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REMARKS ON THE ORDER-THEORETICAL AND ALGEBRAIC PROPERTIES OF QUANTUM STRUCTURES

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Remarks on the order-theoretical and algebraic properties of quantum structures

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Declaration of Authorship

I hereby declare that Chapters 3, 4 and 6 contents are excerpts of [17], [18] and [20], respectively. Chapter 5 si taken from [19]. I take full responsibility of any other content reported in this thesis.

Signed: ______________________________

Date: ______________________________

“We live on a placid island of ignorance in the midst of black seas of infinity, and it was not meant that we should voyage far.”

H. P. Lovecraft

Abstract

This thesis is concerned with the analysis of common features and distinguishing traits of algebraic structures arising in the sharp as well as in the unsharp approaches to quantum theory both from an order-theoretical and an algebraic perspective. Firstly, we recall basic notions of order theory and universal algebra. Furthermore, we introduce fundamental concepts and facts concerning the algebraic structures we deal with, from orthomodular lattices to effect algebras, MV algebras and their non-commutative generalizations. Finally, we present Basic algebras as a general framework in which (lattice) quantum structures can be studied from an universal algebraic perspective.

Taking advantage of the categorical (term-)equivalence between Basic algebras and Łukasiewicz near semirings, in Chapter 3 we provide a structure theory for the latter in order to highlight that, if turned into near-semirings, orthomodular lattices, MV algebras and Basic algebras determine ideals amenable of a common simple description. As a consequence, we provide a rather general Cantor-Bernstein Theorem for involutive left-residuable near semirings.

In Chapter 4, we show that lattice pseudoeffect algebras, i.e. non-commutative generalizations of lattice effect algebras can be represented as near semirings. One one side, this result allows the arithmetical treatment of pseudoeffect algebras as total structures; on the other, it shows that near semirings framework can be really seen as the common "ground" on which (commutative and non commutative) quantum structures can be studied and compared.

In Chapter 5 we show that modular paraorthomodular lattices can be represented as semiring-like structures by first converting them into (left-) residuated structures. To this aim, we show that any modular bonded lattice $\mathbf{A}$ with antitone involution satisfying a strengthened form of regularity can be turned into a left-residuated groupoid. This condition turns out to be a sufficient and necessary for a Kleene lattice to be equipped with a Boolean-like material implication.

Finally, in order to highlight order theoretical peculiarities of orthomodular quantum structures, in Chapter 6 we weaken the notion of orthomodularity for posets by introducing the concept of the generalized orthomodularity property (GO-property) expressed in terms of LU-operators. This seemingly mild generalization of orthomodular posets and its order theoretical analysis yields rather strong applications to effect algebras, and orthomodular structures. Also, for several classes of orthoalgebras, the GO-property yields a completely order-theoretical characterization of the coherence law and, in turn, of proper orthoalgebras.

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To the great blind jumpers.

Introduction

The significant direction of research aimed at exploring algebraic properties of quantum structures grew up in the “cradle” of the so-called logico-algebraic approach to Quantum Theory (QT). This tradition, having as its starting point the famous paper by G. Birkhoff and J. von Neumann “The logic of quantum mechanics” (1936) [5], finds its roots in the idea according to which physical theories are significantly characterized by their abstract mathematical structure. In other words, it develops in a tradition of conceptual analysis of physical explanations in which the ideal is to characterize questions of theoretical relevance so sharply that they admit of formally precise answers which can be found by a deep understanding of physics' mathematical machinery (cf. [79, p. VIII]).

One of the aspects of QT which has attracted the most general attention, is the novelty of the logical notions which it presupposes. It asserts that even a complete mathematical description of a physical system Σ does not in general enable one to predict with certainty the result of an experiment on Σ. Therefore, it clashes with the essentially predictive nature of classical mechanics. An exhaustive and deep discussion on the history and development of quantum theory can be found e.g. in [79].

For a long time, orthomodular lattices have been considered as abstract algebraic representatives of the logic of quantum experimental propositions. Their concrete mathematical counterpart is given by ortholattices of closed subspaces of Hilbert spaces. However, later on it was recognized that this framework needed to be revised. In fact, according to Mackey's axiomatization of QT (see [91]), joins (i.e. disjunctions in the corresponding logic) need not exist provided the elements in question are not orthogonal (see below). For this reason, orthomodular posets were defined with the aim of faithfully formalizing event structures of quantum mechanical systems (see e.g. [3, 87] for a detailed account). This class of orthoposet has been deeply studied in the so called *sharp approach* to QT. In spite of the large amount of researches on the subject, some doubts concerning the relevance of orthomodular lattices for the algebraic investigation of quantum mechanics arose when it was discovered that the lattices of projection operators on Hilbert spaces do not generate the whole variety of orthomodular lattices (see e.g. [33]). This shows that there are equational properties of event-state systems that cannot be captured by the proposed mathematical abstraction. Moreover, in [77] it has been shown that orthomodular posets are not strictly tied to the algebraic properties of Hilbert spaces. In fact, they are strongly related to decompositions of algebras and sets, since in general the set of pairs of "factor" equivalence relations on a set A carries quite naturally the structure of an orthomodular poset. This result seems to limit the extent of the celebrated Amemiya-Araki-Piron Theorem ([1]) stating that the lattice $\mathcal{C}(\mathcal{V})$ of closed subspaces of a normed vector space $\mathcal{V}$ is orthomodular if and only if $\mathcal{V}$ is metrically complete. It proves indeed that the source of orthomodularity cannot be recognized in the particular properties of Hilbert spaces. Therefore, it would not be too far from truth concluding that orthomodularity should be considered an accident rather than the substance of the lattice of quantum properties.

During the last years, alternative approaches have appeared. Let us mention e.g. the so-called weakly orthomodular and dually weakly orthomodular lattices [24] which are able to capture e.g. the order theoretical properties of the lattice of closed subspaces (which need not be orthomodular) of a finite dimensional vector space over a finite field. A different formal counterpart of the set of quantum events has been suggested within the *unsharp* approach to quantum theory by effect algebras [47] and quantum MV-algebras (see e.g. [58]) that ensure the algebraic treatment of quantum effects, namely linear bounded self-adjoint operators on a Hilbert space satisfying the Born's rule. These structures have been the subject of increasing and intensive inquiries. In particular, special attention has been paid to the problem of providing a "*basis for deciding precisely when and exactly how an experimental, observational, or operational situation, either real or idealized and either practical or contrived, gives rise to events, questions, propositions, or observables that can be regarded as fuzzy or unsharp and that are represented by elements of an effect algebra*" (see [49, p. 2]). An answer has been provided in 2001 by D.J. Foulis and S. Gudder who introduced *D-models with parameters* ([49, p.18]). These "toy" models are aimed at interpreting effect algebras as logics of propositions concerning (fuzzy) events related to the calibration of measuring devices for quantum experiments. Roughly speaking, in this framework effects are representatives of sets of statements (events) of the form "the *real* measure of the observable A ranges over the Borel set Δ with probability distribution χ_Δ and the measured value obtained by the measuring device D ranges over Γ with probability distribution χ_Γ". Therefore, studies on effect algebras can be framed within the so-called stochastic approach to quantum mechanics. During the last years, several efforts have been made in order to capture the logic of effect algebras. However, several difficulties have been encountered. In fact, the *canonical* order (CO) induced by the Born's rule (see [33, Cap. 4]) on the set $\mathcal{E}(\mathcal{H})$ of

effects on a Hilbert space $\mathcal{H}$ does not ensure that meets and joins of elements always exist. In fact, a general result by S. Gudder states that any Hilbert space $\mathcal{H}$ of dimension $n \geq 2$ yields an effect algebras $\mathcal{E}(\mathcal{H})$ which is *proper* i.e. it is neither lattice-order nor lattice-orderable via Dedekind-MacNeille completion of its underlying poset. Therefore, there is no hope of finding a semantics of connectives for effect algebras which generalizes properly the orthomodular case. In order to cope with this problem, lattice effect algebras have been introduced. Interestingly enough, these structures turn out to be deeply related to MV algebras and, in general, residuated structures (see e.g. [112, 31]) since they are pastings of MV (effect) algebras (see Chapter 2).
In [59, 60] paraorthomodular lattices, i.e. regular bounded lattices with an antitone involution satisfying the paraorthomodular law have been introduced. They are indeed a natural generalization of the lattice ordering on closed subspaces of a Hilbert space to the whole class of effects by means of the so-called *spectral* ordering introduced by M.P. Olson in 1971 [103]. Such an approach capture an important order theoretical property of quantum structures, i.e. *paraorthomodularity* (see [59] and Chapter 2). This notion turns out to be a cross-cutting concept in quantum structures framework. However, at the best of our knowledge, it is still an open problem if it might be considered as a distinguishing trait of algebras of (quantum) events.

