Hyperbolicity of a model for Polyatomic Gases in Relativistic Extended Thermodynamics

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Abstract

The hyperbolicity of a model for Relativistic Extended Thermodynamics of Polyatomic Gases is here proved for every time direction ξ_α; moreover, it is proposed an expressions for the production terms of the field equation which encloses as particular cases the variant of the Anderson-Witting model already known in literature, but also a new variant of the Marle model. With these expressions, it is shown that the field equations satisfy automatically the requirement of a non negative entropy production.

1 Introduction

This article fits into the context of Extended Thermodynamics, which turned out to be a very fertile research field, as seen in the report [1] of part of its results. But near 2011 it seemed that this field of research was exhausted, until in this year the professors T. Arima, S. Taniguchi, T. Ruggeri and M. Sugiyama extended in [2] its methodology to polyatomic gases. The basic idea was simple, as in all masterpieces, and in this case it was to consider two blocks of field equations, called respectively the mass block and that of energy. We do not go into details so as not to deprive the reader of the pleasure of reading these works. This has given enormous new vitality to this area of research, as can be seen in the partial but exhaustive report given in [3]. The relativistic formulation of this work was found in [4] for the modelization of polyatomic gases in the relativistic context. In the present work we find important properties of the field equations proposed in this last article, i.e., their hyperbolicity and a more general expression for the production terms which encloses as particular cases the variant of the Anderson-Witting model already known in literature.
Briefly, this relativistic model was constructed with the idea of reproducing, in its non relativistic limit, the one already known in the classical

framework [2]; moreover, it has been taken into account that in the relativistic context energy is equivalent to mass and this must be true also for internal energy $\mathcal{I}$. With these assumptions it was built the model that can be described quickly in the following way:
In the beginning the field equations are expressed in terms of the so-called Main Field $\lambda_A \equiv (\lambda\,,\,\lambda_\beta\,,\,\Sigma_{\beta\gamma})$ ($\Sigma_{\beta\gamma}$ is taken symmetric and traceless) and they can be written in compact form as

$$\partial_\alpha\left(\frac{\partial\, h'^\alpha}{\partial\,\lambda_A}\right) = I^A \quad , \quad \text{or} \quad \frac{\partial^2\, h'^\alpha}{\partial\,\lambda_A\partial\,\lambda_B}\partial_\alpha\,\lambda_B = I^A\,. \tag{1}$$

In a second moment they are converted in terms of physical variables. In (1) the 4-potential h'^α is

$$h'^\alpha = -k_B\, c\int_{\Re^3}\int_0^{+\infty} e^{-1-\frac{\chi}{k_B}}\, p^\alpha\,\phi(\mathcal{I})\, d\overrightarrow{P}\, d\,\mathcal{I}\,, \tag{2}$$

where

$$f = e^{-1-\frac{\chi}{k_B}}\,, \quad \text{and} \tag{3}$$
$$\chi = m\,\lambda + \left(1+\frac{\mathcal{I}}{m\,c^2}\right)\lambda_\beta p^\beta + \frac{1}{m}\,\left(1+\frac{2\,\mathcal{I}}{m\,c^2}\right)\Sigma_{\beta\gamma}p^\beta p^\gamma\,.$$

Here $\mathcal{I}$ is the internal energy of a molecule and the function $\phi(\mathcal{I}) = I^a$ measures how much the gas is polyatomic in correspondence to the parameter a (monoatomic gases are obtained as a limiting case for $a \to -1$). In [2] the symbol I^α was used, while in [4] it was replaced by the symbol I^a because, in the relativistic context, Greek indexes have a special meaning which they don' t have in the non relativistic context. We did this change to avoid confusion and here we adopt the notation of [4]. (See [5] for the analytical calculation of this limit). This distribution function f was obtained through the Maximum Entropy Principle (See [6], [7], [1], [8] for historical treatments of this principle) and isn' t the solution of the Boltzmann Equation which was used only to give suggestions on the form of the balance equations and, after that, left out. For this reason the nice property proved in [9], such as the H-Theorem, cannot be used here without further discussions.
Also the expression of the right hand side of eq. (1) was not indicated in [4], but was left for future investigations. A first proposal to this regard was introduced in [9] and [10] as a variant of the Anderson-Witting model [11]. The authors of [9] started by considering two previous expressions for Q, i.e.,

- The Marle model [12] which is an extension of the non-relativistic BGK model in the Eckart frame:

$$Q = -\frac{m}{\tau_M}\,\left(f - f_E\right)\,, \tag{4}$$

  where τ_M is the relaxation time in the rest frame where the momentum of particles is zero, and f_E is the Jüttner equilibrium distribution. (We have substituted here τ with τ_M to distinguish it from the subsequent one).

- The Anderson-Witting model which provides the expression :

$$Q = - \frac{U_{L\mu} p^\mu}{c^2 \tau_{A-W}} \, (f - f_E) \, , \tag{5}$$

where $U_{L\mu}$ indicates the four-velocity according with the Landau-Lifshitz definition.

(We have substituted here τ with τ_{A-W} to distinguish it from τ_M). Both models have been discussed in [9] and the need was felt to introduce another expression, more suited to the system found in [4]. This expression was called "variant of the Anderson-Witting model" because it draws inspiration from the latter. We do not report the result because it is contained, as a special case, in the present that we are now going to describe. Here we propose the following generalization for the right hand side of eq. (1): $I^A \equiv (0\, , \, 0\, , \, I^{\beta\gamma})$ with

$$I^{\beta\gamma} = \frac{c}{m} \int_{\Re^3} \int_0^{+\infty} Q \, p^\beta \, p^\gamma \, \left(1 \, + \, \frac{2 \, \mathcal{I}}{m \, c^2} \right) \, \phi(\mathcal{I}) \, d\overrightarrow{P} \, d\mathcal{I} \, , \tag{6}$$

and

$$Q = \left(m \, a_1 + \frac{T}{c^2} \, a_2 \lambda_\alpha p^\alpha \right) \left(f_E \, - \, f \, e^{- \frac{1}{k_B} \left[m \, \psi + \psi_\mu \, p^\mu \, \left(1 + \frac{\mathcal{I}}{m \, c^2} \right) \right]} \right) \, , \tag{7}$$

while a_1 and a_2 are two non negative scalars such that $(a_1)^2 + (a_2)^2 > 0$. In this way, for $a_1 = 0$, $a_2 = \frac{1}{\tau}$, with T the absolute temperature and τ a relaxation time, we obtain the variant of the Anderson and Witting model reported in [9] and we will denote it with $Q = Q_{A-W}$ to distinguish it from the general one, while for $a_1 = \frac{1}{\tau_M}$, $a_2 = 0$, with τ_M a relaxation time, we obtain the variant of the Marle model [12] and we will denote it with $Q = Q_M$ to distinguish it from $Q = Q_{A-W}$.
Therefore the present work does justice to Marle, showing that his model also allows a variant suitable to the system in [4]. We call it "variant of the Marle model". More than that, the present mathematical proposal allows to treat in a unified manner both models; it also shows that the closure for the production term is not unique.
In every cases, ψ and ψ_μ have to be determined by the conditions that their values at equilibrium are zero and, moreover, that the productions of mass and momentum-energy are zero, i.e.,

$$\begin{aligned} & \psi^E = 0 \quad . \quad \psi^E_\mu = 0 \quad , \\ & \tilde{I} = m \, c \int_{\Re^3} \int_0^{+\infty} Q \, \phi(\mathcal{I}) \, d\overrightarrow{P} \, d\mathcal{I} = 0 \, , \\ & \tilde{I}^\beta = c \int_{\Re^3} \int_0^{+\infty} Q \, p^\beta \, \left(1 \, + \, \frac{\mathcal{I}}{m \, c^2} \right) \, \phi(\mathcal{I}) \, d\overrightarrow{P} \, d\mathcal{I} = 0 \, , \end{aligned} \tag{8}$$

We will see in sect. 2 that these conditions determine in an unique way ψ and ψ_μ without leaving residual constraints; moreover, we will see that this definition implies automatically a non negative entropy

production and some useful consequences of this property.
We observe that at equilibrium eq. (7) implies $Q = 0$, so that the above expression of Q is at least of the first order with respect to equilibrium.
We observe also that the first two components of I^A are zero, so that the first two equations in (1) give the conservation laws of mass and energy-momentum.
In any case, eqs. (1) have the symmetric form (but only because we are using the Main Field as independent variables); we will prove in sect. 3 that the quadratic form

$$K = \xi_\alpha \frac{\partial^2 h'^\alpha}{\partial \lambda_A \partial \lambda_B} \delta \lambda_A \delta \lambda_B \tag{9}$$

is positive definite $\forall$ timelike 4-vector ξ_α. This property is called "Convexity of Entropy". Consequently, we have that the balance equations (1) aren't only symmetric, but they are also hyperbolic. In fact, by applying the results of [13], we see that symmetrization and convexity of entropy $\forall$ time-like 4-vector ξ_α are two sufficient conditions for hyperbolicity and we prove here both of them without assuming that one is a consequence of the other. (To better clarify what we mean by this, let's consider the system $F^{\alpha AB}\partial_\alpha\lambda_B = I^A$. In some literature this system is called symmetric if $F^{\alpha AB} = F^{\alpha BA}$ and if $\xi_\alpha F^{\alpha AB}\delta\lambda_A\delta\lambda_B$ is positive or negative definite for any time-like unitary 4-vector ξ_α (it may be constant or not, because its derivatives don't take a role in this condition). From this point of view any symmetric system is hyperbolic. Here we prefer to follow what the common sense suggests, namely that it is symmetric if $F^{\alpha AB} = F^{\alpha BA}$ and that the convexity of entropy holds if $\xi_\alpha F^{\alpha AB}\delta\lambda_A\delta\lambda_B$ is positive or negative definite. From this point of view hyperbolicity is assured if the 2 independent conditions hold, namely symmetric form and convexity of entropy. Obviously, this is a sufficient condition and not a necessary condition).
So we achieve here 3 objectives:

- Firstly, we prove this hyperbolicity property; this proof is missing in [9] and [10]; in fact, in lines 4 and 5 of page 304 of [10] we read : "Therefore the quadratic form is not negative definite for any value of the field but only near equilibrium". Similarly in its Statement 2 of the same page 304 we read that the symmetry of the system is presented as a requirement and not as a proved property.
- Second, we note that the expression of Q found in [9] and [10] was introduced ad hoc and is valid only at first order with respect to equilibrium. To find its generalization up to any order with respect to equilibrium, we introduce in the expression of Q a scalar ψ and a four-vector ψ_μ to be determined starting from the requirement that we have conservation of mass and energy in any order with respect to equilibrium.
- Finally, the third objective is to generalize the expression of Q by introducing two arbitrary constants a_1 and a_2 so as to obtain, for

their particular values, the expression of Q found in [9] and that obtained as a variant of the Marle model which is here presented. Since the previous objective also must be achieved still for this generalized Q, we will follow the steps described there also for the more general model.

All these objectives are realized here in a mathematically correct way and in a unified manner; for their physical reasons, we rely to those exposed in [11] and [12] by Anderson-Witting and Marle respectively.

