Complex Manifolds 2019; 6:320–334 DE GRUYTER

Research Article Open Access

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Nearly Sasakian manifolds revisited

https://doi.org/10.1515/coma-2019-0017
Received March 30, 2019; accepted July 2, 2019

Abstract: We provide a new, self-contained and more conceptual proof of the result that an almost contact metric manifold of dimension greater than 5 is Sasakian if and only if it is nearly Sasakian.

MSC: Primary 53C25, 53D35

Dedicated to Prof. David E. Blair on the occasion of his 78th birthday

1 Introduction

A Sasakian manifold M is a contact metric manifold that satisfies a normality condition, encoding the integrability of a canonical almost complex structure on the product $M \times \mathbb{R}$. Several equivalent characterizations of this class of manifolds, in terms of Riemannian cone, or transversal structure, or curvature, are also known. In particular one can show that an almost contact metric structure (g, ϕ, ξ, η) is Sasakian if and only if the covariant derivative of the endomorphism ϕ satisfies

$$(\nabla_X \phi)Y - g(X, Y)\xi + \eta(Y)X = 0, \tag{1}$$

for all vector fields $X, Y \in \Gamma(TM)$. A relaxation of this notion was introduced by Blair, Showers and Yano in [2], under the name of nearly Sasakian manifolds, by requiring that just the symmetric part of (1) vanishes. Later on, several important properties of nearly Sasakian manifolds were discovered by Olszak ([6]). Nearly Sasakian manifolds may be considered as an odd-dimensional analogue of nearly Kähler manifolds. In fact, the prototypical example of nearly Sasakian manifold is the 5-sphere as totally umbilical hypersurface of $\mathbb{S}^6$, endowed with the almost contact metric structure induced by the well-known nearly Kähler structure of $\mathbb{S}^6$. Nevertheless, in recent years several differences between nearly Sasakian and nearly Kähler geometry were pointed out. In particular, in [3] it was proved that the 1-form η of any nearly Sasakian manifold is necessarily a contact form, while the fundamental 2-form of a nearly Kähler manifold is never symplectic unless the manifold is Kähler. A peculiarity of nearly Sasakian five dimensional manifolds which are not Sasakian is that upon rescaling the metric one can define a Sasaki-Einstein structure on them. In fact one has an SU(2)-reduction of the frame bundle. Conversely, starting with a five dimensional manifold with a Sasaki-Einstein SU(2)-structure it is possible to construct a one-parameter family of nearly Sasakian non-Sasakian manifolds. Thus the theory of nearly Sasakian non-Sasakian manifolds is essentially equivalent to the one of Sasaki-Einstein manifolds.

Concerning other dimensions, there have been many attempts of finding explicit examples of nearly Sasakian non-Sasakian manifolds until the recent result obtained in [4] showing that every nearly Sasakian

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structure of dimension greater than five is always Sasakian. This result depends on the early work [3] by the first and third authors, which in turn draws many properties proved in [6]. This makes the proof to be spread over several different texts with different notation.

The aim of this note is to provide a complete and streamlined proof of the aforementioned dimensional restriction on nearly Sasakian non-Sasakian manifolds. We will also pinpoint where the positivity of the Riemannian metric is used. For this purpose we work in the more general setting of pseudo-Riemannian geometry. We will always assume that the metric is non-degenerate.

This paper was written on occasion of the conference *RIEMain in Contact*, held in Cagliari (Italy), 18–22 June 2018.

2 Preliminaries

2.1 Tensor calculus notation.

In this section we review the notation for the tensor calculus we use throughout the paper.

Given a permutation $\sigma \in \Sigma_q$, we will denote by the same symbol the (q, q)-tensor $TM^{\otimes q} \to TM^{\otimes q}$ defined by $\sigma(X_1 \otimes \cdots \otimes X_q) = X_{\sigma^{-1}(1)} \otimes \cdots \otimes X_{\sigma^{-1}(q)}$.

Let ∇ be a covariant derivative. It is easy to show that $\nabla\sigma = 0$. If T is an arbitrary (p, q)-tensor, then ∇T can be considered as a $(p, q+1)$-tensor. We define recursively the $(p, q+k)$-tensors $\nabla^k T$ by $\nabla^{k+1} T := \nabla(\nabla^k T)$.

We will use the following convention regarding the arguments of $\nabla^k T$

$$(\nabla^k T)(X_1 \otimes \cdots \otimes X_{q+k}) := (\nabla^k_{X_1,\dots,X_k} T)(X_{k+1} \otimes \cdots \otimes X_{q+k}).$$

Given T_1 and T_2 of valencies (p_1, q_1), (p_2, q_2), respectively, and such that $q_1 \geq p_2$, we define the tensor $T_1 \circ T_2$ of type $(p_1, q_1 - p_2 + q_2)$ by

$$(T_1 \circ T_2)(X_1, \dots X_{q_1-p_2}, Y_1, \dots Y_{q_2}) = T_1(X_1, \dots, X_{q_1-p_2}, T_2(Y_1, \dots, Y_{q_2})).$$

Note that with our convention for ∇T, if T_1 and T_2 are tensors of valencies (p_1, q_1) and (p_2, q_2) respectively, then

$$\nabla(T_1 \otimes T_2) = \nabla T_1 \otimes T_2 + (T_1 \otimes \nabla T_2) \circ (q_1 + 1, \dots, 2, 1),$$

where we used the cycle notation for permutations, as we will do throughout the paper. Moverover, one has

$$\nabla(T_1 \circ T_2) = \nabla T_1 \circ T_2 + T_1 \circ \nabla T_2 \circ (q_1 - p_2 + 1, \dots, 2, 1). \tag{2}$$

Of course if $q_1 = p_2$, then we get just $\nabla(T_1 \circ T_2) = (\nabla T_1) \circ T_2 + T_1 \circ (\nabla T_2)$. Suppose $T_2 = \sigma$ is a permutation in Σ_{q_1}. Then (2) should be used with caution since in the term $\nabla T_1 \circ \sigma$, we have to consider σ as an element of Σ_{q_1}, not as an element of Σ_{q_1+1}. Let us denote by s the inclusion Σ_{q_1} into Σ_{q_1+1} defined by $s(\sigma)(i) = \sigma(i-1)+1$, $i \geq 2$, $s(\sigma)(1) = 1$. Then $\nabla(T \circ \sigma) = \nabla T \circ s(\sigma)$. In the computations below, we will always substitute σ with $s(\sigma)$ when needed, so that if in the composition chain the tensor T of type (p, q) is followed by a permutation σ then σ is always in Σ_q.

2.2 Nearly Sasakian manifolds

The definition of Sasakian manifolds was motivated by study of local properties of Kähler manifolds. Namely, a *Sasakian manifold* is an odd dimensional Riemannian manifold (M, g) such that the metric cone $(M \times \mathbb{R}_+, tg + dt^2)$ is Kähler. Sasakian manifolds can also be characterized as a subclass of almost contact metric manifolds.

Definition 2.1. An *almost contact metric manifold* is a tuple $(M^{2n+1}, g, \phi, \xi, \eta)$, where

1) g is a Riemannian metric;
2) ϕ is a $(1, 1)$-tensor;
3) ξ is a vector field on M;
4) η is a 1-form on M

such that

i) $\phi^2 = -\mathrm{Id} + \xi \otimes \eta$
ii) $\eta(X) = g(X, \xi)$, $g(\xi, \xi) = 1$;
iii) ϕ is skew symmetric, i.e. $g \circ (\phi \otimes \mathrm{Id}) = -g \circ (\mathrm{Id} \otimes \phi)$.

From the definition it follows that $\phi\xi = 0$ and $\eta \circ \phi = 0$.

By [1, Theorem 6.3] the following can be used as an alternative definition of Sasakian manifolds.

Definition 2.2. A *Sasakian* manifold is an almost contact metric manifold (M, g, ϕ, ξ, η) such that

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X. \tag{3}$$

Nearly Sasakian manifolds where introduced in [2] as a generalization of Sasakian manifolds by relaxing the condition (3).

Definition 2.3. A *nearly Sasakian* manifold is an almost contact metric manifold (M, g, ϕ, ξ, η) such that

$$(\nabla_X \phi)X = g(X, X)\xi - \eta(X)X. \tag{4}$$

By polarizing at X the condition (4) can be restated in the form

$$(\nabla\phi - \xi \otimes g + \eta \otimes \mathrm{Id})(1 + (1, 2)) = 0. \tag{5}$$

As explained in the introduction, we will work in the more general setting of pseudo-Riemannian geometry. The definitions of *nearly pseudo-Sasakian* and *pseudo-Sasakian* manifolds are the same as above with only distinction that now g is a pseudo-Riemannian metric.

We start with establishing some simple properties of nearly pseudo-Sasakian manifolds. In the case of nearly Sasakian manifolds they were proved in [2].

