Cohomogeneity one Kähler and Kähler-Einstein manifolds with one singular orbit II

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Abstract

F. Podestà and A. Spiro [22] introduced a class of G-manifolds M with a cohomogeneity one action of a compact semisimple Lie group G which admit an invariant Kähler structure (g, J) ("standard G-manifolds") and studied invariant Kähler and Kähler-Einstein metrics on M.

In the first part of this paper, we gave a combinatoric description of the standard non compact G-manifolds as the total space M_φ of the homogeneous vector bundle $M = G \times_H V \to S_0 = G/H$ over a flag manifold S_0 and we gave necessary and sufficient conditions for the existence of an invariant Kähler-Einstein metric g on such manifolds M in terms of the existence of an interval in the T-Weyl chamber of the flag manifold $F = G \times_H PV$ which satisfies some linear condition. In this paper, we consider standard cohomogeneity one manifolds of a classical simply connected Lie group $G = SU_n, Sp_n.Spin_n$ and reformulate these necessary and sufficient conditions in terms of easily checked arithmetic properties of the Koszul numbers associated with the flag manifold $S_0 = G/H$. If this conditions is fulfilled, the explicit construction of the Kähler-Einstein metric reduces to the calculation of the inverse function to a given function of one variable.

Contents

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1 Introduction

F. Podestà and A. Spiro [22] defined a class of cohomogeneity one G-manifolds M, called standard, of a semisimple compact Lie group G with an invariant complex structure J. It is defined by the condition that the complex structure J restricted to a regular orbit G/L defines a projectable CR structure $(\mathcal{H}, J^{\mathcal{H}})$, so that the restriction $\mu : G/L \to F = G/K$ of the moment map to G/L is a holomorphic map to a flag manifold $F = G/K = G/N_G(L)$ with a fixed invariant complex structure J^F which does not depend on the regular orbit G/L. They gave a nice description of the invariant Kähler metrics on the complex manifold (M, J) in terms of some interval in the T-Weyl chamber ossociated with the complex structure J^F and in the case when M is compact (hence, it has two singular orbits) got necessary and sufficient conditions for M to admit an invariant Kähler-Einstein metric. Similar results had been obtained by A. Dancer and M.Y. Wang [15], who used a different approach.

In the previous paper we showed that non compact standard cohomogeneity one manifolds are exactly the total spaces M_φ of the homogeneous complex vector bundles $M_\varphi = G \times_H V_\varphi \to S_0 = G/H$ over a flag manifold S_0 with an invariant complex structure J^S defined by a representation $\varphi : H \to GL(V_\varphi)$, $V_\varphi = \mathbb{C}^m$ with $\varphi(H) = U(V_\varphi) \simeq U_m$ and gave a description of the invariant Kähler structures in terms of the painted Dynkin diagrams associated with the flag manifolds $S_0 = G/H$ (the singular orbit) and $F = G/K = \mu(G/L) = G \times_H PV_\varphi$ (the projectivisation of the vector bundle M_φ). We also gave necessary and sufficient conditions (similar to the conditions by Podestà-Spiro) for the existence of invariant Kähler-Einstein metrics in terms of an interval in the T-Weyl chamber associated with the complex structure J^F. If this condition is satisfied, the construction of an associated Kähler-Einstein metric is described explicitly in terms of a function $f(t)$ which is the inverse function to a function $t = t(f)$ given by the integral of an explicit function of one variable.

In the present paper, for a non compact standard cohomogeneity one G-manifold $M_\varphi = G \times_H V_\varphi$ of a classical semisimple Lie group

$SU_n, Sp_n, Spin_n$, we reformulate the necessary and sufficient conditions for the existence of invariant Kähler- Einstein metrics on M_φ in terms of easily checked arithmetic properties of the Koszul numbers associated with the flag manifold $S_0 = G/H$, see Theorems 7 and 8.

We will always assume that the group G is simply connected and it acts on M almost effectively.

Remark 1 When the paper was finished we find the two interesting papers [8] and [14], where invariant Ricci-flat metrics on some holomorphic bundles over flag manifold are constructed.
In [14] the author gets a nice general formula for the unique asymptotically conical Ricci-flat Kähler metric on the canonical bundle K_F of a flag manifolds $F = G^{\mathbb{C}}/P$. In [8], the authors describe more explicitly the invariant Ricci-flat Kähler metric on the canonical bundle K_F of the reducible flag manifold $F = SU_n/S(U_{n_1} \times \cdots \times U_{n_s}) \times SU_q/U_{q-1}$ and show that, in the case when the q-root $K_F^{\frac{1}{q}}$ exists, the same formula gives a Ricci flat Kahler metric on the rank q holomorphic vector bundle $qK_F^{\frac{1}{q}}$.

2 Preliminary and statement of the main results

2.1 Cohomogeneity one Kähler manifolds of standard type

Following [22], we focus our attention to cohomogeneity one Kähler G-manifolds (M, J, ω) of the *standard type*, i.e. manifolds which satisfy the following conditions:
(i) a regular orbit $S = Gx = G/L$ is an *ordinary* manifold. This means that the normalizer $K = N_G(L)$ of the stability subgroup L is the centralizer of a torus in G and $\dim K/L = 1$.
In particular, $F = G/K$ is a flag manifold with induced invariant complex structure J^F.
(ii) the CR structure induced by the complex structure J of M on a regular orbit $S = G/L$ is *projectable*, that is the restriction $\pi : S = G/L \to F = G/K$ to S of the momentum map is a holomorphic map of a CR manifold onto the flag manifold $F = G/K$ equipped with a fixed invariant complex structure J^F (which does not depend on S).
(iii) The G- manifold M has only one singular orbit $S_0 = G/H$, which is a complex submanifold, hence M is not compact.
Condition (ii) depends on the complex structure J on M and shows that the CR structure on a regular orbit G/L is determined by the invariant complex structure J_F on the flag manifold F. In particular, all regular orbits are isomorphic as homogeneous CR manifolds.

Such a cohomogeneity one Kähler G-manifold (M, J, ω) is called , shortly, a *standard cohomogeneity one manifold*.

In [2] we have proved that any standard cohomogeneity one manifold M is the total space of the homogeneous vector bundle (called *admissible bundle*)

$$\pi : M_\varphi = G \times_H V_\varphi \to S_0 = G/H$$

over the singular orbit S_0 defined by a representation $\varphi : H \to GL(V_\varphi)$ with $\varphi(H) \simeq U_m$ ($m = \dim(V_\varphi)$), called *admissible representation*.

Note that the singular orbit S_0 is the zero section of the vector bundle π and all the other orbits have the form $S_t = G(tv_0) = G/L$ and are regular, with $0 \neq v_0 \in V$ a fixed vector and $t > 0$. So the set of G-regular points is $M_{reg} := M_\varphi \setminus S_0 = G/L \times \mathbb{R}^+$, where $L \subset H$ is the stabilizer of the ray $\mathbb{R}^+ v_0$.
It is known (see [22], [2]) that the singular orbit $S_0 = G/H \subset M_\varphi$ is a complex submanifold, hence a flag manifold. The induced complex structure J^S on S_0 together with one of the $\varphi(H) = U(V_\varphi)$-invariant complex structures $\pm J^{V_\varphi}$ on V_φ defines the invariant complex structure on the manifold M_φ.

2.2 Examples of standard cohomogeneity one Kähler and Kähler-Einstein manifolds

Let $(F = G/K, g^F, \omega^F, J^F)$ be a homogeneous Kähler manifold of a semisimple compact Lie group G with integral Kähler form ω (a Hodge manifold). Denote by $\omega \in \Lambda^2(\mathfrak{g})^*$ the form defined by ω^F in the Lie algebra $\mathfrak{g}$. It is exact, i.e. $\omega = d\sigma$, $\sigma \in \mathfrak{g}^*$. We set $Z = B^{-1}\sigma$. Then F is identified with the coadjoint orbit of σ and the adjoint orbit of Z: $F = \mathrm{Ad}^*_G \sigma = \mathrm{Ad}_G Z = G/K$. Denote by $\mathfrak{l}$ the kernel of σ in $\mathfrak{k}$. Then $\mathfrak{k} = \mathfrak{l} + \mathbb{R}Z$ is a B-orthogonal decomposition and $\mathfrak{l}$ generates a closed subgroup L of K such that $\pi : S = G/L \to F = G/K$ is a principal T^1 bundle and $S = G/L$ is an ordinary manifold. The form σ is Ad_K-invariant and it extends to an invariant contact form σ^S on S which is a connection form of a G- invariant connection with curvature ω^F. The complex structure J^F defines a projectable invariant CR structure J^S in the contact distribution $\mathcal{H} = \mathrm{ker}\sigma^{\mathrm{S}}$. It is known (see [6], Theorem 2.3) that the Kähler metric g^F is extended to an invariant Sasaki metric $g^S = \sigma^2 + \frac{1}{2}\pi^* g^F$ on S. The G-invariant extension Z^S of the Ad_K-invariant vector Z is the fundamental vector field of the principal bundle π and it is a Killing vector field for g^S.
The Riemannian cone $(M = C(F) := \mathbb{R}^+ \times S, g = dr^2 + r^2 g^S)$ is a cohomogeneity one Kähler G-manifold with complex structure defined by (see [6], Theorem 2.8)

$$J|_{\mathcal{H}} = J^S,\ J\xi = Z^S.$$

where $\xi = r\partial_r$ is the homothetic vector field. Moreover, if $(F = G/K, g^F, J^F)$ is a Kähler-Einstein homogeneous manifold, then $(S = G/L, g^S, Z^S)$ is a Sasaki-Einstein homogeneous manifold and the cone $(M = C(S), g, J)$ is a Ricci flat Kähler cohomogeneity one manifold (see [13], [25]).

Note that the cone manifold M is a cohomogeneity one G-manifold, but it admits a transitive group of homothetic transformations, generated by G and the 1-parameter homothety group $\exp\mathbb{R}\xi$.
We give a generalisation of this construction of Kähler cones associated to a homogeneous Kähler manifold.

2.3 Description of admissible vector bundles

We recall the description of the admissible vector bundles $M_\varphi \to S_0 = G/H$ of rank m over a given flag manifold $S_0 = G/H$, see [2] for details.