2 Entropy Production, its consequences and determination of ψ and ψ_μ from (8).

We calculate now

$$\Sigma = I^{\beta\gamma}\Sigma_{\beta\gamma} = \Sigma_{\beta\gamma}\frac{c}{m}\int_{\Re^3}\int_0^{+\infty}\left(m\,a_1+\frac{T}{c^2}\,a_2\lambda_\alpha p^\alpha\right)\cdot \tag{10}$$
$$\left(f_E\,-\,f\,e^{-\frac{1}{k_B}\left[m\,\psi+\psi_\mu\,p^\mu\,\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\right]}\right)\,p^\beta\,p^\gamma\,\left(1\,+\,\frac{2\mathcal{I}}{m\,c^2}\right)\,\phi(\mathcal{I})\,d\overrightarrow{P}\,d\mathcal{I}\,,$$

and we add to this expression $0=\psi\,\tilde{I}+\psi_\beta\,\tilde{I}^\beta$ so that it becomes

$$\Sigma = k_B\,c\int_{\Re^3}\int_0^{+\infty}\left(m\,a_1+\frac{T}{c^2}\,a_2\lambda_\alpha p^\alpha\right)\cdot$$
$$\left(-\,f_E\,+\,f\,e^{-\frac{1}{k_B}\left[m\,\psi+\psi_\mu\,p^\mu\,\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\right]}\right)\cdot \tag{11}$$
$$\cdot\,\ln\,\frac{f\,e^{-\frac{1}{k_B}\left[m\,\psi+\psi_\mu\,p^\mu\,\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\right]}}{f_E}\,\phi(\mathcal{I})\,d\overrightarrow{P}\,d\mathcal{I}\geq\,0\,.$$

The conclusion is based on the fact that, for every positive values of x and y, we have $(-x+y)\,\ln\,\frac{y}{x}\geq 0$ and it is equal to zero if and only if $x=y$. (In fact,

- If $x>y\,\rightarrow\quad\frac{y}{x}<1\,\rightarrow\quad\ln\,\frac{y}{x}<0\,\rightarrow\quad(-x+y)\,\ln\,\frac{y}{x}>0\,,$
- If $x=y\,\rightarrow\quad(-x+y)\,\ln\,\frac{y}{x}=0\,,$
- If $x<y\,\rightarrow\quad\frac{y}{x}>1\,\rightarrow\quad\ln\,\frac{y}{x}>0\,\rightarrow\quad(-x+y)\,\ln\,\frac{y}{x}>0).$

This proves that with the above expressions of f and Q we have automatically a non negative entropy production. Moreover, it is zero if and only if $\Sigma_{\beta\gamma}=0$.
It is interesting that, if we add to this expression
$0=\left(\frac{k_B}{m}+\lambda\right)\,\tilde{I}\,-\,\lambda_\delta\,\tilde{I}^\delta$, it becomes

$$\Sigma = -\,k_B\,c\int_{\Re^3}\int_0^{+\infty}\,Q\,\ln\,f\,\phi(\mathcal{I})\,d\overrightarrow{P}\,d\mathcal{I}\,,$$

even if f is not a solution of the Boltzmann equation.
From the result that Σ is positive definite in the variables $\Sigma_{\mu\nu}$ we can

deduce 3 important inequalities which in another way are difficult to prove. In fact, from eq. (5) of [10], we have

$$I^{<\beta\gamma>} = B_1^{\pi}\, \pi \left(g^{\beta\gamma} - \frac{4}{c^2} U^\beta U^\gamma \right) + 2\, B_2^q U^{(\beta} q^{\gamma)} + B_3^t\, t^{<\beta\gamma>_3} \,. \qquad (12)$$

But we can define π, q^δ, $t^{<\delta\theta>_3}$ from pages 430 and 431 of [4]. i.e.,

$$\pi = -\,\alpha_1\, U^\mu U^\nu \Sigma_{\mu\nu} \,, \quad q^\beta = \alpha_2\, h^\mu_\delta U^\nu \Sigma_{\mu\nu} \,, \quad t^{<\beta\gamma>_3} = \alpha_3\, h_\mu^{<\beta} h_\nu^{\gamma>_3}\, \Sigma^{\mu\nu} \,. \qquad (13)$$

with

$$\alpha_1 = \frac{\begin{vmatrix} n\,c^2 & \frac{e}{m} & \frac{A_1^0 c^2 + A_{11}^0}{m} \\ \frac{e\,c^2}{m} & c^4 B_5 & \frac{1}{3}\, B_2 c^2 + B_3 c^4 \\ \frac{p}{m} & \frac{1}{3}\, B_4 & \frac{1}{3}\, B_2 + \frac{1}{9} \frac{B_1}{c^2} \end{vmatrix}}{\begin{vmatrix} n\,c^2 & \frac{e}{m} \\ \frac{e\,c^2}{m} & B_5 c^4 \end{vmatrix}} \, \frac{m^2}{k_B} \,,$$

$$\alpha_2 = -\, \frac{\frac{D_3}{p}\, \frac{2\,n^2\,c^6}{9}\, \frac{m^3}{k_B}}{\left(\int_0^{+\infty} J_{2,1}(\gamma^*)\, \phi(\mathcal{I})\, d\,\mathcal{I} \right)^2} \,,$$

$$\alpha_3 = -\, \frac{2}{15}\, B_1\, \frac{m^2}{k_B} \,.$$

So, by using (12) and (13), we obtain

$$\Sigma = \frac{4}{c^2}\, \alpha_1\, B_1^\pi \, \left(U^\beta U^\gamma \Sigma_{\beta\gamma} \right)^2 + 2\, \alpha_2\, B_2^q\, U^\mu \Sigma_{\mu\nu} h^{\nu\beta} \Sigma_{\beta\gamma} U^\gamma +$$
$$+\, \alpha_3\, B_3^t \left(h^{\mu\beta} h^{\gamma\nu} - \frac{1}{3}\, h^{\mu\nu} h^{\beta\gamma} \right) \Sigma_{\mu\nu} \Sigma_{\beta\gamma} = \frac{4}{c^2}\, \alpha_1\, B_1^\pi\, c^2\, \left(\Sigma_{00}\right)^2 +$$
$$+\, 2\, \alpha_2\, B_2^q\, c^2 \left[\left(\Sigma_{01}\right)^2 + \left(\Sigma_{02}\right)^2 + \left(\Sigma_{03}\right)^2 \right] +$$
$$+\, \alpha_3\, B_3^t \left[2\, \left(\Sigma_{12}\right)^2 + 2\, \left(\Sigma_{13}\right)^2 + 2\, \left(\Sigma_{23}\right)^2 + \frac{3}{2}\, \left(X_6\right)^2 + 2\, \left(X_7\right)^2 \right] \,,$$

with $X_6 = \Sigma_{11} - \frac{1}{3}\, \Sigma_{00}$, $X_7 = \Sigma_{22} - \frac{1}{2}\, \Sigma_{00} + \frac{1}{2}\, \Sigma_{11}$. Since Σ is positive definite, it follows automatically that

$$\alpha_1\, B_1^\pi > 0 \quad , \quad \alpha_2\, B_2^q > 0 \quad , \quad \alpha_3\, B_3^t > 0 \,. \qquad (14)$$

Another important property can be obtained by adding $\frac{\partial \psi}{\partial \Sigma_{\beta\gamma}}\, \tilde{I} + \frac{\partial \psi_\delta}{\partial \Sigma_{\beta\gamma}}\, \tilde{I}^\delta =$ 0 to eq. (6) so that it becomes

$$I^{\beta\gamma} = c \int_{\Re^3} \int_0^{+\infty} Q \left[\frac{1}{m} \left(1 + \frac{2\mathcal{I}}{m\, c^2} \right) p^\beta\, p^\gamma + m \frac{\partial \psi}{\partial \Sigma_{\beta\gamma}} + \right. \tag{15}$$
$$\left. + \frac{\partial \psi_\delta}{\partial \Sigma_{\beta\gamma}}\, p^\delta \left(1 + \frac{\mathcal{I}}{m\, c^2} \right) \right] \phi(\mathcal{I})\, d\vec{P}\, d\mathcal{I}\, ,$$

from which it follows

$$\left(\frac{\partial\, I^{\beta\gamma}}{\partial\, \Sigma_{\mu\nu}} \right)_E = \frac{c}{k_B} \int_{\Re^3} \int_0^{+\infty} \left(m\, a_1 + \frac{a_2}{c^2}\, U_\alpha p^\alpha \right) f_E\, \psi^{\beta\gamma} \psi^{\mu\nu}\, \phi(\mathcal{I})\, d\vec{P}\, d\mathcal{I}\, ,$$

with

$$\psi^{\beta\gamma} = \frac{1}{m} \left(1 + \frac{2\mathcal{I}}{m\, c^2} \right) p^\beta p^\gamma + m \frac{\partial \psi}{\partial \Sigma_{\beta\gamma}} + \frac{\partial \psi_\delta}{\partial \Sigma_{\beta\gamma}}\, p^\delta \left(1 + \frac{\mathcal{I}}{m\, c^2} \right) .$$

In other words, we have seen that up to first order with respect to equilibrium, $I^{\beta\gamma} = I^{\beta\gamma\mu\nu}\Sigma_{\mu\nu}$ with $I^{\beta\gamma\mu\nu} = I^{\mu\nu\beta\gamma}$, $\Sigma = I^{\beta\gamma\mu\nu}\Sigma_{\beta\gamma}\Sigma_{\mu\nu}$ and

$$\Sigma = \frac{c}{k_B} \int_{\Re^3} \int_0^{+\infty} \left(m\, a_1 + \frac{a_2}{c^2}\, U_\alpha p^\alpha \right) f_E \left(\psi^{\beta\gamma} \Sigma_{\beta\gamma} \right)^2 \phi(\mathcal{I})\, d\vec{P}\, d\mathcal{I}\, .$$

2.1 Determination of ψ and ψ_μ from (8).

We firstly see that $(8)_{1,2}$ are simply definitions of ψ^E and ψ^E_μ.
By using (7), we see that $(8)_{3,4}$ become

$$0 = \tilde{I} = m\, c \int_{\Re^3} \int_0^{+\infty} \left(m\, a_1 + \frac{T}{c^2}\, a_2 \lambda_\alpha p^\alpha \right) \cdot$$
$$\cdot \left(f_E \, - \, f\, e^{-\frac{1}{k_B} \left[m\, \psi + \psi_\mu\, p^\mu \left(1 + \frac{\mathcal{I}}{m\, c^2} \right) \right]} \right) \phi(\mathcal{I})\, d\vec{P}\, d\mathcal{I}\, ,$$
$$0 = \tilde{I}^\beta = c \int_{\Re^3} \int_0^{+\infty} \left(m\, a_1 + \frac{T}{c^2}\, a_2 \lambda_\alpha p^\alpha \right) \cdot$$
$$\cdot \left(f_E \, - \, f\, e^{-\frac{1}{k_B} \left[m\, \psi + \psi_\mu\, p^\mu \left(1 + \frac{\mathcal{I}}{m\, c^2} \right) \right]} \right) p^\beta \left(1 + \frac{\mathcal{I}}{m\, c^2} \right) \phi(\mathcal{I})\, d\vec{P}\, d\mathcal{I}\, . \tag{16}$$

and they are identically satisfied at equilibrium. Now we see that the following relation holds

$$\left(f_E \, - \, f\, e^{-\frac{1}{k_B} \left[m\, \psi + \psi_\mu\, p^\mu \left(1 + \frac{\mathcal{I}}{m\, c^2} \right) \right]} \right)^{(1)} = \tag{17}$$
$$= -f^{(1)} + \frac{f_E}{k_B} \left[m\, \psi^{(1)} \, + \, \psi^{(1)}_\mu\, p^\mu \left(1 \, + \, \frac{\mathcal{I}}{m\, c^2} \right) \right] .$$