Proposition 2.4. *If (M, g, ϕ, ξ, η) is a nearly pseudo-Sasakian manifold then*

i) for any vector field X, the vector field $\nabla_X \xi$ is orthogonal to ξ, equivalently $\eta \circ \nabla\xi = 0$;
ii) $\nabla_\xi \xi = 0$ and $\nabla_\xi \eta = 0$;
iii) the operators $\nabla_\xi \phi$ and $\phi \circ \nabla_\xi \phi$ are skew-symmetric and anticommute with ϕ;
iv) $\nabla_\xi \phi = \phi(\phi + \nabla\xi)$ and $\phi + \nabla\xi + \phi \circ \nabla_\xi \phi = 0$.
v) $(\nabla\xi)^2 + \mathrm{Id} - \xi \otimes \eta = (\nabla_\xi \phi)^2 = (\phi \nabla_\xi \phi)^2$, in particular, $(\nabla\xi)^2$ commutes with ϕ;
vi) ξ is a Killing vector field or, equivalently, $\nabla\xi$ is a skew-symmetric operator;
vii) $d\eta = 2\nabla\eta = -2g \circ \nabla\xi$.

Proof. Applying ∇_X to $1 = g(\xi, \xi)$, we get

$$0 = g(\nabla_X \xi, \xi) + g(\xi, \nabla_X \xi) = 2g(\nabla_X \xi, \xi) = 2(\eta \circ \nabla\xi)(X),$$

which is equivalent to $\nabla_X \xi \perp \xi$.

To show that $\nabla_\xi \xi = 0$, we proceed as follows. First we substitute $X = \xi$ in $(\nabla_X \phi)X = g(X, X)\xi - \eta(X)X$ and obtain $(\nabla_\xi \phi)\xi = 0$. As $\phi\xi = 0$, this implies $\phi(\nabla_\xi \xi) = 0$. Therefore

$$0 = \phi^2(\nabla_\xi \xi) = -\nabla_\xi \xi + \eta(\nabla_\xi \xi)\xi.$$

Since $\eta \circ \nabla\xi = 0$, the above equation implies

$$\nabla_\xi \xi = \eta(\nabla_\xi \xi)\xi = 0, \qquad \nabla_\xi \eta = \nabla_\xi (g \circ (\xi \otimes \mathrm{Id})) = g \circ (\nabla_\xi \xi \otimes \mathrm{Id}) = 0. \tag{6}$$

To see that $\nabla_\xi \phi$ is skew-symmetric it is enough to apply ∇_ξ to the equation $g \circ (\phi \otimes \mathrm{Id} + \mathrm{Id} \otimes \phi) = 0$. To show that $\nabla_\xi \phi$ anticommutes with ϕ we apply ∇_ξ to the equation $\phi^2 = -\mathrm{Id} + \xi \otimes \eta$ and use $\nabla_\xi \xi = 0$, $\nabla_\xi \eta = 0$. Now, that $\phi \nabla_\xi \phi$ is skew-symmetric and anticommutes with ϕ follows from the following computation

$$g(\phi(\nabla_\xi \phi)X, Y) = -g((\nabla_\xi \phi)X, \phi Y) = g(X, (\nabla_\xi \phi)\phi Y) = -g(X, (\phi \nabla_\xi \phi)Y)$$
$$\phi(\phi \nabla_\xi \phi) = -\phi((\nabla_\xi \phi)\phi) = -(\phi \nabla_\xi \phi)\phi.$$

Next we show that $\nabla_\xi \phi = \phi(\phi + \nabla\xi)$. First we polarize $(\nabla_X \phi)X = g(X, X)\xi - \eta(X)X$ with respect to X, and get that for any two vector fields X and Y

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = 2g(X, Y)\xi - \eta(X)Y - \eta(Y)X. \tag{7}$$

Taking $Y = \xi$ in the above equation, we obtain

$$(\nabla_X \phi)\xi + (\nabla_\xi \phi)X = \eta(X)\xi - X = \phi^2 X. \tag{8}$$

As $\phi\xi = 0$, we have $(\nabla_X \phi)\xi = \nabla_X(\phi\xi) - \phi(\nabla_X \xi) = -\phi(\nabla_X \xi) = -(\phi \circ \nabla\xi)X$. Thus (8) can be rewritten as $\nabla_\xi \phi = \phi(\phi + \nabla\xi)$. Now since $\eta \circ \phi = 0$ and $\eta \circ \nabla\xi = 0$, we get

$$\phi \circ \nabla_\xi \phi = \phi^2(\phi + \nabla\xi) = -\phi + \xi \otimes (\eta \circ \phi) - \nabla\xi + \xi \otimes (\eta \circ \nabla\xi) = -\phi - \nabla\xi.$$

Next we show that $(\nabla_\xi \phi)^2 = (\nabla\xi)^2 + \mathrm{Id} - \xi \otimes \eta$. Since ϕ anticommutes with $\nabla_\xi \phi$, we get

$$(\nabla\xi)^2 = (-\phi - \phi \circ \nabla_\xi \phi)^2 = \phi^2 + \phi^2 \circ \nabla_\xi \phi + \phi \circ \nabla_\xi \phi \circ \phi + \phi \circ \nabla_\xi \phi \circ \phi \circ \nabla_\xi \phi$$
$$= -\mathrm{Id} + \eta \otimes \xi + (\nabla_\xi \phi)^2 - (\nabla_\xi \phi)^2 \xi \otimes \eta = -\mathrm{Id} + \eta \otimes \xi + (\nabla_\xi \phi)^2,$$

where in the last step we used $(\nabla_\xi \phi)\xi = \nabla_\xi(\phi\xi) - \phi(\nabla_\xi \xi) = 0$. Since ϕ anticommutes with $\nabla_\xi \phi$, we get

$$(\phi \nabla_\xi \phi)^2 = -(\nabla_\xi \phi)^2 \phi^2 = (\nabla_\xi \phi)^2 + (\nabla_\xi \phi)\xi \otimes \eta = (\nabla_\xi \phi)^2,$$

where we used in the last step $\phi\xi = 0$ and $\nabla_\xi \xi = 0$.

Next we show that ϕ commutes with $(\nabla\xi)^2$. Since ϕ anticommutes with $\nabla_\xi \phi$, it commutes with $(\nabla_\xi \phi)^2$. Thus to show that $(\nabla\xi)^2$ commutes with ϕ, we only have to check that ϕ commutes with $\xi \otimes \eta$. But, as we saw, $\phi\xi = 0$ and $\eta \circ \phi = 0$. Thus $\phi \circ (\xi \otimes \eta) = 0 = (\xi \otimes \eta) \circ \phi$.

Next we prove that ξ is a Killing vector field, which, in view of

$$\mathcal{L}_\xi g = g \circ (\nabla\xi \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla\xi),$$

is equivalent to the claim that $\nabla\xi$ is skew-symmetric. But $\nabla\xi = -\phi - \phi \circ \nabla_\xi \phi$ is a sum of two skew-symmetric operators, and therefore is skew-symmetric.

Since $\nabla\xi$ is skew-symmetric, we get

$$d\eta(X, Y) = (\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = g(Y, \nabla_X \xi) - g(X, \nabla_Y \xi)$$
$$= -2(g \circ \nabla\xi)(X, Y)$$
$$= 2g(Y, \nabla_X \xi) = 2(\nabla_X (g \circ \xi))(Y) = 2(\nabla_X \eta)(Y).$$

□

Next we establish that the 1-form η of any nearly Sasakian manifold is contact. We use in this proposition that the metric g is positively defined, since this permits to conclude that the square of g-skew-symmetric operator has non-positive spectrum. This is not true for a general pseudo-Riemannian metric.

Theorem 2.5 ([3]). *Let $(M^{2n+1}, g, \phi, \xi, \eta)$ be a nearly Sasakian manifold. Then*

i) the eigenvalues of $(\nabla\xi)^2$ are non-positive and 0 has multiplicity one in the spectrum of $(\nabla\xi)^2$;
ii) the operator $(\nabla\xi)$ has rank $2n$;
iii) η is a contact form.

Proof. By Proposition 2.4, the operator $\nabla_\xi\phi$ is skew-symmetric, and therefore the eigenvalues of $(\nabla_\xi\phi)^2 - \mathrm{Id}$ are negative. By the same proposition $(\nabla\xi)^2 - \xi\otimes\eta = (\nabla_\xi\phi)^2 - \mathrm{Id}$. This shows that the spectrum of $A := (\nabla\xi)^2 - \xi\otimes\eta$ is negative and A has rank $2n+1$. Since $\mathbf{rk}(\xi\otimes\eta) = 1$ and for any two operators $\mathbf{rk}(B+C) \le \mathbf{rk}(B) + \mathbf{rk}(C)$, we conclude that $2n+1 = \mathbf{rk}(A) \le \mathbf{rk}((\nabla\xi)^2) + 1$, i.e. the rank of $(\nabla\xi)^2$ is at least $2n$. This shows also that multiplicity of 0 in the spectrum of $(\nabla\xi)^2$ cannot be greater than one. Since ξ is in the kernel of $\nabla\xi$ we get that the spectrum of $(\nabla\xi)^2$ contains 0, it has multiplicity one, and $(\nabla\xi)^2$ has rank $2n$. As $\nabla\xi$ is skew-symmetric by Proposition 2.4, the rank of $\nabla\xi$ coincides with the rank of $(\nabla\xi)^2$. Therefore $\mathbf{rk}(\nabla\xi) = 2n$. Thus at every point of M, there exists an adapted basis of T_xM of the form $\xi, X_1, \dots, X_n, Y_1, \dots, Y_n$, with the property that $\nabla_{X_k}\xi = \lambda_k Y_k$ and $\nabla_{Y_k}\xi = -\lambda_k X_k$ for some $\lambda_k > 0$. Then

$$(\eta\wedge(d\eta)^n)(\xi, X_1, Y_1, \dots, X_n, Y_n) = n!\cdot 2^n\cdot\prod_{k=1}^n \lambda_k \neq 0.$$ □

3 Curvature properties of nearly Sasakian manifolds

In this section we reestablish curvature properties of nearly Sasakian manifolds obtained by Olszak in [6]. The main consequence of these properties, used in the rest of the paper, is an explicit formula for $\nabla^2\xi$ in terms of $\nabla\xi$.