In spite of the large amount of research on quantum structures and their multiple relationships with many-valued logics, there had never been a serious effort in producing an unifying theory until the very beginning of the 21st century. In particular, *basic algebras* were introduced as a general framework in which lattice effect algebras, orthomodular lattices and MV algebras can be put under a same formal umbrella by observing that any of the aforementioned structures induce an underlying lattice which can be equipped with sectional antitone involutions ([21]). One of the most fruitful obtainments yielded by the theory of basic algebras is in that it allows to establish connections between quantum structures and residuation theory. In fact, it has been shown that any basic algebra can be made into a (partially) residuated lattice-ordered groupoid. This result establishes novel connections between quantum algebras and the algebraic semantics of substructural logics (see [54] for details). Moreover, as residuated lattices are strongly related to semirings, basic algebras can be converted into *near semirings* with involution ([6]). This result allows indeed a smooth arithmetical treatment of basic algebras as well as it links them to structures of prominent importance for applications (see e.g. [64]).

This thesis is concerned with exploring common as well as distinguishing traits of quantum structures and MV algebras. In particular, taking advantage of the semiring-like counterpart of basic algebras, i.e. Łukasiewicz near-semirings, we will provide a general structure theory, i.e. a complete description of kernels of homomorphisms. It will be clear that albeit e.g. orthomodular lattices and MV algebras have different order theoretical as well as algebraic properties, their ideals are amenable of a common description. As an application, we will obtain a rather general proof of the algebraic Cantor-Bernstein-Schroeder Theorem for involutive near semirings which can be equipped with a left-residuated operation. This result generalizes several achievements in orthomodular lattices, MV algebras and Basic algebras theories. Subsequently, we will extend the semiring approach to non-commutative lattice-ordered generalizations of effect algebras, i.e. lattice pseudo effect algebras. This fact will suggest the possibility of considering near-semirings as the common "ground" of algebras arising in contexts which are seemingly different and unrelated. Furthermore, we will address the problem of representing paraorthomodular lattices as near semirings. To this aim, we will study sufficient and necessary conditions for a paraorthomodular lattice to be converted into a partially residuated structure. Then, taking advantage of those results, we extend the Cantor-Bernstein Theorem to modular and distributive lattices with an antitone involution satisfying a strenghtened form of regularity.

In the last chapter of this work, we will generalize orthomodular posets by developing the theory of GO-posets, namely orthoposets enjoying the generalized orthomodularity property. More precisely, GO-posets might be regarded as complemented posets in which l.u.b.'s of pairs of mutually orthogonal elements need not exist. Moreover, they satisfy a poset version of the orthomodular law expressed by means of the LU-operators (see [32]). We will provide a Dedekind-Birkhoff-type theorem for GO-posets by characterizing forbidden configurations under which the generalized orthomodularity property fails. Surprisingly enough, this result will yield the celebrated Greechie's theorems as corollaries (cf. [3]). In fact, It will turn out that the absence of loops of order three is equivalent to the GO-property. Finally, we will put in good use the GO-machinery in order to provide a completely order-theoretical characterization of the coherence law for some classes of effect algebras. As the semiring like approach allows to capture the common algebraic traits of quantum structures, the generalized orthomodular posets framework will shed a new light upon characterizing properties of structures arising in the sharp and unsharp approaches to quantum theory.

Chapter 1

Mathematical Background

In this chapter, basic facts, definitions and notations concerning mathematical tools and structures having a relevant role in this thesis will be recalled.

1.1 Basics on set theory and order theory

In what follows, we assume the reader has knowledge of basic set theoretical and arithmetical notions. We will work in the framework of standard set theory, since no further assumptions are needed.

As regards notation, we will employ customary symbols denoting basic set-theoretical operations and relations, i.e. $\emptyset$ for the empty set, $\subseteq$ for inclusion, $\subset$ (sometimes denoted by $\subsetneq$) for *proper* inclusion, $\cap$ and $\cup$ for intersection resp. union of sets. Moreover, if A and B contain the same elements, we will write $A = B$.

Given a (countable or uncountable) non-empty set of indexes I and an indexed family $\{A_i : i \in I\}$ (or $\{\mathbf{A}_i\}_{i\in I}$) of sets we denote (assuming the Axiom of Choice) by $\Pi_{i\in I}A_i$ the Cartesian product of the A_i's and and we denote its elements (I-tuples) as $(a_0, a_1, ..., a_n, ...)$ or $(a_i : i \in I)$. If $I = \{0, 1, ..., n\}$ we will denote $\Pi_{i\in I}A_i$ by $A_0 \times A_1 \times ... \times A_n$. Moreover, if $A_i = A_j$, for any $i, j \in I$, we will write A^I (A^n, if $I = \{0, 1, ..., n\}$), in order to simplify the above notation and we put $A^0 = \{\emptyset\}$.

If A is a set and $B \subseteq A^I$ (with I countable or uncountable), we say that B is an I-ary relation over A. Moreover, if $R \subseteq A^I$ is an $I - ary$ relation over A and $B \subseteq A$, we will indicate by $R \upharpoonright B^I$ the set of $I - tuples$ whose entries are in B.

If $\{A_i : i \in I\}$ is an indexed family of sets, we indicate as $\bigcup_{i\in I} A_i$ ($\bigcap_{i\in I} A_i$) the set $\{a : a \in A_i, \text{for some } i \in I\}$ ($\{a : a \in A_i, \text{for any } i \in I\}$). Moreover, if A is a set, we denote with $\wp(A)$ the power-set of A, i.e. the set of all its subsets. Finally, if $X \subseteq \wp(A)$, then $\bigcup X = \{a \in A : a \in B, \text{ for some } B \in X\}$ and $\bigcap X = \{a \in A : a \in B, \text{ for any } B \in X\}$.

Functions (mappings) from a set A to a set B will be denoted by $f, g, h, ...$ and e.g. $f : A \to B$. Composition of functions will be denoted, following the standard notation, i.e. if $f : A \to B$ and $g : B \to C$ we will write $g \circ f : A \to C$ (shortly gf) to designate the composition of f and g. Moreover, if $C \subseteq A$ and $f : A \to B$, we will denote by $f \restriction C$ the mapping $g : C \to B$ such that $g(x) = f(x)$, for any $x \in C$. Finally, we call the pre-image $f^{-1}(X)$ of f with respect to $X \subseteq B$, the set $\{x \in A : f(x) \in X\}$.
Moreover, if $B \subseteq A$ and $R \subseteq A^I$, we denote by $R \restriction B^I$ the set of I-tuples in R with entries in B. Furthermore, if $n = 2$, then $(a_0, a_1) \in B \subseteq A^2$ is said to be an *ordered pair* and B a binary relation over A. The following definition summarizes basic properties of binary relations on sets.

