WEAKLY BIHARMONIC MAPS FROM THE BALL TO THE SPHERE

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ABSTRACT. The aim of this paper is to investigate the existence of proper, weakly biharmonic maps within a family of rotationally symmetric maps $u_a : B^n \to \mathbb{S}^n$, where B^n and $\mathbb{S}^n$ denote the Euclidean n-dimensional unit ball and sphere respectively. We prove that there exists a proper, weakly biharmonic map u_a of this type if and only if $n = 5$ or $n = 6$. We shall also prove that these critical points are unstable.

1. Introduction

Harmonic maps are the critical points of the *energy* functional

$$E(u) = \frac{1}{2} \int_M |du|^2 \, dv_g \ , \tag{1.1}$$

where $u : M \to N$ is a smooth map between two Riemannian manifolds (M, g) and (N, h) of dimension m and n respectively (we refer to [5, 6] for background on harmonic maps). In analytical terms, the condition of harmonicity is equivalent to the fact that the map u is a solution of the Euler-Lagrange equation associated to the energy functional (1.1), i.e.

$$\tau(u) = -d^* d\, u = \operatorname{trace} \nabla du = 0 \ . \tag{1.2}$$

The left member $\tau(u)$ of (1.2) is a vector field along the map u or, equivalently, a section of the pull-back bundle $u^{-1}(TN)$: it is called *tension field*. Its expression with respect to local coordinates is given by

$$[\tau(u)]^k = \Delta u^k + g^{ij} \ {}^N\Gamma^k_{\ell p} \frac{\partial u^\ell}{\partial x_i} \frac{\partial u^p}{\partial x_j} \ , \quad 1 \leq k \leq n \ , \tag{1.3}$$

where the Einstein convention on the sum over repeated indices is used, ${}^N\Gamma^k_{\ell p}$ are the Christoffel symbols of (N, h) and the Laplacian on (M, g) is:

$$\Delta u^k = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} \left(\sqrt{|g|} \, g^{ij} \frac{\partial u^k}{\partial x_j} \right) \ . \tag{1.4}$$

A related topic of growing interest deals with the study of the so-called *biharmonic maps*. These maps, which provide a natural generalization of harmonic maps, are the critical points of the *bienergy functional* (as suggested in [6], [7])

$$E_2(u) = \frac{1}{2} \int_M |\tau(u)|^2 \, dv_g \ . \tag{1.5}$$

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There have been extensive studies on biharmonic maps (see [4, 15, 17] for an introduction to this topic and [11, 18, 19, 20, 21] for an approach which is related to this paper). In particular, in 1986 Jiang [15] obtained the first and the second variational formulas for the bienergy functional (1.5). Clearly, any harmonic map is trivially biharmonic and an absolute minimum for the bienergy functional. Therefore, we say that a (weakly) biharmonic map is *proper* if it is *not* (weakly) harmonic. Note that the notion of a *weak* solution requires the introduction of suitable Sobolev's spaces: this will be detailed in Section 2 below.
Let B^n and $\mathbb{S}^n$ denote the $n-$dimensional Euclidean unit ball and sphere respectively. The main aim of this paper is to study the following family of rotationally symmetric maps:

$$\begin{array}{rcl} u_a : \quad B^n & \to & \mathbb{S}^n \subset \mathbb{R}^n \times \mathbb{R} \\ x & \mapsto & (\sin a \; \frac{x}{r}, \, \cos a) \ , \end{array} \tag{1.6}$$

where $r = |x|$ and a is a constant value in the interval $(0, \pi/2)$. Of course, u_a is well-defined and smooth away from the origin O. Note that we do not study the case $a = \pi/2$ because, if $n \geq 3$, that would give rise to the well-known *weakly harmonic* equator map (see [13]). Our main existence result is the following:

Theorem 1.1. *Let u_a be a map as in* (1.6)*. Then u_a is a proper, weakly biharmonic map if and only if either*

(i) $n = 5$ *and* $a = \pi/3$*; or*
(ii) $n = 6$ *and* $a = (1/2)\arccos(-4/5)$*.*

Next, we state our result concerning the stability of these critical points:

Theorem 1.2. *Let u_a be one of the two proper, weakly biharmonic maps of Theorem 1.1. Then u_a is unstable.*

Remark 1.3. It follows from Theorem 1.2 that the biharmonic maps of Theorem 1.1 are not minimizers for the bienergy functional (the notion of minimizing biharmonic maps will be detailed in Section 2).