A flag manifold $S_0 = G/H$ is described by a painted Dynkin diagram, which represents a decomposition $\Pi = \Pi_B \cup \Pi_W$ of the system Π of simple roots of G into the subsystem of *white roots* Π_W, which corresponds to the semisimple part $\mathfrak{h}'$ of the stability subalgebra $\mathfrak{h}$, and the subsystem of *black roots* Π_B. Associated with black roots β_i fundamental weights π_i define a basis $B^{-1}\pi_i$ of the center $Z(\mathfrak{h})$, where B is the Killing form, see Appendix for details.
Now we give a short description of the admissible vector bundles $M_\varphi = G \times_H V_\varphi \to S_0 = G/H$ of rank $m = \dim(V_\varphi)$ over a flag manifold .

2.3.1 Case of line bundles

If $m = 1$, then $M_\varphi = G \times_H \mathbb{C}$ is a complex line bundle defined by a character $\chi : Z(H) = T^k \to T^1 = SO(V_\varphi) = SO_2$ which is naturally extended to the homomorphism $\varphi : H = H^s \cdot T^k \to T^1$, which sends the semisimple part H^s of H into identity, and by identification of the tautological SO_2-module $V_\varphi = \mathbb{R}^2$ with $\mathbb{C}$ by choosing one of the two invariant complex structures $\pm J$. In this case, the singular orbit $S_0 = G/H$ is identified with the projectivisation $PM_\varphi = G \times_H P\mathbb{C} = G/K$ of the vector bundle.

Let $\beta_1, \cdots, \beta_p$ be simple black roots (from Π_B) and $\pi_1, \cdots, \pi_p$ be the associated fundamental weights. Then the character $\chi : Z(H) = T^k \to T^1$ is determined by an infinitesimal T-character $\dot\chi \in P_T := \mathrm{span}_{\mathbb{Z}}\Pi_B$ (see [3]) and has the form

$$\chi(\exp(2\pi t)) = \exp(2\pi\dot\chi(t)),\ t \in Z(\mathfrak{h}).$$

Sometimes, we will identify χ with $\dot\chi$.

2.3.2 Case of vector bundles of rank $m > 1$

The description of standard vector bundles of rank $m > 1$ over a flag manifold $S_0 = G/H$ is similar to the case of line bundles.

Let $(S_0 = G/H, J^S)$ be a flag manifold associated with a painted Dynkin diagram $\Pi = \Pi_B \cup \Pi_W$. We fix a connected component of the white subdiagram Π_W which is a string of length $m-1$, i.e. has the type A_{m-1} and corresponds to a $\mathfrak{su}_m$ ideal of $\mathfrak{h}$. We have the following decomposition of $\mathfrak{h}$ into a direct sum of ideals

$$\mathfrak{h} = \mathfrak{su}_m \oplus \mathfrak{n}' \oplus \mathfrak{t}^k$$

where $\mathfrak{t}^k = Z(\mathfrak{h})$ is the center and $\mathfrak{n}'$ is the semisimple ideal complementary to $\mathfrak{su}_m$.
As in the case $m = 1$, an admissible bundle is defined by a character $\chi : Z(H) = T^k \to T^1 = e^{i\mathbb{R}}$ which determines the homomorphism $\varphi : H = SU_m \cdot N' \cdot T^k \to V_\varphi$ where $V_\varphi = \mathbb{C}^m$ is the tautological SU_m-module extended to an H-module by the conditions

$$\varphi(N') = \mathrm{id}, \quad \varphi|_{T^k} = \chi$$

where $e^{ia} \in \chi(T^k)\,, a \in \mathbb{R}$, acts on $V_\varphi = \mathbb{C}^m$ by complex multiplication. Note that we fix one of the two invariant complex structures J^{V_φ} in the H-module V_φ. Together with a complex structure J^S on the base S_0 of the vector bundle $M_\varphi = G \times_H V_\varphi$, this defines a projectable invariant complex structure in the total space $M_\varphi = G \times_H V_\varphi$, hence also an invariant complex structure J^F on the flag manifold $F = G/K = G/L \cdot T^1$ which is the projectivisation of the vector bundle $M_\varphi = G \times_H V_\varphi$.
Note that the opposite complex structure $-J^{V_\varphi}$ defines another projectable complex structure J' on M_φ and another invariant complex structure $(J')^F$ on F.

The following definition describes the data which determine an admissible homogeneous vector bundle $M_\varphi = G \times_H V_\varphi \to S_0 = G/H$ of rank $m > 1$ together with an invariant complex structure J.

Definition 1 *Let $\Pi = \Pi_B \cup \Pi_W$ be a painted Dynkin diagram which defines a flag manifold $(S_0 = G/H, J^S)$. A triple (A_{m-1}, χ, β) , where $A_{m-1} = \{\alpha_1, \cdots, \alpha_{m-1}\}$ is a string, i.e. a connected component of the white subdiagram Π_W of type A_{m-1}, $\chi : Z(H) = T^k \to T^1$ a character and β is one of the end roots of A_{m-1}, (the left $\beta = \alpha_1$ or the right $\beta = \alpha_{m-1}$) is called a data.*

Proposition 2 *[2] Let $(S_0 = G/H, J^S)$ be a flag manifold with reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, associated with a painted Dynkin diagram $\Pi = \Pi_B \cup \Pi_W$. A data (A_{m-1}, χ, β) defines an admissible homogeneous vector bundle $M_\varphi = G \times_H V_\varphi$ with a complex structure J. The restriction of the complex structure J to $M_{reg} = G/L \times \mathbb{R}^+$*

is defined as follows. The B-orthogonal reductive decomposition of a regular orbit G/L can be written as

$$\mathfrak{g} = \mathfrak{l} + (\mathbb{R}Z_F^0 + \mathfrak{p}) = \mathfrak{l} + (\mathbb{R}Z_F^0 + \mathfrak{q} + \mathfrak{m})$$

where $\mathfrak{h} = \mathfrak{k} + \mathfrak{q}$, $, \mathfrak{k} = \mathfrak{l} + \mathbb{R}Z_F^0$ and Z_F^0 is the fundamental vector of the principal T^1 bundle $G/L \to F = G \times_H PV_\varphi = G/K$, normalised by $B(Z_F^0, Z_F^0) = -1$. The complex structure J^F induces the invariant CR structure $(\mathcal{H}, J^{\mathcal{H}})$ in G/L. It is extended to the invariant complex structure J on $M_{reg} = G/L \times \mathbb{R}^+$ by the formula

$$J\partial_t = \frac{1}{a(t)} Z_F^0, \quad JZ_F^0 = -a(t)\partial_t$$

where $a(t)$ is a non-vanishing function.

The B-orthogonal reductive decomposition of the singular orbit $S_0 = G/H$ and a regular orbit G/L can be rewritten as

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} = (\mathfrak{su}_m \oplus \mathfrak{n}' \oplus \mathfrak{t}^k) + \mathfrak{m} = (\mathfrak{su}_m \oplus \mathbb{R}Z^\chi \oplus \mathfrak{n}' \oplus \ker\chi) + \mathfrak{m}. \quad (1)$$

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{p} = (\mathfrak{u}_{m-1} \oplus \mathfrak{n}' \oplus \ker\chi) + (\mathbb{R}Z_F^0 + \mathfrak{q} + \mathfrak{m}) \quad (2)$$

where $Z^\chi = B^{-1}\dot{\chi}$, $\mathfrak{t}^k = Z(\mathfrak{h}) = \mathbb{R}Z^\chi + \ker\dot{\chi}$, $\mathfrak{k} = \mathfrak{l} + \mathbb{R}Z^\chi$. We identify the subalgebra $\mathfrak{u}_m = \mathfrak{su}_m + \mathbb{R}Z^\chi$ with $\varphi(\mathfrak{u}_m) = \mathfrak{u}(V_\varphi)$ and denote by $\mathfrak{u}_{m-1}$ the stabilizer of a fixed vector $e_0 \in V_\varphi = \mathbb{C}^m$; Finally, $\mathfrak{q}$ is the invariant complement to $\mathfrak{u}_{m-1}$ in $\mathfrak{u}_m = \mathfrak{u}_{m-1} + \mathfrak{q}$.

2.4 Invariant Kähler structures on the total space M_φ of a standard vector bundle

Invariant Kähler structures on a standard cohomogeneity one manifold $M_\varphi = G \times_H V_\varphi$ are described by segments (an interval or a ray) in the T-Weyl chamber

$$C(J^F) = \{\beta > 0, \beta_1 > 0, \cdots, \beta_k > 0\} \subset iZ(\mathfrak{k}) = \mathfrak{t}^k + \mathbb{R}Z^0$$

of the flag manifold $F = G/K$ corresponding to the complex structure J^F (see Theorem 7 in the Appendix). Here

$$Z^0 = -iZ_F^0 \quad (3)$$

where Z_F^0 is the fundamental vector.

We may assume that $\beta(Z^0) > 0$ where β is the new black root in the Dynkin diagram of G/K.

Choose a vector $Z_0 \in iZ(\mathfrak{k})$ such that $\beta(Z_0) = 0$, $\beta_i(Z_0) > 0$, $i = 1, \ldots, k$. Geometrically, the vector Z_0 belongs to the face $\beta = 0$ of the Weyl chamber $C(J^F)$ and its projection to $iZ(\mathfrak{h})$ is in the Weyl chamber $C(J^S)$.

Definition 3 *A segment (an interval or a ray) in $C(J^F)$ of the form (Z_0Z_d), $\beta(Z_0) = 0$, which is parallel to the fundamental vector Z^0 together with a parametrization $Z_0 + f(t)Z^0$ such that $\dot{f}(t) > 0, f(0) = 0, Z_d = Z_0 + f(d)Z^0$, is called an admissible segment.*

Theorem 4 *([2], Proposition 17 , see also [22]) Let (M_φ, J) be an admissible vector bundle associated with a data (A_{m-1}, χ, β). Any admissible segment $(Z_0Z_d) \subset C(J^F)$ defines a Kähler metric in the tubular S_0-punctured neighbourhood $M = (0, d) \times G/L \subset M_\varphi \setminus S_0$ of the zero section S_0 of the vector bundle $M_\varphi \to S_0 = G/H$ given by*

$$g_{reg} = dt^2 + (\dot{f}\theta^0)^2 + \pi_F^* g_0 + f(t)\pi_F^* g^0.$$

Here $\pi_F : M_{reg} = G/L \times \mathbb{R}^+ \to F = G/K$ is the natural projection and $g_0 = -\omega_{Z_0} \circ J^F$, $g^0 = -\omega_{Z^0} \circ J^F$, where ω_{Z_0}, ω_{Z^0} are the closed invariant forms on F associated with Z_0, Z^0 (see Proposition 9 in the Appendix below for the correspondence between vectors in $C(J^F)$ and forms on F). Any invariant Kähler metric of standard type can be obtained by this construction.
The Kähler metric g smoothly extends to the zero section S_0 if and only if the function $f(t)$ is extended to a smooth even function on $\mathbb{R}$ which satisfies the following Verdiani conditions [26] :

$$f(0) = \dot{f}(0) = 0, \ \ddot{f}(0) = \kappa,$$

where

$$\kappa = 2\pi/T_0, \quad T_0 = \min\{t > 0 \mid \exp(tZ_0) \in L\} \tag{4}$$

Moreover, the invariant Kähler metric g is geodesically complete on $M_\varphi = G \times_H V_\varphi$ if and only if the function $f(t)$ is defined on $\mathbb{R}^+$ and satisfies the Verdiani conditions.