Definition 1.1. Let A be a set and $R \subseteq A^2$ a binary relation over A. Then R is

(O1) *reflexive*, provided that $(a, a) \in R$, for any $a \in A$;

(O2) *symmetric*, if $(a, b) \in R$ implies $(b, a) \in R$ for any $a, b \in A$;

(O3) *transitive*, if $(a, b), (b, c) \in R$, then $(a, c) \in R$ for any $a, b, c \in A$;

(O4) *anti-symmetric*, if $(a, b), (b, a) \in R$ implies $a = b$, for any $a, b \in A$;

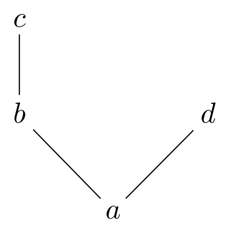
(O5) *connected*, if for any $a, b \in A$, one has $(a, b) \in R$ or $(b, a) \in R$.

Given a binary relation $R \subseteq A^2$, R is said to be

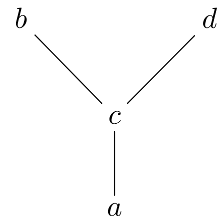
- a *pre-order* provided that R satisfies (O1) and (O3);
- a *partial order*, if it satisfies $(O1), (O3)$ and $(O4)$;
- a *total* order, if it is a partial order satisfying $(O5)$;
- an *equivalence* relation, if it satisfies $(O1) - (O3)$.

It is customary denoting pre- (partial, total) orders by $\leq, \precsim, ...$ *etc.* Moreover, equivalence relations will be often designated by $\backsim, \equiv$ or $\theta, \sigma, \delta, ...$ *etc.*
Given a set A, $a, b \in A$ and an equivalence relation $\backsim$ over A, we will write $a \backsim b$ for saying that $(a, b) \in \backsim$, we set $a/\backsim = \{b \in A : a \backsim b\}$ (or $[a]_\backsim$) for the equivalence class of a modulo $\backsim$ and $A/\backsim = \{a/\backsim : a \in A\}$ for the *quotient* of A modulo $\backsim$. It is well known that equivalence relations over A are into one-to-one corrispondence with *partitions*, being the latter families $X \subseteq \wp(A)$ such that $\bigcup X = A$ and, for any $B, C \in X$, $B \cap C = \emptyset$.
If A is a set and $\leq$ a partial ordered over A, we will call the pair $\mathbf{A} = (A, \leq^{\mathbf{A}})$ a *partially ordered* set (a poset, for short) and A the *universe* of $\leq$. Moreover, if $\mathbf{A}$ is a poset and $B \subseteq A$, then putting $\leq^{\mathbf{B}} = \leq^{\mathbf{A}} \restriction B^2$, we call $\mathbf{B} = (B, \leq^{\mathbf{B}})$ a *sub-poset* of $\mathbf{A}$.

Example 1.1. *Let $A = \{a, b, c, d\}$. We can represent different partial orders over A by Hasse diagrams as e.g. in Fig. 1.1 and 1.2.*



(1.1)



(1.2)

The above graphs must be read as follows: for any $x, y \in A$, put $x \leq y$ if $x = y$ or there exists a bottom-up path of arcs connecting x to y.

Since partially ordered sets will be ubiquitously employed, we further explore their structure and properties in the next section.

1.1.1 Partially ordered sets

in this section we summarize basic facts and notions concerning partially ordered sets that will be crucial for the understanding of subsequent chapters.

Definition 1.2. Given a partially ordered set $\mathbf{A} = (A, \leq)$, then $\mathbf{A}$ is said to be

(a) *bounded*, if it admits a largest upper bound and a least lower bound, namely there are elements usually denoted by 0 resp. 1 such that, for any $x \in A$, $0 \leq x$ and $x \leq 1$;

(b) a *chain*, if $\leq$ is a total order;

(c) *lattice-ordered*, provided that any pair of elements $a, b \in A$ has a *least upper bound* (l.u.b.) and a *greatest lower bound* (g.l.b.) denoted by $a \vee^{\mathbf{A}} b$ resp. $a \wedge^{\mathbf{A}} b$, namely an element c resp. d such that, for any $e \in A$, $e \geq a, b$ implies $c \leq e$ resp. $e \leq a, b$ implies $e \leq d$.

Elements x, y in a poset are said to be *comparable* whenever $x \leq y$ or $y \leq x$ hold true, otherwise they will be said uncomparable and we denote this fact by $x||y$.
If a poset $\mathbf{A} = (A, \leq)$ is lattice-ordered, then it is said to be a lattice. Moreover, if $X \subseteq A$ admits an upper bound and a lower bound, then it is said to be *bounded*. Finally, if X

has a l.u.b. (g.l.b.), then it will be denoted by $\bigvee^{\mathbf{A}} X$ ($\bigwedge^{\mathbf{A}} X$). Whenever the context will be clear, we will omit unnecessary supscripts.
Besides lattices, an important class of posets is represented by the so-called *semilattices.*

Definition 1.3. A poset $\mathbf{A} = (A, \leq)$ is said to be a

- *meet*-semilattice ($\wedge$-semilattice) provided that any pair of elements in A has a g.l.b.;
- *join*-semilattice ($\vee$-semilattice) if any pair of elements in A has a l.u.b. .

Definition 1.4. Let $\mathbf{A} = (A, \leq)$ be a lattice. Then $\mathbf{A}$ is

(i) *conditionally complete*, if whenever X is upper (lower) bounded then $\bigvee X$ ($\bigwedge X$) exists, for any $X \subseteq A$;

(ii) *complete*, if $\bigvee X$ and $\bigwedge X$ exist, for any $X \subseteq A$.

An important example of complete lattice is given by the next lemma. Let $Eq(A)$ be the set of equivalence relations on a set A.

Lemma 1.5. *The poset $Eq(A)$, with $\subseteq$ as the partial ordering, is a complete lattice.*

Obviously, any complete lattice is bounded. Moreover, it can be easily proven that a lattice $\mathbf{A}$ is complete if and only if $\bigvee X$ exists, for any $X \subseteq A$. Clearly, any chain is a lattice but not the other way around. In fact, the poset in Fig. 1.1 is a lattice which is not a chain, since b and d are uncomparable elements. For a lattice-ordered poset $\mathbf{A}$,

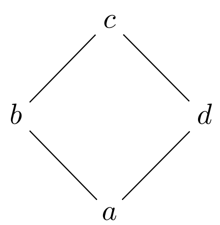


Figure 1.1

an element $a \in A$ is *compact* if whenever $\bigvee X$ exists for some $X \subseteq A$ and $a \leq \bigvee X$,then $a \leq \bigvee Y$ for some finite $Y \subseteq X$. We say that $\mathbf{A}$ is *compactly generated* provided that every element in A is the l.u.b. of compact elements. A lattice is *algebraic* if it is complete and compactly generated.
Among mappings from a partially ordered set A to a partially ordered set B, one can find order preserving functions.

Definition 1.6. Let $\mathbf{A} = (A, \leq)$ and $\mathbf{B} = (B, \leq)$ be partially ordered sets and $f : A \to B$ a mapping. We say that f is

- an *order homomorphism* (*order anti-homomorphism*), if for any $x, y \in A$, one has $x \leq^{\mathbf{A}} y$ implies $f(x) \leq^{\mathbf{B}} f(y)$ ($x \leq^{\mathbf{A}} y$ yields $f(y) \leq^{\mathbf{B}} f(x)$);
- an *order embedding* (*order anti-embedding*), if it is an order homomorphism (order anti-homomorphism) such that, for any $x, y \in A$, one has also $f(x) \leq^{\mathbf{B}} f(y)$ implies $x \leq^{\mathbf{A}} y$ ($f(x) \leq^{\mathbf{B}} f(y)$ implies $y \leq^{\mathbf{A}} x$);
- an *order automorphism* (*order anti-automorphism*), if it is an order embedding (order anti-embedding) which is onto.