We will use the following notation for curvature tensors

$$R_{X,Y} := \nabla^2_{X,Y} - \nabla^2_{Y,X}, \text{ i.e. } R = \nabla^2\circ(1-(1,2))$$
$$\widetilde{R}(X,Y,Z,W) := g\left(R_{X,Y}Z,\ W\right).$$

In particular $R\xi$ denotes the $(1,2)$-tensor on M given by $(R\xi)(X,Y) = R_{X,Y}\xi$. Also

$$(\widetilde{R}\circ(1,4,3,2))(X,Y,Z,W) = \widetilde{R}(Y,Z,W,X) = g(R_{Y,Z}W,X) = g(X,R_{Y,Z}W)$$
$$= (g\circ R)(X,Y,Z,W),$$

that is

$$\widetilde{R}\circ(1,4,3,2) = g\circ R. \tag{9}$$

For every covariant tensor $T\in\Gamma(TM^{\otimes k})$ and endomorphism ϕ, we define $i_\phi T\in\Gamma(TM^{\otimes k})$ by

$$i_\phi T = T\circ(\phi\otimes\mathrm{Id}^{\otimes(k-1)} + \mathrm{Id}\otimes\phi\otimes\mathrm{Id}^{\otimes(k-2)} + \dots + \mathrm{Id}^{\otimes(k-1)}\otimes\phi).$$

In the following series of propositions we show that $i_\phi R$ vanishes on every nearly pseudo-Sasakian manifold. This generalizes the Olszak's result obtained in [6] for nearly Sasakian manifolds.

Proposition 3.1. *Let (M,g) be a pseudo-Riemannian manifold and ϕ a linear endomorphism of TM. Then the tensor $i_\phi\widetilde{R}$ has the following symmetries*

$$(i_\phi\widetilde{R})(1+(1,2)) = 0,\quad (i_\phi\widetilde{R})(1-(1,3)(2,4)) = 0,\quad (i_\phi\widetilde{R})(1+(1,2,3)+(1,3,2)) = 0. \tag{10}$$

Proof. Since $\phi\otimes\mathrm{Id}^{\otimes 3} + \mathrm{Id}\otimes\phi\otimes\mathrm{Id}^{\otimes 2} + \mathrm{Id}^{\otimes 2}\otimes\phi\otimes\mathrm{Id} + \mathrm{Id}^{\otimes 3}\otimes\phi$ commutes with every element of Σ_4, the result follows from the corresponding symmetries of the curvature tensor $\widetilde{R}$. □

The following proposition lists a well-known property of tensors with certain symmetries (see e.g. [5, page 198]).

Proposition 3.2. *Let M be a manifold and T a $(0,4)$-tensor on M such that*

$$T(1+(1,2))=0, \quad T(1-(1,3)(2,4))=0, \quad T(1+(1,2,3)+(1,3,2))=0.$$

If $T(X,Y,X,Y)=0$ for any pair of vector fields X, Y then $T=0$.

In the next proposition we relate the tensors $i_\phi \widetilde{R}$ and $R\phi$.

Proposition 3.3. *Let (M,g) be a pseudo-Riemannian manifold. If $\phi\colon TM \to TM$ is skew-symmetric with respect g then $i_\phi \widetilde{R} = g\circ(R\phi\otimes \mathrm{Id})(1+(1,3)(2,4))$.*

Proof. The result follows from

$$\begin{aligned} g((R_{X,Y}\phi)Z,W) &= g(R_{X,Y}(\phi Z),W) - g(\phi(R_{X,Y}Z),W) \\ &= \widetilde{R}(X,Y,\phi Z,W) + \widetilde{R}(X,Y,Z,\phi W) \end{aligned}$$

and symmetries of $\widetilde{R}$. □

Proposition 3.4. *If (M,g,ϕ,ξ,η) is a nearly pseudo-Sasakian manifold then $i_\phi\widetilde{R}=0$. Equivalently, $g\circ(R\phi\otimes \mathrm{Id})(1+(1,3)(2,4))=0$.*

Proof. By Proposition 3.1 the tensor $i_\phi\widetilde{R}$ has the symmetries which permit to apply Proposition 3.2. Thus it is enough to show that $(i_\phi\widetilde{R})(X,Y,X,Y)=0$ for all $X,Y\in\Gamma(TM)$. By Proposition 3.3, we have $i_\phi\widetilde{R}=g\circ(R\phi\otimes\mathrm{Id})(1+(1,3)(2,4))$. Thus $(i_\phi\widetilde{R})(X,Y,X,Y)=2g((R_{X,Y}\phi)X,Y)$. By definition $R_{X,Y}\phi=\nabla^2_{X,Y}\phi-\nabla^2_{Y,X}\phi$. Since $\nabla^2_{Y,X}\phi$ is a skew-symmetric operator, we get

$$(i_\phi\widetilde{R})(X,Y,X,Y) = -2(g((\nabla^2_{X,Y}\phi)Y,X)+g((\nabla^2_{Y,X}\phi)X,Y)).$$

From the above expression it follows that $(i_\phi\widetilde{R})(X,Y,X,Y) = 0$ if and only if the form $Q(X,Y) := g((\nabla^2_{Y,X}\phi)X,Y)$ satisfies $Q(X,Y)=-Q(Y,X)$. In the remaining part of the proof we will show that $Q(X,Y)=(1/2)d\eta(X,Y)g(X,Y)$. Then the result follows since $d\eta$ is skew-symmetric and g is symmetric.

Applying ∇ to the defining condition for nearly pseudo-Sasakian structure

$$(\nabla\phi - \xi\otimes g + \eta\otimes\mathrm{Id})(1+(1,2))=0,$$

we get

$$(\nabla^2\phi - \nabla\xi\otimes g + \nabla\eta\otimes \mathrm{Id})(1+(2,3))=0. \tag{11}$$

Substituting (Y,X,X) in (11) and then applying $g(-,Y)$ to the result, we get

$$2(Q(X,Y)-g(\nabla_Y\xi,Y)g(X,X)+(\nabla_Y\eta)(X)g(X,Y))=0.$$

By Proposition 2.4, $(\nabla_Y\eta)(X)=(1/2)d\eta(Y,X)$ and $\nabla\xi$ is skew-symmetric, which implies that $g(\nabla_Y\xi,Y)=0$. Hence $Q(X,Y)=(1/2)d\eta(X,Y)g(X,Y)$ as promised. □

Proposition 3.5. *Let (M,g,ϕ,ξ,η) be a nearly pseudo-Sasakian manifold. Then $\widetilde{R}\circ\xi|_{\xi^\perp}=0$.*

Proof. Let $X,Y,Z\in\xi^\perp$. We evaluate $i_\phi\widetilde{R}=0$ on the quadruples $(\phi X,Y,Z,\xi)$, $(X,\phi Y,Z,\xi)$, $(X,Y,\phi Z,\xi)$, and $(\phi X,\phi Y,\phi Z,\xi)$. As $\phi^2|_{\xi^\perp}=-\mathrm{Id}$ and, by Proposition 2.4, $\phi\xi=0$, this gives the relations

$$\begin{aligned} -(\widetilde{R}\circ\xi)(X,Y,Z)+(\widetilde{R}\circ\xi)(\phi X,\phi Y,Z)+(\widetilde{R}\circ\xi)(\phi X,Y,\phi Z)&=0\\ (\widetilde{R}\circ\xi)(\phi X,\phi Y,Z)-(\widetilde{R}\circ\xi)(X,Y,Z)+(\widetilde{R}\circ\xi)(X,\phi Y,\phi Z)&=0\\ (\widetilde{R}\circ\xi)(\phi X,Y,\phi Z)+(\widetilde{R}\circ\xi)(X,\phi Y,\phi Z)-(\widetilde{R}\circ\xi)(X,Y,Z)&=0\\ -(\widetilde{R}\circ\xi)(X,\phi Y,\phi Z)-(\widetilde{R}\circ\xi)(\phi X,Y,\phi Z)-(\widetilde{R}\circ\xi)(\phi X,\phi Y,Z)&=0 \end{aligned}$$