Finally, we recall the conditions for the Kähler metric associated with an admissible segment (Z_0Z_d) to be a Kähler-Einstein metric.

Theorem 5 *(Theorem 34 in [2]) Let M_φ be a standard cohomogeneity one manifold, i.e. the total space of an admissible bundle $M_\varphi = G \times_H V_\varphi \to S_0$ over the singular orbit $(S_0 = G/H, J^S)$ and $(F = G \times_H PV_\varphi = G/K, J^F)$ be the flag manifold associated with regular orbits. The invariant Kähler metric g in M_φ associated with an admissible segment $(Z_0Z_d) \subseteq C = C(J^F)$ in the T-Weyl chamber $C(J^F)$ is a Kähler-Einstein metric with Einstein constant λ if and only if*
(i) the Koszul vector $Z^{Kos} \in C(J^F)$ (which defines the invariant Kähler-Einstein metric on the flag manifold (F, J^F), see the Appendix), the initial vector Z_0 of the segment and the fundamental vector Z^0 are related by

$$Z^{Kos} = \lambda Z_0 + \kappa m Z^0 \tag{5}$$

where $m = \dim(V_\varphi)$ and κ is defined by (4);
(ii) the function $f(t)$ satisfies the equation

$$\ddot{f}(t) + \frac{1}{2}A(f)\dot{f}^2 + \lambda f = \kappa m \tag{6}$$

with the initial conditions
$\lim_{t\to 0} f(t) = \lim_{t\to 0} \dot{f}(t) = 0$, $\lim_{t\to 0} \ddot{f}(t) = \kappa$,
where $A(f) = \sum_{\alpha\in R^+_{\mathfrak{m}}} \frac{\alpha(Z^0)}{\alpha(Z_0)+f\alpha(Z^0)}$ *and* $R^+_{\mathfrak{m}}$ *is the set of the positive black roots of* G/K*, see Appendix.*
Moreover, the Kähler-Einstein metric can be extended to a complete metric if and only if $\lambda \leq 0$*, and the segment extends to a ray* $Z_0+\mathbb{R}^+Z^0$ *in* $C(J^F)$*.*

The proof, given in [2], is based on the following

Theorem 6 *If the condition (5) of Theorem 5 is fulfilled, then the function* $f(t)$ *parametrizing the segment* (Z_0Z_d) *which gives the Kähler-Einstein metric is the inverse to the function*

$$t(f) = \int_0^f \sqrt{\frac{P(s)}{2\int_0^s(\kappa m - \lambda v)P(v)dv}}ds \tag{7}$$

where P *is the polynomial defined by* $P(x) = \Pi_{\alpha\in R^+_{\mathfrak{m}}}(\alpha(Z_0)+x\ \alpha(Z^0))$*.*

Remark If the necessary and sufficient conditions are fulfilled this theorem reduces the explicit construction of a Kähler-Einstein metric to the construction of the inverse function $f(t)$ to the function $t(f)$.

2.5 The main results

Let $(F = G/K, J^F)$ be the flag manifold with an invariant complex structure associated with a painted Dynkin diagram $\Pi = \Pi^F_B \cup \Pi^F_W$. Denote by $\{\beta_0, \beta_1, \cdots, \beta_p\} = \Pi^F_B$ the simple black roots and by $\pi_0, \pi_1, \cdots, \pi_p$ the associated black fundamental weights. The Koszul form $\sigma_F = B \circ Z^{Kos}$ associated with the Koszul vector Z^{Kos} admits a decomposition

$$B \circ Z^{Kos} = n_0\pi_0 + n_1\pi_1 + \cdots + n_p\pi_p \tag{8}$$

where the natural numbers n_i are called the *Koszul numbers of the complex flag manifold* (F, J^F).

Now we are ready to state our main theorems which give necessary and sufficient conditions in order that the admissible vector bundle $M_\varphi = G \times_H V_\varphi \to S_0 = G/H$ over a flag manifold $(S_0 = G/H, J^S)$ associated with a painted Dynkin diagram $\Pi = \Pi_B \cup \Pi_W$ admits an invariant Kähler-Einstein metric, where G is one of the classical compact Lie groups SU_n, Sp_n, SO_n. Recall that a cohomogeneity one G-manifold M_φ having G/H as singular orbit and endowed with a complex structure J is defined by the data $(\mathfrak{su}_m, \chi, \beta)$, where $\mathfrak{su}_m$ is a connected component of the white subdiagram Π_W of G/H, $\beta = \beta_0$ is one of the end roots of the string $\mathfrak{su}_\mathfrak{m}$ and $\chi : Z(H) = T^k \to T^1$ is a

character.
These data define a complex structure on the flag manifold $F = G\times_H PV_\varphi = G/K$ such that (F, J^F) corresponds to the painted Dynkin diagram $\Pi = \Pi_B^F \cup \Pi_W^F$ obtained form the painted Dynkin diagram $\Pi = \Pi_B \cup \Pi_W$ of $S_0 = G/H$ by painting the simple root $\beta = \beta_0$ into black. So $\Pi_B^F = \Pi_B \cup \{\beta_0\}$.

For classical simple Lie algebras $\mathfrak{g}$ of types A, B, C, D we use the standard notation for the root system $R_\mathfrak{g}$ and the simple root system $\Pi_\mathfrak{g}$ as in [17]:

$$\begin{array}{llll} R_A = & \{\varepsilon_i - \varepsilon_j\}, & R_B = & \{\varepsilon_i - \varepsilon_j, \pm\varepsilon_i\}, \\ \Pi_A = & \{\alpha_i := \varepsilon_i - \varepsilon_{i+1}\} & \Pi_B = & \{\alpha_i := \varepsilon_i - \varepsilon_{i+1}; \varepsilon_\ell\} \\ R_C = & \{\varepsilon_i - \varepsilon_j, \pm 2\varepsilon_i\} & R_D = & \{\pm\varepsilon_i \pm \varepsilon_j\} \\ \Pi_C = & \{\alpha_i := \varepsilon_i - \varepsilon_{i+1}; 2\varepsilon_\ell\} & \Pi_D = & \{\alpha_i := \varepsilon_i - \varepsilon_{i+1}; \alpha_\ell := \varepsilon_{\ell-1} + \varepsilon_\ell\} \end{array}$$

where ℓ is the rank. Now we are ready to state the main results of the paper. The following Theorem 7 (resp. Theorem 8) gives necessary and sufficient conditions for an admissible vector bundle $M_\varphi = G \times_H V_\varphi$ of rank $m = \dim V_\varphi > 1$ (resp. $m = 1$) to admit a Kähler-Einstein standard invariant metric.

Theorem 7 *Let $(S_0 = G/H, J^S)$ be the flag manifold of one of the classical simply connected Lie groups $G = SU_n, Sp_n, \widetilde{SO}_n = Spin_n$ defined by the painted Dynkin diagram $\Pi = \Pi_W \cup \Pi_B$, $\Pi_B = \{\beta_1, \cdots, \beta_p\}$, and let $n_1, \ldots, n_p$ be the Koszul numbers of G/H.*

Let $m > 1$ an integer, let $A_{m-1} = \{\alpha_1, \cdots, \alpha_{m-1}\}$ be a white string of Π_W and $\beta = \alpha_1$ (resp. $\beta = \alpha_{m-1}$). Then, we have the following

(i) the admissible vector bundle $M_\varphi = G \times_H V_\varphi \to S_0$ of rank m associated with the data (A_{m-1}, χ, β) admits a Ricci- flat Kähler standard metric if and only if $m|n_j$ for $j = 1, \ldots, p$ and

$$\chi = \sum_{j=1}^{p} \frac{n_j}{m}\pi_j \quad (\text{resp. } \ \chi = -\sum_{j=1}^{p} \frac{n_j}{m}\pi_j) \tag{9}$$

(ii) the admissible vector bundle $M_\varphi = G \times_H V_\varphi \to S_0$ of rank m associated with the triple (A_{m-1}, χ, β), where $\chi = k_1\pi_1 + \cdots + k_p\pi_p$ and $\beta = \alpha_1$ (resp. $\beta = \alpha_{m-1}$) admits a unique Kähler-Einstein standard metric g defined in a neighborhood of the singular section with Einstein constant $\lambda > 0$ if and only if $k_j < \frac{n_j}{m}$ (resp. $k_j > -\frac{n_j}{m}$) and with Einstein constant $\lambda < 0$ if and only if $k_j > \frac{n_j}{m}$ (resp. $k_j < -\frac{n_j}{m}$).
In the case $\lambda < 0$ the metric is extended to a globally defined complete metric in M_φ.

Theorem 8 *Let G and S_0 be as in Theorem 7. Then the admissible vector bundle $M_\varphi = G \times_H V_\varphi \to S_0$ of rank $m = 1$ with S_0 as only singular orbit associated to the infinitesimal character $\dot\chi = k_1\pi_1 + \cdots + k_p\pi_p$ admits a Kähler-Einstein standard cohomogeneity one structure* *with Einstein constant $\lambda = 0$ (resp. $\lambda > 0$, $\lambda < 0$) if and only if $k_j = n_j$ (resp. $k_j > n_j$, $k_j < n_j$), where the n_j's, $j = 1, \ldots, p$, denote as above the Koszul numbers of G/H.*

Note that the last theorem includes the case when S_0 is the manifold of full flags (i.e. $H = T^\ell$ is a maximal torus).

