $A \in \mathcal{A}$, let us consider the set of relative congruences of A, $\mathrm{Rel}(A) = \{\theta \in \mathrm{Con}(A)\colon A/\theta \in \mathcal{A}\}$. Let us define the class operator $\mathcal{A} \ni A \mapsto \mathrm{Rel}(A)$. It is not difficult to prove that $\mathrm{Rel}(-)$ is a congruences operator and that $\mathcal{A}_{\mathrm{Rel}} = \langle Ob(\mathcal{A}), Hom_{\mathcal{A}_{\mathrm{Rel}}}\rangle$ is a category. We shall prove that if $f\colon A \to B \in Hom_{\mathcal{A}_{\mathrm{Rel}}}$ then $\mathrm{Imag}(f^*) \subseteq \mathrm{Rel}(A)$ which is equivalent to prove that if $f\colon A \to B \in Hom_{\mathcal{A}_{\mathrm{Rel}}}$ then, for each $\theta \in \mathrm{Rel}(B)$, $A/f^*(\theta) \in \mathcal{A}$. Indeed: Let us consider a quasi equation $(\&_{i=1}^n r_i(\overline{x}) = s_i(\overline{x})) \implies r(\overline{x}) = s(\overline{x})$ holding in $\mathcal{A}$ where $\overline{x}$ is a vector of k variables. Let $\overline{a}_{/_{f^*(\theta)}}$ be a vector of k elements of the algebra $A/f^*(\theta)$ such that $A/f^*(\theta) \models \&_{i=1}^n r_i(\overline{a}_{/_{f^*(\theta)}}) = s_i(\overline{a}_{/_{f^*(\theta)}})$. Thus, by definition of f^* in Eq. (3.1), we have that $(f(s_i(\overline{a})), f(r_i(\overline{a}))) = (s_i(f(\overline{a})), r_i(f(\overline{a}))) \in \theta$ for $1 \le i \le n$ and then $B/\theta \models \&_{i=1}^n r_i(f(\overline{a}))_{/_\theta} = s_i(f(\overline{a}))_{/_\theta}$. Since $B/\theta \in \mathcal{A}$ and the quasi equation holds in $\mathcal{A}$, $B/\theta \models s(f(\overline{a}))_{/_\theta} = r(f(\overline{a}))_{/_\theta}$. It implies that $(f(s(\overline{a})), f(r(\overline{a}))) \in \theta$ and then $(s(\overline{a}), r(\overline{a})) \in f^*(\theta)$. Hence $A/f^*(\theta) \models r(\overline{a}_{/_{f^*(\theta)}}) = s(\overline{a}_{/_{f^*(\theta)}})$. It proves that $A/f^*(\theta) \in \mathcal{A}$ and, by Proposition 3.2, $\mathrm{Rel}(-)$ is a congruences presheaf.

Proposition 3.9. *Let $\mathcal{A}$ be a category of algebras, $\mathcal{K}$ be a congruences presheaf and $A \in \mathcal{A}$. If $\sigma \in \mathcal{K}(A)$, $\theta \in \mathcal{K}(A) \cap [\sigma, \nabla_A]_{\mathrm{Con}(A)}$ and $A \cong_{\mathcal{A}} A/\theta$ then there exists $\theta' \in \mathcal{K}(A/\sigma)$ such that $A \cong (A/\sigma)/\theta'$.*

Proof. Since $\theta \in \mathcal{K}(A) \cap [\sigma, \nabla_A]_{\mathrm{Con}(A)}$, by Proposition 3.5(3), $\theta' = \theta/\sigma \in \mathcal{K}(A/\sigma)$. Then, by Theorem 2.3(1), $(A/\sigma)/\theta' = (A/\sigma)/(\theta/\sigma) \cong_{\mathcal{A}} A/\theta \cong A$. □

Definition 3.10. Let $\mathcal{A}$ be a category of algebras and $\mathcal{K}$ be a congruences presheaf. An algebra $A \in \mathcal{A}$ has the *Cantor–Bernstein–Schröder property* with respect to $\mathcal{K}$ (*$CBS_{\mathcal{K}}$-property* for short) if and only if the following holds: given $B \in \mathcal{A}$ and $\theta_B \in \mathcal{K}(B)$ such that there is $\theta_A \in \mathcal{K}(A)$ with $A \cong_{\mathcal{A}} B/\theta_B$ and $B \cong_{\mathcal{A}} A/\theta_A$ then $A \cong_{\mathcal{A}} B$.

As we will see in Example 5.4, in the above definition if we assume that $\mathcal{A}$ is the variety of Boolean algebras and the congruences presheaf $\mathcal{K}$ satisfies $\mathcal{K}(A) = \mathrm{FC}(A)$ for each $A \in \mathcal{A}$ then the $CBS_{\mathcal{K}}$-property, attributed to a Boolean algebra, rephrases the Sikorski–Tarski version of the CBS-theorem when the σ-completeness is considered in $\mathrm{FC}(A)$. A very useful equivalence of the $CBS_{\mathcal{K}}$-property is given by the following theorem.

Theorem 3.11. *Let $\mathcal{A}$ be a category of algebras and let $\mathcal{K}$ be a congruences presheaf. Then the following conditions are equivalent for each $A \in \mathcal{A}$:*

(1) *A has the $CBS_{\mathcal{K}}$-property.*
(2) *If $\theta \in \mathcal{K}(A)$ and $A \cong_{\mathcal{A}} A/\theta$ then for all $\sigma \in \mathcal{K}(A)$ such that $\sigma \subseteq \theta$ we have that $A \cong_{\mathcal{A}} A/\sigma$.*

Proof. 1 $\Longrightarrow$ 2. Let $\sigma, \theta \in \mathcal{K}(A)$ such that $\sigma \subseteq \theta$ and $A \cong_{\mathcal{A}} A/\theta$. Let $B = A/\sigma$. Note that $\theta \in \mathcal{K}(A) \cap [\sigma, \nabla_A]_{\mathrm{Con}(A)}$ then, by Proposition 3.9, there exists $\theta_B \in \mathcal{K}(A/\sigma) = \mathcal{K}(B)$ such that $A \cong_{\mathcal{A}} B/\theta_B$. Since A has the $CBS_{\mathcal{K}}$-property we have that $A \cong_{\mathcal{A}} B = A/\sigma$.

2 $\Longrightarrow$ 1. Let $B \in \mathcal{A}$, $\sigma_A \in \mathcal{K}(A)$ and $\sigma_B \in \mathcal{K}(B)$. Suppose that there exist two $\mathcal{A}$-isomorphisms $f\colon A \to B/\sigma_B$ and $g\colon B \to A/\sigma_A$.

By Proposition 3.2(4), we have that $g_*(\sigma_B) \in \mathcal{K}(A/\sigma_A)$ and there exists a $\mathcal{A}$-isomorphism $g'\colon B/\sigma_B \to (A/\sigma_A)/g_*(\sigma_B)$. Let us consider the following composition of $\mathcal{A}$-isomorphisms:

$$A \xrightarrow{f} B/\sigma_B \xrightarrow{g'} (A/\sigma_A)/g_*(\sigma_B). \tag{3.4}$$

Note that $g_*(\sigma_B) = \theta/\sigma_A$ for some $\theta \in \mathrm{Con}(A)$ and, by Proposition 3.5(3), $\theta \in \mathcal{K}(A) \cap [\sigma_A, \nabla_A]_{\mathrm{Con}(A)}$. Thus, by Theorem 2.3(1), $(A/\sigma_A)/g_*(\sigma_B) = (A/\sigma_A)/(\theta/\sigma_A) \cong_{\mathcal{A}} A/\theta$ and the diagram of $\mathcal{A}$-isomorphisms given in Eq. (3.4) can be seen as

$$A \xrightarrow{f} B/\sigma_B \xrightarrow{g'} A/\theta.$$

Therefore $A \cong_{\mathcal{A}} A/\theta$ where $\theta \in \mathcal{K}(A) \cap [\sigma_A, \nabla_A]_{\mathrm{Con}(A)}$. Since $\sigma_A \subseteq \theta$, by hypothesis, $A \cong_{\mathcal{A}} A/\sigma_A \cong_{\mathcal{A}} B$. Hence A has the $CBS_{\mathcal{K}}$-property. □

Remark 3.12. Let us notice that, by condition 2 of Theorem 3.11, if there are not $\theta \in \mathcal{K}(A)$ such that $A \cong_{\mathcal{A}} A/\theta$ then the algebra A trivially has the $CBS_{\mathcal{K}}$-property. Then we say that A *satisfies the $CBS_{\mathcal{K}}$-property in a non trivial way* whenever this property is satisfied and there exists $\theta \in \mathcal{K}(A)$ such that $A \cong_{\mathcal{A}} A/\theta$.

We conclude this section with a concrete example showing our abstract framework for the CBS-theorem formulated in terms of the congruence presheaf Con introduced in Example 3.7.