(Here and in what follows, an apex (i) denotes the homogeneous part of this expression at the order i with respect to equilibrium). By using it, (16) becomes

$$
\begin{aligned}
0 = \tilde{I} = a_1\, m^2\, c\, & \left[- \frac{V^{(1)}}{m\, c} + \frac{1}{k_B\, c}\, V_E\, \psi^{(1)} + \frac{1}{k_B}\, \psi_\theta^{(1)} T_E^\theta \right] + \\
& + a_2\, \frac{U_\mu}{c^2} \left[- V^{(1)\mu} + \frac{m}{k_B}\, V_E^\mu\, \psi^{(1)} + \frac{m}{k_B}\, \psi_\theta^{(1)} T_E^{\mu\theta} \right] , \\
0 = \tilde{I}^\beta = a_1\, m\, c\, & \left[- T^{(1)\beta} + \frac{m}{k_B}\, \psi^{(1)} T_E^\beta + \frac{1}{k_B}\, \psi_\theta^{(1)}\, A_E^{\beta\theta} \right] + \\
& + a_2\, \frac{U_\mu}{c^2} \left[- T^{(1)\mu\beta} + \frac{m}{k_B}\, \psi^{(1)} T_E^{\mu\beta} + \frac{m^2}{k_B}\, \psi_\theta^{(1)}\, A_{11}^{\mu\beta\theta} \right] ,
\end{aligned}
\tag{18}
$$

which is expressed in terms of the tensors U^μ, $V^\mu = m\, n\, U^\mu$, $T^{\mu\theta}$, $A_{11}^{\mu\beta\theta}$ (which are reported in [4]) and V, T^θ, $A_E^{\beta\theta}$ (which are reported below in Appendix A).
By recalling that $V^{(1)\mu} = 0$, $U_\mu T^{(1)\mu\beta} = 0$, the first one of these relations and the second one contracted by $\frac{U_\beta}{c^2}$ become

$$
\begin{aligned}
0 = a_1\, \frac{m^2}{k_B} & \left[V_E \left(\psi^{(1)} + \lambda - \lambda_E \right) + c\, T_0\, U^\theta \left(\psi_\theta^{(1)} + \lambda_\theta - \lambda_{E\theta} \right) - \right. \\
& \left. \frac{R_0 + R_1}{m\, c^2}\, k_B\, U^\beta U^\gamma \Sigma_{\beta\gamma} \right] + a_2\, \frac{m^2}{k_B\, c^2} \left(n\, c^2\, \psi^{(1)} + \frac{e}{m}\, \psi_\theta^{(1)}\, U^\theta \right) , \\
0 = a_1\, \frac{m^2}{k_B} & \left[c\, T_0 \left(\psi^{(1)} + \lambda - \lambda_E \right) + \frac{T_3}{m\, c}\, U^\theta \left(\psi_\theta^{(1)} + \lambda_\theta - \lambda_{E\theta} \right) - \right. \\
& \left. \frac{k_B}{m\, c} \left(T_1\, c^2 + T_2 \right) U^\mu U^\nu \Sigma_{\mu\nu} \right] + \frac{a_2}{c^2}\, \frac{m^2}{k_B} \left(\frac{e}{m}\, \psi^{(1)} + \psi_\theta^{(1)}\, U^\theta\, B_5\, c^2 \right) ,
\end{aligned}
\tag{19}
$$

while the second one contracted by h_β^δ is

$$
\begin{aligned}
& \frac{a_1\, m\, c}{k_B}\, T_4\, h^{\delta\theta} \left(\psi_\theta^{(1)} + \lambda_\theta - \lambda_{E\theta} - 2\, k_B\, \frac{T_2}{T_4}\, \Sigma_{\theta\mu}\, U^\mu \right) + \\
& + a_2\, \frac{m^2}{k_B\, c^2} \left(\frac{1}{3}\, B_4\, c^2\, \psi_\theta^{(1)}\, h^{\delta\theta} - \frac{k_B}{m^2}\, q^\delta \right) = 0\, .
\end{aligned}
\tag{20}
$$

Now, in (16) of [10] (which here we confirm below in (25)) and (35) there is proved that the matrixes M_{33} and $\tilde{M}_1$ are positive definite, i.e., $\underline{x}^T M_{33}\, \underline{x} > 0$, $\underline{x}^T \tilde{M}_1\, \underline{x} > 0$ for every vector $\underline{x} \neq \underline{0}$. It follows that $\underline{x}^T \left(a_1\, \frac{m^2}{k_B}\, \tilde{M}_1 + a_2\, \frac{m^2}{k_B\, c^2}\, M_{33} \right) \underline{x} \geq 0$ because a_1 and a_2 are both non negative numbers; if the result is zero, then we have $a_1\, \frac{m^2}{k_B}\, \underline{x}^T\, \tilde{M}_1\, \underline{x} = 0$ and $a_2\, \frac{m^2}{k_B\, c^2}\, \underline{x}^T\, M_{33}\, \underline{x} = 0$ from which it follows $a_1 = 0$ and $a_2 = 0$ against the ipothesis $(a_1)^2 + (a_2)^2 > 0$. It follows that the matrix $a_1\, \frac{m^2}{k_B}\, \tilde{M}_1 + a_2\, \frac{m^2}{k_B\, c^2}\, M_{33}$ is positive definite and has a positive determinant. Thanks to this property, we see that (19) gives $\psi^{(1)}$ and $U^\theta \psi_\theta^{(1)}$ because $\left| a_1\, \frac{m^2}{k_B}\, \tilde{M}_1 + a_2\, \frac{m^2}{k_B\, c^2}\, M_{33} \right|$ is its determinant of coefficients of

the unknowns $\psi^{(1)}$ and $U^\theta \psi_\theta^{(1)}$. But, from the results of [4], we have that $\lambda - \lambda_E$, $U^\theta \left(\lambda_\theta - \lambda_{E\theta}\right)$ and $U^\beta U^\gamma \Sigma_{\beta\gamma}$ are proportional to π; so also $\psi^{(1)}$ and $U^\theta \psi_\theta^{(1)}$ are proportional to π.
Similarly, (20) gives $h^{\theta\delta} \psi_\theta^{(1)}$ because $\frac{a_1\, m\, c}{k_B} T_4 \, + \, \frac{1}{3}\, a_2 \, \frac{m^2}{k_B} \, B_4 > 0$. But, from the results of [4], we have that $h^{\theta\delta} \left(\lambda_\theta - \lambda_{E\theta}\right)$ and $h^{\theta\delta} U^\gamma \, \Sigma_{\theta\gamma}$ are proportional to q^δ; so also $h^{\theta\delta} \psi_\theta^{(1)}$ is proportional to q^δ.
We observe that, in the case $a_1 = 0$, these equations give

$$\psi^{(1)} = 0 \quad , \quad \psi_\theta^{(1)} U^\theta = 0 \quad , \quad \psi_\delta^{(1)} = - \, \frac{3\, k_B}{m^2 \, c^2 \, B_4} \, q_\delta \, ,$$

which leads to the expression for Q proposed in [9]. There it was introduced ad hoc, like a rabbit pulled out of a magician 's cylinder, while here we deduced it axiomatically.

- Now that we have desumed $\psi^{(1)}$ and $\psi_\theta^{(1)}$ from eqs. $(8)_{3,4}$ at

first order with respect to equilibrium, let us suppose with an iterative procedure, to have imposed eqs. $(8)_{3,4}$ up to the order $i-1$ with respect to equilibrium obtaining as result the expressions of $\psi^{(h)}$ and $\psi_\theta^{(h)}$ for $h = 0, \, 1, \, \cdots, i-1$; we prove now that, by imposing them at the order i we obtain $\psi^{(i)}$ and $\psi_\theta^{(i)}$. We consider now the expansions

$$\begin{aligned}
& e^{-\frac{1}{k_B}\left[m\, \psi + \psi_\mu \, p^\mu \, \left(1 + \frac{\mathcal{I}}{m\, c^2} \right) \right]} = \\
& \quad = \sum_{h=0}^{+\infty} \frac{1}{h!} \left(\frac{-1}{k_B} \right)^h \left[m\, \psi \, + \, \psi_\mu \, p^\mu \, \left(1 \, + \, \frac{\mathcal{I}}{m\, c^2} \right) \right]^h = \\
& \quad = 1 \, - \, \frac{1}{k_B} \left[m\, \psi \, + \, \psi_\mu \, p^\mu \, \left(1 \, + \, \frac{\mathcal{I}}{m\, c^2} \right) \right] + \\
& \quad + \left(\frac{1}{k_B} \right)^2 \left[m\, \psi \, + \, \psi_\mu \, p^\mu \, \left(1 \, + \, \frac{\mathcal{I}}{m\, c^2} \right) \right]^2 \; \sum_{h=2}^{+\infty} \frac{1}{h!} \left(\frac{-1}{k_B} \right)^{h-2} \cdot \\
& \quad \cdot \left[m\, \psi \, + \, \psi_\mu \, p^\mu \, \left(1 \, + \, \frac{\mathcal{I}}{m\, c^2} \right) \right]^{h-2} \, ,
\end{aligned}$$

$$\begin{aligned}
\text{and} \quad & \left(f \, e^{-\frac{1}{k_B}\left[m\, \psi + \psi_\mu \, p^\mu \, \left(1 + \frac{\mathcal{I}}{m\, c^2} \right) \right]} \right)^{(i)} = \\
& = f_E \, \left(e^{-\frac{1}{k_B}\left[m\, \psi + \psi_\mu \, p^\mu \, \left(1 + \frac{\mathcal{I}}{m\, c^2} \right) \right]} \right)^{(i)} + \\
& + \sum_{r=1}^{i} f^{(r)} \, \left(e^{-\frac{1}{k_B}\left[m\, \psi + \psi_\mu \, p^\mu \, \left(1 + \frac{\mathcal{I}}{m\, c^2} \right) \right]} \right)^{(i-r)} \, .
\end{aligned}$$

Moreover, we note that $\left(e^{-\frac{1}{k_B}\left[m\, \psi + \psi_\mu \, p^\mu \, \left(1 + \frac{\mathcal{I}}{m\, c^2} \right) \right]} \right)^{(i)} =$

$$\begin{aligned}
& = - \, \frac{1}{k_B} \, \left[m\, \psi^{(i)} \, + \, \psi_\mu^{(i)} \, p^\mu \, \left(1 \, + \, \frac{\mathcal{I}}{m\, c^2} \right) \right] + \\
& + \text{ terms involving only } \psi^{(s)} \text{ and } \psi_\mu^{(s)} \text{ with } s < i \text{ which are known} \, ,
\end{aligned}$$

for the iterative procedure. Consequently, eqs. $(8)_{3,4}$ at the order i with respect to equilibrium, can be expressed as

$$\frac{m^2 c}{k_B\,\tau_M}\int_{\Re^3}\int_0^{+\infty} f_E\left[m\,\psi^{(i)} + \psi^{(i)}_\mu\,p^\mu\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\right]\phi(\mathcal{I})\,d\overrightarrow{P}\,d\mathcal{I} = X\,,$$

$$\frac{mc}{k_B\,\tau_M}\int_{\Re^3}\int_0^{+\infty} f_E\left[m\,\psi^{(i)} + \psi^{(i)}_\mu\,p^\mu\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\right]p^\beta\cdot$$
$$\cdot\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\phi(\mathcal{I})\,d\overrightarrow{P}\,d\mathcal{I} = X^\beta\,,$$

where we have transferred to the right hand sides all the terms of order less than i, which are known, and we have called them X and X^β respectively.