Our work is organized as follows: in order to make this work reasonably self-contained, in Section 2 we recall some basic facts about Sobolev's spaces, weak solutions and stability. In Section 3 we shall prove Theorems 1.1 and 1.2.

2. Preliminaries

First, we introduce the most convenient setting to study maps of the type (1.6). Let (M, g) be an m-dimensional compact Riemannian manifold with boundary ∂M and $u : M \to \mathbb{S}^n$. We consider the canonical embedding $i : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ and still write $u = (u_1, \ldots, u_{n+1})$ for $i \circ u$. We shall use the following notation:

$$\nabla u = (\nabla u_1, \ldots, \nabla u_{n+1}) \quad \text{and} \quad \Delta u = (\Delta u_1, \ldots, \Delta u_{n+1}) \ , \tag{2.1}$$

where ∇ is the gradient on (M, g) and the Laplacian Δ acts on functions as specified in (1.4) (note that each entry of ∇u is an m-dimensional vector). Next, let p denote a positive integer. In this context we introduce the Sobolev spaces (see [1, 8])

$$W^{p,2}\left(M, \mathbb{S}^n\right) = \left\{ u \in W^{p,2}\left(M, \mathbb{R}^{n+1}\right) \ : \ u(x) = (u_1(x), \ldots, u_{n+1}(x)) \in \mathbb{S}^n \text{ a.e.} \right\} \ . \tag{2.2}$$

The energy functional (1.1) becomes

$$E(u) = \frac{1}{2} \int_M |\nabla u|^2 \, dv_g \tag{2.3}$$

and its Euler-Lagrange equation (1.2) takes the form

$$\Delta u + |\nabla u|^2 \, u = 0 \ . \tag{2.4}$$

Now, we say that a map $u \in W^{1,2}(M, \mathbb{S}^n)$ is *weakly* harmonic if it is a critical point of (2.3) in $W^{1,2}(M, \mathbb{S}^n)$, i.e., if it is a solution of (2.4) in the sense of distributions. Let $u_0 \in W^{1,2}(M, \mathbb{S}^n)$: we define

$$W^{p,2}_{u_0}(M, \mathbb{S}^n) = \left\{ u \in W^{p,2}(M, \mathbb{S}^n) \ : \ \nabla^k (u - u_0)\big|_{\partial M} \equiv 0, \, 0 \leq k \leq p-1 \right\} \ , \tag{2.5}$$

where the boundary condition in (2.5) is understood in the sense of traces. A typical class of weakly harmonic maps is that of *minimizers* for the energy functional. More precisely, we say that $u_0 \in W^{1,2}(M, \mathbb{S}^n)$ is a minimizer if it satisfies

$$E(u_0) \leq E(v) \quad \forall \, v \in W^{1,2}_{u_0}(M, \mathbb{S}^n) \ .$$

Existence and regularity of weakly harmonic maps is an important area of research. For instance, F. Hélein [9] has shown that, if $m = 2$, then any weakly harmonic map is smooth. By contrast, if $m \geq 3$, there exist weakly harmonic maps into the sphere which are everywhere discontinuous (see [22]). Let $\mathbb{S}^n_+ = \{ y \in \mathbb{S}^n \ : \ y_{n+1} > 0 \}$ be the open hemisphere. If $u : M \to \mathbb{S}^n_+$ is weakly harmonic and $u(M)$ is contained in a compact set of $\mathbb{S}^n_+$, then u is smooth (see [10]). In particular, no map of the type (1.6) can be weakly harmonic if $0 < a < \pi/2$. The previous regularity result does not hold for the closed hemisphere $\overline{\mathbb{S}^n_+}$. Indeed, the equator map (i.e., u_a defined as in (1.6) with $a = \pi/2$) is discontinuous and weakly harmonic if $n \geq 3$. Moreover, we know that the equator map is a minimizer if and only if $n \geq 7$ (see [13]).
As for the bienergy functional (1.5), in our context its expression becomes (see [2, 23])