Example 3.13 (Pseudo-simple algebras). An algebra A is called *pseudo-simple* [37] if and only if $Card(A) > 1$ and for every $\sigma \in \mathrm{Con}(A) - \{\nabla_A\}$, $A/\sigma \cong A$. Let $\mathcal{A}$ be a category of algebras closed under homomorphic images and let us consider the congruences presheaf Con. Then, by Theorem 3.11, pseudo-simple algebras of $\mathcal{A}$ satisfy the CBS_{Con}-property.

Concrete examples of these algebras can be found in the variety $\mathcal{G}rp$ of groups. Indeed, a *quasi-cyclic group* is an Abelian group which is isomorphic to $Z(p^\infty)$ for some prime number p. They are pseudo-simple algebras in $\mathcal{G}rp$. In this way quasi-cyclic groups have the CBS_{Con}-property.

4. Factor congruences presheaves

In this section we introduce and study a special case of congruences presheaf that allow us to formulate versions of the CBS-theorem based on factor congruences. In this particular framework necessary and sufficient conditions for the validity of CBS-theorem are established.

Definition 4.1. Let $\mathcal{A}$ be a category of algebras. A *factor congruences presheaf* is a congruences presheaf $\mathcal{K}$ such that for each $A \in \mathcal{A}$,

(1) $\mathcal{K}(A) \subseteq \mathrm{FC}(A)$.
(2) For each $\theta \in \mathcal{K}(A)$ there exists $\neg\theta \in \mathcal{K}(A)$, such that $(\theta, \neg\theta)$ is a pair of factor congruences on A.
(3) If $\sigma \in \mathcal{K}(A)$, $\theta \in \mathcal{K}(A) \cap [\sigma, \nabla_A]_{\mathrm{Con}(A)}$ and $(\theta, \neg\theta)$ is a pair of factor congruences in $\mathcal{K}(A)$ then $(\theta/\sigma, (\neg\theta \vee \sigma)/\sigma)$ is a pair of factor congruences in $\mathcal{K}(A/\sigma)$.

By item 2 of the above definition, $\nabla_A \in \mathcal{K}(A)$ because $\Delta_A \in \mathcal{K}(A)$ and, by Proposition 3.5, the following result is immediate.

Proposition 4.2. *Let $\mathcal{A}$ be a category of algebras and $\mathcal{K}$ be a factor congruences presheaf. Let $A \in \mathcal{K}$, $\sigma \in \mathcal{K}(A)$, $\theta \in \mathcal{K}(A) \cap [\sigma, \nabla_A]_{\mathrm{Con}(A)}$ and $(\theta, \neg\theta)$ be a pair of factor congruences in $\mathcal{K}(A)$. Then $\neg\theta \vee \sigma \in \mathcal{K}(A) \cap [\sigma, \nabla_A]_{\mathrm{Con}(A)}$.*

Let $\mathcal{A}$ be a category of algebras such that for each $A \in \mathcal{A}$ and $\sigma \in \mathrm{FC}(A)$, $A/\sigma \in \mathcal{A}$. Then, by Proposition 3.2(5), it is immediate that the class operator

$$\mathcal{A} \ni A \mapsto \mathrm{FC}(A) \tag{4.1}$$

is a congruence operator. The following proposition provides a sufficient condition for FC to be a congruences presheaf.

Proposition 4.3. *Let $\mathcal{A}$ be a category of algebras such that for each $A \in \mathcal{A}$ and $\sigma \in \mathrm{FC}(A)$, $A/\sigma \in \mathcal{A}$. If $\mathcal{A}$ is congruence modular or congruence permutable then* FC *is a congruences presheaf.*

Proof. Let us assume that $\mathcal{A}$ is congruence modular. Let $A \in \mathcal{A}$, $\sigma \in \mathrm{FC}(A)$, $\theta \in \mathrm{FC}(A) \cap [\sigma, \nabla_A]_{\mathrm{Con}(A)}$ and $(\theta, \neg\theta)$ be a pair of factor congruences in $\mathrm{FC}(A)$. We first prove that $(\theta/\sigma, \neg\theta \vee \sigma/\sigma)$ is a pair of factor congruences in $\mathrm{FC}(A/\sigma)$. By modularity $\theta \cap (\sigma \vee \neg\theta) = \sigma \vee (\theta \cap \neg\theta) = \sigma \vee \Delta_A = \sigma$ because $\sigma \subseteq \theta$. Then, by Theorem 2.3(2), $\theta/\sigma \cap (\neg\theta \vee \sigma)/\sigma = \Delta_{A/\sigma}$. We also note that $\nabla_A = \theta \circ \neg\theta \subseteq \theta \circ (\neg\theta \vee \sigma)$. Then, by Theorem 2.3(3), $\theta/\sigma \circ (\neg\theta \vee \sigma)/\sigma = \nabla_{A/\sigma}$. Thus, $(\theta/\sigma, (\neg\theta \vee \sigma)/\sigma)$ is a pair of factor congruences on A/σ and $\theta/\sigma \in \mathrm{FC}(A/\sigma)$. Now if we suppose that $\theta/\sigma \in \mathrm{FC}(A/\sigma)$ then, by Proposition 2.4, $\theta \in \mathrm{FC}(A)$. Hence, by Proposition 3.5, FC is a factor congruences presheaf. Let us notice that if $\mathcal{A}$ is a category of congruence permutable algebras then, by the Birkhoff theorem (see [7, Proposition 5.10]), $\mathcal{A}$ is congruence modular. Hence our claim. □

Example 4.4 [*CBS*$_{\mathrm{FC}}$-property: injective modules and divisible groups]. Let $\mathcal{M}od_R$ be the variety of modules over the ring R and $\mathcal{A}b$ be the variety of Abelian groups. Let us notice that divisible groups are the injective objects in

$\mathcal{A}b$. We will denote by $\mathcal{A}$ both the varieties $\mathcal{M}od_R$ and $\mathcal{A}b$. In the variety $\mathcal{A}$, the notions of finite direct sum and finite direct product coincide. Thus, for each $A \in \mathcal{A}$, $\langle \mathrm{FC}(A), \subseteq \rangle$ is order reverse isomorphic to the set of direct factor subalgebras of A denoted by $\langle \mathrm{DF}(A), \subseteq \rangle$. It is well known that $\mathcal{A}$ is a congruence permutable variety and then, by Proposition 4.3, FC is a congruences presheaf.

Let A be an injective object in $\mathcal{A}$. We shall prove that A has the CBS_{FC}-property. In order to do this, by Theorem 3.11, we have to show that: for $I, K \in \mathrm{DF}(A)$ such that I is a subalgebra of K, if $A \cong_{\mathcal{A}} I$ then $A \cong_{\mathcal{A}} K$.

Indeed, let $f \colon I \to A$ be a $\mathcal{A}$-isomorphism. Since A is an injective object, there exists a $\mathcal{A}$-homomoprhism $g \colon K \to A$ such that the following diagram commutes

$$\begin{array}{ccc} I & \xrightarrow{f} & A \\ {\scriptstyle 1_I}\downarrow & \equiv & \nearrow {\scriptstyle g} \\ K & & \end{array}$$

Let us notice that the composition $g1_I$ is an injective $\mathcal{A}$-homomoprhism. Thus, if we consider the following composition $K \overset{f^{-1}\restriction_K}{\rightarrowtail} I \overset{g1_I}{\rightarrowtail} A$, by commutativity of the above diagram, we have that $A \supseteq K \ni x = f(f^{-1}(x)) = g1_I(f^{-1}(x))$. It proves that the diagram $K \overset{f^{-1}\restriction_K}{\rightarrowtail} I \overset{g1_I}{\rightarrowtail} K$ is the identity 1_K. Therefore, $g1_I$ is also a surjective $\mathcal{A}$-homomoprhism and $I \cong_{\mathcal{A}} K$. Hence $A \cong_{\mathcal{A}} K$ and A has the CBS_{FC}-property. Since A is an injective object then the denumerable direct product $B = \prod_{\mathbb{N}} A$ is injective in $\mathcal{A}$. Thus, by Proposition 2.5, there exists $\sigma \in \mathrm{FC}(B)$ such that $B \cong_{\mathcal{A}} B/\sigma$. In this way B satisfies the CBS_{FC}-property in a non trivial way.

Now we study a necessary a sufficient condition for the validity of the CBS-property with respect to a factor congruences presheaf.