We see that here, the coefficients of the unknowns $\psi^{(i)}$ and $\psi^{(i)}_\mu$ are the same coefficients of the unknowns $\psi^{(1)}$ and $\psi^{(1)}_\mu$ in eq. (18). Since the system (18) has given an unique solution for its unknowns, we can conclude that also this last system gives an unique solution for its unknowns, i.e., $\psi^{(i)}$ and $\psi^{(i)}_\mu$.

2.2 Convergenze of the expression of Q to the classical BGK.

We prove now that the present BGK relativistic variant converges to the classical BGK. To this end, let us note firstly that, up to first order with respect to equilibrium, we have

$$f_E - f\,e^{-\frac{1}{k_B}\left[m\,\psi + \psi_\mu\,p^\mu\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\right]} = f_E - f +$$
$$+\frac{f_E}{k_B}\left[m\,\psi^{(1)} + \psi^{(1)}_\mu\,p^\mu\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\right],$$

so that the first order deviation of eq. (7) with respect to equilibrium is

$$Q = \left(m\,a_1 + \frac{1}{c^2}\,a_2 U_\alpha p^\alpha\right)\left(f_E - f + \right.$$
$$\left. + \frac{f_E}{k_B}\left[m\,\psi^{(1)} + \psi^{(1)}_\mu\,p^\mu\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\right]\right).$$

In the reference frame with $U^\alpha \equiv (c,\,0,\,0,\,0)$, $p^\alpha \equiv m\Gamma(c,\,\xi^i)$ where $\Gamma = \left(1-\frac{|\vec{\xi}|^2}{c^2}\right)^{-1/2}$ is the Lorentz factor, this expression becomes

$$Q = m\,(a_1 + a_2\,\Gamma)\,\{f_E - f +$$
$$+\frac{m f_E}{k_B}\left[\psi^{(1)} + \Gamma\left(\psi^{(1)}_\mu\,U^\mu - \psi^{(1)}_\mu\,h^\mu_\delta p^\delta\right)\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\right]\right\}.$$

We will prove in the sequel that $\psi^{(1)}$, $\psi^{(1)}_\mu\, U^\mu$ have a finite non relativistic limit and, moreover,

$$\lim_{c\to+\infty} \psi^{(1)} + \psi^{(1)}_\mu\, U^\mu = 0\,, \quad \lim_{c\to+\infty} \psi^{(1)}_\mu\, h^\mu_\delta = 0\,. \tag{21}$$

By adding to these properties, those reported on page 7 of [9], i.e.,

$$\lim_{c\to\infty} \Gamma = 1, \quad \lim_{c\to\infty} f_E = \frac{1}{m^3} f_M, \quad \lim_{c\to\infty} f = \frac{1}{m^3} f^C,$$

where f_M is the Maxwellian and f^C denotes the classical distribution function solution of the classical Boltzmann equation,
we obtain:

$$Q^C = \lim_{c\to\infty} m^2 Q = \frac{1}{\tau}(f_M - f^C)\,, \quad \text{with}\, \frac{1}{\tau} = a_1 + a_2\,.$$

So, to complete our discussion, there remains to prove (21). In particular, $(21)_2$ is the limit of (20), taking into account that

$$\lim_{c\to\infty} mcT_4 = mnk_BT, \quad \lim_{c\to\infty} B_4 = 3n\frac{k_BT}{m},$$
$$\lim_{c\to\infty} \frac{A^0_{11}}{m} = n\frac{k_BT}{m}, \quad \lim_{c\to\infty} -k_B\,\frac{T_2}{T_4} = 1$$

and that, from $(A.14)_1$ of [4], it follows

$$\lim_{c\to\infty} h^{\delta\theta}\,(\lambda_\theta - \lambda_{E\theta} + 2\,\Sigma_{\theta\mu}\, U^\mu) = 0\,.$$

Equation $(21)_1$ is the limit of $(19)_2$, taking into account that cT_0, $\frac{T_3}{mc}$, $-k_B\frac{R_0+R_1}{mc^2}$, $\frac{e}{mc^2}$, B_5 and $\frac{A^0_1}{m}$ have all limit equal to n and that, from $(A.10)_1$ of [4] divided by nc^2, it follows

$$\lim_{c\to\infty} \left[\lambda - \lambda_E + U^\theta\,(\lambda_\theta - \lambda_{E\theta}) + \Sigma_{\theta\mu} U^\theta U^\mu\right] = 0\,.$$

Finally, to prove that $\psi^{(1)}$, $\psi^{(1)}_\mu\, U^\mu$ have a finite non relativistic limit, we use the measure $\varphi(\mathcal{I}) = \mathcal{I}^a$ and consider $(19)_1$ multiplied by $-\gamma^2 + \left(a+\frac{5}{2}\right)\gamma$ and we add to it $(19)_2$ multiplied by γ^2. We replace $(19)_1$ with this new equation of which we now take the non relativistic limit. In the result we take into account also of the non relativistic limit of $(A.10)_{1,2}$ of [4]. The calculations are long and we don't report them for the sake of brevity. We say only that they make use of the expansions in page 434 of [4]. In addition to them, we often use also the following property

$$\lim_{\gamma\to\infty} \frac{\int_0^{+\infty} e^{-\frac{\mathcal{I}}{k_BT}} \left(\frac{\gamma}{\gamma^*}\right)^{1/2} \left(\frac{\mathcal{I}}{k_BT} - a - 1\right) \mathcal{I}^a\, d\mathcal{I}}{\int_0^{+\infty} e^{-\frac{\mathcal{I}}{k_BT}} \mathcal{I}^a\, d\mathcal{I}} = -\,\frac{1}{2}\,(a+1)$$

with $\gamma^* = \gamma\left(1 + \frac{\mathcal{I}}{mc^2}\right)$ from which it follows $\left(\frac{\gamma}{\gamma^*}\right)^{1/2} = \left(1 + \frac{\mathcal{I}}{mc^2}\right)^{-1/2} = 1 - \frac{1}{2}\frac{\mathcal{I}}{\gamma k_B T}$ plus higher order terms in $\frac{1}{\gamma}$. So the zero order term is 1 and, in correspondence with it we obtain zero before to take the limit; the first order term is $-\frac{1}{2}\frac{\mathcal{I}}{\gamma k_B T}$ and, in correspondence with it we obtain our result after taking the limit.

In this way we obtain another non relativistic equation for the unknowns $\psi^{(1)}$, $\psi^{(1)}_\mu$ U^μ besides $(21)_1$. It is interesting that the determinant of the coefficients results of the type $(a_1 + a_2)\,[a_1 f(a) + a_2 g(a)]$ with $f(a)$ and $g(a)$ polinomials in a of degree 2 and 1 respectively. But now we refrain to speak about other details.

2.3 Comparison with different models of BGK.

Various comparisons have already been made above with different models of BGK, such a with [11], [12], [10]. We add here a last comparison with [14]. We begin with the following observation: Here the authors use the variable U^α_L which is the four-velocity in the Landau and Lifshitz description. By comparing their eq. $(3)_1$ with $(16)_1$ of [4], we deduce the link $N^\alpha = N\,U^\alpha$. Moving on to their eq. $(4)_1$, we can get $U^\alpha_L = U^\alpha + \frac{q^\alpha}{nh}$, where h is the entalpy which they define 2 lines after eq. (4) as $h = e + p/n$. After that, in their eq. (7) the authors assert that $\left(T^{\alpha\beta} - T^{\alpha\beta}_E\right) U_{L\beta} = 0$. Instead of this, if we use for $T^{\alpha\beta}$ and $T^{\alpha\beta}_E$ their expressions reported in [4], we find

$$\left(T^{\alpha\beta} - T^{\alpha\beta}_E\right) U_{L\beta} = q^\alpha + \frac{1}{nh}\left(-\pi q^\alpha + \frac{1}{c^2} U^\alpha q^\beta q_\beta + t^{<\alpha\beta>_3} q_\beta\right) ,$$

which isn't zero. A possible explanation is that they define the equilibrium differently from [4]. It is obvious that they define equilibrium as the state described by the independent variables n, e, $U_{L\beta}$ constrained by $U_{L\beta} U^\beta_L = c^2$. After that, at this equilibrium we have

$$V^\alpha_E = mnU^\alpha_L\,, \quad T^{\alpha\beta}_E = -p\Delta^{\alpha\beta} + \frac{en}{c^2}\, U^\alpha_L U^\beta_L\,,$$
$$\text{with} \quad \Delta^{\alpha\beta} = g^{\alpha\beta} - \frac{1}{c^2}\, U^\alpha_L U^\beta_L\,.$$

Subsequently, the non equilibrium variables can be defined as π, q^α, $P^{<\alpha\beta>_3}$ constrained by $0 = q^\alpha U_{L\alpha} = P^{<\alpha\beta>_3} U_{L\alpha} = 0 = P^{<\alpha\beta>_3}\Delta_{\alpha\beta}$. From these definitions it follows that at first order with repect to equilibrium we have

$$V^\alpha - V^\alpha_E = -\frac{m}{h} q^\alpha\,, \quad T^{\alpha\beta} - T^{\alpha\beta}_E = P^{<\alpha\beta>_3} - \pi\Delta^{\alpha\beta}\,,$$

from which it follows

$$U^\alpha_E = U^\alpha_{LE}\,, \quad U^\alpha - U^\alpha_E = -\frac{1}{nh} q^\alpha\,,\, U^\alpha U_\alpha = c^2 + q^\alpha q_\alpha\,.$$

From these results it is evident that eqs. from (3) to (11) of [14] are consistent. But how strange is, from the view point of [4], to have a V^α of non-equilibrium! And a U^α of non-equilibrium! And a $U^\alpha U_\alpha$ which is not constant and can even be negative for large values of q^α!
From these comparison we can deduce that

- The article [14] doesn't study the hyperbolicity requirement and therefore the results of the present work on this subject do not find a counterpart in [14].
- There is a counterpart only for the production terms. But for these they use a definition of equilibrium that contrasts completely with that of [4]; therefore, to find such usable terms for the equations of [4] we need to use an alternative method that is valid in this context. This has already been achieved in [10] following the variant of the Anderson and Witting model. Here the same result is obtained following the variant of the Marle model, thus broadening the understanding of the subject.