2.6 Calculation of Koszul numbers and examples

Let us recall the flag manifolds $F = G/K$ of the classical groups G (see, for example, [5], [7]):

- $SU(n)/S(U(n_1) \times \cdots \times U(n_s) \times U(1)^m)$
  $n = n_1 + \cdots + n_s + m,\ s, m \geq 0$
- $SO(2n+1)/U(n_1) \times \cdots \times U(n_s) \times SO(2r+1) \times U(1)^m$
- $Sp(n)/U(n_1) \times \cdots \times U(n_s) \times Sp(r) \times U(1)^m$
- $SO(2n)/U(n_1) \times \cdots \times U(n_s) \times SO(2r) \times U(1)^m$
  $n = n_1 + \cdots + n_s + m + r,\ s, m, r \geq 0, r \neq 1$

The Koszul numbers n_j for F endowed with a G-invariant complex structure J^F are determined by the corresponding painted Dynkin diagram as follows ([5]):
$n_j = b_j + 2$, where b_j equals the number of white roots connected to the black root β_j , with the following exceptions.
For the group $G = SO_{2n+1}$ of type B_n, each long root of the last white chain which defines the root system of the type so_{2r+1} is counted as two.
For G of type C_n, each root of the last white chain of type sp_r is counted as 2.
For G of type D_n, the last white chain of type so_{2r} is considered as a chain of length 2(r-1).
If r = 0 and one of the two end roots is white and the other one is black, the Koszul number associated to the end black root β is 2(k-1), where k is the number of white roots connected with β.

Example 9 Let us consider for example the flag manifold $(G/H, J)$ given by the following painted diagram:

$$\circ - \circ - \bullet - \circ - \circ - \bullet - \circ - \circ - \circ - \circ - \circ \tag{10}$$

The Koszul numbers associated to the black roots are

$$n_1 = 6,\ n_2 = 9$$

For the first white string on the left $\mathfrak{su}_m$, $m = 3$, condition $m|n_j$ in Theorem 7 is satisfied, so there exists a Kahler-Einstein admissible vector bundle of rank 3 with Einstein constant $\lambda = 0$ if we choose data $(\mathfrak{su}_m, \chi, \beta)$ with $\mathfrak{su}_m$ being this string (both when the new black root β is the first and the second node of the string). For the white string $\mathfrak{su}_m$, $m = 6$ on the right, the condition $m|n_j$ in Theorem 7 is not satisfied since 6 does not divide 9, so the admissible vector bundle corresponding to the choice of this string does not admit a Ricci-flat structure.

If G/H is given by the following painted diagram:

$$\bullet - \circ - \circ - \circ - \bullet \tag{11}$$

the Koszul numbers associated to the black roots are

$$n_1 = n_2 = 5$$

For the central white string $\mathfrak{su}_m$, $m = 4$, condition $m|n_j$ in Theorem 7 is not satisfied, so the admissible vector bundle corresponding to the choice of this string does not admit a Ricci-flat structure.

Example 10 The conditions for the existence of a Kahler-Einstein metric (with Einstein constant $\lambda = 0, \lambda > 0$ or $\lambda < 0$) are satisfied in particular when the painted Dynkin diagram of the singular orbit $S_0 = G/H$ consists of a white A_{m-1} string only (i.e. $S_0 = SU(m)/SU(m)$ is a point), since in this case the Koszul numbers n_j, $j = 1, \ldots, p$ of G/H all vanish. So there exists a Kahler-Einstein standard cohomogeneity one $SU(m)$-manifold having a point as singular orbit for any value of the Einstein constant λ.
In order to determine explicitly the metric for any case, assume for example that $\beta = \alpha_1 = \varepsilon_1 - \varepsilon_2$ (the case when $\beta = \alpha_{m-1} = \varepsilon_{m-1} - \varepsilon_m$ is similar) and observe that $Z(\mathfrak{h}) = \{\bar{0}\}$ and that the set of positive black roots of G/K is

$$R^+_{\mathfrak{m}} = \{\beta, \beta + (\varepsilon_2 - \varepsilon_3), \ldots, \beta + \cdots + (\varepsilon_{m-1} - \varepsilon_m)\}$$

Then we have that the polynomial $P(x)$ is

$$P(x) = \Pi_{\alpha \in R^+_{\mathfrak{m}}}(\alpha(Z_0) + x\ \alpha(Z^0)) = \Pi_{\alpha \in R^+_{\mathfrak{m}}} x\ \alpha(Z^0) = \beta^0 x^{m-1}$$

(where we are denoting $\beta^0 = \beta(Z^0)^{m-1}$) and then

$$\int_0^s (km - \lambda v)P(v)dv = \beta^0 k s^m - \lambda \frac{\beta^0}{m+1} s^{m+1}.$$

So, by Theorem 6, the function $f(t)$ which determines the metric is the inverse to

$$t(f) = \int_0^f \sqrt{\frac{P(s)}{2\int_0^s (\kappa m - \lambda v)P(v)dv}} ds = \int_0^f \sqrt{\frac{1}{2\kappa s - \lambda \frac{2}{m+1} s^2}} ds$$

By a straight calculation, one then sees

- $\lambda = 0$:

$$t(f) = \int_0^f \sqrt{\frac{1}{2\kappa s}} ds = \frac{\sqrt{2}}{\kappa}\sqrt{f}$$

so that $t \in [0, +\infty)$ and

$$f(t) = \frac{k^2}{2} t^2$$

This is the flat metric on $\mathbb{C}^m$, endowed with the canonical $SU(m)$ action.

- $\lambda > 0$:

$$t(f) = \int_0^f \sqrt{\frac{1}{2\kappa s - \lambda \frac{2}{m+1} s^2}} ds = -\frac{2}{\sqrt{-b}} artg \frac{1}{\sqrt{-b}} \sqrt{\frac{bf}{f + \frac{a}{b}}}$$

so that $t \in [0, \frac{\pi}{\sqrt{-b}}]$ and

$$f(t) = -\frac{a}{b} \sin^2(\frac{\sqrt{-b}}{2} t)$$

where we are denoting $a = 2\kappa$, $b = -\lambda \frac{2}{m+1}$: this is the (non complete) Fubini-Study metric on $\mathbb{C}^m$.

- $\lambda < 0$:

$$t(f) = \int_0^f \sqrt{\frac{1}{2\kappa s - \lambda \frac{2}{m+1} s^2}} ds = \frac{1}{\sqrt{b}} ln \frac{1 + \sqrt{\frac{f}{f + \frac{a}{b}}}}{1 - \sqrt{\frac{f}{f + \frac{a}{b}}}}$$

so that $t \in [0, +\infty)$ and

$$f(t) = \frac{a}{b} \sinh^2(\frac{\sqrt{b}}{2} t)$$

where we are denoting $a = 2\kappa$, $b = -\lambda \frac{2}{m+1}$: this is the hyperbolic metric on the open disk endowed with the canonical action of $SU(m)$.

3 Proofs

The proofs of Theorem 7 and Theorem 8 consist in finding the conditions under which there exists a Lie algebra character $\chi : Z(\mathfrak{h}) \to \mathbb{C}$ such that the above algebraic condition $Z^{Kos} = \lambda Z_0 + \kappa m Z^0$ (5) in Theorem 5 is satisfied. In order to do this, we need to calculate Z^0 and Z^{Kos}.

Lemma 11 *Let G be a simply connected group with Lie algebra $\mathfrak{su}_n$, $\mathfrak{sp}_n$, $\mathfrak{so}_{2n}$, $\mathfrak{so}_{2n+1}$ and let $S_0 = G/H$ be a flag manifold with painted Dynkin diagram $\Pi = \Pi_B^H \cup \Pi_W^H$.*
Let $G \times_H V$ be the standard admissible bundle of rank $m > 1$ defined by the data (A_{m-1}, χ, β), where $A_{m-1} = \{\alpha_1, \ldots, \alpha_{m-1}\}$ is a white string in Π_W^H, $\chi : Z(\mathfrak{h}) \to \mathbb{C}$ is a Lie algebra character and $\beta = \alpha_1$ (resp.

$\beta = \alpha_{m-1}$) is the new black root in the painted Dynkin diagram of the flag G/K associated to the regular orbits.
Let Z^0, κ be defined by (3) and (4). If π_s, π_{s+1} denote the fundamental weights of the black roots β_s, β_{s+1} of the diagram of G/H connected to α_1 and α_{m-1} respectively, then, up to sign, we have

$$\kappa Z^0 = B^{-1}(\chi + \pi_0 - \frac{m-1}{m}\pi_s - \frac{1}{m}\pi_{s+1})$$
$$(\text{resp.}\quad \kappa Z^0 = B^{-1}(\pi_0 - \chi - \frac{m-1}{m}\pi_{s+1} - \frac{1}{m}\pi_s)) \tag{12}$$

with the exception of the following two cases:

(1) $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and the painted Dynkin diagram of G/H is

$$\cdots - \underset{\beta_s}{\bullet} - \underset{\alpha_1}{\circ} - \cdots - \underset{\alpha_{m-1}}{\circ} \Rightarrow \underset{\beta_{s+1}}{\bullet} \tag{13}$$

then

$$\kappa Z^0 = B^{-1}(\chi + \pi_0 - \frac{m-1}{m}\pi_s - \frac{2}{m}\pi_{s+1})$$
$$(\text{resp.}\quad \kappa Z^0 = B^{-1}(\pi_0 - \chi - \frac{2(m-1)}{m}\pi_{s+1} - \frac{1}{m}\pi_s)) \tag{14}$$

(2) $\mathfrak{g} = \mathfrak{so}_{2n}$ and the painted Dynkin diagram of G/H is

$$\begin{array}{ccccccccc} & & & & & & \bullet\beta_{s+1} & & \\ & & & & & & | & & \\ \cdots - & \underset{\beta_s}{\bullet} & - & \underset{\alpha_1}{\circ} & - & \cdots - & \circ & - & \underset{\alpha_{m-1}}{\circ} \end{array} \tag{15}$$

$$\tag{16}$$

then

$$\kappa Z^0 = B^{-1}(\chi + \pi_0 - \frac{m-1}{m}\pi_s - \frac{2}{m}\pi_{s+1})$$
$$(\text{resp.}\quad \kappa Z^0 = B^{-1}(\pi_0 - \chi - \frac{m-2}{m}\pi_{s+1} - \frac{1}{m}\pi_s)) \tag{17}$$

where we are denoting $B^{-1}(\xi)$ the dual of ξ with respect to the Killing form B, that is $\xi(X) := B(B^{-1}(\xi), X)$.
For the admissible vector bundle $M_\varphi = G \times_H V_\varphi \to S_0$ of rank $m = 1$ with S_0 as only singular orbit (i.e. $G/K = G/H$) defined by the pair (A_{m-1}, χ) we have

$$\kappa Z^0 = B^{-1}(\chi) \tag{18}$$

Remark 2 If α_1 is the first (resp. α_{m-1} is the last) node of the diagram, then we have no black root β_s (resp. β_{s+1}) and in formulas (12), (14) and (17) the term in π_s (resp. in π_{s+1}) cancels.