Let $\mathcal{A}$ be a category of algebras and $\mathcal{K}$ be a factor congruences presheaf. Let $A \in \mathcal{A}$, $\theta \in \mathcal{K}(A)$ and let us suppose that there exists a $\mathcal{A}$-isomorphism $f \colon A \to A/\theta$. By Theorem 2.3(2) and Proposition 3.2(4) let us consider the $\langle \nabla, \Delta, \subseteq \rangle$-isomorphism $\hat{f} = u_\theta^{-1} f_*$ i.e.,

$$\hat{f} \colon \mathcal{K}(A) \overset{f_*}{\to} \mathcal{K}(A/\theta) \overset{u_\theta^{-1}}{\to} \mathcal{K}(A) \cap [\theta, \nabla_A]_{\mathrm{Con}(A)}. \tag{4.2}$$

If $\sigma \in \mathcal{K}(A)$ such that $\sigma \subseteq \theta$ then we define the following set:

$$\langle \sigma \rangle_\theta = \{\zeta \in [\Delta_A, \theta]_{\mathrm{Con}(A)} \cap \mathcal{K}(A) : A/\sigma \cong_{\mathcal{A}} A/\zeta\}. \tag{4.3}$$

If $\zeta \in \langle \sigma \rangle_\theta$ then we recursively define the following sequences of congruences:

$$\begin{array}{ll} \sigma_0 = \Delta_A\,, & \\ \sigma_1 = \zeta\,, & \theta_1 = f_*(\sigma_0) = \theta/\theta\,, \\ \sigma_2 = u_\theta^{-1}(\theta_1) = \theta\,, & \theta_2 = f_*(\sigma_1)\,, \\ \sigma_3 = u_\theta^{-1}(\theta_2)\,, & \theta_3 = f_*(\sigma_2)\,, \end{array}$$

$$\vdots \qquad\qquad \vdots$$

$$\sigma_{n+1} = u_\theta^{-1}(\theta_n)\,, \qquad \theta_{n+1} = f_*(\sigma_n)\,. \tag{4.4}$$

Let us notice that, by Eq. (4.2), $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}(A)$.

Proposition 4.5. *Let $\mathcal{A}$ be a category of algebras, $\mathcal{K}$ be a factor congruences presheaf, $A \in \mathcal{A}$ and $\theta \in \mathcal{K}(A)$ such that there exists a $\mathcal{A}$-isomorphism $f\colon A \to A/\theta$. Let us consider the sequence $(\sigma_n)_{n\in\mathbb{N}}$ in $\mathcal{K}(A)$ given in* Eq. (4.4)*. Then:*

(1) $\hat{f}(\sigma_n) = \sigma_{n+2}$,
(2) $(\sigma_n)_{n\in\mathbb{N}}$ *is an increasing sequence in $\mathcal{K}(A)$. In particular, if $\Delta_A < \zeta$ then $(\sigma_n)_{n\in\mathbb{N}}$ is strictly increasing.*

Proof. (1) If $k \geq 2$ then $\sigma_k = u_\theta^{-1}(\theta_{k-1}) = u_\theta^{-1} f_*(\sigma_{k-2}) = \hat{f}(\sigma_{k-2})$. Thus, if $k = n+2$ then we have that $\hat{f}(\sigma_n) = \sigma_{n+2}$.

(2) Suppose that $\sigma_0 = \Delta_A = \zeta = \sigma_1$. Then it is not very hard to see that $\sigma_n = \theta$ for $n \geq 2$. Thus $(\sigma_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathcal{K}(A)$. Let us assume that $\sigma_0 = \Delta_A < \zeta = \sigma_1$. By induction, let us assume that $\sigma_i < \sigma_j$ whenever $1 < i < j < n$. Since the function $\hat{f}$ is an order isomorphism and $n \geq 2$, by item 1, we have that $\sigma_n = \hat{f}(\sigma_{n-2}) < \hat{f}(\sigma_{n-1}) = \sigma_{n+1}$. Hence $(\sigma_n)_{n\in\mathbb{N}}$ is strictly increasing. □

Definition 4.6. Let $\mathcal{A}$ be a category of algebras, $\mathcal{K}$ be a factor congruences presheaf, $A \in \mathcal{A}$ and $\theta \in \mathcal{K}(A)$ such that there exists a $\mathcal{A}$-isomorphism $f\colon A \to A/\theta$. Let us consider the sequence $(\sigma_n)_{n\in\mathbb{N}}$ in $\mathcal{K}(A)$ given in Eq. (4.4). Then a *CBS-sequence* is a sequence of the form $(\sigma_{2n} \vee \neg\sigma_{2n+1})_{n\geq 0}$ such that

(1) $\neg\sigma_1 = \neg\zeta = \hat{f}^{-1}(\neg\hat{f}(\zeta))$ where $(\hat{f}(\zeta), \neg\hat{f}(\zeta))$ is a pair of factor congruences in $\mathcal{K}(A)$.
(2) $\neg\sigma_{2n+3} = \hat{f}(\neg\sigma_{2n+1})$ for $n \geq 1$.

Let us note that $(\zeta, \neg\zeta)$ is a pair of factor congruences because $\hat{f}$ preserves order and permutability in view of Proposition 3.2(5).

Proposition 4.7. *Let $\mathcal{A}$ be a category of algebras, $\mathcal{K}$ be a factor congruences presheaf, $A \in \mathcal{A}$ and $\theta \in \mathcal{K}(A)$ such that there exists a $\mathcal{A}$-isomorphism $f\colon A \to A/\theta$. Let us consider the sequence $(\sigma_n)_{n\in\mathbb{N}}$ in $\mathcal{K}(A)$ given in* Eq. (4.4) *and a CBS-sequence $(\sigma_{2n} \vee \neg\sigma_{2n+1})_{n\geq 0}$. Then:*

(1) $\sigma_{2n+1} \vee \neg\sigma_{2n+1} = \nabla_A$.
(2) $(\sigma_{2n} \vee \neg\sigma_{2n+1})_{n\geq 0}$ *is a dual orthogonal sequence in $\mathcal{K}(A)$.*
(3) $\hat{f}(\sigma_{2n} \vee \neg\sigma_{2n+1}) = \sigma_{2n+2} \vee \neg\sigma_{2n+3}$ *for $n \geq 0$.*

Proof. (1) By definition of CBS-sequence, $(\sigma_1, \neg\sigma_1)$ is a pair of factor congruences in $\mathcal{K}(A)$ and then $\sigma_1 \vee \neg\sigma_1 = \nabla_A$. Since $\hat{f}$ is an order isomorphism, if $n > 0$ and $\sigma_{2(n-1)+1} \vee \neg\sigma_{2(n-1)+1} = \nabla_A$ then $\nabla_A = \hat{f}(\sigma_{2(n-1)+1} \vee \neg\sigma_{2(n-1)+1}) = \hat{f}(\sigma_{2(n-1)+1}) \vee \hat{f}(\neg\sigma_{2(n-1)+1}) = \sigma_{2(n-1)+3} \vee \neg\sigma_{2(n-1)+3} = \sigma_{2n+1} \vee \neg\sigma_{2n+1}$.

(2) By Proposition 4.5(2), for each natural number n we have that $\sigma_{2n} \leq \sigma_{2n+1}$ and then $\sigma_{2n+1} \in \mathcal{K}(A) \cap [\sigma_{2n}, \nabla_A]_{\mathrm{Con}(A)}$. Thus, by Definition 4.1(3) and Proposition 4.2, $\sigma_{2n} \vee \neg\sigma_{2n+1} \in \mathcal{K}(A)$. In this way, $(\sigma_0 \vee \neg\sigma_1, \sigma_2 \vee \neg\sigma_3, \ldots) =$

$(\sigma_{2n} \vee \neg\sigma_{2n+1})_{n\geq 0}$ is a sequence in $\mathcal{K}(A)$. Suppose that $m < n$. Since $(\sigma_n)_{n\in\mathbb{N}}$ is an increasing sequence, $\sigma_{2n} \geq \sigma_{2m+1}$ then, by item 1, we have that

$$\begin{aligned}(\sigma_{2m} \vee \neg\sigma_{2m+1}) \vee (\sigma_{2n} \vee \neg\sigma_{2n+1}) &\geq \sigma_{2m} \vee (\neg\sigma_{2m+1} \vee \sigma_{2m+1}) \vee \neg\sigma_{2n+1} \\ &= \sigma_{2m} \vee \nabla_A \vee \neg\sigma_{2n+1} = \nabla_A.\end{aligned}$$

Hence $(\sigma_{2n} \vee \neg\sigma_{2n+1})_{n\geq 0}$ is a dual orthogonal sequence in $\mathcal{K}(A)$.

(3) Since $\hat{f}$ is an order isomorphism, by Proposition 4.5(1), $\hat{f}(\sigma_{2n} \vee \neg\sigma_{2n+1}) = \hat{f}(\sigma_{2n}) \vee \hat{f}(\neg\sigma_{2n+1}) = \sigma_{2n+2} \vee \neg\sigma_{2n+3}$. □

In what follows, the infimum in $\mathcal{K}(A)$ of a family $(\sigma_i)_{i\in I}$ of $\mathcal{K}(A)$, if it exists, will be denoted by $\prod_{i\in I}^{\mathcal{K}(A)} \sigma_i$, to distinguish it from the infimum $\bigcap_{i\in I} \sigma_i$ in $\mathrm{Con}(A)$, which does not necessarily belong to $\mathcal{K}(A)$.