In what follows and in the rest of this thesis, order (anti-)homomorphisms will be denoted as $\phi, \psi, ...etc.$
Given a bounded poset $(A, \leq, 0, 1)$ with $B \subseteq A$, we denote by

$$\begin{aligned} U(B) =& \{a \in A : \forall b \in B (b \leq a)\} \\ L(B) =& \{a \in A : \forall b \in B (a \leq b)\} \end{aligned}$$

the *upper* and the *lower set* of B, respectively. In order to ease notation, if A, B are sets, we will just write $U(A, B)$ and $L(A, B)$ for $U(A \cup B)$ and $L(A \cup B)$, respectively. In what follows, for $M \subseteq A$, $U_M(X), L_M(X)$ will stand for the upper and lower sets of X in M, respectively. In any poset $(A, \leq)$, with $C, B \subseteq A$, the following properties hold (see e.g. [32]):

Lemma 1.7. *(i) If $C \subseteq B$, then $U(B) \subseteq U(C)$ and $L(B) \subseteq L(C)$;*

(ii) $U(L(U(B))) = U(B)$ and $L(U(L(B))) = L(B)$;

(iii) $U(B \cup C) = U(B) \cap U(C)$ and $L(B \cup C) = L(B) \cap L(C)$;

(iv) $a \leq b$ is equivalent to $U(b) \subseteq U(a)$, which is equivalent to $L(a) \subseteq L(b)$.

A moment's reflection shows that $a = b$ if and only if $U(a) = U(b)$ if and only if $L(a) = L(b)$, and that $U(L(a)) = U(a)$, $L(U(a)) = L(a)$. Moreover, if $\bigvee B$ exists in $(A, \leq)$, then $U(B) = U(\bigvee B)$ and if $\bigwedge B$ exists in $(A, \leq)$, then $L(B) = L(\bigwedge B)$.
It is well known that given a bounded poset $\mathbf{A}$, its *Dedekind-MacNeille completion* will be the complete bounded lattice

$$\mathrm{DM}(\mathbf{A}) = (\mathrm{DM}(A), \wedge^{\mathrm{DM}(\mathbf{A})}, \vee^{\mathrm{DM}(\mathbf{A})}, \{0\}, A),$$

such that $\mathrm{DM}(A) = \{X \subseteq A : L(U(X)) = X\}$, $X \wedge^{\mathrm{DM}(\mathbf{A})} Y = X \cap Y$ and $X \vee^{\mathrm{DM}(\mathbf{A})} Y = L(U(X, Y))$, for any $X, Y \in \mathrm{DM}(A)$. When no danger of confusion will be impending, we will omit unnecessary supscripts. Any poset $\mathbf{A}$ can be embedded into $\mathrm{DM}(\mathbf{A})$ by the order homomorphism $\phi^{\mathrm{DM}(\mathbf{A})} : \mathbf{A} \to \mathrm{DM}(\mathbf{A})$ such that $x \mapsto L(x)$, for any $x \in A$. Moreover, $\phi^{\mathrm{DM}(\mathbf{A})}$ is a join- and meet- dense embedding. This means that any element in $\mathrm{DM}(\mathbf{A})$ is the supremum and the infimum of (images of) elements in A. In other words, for any $a \in \mathrm{DM}(\mathbf{A})$, there exist $X, Y \subseteq A$ such that $a = \bigvee^{\mathrm{DM}(\mathbf{A})} \phi^{\mathrm{DM}(\mathbf{A})}(X) = \bigwedge^{\mathrm{DM}(\mathbf{A})} \phi^{\mathrm{DM}(\mathbf{A})}(Y)$. The interested reader may refer to [123] for further details. A poset $(A, \leq)$ will be called *distributive* if it satisfies the *LU-identity*:

$$U(L(x, y), z) = U(L(U(x, z), U(y, z))) = U(L(U(x, z) \cup U(y, z))),$$

which is equivalent to

$$L(U(x, y), z) = L(U(L(x, z), L(y, z))),$$

see [32] for further details. Let us recall that a poset $(A, \leq)$ is said to be *modular* in case, for all $a, b, c \in A$:

$$\text{if } a \leq b, \text{ then } L(U(a, L(b, c))) = L(U(a, c), b),$$

or, dually,

$$\text{if } a \leq b, \text{ then } U(L(b, U(a, c))) = L(U(b, c), a).$$

Many well known partially ordered structures of prominent importance for logic and mathematics are often equipped with operations on their universes which interact with their characterizing order. In particular, in this thesis we will deal with posets with an antitone involution.

Definition 1.8. A *poset with antitone involution* (an *involution* or *involutive poset*, for short) is a structure $\mathbf{A} = (A, \leq, ')$ such that

(i) $(A, \leq)$ is a partially ordered set, and

(ii) $' : A \to A$ is a unary operation over A (cf. Subsection 1.2.1) such that

$$x \leq y \text{ implies } y' \leq x' \text{ and } (x')' = x, \text{ for any } x, y \in A,$$

i.e. it is an order anti-automorphism which is its own inverse.

Moreover, if $\mathbf{A}$ is bounded and $'$ satisfies $L(x, x') = \{0\}$ and $U(x, x') = \{1\}$ for any $x \in A$ (we call it a *complementation*), then $\mathbf{A}$ is said to be an *orthoposet*.

Given an involution poset $\mathbf{A}$ and $x, y \in A$, if $x \leq y'$ then we call $\{x, y\}$ a pair of *orthogonal elements*, written $x \perp y$.

It is clear that, if an involution poset $\mathbf{A}$ has a least lower bound 0, then it has also a greatest lower bound, since for any $x \in A$ one has $x \leq 0'$. We will denote a bounded involutive poset $\mathbf{A}$ as $(A, \leq, ', 0, 1)$. Moreover, if $\mathbf{A}$ is an involution poset (orthoposet) which is a lattice, then we will call it an involution lattice (ortholattice).

Definition 1.9. Let $\mathbf{A} = (A, \leq, ')$ be an involutive poset. A sub-poset $\mathbf{B}$ of $\mathbf{A}$ is said to be a sub-involutive poset if $x \in B$ implies $x' \in B$, for any $x \in A$. Moreover, if $\mathbf{A}$ is bounded, then $\mathbf{B}$ is a bounded involution sub-poset of $\mathbf{A}$ if it contains 0.

Definition 1.10. Let $\mathbf{A} = (A, \leq, ', 0, 1)$ be an orthoposet. A sub-poset $\mathbf{B}$ of $\mathbf{A}$ is a sub-orthoposet if it is a bounded involutive sub-poset of $\mathbf{A}$.

Note that if $\mathbf{A}$ is an orthoposet and $\mathbf{B}$ a sub-orthoposet of $\mathbf{A}$, then $L_B(x, x') = \{0\}$ and $U_B(x, x') = \{1\}$, for any $x \in B$. Therefore, $\mathbf{B}$ is an orthoposet as well. This remark motivates Definition 1.10.