Summing up the first three equations with the last one taken twice, we obtain that $-3(\widetilde{R}\circ\xi)(X,Y,Z)=0$, and thus $\widetilde{R}\circ\xi|_{\xi^\perp}=0$. □

Proposition 3.6. *Let (M,g) be a pseudo-Riemannian manifold and ξ a Killing vector field on M. Then $\nabla^2\xi$ can be determined from $R\xi$, namely*

$$g\circ\nabla^2\xi = g\circ R\xi\circ(1,2).$$

Proof. Since ξ is Killing, the operator $\nabla\xi$ is skew-symmetric, i.e. $g\circ(\nabla\xi\otimes\mathrm{Id}+\mathrm{Id}\otimes\nabla\xi)=0$. Applying ∇ to this equation we get $g\circ(\nabla^2\xi\otimes\mathrm{Id}+\mathrm{Id}\otimes\nabla^2\xi\circ(1,2)))=0$. Since $g\circ(\nabla^2\xi\otimes\mathrm{Id})=g\circ\nabla^2\xi\circ(1,2,3)$, we get

$$0=g\circ\nabla^2\xi\circ((1,2,3)+(1,2))=g\circ\nabla^2\xi\circ((1,3)+1)(1,2).$$

Thus

$$g\circ\nabla^2\xi=-g\circ\nabla^2\xi\circ(1,3). \tag{12}$$

Next denote $g\circ\xi$ by η. Since ξ is Killing, by repeating the computation in the last step of the proof of Proposition 2.4, we get $d\eta=-2g\circ\nabla\xi$. This implies

$$\begin{aligned}0=d^2\eta&=(\nabla d\eta)(1+(1,2,3)+(1,3,2))=-2(g\circ\nabla^2\xi\circ(1,2)))(1+(1,2,3)+(1,3,2))\\&=-2g\circ\nabla^2\xi\circ(1+(1,2,3)+(1,3,2))(1,2).\end{aligned} \tag{13}$$

Now from (12) and (13), we get

$$\begin{aligned}g\circ R\xi&=g\circ\nabla^2\xi\circ(1-(2,3))=-g\circ\nabla^2\xi\circ((1,3)+(2,3))\\&=-g\circ\nabla^2\xi\circ(1+(1,2,3))(1,3)=g\circ\nabla^2\xi\circ(1,3,2)(1,3)=g\circ\nabla^2\xi\circ(1,2).\end{aligned}$$

□

In the next proposition we collect several partial results on the curvature tensor of a nearly pseudo-Sasakian manifold.

Proposition 3.7. *Let (M,g,ϕ,ξ,η) be a nearly pseudo-Sasakian manifold. Then*

$$R\xi=\eta\wedge(\nabla\xi)^2,\quad \nabla^2\xi=-(\nabla\xi)^2\otimes\eta+(g\circ(\nabla\xi)^2)\otimes\xi$$
$$R_\xi=(\nabla\xi)^2\otimes\eta-\xi\otimes g\circ(\nabla\xi)^2$$
$$(R\phi)\xi=-\eta\wedge\phi(\nabla\xi)^2,\quad R_\xi\phi=-(\nabla\xi)^2\phi\otimes\eta-(g\circ\phi(\nabla\xi)^2)\otimes\xi.$$

Proof. From Proposition 3.5, we know that $\widetilde{R}(X,Y,Z,\xi)=0$ for any $X,Y,Z\in\xi^\perp$. As $\widetilde{R}$ is skew-symmetric on the last two arguments, we conclude that $g(R_{X,Y}\xi,Z)=\widetilde{R}(X,Y,\xi,Z)=0$. Thus $R_{X,Y}\xi$ is proportional to ξ. Hence $R_{X,Y}\xi=\eta(R_{X,Y}\xi)\xi=\widetilde{R}(X,Y,\xi,\xi)\xi=0$ for $X,Y\in\xi^\perp$. This implies

$$R_{X,Y}\xi=\eta(X)R_{\xi,Y}\xi-\eta(Y)R_{\xi,X}\xi. \tag{14}$$

Thus it is enough to compute $R_{\xi,X}\xi$ or, equivalently, $\widetilde{R}(\xi,X,\xi,Y)$. Since $\widetilde{R}(\xi,X,\xi,Y)$ is symmetric with respect to the swap of X and Y, it suffices to find formula for $\widetilde{R}(\xi,X,\xi,X)$. By Proposition 2.4 the operator $\nabla\xi$ is skew-symmetric, and thus also $\nabla^2_\xi\xi$ is skew-symmetric. This implies

$$\begin{aligned}\widetilde{R}(\xi,X,\xi,X)&=g(\nabla^2_{\xi,X}\xi,X)-g(\nabla^2_{X,\xi}\xi,X)=0-g(\nabla_X(\nabla_\xi\xi),X)+g(\nabla_{\nabla_X\xi}\xi,X)\\&=g((\nabla\xi)^2X,X).\end{aligned}$$

Polarizing at X, we get $\widetilde{R}(\xi,X,\xi,Y)=g((\nabla\xi)^2X,Y)$. Therefore $R_{\xi,X}\xi=(\nabla\xi)^2X$. Now (14) can be written in the form

$$R\xi=\eta\wedge(\nabla\xi)^2.$$

To compute $\nabla^2\xi$, we use the expression $g \circ \nabla^2\xi = (g \circ R\xi) \circ (1, 2)$ obtained in Proposition 3.6. We get that for any $X, Y, Z \in \Gamma(TM)$

$$\begin{aligned} g(X, \nabla^2_{Y,Z}\xi) &= g(Y, R_{X,Z}\xi) = \eta(X)g(Y, (\nabla\xi)^2 Z) - \eta(Z)g(Y, (\nabla\xi)^2 X) \\ &= g(Y, (\nabla\xi)^2 Z)g(X, \xi) - g(X, (\nabla\xi)^2 Y)\eta(Z). \end{aligned}$$

The above formula is equivalent to the formula for $\nabla^2\xi$ in the statement of the proposition since g is non-degenerate.

Now let X, Y, Z be arbitrary vector fields on M. Then

$$\begin{aligned} g(R_{\xi,X}Y, Z) &= \widetilde{R}(\xi, X, Y, Z) = \widetilde{R}(Y, Z, \xi, X) \\ &= g(R_{Y,Z}\xi, X) = \eta(Y)g((\nabla\xi)^2 Z, X) - \eta(Z)g((\nabla\xi)^2 Y, X). \end{aligned}$$

Since $(\nabla\xi)^2$ is self-adjoint and g is non-degenerate, we get

$$R_{\xi,X}Y = \eta(Y)(\nabla\xi)^2 X - g(X, (\nabla\xi)^2 Y)\xi$$

which is equivalent to the formula in the statement.

To compute $(R\phi)\xi$ we use the already established formula for $R\xi$

$$(R_{X,Y}\phi)\xi = R_{X,Y}(\phi\xi) - \phi(R_{X,Y}\xi) = -(\eta \wedge \phi(\nabla\xi)^2)(X, Y).$$

To find $R_\xi\phi$ we use the symmetry property of $g \circ (R\phi \otimes \mathrm{Id})$ that was proved in Proposition 3.4. We get

$$\begin{aligned} g((R_{\xi,X}\phi)Y, Z) &= -g((R_{Y,Z}\phi)\xi, X) = g(\eta(Y)\phi(\nabla\xi)^2 Z, X) - g(\eta(Z)\phi(\nabla\xi)^2 Y, X) \\ &= -g((\nabla\xi)^2\phi X, Z)\eta(Y) - g(\xi, Z)g(X, \phi(\nabla\xi)^2 Y). \end{aligned}$$

Since g is non-degenerate it is equivalent to $R_\xi\phi = -(\nabla\xi)^2\phi \otimes \eta - \xi \otimes (g \circ \phi(\nabla\xi)^2)$. □

Theorem 3.8. *Suppose (M, g, ϕ, ξ, η) is a nearly pseudo-Sasakian manifold. Then the characteristic polynomial of $(\nabla\xi)^2$ has constant coefficients.*

Proof. Throughout the proof we use that $\nabla\xi$ and $\nabla^2_Y\xi$ are skew-symmetric operators. The first fact was proved in Proposition 2.4, and the second is its consequence.