Before starting to prove the Lemma, let us fix some notation which will be fundamental in the proof.
As we have recalled above, the stability subgroup L of a regular orbit in $M_\varphi = G \times_H V_\varphi$ can be identified with the stability subgroup H_e of a non-zero vector $e \in V_\varphi$ and the corresponding stability subgroup K of the flag manifold $F = G \times_H PV_\varphi$ with the stabilizer $H_{[e]}$ of the line $[e] \in PV_\varphi$.
This holds true also when $m = \dim(V_\varphi) = 1$, in which case $G/K = G/H$, that is we have no new black root β.

Given the Lie algebra character $\chi : Z(\mathfrak{h}) \to \mathbb{C}$, let us denote $\mathfrak{a} = \ker \chi$ and define the following direct sum orthogonal decomposition

$$\mathfrak{h} = \mathfrak{su}_m + \mathfrak{n}' + Z(\mathfrak{h}) = \mathfrak{su}_m + \mathfrak{n}' + \mathfrak{a} + \mathbb{R}Z^\chi \tag{19}$$

where Z^χ is the vector in $\mathfrak{a}^\perp$ such that $\chi(\exp tZ^\chi) = e^{it}$.

We identify V_φ with the Hermitian space $\mathbb{C}^m$ such that the standard basis e_j, $j = 1, \dots, m$ consists of weight vectors with weights ε_j w.r.t. the Cartan subalgebra $\varphi(\mathfrak{c})$ and the simple roots $\alpha_j \in A_{m-1}$ satisfy $\alpha_j|_\mathfrak{c} = \varepsilon_i - \varepsilon_{i+1}$.
In the case $m > 1$, we have either $\beta = \alpha_1$, in which case we take $e = e_1$ and set

$$Z^\beta := i\mathrm{diag}((m-1), -\mathrm{id}_{m-1}) \in \mathfrak{su}_m \subset \mathfrak{h} \tag{20}$$

or $\beta = \alpha_{m-1}$, in which case we choose $e = e_m$ and set

$$Z^\beta := i\mathrm{diag}(-\mathrm{id}_{m-1}, m-1) \in \mathfrak{su}_m \subset \mathfrak{h} \tag{21}$$

(when there is no risk of confusion, with a slight abuse of notation in the following we will denote by Z^β both the element of $\mathfrak{su}_m$ and its immersion in $\mathfrak{h}$, see also (22)-(25) below).

Since Z^χ goes under the Lie algebra representation to $i\mathrm{id}_m$, in both cases the element

$$Z^\mathfrak{l} = Z^\beta - (m-1)Z^\chi$$

annihilates e, hence it belongs to the stability subalgebra $\mathfrak{l}$. Then the fundamental vector Z_F^0 coincides with the vector of the plane $\mathrm{span}(Z^\beta, Z^\chi)$ orthogonal to $Z^\mathfrak{l}$ and normalized by $B(Z_F^0, Z_F^0) = -1$. Recall that the relations between the stability subalgebras $\mathfrak{h}, \mathfrak{k}$, $\mathfrak{l}$ of the flag manifolds $S_0 = G/H, F = G/K$, the CR manifold G/L and their centers are given by

$$\begin{array}{lll}
\mathfrak{h} = & \mathfrak{su}_m + \mathfrak{n}' + Z(\mathfrak{h}), & Z(\mathfrak{h}) = \mathfrak{a} + \mathbb{R}Z^\chi \\
\mathfrak{l} = & \mathfrak{su}_{m-1} + \mathfrak{n}' + Z(\mathfrak{l}), & Z(\mathfrak{l}) = \mathfrak{a} + \mathbb{R}Z^\mathfrak{l} \\
\mathfrak{k} = & \mathfrak{su}_{m-1} + \mathfrak{n}' + Z(\mathfrak{k}) = \mathfrak{l} + \mathbb{R}Z^0 & Z(\mathfrak{k}) = Z(\mathfrak{l}) + \mathbb{R}Z^0 = Z(\mathfrak{h}) + \mathbb{R}Z^0
\end{array}$$

where we denote by $\mathfrak{su}_{\mathfrak{m}-\mathfrak{1}}$ the stability subalgebra of the vector e in $\mathfrak{su}_m$.
In the case $m = 1$, where as we have observed above $G/H = G/K$ we have no new black root β, we have $Z(\mathfrak{h}) = Z(\mathfrak{k})$, $Z^\mathfrak{l} = \bar{0}$ and $Z(\mathfrak{l}) = \mathfrak{a} = \ker \chi$.

Proof of Lemma 11: As we have seen above, the vector Z^0_F is given by the B-orthogonal decomposition $Z(\mathfrak{k}) = Z(\mathfrak{l}) + \mathbb{R}Z^0_F$, and $B(Z^0_F, Z^0_F) = -1$, being $Z(\mathfrak{l}) = ker(\chi)$ in the case $m = 1$, while for $m > 1$

$$Z(\mathfrak{l}) = \ker(\chi) + \mathbb{R}Z^{\mathfrak{l}}, \quad Z^{\mathfrak{l}} = Z^{\beta} - (m-1)Z^{\chi}$$

where $Z^{\chi} \in Z(\mathfrak{h})$ is orthogonal to $ker(\chi)$ and $\chi(Z^{\chi}) = i$.

Let $m > 1$: by the well-known structure of classical semi-simple Lie algebras, in the case $\mathfrak{g} = \mathfrak{su}_n$, if $\beta = \alpha_1$ (resp. $\beta = \alpha_2$), then by (20) (resp. (21)) above we have

$$Z^{\beta} = D_m := i\ diag(O, m-1, -id_{m-1}, O) \tag{22}$$

$$(\text{ resp. } Z^{\beta} = D_m := i\ diag(O, -id_{m-1}, m-1, O)\ ) \tag{23}$$

where the order of the zero matrices O depends on the position of the A_{m-1} component in the Dynkin diagram, while for the other classical Lie algebras we have

$$Z^{\beta} = \begin{pmatrix} D_m & 0 \\ 0 & -D_m \end{pmatrix} \text{ for } \mathfrak{g} = \mathfrak{sp}_{2\mathrm{n}}, \mathfrak{so}_{2\mathrm{n}} \tag{24}$$

$$Z^{\beta} = \begin{pmatrix} D_m & 0 & 0 \\ 0 & -D_m & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ for } \mathfrak{g} = \mathfrak{so}_{2\mathrm{n}+1} \tag{25}$$

where D_m is given either by (22) or (23) depending on the choice of β. We are going to show that

$$Z^0_F = \frac{1}{\sqrt{-\frac{1}{\|Z^{\chi}\|^2} - \frac{(m-1)^2}{\|Z^{\beta}\|^2}}} \left(\frac{Z^{\chi}}{\|Z^{\chi}\|^2} + (m-1)\frac{Z^{\beta}}{\|Z^{\beta}\|^2} \right) \tag{26}$$

where we are using the notation $\|Z\|^2 = B(Z, Z)$. Indeed, Z^{χ} and Z^{β} are orthogonal since Z^{χ} belongs to $Z(\mathfrak{h})$ which consists of matrices of the kind

$$X_m := i\ diag(O, \theta\ id_m, O) \tag{27}$$

for $\mathfrak{g} = \mathfrak{su}_n$ and

$$\begin{pmatrix} X_m & 0 \\ 0 & -X_m \end{pmatrix} \text{ for } \mathfrak{g} = \mathfrak{sp}_{\mathrm{n}}, \mathfrak{so}_{2\mathrm{n}} \tag{28}$$

$$\begin{pmatrix} X_m & 0 & 0 \\ 0 & -X_m & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ for } \mathfrak{g} = \mathfrak{so}_{2\mathrm{n}+1} \tag{29}$$

so the claim is true by (22)-(25) and by recalling that the Killing form B is given by $B(X,Y) = 2n \cdot tr(XY), 2(n+1) \cdot tr(XY), 2(n-1) \cdot tr(XY), (2n-1) \cdot tr(XY)$ for $\mathfrak{g} = \mathfrak{su}_n, \mathfrak{sp}_n, \mathfrak{so}_{2n}, \mathfrak{so}_{2n+1}$ respectively.