Example 1.2. *Consider the real interval* $[0, 1]$ *and the unary operation* $'$ *such that* $x' = 1 - x$. *It can be easily seen that* $([0, 1], \leq, ', 0, 1)$ *where* $\leq$ *is the natural order on the real line, is a bounded involutive chain.*

Example 1.3. *Let* $\mathbf{A} = (\{0, 1, b, z, c, u, b', z', c', u'\}, \leq, ', 0, 1)$ *be the bounded involution poset depicted in Fig. 1.2. It can be directly verified it is an orthoposet which is not a lattice.*

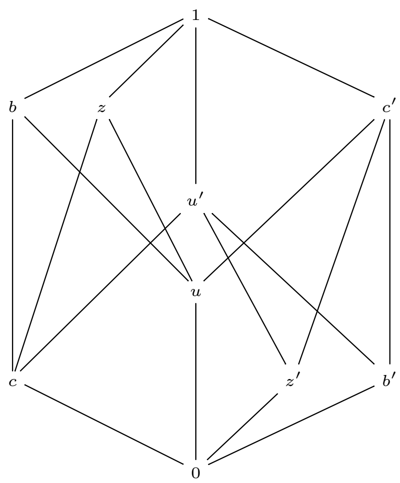


FIGURE 1.2

Definition 1.11. Let $\mathbf{A}$ and $\mathbf{B}$ be bounded involutive posets, and $\phi : \mathbf{A} \to \mathbf{B}$ an order homomorphism Then ϕ is an *ortho-homomorphism* (ortho-embedding, ortho-isomorphism) if the following hold:

(a) ϕ is $'-$preserving, i.e. for any $x \in A$, one has $\phi(x'^{\mathbf{A}}) = \phi(x)'^{\mathbf{B}}$.

(b) $\phi(0^{\mathbf{A}}) = 0^{\mathbf{B}}$

Let us introduce the following notation, for any subset B of an involution poset $\mathbf{A}$:

$$B' = \{b' : b \in B\}.$$

It is well known (see e.g. [33] for details.) that if $\mathbf{A}$ is an involution poset, then $\mathrm{DM}(\mathbf{A})$ is complete bounded involution lattice by putting, for any $X \in \mathrm{DM}(A)$, $X'^{\mathrm{DM}(\mathbf{A})} = L(X')$. Moreover, if $\mathbf{A}$ is also an orthoposet, then $\mathrm{DM}(\mathbf{A})$ is a complete ortholattice ([92]). Orthoposets as well as ortholattices will play a central role in this work and they will be furtherly investigated in subsequent sections. Moreover, from now on, we will deal only and exclusively with bounded involution posets (lattices). Therefore, we will safely call them simply involution posets (lattices).

1.2 Universal algebra and lattice theory

In this section we will deal with the basic machinery that will be heavily employed for the development of our arguments in subsequent chapters. The interested reader is referred to [11] for an exhaustive textbook on the subject.

1.2.1 Basic tools of Universal algebra

Given a non-empty set A, an n-ary operation over A is a function $f : A^n \to A$. We call n the arity (rank) of f. Moreover, if $n = 0$, then f is uniquely determined by its image $f(\emptyset)$ and therefore it can be identified with a distinguished element of A.
A *language* (type) of algebras is a set ν of function symbols indexed by a non-negative integer n. A member $f^n \in \nu$ is said to be an n-ary function symbol. The subset of all n-ary function symbols of ν is denoted by ν_n. Any finite language ν will be identified with the n-tuple of arieties of its elements listed in decreasing order, e.g. if $\nu = \{f_0^2, f_1^2, f_2^1, f_3^0\}$ then we identify ν with the 4-tuple $(2, 2, 1, 0)$. For ν a given language of algebras, an algebra $\mathbf{A}$ of type ν is an ordered pair (A, ν) where A is a nonempty set and ν is a family of operations on A, indexed by the language ν such that in correspondence with each n-ary function symbol $f \in \nu$ there is an n-ary operation $f^{\mathbf{A}}$ on A. The set A is called the *universe* of $\mathbf{A} = (A, \nu)$ and the $f^{\mathbf{A}}$'s are called the *fundamental operations* of $\mathbf{A}$. If ν is finite, say $\nu = \{f_1, ..., f_n\}$, we often write $(A, f_1, ..., f_n)$ for (A, ν).
Any algebra can be *axiomatized* by identities. In accordance with the algebraic literature,

in this thesis we will denote by $\approx$ or, alternatively, by $=$, that two expressions name the same object.
Before discussing some of the algebras that will be employed in this work, it will be useful introducing the notion of subalgebra.
Let $\mathbf{A}$ and $\mathbf{B}$ be algebras of the same type. We say that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ if $B \subseteq A$, every fundamental operation of $\mathbf{B}$ is the restriction of the corresponding operation of $\mathbf{A}$, i.e. for any operation f^n of $\mathbf{A}$, $f^{\mathbf{B}} = f \upharpoonright B^n$. Note that if $\mathbf{A}$ has nullary operations, $\mathbf{B}$ contains them as well. We will write $\mathbf{B} \leq \mathbf{A}$ if $\mathbf{B}$ is a subalgebra of $\mathbf{A}$. Moreover, we call $B \subseteq A$ simply a *subuniverse* of $\mathbf{A}$ if for any $a_1, ..., a_n \in B$, one has $f^{\mathbf{A}}(a_1, ..., a_n) \in B$ for any n-ary operation on $\mathbf{A}$.
If $\mathbf{A}$ is an algebra and $X \subseteq A$, we call the smallest subalgebra of $\mathbf{A}$ containing X, i.e. the intersection of all subalgebras containing X (which is provably again a subalgebra), the subalgebra *generated* by X.

Definition 1.12. A *groupoid* is an algebra $\mathbf{A} = (A, \cdot)$ of type (2), i.e. it can be seen as a set A closed under a binary operation $\cdot$.

Definition 1.13. Let $\mathbf{A} = (A, \cdot)$ be a groupoid. Then $\mathbf{A}$ is said to be

- *commutative* (or *abelian*), if it holds that

  (A1) $x \cdot y \approx y \cdot x$

- *associative*, if it satisfies the identity

  (A2) $(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$

- *idempotent*, if it satisfies

  (A3) $x \cdot x \approx x$.

Definition 1.14. A call an algebra $\mathbf{A} = (A, \cdot, e)$ of type $(2, 0)$ such that $(A, \cdot)$ is a groupoid

- *unital* if it satisfies

  (A4) $x \cdot e \approx x \approx e \cdot x$,

  i.e. e is both a right and left neutral element with respect to $\cdot$;

- a *groupoid with* 0, if it satisfies

  (A5) $x \cdot 0 \approx 0 \approx 0 \cdot x$.

If a groupoid $\mathbf{A}$ satisfies (A2), then it is said to be a *semigroup.* Moreover, if it satisfies also (A3) then it is also a *monoid.*
Well known examples of algebras are the following

Example 1.4. *A group* $\mathbf{A}$ *is an algebra* $(A, \cdot, ^{-1}, e)$*, of type* $(2, 1, 0)$ *such that* $(A, \cdot, e)$ *is a monoid and it holds that*

(A6) $x \cdot x^{-1} \approx e \approx x^{-1} \cdot x.$

As usually, if a group $\mathbf{A}$ is commutative, we employ the additive notation, i.e. we replace $\cdot$ by $+$ and $^{-1}$ by $-$. Moreover, it is customary denoting the neutral element of a (abelian) group $\mathbf{A}$ by 1 (0).

Example 1.5. *A ring* $\mathbf{A}$ *is an algebra* $(R, +, \cdot, 1, 0)$ *of type* $(2, 2, 1, 0)$*, which satisfies the following conditions:*

(i) $(R, +, -, 0)$ *is a commutative group,*

(ii) $(R, \cdot)$ *is a semigroup,*

and the following identities hold true:

(A7) $x \cdot (y + z) \approx (x \cdot y) + (x \cdot z),$

(A8) $(x + y) \cdot z \approx (x \cdot z) + (y \cdot z).$

1.2.2 Fundamentals of lattice theory

In this subsection we will recall basic definitions and facts concerning lattices (see Definition 1.2(c)) in their algebraic formulation. It will turn out that many order-theoretical properties can be axiomatized in terms of simple identities.