$$E_2(u) = \frac{1}{2} \int_M \left(|\Delta u|^2 \, - \, |\nabla u|^4 \right) \, dv_g \tag{2.6}$$

and its Euler-Lagrange equation is given by

$$\Delta^2 u + 2 \, \mathrm{div} \left(|\nabla u|^2 \, \nabla u \right) + \left(|\Delta u|^2 + \Delta \, |\nabla u|^2 + 2 \, \nabla u \cdot \nabla \Delta u + 2 \, |\nabla u|^4 \right) \, u = 0 \ , \tag{2.7}$$

where the divergence operator div is applied to each component and $\cdot$ denotes scalar product in the following sense:

$$\nabla u \cdot \nabla \Delta u = \sum_{j=1}^{n+1} \nabla u_j \cdot \nabla \Delta u_j \ .$$

Next, we say that a map $u \in W^{2,2}(M, \mathbb{S}^n)$ is *weakly* biharmonic if it is a critical point of (2.6) in $W^{2,2}(M, \mathbb{S}^n)$, i.e., if it is a solution of (2.7) in the sense of distributions. Again, a typical class of weakly biharmonic maps is that of *minimizers* for the bienergy functional. Indeed, we say that $u_0 \in W^{2,2}(M, \mathbb{S}^n)$ is a minimizer if it satisfies

$$E_2(u_0) \leq E_2(v) \quad \forall \, v \in W^{2,2}_{u_0}(M, \mathbb{S}^n) \ .$$

The regularity of weakly biharmonic maps is an interesting topic. In particular, when $n \leq 3$, every biharmonic map is smooth as a consequence of the injection theorem of Sobolev. More generally, in this case Uhlenbeck [24] has proved regularity for biharmonic maps which belong to the Sobolev spaces $W^{2,p}$ for some $p > n/2$. When $n = 4$, the regularity of weakly biharmonic maps was proved in [14, 25]. In the case that $n \geq 5$ there is not a general theorem on the regularity of weakly biharmonic maps. We cite [12] and the references therein for a study of the regularity of minimizing biharmonic maps: in this paper it is shown that every minimizing biharmonic map from a domain $\Omega \subset \mathbb{R}^n$ to $\mathbb{S}^k$ ($n \geq 5$) is smooth away from a closed set whose Hausdorff dimension is at most $n-5$. An important step towards the understanding whether a given weakly biharmonic map is a minimizer consists in studying its *stability*. More precisely, let $u \in W^{2,2}(M, \mathbb{S}^n)$ be a weakly biharmonic map and denote by u_s ($s \geq 0$) a variation of u through maps in $W^{2,2}_u(M, \mathbb{S}^n)$. We say that u is *stable* if

$$\frac{d^2}{ds^2} E_2(u_s)\Big|_{s=0} \geq 0 \tag{2.8}$$

for all variations u_s. In particular, if u is *not* stable, then it cannot be a minimizer.

3. Proofs of the results

Proof of Theorem 1.1. A map of type (1.6) is smooth and not harmonic on $B^n \backslash \{O\}$. If u_a is weakly biharmonic on B^n, then it must be a strong solution of (2.7) on $B^n \backslash \{O\}$. First, we observe that the $(n+1)$ component of u_a is a non-zero constant. Thus, it is immediate to conclude that the $(n+1)$ component of the first two terms in (2.7) vanishes. It follows that, if u_a is a solution of (2.7), then

$$|\Delta u_a|^2 + \Delta |\nabla u_a|^2 + 2\, \nabla u_a \cdot \nabla \Delta u_a + 2\, |\nabla u_a|^4 = 0. \tag{3.1}$$

Now, we want to compute directly the single terms in (3.1). To this purpose, first we establish a general lemma which will also be useful in the study of the second variation.