The coefficients of the characteristic polynomial of $(\nabla\xi)^2$ are constant if and only if the traces of the operators $(\nabla\xi)^{2s}$ for $0 \le s \le 2n+1$ are constant. In fact, if at some point p of M the spectrum (over $\mathbb{C}$) of $(\nabla\xi)^2$ is $(\lambda_1, \dots, \lambda_{2n+1})$ then the s-th coefficient of the characteristic polynomial of $(\nabla\xi)^2$ is up to the sign an elementary symmetric polynomial

$$e_s = \sum_{j_1 < \dots < j_s} \lambda_{j_1} \cdot \lambda_{j_2} \dots \lambda_{j_s}$$

and the trace of $(\nabla\xi)^{2s}$ is the power sum symmetric polynomial $p_s = \lambda_1^s + \dots + \lambda_{2n+1}^s$. Now the claim follows from the Newton identities

$$e_1 = p_1, \quad se_s = \sum_{j=1}^{s} (-1)^{j-1} e_{s-j} p_j, \; s \ge 2.$$

Next, we show that the traces $\mathrm{tr}((\nabla\xi)^{2s})$ are constant functions for all $s \ge 1$. Since ∇ commutes with contraction, we get that for any vector field Y on M

$$Y(\mathrm{tr}\,(\nabla\xi)^{2s}) = \mathrm{tr}(\nabla_Y(\nabla\xi)^{2s}) = \sum_{k+\ell=2s-1} \mathrm{tr}\left((\nabla\xi)^k (\nabla^2_Y\xi)(\nabla\xi)^\ell \right).$$

By Proposition 3.7 we know that $\nabla^2\xi = -(\nabla\xi)^2 \otimes \eta + \xi \otimes (g \circ (\nabla\xi)^2)$. Since $\nabla_\xi\xi = 0$ and $\eta \circ \nabla\xi = 0$ by Proposition 2.4, we get $(\nabla\xi) \circ (\nabla^2_Y\xi) \circ \nabla\xi = 0$. Thus

$$Y(\mathrm{tr}\,(\nabla\xi)^{2s}) = \mathrm{tr}\left((\nabla^2_Y\xi)(\nabla\xi)^{2s-1} \right) + \mathrm{tr}\left((\nabla\xi)^{2s-1}(\nabla^2_Y\xi) \right). \tag{15}$$

Since the trace of a nilpotent operator is always zero and

$$\big((\nabla^2_Y\xi)(\nabla\xi)^{2s-1}\big)^2 = (\nabla^2_Y\xi)(\nabla\xi)^{2s-1}(\nabla^2_Y\xi)(\nabla\xi)^{2s-1} = 0$$
$$\big((\nabla\xi)^{2s-1}(\nabla^2_Y\xi)\big)^2 = (\nabla\xi)^{2s-1}(\nabla^2_Y\xi)(\nabla\xi)^{2s-1}(\nabla^2_Y\xi) = 0,$$

we conclude that the both traces in (15) are zero and therefore $\mathrm{tr}(\nabla\xi)^{2s}$ is a constant function for all s. □

In the case of nearly Sasakian manifolds, Theorem 3.8 implies the existence of a tangent bundle decomposition into a direct sum of subbundles. This decomposition will be crucial in our proof of Theorem 4.6, which gives an explicit formula for $\nabla\phi$ on a nearly Sasakian manifold. Recall that by Theorem 2.5 the spectrum of $(\nabla\xi)^2$ on a nearly Sasakian manifold is non-positive.

Proposition 3.9. *Let (M, g, ϕ, ξ, η) be a nearly Sasakian manifold. Suppose $0 = \lambda_0 > -\lambda_1 > \cdots > -\lambda_\ell$ are the roots of the characteristic polynomial of $(\nabla\xi)^2$. Then TM can be written as a direct sum of pair-wise orthogonal subbundles $V_k \subset TM$ such that, for every $0 \le k \le \ell$, the restriction of $(\nabla\xi)^2$ to V_k equals $-\lambda_k \cdot \mathrm{Id}$.*

Proof. By Proposition 2.4 the operator $\nabla\xi$ is skew-symmetric, and therefore $(\nabla\xi)^2$ is symmetric. As g is positively defined this implies that $(\nabla\xi)^2$ is diagonalizable. Denote by a_k the multiplicity of $-\lambda_k$ in the characteristic polynomial of $(\nabla\xi)^2$. Then, by examining the diagonal form of $(\nabla\xi)^2$, one can see that $\mathbf{rk}((\nabla\xi)^2 + \lambda_k \cdot \mathrm{Id}) = 2n + 1 - a_k$ and that TM can be written as a direct sum of the subbundles $V_k = \ker((\nabla\xi)^2 + \lambda_k \cdot \mathrm{Id})$. It is a standard fact that these subbundles are mutually orthogonal and clearly the restriction of $(\nabla\xi)^2$ to V_k equals $-\lambda_k \cdot \mathrm{Id}$. □

4 Covariant derivative of ϕ

In this section we derive a rather explicit formula for $\nabla_X\phi$ on a nearly pseudo-Sasakian manifold. We achieve this by computing separately $\nabla_X\phi$ on subspaces $\langle\xi\rangle$, $\mathrm{Im}(\nabla_\xi\phi)$, and $\mathrm{Im}(\nabla_\xi\phi)^\perp \cap \xi^\perp$. Then, we will use the formula to prove Theorem 4.9.

Proposition 4.1. *Let (M, g, ϕ, ξ, η) be a nearly pseudo-Sasakian manifold. Then*

$$\nabla^2_\xi\phi = \eta \wedge (\nabla_\xi\phi \circ \nabla\xi) - \xi \otimes \Big(g \circ (\nabla_\xi\phi \circ \nabla\xi)\Big).$$

Proof. Applying ∇ to the defining relation of nearly pseudo-Sasakian structure $(\nabla\phi - \xi\otimes g + \eta\otimes\mathrm{Id})(1 + (1, 2)) = 0$ we get

$$(\nabla^2\phi - \nabla\xi \otimes g + \nabla\eta \otimes \mathrm{Id})(1 + (2, 3)) = 0. \tag{16}$$

Denote $(\nabla\xi \otimes g - \nabla\eta \otimes \mathrm{Id})(1 + (2, 3))$ by T. Then (16) becomes $(\nabla^2\phi)(1 + (2, 3)) = T$. By definition of R we have $(\nabla^2\phi)(1 - (1, 2)) = R\phi$. We have the following equality in $\mathbb{R}\Sigma_3$

$$2 \cdot \mathrm{id} = (1 - (1, 2))(1 + (1, 2, 3) - (1, 3, 2)) + (1 + (2, 3))(1 - (1, 2, 3) + (1, 3, 2)). \tag{17}$$

Therefore

$$2\nabla^2\phi = R\phi(1 + (1, 2, 3) - (1, 3, 2)) + T(1 - (1, 2, 3) + (1, 3, 2)). \tag{18}$$

Now we substitute (ξ, X, Y) in (18)

$$2(\nabla^2_{\xi,X}\phi)Y = (R_{\xi,X}\phi)Y + (R_{Y,\xi}\phi)X - (R_{X,Y}\phi)\xi + T(\xi, X, Y) - T(Y, \xi, X) + T(X, Y, \xi). \tag{19}$$

By Proposition 3.7, we have $R_\xi\phi = -(\nabla\xi)^2\phi\otimes\eta - (g\circ\phi(\nabla\xi)^2)\otimes\xi$ and $(R\phi)\xi = -\eta\wedge\phi(\nabla\xi)^2$. Therefore the R-part of (19) evaluates to

$$\begin{aligned}
&-g(X,\phi(\nabla\xi)^2Y)\xi - \eta(Y)(\nabla\xi)^2\phi X\\
&\qquad + g(Y,\phi(\nabla\xi)^2X)\xi + \eta(X)(\nabla\xi)^2\phi Y\\
&\qquad\qquad + \eta(X)\phi(\nabla\xi)^2Y - \eta(Y)\phi(\nabla\xi)^2X\\
&\qquad = 2\Big(-g(X,(\nabla\xi)^2\phi Y)\xi - \eta(Y)\phi(\nabla\xi)^2X + \eta(X)\phi(\nabla\xi)^2Y\Big),
\end{aligned}$$

where we use that ϕ and $(\nabla\xi)^2$ commute by Proposition 2.4. Next,

$$T(\xi,X,Y) = 0$$

$$\begin{aligned}
T(Y,\xi,X) &= 2(\nabla\xi)(Y)\eta(X) - (\nabla_Y\eta)(\xi)X - (\nabla_Y\eta)(X)\xi\\
&= -g(X,(\nabla\xi)Y)\xi + 2\eta(X)(\nabla\xi)Y
\end{aligned}$$

$$T(X,Y,\xi) = T(X,\xi,Y) = -g(Y,(\nabla\xi)X)\xi + 2\eta(Y)(\nabla\xi)X.$$

Thus the T-part of the right side of (19) is

$$2g(X,(\nabla\xi)Y)\xi + 2\eta(Y)(\nabla\xi)X - 2\eta(X)(\nabla\xi)Y.$$

As a result we get

$$\nabla^2_\xi\phi = \xi\otimes g\circ(\nabla\xi)(\mathrm{Id}-(\nabla\xi)\phi) - \eta\wedge(\nabla\xi)(\mathrm{Id}-(\nabla\xi)\phi). \tag{20}$$

By Proposition 2.4 the operator $(\nabla\xi)^2$ commutes with ϕ, $\eta\circ\nabla\xi$ vanishes, and $\phi(\phi+\nabla\xi) = \nabla_\xi\phi$. Therefore

$$\begin{aligned}
(\nabla\xi)(\mathrm{Id}-(\nabla\xi)\phi) &= (\mathrm{Id}-\phi\nabla\xi)\nabla\xi = (-\phi^2+\xi\otimes\eta-\phi\nabla\xi)\nabla\xi\\
&= -\phi(\phi+\nabla\xi)\nabla\xi = -\nabla_\xi\phi\circ\nabla\xi.
\end{aligned}\tag{21}$$

Substituting (21) in (20), we get the claim of the proposition. □

Given two tensor fields T_1 and T_2 on a manifold M such that both products $T_1\circ T_2$ and $T_2\circ T_1$ make sense, we define *commutator* and *anticommutator* of T_1 and T_2 by $[T_1,T_2] = T_1\circ T_2 - T_2\circ T_1$ and $\{T_1,T_2\} = T_1\circ T_2 + T_2\circ T_1$, respectively. The aim of the next three propositions is to find $(\nabla_X\phi)Y$ on a nearly pseudo-Sasakian manifold in the case Y is in the image of $\nabla_\xi\phi$. For this we compute $(\nabla\phi)(\nabla_\xi\phi)$. The later tensor can be written as a half-sum of $\{\nabla\phi,\nabla_\xi\phi\}$ and $[\nabla\phi,\nabla_\xi\phi]$.