So we have

$$\left\langle \frac{Z^\chi}{\|Z^\chi\|^2} + (m-1)\frac{Z^\beta}{\|Z^\beta\|^2},\quad Z^\beta - (m-1)Z^\chi \right\rangle = -(m-1)+(m-1) = 0$$

which shows that the vector given by (26) is orthogonal to $Z^{\mathfrak{l}}$.
Moreover, (26) is orthogonal to $ker(\chi)$ since both Z^β and Z^χ are (Z^χ by definition, Z^β since, as we observed above, it is orthogonal to $Z(\mathfrak{h})$ and $ker(\chi) \subseteq Z(\mathfrak{h})$). Finally, it is easy to verify that $B(Z_F^0, Z_F^0) = -1$.
If $m = 1$, then by $Z(\mathfrak{l}) = \ker \chi$ and the orthogonality condition we immediately see that $Z_F^0 = \frac{Z^\chi}{\sqrt{-\|Z^\chi\|^2}}$

We now calculate κ defined by (4).
To this aim, recall that L is the isotropy subgroup of a non-zero vector $e \in \mathbb{C}^m$ (with respect to the action of H on $\mathbb{C}^m$ defined through χ).
In the case $m > 1$, let us consider just the case when $\beta = \alpha_1$, $e = e_1$ and Z^β is given by (20) (the calculation being similar in the case $\beta = \alpha_{m-1}$, $e = e_m$).
Since through the (Lie algebra) representation tZ^χ corresponds to $itId_m$ and tZ^β to $it\ diag(m-1, -1, \ldots, -1)$, by (26) we have that tZ_F^0 goes in the Lie algebra representation to

$$\frac{1}{\sqrt{-\frac{1}{\|Z^\chi\|^2} - \frac{(m-1)^2}{\|Z^\beta\|^2}}}\left(\frac{it}{\|Z^\chi\|^2}Id + \frac{m-1}{\|Z^\beta\|^2}it\ diag(m-1,\ldots)\right) =$$

$$= diag\left(-it\sqrt{-\frac{1}{\|Z^\chi\|^2} - \frac{(m-1)^2}{\|Z^\beta\|^2}}, \ldots\right)$$

so that $exp(tZ_F^0)$ goes to

$$diag\left(e^{-it\sqrt{-\frac{1}{\|Z^\chi\|^2} - \frac{(m-1)^2}{\|Z^\beta\|^2}}}, \ldots\right).$$

So, in order for $exp(tZ^0)$ to fix e_1 we must have $-t\sqrt{-\frac{1}{\|Z^\chi\|^2} - \frac{(m-1)^2}{\|Z^\beta\|^2}} = 2\pi k$ for some $k \in \mathbb{Z}$, and the first positive value for which this holds true is $T_0 = \frac{2\pi}{\sqrt{-\frac{1}{\|Z^\chi\|^2} - \frac{(m-1)^2}{\|Z^\beta\|^2}}}$, from which we finally deduce that

$$\kappa = \frac{2\pi}{T_0} = \sqrt{-\frac{1}{\|Z^\chi\|^2} - \frac{(m-1)^2}{\|Z^\beta\|^2}}$$

that is

$$\kappa Z_F^0 = \frac{Z^\chi}{\|Z^\chi\|^2} + (m-1)\frac{Z^\beta}{\|Z^\beta\|^2}$$

(notice that this equality holds true both when $\beta = \alpha_1$ and $\beta = \alpha_{m-1}$).
Moreover, by the definition of $B(X, Y)$ in each of the classical groups recalled above and by (22)-(25) we have $\|Z_\beta\|^2 = -2cm(m-1)$ where

$c = n, 2(n+1), 2(n-1), 2n-1$ for $\mathfrak{g} = \mathfrak{su}_n, \mathfrak{sp}_n, \mathfrak{so}_{2n}, \mathfrak{so}_{2n+1}$ respectively, so we finally get

$$\kappa Z_F^0 = \frac{Z^\chi}{\|Z^\chi\|^2} - \frac{Z^\beta}{2cm} \tag{30}$$

In the case $m = 1$, the same argument shows that $\kappa = \frac{1}{\sqrt{-\|Z^\chi\|^2}}$, so that we get

$$\kappa Z_F^0 = -\frac{Z^\chi}{\|Z^\chi\|^2} \tag{31}$$

Now, we are going to rewrite this by B-duality, i.e. to calculate the dual form $\xi^0 = B^{-1}(\kappa Z_F^0)$. For the sake of brevity, from now on we will denote $Z \simeq \xi$ to mean $Z = B^{-1}(\xi)$.

First, the fact that $B(Z^\chi, Z) = 0$ for every $Z \in ker(\chi)$ means that $Z^\chi \simeq D\chi$, for some $D \in \mathbb{C}$; then, by $\|Z^\chi\|^2 = B(Z^\chi, Z^\chi) = D\chi(Z^\chi) = D\ i$ we have

$$\frac{Z^\chi}{\|Z^\chi\|^2} \simeq \frac{\chi}{i} \tag{32}$$

By (31), this immediately yields $\kappa Z_F^0 \simeq i\chi$ in the case $m = 1$.

In the case $m > 1$, we need to calculate $B^{-1}(Z^\beta)$.

Recall that, if we denote by E_{ij} the square matrix having 1 at position $i\ j$ and zero otherwise, then

$$\frac{1}{2c}(E_{ii} - E_{jj}) \simeq \varepsilon_i - \varepsilon_j \text{ for } \ \mathfrak{g} = \mathfrak{su}_\mathrm{n} \ \ (\mathrm{c = n})$$

$$\frac{1}{2c}\begin{pmatrix} E_{ii} - E_{jj} & 0 \\ 0 & E_{jj} - E_{ii} \end{pmatrix} \simeq \varepsilon_i - \varepsilon_j \text{ for } \ \mathfrak{g} = \mathfrak{sp}_{2\mathrm{n}}, \mathfrak{so}_{2\mathrm{n}} \ \ (\mathrm{c = 2(n+1), 2(n-1) \ respectively})$$

$$\frac{1}{2c}\begin{pmatrix} E_{ii} - E_{jj} & 0 & 0 \\ 0 & E_{jj} - E_{ii} & 0 \\ 0 & 0 & 0 \end{pmatrix} \simeq \varepsilon_i - \varepsilon_j \text{ for } \ \mathfrak{g} = \mathfrak{so}_{2\mathrm{n}+1} \ \ (\mathrm{c = 2n-1})$$

Then, if $\alpha_1 = \varepsilon_k - \varepsilon_{k+1}$ and $\alpha_{m-1} = \varepsilon_{k+m-2} - \varepsilon_{k+m-1}$, combining these identities with (22)-(25) above we get

$$Z^\beta \simeq 2c\ i((m-1)\varepsilon_k - (\varepsilon_{k+1} + \cdots + \varepsilon_{k+m-1})) \tag{33}$$

$$(\text{ resp. } Z^\beta \simeq 2c\ i((m-1)\varepsilon_{k+m-1} - (\varepsilon_k + \cdots + \varepsilon_{k+m-2}))\ ) \tag{34}$$

for $\beta = \alpha_1$ (resp. $\beta = \alpha_{m-1}$).

Now, if we are not in one of the cases (1) or (2) of the statement of the lemma, then $\beta_s = \varepsilon_{k-1} - \varepsilon_k$, $\beta_{s+1} = \varepsilon_{k+m-1} - \varepsilon_{k+m}$ and the corresponding fundamental weights are

$$\pi_s = \varepsilon_1 + \cdots + \varepsilon_{k-1}, \quad \pi_{s+1} = \varepsilon_1 + \cdots + \varepsilon_{k+m-1}$$

Moreover, the fundamental weight associated to the new black node $\beta = \alpha_1$ (resp. $\beta = \alpha_{m-1}$) is given by $\pi_0 = \varepsilon_1 + \cdots + \varepsilon_k$ (resp. $\pi_0 = \varepsilon_1 + \cdots + \varepsilon_{k+m-2}$) and then, by (33) and (34) we get

$$Z^\beta \simeq 2c\, i(m\pi_0 - (m-1)\pi_s - \pi_{s+1}) \tag{35}$$

$$(\text{ resp. } Z^\beta \simeq 2c\, i((m-1)\pi_{s+1} - m\pi_0 + \pi_s)\ ) \tag{36}$$

If we are in case (1) of the statement of the lemma, we have $\pi_{s+1} = \frac{\epsilon_1 + \cdots + \epsilon_n}{2}$ and then

$$Z^\beta \simeq 2c\, i(m\pi_0 - (m-1)\pi_s - 2\pi_{s+1}) \tag{37}$$

$$(\text{ resp. } Z^\beta \simeq 2c\, i(2(m-1)\pi_{s+1} - m\pi_0 + \pi_s)\ ) \tag{38}$$

for $\beta = \alpha_1$ (resp. $\beta = \alpha_{m-1}$).
If we are in case (2) of the statement of the lemma, then $\pi_{s+1} = \frac{\epsilon_1 + \cdots + \epsilon_{n-1} - \epsilon_n}{2}$ and one sees that

$$Z^\beta \simeq 2c\, i(m\pi_0 - (m-1)\pi_s - 2\pi_{s+1}) \tag{39}$$

$$(\text{ resp. } Z^\beta \simeq 2c\, i((m-2)\pi_{s+1} - m\pi_0 + \pi_s)\ ) \tag{40}$$

Then the lemma follows by substitution in (30) and by $Z^0 = -iZ^0_F$ (recall also that both Z^0 and Z^0_F are determined up to the sign). $\square$

Now we are ready to prove Theorems 7 and 8. Let us recall that, assuming that the group G is simply connected, it is known that the lattice of characters coincide with the lattice of weights ([17], [24]) so that the Lie algebra character χ is given by a linear combination *with integer coefficients* of the fundamental weights $\pi_1, \ldots, \pi_p$ associated to the black nodes of the diagram of G/H:

$$\chi = \sum_{j=1}^{p} k_j \pi_j, \quad k_j \in \mathbb{Z}$$

while the Koszul form $\sigma = B^{-1}(Z^{Kos}) = \sum_{\alpha \in R^+_{\mathfrak{m}}} \alpha$ (where $R^+_{\mathfrak{m}} = R^+ \setminus R^+_{\mathfrak{h}}$ denotes the set of complementary to $R^+_{\mathfrak{h}}$ positive roots). In what follows, when necessary to avoid ambiguity we will write $Z^{Kos}_{G/H}$ to denote the Koszul vector of the flag manifold G/H.