Definition 4.8. Let $\mathcal{A}$ be a category of algebras and $\mathcal{K}$ be a factor congruences presheaf. An algebra $A \in \mathcal{A}$ is called *$CBS_{\mathcal{K}}$-complete* if and only if for all $\mathcal{A}$-isomorphism $f\colon A \to A/\theta$, where $\theta \in \mathcal{K}(A)$, and for all $\sigma \in \mathcal{K}(A)$ such that $\sigma \subseteq \theta$, there exists $\zeta \in \langle\sigma\rangle_\theta$ and a CBS-sequence $(\sigma_{2n} \vee \neg\sigma_{2n+1})_{n\geq 0}$ satisfying the following conditions:

(1) $\sigma_\zeta = \prod_{n\geq 1}^{\mathcal{K}(A)} (\sigma_{2n} \vee \neg\sigma_{2n+1})$ exists.
(2) There exists $\neg\sigma_\zeta \in \mathcal{K}(A)$ such that $(\sigma_\zeta, \neg\sigma_\zeta)$ and $(\neg\zeta \cap \sigma_\zeta, \zeta \vee \neg\sigma_\zeta)$ are two pairs of factor congruences in $\mathcal{K}(A)$.

Theorem 4.9. *Let $\mathcal{A}$ be a category of algebras, $\mathcal{K}$ be a factor congruences presheaf and $A \in \mathcal{A}$. Then the following conditions are equivalent:*

(1) *A is $CBS_{\mathcal{K}}$-complete.*
(2) *A has the $CBS_{\mathcal{K}}$-property.*

Proof. (1) $\Longrightarrow$ (2). Let us assume that A is $CBS_{\mathcal{K}}$-complete. Let $\sigma, \theta \in \mathcal{K}(A)$ such that $\sigma \subseteq \theta$ and $f\colon A \to A/\theta$ be a $\mathcal{A}$-isomorphism. By Theorem 3.11 we shall prove that $A \cong_{\mathcal{A}} A/\sigma$. Let us suppose that $(\sigma_{2n} \vee \neg\sigma_{2n+1})_{n\geq 0}$ is a CBS-sequence satisfying the conditions introduced in Definition 4.8.

By hypothesis $\sigma_\zeta = \prod_{n\geq 1}^{\mathcal{K}(A)} (\sigma_{2n} \vee \neg\sigma_{2n+1}) \in \mathcal{K}(A) \cap [\zeta, \nabla_A]_{\mathrm{Con}(A)}$. Further, there exists $\neg\sigma_\zeta \in \mathcal{K}(A)$ such that $(\sigma_\zeta, \neg\sigma_\zeta)$ and $(\neg\zeta \cap \sigma_\zeta, \zeta \vee \neg\sigma_\zeta)$ are two pairs of factor congruences in $\mathcal{K}(A)$. If we define $\chi = \neg\zeta \cap \sigma_\zeta$ and $\neg\chi = \neg\sigma_\zeta \vee \zeta$ then

$$A \cong A/\neg\chi \times A/\chi. \tag{4.5}$$

Since $\sigma_\zeta \in \mathcal{K}(A) \cap [\zeta, \nabla_A]_{\mathrm{Con}(A)}$, by Proposition 4.2 and by hypothesis, we have that

$$\begin{aligned}A/\zeta &\cong_{\mathcal{A}} A/(\neg\sigma_\zeta \vee \zeta) \times A/\sigma_\zeta \\ &= A/\neg\chi \times A/\sigma_\zeta.\end{aligned} \tag{4.6}$$

Since $f_*(\chi) \in \mathcal{K}(A/\theta)$, by Theorem 3.5(3), there exists a congruence ρ in $\mathcal{K}(A) \cap [\theta, \nabla_A]_{\mathrm{Con}(A)}$ such that $f_*(\chi) = \rho/\theta$. Therefore, $\hat{f}(\chi) = u_\theta^{-1} f_*(\chi) = u_\theta^{-1}(\rho/\theta) = \rho$ and, by Proposition 3.2(4), we have that

$$A/\chi \cong_{\mathcal{A}} (A/\theta)/f_*(\chi) = (A/\theta)/(\rho/\theta) \cong_{\mathcal{A}} A/\rho = A/\hat{f}(\chi). \tag{4.7}$$

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Since $\hat{f}\colon \mathcal{K}(A) \to \mathcal{K}(A) \cap [\theta, \nabla_A]_{\mathrm{Con}(A)}$ is a $\langle \nabla, \Delta, \subseteq \rangle$-isomorphism, by Definition 4.6, we have that

$$\begin{aligned}
\hat{f}(\chi) &= \hat{f}(\sigma_\zeta \cap \neg\zeta) \\
&= \hat{f}(\textstyle\prod_{n\geq 1}^{\mathcal{K}(A)}(\sigma_{2n} \vee \neg\sigma_{2n+1})) \cap \hat{f}(\neg\zeta) \\
&= \hat{f}(\textstyle\prod_{n\geq 1}^{\mathcal{K}(A)}(\sigma_{2n} \vee \neg\sigma_{2n+1})) \cap \hat{f}(\hat{f}^{-1}(\neg\hat{f}(\zeta)) \\
&= \hat{f}(\textstyle\prod_{n\geq 1}^{\mathcal{K}(A)}(\sigma_{2n} \vee \neg\sigma_{2n+1})) \cap \hat{f}(\Delta_A \vee \hat{f}^{-1}(\neg\hat{f}(\zeta)) \\
&= \hat{f}(\textstyle\prod_{n\geq 1}^{\mathcal{K}(A)}(\sigma_{2n} \vee \neg\sigma_{2n+1})) \cap (\hat{f}(\Delta_A) \vee \hat{f}(\hat{f}^{-1}(\neg\hat{f}(\zeta))) \\
&= \textstyle\prod_{n\geq 1}^{\mathcal{K}(A)}(\sigma_{2n+2} \vee \neg\sigma_{2n+3}) \cap (\theta \vee \neg\hat{f}(\zeta)) \\
&= \textstyle\prod_{n\geq 1}^{\mathcal{K}(A)}(\sigma_{2n+2} \vee \neg\sigma_{2n+3}) \cap (\sigma_2 \vee \neg\sigma_3) \\
&= \textstyle\prod_{n\geq 1}^{\mathcal{K}(A)}(\sigma_{2n} \vee \sigma_{2n+1}) \\
&= \sigma_\zeta.
\end{aligned} \tag{4.8}$$

Therefore, by Eqs. (4.7) and (4.8), $A/\chi \cong_{\mathcal{A}} A/\sigma_\zeta$. Then, by Eq. (4.6), $A/\zeta \cong_{\mathcal{A}} A/\neg\chi \times A/\chi$ and, by equation Eq. (4.5), $A \cong_{\mathcal{A}} A/\zeta \cong_{\mathcal{A}} A/\sigma$ since $\zeta \in \langle\sigma\rangle_\theta$. Hence A has the $CBS_{\mathcal{K}}$-property.

(2) $\Longrightarrow$ (1). Let us assume that A has the $CBS_{\mathcal{K}}$-property. Let $f\colon A \to A/\theta$ be a $\mathcal{A}$-isomorphism where $\theta \in \mathcal{K}(A)$, and $\sigma \in [\Delta_A, \theta]_{\mathrm{Con}(A)} \cap \mathcal{K}(A)$. Then, by hypothesis, $A/\Delta_A \cong_{\mathcal{A}} A \cong_{\mathcal{A}} A/\sigma$ and $\Delta_A \in \langle\sigma\rangle_\theta$ (see Eq. (4.3)). Thus, we consider the sequence $(\sigma_n)_{n\in\mathbb{N}}$ given by

$$\begin{aligned}
\sigma_0 &= \Delta_A\,, & & \\
\sigma_1 &= \Delta_A\,, & \theta_1 &= f_*(\sigma_0) = \theta/\theta\,, \\
\sigma_2 &= u_\theta^{-1}(\theta_1) = \theta\,, & \theta_2 &= f_*(\sigma_1) = f_*(\Delta_A) = \theta/\theta\,, \\
\sigma_3 &= u_\theta^{-1}(\theta_2) = \theta\,, & \theta_3 &= f_*(\sigma_2)\,, \\
&\vdots & &\vdots \\
\sigma_{n+1} &= u_\theta^{-1}(\theta_n)\,, & \theta_{n+1} &= f_*(\sigma_n)\,.
\end{aligned}$$

By induction, we show that $\sigma_{2n} = \sigma_{2n+1}$ for all $n \geq 1$. Indeed $\sigma_2 = \sigma_3 = \theta/\theta$. Let us suppose that $\sigma_{2k} = \sigma_{2k+1}$. Then

$$\begin{aligned}
\sigma_{2(k+1)} &= u_\theta^{-1}(\theta_{2k+1}) = u_\theta^{-1} f_*(\sigma_{2k}) \\
&= u_\theta^{-1} f_*(\sigma_{2k+1}) = u_\theta^{-1}(\theta_{2(k+1)}) \\
&= \sigma_{2(k+1)+1}.
\end{aligned}$$

In this way, $(\sigma_{2n} \vee \neg\sigma_{2n+1})_{n\geq 1} = (\nabla_A, \nabla_A, \nabla_A, \ldots)$ and consequently $\sigma_{\Delta_A} = \prod_{n\geq 1}^{\mathcal{K}(A)}(\sigma_{2n} \vee \neg\sigma_{2n+1}) = \nabla_A$. Hence A is $CBS_{\mathcal{K}}$-complete. □

In the rest of the section we study a special framework for the CBS-theorem based on congruences presheaves defined by sets of factor congruences with a Boolean structure. For this aim we first introduce the following definition.