3 On the convexity of entropy and some of its consequences

By using the definition (2) of the 4-potential, we have:

$$\frac{\partial\, h'^\alpha}{\partial\, \lambda_A} = c \int_{\Re^3} \int_0^{+\infty} e^{-1-\frac{\chi}{k_B}} \frac{\partial\, \chi}{\partial\, \lambda_A} \, p^\alpha \phi(\mathcal{I}) \, d\overrightarrow{P} \, d\mathcal{I} \, .$$

Since $\frac{\partial \chi}{\partial \lambda_A}$ does not' t depend on λ_B, it follows

$$\frac{\partial^2\, h'^\alpha}{\partial\, \lambda_B \, \partial\, \lambda_A} = - \frac{c}{k_B} \int_{\Re^3} \int_0^{+\infty} e^{-1-\frac{\chi}{k_B}} \frac{\partial\, \chi}{\partial\, \lambda_B} \frac{\partial\, \chi}{\partial\, \lambda_A} \, p^\alpha \phi(\mathcal{I}) \, d\overrightarrow{P} \, d\mathcal{I} \, .$$

By using these results, the quadratic form (9) becomes:

$$K = - \frac{c}{k_B} \, \xi_\alpha \int_{\Re^3} \int_0^{+\infty} e^{-1-\frac{\chi}{k_B}} \, (\delta\, \chi\,)^2 \, p^\alpha \phi(\mathcal{I}) \, d\overrightarrow{P} \, d\mathcal{I} \, \leq \, 0 \, , \qquad (22)$$

$\forall$ time-like unitary 4-vector ξ_α and this is true even if ξ_α is constant or depending also on x^α. In fact, it isn't possible that $\xi_\alpha p^\alpha = 0$ because they are both time-like 4-vectors (otherwise, in the reference frame where ξ_α has the components $\xi_\alpha \equiv (1\,,\,0\,,\,0\,,\,0)$ we would have $p^0 = 0$ and, consequently, $p^\beta p_\beta < 0$ against the fact that $p^\beta p_\beta = m^2\, c^2$). So, for the theorem of existence of zeros for continuos functions, we have that $\xi_\alpha p^\alpha$ has always the same sign. If $\xi_\alpha p^\alpha > 0$, then ξ_α and p^α don't only belong to the same light cone, but they are also both directed towards the future. (Usually in literature, when calculating integrals like those in (2), (6), (8) a change of variables is used with $p^0 = mc \cosh s$, so that $p^0 > 0$ implies that p^α has been chosen directed towards the future). Consequently $\xi_\alpha p^\alpha > 0$ means that we have used the same choice for ξ_α

and p^α.
In any case, even if we choose $\xi_\alpha p^\alpha < 0$, it suffices to change sign in the definition (2) and we go back to the previous case. It is true that in this way also the balance equations change sign; but it suffices to multiply them by -1 to go back to the previous case.
The condition (22) ensures that the matrix $\frac{\partial^2 h'^\alpha}{\partial \lambda_B \partial \lambda_A}$ is negative semi-definite and we prove now that it is negative definite. In fact, we have $K=0$ if and only if $\delta\chi$ is identically zero, i.e.,

$$m\,\delta\lambda + \left(1+\frac{\mathcal{I}}{mc^2}\right)p^\beta\,\delta\lambda_\beta + \frac{1}{m}\left(1+\frac{2\mathcal{I}}{mc^2}\right)p^\beta p^\gamma\,\delta\Sigma_{\beta\gamma} = 0\,, \qquad \forall p^\beta\,,\,\mathcal{I}\,, \tag{23}$$

which are constrained only by $p^\beta p_\beta = m^2 c^2$. Now, it can be easily proved that this condition (23) implies that $\delta\lambda = 0$, $\delta\lambda_\beta = 0$, $\delta\Sigma_{\beta\gamma} = 0$, except for the 15 moments model of monoatomic case where we have the general solution $\delta\lambda = 0$, $\delta\lambda_\beta = 0$, $\delta\Sigma_{\beta\gamma} = g_{\beta\gamma}\,\delta\Sigma$, $\quad\forall\delta\Sigma$. But this case is trivial because for this model the trace of the last field equation is proportional to the first one so that $\Sigma_{\beta\gamma}$ is taken traceless and the present proof is correct also for this case. (See [1], [15] for this 14 moments model of monoatomic case).
Consequently, we have proved that the matrix $\frac{\partial^2 h'^\alpha}{\partial \lambda_B \partial \lambda_A}$ is negative definite, i.e., the convexity of entropy. This result is important because ensures that the differential system is in the symmetric form and the hyperbolicity requirement is automatically satisfied.
We note that in other contexts the field equations are approximated around equilibrium; after this, the hyperbolicity of the resulting equations is studied and it is no longer ensured for any value of the independent variables as occurs here, but only within a domain called the hyperbolicity zone. See, for example, [16] , [17] where equations developed at first order were used. From the present demonstrations it is evident that the loss of hyperbolicity arises as a consequence of the approximations. This is confirmed by [18] which shows how the hyperbolicity zone increases when the equations are developed until to the second order, rather than the first one; moreover, this was explicitly stated in the conference [19].
This problem is avoided in the present article because the exact solution is presented without using any approximation.

3.1 Consequences of the above results

The inequality (22) is ensured also in the non linear case but, for reason of simplicity, in [4] the closure was obtained only near equilibrium. Therefore we are now interested in its implications near this state. For this aim we rewrite K as

$$K = \xi_\alpha \left[\frac{\partial^2 h'^\alpha}{\partial \lambda^2} (\delta \lambda)^2 + 2 \frac{\partial^2 h'^\alpha}{\partial \lambda \partial \lambda_\mu} \delta \lambda \delta \lambda_\mu + 2 \frac{\partial^2 h'^\alpha}{\partial \lambda \partial \Sigma_{\mu\nu}} \delta \lambda \delta \Sigma_{\mu\nu} + \right.$$
$$\left. + \frac{\partial^2 h'^\alpha}{\partial \lambda_\beta \partial \lambda_\mu} \delta \lambda_\beta \delta \lambda_\mu + 2 \frac{\partial^2 h'^\alpha}{\partial \lambda_\beta \partial \Sigma_{\mu\nu}} \delta \lambda_\beta \delta \Sigma_{\mu\nu} + \frac{\partial^2 h'^\alpha}{\partial \Sigma_{\beta\gamma} \partial \Sigma_{\mu\nu}} \delta \Sigma_{\beta\gamma} \delta \Sigma_{\mu\nu} \right] .$$

By calculating the coefficients at equilibrium, it becomes

$$K_E = - \frac{m}{k_B} \xi_\alpha \left[V_E^\alpha (\delta \lambda)^2 + 2 T_E^{\alpha\mu} \delta \lambda \delta \lambda_\mu + 2 A_E^{\alpha\mu\nu} \delta \lambda \delta \Sigma_{\mu\nu} \right.$$
$$\left. + m A_{11}^{\alpha\beta\delta} \delta \lambda_\beta \delta \lambda_\delta + 2m A_{12}^{\alpha\beta\mu\nu} \delta \lambda_\beta \delta \Sigma_{\mu\nu} + m A_{22}^{\alpha\beta\gamma\mu\nu} \delta \Sigma_{\beta\gamma} \delta \Sigma_{\mu\nu} \right] ,$$

where the expressions of the tensors in the right hand side are reported in [4]. For the sake of simplicity, we calculate also the coefficients of the differentials in the reference frame where U^α and ξ^α have the components $U^\alpha \equiv (c\,,\, 0\,,\, 0\,,\, 0)$ and $\xi^\alpha(\xi_0\,,\, \xi_1\,,\, 0\,,\, 0)$ with $\xi_0 = \sqrt{1 + (\xi_1)^2}$; in any case, we can at the end express again all the results in covariant form replacing ξ_0 and ξ_1 with $\xi_0 = \frac{1}{c} \xi^\alpha U_\alpha$ and $(\xi_1)^2 = \xi_\alpha \xi_\beta h^{\alpha\beta}$.
We define also $X_1 = \delta \lambda$, $X_2 = c\, \delta \lambda_0$, $X_3 = c^2\, \delta \Sigma_{00}$, $X_4 = \delta \lambda_1$, $X_5 = c\, \delta \Sigma_{01}$, $X_6 = \delta \Sigma_{11} - \frac{1}{3} \delta \Sigma_{00}$, $X_9 = \delta \Sigma_{22} - \frac{1}{2} \delta \Sigma_{00} + \frac{1}{2} \delta \Sigma_{11}$, $Y_1 = \delta \lambda_2$, $Y_2 = c\, \delta \Sigma_{02}$, $Y_3 = \delta \Sigma_{12}$, $Z_1 = \delta \lambda_3$, $Z_2 = c\, \delta \Sigma_{03}$, $Z_3 = \delta \Sigma_{13}$ from which it follows

$$\delta \Sigma_{11} = X_6 + \frac{1}{3} \delta \Sigma_{00} ,$$
$$\delta \Sigma_{22} = X_9 - \frac{1}{2} X_6 + \frac{1}{3} \delta \Sigma_{00} ,$$
$$\delta \Sigma_{33} = - X_9 - \frac{1}{2} X_6 + \frac{1}{3} \delta \Sigma_{00} ,$$

which take into account that $\Sigma_{\beta\gamma}$ is traceless. In this way our quadratic form becomes

$$- \frac{k_B\, c}{m^2} K_E = \sum_{a,b=1}^{6} \tilde{M}^{ab} X_a X_b + \sum_{a,b=1}^{3} \tilde{N}^{ab} Y_a Y_b + \sum_{a,b=1}^{3} \tilde{N}^{ab} Z_a Z_b +$$
$$+ \frac{4}{15} B_6\, c^2\, \xi_0 \left[(\delta \Sigma_{23})^2 + (X_9)^2 \right] , \tag{24}$$

and, moreover, the matrices $\tilde{M}^{ab}$ and $\tilde{N}^{ab}$, written in compact form, are

$$\tilde{M} = \begin{pmatrix} \xi_0\, A & \xi_1\, B \\ \xi_1\, B^T & C \end{pmatrix} ,$$

$$\tilde{N} = \begin{pmatrix} \frac{1}{3} B_4\, c^2\, \xi_0 & \frac{2}{3} B_2\, c^2\, \xi_0 & \frac{2}{15} B_1\, c\, \xi_1 \\ \frac{2}{3} B_2\, c^2\, \xi_0 & \frac{4}{3} B_7\, c^2\, \xi_0 & \frac{4}{15} B_6\, c\, \xi_1 \\ \frac{2}{15} B_1\, c\, \xi_1 & \frac{4}{15} B_6\, c\, \xi_1 & \frac{4}{15} B_6\, c^2\, \xi_0 \end{pmatrix} ,$$

where

$$A = \begin{pmatrix} n\,c^2 & \frac{e}{m} & \frac{A_1^0\,c^2 + A_{11}^0}{m} \\ \frac{e}{m} & B_5\,c^2 & B_3\,c^2 \,+ \frac{1}{3}\,B_2 \\ \frac{A_1^0\,c^2 + A_{11}^0}{m} & B_3\,c^2 \,+ \frac{1}{3}\,B_2 & B_8\,c^2 \,+ \frac{2}{3}\,B_7 \,+ \frac{1}{9}\,\frac{B_6}{c^2} \end{pmatrix},$$

$$B = \begin{pmatrix} \frac{c}{m}\,p & 2\,\frac{A_{11}^0\,c}{m} & 0 \\ \frac{1}{3}\,B_4\,c & \frac{2}{3}\,B_2\,c & 0 \\ \frac{1}{9}\,\frac{B_1}{c} + \frac{1}{3}\,B_2\,c & \frac{2}{9}\,\frac{B_6}{c} + \frac{2}{3}\,B_7\,c & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{1}{3}\,B_4\,c^2\,\xi_0 & \frac{2}{3}\,B_2\,c^2\,\xi_0 & \frac{2}{15}\,B_1\,c\,\xi_1 \\ \frac{2}{3}\,B_2\,c^2\,\xi_0 & \frac{4}{3}\,B_7\,c^2\,\xi_0 & \frac{4}{15}\,B_6\,c\,\xi_1 \\ \frac{2}{15}\,B_1\,c\,\xi_1 & \frac{4}{15}\,B_6\,c\,\xi_1 & \frac{1}{5}\,B_6\,c^2\,\xi_0 \end{pmatrix}.$$

We note that $\tilde{N}$ and C differs only for the element in their third line and third coulumn; moreover, the algebraic complement of this element, divided by ξ_0 is the matrix N in [10]. Similarly, the matrix A is the matrix M in [10].