Proof of Theorem 7:

Let $\pi_1, \ldots, \pi_p$ be the fundamental weights of the black roots $\beta_1, \ldots, \beta_p$ of the Dynkin diagram of G/H, and $n_1, \ldots, n_p$ be the corresponding Koszul numbers, so that $B^{-1}(Z^{Kos}_{G/H}) = \sum_{j=1}^{p} n_j \pi_j$.
Let $m > 1$, let $A_{m-1} = \{\alpha_1, \ldots, \alpha_{m-1}\}$ be the white string given by the data which define the admissible vector bundle and let π_0 be

the fundamental weight of the new black node β (with $\beta = \alpha_1$ or $\beta = \alpha_{m-1}$), so that we have $B^{-1}(Z^{Kos}_{G/K}) = n'_0\pi_0 + \sum_{j=1}^{p} n'_j\pi_j$.
By using the description of the Koszul numbers given in subsection 2.6, one easily verifies that if $\beta = \alpha_1$ (resp. $\beta = \alpha_{m-1}$), then

$$n'_0 = m, \quad n'_s = n_s - (m-1), \quad n'_{s+1} = n_{s+1} - 1 \tag{41}$$

$$(\text{resp. } n'_0 = m, \quad n'_s = n_s - 1, \quad n'_{s+1} = n_{s+1} - (m-1) \ ) \tag{42}$$

with the exception of the same two cases seen in Lemma 11, that is

$$\mathfrak{g} = \mathfrak{so}_{2n+1}, \qquad \cdots - \underset{\beta_s}{\bullet} - \underset{\alpha_1}{\circ} - \cdots - \underset{\alpha_{m-1}}{\circ} \Rightarrow \underset{\beta_{s+1}}{\bullet} \tag{43}$$

where, if $\beta = \alpha_1$ (resp. $\beta = \alpha_{m-1}$), we have

$$n'_0 = m, \quad n'_s = n_s - (m-1), \quad n'_{s+1} = n_{s+1} - 2 \tag{44}$$

$$(\text{resp. } n'_0 = m, \quad n'_s = n_s - 1, \quad n'_{s+1} = n_{s+1} - 2(m-1) \ ) \tag{45}$$

and

$$\mathfrak{g} = \mathfrak{so}_{2n}, \qquad \cdots - \underset{\beta_s}{\bullet} - \underset{\alpha_1}{\circ} - \cdots - \overset{\overset{\bullet \beta s+1}{|}}{\circ} - \underset{\alpha_{m-1}}{\circ} \tag{46}$$

(47)

where

$$n'_0 = m, \quad n'_s = n_s - (m-1), \quad n'_{s+1} = n_{s+1} - 2 \tag{48}$$

$$(\text{resp. } n'_0 = m, \quad n'_s = n_s - 1, \quad n'_{s+1} = n_{s+1} - (m-2) \ ) \tag{49}$$

for $\beta = \alpha_1$ (resp. $\beta = \alpha_{m-1}$).
We point out that formulas (41) and (42) hold true also for $G = Sp(n)$ in the case when the painted Dynkin diagram of G/H is

$$\cdots - \underset{\beta_s}{\bullet} - \underset{\alpha_1}{\circ} - \cdots - \underset{\alpha_{m-1}}{\circ} \Leftarrow \underset{\beta_{s+1}}{\bullet} \tag{50}$$

Now, if $\lambda = 0$, the algebraic condition for the existence of the Einstein metric is

$$Z^{Kos} = \kappa m Z^0 \tag{51}$$

By using Lemma 11 and (41),(42), (44), (45), (48), (49) if $\beta = \alpha_1$ (resp. $\beta = \alpha_{m-1}$), one rewrites condition (51) as

$$m\pi_0+\sum_{j\neq s,s+1} n_j\pi_j+[n_s-(m-1)]\pi_s+[n_{s+1}-1]\pi_{s+1} = m\chi+m\pi_0-(m-1)\pi_s-\pi_{s+1} \tag{52}$$

(resp.

$$m\pi_0+\sum_{j\neq s,s+1} n_j\pi_j+(n_s-1)\pi_s+[n_{s+1}-(m-1)]\pi_{s+1} = m\pi_0-m\chi-(m-1)\pi_{s+1}-\pi_s\ ) \tag{53}$$

or

$$m\pi_0+\sum_{j\neq s,s+1} n_j\pi_j+[n_s-(m-1)]\pi_s+[n_{s+1}-2]\pi_{s+1} = m\chi+m\pi_0-(m-1)\pi_s-2\pi_{s+1} \tag{54}$$

(resp.

$$m\pi_0+\sum_{j\neq s,s+1} n_j\pi_j+(n_s-1)\pi_s+[n_{s+1}-2(m-1)]\pi_{s+1} = m\pi_0-m\chi-2(m-1)\pi_{s+1}-\pi_s\ ) \tag{55}$$

$$m\pi_0+\sum_{j\neq s,s+1} n_j\pi_j+[n_s-(m-1)]\pi_s+[n_{s+1}-2]\pi_{s+1} = m\chi+m\pi_0-(m-1)\pi_s-2\pi_{s+1} \tag{56}$$

(resp.

$$m\pi_0+\sum_{j\neq s,s+1} n_j\pi_j+(n_s-1)\pi_s+[n_{s+1}-(m-2)]\pi_{s+1} = m\pi_0-m\chi-(m-2)\pi_{s+1}-\pi_s\ ) \tag{57}$$

in the exceptional cases of $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $\mathfrak{g} = \mathfrak{so}_{2n}$.

In all the cases (52), (53), (54), (55), (56), (57), one immediately sees that after simplifications one gets

$$\sum_{j=1}^{p} n_j\pi_j = m\chi$$

from which (i) of Theorem 7 follows.

In order to prove (ii), recall that in the case $\lambda \neq 0$, the necessary and sufficient conditions to have a standard Kahler-Einstein metric in a neighbourhood of the singular section are (see Theorem 5)

(1) the vector $Z_0 = \frac{1}{\lambda}(Z^{Kos} - kmZ^0)$ satisfies $\beta_j(Z_0) > 0$, $j = 1,\ldots,p$ and $\beta(Z_0) = 0$

(2) for at least small values of $s > 0$, the segment $Z_0 + sZ^0$ satisfies $\beta_j(Z_0 + sZ^0) > 0$, $j = 1,\ldots,p$ and $\beta(Z_0 + sZ^0) > 0$

In fact, if (1) is satisfied, then $\beta(Z_0 + sZ^0) > 0$ reduces to $\beta(Z^0) > 0$ and $\beta_j(Z_0 + sZ^0) > 0$ is always satisfied for small values of s.
So we need just to check $\beta(Z^0) > 0$, $\beta_j(Z_0) > 0$, $\beta(Z_0) = 0$.

The first condition is easily verified by (12), (14), (17) in Lemma 11.

As for the other two conditions, by (52), (53), (54), (55), (56), (57), one sees immediately that in any case

$$B^{-1}(Z_0) = \frac{1}{\lambda}(\sum_{j=1}^{p} n_j\pi_j - m\chi)$$

$$\text{(resp. } B^{-1}(Z_0) = \frac{1}{\lambda}(\sum_{j=1}^{p} n_j\pi_j + m\chi)\ ) \tag{58}$$

for $\beta = \alpha_1$ (resp. $\beta = \alpha_{m-1}$), so $\beta(Z_0) = 0$ is clear, while $\beta_j(Z_0) > 0$ writes

$$\tfrac{1}{\lambda}\tfrac{\|\beta_j\|^2}{2}[n_j - mk_j] > 0, \quad j = 1, \dots, p \tag{59}$$

$$\text{(resp. } \tfrac{1}{\lambda}\tfrac{\|\beta_j\|^2}{2}[n_j + mk_j] > 0, \quad j = 1, \dots, p \tag{60}$$

from which the assertions in (ii) of Theorem 7 immediately follow. □

Proof of Theorem 8: The proof is the same as for Theorem 7, but now since $m = 1$ we have $G/K = G/H$ and the Koszul forms σ of G/K and G/H coincide: then, for $\lambda = 0$ we use (18) to rewrite condition (51) for the existence of a Ricci-flat metric as

$$\chi = \sigma = \sum_{j=1}^{p} n_j\pi_j \tag{61}$$

while, in the case $\lambda \neq 0$, since there is no new black root β and the only condition to have a standard Kähler-Einstein metric in a neighbourhood of the singular section is $\beta_j(Z_0) > 0$, the assertions of the theorem immediately follow from $B^{-1}(Z_0) = \frac{1}{\lambda}(\sum_{j=1}^{p} n_j\pi_j - \chi)$. □

Remark 3 The above proofs show that, once the conditions for the existence of a Kähler-Einstein standard cohomogeneity one structure on an admissible vector bundle $M_\varphi = G \times_H V_\varphi$ having S_0 as only singular orbit are satisfied, then this structure is unique provided the Einstein constant $\lambda \neq 0$, while for $\lambda = 0$ the metrics are parametrized by the vectors $Z_0 \in C(J^S)$. Indeed, in the case $\lambda = 0$ condition (51) for the existence of the metric does not depend on the choice of the initial vector $Z_0 \in C(J^S)$ of the segment in $C(J^F)$, while for $\lambda \neq 0$ the vector Z_0 is completely determined by $Z_0 = \frac{1}{\lambda}(Z^{Kos} - kmZ^0)$. This is consistent with the results in [15].

Remark 4 By the same remarks made to prove (ii) of Theorem 7, one sees that, if a metric with $\lambda \neq 0$ exists, then the segment $Z_0 + xZ^0$ can be extended to a whole ray in the T-Weyl chamber $C(J^F)$ of $F = G/K$ if and only if $\beta_j(Z^0) > 0$ for every $j = 1, \dots, p$.

If we are not in one of the exceptional cases of the statement of Lemma 11, by (12) this condition both for $\lambda > 0$ and $\lambda < 0$ reads

$$k_j > 0, \quad k_s > \frac{m-1}{m}, \quad k_{s+1} > \frac{1}{m}$$
$$(\text{ resp. } k_j < 0, \quad k_s < -\frac{1}{m}, \quad k_{s+1} < -\frac{m-1}{m} \text{ }) \tag{62}$$

if $\beta = \alpha_1$ (resp. $\beta = \alpha_{m-1}$).
By Theorem 7, these conditions are always compatible with those for the existence of the Kähler-Einstein metric in the case $\lambda < 0$, i.e. $k_j > \frac{n_j}{m}$ (resp. $k_j < -\frac{n_j}{m}$), while this is not true in the case $\lambda > 0$, where we must find *integer* k_j's, $j = 1, \dots, p$, such that $0 < k_j < \frac{n_j}{m}$ for $j \neq s, s+1$ and $\frac{m-1}{m} < k_s < \frac{n_s}{m}$, $\frac{1}{m} < k_{s+1} < \frac{n_{s+1}}{m}$ (resp. $-\frac{n_j}{m} < k_j < 0$ for $j \neq s, s+1$ and $-\frac{n_s}{m} < k_s < -\frac{1}{m}$, $-\frac{n_{s+1}}{m} < k_{s+1} < -\frac{m-1}{m}$).
In the case $\lambda = 0$, the algebraic condition (51) implies that $Z^0 = \frac{1}{\kappa m} Z^{Kos} \in C(J^F)$, and then for any choice of the starting point $Z_0 \in C(J^F) \cap \{\beta = 0\}$, the segment $Z_0 + xZ^0$ extends to a ray in $C(J^F)$. This is in accordance with the last assertion in Theorem 5 (see also the end of the proof of Theorem 36 in [2]).
We leave the details about the exceptional cases of the statement of Lemma 11 to the reader.