Proposition 4.2. *Let (M,g,ϕ,ξ,η) be a nearly pseudo-Sasakian manifold. Then*

$$\{\nabla\phi,\nabla_\xi\phi\} = 2\eta\otimes(\nabla_\xi\phi)^2 - (\nabla_\xi\phi)(\mathrm{Id}+\nabla_\xi\phi)\otimes\eta + \xi\otimes\Big(g\circ(\nabla_\xi\phi)(\mathrm{Id}-\nabla_\xi\phi)\Big).$$

Proof. Recall that by Proposition 2.4 we have $\nabla_\xi\xi = 0$ and $\nabla_\xi\eta = 0$. Applying ∇^2_ξ to the almost contact structure condition $\phi^2+\mathrm{Id}-\xi\otimes\eta = 0$ we get

$$(\nabla^2_\xi\phi)\circ\phi + (\nabla_\xi\phi)\circ(\nabla\phi) + (\nabla\phi)\circ(\nabla_\xi\phi) + \phi(\nabla^2_\xi\phi) - (\nabla^2_\xi\xi)\otimes\eta - \xi\otimes g\circ(\nabla^2_\xi\xi) = 0.$$

Applying the formula $\nabla^2\xi = -(\nabla\xi)^2\otimes\eta + (g\circ(\nabla\xi)^2)\otimes\xi$ obtained in Proposition 3.7, we get

$$(\nabla^2_{\xi,Y}\xi) = -(\nabla\xi)^2\xi\cdot\eta(Y) + g(Y,(\nabla\xi)^2\xi)\xi = 0.$$

Therefore

$$(\nabla_\xi\phi)\circ(\nabla\phi) + (\nabla\phi)\circ(\nabla_\xi\phi) = -(\nabla^2_\xi\phi)\circ\phi - \phi\circ(\nabla^2_\xi\phi). \tag{22}$$

We showed in Proposition 4.1 that

$$\nabla^2_\xi\phi = \eta\wedge(\nabla_\xi\phi\circ\nabla\xi) - \xi\otimes\Big(g\circ(\nabla_\xi\phi\circ\nabla\xi)\Big).$$

Since $\phi\xi = 0$ and $\eta \circ \phi = 0$, we conclude

$$\begin{aligned}\nabla^2_\xi\phi \circ \phi &= \eta \otimes (\nabla_\xi\phi \circ \nabla\xi \circ \phi) - \xi \otimes \Big(g \circ (\nabla_\xi\phi \circ \nabla\xi \circ \phi)\Big)\\ \phi \circ \nabla^2_\xi\phi &= \eta \wedge (\phi \circ \nabla_\xi\phi \circ \nabla\xi).\end{aligned} \tag{23}$$

Next, we use that by Proposition 2.4 the operators ϕ and $(\nabla_\xi\phi)$ anticommute, and $\nabla\xi = -\phi(\mathrm{Id} + \nabla_\xi\phi)$ to get

$$\begin{aligned}\nabla_\xi\phi \circ \nabla\xi \circ \phi &= -\nabla_\xi\phi \circ \phi(\mathrm{Id} + \nabla_\xi\phi) \circ \phi = -\nabla_\xi\phi \circ \phi^2(\mathrm{Id} - \nabla_\xi\phi) = \nabla_\xi\phi(\mathrm{Id} - \nabla_\xi\phi)\\ \phi \circ \nabla_\xi\phi \circ \nabla\xi &= -\phi \circ \nabla_\xi\phi \circ \phi(\mathrm{Id} + \nabla_\xi\phi) = \nabla_\xi\phi \circ \phi^2(\mathrm{Id} + \nabla_\xi\phi) = -\nabla_\xi\phi(\mathrm{Id} + \nabla_\xi\phi).\end{aligned} \tag{24}$$

Combining (22), (23), and (24) we get the statement of the proposition. □

Proposition 4.3. *Let (M, g, ϕ, ξ, η) be a nearly pseudo-Sasakian manifold. Then*

$$\big[\nabla\phi, \nabla_\xi\phi\big] = (\nabla_\xi\phi)(\mathrm{Id} + \nabla_\xi\phi) \otimes \eta + \xi \otimes \Big(g \circ (\nabla_\xi\phi)(\mathrm{Id} - \nabla_\xi\phi)\Big).$$

Proof. By Proposition 2.4, we know that $\nabla_\xi\phi = \phi(\phi + \nabla\xi)$. Notice that for any three tensors A, B, and C, such that all pair-wise compositions are defined, we have

$$[A, B \circ C] = (A \circ B + B \circ A) \circ C - B \circ (A \circ C + C \circ A) = \{A, B\} \circ C - B \circ \{A, C\}.$$

Thus to find the commutator of $\nabla\phi$ with $\nabla_\xi\phi$, we only have to compute the anti-commutators of $\nabla\phi$ with ϕ and $\nabla\xi$.

We start with the anticommutator between $\nabla\phi$ and ϕ. For this we apply ∇ to the almost contact metric condition $\phi^2 = -\mathrm{Id} + \xi \otimes \eta$, which gives

$$\big\{\nabla\phi, \phi\big\} = (\nabla\phi)\phi + \phi(\nabla\phi) = (\nabla\xi) \otimes \eta - \xi \otimes (g \circ \nabla\xi), \tag{25}$$

where we are using $\nabla\eta = -g \circ \nabla\xi$ from Proposition 2.4.

To find the anticommutator between $\nabla\phi$ and $\nabla\xi$, we first compute the anticommutator between ϕ and $\nabla\xi$ and then apply ∇ to the resulting formula. By Proposition 2.4, we know that $\nabla\xi = -\phi - \phi \circ \nabla_\xi\phi$ and that ϕ anticommutes with $\phi \circ \nabla_\xi\phi$. Therefore,

$$\phi \circ \nabla\xi + \nabla\xi \circ \phi = -2 \cdot \phi^2 = 2 \cdot \mathrm{Id} - 2 \cdot \xi \otimes \eta$$

and hence

$$(\nabla\phi) \circ (\nabla\xi) + \phi \circ \nabla^2\xi + \nabla^2\xi \circ \phi + (\nabla\xi) \circ (\nabla\phi) = -2\nabla\xi \otimes \eta + 2\xi \otimes (g \circ \nabla\xi). \tag{26}$$

By Proposition 3.7, we know that $\nabla^2\xi = -(\nabla\xi)^2 \otimes \eta + \xi \otimes (g \circ (\nabla\xi)^2)$. Since $\phi\xi = 0$ and $\eta \circ \phi = 0$, we get

$$\begin{aligned}\phi \circ \nabla^2\xi &= -\phi(\nabla\xi)^2 \otimes \eta\\ \nabla^2\xi \circ \phi &= \xi \otimes \big(g \circ (\nabla\xi)^2 \circ \phi\big).\end{aligned} \tag{27}$$

Combining (26) with (27) and then adding the result to (25), we get

$$\big\{\nabla\phi, \phi + \nabla\xi\big\} = (\phi(\nabla\xi)^2 - \nabla\xi) \otimes \eta + \xi \otimes \Big(g \circ (\nabla\xi - (\nabla\xi)^2\phi)\Big).$$

Thus

$$\begin{aligned}\big[\nabla\phi, \phi(\phi + \nabla\xi)\big] &= \big\{\nabla\phi, \phi\big\} \circ (\phi + \nabla\xi) - \phi \circ \big\{\nabla\phi, \phi + \nabla\xi\big\}\\ &= -\xi \otimes \Big(g \circ \nabla\xi \circ (\phi + \nabla\xi)\Big) - (\phi^2(\nabla\xi)^2 - \phi\nabla\xi) \otimes \eta.\end{aligned}$$