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Definition 4.10. Let $\mathcal{A}$ be a category of algebras. A *Boolean factor congruences presheaf* is a factor congruences presheaf $\mathcal{K}$ such that, for each $A \in \mathcal{A}$, $\langle \mathcal{K}(A), \vee, \cap, \neg, \Delta_A, \nabla_A \rangle$ is a Boolean sublattice of $\mathrm{Con}(A)$ where $\neg$ is the factor complement.

By Proposition 2.1(2) and by item 3 of Definition 4.1 we can see that for each $\sigma \in \mathcal{K}(A)$, the Boolean structure of $\mathcal{K}(A/\sigma)$ is given by

$$\langle \mathcal{K}(A/\sigma), \vee, \cap, \neg, \Delta_{A/\sigma}, \nabla_{A/\sigma} \rangle \text{ where } \neg(\theta/\sigma) = (\neg_\sigma \theta)/\sigma. \tag{4.9}$$

The following proposition allows us to provide examples of Boolean factor congruences presheaves from the centers of the congruence lattices of algebras in a category of algebras.

Proposition 4.11. *Let $\mathcal{A}$ be a category of algebras such that for each $A \in \mathcal{A}$ and $\sigma \in Z(\mathrm{Con}(A))$, $A/\sigma \in \mathcal{A}$. Then the class operator $\mathcal{A} \ni A \mapsto Z(\mathrm{Con}(A))$ is a congruences operator over $\mathcal{A}$ and the following statements are equivalent:*

1. *$Z(\mathrm{Con}(-))$ is a Boolean factor congruences presheaf.*
2. *For each $A \in \mathcal{A}$, and $\theta \in Z(\mathrm{Con}(A))$, $\theta \circ \neg\theta = \nabla_A$ where $\neg\theta$ is the Boolean complement of θ in $Z(\mathrm{Con}(A))$.*

Proof. By Proposition 3.2 it is immediate to see that $Z(\mathrm{Con}(-))$ is a congruences operator over $\mathcal{A}$.

$1 \Longrightarrow 2$. Let us assume that $Z(\mathrm{Con}(-))$ is a Boolean factor congruences presheaf. Then, for each $A \in \mathcal{A}$, $Z(\mathrm{Con}(A)) \subseteq \mathrm{FC}(A)$. Since $Z(\mathrm{Con}(A))$ is a Boolean algebra, the complement of an element in $Z(\mathrm{Con}(A))$ is unique. Consequently, by condition 2 of Definition 4.1, for each $\theta \in Z(\mathrm{Con}(A))$ we have that $\theta \circ \neg\theta = \nabla_A$.

$2 \Longrightarrow 1$. Let us assume that for each $\theta \in Z(\mathrm{Con}(A))$, $\theta \circ \neg\theta = \nabla_A$. Then $Z(\mathrm{Con}(A)) \subseteq \mathrm{FC}(A)$ for each $A \in \mathcal{A}$. Let $\sigma \in Z(\mathrm{Con}(A))$. By Proposition 2.1 and Proposition 2.3(2) we have that $\theta \in [\sigma, \nabla_A]_{\mathrm{Con}(A)} \cap Z(\mathrm{Con}(A))$ if and only if $\theta \in Z([\sigma, \nabla_A])$ if and only if θ/σ in $Z(\mathrm{Con}(A/\sigma))$. Thus, by Proposition 3.5(3), $Z(\mathrm{Con}(-))$ is a congruences presheaf. Hence our claim. ☐

Example 4.12. Let $\mathcal{A}$ be a congruence permutable variety. Let us notice that for each $A \in \mathcal{A}$ and $\theta \in Z(\mathrm{Con}(A))$, $\theta \cap \neg\theta = \Delta_A$ and $\theta \circ \neg\theta = \theta \vee \neg\theta = \nabla_A$ because of the permutability of θ. Then $Z(\mathrm{Con}(A)) \subseteq \mathrm{FC}(A)$ and, by Proposition 4.11, $Z(\mathrm{Con}(-))$ is a Boolean factor congruences presheaf.

Example 4.13. Let $\mathcal{A}$ be an arithmetical variety i.e., $\mathcal{A}$ is a congruence distributive and congruence permutable variety. By Example 4.12, for each $A \in \mathcal{A}$, $Z(\mathrm{Con}(A)) \subseteq \mathrm{FC}(A)$ and $Z(\mathrm{Con}(-))$ is a Boolean factor congruences presheaf. Since A is congruence distributive, $\mathrm{FC}(A)$ is a Boolean sublattice of $\mathrm{Con}(A)$ and then $\mathrm{FC}(A) \subseteq Z(\mathrm{Con}(A))$. Thus $Z(\mathrm{Con}(A)) = \mathrm{FC}(A)$. In this way $\mathrm{FC}(-) = Z(\mathrm{Con}(-))$ is a Boolean factor congruences presheaf. Other interesting categories of algebras in which $\mathrm{FC}(-) = Z(\mathrm{Con}(-))$ is a Boolean factor congruences presheaf are discriminator varieties since they are arithmetical varieties.

Theorem 4.14. *Let $\mathcal{A}$ be a category of algebras and $\mathcal{K}$ be a Boolean factor congruences presheaf. Then the following conditions are equivalent:*

(1) *A has the* $CBS_{\mathcal{K}}$*-property.*
(2) *For each* $\mathcal{A}$*-isomorphism* $f\colon A \to A/\theta$*, where* $\theta \in \mathcal{K}(A)$*, and for each* $\sigma \in [\Delta_A, \theta]_{\mathrm{Con}(A)} \cap \mathcal{K}(A)$ *there exists* $\zeta \in \langle\sigma\rangle_\theta$ *and a CBS-sequence* $(\sigma_{2n} \vee \neg\sigma_{2n+1})_{n\geq 0}$ (see Definition 4.6) *such that* $\sigma_\zeta = \bigsqcap_{n\geq 1}^{\mathcal{A}(A)}(\sigma_{2n} \vee \neg\sigma_{2n+1})$ *exists.*

Proof. Since for each $A \in \mathcal{A}$, $\mathcal{K}(A)$ is a Boolean sublattice of $\mathrm{Con}(A)$, for all $\zeta, \sigma \in \mathcal{K}(A)$ we have that $(\neg\zeta \cap \sigma, \neg(\neg\zeta \cap \sigma)) = (\neg\zeta \cap \sigma, \zeta \vee \neg\sigma)$ is a pair of factor congruences in $\mathcal{K}(A)$. Hence, by Theorem 4.9, our claim. □

By Theorem 4.14 and Proposition 4.7(2) we can immediate establish the following instance of the CBS-theorem formulated in a language closer to the algebraic versions already known in literature.

Proposition 4.15. *Let* $\mathcal{A}$ *be a category of algebras,* $\mathcal{K}$ *be a Boolean factor congruences presheaf and* $A \in \mathcal{A}$ *such that* $\mathcal{K}(A)$ *is dual orthogonal* σ*-complete Boolean lattice. Then A has the* $CBS_{\mathcal{K}}$*-property.*

5. Boolean factor congruences and CBS-property

An algebra A has *Boolean factor congruences* (BFC for short) if and only if $\mathrm{FC}(A)$ is a Boolean sublattice of $\mathrm{Con}(A)$. We say that a *category of algebras has BFC* if and only if each algebra of the category has BFC.

Categories of algebras having BFC are examples of categories where the class operator FC defines a Boolean factor congruences presheaf. In virtue of Proposition 4.15 it is possible to establish several examples of the CBS-theorem for these categories. Indeed, most of the versions of the CBS-theorem related to classes of algebras having an underling lattice structure can be formulated in terms of the congruences presheaf FC. In this section we deal with this argument and we establish new examples of algebras having the CBS_{FC}-property.

Proposition 5.1. *Let* $\mathcal{A}$ *be a category of algebras having BFC such that for each* $A \in \mathcal{A}$ *and* $\sigma \in \mathrm{FC}(A)$*,* $A/\sigma \in \mathcal{A}$*. Then* FC *is a Boolean factor congruences presheaf.*

Proof. Let $A \in \mathcal{A}$ and $\sigma \in \mathrm{FC}(A)$. Let us suppose that $\theta \in \mathrm{FC}(A) \cap [\sigma, \nabla_A]_{\mathrm{Con}(A)}$. We want to prove that $\theta/\sigma \in \mathrm{FC}(A/\sigma)$. We first note that $\theta/\sigma \cap (\neg\theta \vee \sigma)/\sigma = \Delta_{A/\sigma}$. Moreover, $\nabla_A = \theta \circ \neg\theta \subseteq \theta \circ (\neg\theta \vee \sigma)$ and, by Theorem 2.3(3), $\theta/\sigma \circ (\neg\theta \vee \sigma)/\sigma = \nabla_{A/\sigma}$. Thus, $(\theta/\sigma, (\neg\theta \vee \sigma)/\sigma)$ is a pair of factor congruences of A/σ and $\theta/\sigma \in \mathrm{FC}(A/\sigma)$. Now, if we suppose that $\theta/\sigma \in \mathrm{FC}(A/\sigma)$ then, by Proposition 2.4, $\theta \in \mathrm{FC}(A)$. Hence, by Proposition 3.5, FC is a Boolean factor congruences presheaf. □

The next proposition provides a general method to obtain algebras satisfying the CBS_{FC}-property in categories of algebras having BFC.