Now we have proved above in (22) that K is negative definite $\forall$ time-like unitary 4-vector ξ_α and for every value of the variables; so this property holds also if these variables assume their values at equilibrium. In other words, we have now that $-K_E$ is positive defined. This was true in its integral form (22), so it must be true also after calculations of the integrals. So we are sure that the above matrixes $\tilde{M}$ and $\tilde{N}$ are positive defined (There is no problem for the remaining parts of $-K_E$ because $B_6 > 0$ is evident from its expression in $(A.8)_1$ of [4]) $\forall$ time-like ξ_α. This isn't a new requirement as it may seem apparently from the description in [10]. It is a proved theorem.

In the case $\xi_1 = 0$ (and then $\xi_0 = 1$), this result says that the matrixes M and N in [10] are positive defined, i.e., we are sure that

$$|\,N\,| > 0 \quad , \quad |\,M\,| > 0 \quad , \quad M_{33} = \begin{vmatrix} n\,c^2 & \frac{e}{m} \\ \frac{e}{m} & B_5\,c^2 \end{vmatrix} > 0\,, \tag{25}$$

since $n\,c^2 > 0$, $\frac{1}{3}\,B_4\,c^2 > 0$. Obviously, here $|\,N\,|$ and $|\,M\,|$ are the determinants of N and M respectively, while M_{33} is the algebraic complement of the element in line 3, coulumn 3 of the matrix M.

It is interesting to see that the last one of these conditions is equivalent to $\left(\frac{\partial e}{\partial T}\right)_n > 0$, as it can be seen from (42), (36) and $(A7)_2$ of [4].

But we have now more than this result because we have that the matrixes $\tilde{M}$ and $\tilde{N}$ are positive defined also for every value of $\xi_1 \neq 0$.

- Let us begin with $\tilde{M}$.

For a well know theorem on positive defined matrixes, we have that $H_p > 0$ for $p = 1, \cdots, 6$ and where H_p denotes the subdeterminant obtained from $\tilde{M}$ by eliminating its last $6-p$ lines and its last $6-p$ coulumns. Now we have already obtained that $H_p > 0$ for $p = 1, \cdots, 3$. So there remains to exploit the results $H_p > 0$ for $p = 4, \cdots, 6 \ \forall \, \xi_1 \neq 0$. For example, we have that H_4 is equal to

$$\begin{vmatrix} n\, c^2\, \xi_0 & \frac{e}{m}\, \xi_0 & \frac{A_1^0\, c^2 + A_{11}^0}{m}\, \xi_0 & \frac{c}{m}\, p\, \xi_1 \\ \frac{e}{m}\, \xi_0 & B_5\, c^2\, \xi_0 & B_3\, c^2 \, + \frac{1}{3}\, B_2\, \xi_0 & \frac{1}{3}\, B_4\, c\, \xi_1 \\ \frac{A_1^0 c^2 + A_{11}^0}{m} \xi_0 & \left(B_3 c^2 + \frac{1}{3} B_2\right) \xi_0 & \left(B_8 c^2 + \frac{2}{3} B_7 + \frac{1}{9} \frac{B_6}{c^2}\right) \xi_0 & \left(\frac{1}{9} \frac{B_1}{c} + \frac{1}{3} B_2 c\right) \xi_1 \\ \frac{c}{m}\, p\, \xi_1 & \frac{1}{3} B_4 c \xi_1 & \left(\frac{1}{9} \frac{B_1}{c} + \frac{1}{3} B_2 c\right) \xi_1 & \frac{1}{3} B_4 c^2 \xi_0 \end{vmatrix}$$

that is $H_4 = (\xi_0)^2 \left[a_0 (\xi_1)^2 + a_1 (\xi_0)^2\right]$ with

$$a_1 = \frac{1}{3} B_4 c^2 \, |A| \quad ,$$

$$a_0 = \begin{vmatrix} n\, c^2 & \frac{e}{m} & \frac{A_1^0 c^2 + A_{11}^0}{m} & \frac{c}{m}\, p \\ \frac{e}{m} & B_5 c^2 & B_3 c^2 + \frac{1}{3} B_2 & \frac{1}{3} B_4 c \\ \frac{A_1^0 c^2 + A_{11}^0}{m} & B_3 c^2 + \frac{1}{3} B_2 & B_8 c^2 + \frac{2}{3} B_7 + \frac{1}{9} \frac{B_6}{c^2} & \frac{1}{9} \frac{B_1}{c} + \frac{1}{3} B_2 c \\ \frac{c}{m}\, p & \frac{1}{3} B_4 c & \frac{1}{9} \frac{B_1}{c} + \frac{1}{3} B_2 c & 0 \end{vmatrix} .$$

We appreciate that these values of a_0 and a_1 are in covariant form, even if we used for the sake of simplicity a particular reference frame. Now we can use the theorem proved in Appendix B and conclude that the property $H_4 > 0 \ \forall \, \xi_1$ is equivalent to

$$a_1 > 0 \quad , \quad a_1 + a_0 \geq 0 \, . \tag{26}$$

The first one of these properties was already found in [10], while the second one is new.
Similarly, let us consider now the implications of the property $H_5 > 0$ $\forall \, \xi_1$. We have that $H_5 = \xi_0 \, \left[b_0 (\xi_1)^4 + b_1 (\xi_1)^2 (\xi_0)^2 + b_2 (\xi_0)^4\right]$ with

$b_0 =$

$$\begin{vmatrix} n\,c^2 & \frac{e}{m} & \frac{A_1^0 c^2 + A_{11}^0}{m} & \frac{c}{m}\,p & 2\frac{A_{11}^0 c}{m} \\ \frac{e}{m} & B_5 c^2 & B_3 c^2 + \frac{1}{3}B_2 & \frac{1}{3}B_4 c & \frac{2}{3}B_2 c \\ \frac{A_1^0 c^2 + A_{11}^0}{m} & B_3 c^2 + \frac{1}{3}B_2 & B_8 c^2 + \frac{2}{3}B_7 + \frac{1}{9}\frac{B_6}{c^2} & \frac{1}{9}\frac{B_1}{c} + \frac{1}{3}B_2 c & \frac{2}{5}\frac{B_6}{c} + \frac{2}{3}B_7 \\ \frac{c}{m}\,p & \frac{1}{3}B_4 c & \frac{1}{9}\frac{B_1}{c} + \frac{1}{3}B_2 c & 0 & 0 \\ 2\frac{A_{11}^0\,c}{m} & \frac{2}{3}B_2 c & \frac{2}{5}\,\frac{B_6}{c} + \frac{2}{3}\,B_7\,c & 0 & 0 \end{vmatrix}\,.$$

$b_1 =$

$$\begin{vmatrix} n\,c^2 & \frac{e}{m} & \frac{A_1^0 c^2 + A_{11}^0}{m} & \frac{c}{m}\,p & 2\frac{A_{11}^0 c}{m} \\ \frac{e}{m} & B_5 c^2 & B_3 c^2 + \frac{1}{3}B_2 & \frac{1}{3}B_4 c & \frac{2}{3}B_2 c \\ \frac{A_1^0 c^2 + A_{11}^0}{m} & B_3 c^2 + \frac{1}{3}B_2 & B_8 c^2 + \frac{2}{3}B_7 + \frac{1}{9}\frac{B_6}{c^2} & \frac{1}{9}\frac{B_1}{c} + \frac{1}{3}B_2 c & \frac{2}{5}\frac{B_6}{c} + \frac{2}{3}B_7 \\ 0 & 0 & 0 & \frac{1}{3}B_4 c^2 & \frac{2}{3}B_2 c^2 \\ 2\frac{A_{11}^0\,c}{m} & \frac{2}{3}B_2 c & \frac{2}{5}\,\frac{B_6}{c} + \frac{2}{3}\,B_7\,c & 0 & 0 \end{vmatrix} +$$

$$+\begin{vmatrix} n\,c^2 & \frac{e}{m} & \frac{A_1^0 c^2 + A_{11}^0}{m} & \frac{c}{m}\,p & 2\frac{A_{11}^0 c}{m} \\ \frac{e}{m} & B_5 c^2 & B_3 c^2 + \frac{1}{3}B_2 & \frac{1}{3}B_4 c & \frac{2}{3}B_2 c \\ \frac{A_1^0 c^2 + A_{11}^0}{m} & B_3 c^2 + \frac{1}{3}B_2 & B_8 c^2 + \frac{2}{3}B_7 + \frac{1}{9}\frac{B_6}{c^2} & \frac{1}{9}\frac{B_1}{c} + \frac{1}{3}B_2 c & \frac{2}{5}\frac{B_6}{c} + \frac{2}{3}B_7 \\ \frac{c}{m}\,p & \frac{1}{3}B_4 & \frac{2}{9}\frac{B_1}{c} + \frac{1}{3}B_2 c & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3}B_2 c^2 & \frac{4}{3}B_7 c^2 \end{vmatrix},$$

$$b_2 = |A| \cdot \begin{vmatrix} \frac{1}{3}\,B_4\,c^2 & \frac{2}{3}\,B_2\,c^2 \\ \frac{2}{3}\,B_2\,c^2 & \frac{4}{3}\,B_7\,c^2 \end{vmatrix}\,.$$

Also these values of b_0, b_1 and b_2 are in covariant form. Now we can use the theorem proved in Appendix B and conclude that the property $H_5 > 0\ \forall \xi_1$ is equivalent to

$$\begin{cases} b_2 > 0 \quad , \quad b_0 + b_1 + b_2 \geq 0 \quad , \quad b_1 + 2\,b_2 \geq 0 & \text{if} \quad (b_1)^2 - 4b_0 b_2 \geq 0\,, \\ \qquad\qquad b_2 > 0 \quad , & \text{if} \quad (b_1)^2 - 4b_0 b_2 < 0\,. \end{cases} \tag{27}$$

The property $b_2 > 0$, thanks to the already found $|A| > 0$ gives the result $|N| > 0$ of [10], while the others are new.
Finally, for $\tilde{M}$, we have now to see the implications of the property $H_6 = |\tilde{M}| > 0 \; \forall \, \xi_1$. We have that

$$H_6 = |\tilde{M}| = (\xi_0)^2 \left[c_0 (\xi_1)^4 + c_1 (\xi_1)^2 (\xi_0)^2 + c_2 (\xi_0)^4 \right] , \quad \text{with} \tag{28}$$

$$c_0 = \begin{vmatrix} A & \vec{b}_1 & \vec{b}_2 \\ \vec{b}_2^{\,T} & 0 & 0 \\ \vec{0}^{\,T} & c_{13} & c_{23} \end{vmatrix} c_{13} -$$

$$\begin{vmatrix} A & \vec{b}_1 & \vec{b}_2 \\ \vec{b}_1^{\,T} & 0 & 0 \\ \vec{0}^{\,T} & c_{13} & c_{23} \end{vmatrix} c_{23} + \begin{vmatrix} A & \vec{b}_1 & \vec{b}_2 \\ \vec{b}_1^{\,T} & 0 & 0 \\ \vec{b}_2^{\,T} & 0 & 0 \end{vmatrix} c_{33} ,$$

$$\vec{b}_1 = \begin{pmatrix} \frac{c}{m} \, p \\ \frac{1}{3} \, B_4 \, c \\ \frac{1}{9} \, \frac{B_1}{c} + \frac{1}{3} \, B_2 \, c \end{pmatrix} , \vec{b}_2 = \begin{pmatrix} 2 \, \frac{A^0_{11} \, c}{m} \\ \frac{2}{3} \, B_2 \, c \\ \frac{2}{5} \, \frac{B_6}{c} + \frac{2}{3} \, B_7 \, c \end{pmatrix} ,$$