Definition 1.15. A lattice $\mathbf{L}$ is an algebra $(L, \wedge, \vee)$ of type $(2, 2)$ which satisfies

(L1a) $x \wedge x \approx x$ (L1b) $x \vee x \approx x$

(L2a) $x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$ (L2b) $x \vee (y \vee z) \approx (x \vee y) \vee z$

(L3a) $x \wedge y \approx y \wedge x$ (L3b) $x \vee y \approx y \vee x$

(L4a) $x \wedge (x \vee y) \approx x$ (L4b) $x \vee (x \wedge y) \approx x$

Any subalgebra of a lattice will be called a *sublattice*. Moreover, is easily seen that, given a lattice $\mathbf{L} = (L, \wedge, \vee)$, both $(L, \wedge)$ and $(L, \vee)$ are commutative idempotent semigroups. is is not difficult to verify that Definition 1.15 and Definition 1.2(c) are indeed equivalent. In fact, given a lattice $\mathbf{L}$, putting $x \leq y$, if $x \wedge y = x$, then $(L, \leq)$ turns out to be a lattice-ordered poset. *Viceversa*, if we have a lattice ordered poset $\mathbf{L}$, putting, for any $x, y \in L$, $x \wedge y = c$ ($x \vee y = d$) where c is the g.l.b. of x and y (d is the l.u.b. of x and y), then $(L, \wedge, \vee)$ satisfies the above axioms.

Definition 1.16. A lattice $\mathbf{L} = (L, \wedge, \vee)$ is said to be:

- *distributive*, if it satisfies

  (L5) $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$;

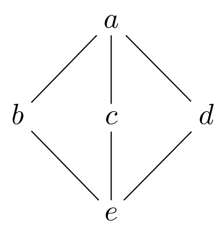
- *modular*, if it holds that

  (L6) $x \leq y$ implies $x \vee (y \wedge z) \approx y \wedge (x \vee z)$

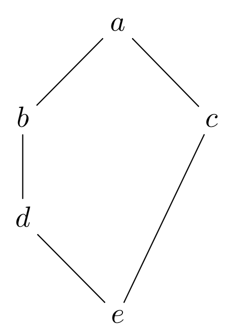
It can be proven that any distributive lattice is also modular. Moreover, It is not difficult to see that, since reversing the order induced by a lattice yields again a lattice, then if a lattice identity holds, of course its *dual* (i.e. the identity obtained by replacing $\wedge$ by $\vee$ and *viceversa*) holds as well. In light of the above definitions and considerations, it is now clear the meaning of the LU-identities characterizing distributive and modular posets in Subsection 1.1.1: a poset $\mathbf{A}$ is distributive (modular) if its elements satisfy distributivity (modularity) in DM($\mathbf{A}$).
The next theorem is well known.

Theorem 1.17. *Let* $\mathbf{L}$ *be a lattice. Then:*

- $\mathbf{L}$ *is modular if and only if it does not contain* N_5 *as a sublattice;*
- $\mathbf{L}$ *is distributive if and only if it does not contain neither* N_5 *nor* M_5 *as its sublattices.*



(M_5)



(N_5)

We say that a lattice $\mathbf{L}$ satisfies the *join infinite distribution property* if, for any $\{a\} \cup X \subseteq L$, if $\bigvee X$ exists than it holds that

$$a \wedge \bigvee X = \bigvee_{x \in X} (a \wedge x). \tag{JID}$$

Proposition 1.18 (Exercise 4.20, [65]). *Let* $\mathbf{L}$ *be a distributive algebraic lattice. Then* $\mathbf{L}$ *has the* (JID).

Finally, we give some definitions in the algebraic framework of some structures that we have already encountered in Subsection 1.1.1.

Definition 1.19. A bounded lattice $\mathbf{L}$ is an algebra $(L, \wedge, \vee, 0, 1)$ of type $(2, 2, 0, 0)$ such that

- $(L, \wedge, \vee)$ is a lattice

and the following are satisfied:

- $x \wedge 0 \approx 0$;
- $x \vee 1 \approx 1$.

Definition 1.20. An involution lattice is an algebra $(L, \wedge, \vee, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that

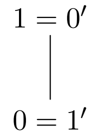
- $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;
- It holds that
  (I1) $x' \vee y' \approx (x \wedge y)'$ (the *De Morgan law*), and
  (I2) $(x')' \approx x$.

Moreover $\mathbf{L}$ is an ortholattice if $'$ is a complementation (cf. Definition 1.8), i.e. it holds that $x \wedge x' = 0$ and $x \vee x' = 1$.

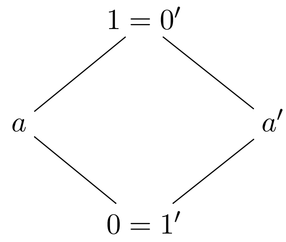
It is a simple exercise verifying that any unary operation on a lattice satisfying conditions of Definition 1.20 is an antitone involution in the sense of Definition 1.8.
We close this subsection by defining a class of algebras which will play a crucial role in the next chapters.

Definition 1.21. A Boolean algebra $\mathbf{B}$ is a distributive ortholattice.

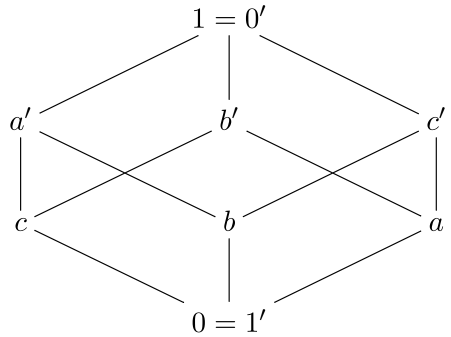
Example 1.6. *The ortholattices depicted below are prototypical examples of Boolean algebras.*



($\mathbf{B}_2$)



($\mathbf{B}_4$)



($\mathbf{B}_8$)

Facts and definitions concerning Boolean algebras which will be expedient for the development of our discussions will be provided whenever necessary in subsequent chapters. The interested reader may refer to [61] for a nice dissertation on the subject.

1.2.3 Congruences, homomorphisms and products

In this subsection we will recall basic universal algebraic tools that will be crucial for the development of our arguments.
Let $\mathbf{A}$ and $\mathbf{B}$ be algebras of the same type ν. An homomorphism $\phi : \mathbf{A} \to \mathbf{B}$ is a mapping from A to B which is "operation-preserving", namely for any $f^n \in \nu$ one has

$$\phi(f^{\mathbf{A}}(a_1, ..., a_n)) = f^{\mathbf{B}}(\phi(a_1), ..., \phi(a_n)).$$

We call a homomorphism $\phi : \mathbf{A} \to \mathbf{B}$ an *embedding* if it is injective. Moreover if ϕ is an onto homomorphism then $\mathbf{B}$ is said to be an *homomorphic image* of $\mathbf{A}$. An injective and surjective homomorphism is an *isomorphism*, written $\mathbf{A} \cong \mathbf{B}$.

Lemma 1.22. *If* $\phi : \mathbf{A} \to \mathbf{B}$ *is an embedding, then* $\phi(A)$ *is a sub-universe of* $\mathbf{B}$.