Proposition 5.2. *Let $\mathcal{A}$ be a category of algebras closed under direct products having BFC and let us consider a family $(A_i)_{i\in I}$ of directly indecomposable algebras in $\mathcal{A}$. Then*

$$B = \prod_{i\in I} A_i \text{ satisfies the } CBS_{\mathrm{FC}}\text{-property.}$$

In particular, if $I = \mathbb{N}$ and $A_i = A$ for each $i \in \mathbb{N}$ then B satisfies the CBS_{FC}-property in a non trivial way (see Remark 3.12).

Proof. Note that for each $i \in I$, $\mathrm{FC}(A) = \{\Delta_{A_i}, \nabla_{A_i}\}$. Then, by [23, Theorem 2 and Theorem 11], we can see that $\mathrm{FC}(B)$ is lattice isomorphic to $\prod_{i\in I} \mathrm{FC}(A_i) = \mathbf{2}^I$. Since $\mathbf{2}^I$ is a complete Boolean algebra, by Proposition 4.15, B satisfies the CBS_{FC}-property. The second part follows from Proposition 2.5. □

The rest of the section is devoted to rephrasing several versions of the CBS-theorem already known in literature in terms of Boolean factor congruences presheaves. Moreover we establish new versions of the theorem in categories of algebras having BFC.

Example 5.3 (Lattice ordered groups). A *lattice ordered group* (*l-group* for short) is an algebra $\langle A, +, \vee, \wedge, -, 0\rangle$ of type $\langle 2, 2, 2, 1, 0\rangle$ such that

(1) $\langle A, +, -, 0\rangle$ is a group,
(2) $\langle A, \vee, \wedge\rangle$ is a lattice,
(3) $x + (s \wedge t) + y = (x + s + y) \wedge (x + t + y)$,
(4) $x + (s \vee t) + y = (x + s + y) \vee (x + t + y)$.

Thus, l-groups define a variety of algebras denoted by $\mathcal{LG}$. Let $A \in \mathcal{LG}$. If $x \in A$ then we define $|x| = x \vee -x$. The positive cone of A is given by $A^+ = \{x \in A : x \geq 0\}$. A set $G \subseteq A$ is said to be orthogonal if and only if $G \subseteq A^+$ and $x \wedge y = 0$ for any pair of distinct elements $x, y \in G$. The l-group A is said to be *orthogonal σ-complete* if and only if each denumerable orthogonal subset of A has a supremum in A. It is well known that $\mathrm{Con}(A)$ is lattice isomorphic to the lattice $I_l(A)$ of all convex normal subgroups (also called l-ideals) of A. Moreover $\mathrm{FC}(A)$ is a Boolean sublattice of $\mathrm{Con}(A)$ (see [4, §XIII-9]) identified with a Boolean sublattice of $I_l(A)$, denoted by $\mathrm{FCI}_l(A)$, whose elements are called *direct factors* of A. Thus, $\mathcal{LG}$ has BFC and, by Proposition 5.1, FC is a Boolean factor congruences presheaf. If $I \in \mathrm{FCI}_l(A)$ then the set $\neg I$ defined by $\neg I = \{a \in A : |a| \wedge |x| = 0 \text{ for each } x \in I\}$ is the complement of I in $\mathrm{FCI}_l(A)$ (see [26, Eq. (1.3)]). To establish a CBS-theorem for l-groups we need to prove the following result:

> Let A be an orthogonal σ-complete l-group. Then $\mathrm{FC}(A)$ is a σ-complete Boolean algebra.

Indeed, if $(I_n)_{n\in\mathbb{N}}$ is a dual orthogonal sequence in $\mathrm{FCI}_l(A)$ then the sequence $(\neg I_n)_{n\in\mathbb{N}}$ is an orthogonal sequence in $\mathrm{FCI}_l(A)$ because $\mathrm{FCI}_l(A)$ is a Boolean algebra. By [26, Lemma 1.5] $\neg\bigcup_{n\in\mathbb{N}} \neg I_n \in \mathrm{FCI}_l(A)$ and in [41, Theorem 2.2.5] it is proved that $\neg\bigcup_{n\in\mathbb{N}} \neg I_n = \bigcap_{n\in\mathbb{N}} \neg\neg I_n = \bigcap_{n\in\mathbb{N}} I_n$. Thus,

$FCI_l(A)$ is a dual orthogonal σ-complete Boolean algebra and, by Proposition 2.2, $FCI_l(A)$ is a σ-complete Boolean algebra. Hence $FC(A)$ is a σ-complete Boolean algebra.

Therefore, by the above result and by Proposition 4.15, we can rephrase the CBS-theorem for l-groups (given in [26]) in terms of the Boolean factor congruences presheaf FC as follows.

CBS-theorem If A is an orthogonal σ-complete l-group then A has the $CBS_{\rm FC}$-property.

Example 5.4 [$\mathcal{L}$-varieties]. $\mathcal{L}$-varieties were introduced in [18] as a general lattice ordered structure in which several versions of the CBS-theorem can be formulated. A variety $\mathcal{A}$ of algebras is a *$\mathcal{L}$-variety* if and only if

(1) there are terms of the language of $\mathcal{A}$ defining on each $A \in \mathcal{A}$ operations $\vee$, $\wedge$, 0, 1 such that $L(A) = \langle A, \vee, \wedge, 0, 1\rangle$ is a bounded lattice;
(2) for all $A \in \mathcal{A}$ and for all $z \in Z(L(A))$, the binary relation Θ_z on A defined by $(a, b) \in \Theta_z$ if and only if $a \wedge z = b \wedge z$ is a congruence on A such that $A \cong A/\Theta_z \times A/\Theta_{\neg z}$.

Examples of $\mathcal{L}$-varieties are the following (see [18, §2])

- The variety $\mathcal{L}_{01}$ of bounded lattices and its subvarieties. In particular, distributive lattices and modular lattices.
- The variety $\mathcal{LI}_{01}$ of bounded lattices with involution "$\sim$" [30] satisfying the *Kleene equation* $x \wedge \sim x = (x \wedge \sim x) \wedge (y \vee \sim y)$. Subvarieties of $\mathcal{LI}_{01}$ are the variety $\mathcal{OL}$ of *ortholattices* [4,35], characterized by the equation $x \wedge \sim x = 0$, and the variety $\mathcal{KL}$ of *Kleene algebras* [1], characterized by the distributive law. The intersection $\mathcal{OL} \cap \mathcal{KL}$ is the variety $\mathcal{B}$ of *Boolean algebras*. An important subvariety of $\mathcal{OL}$ is the variety $\mathcal{OML}$ of *orthomodular lattices* [4,35].
- The variety $\mathcal{B}_\omega$ of *pseudocomplemented distributive lattices* [1] and the subvariety of Stone algebras $\mathcal{ST}$ defined as

$$\mathcal{ST} = \mathcal{B}_\omega + \{(x \wedge y)^* = x^* \vee y^*\}$$

  where * is the pseudocomplement (see [1, §VIII]).
- The variety $\mathcal{RL}$ of *residuated lattices* [29] also called *commutative integral residuated* $0,1$*-lattices* [33] defined by algebras $\langle A, \vee, \wedge, \odot, \to, 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$ satisfying:
  (1) $\langle A, \odot, 1\rangle$ is an abelian monoid,
  (2) $L(A) = \langle A, \vee, \wedge, 0, 1\rangle$ is a bounded lattice,
  (3) $(x \odot y) \to z = x \to (y \to z)$,
  (4) $((x \to y) \odot x) \wedge y = (x \to y) \odot x$,
  (5) $(x \wedge y) \to y = 1$.

  Very important subvarieties of $\mathcal{RL}$ are: the variety of *Heyting algebras* [1] given by $\mathcal{H} = \mathcal{RL} + \{x \odot y = x \wedge y\}$ and the variety of *BL-algebras*, characterized by

$$\mathcal{BL} = \mathcal{RL} + \{x \wedge y = x \odot (x \to y),\ (x \to y) \vee (y \to x) = 1\}.$$

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BL-algebras are the algebraic counterpart of the fuzzy logic related to continuous t-norms [21]. Important subvarieties of $\mathcal{BL}$ are: the variety of *MV-algebras*, representing the algebraic counterpart of the infinite-valued Łukasiewicz logic [9,21] given by $\mathcal{MV} = \mathcal{BL} + \{\neg\neg x = x\}$, the variety of *linear Heyting algebras*, also known as *Gödel algebras*, given by

$$\mathcal{HL} = \mathcal{H} + \{(x \to y) \vee (y \to x) = 1\}$$

and the variety of *Product logic algebras* [10,11] given by

$$\mathcal{PL} = \mathcal{BL} + \{\neg\neg x \to ((x \to (x \odot y)) \to (y \odot \neg\neg y))\}.$$

- The varieties of *Łukasiewicz* and of *Post algebras* of order $n \geq 2$ [1], as well as the various types of *Łukasiewicz–Moisil* algebras which are considered in [5].
- $\mathcal{PMV}$, the variety of *pseudo MV-algebras* [15,20].