Lemma 3.1. *Let $r = |x|$. Let $u : B^n \backslash \{O\} \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be a map of the following form:*

$$x = (x_1, \dots, x_n) \mapsto (p(r)\, x, q(r)) = (p(r)\, x_1, \dots, p(r)\, x_n, q(r)) \ , \tag{3.2}$$

where $p(r)$ and $q(r)$ are smooth functions for $r > 0$. Then

$$\begin{aligned}
\Delta u &= \left(\left[p'' + \frac{(n+1)}{r} p' \right] x_1, \dots, \left[p'' + \frac{(n+1)}{r} p' \right] x_n, \left[q'' + \frac{(n-1)}{r} q' \right] \right) \\
|\Delta u|^2 &= \left[p'' + \frac{(n+1)}{r} p' \right]^2 r^2 + \left[q'' + \frac{(n-1)}{r} q' \right]^2 \\
\nabla u &= \left(\frac{p'\, x_1}{r} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} p \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \frac{p'\, x_2}{r} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ p \\ \vdots \\ 0 \end{bmatrix}, \dots, \frac{p'\, x_n}{r} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ p \end{bmatrix}, \frac{q'}{r} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right) \\
|\nabla u|^2 &= r^2\, p'^2 + n\, p^2 + 2r\, p\, p' + q'^2 \ .
\end{aligned} \tag{3.3}$$

Proof. The proof is a straightforward computation which can be carried out by using the following standard equalities:

$$\begin{aligned}
\nabla p(r) &= p'(r)\,\frac{x}{r} \\
\Delta p(r) &= p''(r) + \frac{(n-1)}{r}\,p'(r) \\
\Delta(fg) &= f\Delta g + g\Delta f + 2\langle \nabla f, \nabla g\rangle\ .
\end{aligned}$$

$\square$

Since u_a in (1.6) and Δu_a are maps of type (3.2), using Lemma 3.1 and computing we find:

$$\begin{aligned}
&|\Delta u_a|^2 = (n-1)^2\ \frac{\sin^2 a}{r^4} \\
&\Delta\,|\nabla u_a|^2 = (n-1)(8-2n)\ \frac{\sin^2 a}{r^4} \\
&2\,\nabla u_a \cdot \nabla\Delta u_a = -\,2\,(n-1)^2\ \frac{\sin^2 a}{r^4} \\
&2\ |\nabla u_a|^4 = 2\,(n-1)^2\ \frac{\sin^4 a}{r^4}
\end{aligned} \tag{3.4}$$

By using (3.4) we find that the vanishing of the expression (3.1) is equivalent to

$$(n-1)\,\frac{\sin^2 a}{r^4}\ \left[(n-1)+(8-2n)-2(n-1)+2(n-1)\sin^2 a\right] \equiv 0\ ,$$

i.e.,

$$\cos(2a) = \frac{2(n-4)}{(1-n)}\ . \tag{3.5}$$

By way of summary, a map of the type (1.6) can be a solution of (2.7) on $B^n\backslash\{O\}$ only if (3.5) holds. Since $0 < a < (\pi/2)$, the only possibilities are:

$$\text{(i)}\ n=4 \text{ and } a=\frac{\pi}{4}\ ;\ \text{(ii)}\ n=5 \text{ and } a=\frac{\pi}{3}\ ;\ \text{(iii)}\ n=6 \text{ and } a=\frac{1}{2}\arccos(-4/5)\ . \tag{3.6}$$

Next, assume that u_a satisfies (3.1), that is (3.5) holds. Then, according to (2.7), u_a is biharmonic if and only if

$$\Delta^2 u_a + 2\,\mathrm{div}\left(|\nabla u_a|^2\ \nabla u_a\right) = 0\ . \tag{3.7}$$

By using again Lemma 3.1 we compute the two terms in (3.7) and find:

$$\begin{aligned}
&\Delta^2 u_a = 3\,(n-1)(n-3)\ \frac{u}{r^4} \\
&2\,\mathrm{div}\left(|\nabla u_a|^2\ \nabla u_a\right) = -2\,(n-1)^2\,\sin^2 a\ \frac{u}{r^4}
\end{aligned} \tag{3.8}$$