$\vec{b}_1^{\,T}$ and $\vec{b}_2^{\,T}$ are the transposite of $\vec{b}_1$ and $\vec{b}_2$, respectively, and c_{ij} is the element in line i, coulumn j of the matrix C divided by ξ_0. For the other values we have

$$c_1 = |A| \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & 0 \end{vmatrix} +$$

$$+ \begin{vmatrix} A & \vec{b}_1 & \vec{b}_2 \\ \vec{b}_1^{\,T} & 0 & 0 \\ \vec{0}^{\,T} & c_{12} & c_{22} \end{vmatrix} c_{33} + \begin{vmatrix} A & \vec{b}_1 & \vec{b}_2 \\ \vec{0}^{\,T} & c_{11} & c_{12} \\ \vec{b}_2^{\,T} & 0 & 0 \end{vmatrix} c_{33} ,$$

$$c_2 = |A| \begin{vmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{vmatrix} .$$

Also these values of c_0, c_1 and c_2 are in covariant form. Now we can use the theorem proved in Appendix B and conclude that the property $H_6 > 0 \; \forall \xi_1$ is equivalent to

$$\begin{cases} c_2 > 0 \quad , \quad c_0 + c_1 + c_2 \geq 0 \quad , \quad c_1 + 2\, c_2 \geq 0 & \text{if} \quad (c_1)^2 - 4 c_0 c_2 \geq 0 \, , \\ \qquad\qquad c_2 > 0 \quad , & \text{if} \quad (c_1)^2 - 4 c_0 c_2 < 0 \, . \end{cases} \tag{29}$$

The property $c_2 > 0$, thanks to the already found $|A| > 0$ gives the result $|N| > 0$ of [10], while the others are new.

- Let us consider now $\tilde{N}$.

Thanks to the result (25), we have only to exploit the property $|\,\tilde{N}\,| > 0$. But

$$|\,\tilde{N}\,| = |\,C| + \frac{1}{15} B_6\, c^2 \, (\xi_0)^2 \begin{vmatrix} \frac{1}{3} B_4\, c^2 & \frac{2}{3} B_2\, c^2 \\ \frac{2}{3} B_2\, c^2 & \frac{4}{3} B_7\, c^2 \end{vmatrix} > 0 \, , \tag{30}$$

because both determinants here appearing are principal minors of the matrix $\tilde{M}$ which is positive definite. So there is nothing else to exploit.

4 Conclusions

In this work, we have given a strong support to the field equations of [4], proving their hyperbolicity for any value of the independent variables, at least until no approximations around equilibrium is introduced. We found an expression for the production terms that improves that of [10], as the present ones are valid up to any order with respect to equilibrium, while the previous one was valid only for the first order. For this latter ones the previous expression is confirmed. We finally expanded this possible expressions, finding for them a counterpart to the Marle model, while those of [10] had a counterpart only to the Anderson-Witting model.

A Some useful integrals

In the principal text of this article we need some integrals. they are the following ones:

$$V = m\,c \int_{\Re^3} \int_0^{+\infty} f\,\phi(\mathcal{I})\,d\overrightarrow{P}\,d\mathcal{I}\,,$$
$$T^\theta = \int_{\Re^3} \int_0^{+\infty} f\,\left(1 + \frac{\mathcal{I}}{m\,c^2}\right) p^\theta\,\phi(\mathcal{I})\,d\overrightarrow{P}\,d\mathcal{I}\,,$$
$$A_E^{\beta\theta} = \int_{\Re^3} \int_0^{+\infty} f_E\,\left(1 + \frac{\mathcal{I}}{m\,c^2}\right)^2 p^\beta p^\theta\,\phi(\mathcal{I})\,d\overrightarrow{P}\,d\mathcal{I}\,,$$
$$A_{22}^{\mu\nu\beta\gamma} = \frac{c}{m^3} \int_{\Re^3} \int_0^{+\infty} f_E\,\left(1 + \frac{2\,\mathcal{I}}{m\,c^2}\right)^2 p^\mu p^\nu p^\beta p^\gamma\,\phi(\mathcal{I})\,d\overrightarrow{P}\,d\mathcal{I}\,.$$

We can integrate these expressions by using the Representation Theorems and with the same methodology used in [4]. So we obtain, for example:

$$V_E = 4\,\pi\,m^3\,c^3\,e^{-1-\frac{m}{k_B}\,\lambda_E} \int_0^{+\infty} J_{2,0}(\gamma^*)\,\phi(\mathcal{I})\,d\mathcal{I}\,,$$

where $\gamma^* = \gamma\,\left(1 + \frac{\mathcal{I}}{m\,c^2}\right)$. But from eq. (26) of [4] we have

$$n = 4\,\pi\,m^3\,c^3\,e^{-1-\frac{m}{k_B}\,\lambda_E} \int_0^{+\infty} J_{2,1}(\gamma^*)\,\phi(\mathcal{I})\,d\mathcal{I}\,,$$

so that the above expression becomes

$$V_E = n\,\frac{\int_0^{+\infty} J_{2,0}(\gamma^*)\,\phi(\mathcal{I})\,d\mathcal{I}}{\int_0^{+\infty} J_{2,1}(\gamma^*)\,\phi(\mathcal{I})\,d\mathcal{I}}\,. \tag{31}$$

By proceeding in the same way for the other quantities, we obtain

$$V^{(1)} = -\frac{m}{k_B}\,V_E\,(\lambda - \lambda_E) - \frac{m\,c}{k_B}\,T_E^\theta\,(\lambda_\theta - \lambda_{E\theta}) + R^{\beta\gamma}\,\Sigma_{\beta\gamma}\,,$$
$$T_E^\theta = T_0\,U^\theta\,,\; R_{\beta\gamma} = -\frac{c}{k_B} \int_{\Re^3} \int_0^{+\infty} f_E\,\left(1 + \frac{2\mathcal{I}}{m\,c^2}\right) p^\beta p^\gamma\,\phi(\mathcal{I})\,d\overrightarrow{P}\,d\mathcal{I}\,,$$
$$T_0 = \frac{n}{c}\,\frac{\int_0^{+\infty} J_{2,1}(\gamma^*)\,\left(1 + \frac{\mathcal{I}}{m\,c^2}\right)\,\phi(\mathcal{I})\,d\mathcal{I}}{\int_0^{+\infty} J_{2,1}(\gamma^*)\,\phi(\mathcal{I})\,d\mathcal{I}}\,,\; R^{\beta\gamma} = R_0\frac{U^\beta U^\gamma}{c^2} + R_1\,h^{\beta\gamma}\,,$$
$$R_0 = -\frac{m\,n\,c^2}{k_B}\,\frac{\int_0^{+\infty} J_{2,2}(\gamma^*)\,\left(1 + \frac{2\mathcal{I}}{m\,c^2}\right)\,\phi(\mathcal{I})\,d\mathcal{I}}{\int_0^{+\infty} J_{2,1}(\gamma^*)\,\phi(\mathcal{I})\,d\mathcal{I}}\,,$$
$$R_1 = -\frac{m\,n\,c^2}{3\,k_B}\,\frac{\int_0^{+\infty} J_{4,0}(\gamma^*)\,\left(1 + \frac{2\mathcal{I}}{m\,c^2}\right)\,\phi(\mathcal{I})\,d\mathcal{I}}{\int_0^{+\infty} J_{2,1}(\gamma^*)\,\phi(\mathcal{I})\,d\mathcal{I}}\,,$$
$$T^{(1)\theta} = -\frac{m}{k_B}\,T_E^\theta\,(\lambda - \lambda_E) - \frac{1}{k_B}\,A_E^{\theta\beta}\,(\lambda_\beta - \lambda_{E\beta}) + T^{\theta\beta\gamma}\,\Sigma_{\beta\gamma}\,,$$

$$
\begin{aligned}
&T^{\beta\gamma\theta} = \frac{-1}{m\,k_B} \int_{\Re^3} \int_0^{+\infty} f_E \left(1 + \frac{\mathcal{I}}{m\,c^2}\right) \left(1 + \frac{2\mathcal{I}}{m\,c^2}\right) p^\beta p^\gamma p^\theta\, \phi(\mathcal{I})\, d\overrightarrow{P}\, d\mathcal{I}\,,\\
&A_E^{\beta\gamma} = T_3 \frac{U^\beta U^\gamma}{c^2} + T_4\, h^{\beta\gamma} \quad , \quad T^{\beta\gamma\theta} = T_1\, U^\beta U^\gamma U^\theta + 3\, T_2\, h^{(\beta\gamma} U^{\theta)}\,,\\
&T_3 = m\,n\,c\, \frac{\int_0^{+\infty} J_{2,2}(\gamma^*) \left(1 + \frac{\mathcal{I}}{m\,c^2}\right)^2 \phi(\mathcal{I})\, d\mathcal{I}}{\int_0^{+\infty} J_{2,1}(\gamma^*)\, \phi(\mathcal{I})\, d\mathcal{I}}\,,\\
&T_4 = \frac{1}{3}\, m\,n\,c\, \frac{\int_0^{+\infty} J_{4,0}(\gamma^*) \left(1 + \frac{\mathcal{I}}{m\,c^2}\right)^2 \phi(\mathcal{I})\, d\mathcal{I}}{\int_0^{+\infty} J_{2,1}(\gamma^*)\, \phi(\mathcal{I})\, d\mathcal{I}}\,,\\
&T_1 = -\,\frac{m\,n}{k_B c}\, \frac{\int_0^{+\infty} J_{2,3}(\gamma^*) \left(1 + \frac{\mathcal{I}}{m\,c^2}\right) \left(1 + \frac{2\mathcal{I}}{m\,c^2}\right) \phi(\mathcal{I})\, d\mathcal{I}}{\int_0^{+\infty} J_{2,1}(\gamma^*)\, \phi(\mathcal{I})\, d\mathcal{I}}\,,\\
&T_2 = -\,\frac{m\,n\,c}{3\,k_B}\, \frac{\int_0^{+\infty} J_{4,1}(\gamma^*) \left(1 + \frac{\mathcal{I}}{m\,c^2}\right) \left(1 + \frac{2\mathcal{I}}{m\,c^2}\right) \phi(\mathcal{I})\, d\mathcal{I}}{\int_0^{+\infty} J_{2,1}(\gamma^*)\, \phi(\mathcal{I})\, d\mathcal{I}}\,,
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
&A_{22}^{\mu\nu\beta\gamma} = \frac{1}{5}\, C_1\, h^{(\mu\nu} h^{\beta\gamma)} + 2\, C_2\, h^{(\mu\nu} U^\beta U^{\gamma)} + C_3\, U^\mu U^\nu U^\beta U^\gamma\,,\\
&C_1 = n\, c^4\, \frac{\int_0^{+\infty} J_{6,0}(\gamma^*) \left(1 + \frac{2\mathcal{I}}{m\,c^2}\right)^2 \phi(\mathcal{I})\, d\mathcal{I}}{\int_0^{+\infty} J_{2,1}(\gamma^*)\, \phi(\mathcal{I})\, d\mathcal{I}}\,,\\
&C_2 = n\, c^2\, \frac{\int_0^{+\infty} J_{4,2}(\gamma^*) \left(1 + \frac{2\mathcal{I}}{m\,c^2}\right)^2 \phi(\mathcal{I})\, d\mathcal{I}}{\int_0^{+\infty} J_{2,1}(\gamma^*)\, \phi(\mathcal{I})\, d\mathcal{I}}\,,\\
&C_3 = n\, \frac{\int_0^{+\infty} J_{2,4}(\gamma^*) \left(1 + \frac{2\mathcal{I}}{m\,c^2}\right)^2 \phi(\mathcal{I})\, d\mathcal{I}}{\int_0^{+\infty} J_{2,1}(\gamma^*)\, \phi(\mathcal{I})\, d\mathcal{I}}\,.
\end{aligned}
$$

In the next subsection we will find some useful consequences of this expressions.