Let $\mathcal{A}$ be a $\mathcal{L}$-variety. In [18, Proposition 1.4] it is proved that $\mathcal{A}$ has BFC. Then, by Proposition 5.1, FC is a Boolean factor congruences presheaf. Thus, the CBS-theorem given in [18, Corollary 3.8] can be rephrased as follows.

CBS-theorem Let $\mathcal{A}$ be a $\mathcal{L}$-variety and let $A \in \mathcal{A}$ such that $Z(L(A))$ is a σ-complete Boolean algebra. Then A has the CBS_{FC}-property.

Indeed, if $Z(L(A))$ is a σ-complete Boolean algebra then $\mathrm{FC}(A)$ is a σ-complete Boolean algebra too. Therefore, by Proposition 4.15, A has the CBS_{FC}-property.

Let $\mathcal{A}$ be a $\mathcal{L}$-variety and $A \in \mathcal{A}$. Let us notice that the σ-completeness of $L(A)$ does not generally imply the σ-completeness of $Z(L(A))$ (see [18, Example 4.1]). However, there are $\mathcal{L}$-varieties where the σ-completeness, orthogonal σ-completeness or dual orthogonal σ-completeness condition on the algebras guarantee the corresponding σ-completeness of their centers. In these particular cases an algebra $A \in \mathcal{A}$ such that $L(A)$ is σ-complete satisfies the CBS_{FC}-property. Examples of these particular $\mathcal{L}$-varieties are: Boolean algebras (where the CBS_{FC}-property was obtained by Sikorski and Tarski), orthomodular lattices (where the CBS_{FC}-property was obtained in [13]), MV-algebras (where the CBS_{FC}-property was obtained in [12]), pseudo MV-algebras (where the CBS_{FC}-property was obtained in [25]), Stone algebras [18, Proposition 4.3], BL-algebras [18, Corollary 4.8], Łukasiewicz and Post algebras of order n [8, Lemma 3.1].

Example 5.5 [Semigroups with $0,1$ and bounded semilattices]. A *semigroup with* $0,1$ is an algebra $\langle A, \cdot, 0, 1\rangle$ of type $\langle 2,0,0\rangle$ such that the operation $\cdot$ is associative, $0{\cdot}x = x{\cdot}0 = 0$ and $1{\cdot}x = x{\cdot}1 = x$. Thus, semigroups with $0,1$ define a variety denoted by $\mathcal{SG}_{0,1}$. An important subvariety of $\mathcal{SG}_{0,1}$ is the variety of *bounded semilattices* defined as $\mathcal{SL}_{0,1} = \mathcal{SG}_{0,1} + \{x^2 = x, \ \ x \cdot y = y \cdot x\}$. Let $\mathcal{A}$ be a subvariety of $\mathcal{SG}_{0,1}$ and $A \in \mathcal{A}$. An element $z \in A$ is called *central* if and only if there exist $A_1, A_2 \in \mathcal{A}$ and a $\mathcal{SG}_{0,1}$-isomorphism $f\colon A \to A_1 \times A_2$ such that $f(z) = (1,0)$. In [42,43] it is proved that the set of all central elements $Z(A)$ can be identified with $\mathrm{FC}(A)$. Thus, by Proposition 5.1, FC is a Boolean factor congruences presheaf.

Hence, if $A \in \mathcal{A}$ is an algebra such that $Z(A)$ is a σ-complete Boolean algebra then, by Proposition 4.15, A has the CBS_{FC}-property. By Proposition 5.2, denumerable direct product of directly indecomposable semigroups with $0, 1$ are concrete examples of algebras satisfying the CBS_{FC}-property in a non trivial way.

Example 5.6 [Commutative pseudo BCK-algebras]. A *commutative pseudo BCK-algebras* (${}^{cp}BCK$-algebra for short) [20] is an algebra $\langle A, \rightarrow, \rightsquigarrow, 1\rangle$ of type $\langle 2, 2, 0\rangle$ satisfying the following equations:

(1) $x \rightarrow (y \rightsquigarrow z) = y \rightarrow (x \rightsquigarrow z)$,
(2) $x \rightarrow x = x \rightsquigarrow x = 1$,
(3) $1 \rightarrow x = 1 \rightsquigarrow x = x$,
(4) $(x \rightarrow y) \rightsquigarrow y = (y \rightarrow x) \rightsquigarrow x$,
(5) $(x \rightsquigarrow y) \rightarrow y = (y \rightsquigarrow x) \rightarrow x$.

Thus ${}^{cp}BCK$-algebras define a variety denoted by ${}^{cp}\mathcal{BCK}$. Let A be a ${}^{cp}BCK$-algebra. The relation $x \leq y$ if and only if $x \rightarrow y = 1$ if and only if $x \rightsquigarrow y = 1$ defines a join semi-lattice order where $x \vee y = (x \rightarrow y) \rightsquigarrow y = (x \rightsquigarrow y) \rightarrow y$. Let us notice that in [34] a dually equivalent definition for ${}^{cp}BCK$-algebras, based on the reverse order, is introduced. In [17, Corollary 4.4] it is proved that ${}^{cp}\mathcal{BCK}$ is a congruence distributive variety. Then, for each $A \in^{cp} \mathcal{BCK}$, $\mathrm{FC}(A)$ is a Boolean sublattice of $\mathrm{Con}(A)$. Thus ${}^{cp}\mathcal{BCK}$ has BFC and, by Proposition 5.1, FC is a Boolean factor congruences presheaf. By [34, Lemma 4.1] we can dually prove that if A is a dual orthogonal σ-complete ${}^{cp}BCK$-algebra then each dual orthogonal sequences $(\theta_n)_{n\in\mathbb{N}}$ in $\mathrm{FC}(A)$ admits the infimum $\bigcap_{n\in\mathbb{N}} \theta_n \in \mathrm{FC}(A)$. Hence, by Proposition 2.2, if A is a dual orthogonal σ-complete ${}^{cp}BCK$-algebra then $\mathrm{FC}(A)$ is a σ-complete Boolean algebra. Thus, by Proposition 4.15, the version of CBS-theorem for ${}^{cp}BCK$-algebras given in [34], can be rephrased as follows.

CBS-theorem If A is a dual orthogonal σ-complete ${}^{cp}BCK$-algebra then A has the CBS_{FC}-property.

Example 5.7 [Church algebras]. An algebra A is called *Church algebra* [36] if and only if there are two constants $0, 1 \in A$ and a ternary term $t(z, x, y)$ called *if-then-else term* in the language of A such that $t(1, x, y) = x$ and $t(0, x, y) = y$. A variety of algebras $\mathcal{A}$ is called a *Church variety* if and only if every algebra in $\mathcal{A}$ is a Church algebra with respect to the same term $t(z, x, y)$ and constants $0, 1$. Let $\mathcal{A}$ be a Church variety and $A \in \mathcal{A}$. An element $e \in A$ is called *central* if and only if the generated congruences $\theta(1, e)$ and $\theta(e, 0)$ defines a pair of factor congruences of A. It is proved that central elements are equationally characterized in the following way: $e \in A$ is a central element if and only if whenever φ is an operation symbol of arity n in the language of $\mathcal{A}$ and $\overline{a}, \overline{b} \in A^n$, the following equations are satisfied

$$t(e, x, x) = x, \quad t(e, t(e, x, y), z) = t(e, x, z) = t(e, x, t(e, y, z)),$$

$$t(e, 1, 0) = e, \quad t(e, \varphi^A(\overline{a}), \varphi^A(\overline{b})) = \varphi^A(t(e, a_1, b_1) \ldots t(e, a_n, b_n)).$$

Moreover the set $Z(A)$ of all central elements endowed with the operations $x \vee y = t(x, 1, y)$, $x \wedge y = t(x, y, 0)$ and $\neg x = t(x, 0, 1)$ is a Boolean algebra isomorphic to FC(A). Thus, $\mathcal{A}$ has BFC and, by Proposition 5.1, FC is a Boolean factor congruences presheaf. In what follows we shall study concrete examples of Church algebras satisfying the CBS_{FC}-property.

- *Rings with identity* define a Church variety denoted by $\mathcal{R}_1$ where the if-then-else term is given by $t(z,x,y) = (y+z-zy)\cdot(1-z+zx)$. If $A \in \mathcal{R}_1$ then $Z(A)$ is the set of central idempotent elements of A. Two interesting examples of rings with identity whose central idempotent elements define a complete Boolean algebra are the following:
  - *Division rings* because they are simple algebras. Then, by Proposition 5.2, denumerable direct products of division rings satisfy the CBS_{FC}-property in a non trivial way.
  - *Baer rings* i.e., a ring with identity A such that for every subset $S \subseteq A$ the right annihilator $Ann_r(S) = \{r \in A : \forall s \in S, r\cdot s = 0\}$ is the principal right ideal generated by an idempotent element. In [2, §3, 3.3] it is proved that $Z(A)$ is a complete Boolean algebra. Then, by Proposition 4.15, Baer rings have the CBS_{FC}-property.
- $*$-*Rings*. They are rings with identity having an involution operation $x \mapsto x^*$ such that $x^{**} = x$, $(x+y^*) = x^* + y^*$ and $(x\cdot y)^* = y^*\cdot x^*$. By the underling ring with unity structure, $*$-rings define a Church variety denoted by $\mathcal{R}_1^*$. Examples of $*$-rings having the CBS_{FC}-property are the Baer $*$-rings. Indeed: A *Baer* $*$-*ring* is a $*$-rings A such that for every subset $S \subseteq A$, $Ann_r(S) = eA$ where e is a projection (i.e. $e^2 = e^* = e$). By [3, P18, 4A] we can see that $Z(A)$ is determined by the central projections. Moreover, in a Baer $*$-rings their central projections define a complete Boolean algebra [31, p.30, Corollary]. Thus, by Proposition 4.15, Baer $*$-rings have the CBS_{FC}-property.