A crucial notion in universal algebra is the concept of *congruence*. Given an algebra $\mathbf{A}$ of type ν, an equivalence relation θ over $\mathbf{A}$ is a congruence whenever it has the *compatibility property*, i.e. for any $f^n \in \nu$ and $a_i \theta b_i$ $(1 \leq i \leq n)$, we have $f^{\mathbf{A}}(a_1, ..., a_n) \theta f^{\mathbf{A}}(b_1, ..., b_n)$. In what follows, $\mathrm{Con}(\mathbf{A})$ stands for the set of congruence relations on $\mathbf{A}$.

Example 1.7. *Let $\phi : \mathbf{A} \to \mathbf{B}$ be an homomorphism. Then $ker(\phi) = \{(a, b) \in A^2 : \phi(a) = \phi(b)\}$ is a congruence relation on $\mathbf{A}$. It will be called the* kernel *of ϕ.*

We will call the congruence class of an element $a \in A$ a *coset*.
Let $\mathbf{A}$ be an algebra and $(\mathrm{Con}(\mathbf{A}), \subseteq)$ the poset of its congruences under $\subseteq$. It can be seen that $\mathrm{Con}(\mathbf{A})$ is a lattice where meets consist of intersections of congruences and joins are the generated congruences, namely the smallest congruences containing the given ones.
Let us denote by $\{(x, x) : x \in A\}$, the identity congruence, by Δ, and the universal relation by ∇. A little thought shows that they are the bottom and the top element in $\mathbf{Con}(\mathbf{A})$, respectively.

Theorem 1.23. *$\mathbf{Con}(\mathbf{A}) = (\mathrm{Con}(\mathbf{A}), \subseteq, \Delta, \nabla)$ is a complete sublattice of $Eq(A) = (Eq(A), \subseteq, \Delta, \nabla)$ (cf. Lemma 1.5).*

Moreover, we have the following

Lemma 1.24. *For an algebra $\mathbf{A}$, $\mathbf{Con}(\mathbf{A})$ is an algebraic lattice.*

Let us now introduce quotient algebras. Given an algebra $\mathbf{A}$ of type ν and $\theta \in \mathrm{Con}(\mathbf{A})$, the *quotient algebra* $\mathbf{A}/\theta$ of $\mathbf{A}$ modulo θ is the algebra having the quotient set A/θ as universe and operations defined as follows: for any $f^n \in \nu$,

$$f^{\mathbf{A}/\theta}(a_1/\theta, ..., a_n/\theta) = f^{\mathbf{A}}(a_1, ..., a_n)/\theta.$$

Let $\mathbf{A}$ be an algebra and $\theta \in \mathrm{Con}(\mathbf{A})$. Then the map $\eta_\theta : \mathbf{A} \to \mathbf{A}/\theta$, defined by $\eta_\theta(a) = a/\theta$ is called the *natural map*. when no danger of confusion will be impending, we will omit unnecessary subscripts.

Lemma 1.25. *Let $\mathbf{A}$ be an algebra and $\theta \in \mathrm{Con}(\mathbf{A})$. Then the natural map $\eta : \mathbf{A} \to \mathbf{A}/\theta$ is an onto homomorphism.*

The above lemma has an important consequence: it allows us to prove that any homomorphism can be "factorized" in terms of the composition of an embedding and the natural map.

Theorem 1.26. *Let $\phi : \mathbf{A} \to \mathbf{B}$ an homomorphism. Then there exists an embedding $\psi : \mathbf{A}/\theta \to \mathbf{B}$ such that $\phi = \psi \circ \eta$.*

Other relevant constructions in universal are direct products of families of algebras of the same type.
Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be algebras of the same type. The *direct product* of $\mathbf{A}_1$ and $\mathbf{A}_2$, written $\mathbf{A}_1 \times \mathbf{A}_2$ is the algebra having as universe the cartesian product $A_1 \times A_2$ and operations defined as follows: for any $f^n \in \nu$

$$f^{\mathbf{A}_1 \times \mathbf{A}_2}((a_1, b_1), ..., (a_n, b_n)) = (f(a_1, ..., a_n), f(b_1, ..., b_n)),$$

for any $a_1, ..., a_n \in A_1$ and $b_1, ..., b_n \in A_2$. The above construction can be easily generalized to families of algebras of arbitrary cardinality. We will denote by $\Pi_{i \in I}\mathbf{A}_i$ the direct product of the family $\{\mathbf{A}_i\}_{i \in I}$.
We call the (onto) homomorphism $\pi_i : \Pi_{i \in I}\mathbf{A}_i \to \mathbf{A}_i$ assigning to any I-tuple its i-th coordinate, is often called a *projection function*.
Let $\mathbf{A}$ be an algebra and $\theta, \delta \in \mathrm{Con}(\mathbf{A})$. We denote by $\theta \circ \delta$ the *composition* of θ and δ, i.e. the congruence obtained by taking the transitive closure of $\theta \cup \delta$.
If $\theta_1, \theta_2 \in \mathrm{Con}(\mathbf{A})$ and $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$, we say that θ_1, θ_2 *permute*. It can be seen that, if a pair of congruences θ and δ in $\mathrm{Con}(\mathbf{A})$ permute, then $\theta \vee \delta = \theta \circ \delta$.
A congruence $\theta \in \mathrm{Con}(\mathbf{A})$ is said to be a *factor congruence* if there exists θ^* such that

$$\theta \cap \theta^* = \Delta, \quad \theta \circ \theta^* = \theta^* \circ \theta, \quad \theta \vee \theta^* = \nabla.$$

We call $\{\theta, \theta^*\}$ a pair of complementary factor congruences. Moreover, it can be seen that factor congruences are strictly related to the decomposition of algebras. In fact, we have the following

Theorem 1.27. *Let* $\mathbf{A}$ *be an algebra and* (θ, θ^*) *a pair of factor congruences on* $\mathbf{A}$. *Then* $\mathbf{A} \cong \mathbf{A}/\theta \times \mathbf{A}/\theta^*$.

We say that an algebra $\mathbf{A}$ is *directly indecomposable* if the only pair of factor congruences on $\mathbf{A}$ is $\{\Delta, \nabla\}$.

Theorem 1.28. *Every finite algebra is the direct product of directly indecomposable algebras.*

Therefore, any finite algebra can be represented in terms of a direct product of “simpler” algebras.
Theorem 1.28 can be generalized by introducing the notion of *subdirect irreducibility*. We start by defining subdirect products.
Given a family $\{\mathbf{A}\} \cup \{\mathbf{A}_i\}_{i \in I}$ of algebras of the same type, we say that $\mathbf{A}$ is a subdirect product of the $\mathbf{A}_i$'s if

- $\mathbf{A} \leq \Pi_{i\in I}\mathbf{A}_i$;
- $\pi_i(\mathbf{A}) = \mathbf{A}_i$, for any $i \in I$.

An embedding $\phi : \mathbf{A} \to \Pi_{i\in I}\mathbf{A}_i$ is subdirect if $\phi(\mathbf{A})$ is a subdirect product of the $\mathbf{A}_i$. Finally, we say that an algebra $\mathbf{A}$ is *subdirectly irreducible* if for any subdirect embedding

$$\phi : \mathbf{A} \to \Pi_{i\in I}\mathbf{A}_i,$$

there exists $i \in I$ such that $\pi_i \circ \phi : \mathbf{A} \to \mathbf{A}_i$ is an isomorphism.
It can be seen that subdirect irreducibility is a stronger form of direct indecomposability.

Theorem 1.29. *Any subdirectly irreducible algebra is directly indecomposable.*

Moreover, subdirect irreducibility can be characterized in terms of properties of congruence lattices.

Theorem 1.30. *An algebra* $\mathbf{A}$ *is subdirectly irreducible if* $\mathbf{A}$ *is trivial or there is a minimum congruence in* $\boldsymbol{Con}(\mathbf{A}) - \Delta$.