By replacing (3.8) in (3.7) we find that a map u_a which satisfies (3.5) is biharmonic on $B^n\backslash\{O\}$ if and only if

$$3\,(n-1)(n-3) - 2\,(n-1)^2\,\sin^2 a = 0$$

which turns out to be equivalent to (3.5). By way of conclusion, we have verified that a map u_a of type (1.6) is a smooth, proper biharmonic map on $B^n \backslash \{O\}$ if and only if one of the instances in (3.6) holds. Since maps of this type are strong solutions on B^n except at zero, we conclude that they are proper, weakly biharmonic on B^n if and only if they belong to the Sobolev space $W^{2,2}\left(B^n, \mathbb{S}^n\right)$. Next, we observe that the requirement $u_a \in W^{2,2}\left(B^n, \mathbb{S}^n\right)$ is equivalent to

$$\int_{B^n} |\nabla u_a|^2 \, dv_g < +\infty \quad \text{and} \quad \int_{B^n} |\Delta u_a|^2 \, dv_g < +\infty \, . \tag{3.9}$$

By using (3.4) it is easy to verify that the conditions in (3.9) become:

$$\mathrm{Vol}\left(\mathbb{S}^{n-1}\right) \int_0^1 (n-1)(\sin a)^2 \, r^{n-3} \, dr < +\infty \, ; \quad \mathrm{Vol}\left(\mathbb{S}^{n-1}\right) \int_0^1 (n-1)^2(\sin a)^2 \, r^{n-5} \, dr < +\infty \, .$$

It follows that (3.9) is verified if and only if $n \geq 5$. We deduce that the solution in (3.6) (i) is not acceptable and the conclusion of Theorem 1.1 follows immediately. □

Remark 3.2. The notion of biharmonicity that we study in this paper is *intrinsic*, i.e., it does not depend on the choice of the embedding of $\mathbb{S}^n$ into $\mathbb{R}^{n+1}$. We point out that, in the recent literature, several authors have considered an extrinsic version of the bienergy (often called the *Hessian energy*), that is

$$H(u) = \frac{1}{2} \int_M |\Delta u|^2 \, dv_g \ . \tag{3.10}$$

The study of existence, regularity and minimizing properties of the critical points of (3.10) is a difficult topic of rapidly growing interest. For instance, see [3, 11, 12, 23, 25] and references therein for more details. Here we limit ourselves to say that, by using (3.4) and (3.7), it is easy to verify that a map u_a of the type (1.6) is a smooth critical point for the Hessian energy (3.10) on $B^n \backslash \{O\}$ if and only if $n = 3$ (for all $a \in (0, \pi/2)$), but these solutions are *not* weak critical points on B^3 because they do not belong to $W^{2,2}\left(B^3, \mathbb{S}^3\right)$.

Proof of Theorem 1.2. In order to prove that u_a is unstable it suffices to exhibit a variation $u_{a,s}$ of u_a $(u_{a,0} = u_a)$ such that

$$\left. \frac{d^2}{ds^2} E_2\left(u_{a,s}\right) \right|_{s=0} < 0 \ . \tag{3.11}$$

For our purposes, we can use variations of the following type:

$$u_{a,s} = \Big(\sin(a + sV(r)) \, \frac{x}{r}, \, \cos(a + sV(r)) \Big) \qquad (s \geq 0) \ , \tag{3.12}$$

where $V(r)$ is a smooth function on $[0,1]$ such that $V(1) = V'(1) = 0$. For each fixed s, a map $u_{a,s}$ as in (3.12) is of the type (3.2) with $p(r) = \sin(a + sV(r))/r$ and $q(r) = \cos(a + sV(r))$.