Next we use that $\nabla\xi + \phi + \phi \circ \nabla_\xi\phi = 0$ and $(\nabla\xi)^2 = (\nabla_\xi\phi)^2 - \mathrm{Id} + \xi \otimes \eta$ established in Proposition 2.4 to bring the above expression to the form of the proposition statement

$$\begin{aligned}\nabla\xi \circ (\phi + \nabla\xi) &= \phi \circ (\mathrm{Id} + \nabla_\xi\phi) \circ \phi \circ (\nabla_\xi\phi) = \phi^2(\nabla_\xi\phi)(\mathrm{Id} - \nabla_\xi\phi)\\ &= -(\nabla_\xi\phi)(\mathrm{Id} - \nabla_\xi\phi)\\ \phi^2(\nabla\xi)^2 - \phi\nabla\xi &= \phi^2((\nabla_\xi\phi)^2 - \mathrm{Id} + \xi \otimes \eta) + \phi^2(\mathrm{Id} + \nabla_\xi\phi)\\ &= -(\nabla_\xi\phi)(\mathrm{Id} + \nabla_\xi\phi).\end{aligned}$$

This completes the proof. □

Proposition 4.4. *Let (M, g, ϕ, ξ, η) be a nearly pseudo-Sasakian manifold. Then for any Y in the image of $\nabla_\xi \phi$, the following equation holds*

$$(\nabla\phi) \circ Y = \eta \otimes ((\nabla_\xi \phi)Y) + \xi \otimes (g \circ (\mathrm{Id} - \nabla_\xi \phi)Y).$$

Proof. Let Z be such that $(\nabla_\xi\phi)Z = Y$. Since $(\nabla_\xi\phi)\xi = 0$ we can assume that $\eta(Z) = 0$ by replacing Z with $Z - \eta(Z)\xi$ if necessary. By Proposition 4.2, we get

$$\begin{aligned}\left\{\nabla\phi, \nabla_\xi\phi\right\} \circ Z &= 2\eta \otimes ((\nabla_\xi\phi)^2 Z) + \xi \otimes \left(g \circ (\nabla_\xi\phi) \circ (\mathrm{Id} - \nabla_\xi\phi) \circ Z\right)\\ &= 2\eta \otimes ((\nabla_\xi\phi)Y) + \xi \otimes \left(g \circ (\mathrm{Id} - \nabla_\xi\phi) \circ Y\right).\end{aligned}$$

Next, by Proposition 4.3, we have

$$\left[\nabla\phi, \nabla_\xi\phi\right] \circ Z = \xi \otimes \left(g \circ (\nabla_\xi\phi) \circ (\mathrm{Id} - \nabla_\xi\phi) \circ Z\right) = \xi \otimes \left(g \circ (\mathrm{Id} - \nabla_\xi\phi) \circ Y\right).$$

Thus

$$\begin{aligned}(\nabla\phi) \circ Y = (\nabla\phi) \circ (\nabla_\xi\phi) \circ Z &= (1/2)\Big(\left\{\nabla\phi, \nabla_\xi\phi\right\} \circ Z + \left[\nabla\phi, \nabla_\xi\phi\right] \circ Z\Big)\\ &= \eta \otimes ((\nabla_\xi\phi)Y) + \xi \otimes \left(g \circ (\mathrm{Id} - \nabla_\xi\phi) \circ Y\right).\end{aligned}$$

This finishes the proof. □

In the next proposition we use that g is positively defined to conclude that $(\nabla_\xi\phi)^2 Y = 0$ implies $(\nabla_\xi\phi)Y = 0$. This can be false for a general nearly pseudo-Sasakian manifold.

Proposition 4.5. *Let (M, g, ϕ, ξ, η) be a nearly Sasakian manifold. Then for any $Y \in \Gamma(\ker((\nabla\xi)^2 + \mathrm{Id}))$, one has $(\nabla\phi) \circ Y = \xi \otimes (g \circ Y)$.*

Proof. Throughout the proof we will use that by Proposition 3.7, we have

$$\nabla^2\xi = -(\nabla\xi)^2 \otimes \eta + \xi \otimes (g \circ (\nabla\xi)^2). \tag{28}$$

First we show that $\mathrm{Im}(\nabla Y) \subset \ker(\phi\nabla_\xi\phi)$. Since $(\nabla\xi)^2 Y = -Y$, we have

$$\nabla Y = -\nabla((\nabla\xi)^2 Y) = -\nabla^2\xi \circ \nabla\xi \circ Y - \nabla\xi \circ \nabla^2\xi \circ Y - (\nabla\xi)^2 \circ \nabla Y.$$

Since $\eta \circ \nabla\xi = 0$ and $(\nabla\xi)\xi = 0$ by Proposition 2.4, using (28), we get

$$\nabla Y = -\xi \otimes \left(g \circ (\nabla\xi)^3 \circ Y\right) + \eta(Y) \otimes (\nabla\xi)^3 - (\nabla\xi)^2 \circ \nabla Y.$$

Notice that

$$\eta(Y) = g(\xi, Y) = -g(\xi, (\nabla\xi)^2 Y) = 0$$

thus, taking into account $(\nabla\xi)^2 Y = -Y$, we get

$$\nabla Y = \xi \otimes (g \circ (\nabla\xi)Y) - (\nabla\xi)^2 \circ \nabla Y.$$

Applying ∇ to $0 = \eta(Y) = g \circ (\xi \otimes Y)$, we get $g \circ (\nabla\xi \otimes Y) + g \circ (\xi \otimes \nabla Y) = 0$. Since $\nabla\xi$ is skew-symmetric, this implies that $g \circ (\nabla\xi)Y = \eta \circ \nabla Y$. Thus

$$\nabla Y = (\xi \otimes \eta) \circ (\nabla Y) - (\nabla\xi)^2 \circ \nabla Y.$$

The above equation means that the image of ∇Y is a subset of the kernel of the operator $(\nabla\xi)^2 - \xi \otimes \eta + \mathrm{Id}$. By Proposition 2.4 this operator equals to $(\phi\nabla_\xi\phi)^2$. Since $\phi\nabla_\xi\phi$ is skew-symmetric by the same proposition and g is positively defined by assumption, we get that $\mathrm{Im}(\nabla Y) \subset \ker(\phi\nabla_\xi\phi) = \ker(\phi + \nabla\xi)$. Thus $(\phi + \nabla\xi) \circ \nabla Y = 0$.

Next, we claim that $(\phi\nabla_\xi\phi)Y = 0$. For this we compute

$$(\phi\nabla_\xi\phi)^2 Y = \left((\nabla\xi)^2 + \mathrm{Id} - \xi \otimes \eta\right) Y = -Y + Y - 0 = 0.$$

Therefore, arguing as before, we have $(\phi + \nabla\xi)Y = 0$. Applying ∇ to this equation, we get

$$0 = (\nabla\phi + \nabla^2\xi) \circ Y + (\phi + \nabla\xi) \circ \nabla Y = (\nabla\phi) \circ Y + \xi \otimes \left(g \circ (\nabla\xi)^2 Y\right) = (\nabla\phi) \circ Y - \xi \otimes (g \circ Y).$$

This concludes the proof. □

Theorem 4.6 ([4]). *On every nearly Sasakian manifold* (M, g, ϕ, ξ, η)

$$\begin{aligned}(\nabla_X\phi)Y &= g(X, Y)\xi - \eta(Y)X \\ &\quad + \eta(X)(\nabla_\xi\phi)Y - \eta(Y)(\nabla_\xi\phi)X - g\left(X, (\nabla_\xi\phi)Y\right)\xi.\end{aligned} \tag{29}$$

Equivalently

$$\nabla\phi = \xi \otimes g - \mathrm{Id} \otimes \eta + \eta \otimes (\nabla_\xi\phi) - (\nabla_\xi\phi) \otimes \eta - \xi \otimes \left(g \circ (\nabla_\xi\phi)\right).$$

Proof. By Proposition 2.5 the spectrum of $(\nabla\xi)^2$ is non-positive and the multiplicity of 0 is one. Let $0 < \lambda_1 < \cdots < \lambda_\ell$ be such that $(0, -\lambda_1, \ldots, -\lambda_\ell)$ is the spectrum of $(\nabla\xi)^2$. By Proposition 3.9 the vector bundle TM can be written as a direct orthogonal sum of the subbundles $V_0, V_1, \ldots, V_\ell$ such that $(\nabla\xi)^2|_{V_0} = 0$ and $(\nabla\xi)^2|_{V_k} = -\lambda_k \cdot \mathrm{Id}$ with positive λ_k's. Thus every vector field Y on M can be written as a sum $\eta(Y)\xi + \sum_{k=1}^{\ell} Y_k$, where Y_k are such that $(\nabla\xi)^2 Y_k = -\lambda_k Y_k$ and $\eta(Y_k) = 0$.

Since both sides of (29) are linear over $C^\infty(M)$ with respect to Y, we have to check the validity of (29) only for ξ and Y_k's such that $(\nabla\xi)^2 Y_k = -\lambda_k Y_k$ and $\eta(Y_k) = 0$.