A.1 Some useful properties of the above integrals.

Let us define the function

$$
h' = -\,k_B\, c \int_{\Re^3} \int_0^{+\infty} e^{-1-\frac{\chi}{k_B}}\, \phi(\mathcal{I})\, d\overrightarrow{P}\, d\mathcal{I}\,, \tag{33}
$$

with χ given by $(3)_2$, and the quadratic form in the variables $\delta\,\lambda_A$:

$$
K = \frac{\partial^2 h'}{\partial\,\lambda_B\, \partial\,\lambda_A}\, \delta\,\lambda_A\, \delta\,\lambda_B\,.
$$

Altough they have no physical meaning, they will be useful to find properties which will be used below. Now, introducing the multindex notation λ_A where $A = 0, 1, 2$ indicates the number of index: $\lambda_0 = \lambda, \lambda_1 = \lambda_\beta, \lambda_2 = \Sigma_{\beta\gamma}$, we have:

$$
\frac{\partial\, h'}{\partial\,\lambda_A} = c \int_{\Re^3} \int_0^{+\infty} e^{-1-\frac{\chi}{k_B}}\, \frac{\partial\,\chi}{\partial\,\lambda_A}\, \phi(\mathcal{I})\, d\overrightarrow{P}\, d\mathcal{I}\,.
$$

Since $\frac{\partial \chi}{\partial \lambda_A}$ does not' t depend on λ_B, it follows

$$\frac{\partial^2 h'}{\partial \lambda_B \partial \lambda_A} = - \frac{c}{k_B} \int_{\Re^3} \int_0^{+\infty} e^{-1-\frac{\chi}{k_B}} \frac{\partial \chi}{\partial \lambda_B} \frac{\partial \chi}{\partial \lambda_A} \phi(\mathcal{I}) \, d\overrightarrow{P} \, d\mathcal{I} \, .$$

and we have

$$K = - \frac{c}{k_B} \int_{\Re^3} \int_0^{+\infty} e^{-1-\frac{\chi}{k_B}} (\delta \chi)^2 \phi(\mathcal{I}) \, d\overrightarrow{P} \, d\mathcal{I} < 0 \, . \qquad (34)$$

We want now to see its consequences near the equilibrium state. For this aim we rewrite K as

$$K = \frac{\partial^2 h'}{\partial \lambda^2} (\delta \lambda)^2 + 2 \frac{\partial^2 h'}{\partial \lambda \partial \lambda_\mu} \delta \lambda \delta \lambda_\mu + 2 \frac{\partial^2 h'}{\partial \lambda \partial \Sigma_{\mu\nu}} \delta \lambda \delta \Sigma_{\mu\nu} +$$
$$+ \frac{\partial^2 h'}{\partial \lambda_\beta \partial \lambda_\mu} \delta\lambda_\beta \delta\lambda_\mu + 2 \frac{\partial^2 h'}{\partial \lambda_\beta \partial \Sigma_{\mu\nu}} \delta\lambda_\beta \delta\Sigma_{\mu\nu} + \frac{\partial^2 h'}{\partial \Sigma_{\beta\gamma} \partial \Sigma_{\mu\nu}} \delta\Sigma_{\beta\gamma} \delta\Sigma_{\mu\nu} \, .$$

By calculating the coefficients of the differentials at equilibrium, it becomes

$$K_E = - \frac{m}{k_B} \left[V_E (\delta \lambda)^2 + 2 c T_E^\mu \delta \lambda \delta \lambda_\mu - 2 \frac{k_B}{m} R^{\mu\nu} \delta \lambda \delta \Sigma_{\mu\nu} + \right.$$
$$\left. + \frac{c}{m} A_E^{\beta\mu} \delta\lambda_\beta \delta\lambda_\mu - 2 \frac{k_B c}{m} T^{\beta\mu\nu} \delta \lambda_\beta \delta \Sigma_{\mu\nu} + A_{22}^{\beta\gamma\mu\nu} \delta \Sigma_{\beta\gamma} \delta \Sigma_{\mu\nu} \right] ,$$

By using (31)-(32), our expression becomes

$$- \frac{k_B}{c} K_E = \sum_{a,b=1}^{3} P^{ab} X_a X_b + \sum_{a,b=1}^{2} Q^{ab} X_{a\mu} X^{b\mu} + \frac{2}{15} m C_1 X_{\mu\nu} X^{\mu\nu},$$

where $X_1 = \delta \lambda$, $X_2 = U^\mu \delta \lambda_\mu$, $X_3 = U^\mu U^\nu \delta \Sigma_{\mu\nu}$, $X_{1\mu} = h_\mu^\nu \delta \lambda_\nu$, $X_{2\mu} = h_\mu^\nu U^\delta \delta \Sigma_{\nu\delta}$, $X_{\mu\nu} = \delta \Sigma_{<\mu\nu>_3}$ and, moreover, the matrices P^{ab} and Q^{ab} are

$$P = \begin{pmatrix} \frac{m}{c} V_E & m T_0 & - \frac{k_B}{c^3} (R_0 + R_1) \\ m T_0 & \frac{T_3}{c^2} & - k_B \left(T_1 + \frac{1}{c^2} T_2 \right) \\ - \frac{k_B}{c^3} (R_0 + R_1) & - k_B \left(T_1 + \frac{1}{c^2} T_2 \right) & \frac{m}{c} \left(\frac{1}{9} \frac{C_1}{c^4} + \frac{2}{3} \frac{C_2}{c^2} + C_3 \right) \end{pmatrix} ,$$

$$Q = \begin{pmatrix} T_4 & -2 k_B T_2 \\ -2 k_B T_2 & \frac{4}{3} \frac{m}{c} C_2 \end{pmatrix} .$$

Since we have $K < 0$, we have also that P and Q are definite positive. But $V_E > 0$, $T_4 > 0$ are immediate consequences of (31) and following equations; so these properties are equivalent to the following ones

$$\tilde{M}_1 = \begin{vmatrix} V_E & c T_0 \\ c T_0 & \frac{T_3}{m c} \end{vmatrix} > 0 \, , \qquad (35)$$

$$\tilde{M}_3 = \begin{vmatrix} V_E & c\,T_0 & \frac{k_B}{m\,c^2}\,(R_0+R_1) \\ c\,T_0 & \frac{T_3}{m\,c} & \frac{k_B\,c}{m}\,\left(T_1+\frac{1}{c^2}\,T_2\right) \\ \frac{k_B}{m\,c^2}\,(R_0+R_1) & \frac{k_B\,c}{m}\,\left(T_1+\frac{1}{c^2}\,T_2\right) & \frac{1}{9}\,\frac{C_1}{c^4}+\frac{2}{3}\,\frac{C_2}{c^2}+C_3 \end{vmatrix} > 0\,,$$

$$\tilde{M}_2 = \begin{vmatrix} T_4 & 2\,k_B\,T_2 \\ 2\,k_B\,T_2 & \frac{4}{3}\,\frac{m}{c}\,C_2 \end{vmatrix} > 0\,.$$

The first one of these eqs. have been used above, after (20).

B A theorem useful in the above considerations

In the main part of this article we had to exploit the condition

$$\sum_{h=0}^{n} a_h\,(\xi_0)^{2h}\,(\xi_1)^{2n-2h} > 0 \quad ,\forall\,(\xi_1)^2\,, \tag{36}$$

and with $\xi_0=\sqrt{1+(\xi_1)^2}$. We prove now the following
Theorem: "The above condition is equivalent to $a_n>0$ and $x_i \leq 1$ for all real roots of $f(x)=\sum_{h=0}^{n} a_h\,x^h$".
In fact, in the particular case $\xi_1=0$, the above condition becomes $a_n>0$ and there remain to exploit it for $\xi_1\neq 0$. For this case we define

$$x = \frac{(\xi_0)^2}{(\xi_1)^2} = 1+\frac{1}{(\xi_1)^2}\,,$$

and see that x takes all the values belonging to the interval $]\,1\,,+\infty[$ because it is a decreasing function of $(\xi_1)^2$ and its limits for $(\xi_1)^2$ going to zero or to $+\infty$ are $+\infty$ and 1, respectively.
So, for $\xi_1\neq 0$, the condition (36) can be written as

$$f(x)=\sum_{h=0}^{n} a_h\,x^h > 0 \quad ,\forall\,x>1\,. \tag{37}$$

We see now that our condition $x_i\leq 1$ is necesary because, if there is a real root $\bar{x}$ of $f(x)$ with $\bar{x}>1$, then (37) is violated in $x=\bar{x}$.
Our condition $x_i\leq 1$ is also sufficient because in this case in $]\,1\,,+\infty[$ the function $f(x)$ has always the same sign and, moreover, $\lim_{x\to+\infty} f(x)=+\infty>0$ (thanks to $a_n>0$). This fact confirms that $f(x)>0$ in $]\,1\,,+\infty[$.

- The particular case $n=1$.

The condition $a_n > 0$ becomes $a_1 > 0$; moreover, $f(x)$ has only the root $\bar{x} = -\frac{a_0}{a_1}$ so that the second condition $x_i \leq 1$ becomes $a_1 + a_0 \geq 0$. We can conclude that, in the case $n = 1$ the condition (36) becomes

$$a_1 > 0 \quad , \quad a_1 + a_0 \geq 0 \, .$$

- The particular case $n = 2$.

The condition $a_n > 0$ becomes $a_2 > 0$; moreover, if $(a_1)^2 - 4\, a_0\, a_2 < 0$, then $f(x)$ has no real roots and also our second condition is satisfied. If $(a_1)^2 - 4\, a_0\, a_2 \geq 0$, then $f(x)$ has two real roots (or only one double root) $x_1 \leq x_2$. Since $x_1 \leq 1$ and $x_2 \leq 1$, we have $\frac{x_1+x_2}{2} \leq 1$, i.e., $2\, a_2 + a_1 \geq 0$. Moreover, we have $f(1) \geq 0$ (because, if $f(1) < 0$, we have also $f(x) < 0$ in a right neighboourd of 1) and this condition can be expressed as $a_2 + a_1 + a_0 \geq 0$. These conditions are also sufficient because $f(1) \geq 0$ implies that $1 \in]-\infty\,, x_1\,[$ or $1 \in]\, x_2\,, +\infty\,[$ and the first one of these eventualities cannot occur because $\frac{x_1+x_2}{2} \leq 1$. We can conclude that, in the case $n = 2$ the condition (36) becomes

$$\begin{cases} a_2 > 0\,,\; a_0 + a_1 + a_2 \geq 0\,,\; a_1 + 2\, a_2 \geq 0 & \text{if} \quad (a_1)^2 - 4a_0a_2 \geq 0\,, \\ \\ \qquad\qquad a_2 > 0 \quad , & \text{if} \quad (a_1)^2 - 4a_0a_2 < 0\,. \end{cases}$$

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