Therefore, after a straightforward computation based again on Lemma 3.1 we obtain

$$
\begin{aligned}
|\Delta u_{a,s}|^2 =& \frac{1}{r^4}\Big((n-1)\sin(a+sV)-(n-1)rs\cos(a+sV)V' \\
&+r^2s^2\sin(a+sV)V'^2-r^2s\cos(a+sV)V''\Big)^2 \\
&+\frac{r^2s^2\Big((n-1)\sin(a+sV)V'+rs\cos(a+sV)V'^2+r\sin(a+sV)V''\Big)^2}{r^4}
\end{aligned}
$$

and

$$
|\nabla u_{a,s}|^4 = \frac{\Big((n-1)\sin^2(a+tV)+r^2s^2V'^2\Big)^2}{r^4}\,.
$$

Using these expressions and simplifying we find the expression for the bienergy:

$$
\begin{aligned}
E_2\left(u_{a,s}\right) &= \frac{1}{2}\int_{B^n}\left(|\Delta u_{a,s}|^2 - |\nabla u_{a,s}|^4\right)dv_g \\
&= \int_{B^n}\frac{\left[(n-1)\sin(a+sV)\cos(a+sV)-(n-1)s\,r\,V'-s\,r^2\,V''\right]^2}{2\,r^4}\,dv_g\,. \qquad (3.13)
\end{aligned}
$$

By using (3.13) we find:

$$
\begin{aligned}
\frac{d^2}{ds^2}E_2\left(u_{a,s}\right)\Big|_{s=0} = \frac{1}{4}\mathrm{Vol}\left(\mathbb{S}^{n-1}\right) \quad & \int_0^1\Big(\left[2(n-1)\cos(2a)V-2(n-1)rV'-2r^2V''\right]^2 \\
& -4(n-1)^2\sin^2(2a)V^2\Big)r^{n-5}dr\,. \qquad (3.14)
\end{aligned}
$$

Now we are in the right position to complete the proof of Theorem 1.2. We study the two cases separately. First, we assume $n=5$, $a=\pi/3$ and use $V(r)=(1-r^2)^4$ in (3.14). We obtain

$$
\begin{aligned}
\frac{d^2}{ds^2}E_2\left(u_{a,s}\right)\Big|_{s=0} &= 8\,\mathrm{Vol}\left(\mathbb{S}^4\right)\int_0^1\left(1-r^2\right)^4\left(1011r^8-984r^6+278r^4-16r^2-1\right)\,dr \\
&= -\frac{32768}{17017}\,\mathrm{Vol}\left(\mathbb{S}^4\right)<0\,.
\end{aligned}
$$

Next, we assume $n=6$, $a=(1/2)\arccos(-4/5)$ and use $V(r)=(1-r^2)^{18}$ in (3.14). We obtain

$$
\begin{aligned}
\frac{d^2}{ds^2}E_2\left(u_{a,s}\right)\Big|_{s=0} &= \mathrm{Vol}\left(\mathbb{S}^5\right)\int_0^1 r\left(1-r^2\right)^{32}\left(2085127r^8-646876r^6+61674r^4-1756r^2+7\right)dr \\
&= -\frac{9}{28490}\,\mathrm{Vol}\left(\mathbb{S}^5\right)<0
\end{aligned}
$$

so that the proof of Theorem 1.2 is complete. □

Remark 3.3. It was proved in Theorem 1.1.1 of [23] that every extrinsic biharmonic map with values in a compact set of $\mathbb{S}^n_+$ such that $\Delta u_{n+1}\leq 0$ a.e. is a minimizer for the Hessian energy. By contrast, since the examples of our Theorem 1.1 satisfy these conditions but they are unstable, we see that the conclusion of Theorem 1.1.1 of [23] does not hold for the case of the intrinsic energy.

Remark 3.4. Let $u_a : B^5 \to \mathbb{S}^5$ be the proper, weakly biharmonic map of Theorem 1.1. For any fixed positive integer p we can define a new map $U_a : B^5 \times \mathbb{S}^p \to \mathbb{S}^5 \times \mathbb{S}^p$ by setting

$$U_a(x, w) = (u_a(x), w)$$

for all $x \in B^5$, $w \in \mathbb{S}^p$. Then U_a is a proper, weakly biharmonic map from an n-dimensional manifold ($n = p + 5$) which is discontinuous on a set of Hausdorff dimension $n - 5$. By the same argument of Theorem 1.2 these maps are unstable.

Remark 3.5. A detailed study of the second variation operator associated to smooth biharmonic maps into $\mathbb{S}^n$ can be found in [16].

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