Remark 5 Notice that $k_j \in \mathbb{Z}$ assures that the vector Z^0 fulfills the condition that $\{\exp(tZ^0) \mid t \in \mathbb{R}\}$ is compact (more precisely, a circle S^1)

Remark 6 Under the conditions given by the proposition, there always exists at least a non-complete Kahler-Einstein metric, given by a segment $Z_0 + sZ^0$ staying in the interior of the chamber C and parametrized with $s = f(t)$, being $f(t)$ the solution to the ODE (6).
As we have proved in the first part of this paper, if the conditions of Proposition 7 are satisfied, there exists a complete Kahler-Einstein metric for $\lambda \leq 0$, while for $\lambda > 0$ the metric is never complete.

A Appendix. Basic facts on flag manifolds

Let $F = G/K = \mathrm{Ad}_G Z$, where $Z \in \mathfrak{g}$, be a flag manifold, i.e. an adjoint orbit of a compact semisimple Lie group G with the B-orthogonal (where B is the Killing form) reductive decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m} = C_{\mathfrak{g}}(Z) + \mathfrak{m}.$$

We can decompose $\mathfrak{k}$ as

$$\mathfrak{k} = Z(\mathfrak{k}) \oplus \mathfrak{k}'$$

where $\mathfrak{k}'$ is the semisimple part and $Z(\mathfrak{k})$ is the center. We fix a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{k}$ (hence also of $\mathfrak{g}$) and denote by R the root system of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ w.r.t. the Cartan subalgebra $\mathfrak{c}^{\mathbb{C}}$. We set

$$R_{\mathfrak{k}} := \{\alpha \in R,\, \alpha(Z(\mathfrak{k})) = 0\},\, R_{\mathfrak{m}} := R \setminus R_{\mathfrak{k}}.$$

Then

$$\mathfrak{k} = \mathfrak{c} + \mathfrak{g}(R_{\mathfrak{k}})^{\tau},\ \mathfrak{m} = \mathfrak{g}(R_{\mathfrak{m}})^{\tau},$$

where for a subset $P \subset R$, we set

$$\mathfrak{g}(P) = \sum_{\alpha \in P} \mathfrak{g}_{\alpha}$$

being $\mathfrak{g}_{\alpha}$ the root space with root α and V^{τ} means the fix point set in $V \subset \mathfrak{g}^{\mathbb{C}}$ of the complex conjugation τ. Recall that the Killing form induces an Euclidean metric in the real vector space $i\mathfrak{c}$ and roots are identified with real linear forms on $i\mathfrak{c}$. We set $\mathfrak{t} := iZ(\mathfrak{k}) \subset i\mathfrak{c}$ and denote by

$$\rho : R \to R|_{\mathfrak{t}},\ \alpha \mapsto \bar{\alpha} := \alpha|_{\mathfrak{t}}$$

the restriction map.

Definition 7 *The set $R_T = \rho(R_{\mathfrak{m}}) = R_{\mathfrak{m}}|\mathfrak{t}$ of linear forms on $\mathfrak{t}$ which are restriction of roots from $R_{\mathfrak{m}}$ is called the system of T-**roots** and connected components C of the set $\mathfrak{t} \setminus \{\ker \bar{\alpha}, \bar{\alpha} \in R_T\}$ are called T-**Weyl chambers**.*

Sets of T-roots ξ bijectively correspond to irreducible $\mathfrak{k}$-submodules $\mathfrak{m}(\xi) := \mathfrak{g}(\rho^{-1}(\xi))$ of the complexified isotropy module $\mathfrak{m}^{\mathbb{C}}$ of the flag manifold $F = G/K$.

So a decomposition of the $\mathfrak{k}$-modules $\mathfrak{m}^{\mathbb{C}}$ and $\mathfrak{m}$ into irreducible submodules can be written as

$$\mathfrak{m}^{\mathbb{C}} = \sum_{\xi \in R_T} \mathfrak{m}(\xi),\ \mathfrak{m} = \sum_{\xi \in R_T^+} [\mathfrak{m}(\xi) + \mathfrak{m}(-\xi)]^{\tau}$$

where $R_T^+ := \rho(R_{\mathfrak{m}}^+)$ is the system of positive T-roots associated with a system of positive roots R^+, see [5], [1].

We fix a system of simple roots Π_W of $R_{\mathfrak{k}}$ and denote by $\Pi = \Pi_W \cup \Pi_B$ its extension to a system of simple roots of R. Let $R^+ = R^+(\Pi)$ be the associated system of positive roots and $R_{\mathfrak{m}}^+ := R^+ \cap R_{\mathfrak{m}}$. The set $R_T^+ := \rho(R_{\mathfrak{m}}^+)$ is called positive T-root set.

We need the following

Theorem 8 *[5] There exists a one-to-one correspondence between extensions $\Pi = \Pi_W \cup \Pi_B$ of the system Π_W of simple system of $R_{\mathfrak{k}}$, T-Weyl chambers $C \subset \mathfrak{t}$ and invariant complex structures (ICS) J on $F = G/K$. If $\Pi_B = \{\beta_1, \ldots, \beta_k\}$, then the corresponding T-Weyl chamber is defined by $C = \{\bar{\beta}_1 > 0, \ldots, \bar{\beta}_k > 0\}$ where $\bar{\beta} = \rho(\beta)$ and the complex structure is defined by $\pm i$-eigenspace decomposition*

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ + \mathfrak{m}^- = \mathfrak{g}(R_{\mathfrak{m}}^+) + \mathfrak{g}(-R_{\mathfrak{m}}^+) \tag{63}$$

of the complexified tangent space $\mathfrak{m}^{\mathbb{C}} = T_{eK}(G/K)$.

The extension $\Pi = \Pi_W \cup \Pi_B$ can be graphically described by a painted Dynkin diagram, i.e. the Dynkin diagram which represents the system Π with the nodes representing Π_B painted in black. Such a diagram, which we sometimes identify with the pair (Π_W, Π_B), allows to reconstruct the flag manifold $F = G/K$ with invariant complex structure J^F as follows: the semisimple part $\mathfrak{k}'$ of the (connected) stability subalgebra $\mathfrak{k}$ is defined as the regular semisimple subalgebra associated with the closed subsystem $R_{\mathfrak{k}} = R \cap \mathrm{span}(\Pi_W)$ and the vectors ih_j defined by condition

$$\beta_k(h_j) = \delta_{kj},\ \alpha_i(h_j) = 0,\ \beta_j \in \Pi_B, \alpha_i \in \Pi_W$$

form a basis of the center $Z(\mathfrak{k})$. The complex structure is defined by (63).
Now, an element $Z \in \mathfrak{t}$ is called to be K-**regular** if its centralizer $C_G(Z) = K$ or, equivalently, any T-root has a non-zero value on Z. Then we have the following

Proposition 9 *([12], [5]) There exists a natural one-to-one correspondence between elements $Z \in \mathfrak{t}$ and closed invariant 2-forms ω_Z on G/K, given by*

$$Z \leftrightarrow \omega_Z|_o = i\, d(B \circ Z),$$

where d is the exterior differential in the Lie algebra $\mathfrak{g}$ defined by $d\alpha(X,Y) = -1/2\alpha([X,Y])$ and $o = eK \in G/K$.
Moreover, regular elements $Z \in C$ from a T-Weyl chamber C correspond to the Kähler forms ω_Z with respect to the complex structure $J(C)$ associated to C, that is they define an invariant Kähler structure $(\omega_Z, J(C))$. The 2-form $\frac{1}{2\pi}\omega_Z$ is integral if the 1-form $B \circ Z$ has integer coordinates with respect to the fundamental weights π_i associated with the system of black simple roots $\beta_i \in \Pi_B$.

Recall that if $\Pi_W = \{\alpha_1, \dots, \alpha_m\}$ (resp. $\Pi_B = \{\beta_1, \dots, \beta_k\}$) is the set of white (resp. black) simple roots, then the fundamental weight π_i associated with β_i, $i = 1, \dots, k$, is the linear form defined by

$$\frac{2\langle \pi_i, \beta_j \rangle}{\|\beta_j\|^2} = \delta_{ij}, \quad \langle \pi_i, \alpha_j \rangle = 0. \tag{64}$$

where $< .,. >$ is the scalar product in $i\mathfrak{c}^* = \mathrm{span}(R)$ induced by the Killing form. The B-dual to π_i vectors h_i form a basis of $\mathfrak{t}$.
Let $E_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in R$, be the *Chevalley basis* of $\mathfrak{g}(R)$ such that $B(E_\alpha, E_{-\alpha}) = \frac{2}{<\alpha,\alpha>}$ We denote by $\omega_\alpha = B \circ E_\alpha$ the dual basis of 1-forms. Then for $Z \in \mathfrak{t}$

$$\omega_Z = -i \sum_{\alpha \in R^+_{\mathfrak{m}}} \frac{2\alpha(Z)}{< \alpha, \alpha >} \omega_\alpha \wedge \omega_{-\alpha} \tag{65}$$

Indeed,

$$\begin{aligned}
i\, d(B \circ Z)(E_\alpha, E_{-\alpha}) = \quad & -\tfrac{i}{2} B(Z, [E_\alpha, E_{-\alpha}]) \\
= \quad & -\tfrac{i}{2} B([Z, E_\alpha], E_{-\alpha}) \\
= \quad & -\tfrac{i}{2} \alpha(Z) B(E_\alpha, E_{-\alpha}) \\
= \quad & -\tfrac{i\alpha(Z)}{<\alpha,\alpha>} \\
= \quad & -2i \tfrac{\alpha(Z)}{<\alpha,\alpha>} \omega_\alpha \wedge \omega_{-\alpha}(E_\alpha, E_{-\alpha}).
\end{aligned}$$

Definition 10 *The 1-form*

$$\sigma = \sum_{\beta \in R_{\mathfrak{m}}^{+}} \beta \in \mathfrak{t}^{*} \subset i\mathfrak{c}^{*}$$

is called the Koszul form and the dual vector $Z^{Kos} := B^{-1} \circ \sigma$ is called the Koszul vector.

Proposition 11 *[5] The* **Koszul vector** *Z^{Kos} defines the invariant Kähler-Einstein structure $(\omega_{Z^{Kos}}, J(C))$ on $F = G/K$, where $J(C)$ is the invariant complex structure associated with the T-Weyl chamber C which is defined by Π_B.*

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