For $Y = \xi$ the formula (29) reduces to

$$(\nabla_X\phi)\xi = \eta(X)\xi - X - (\nabla_\xi\phi)X.$$

We can see that it holds on every nearly Sasakian manifold by substituting (ξ, X) into the defining relation $(\nabla\phi - \xi \otimes g + \eta \otimes \mathrm{Id})(1 + (1, 2)) = 0$.

Now suppose Y is such that $(\nabla\xi)^2 Y = -Y$ and $\eta(Y) = 0$. By Proposition 4.5 we know that $(\nabla_X\phi)Y = g(X, Y)\xi$. Next, from the equality

$$(\nabla\xi)^2 - \xi \otimes \eta + \mathrm{Id} = (\nabla_\xi\phi)^2 \tag{30}$$

proved in Proposition 2.4, we get that $(\nabla_\xi\phi)^2 Y = 0$. Since $\nabla_\xi\phi$ is skew-symmetric and g is positively defined, we conclude that $(\nabla_\xi\phi)Y = 0$. Thus evaluating the right side of (29) we also get $g(X, Y)\xi$.

Now assume Y is such that $\eta(Y) = 0$ and $(\nabla\xi)^2 Y = -\lambda Y$ with $\lambda \notin \{0, 1\}$. Then from (30), we get $(\nabla_\xi\phi)^2 Y = (1-\lambda)Y$ and $(1-\lambda) \neq 0$. This shows that Y is in the image of $\nabla_\xi\phi$ and we can apply Proposition 4.4 to compute $(\nabla_X\phi)Y$. We get

$$(\nabla_X\phi)Y = \eta(X)(\nabla_\xi\phi)Y + g(X, Y - (\nabla_\xi\phi)Y)\xi.$$

Since $\eta(Y) = 0$ the right side of (29) evaluates to the same expression. This concludes the proof. □

Remark 4.7. It follows from (29) that a nearly Sasakian manifold is Sasakian if and only if $\nabla_\xi\phi = 0$. In fact, if $\nabla_\xi\phi = 0$, then (29) implies

$$(\nabla_X\phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{31}$$

which is the defining condition of Sasakian structures. In the opposite direction, if M is a Sasakian manifold, then computing $\nabla_\xi\phi$ by (31) we get zero.

Proposition 4.8. *Let* (M, g, ϕ, ξ, η) *be a nearly Sasakian manifold. Denote* $g \circ (\phi \otimes \mathrm{Id})$ *by* Φ *and* $g \circ (\nabla_\xi\phi \otimes \mathrm{Id})$ *by* Ψ. *Then* Φ *and* Ψ *are differential forms and*

$$d\Phi = 3\eta \wedge \Psi, \quad \eta \wedge d\Psi = 0, \quad d\eta \wedge \Psi = 0.$$

Proof. The operator ϕ is skew-symmetric by definition of an almost contact metric structure, and $\nabla_\xi\phi$ is skew-symmetric by Proposition 2.4. This implies that both Φ and Ψ are two forms.

By definition of the exterior differential we have

$$d\Phi = g \circ (\nabla\phi \otimes \mathrm{Id}) \circ (1 + (1, 2, 3) + (1, 3, 2)).$$

By Theorem 4.6, we have

$$\nabla\phi = \xi \otimes g - \mathrm{Id} \otimes \eta + \eta \otimes (\nabla_\xi \phi) - (\nabla_\xi \phi) \otimes \eta - \xi \otimes \left(g \circ (\nabla_\xi \phi) \right). \tag{32}$$

Notice that

$$\begin{aligned}
&g \circ (\xi \otimes g \otimes \mathrm{Id}) = g \otimes \eta \\
&g \circ (-\mathrm{Id} \otimes \eta \otimes \mathrm{Id}) = -(g \otimes \eta)(2,3) \\
&g \circ (\eta \otimes (\nabla_\xi \phi) \otimes \mathrm{Id}) = \eta \otimes \Psi \\
&g \circ (-(\nabla_\xi \phi) \otimes \eta \otimes \mathrm{Id}) = -(\eta \otimes \Psi)(1,2) \\
&g \circ (-\xi \otimes \left(g \circ (\nabla_\xi \phi)\right) \otimes \mathrm{Id} \big) = -(\eta \otimes \Psi)(1,3).
\end{aligned}$$

Next observe that for every $\sigma \in \{(1,2),(2,3),(1,3)\}$ we have $\sigma(1+(1,2,3)+(1,3,2)) = (1,2)+(2,3)+(1,3)$. Hence

$$(g \otimes \eta)(1-(2,3))(1+(1,2,3)+(1,3,2)) = (g \otimes \eta)(1-(1,2))(1+(1,2,3)+(1,3,2))$$

vanishes, since g is symmetric. Therefore

$$\begin{aligned}
d\Phi &= (\eta \otimes \Psi)(1-(1,2)-(1,3))(1+(1,2,3)+(1,3,2)) \\
&= (\eta \otimes \Psi)(1-2\cdot(2,3))(1+(1,2,3)+(1,3,2)) \\
&= 3(\eta \otimes \Psi)(1+(1,2,3)+(1,3,2)) = 3\eta \wedge \Psi,
\end{aligned}$$

where we used $(\eta \otimes \Psi)(2,3) = -\eta \otimes \Psi$. Now $0 = d^2\Phi = 3(d\eta \wedge \Psi + \eta \wedge d\Psi)$ implies that $d\eta \wedge \Psi = -\eta \wedge d\Psi$. Thus it is enough to show only $\eta \wedge d\Psi = 0$. For this we have to check that for any $X, Y, Z \in \ker(\eta)$ one has $d\Psi(X,Y,Z) = 0$. In fact we will show that $(\nabla_X(\nabla_\xi \phi))Y$ is proportional to ξ for any $X, Y \in \ker(\eta)$. Then the result will follow from the definitions of Ψ and the exterior derivative d. We have

$$\nabla(\nabla_\xi \phi) = \nabla((\nabla\phi) \circ (\xi \otimes \mathrm{Id})) = \nabla^2\phi \circ (\xi \otimes \mathrm{Id}) + (\nabla\phi) \circ (\nabla\xi \otimes \mathrm{Id}).$$

Applying (32), we get

$$(\nabla\phi) \circ (\nabla\xi \otimes \mathrm{Id}) = \xi \otimes (g \circ (\nabla\xi \otimes \mathrm{Id})) - (\nabla\xi) \otimes \eta - (\nabla_\xi \phi)(\nabla\xi) \otimes \eta - \xi \otimes (g \circ (\nabla\xi \otimes \nabla_\xi \phi)).$$

Evaluating the right side of the above equation on (X, Y) with $Y \in \ker(\eta)$ we get a vector field proportional to ξ. Thus it is left to show that $(\nabla^2_{X,\xi}\phi)Y$ is proportional to ξ. We have

$$(\nabla^2_{X,\xi}\phi)Y = -(R_{\xi,X}\phi)Y + (\nabla^2_{\xi,X}\phi)Y,$$

and therefore we can use the expressions for $R_\xi \phi$ and $\nabla^2_\xi \phi$ obtained in Proposition 3.7 and in Proposition 4.1, respectively. Namely, we have $R_\xi\phi = -(\nabla\xi)^2\phi \otimes \eta - (g \circ \phi(\nabla\xi)^2) \otimes \xi$, which implies that $(R_{\xi,X}\phi)Y$ is proportional to ξ for $Y \in \ker(\eta)$. Further, $\nabla^2_\xi\phi = \eta \wedge (\nabla_\xi\phi \circ \nabla\xi) - \xi \otimes \left(g \circ (\nabla_\xi \phi \circ \nabla\xi)\right)$ implies that $(\nabla^2_{\xi,X}\phi)Y$ is proportional to ξ for $X, Y \in \ker(\eta)$. This concludes the proof. □

Notice that we did not use $\dim M \geq 7$ in the above proposition.

Theorem 4.9. *Let (M, g, ϕ, ξ, η) be a nearly Sasakian manifold of dimension greater or equal to 7. Then M is a Sasakian manifold.*

Proof. In view of Remark 4.7 it is enough to show $\nabla_\xi\phi = 0$. As g is non-degenerate this is equivalent to $\Psi = 0$. By Proposition 2.5 η is a contact form on M. Therefore $d\eta$ is a symplectic form on the distribution $\ker(\eta)$. The dimension of this distribution is greater than or equal to six. Thus the wedge product by $d\eta$ induces an injective map $\bigwedge^2 \ker(\eta) \to \bigwedge^4 \ker(\eta)$. By Proposition 4.8 we know that $d\eta \wedge \Psi = 0$. Therefore the restriction of Ψ to $\bigwedge^2 \ker(\eta)$ is zero. It is left to show that $i_\xi\Psi = 0$. This follows from the definition of Ψ and $(\nabla_\xi\phi)\xi = 0$, which in turn follows from Proposition 2.4. □

Acknowledgement: This work was partially supported by Fondazione di Sardegna and Regione Autonoma della Sardegna, Project GESTA and KASBA, by MICINN (Spain) grant MTM2015- 64166-C2-2-P, by CMUC – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020 and by the exploratory research project in the frame of Programa Investigador FCT IF/00016/2013.

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