Example 5.8 [Effect and pseudo-effect algebras]. Although there are versions of the CBS-theorem related to these structures [16,27], from a strictly formal viewpoint, these versions cannot be framed in our formalism because these algebras are defined by a binary partial operation. However, we can easily extend the notion of Boolean factor congruences presheaf and the CBS-property to these particular algebraic structures. A *pseudo-effect algebra* is a partial algebra $\langle E, +, 0, 1\rangle$ of type $\langle 2, 0, 0\rangle$ such that

(1) $a+b$ and $(a+b)+c$ exist if and only if $b+c$ and $a+(b+c)$ exist and in this case $(a+b)+c = a+(b+c)$,
(2) for each $a \in E$ there is exactly one $a^- \in E$ and exactly one $a^\sim \in E$ such that $a^- + a = a + a^\sim = 1$,
(3) if $a+b$ exists, there are elements $d, e \in E$ such that $a+b = d+a = b+e$,
(4) if $1+a$ or $a+1$ exists then $a = 0$.

We denote by $\mathcal{PE}$ the category whose objects are pseudo-effect algebras and whose arrows, called $\mathcal{PE}$-*homomorphisms*, are functions $f\colon E \to F$ between pseudo-effect algebras such that $f(0) = 0$, $f(1) = 1$ and $f(a+b) = f(a) + f(b)$ whenever $a+b$ exists in E. If $+$ is commutative then E is said to be an *effect algebra* and we denote by $\mathcal{E}$ the subcategory of effect algebras. Let $E \in \mathcal{PE}$. If we define $a \leq b$ if and only if there exists $x \in E$ such that

$a + x = b$ then $\langle E, \leq \rangle$ is a partial order such that $0 \leq a \leq 1$ for any $a \in E$. For a given $e \in E$ the interval $[0, e]_E$ endowed with $+$ restricted to $[0, e]_E^2$ is a pseudo effect algebra $\langle [0, e]_E, +, 0, e \rangle$. An element $e \in E$ is said to be *central* if and only if there exists a $\mathcal{PE}$-isomorphism $f_e \colon E \to [0, e]_E \times [0, e^\sim]_E$ such that $f_e(e) = (e, 0)$ and, if $f_e(x) = (x_1, x_2)$ then $x = x_1 + x_2 = x_1 \vee x_2$. We denote by $Z(E)$ the set of all central elements of E. In [16, Proposition 2.2] it is proved that for any $x \in E$ and $e \in Z(E)$, $x \wedge e$ and $x \wedge e^\sim$ are defined in E and, moreover, $\pi_e \colon E \to [0, e]_E$ such that $\pi_e(x) = x \wedge e$ is a surjective $\mathcal{PE}$-homomorphism. Furthermore, in [16, Theorem 2.3], it is proved that $\langle Z(E), \wedge, ^\sim, 0, 1 \rangle$ is a Boolean algebra. Let us notice that for each $e \in Z(E)$, $\theta_e = \{(x, y) \in E^2 : x \wedge e = y \wedge e\}$ defines a congruence on E such that $E/\theta_e \cong_{\mathcal{PE}} [0, e]_E$. Let us consider the set $\mathrm{FC}(E) = \{\theta_e : e \in Z(E)\}$. It is not very hard to see that for each $e_1, e_2 \in Z(E)$, $\theta_{e_1} \cap \theta_{e_2} = \theta_{e_1 \vee e_2}$. Moreover, the ordered set $\langle \mathrm{FC}(E), \subseteq \rangle$ defines a Boolean algebra $\langle \mathrm{FC}(E), \cap, \vee, \neg, \Delta_E \nabla_E \rangle$ where, $\theta_{e_1} \vee \theta_{e_2} = \theta_{e_1 \wedge e_2}$, $\neg \theta_e = \theta_{e^\sim}$ and the function $e \mapsto \theta_e$ is an order reverse isomorphism from $Z(E)$ to $\mathrm{FC}(E)$. We also note that the class operator $E \mapsto \mathrm{FC}(E)$ defines a congruence operator over $\mathcal{PE}$ in the meaning of Definition 3.4 and, taking into account Eq. (3.3), we can define the class $Hom_{\mathcal{PE}_{\mathrm{FC}}}$ in the following way:

$$Hom_{\mathcal{PE}_{\mathrm{FC}}} = \bigcup_{E \in \mathcal{PE}} \{E \xrightarrow{f_e} [0, e]_E \colon f_e(x) = x \wedge e \ \ and \ \ e \in Z(E)\}. \tag{5.1}$$

In [16, Proposition 2.8] it is proved that:

$$\text{for each } e \in Z(E) \text{ and } x \leq e,\ x \in Z([0, e]_E) \text{ if and only if } x \in Z(E). \tag{5.2}$$

Therefore, by Eq. (5.2), it immediately follows that $Hom_{\mathcal{PE}_{\mathrm{FC}}}$ is closed under composition of $\mathcal{PE}$-homomorphisms and then $\mathcal{PE}_{\mathrm{FC}} = \langle Ob(\mathcal{PE}), Hom_{\mathcal{PE}_{\mathrm{FC}}} \rangle$ defines a category. Let us notice that Eq. (5.2) also implies that if $E \xrightarrow{f_e} [0, e]_E \in Hom_{\mathcal{PE}_{\mathrm{FC}}}$ and if $\theta_a \in \mathrm{FC}([0, e]_E)$ then $[\mathrm{FC}(f_e)](\theta_a) = f_e^*(\theta_a) = \{(x, y) \in E^2 \colon x \wedge a = y \wedge a\} \in \mathrm{FC}(E)$. Consequently, it is not hard to see that $\mathrm{FC} \colon \mathcal{PE}_{\mathrm{FC}} \to Set$ is a presheaf. Thus, following Definition 4.10, we can refer to FC as a Boolean factor congruences presheaf for pseudo-effect algebras.

Now, taking into account Definition 3.10, it is possible to analogously introduce the notion of CBS_{FC}-property for these partial structures. Indeed,

> A pseudo-effect algebra E has the CBS_{FC}-property the following holds: Given a pseudo-effect algebra F, and $\theta_f \in \mathrm{FC}(F)$ such that there is $\theta_e \in \mathrm{FC}(E)$ with $E \cong_{\mathcal{PE}} F/\theta_f$ and $F \cong_{\mathcal{PE}} E/\theta_e$, it follows that $E \cong_{\mathcal{PE}} F$.

In [16, Proposition 6.2] it is proved that if $E, F \in \mathcal{PE}$ and $h \colon E \to [0, f]_F$ is a $\mathcal{PE}$-isomorphism where $f \in Z(F)$ then, for each $e \in Z(E)$, $h(e) \in Z(F)$. This result and the order reverse identification $Z(E) \cong \mathrm{FC}(E)$ allow us to establish the useful equivalence of the CBS_{FC}-property given in Theorem 3.11 for pseudo-effect algebras. More precisely, following the proof of Theorem 3.11, we can also prove that for each pseudo-effect algebra E the following conditions are equivalent

(1) E has the CBS_{FC}-property.

(2) If $\theta \in \mathrm{FC}(E)$ and $E \cong_{\mathcal{PE}} E/\theta$ then for all $\sigma \in \mathrm{FC}(E)$ such that $\sigma \subseteq \theta$ we have that $E \cong_{\mathcal{PE}} E/\sigma$.

The CBS-theorem for pseudo-effect algebras given in [16] is formulated under the hypothesis of orthogonal σ-completeness (referred as central decomposition property in [16]) of the center of the algebras. Since the center of a pseudo-effect algebra E is a Boolean algebra then, by Proposition 2.2, the central decomposition property turns out to be equivalent to the σ-completeness of $Z(E)$. Hence, by the order reverse identification $Z(E) \cong \mathrm{FC}(E)$ for each $E \in \mathcal{PE}$, the CBS-theorem for pseudo-effect algebras given in [16, Theorem 6.3] and the CBS-theorem for effect algebras given in [27, Theorem 1.6] can be rephrased as follows:

CBS-theorem Let E be a pseudo-effect algebra such that $Z(E)$ is a σ-complete Boolean algebra. Then E has the CBS_{FC}-property.

In this way, we have extended our abstract framework for the CBS-theorem to these partial algebraic structures.

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