Finally, we can state one of the most important theorems of universal algebra stating that any algebra can be represented as a subdirect product of directly indecomposable algebra. This shows that subdirect irreducible algebras are the very building blocks of algebras.

Theorem 1.31. *Every algebra* $\mathbf{A}$ *is isomorphic to a subdirect product of subdirectly irreducible algebras (which are homomorphic images of* $\mathbf{A}$*).*

1.2.4 Varieties and congruence properties

One of the main research topics in universal algebra is the study of closure properties of classes of algebras of the same type. We recall some of the basic so-called *class operators*. Let $\mathcal{C}$ and $\mathbf{A}$ be a class of algebras and an algebra, respectively. Then:
$\mathbf{A} \in \mathbf{I}(\mathcal{C})$, if $\mathbf{A}$ is isomorphic to some element in $\mathcal{C}$;
$\mathbf{A} \in \mathbf{H}(\mathcal{C})$, if $\mathbf{A}$ is an homomorphic image of some member in $\mathcal{C}$;
$\mathbf{A} \in \mathbf{S}(\mathcal{C})$, if $\mathbf{A}$ is a subalgebra of some member in $\mathcal{C}$;
$\mathbf{A} \in \mathbf{P}(\mathcal{C})$, if $\mathbf{A}$ is a direct product of some family of members in $\mathcal{C}$;
$\mathbf{A} \in \mathbf{P}_s(\mathcal{C})$, if $\mathbf{A}$ is a subdirect product of some family of elements in $\mathcal{C}$.
We say that a class $\mathcal{C}$ is closed under the a class operator $\mathbf{O}$ if $\mathbf{O}(\mathcal{C}) \subseteq \mathcal{C}$. Moreover, we call a *variety* a class of algebras of the same type which is closed under direct products, subalgebras and homomorphic images. Denoting by $\mathbf{V}(\mathcal{C})$ the smallest variety containing $\mathcal{C}$, we can now recall the first important result of this subsection:

Theorem 1.32. *Let $\mathcal{C}$ be a class of algebras of the same type. Then $\mathbf{V}(\mathcal{C}) = \mathbf{HSP}(\mathcal{C})$.*

We say that a class of algebras is *equational* if it is closed with respect to satisfying a given class of equations.

Theorem 1.33. *A class of algebras $\mathcal{C}$ is a variety if and only if it is an equational class.*

Thus, any variety can be "axiomatized" by means of a given (finite or infinite) set of identities. The examples of algebras already treated in Subsection 1.2.1 and 1.2.2 obviously form a variety.

An interesting direction of research in Universal Algebra is the study of the properties of congruence lattices of algebras. We say that an algebra $\mathbf{A}$ is *congruence-permutable* if any pair $\theta_1, \theta_2 \in \mathrm{Con}(\mathbf{A})$ of distinct congruences on $\mathbf{A}$ permute. Moreover, it is *congruence-distributive* (*congruence-modular*) if its congruence lattice is distributive (modular), and it is said to be *arithmetical* if it is both congruence-distributive and permutable. Finally, we say that an algebra $\mathbf{A}$ is *congruence regular* if any congruence is uniquely determined by any of its cosets, i.e. for any $\theta, \delta \in \mathrm{Con}(\mathbf{A})$, one has that if $a/\theta = a/\delta$, for some $a \in A$, then $\theta = \delta$. A variety V is congruence-distributive (modular, permutable, regular) if each member in V has this property as well. The following proposition will be useful in Chapter 3.

Proposition 1.34 (Theorem 1, [127])**.** *For any variety $\mathcal{V}$ the following conditions are equivalent:*

1. *For any $\mathbf{A} \in \mathcal{V}$ each reflexive relation having the compatibility property is symmetric;*
2. *For any $\mathbf{A} \in \mathcal{V}$ each reflexive relation having the compatibility property is transitive;*
3. *For any $\mathbf{A} \in \mathcal{V}$ each reflexive relation having the compatibility property is a congruence of $\mathbf{A}$;*
4. *$\mathcal{V}$ is congruence permutable.*

For sake of completeness, we close this subsection by listing some theorems showing that the congruence lattice of an algebra $\mathbf{A}$ enjoys certain properties whenever the latter allows the definition of certain terms by means of compositions of fundamental operations. This fact suggests that there indeed are deep connections between congruences and properties of algebras.
Recall that a variety V satisfies an identity $t \approx s$ if any of its members does.

Theorem 1.35. *Let V be a variety. V is congruence-permutable iff there exists a term $p(x, y, z)$ such that V satisfies the identities*

$$p(x, x, y) \approx y \quad \textit{and} \quad p(x, y, y) = x.$$

Theorem 1.36. *A variety V is congruence-distributive iff there is a finite n and terms $p_0(x, y, z), ..., p_n(x, y, z)$ such that V satisfies the following identities:*

$$\begin{gathered} p_i(x, y, x) \approx x, \quad \textit{for } (0 \leq i \leq n); \\ p_0(x, y, z) \approx x; \\ p_n(x, y, z) = z; \\ p_i(x, x, y) \approx p_{i+1}(x, x, y) \ \textit{(for } i \textit{ even)} \\ p_i(x, y, y) \approx p_{i+1}(x, y, y) \ \textit{(for } i \textit{ odd).} \end{gathered}$$

Theorem 1.37. *A variety V is arithmetical iff there is a term $m(x, y, z)$ such that V satisfies*

$$m(x, y, x) \approx m(x, y, y) \approx m(y, y, x) = x.$$

Theorem 1.38. *A variety V is congruence regular if and only if there exists a set of ternary terms $t_i(x, y, z)$ with $1 \leq i$ such that*

$$t_i(x, y, z) = z \textit{ for any } i \textit{ if and only if } x = y.$$

Having the basic algebraic tools ready at hand, in the next chapter we will recall fundamentals of quantum structures.

Chapter 2

Quantum structures and their generalizations

In this chapter we first recall basic facts and definitions concerning quantum structures, i.e. orthomodular lattices, orthomodular posets, effect algebras and paraorthomodular lattices. We will pay attention to the concrete models they are abstraction of. To this aim, we will often make reference to algebras of linear bounded self adjoint operators of a complex separable Hilbert space. An introductory textbook on the mathematics of Hilbert spaces can be found in [129]. A friendly introduction to quantum structures can be found instead in [33]. It will turn out that many of the aforementioned structures can be neatly described in terms of their block structure, i.e. as *pastings* of Boolean algebras. Furthermore, we will highlight their order theoretical common features as well as their algebraic distinguishing traits.

Subsequently, we focus on lattice extensions of effect algebras and their non-commutative generalizations, i.e. lattice pseudoeffect algebras. In passing, we will mention their multiple connections with partially ordered groups theory. Furthermore, we will recall basic facts concerning MV algebras and pseudo-MV algebras as algebraic counterparts of (non-commutative) infinite-valued logics. Finally, we will introduce basic algebras as a common framework in which both lattice effect algebras and MV algebras can be represented.

2.1 Orthomodular lattices and posets

2.1.1 Orthomodular lattices

In 1936, G. Birkhoff and J. von Neumann identified the "logic of quantum properties" as the set $\mathcal{C}(\mathcal{H})$ of closed subspaces of a Hilbert space $\mathcal{H}$ which is into one-to-one correspondence with the set $\Pi(\mathcal{H})$ of projectors on $\mathcal{H}$. $\Pi(\mathcal{H})$, under the order induced by $\subseteq$ on $\mathcal{C}(\mathcal{H})$, turns into the ortholattice

$$\mathbf{\Pi}(\mathcal{H}) = (\Pi(\mathcal{H}), \wedge, \vee, ^{\perp}, \mathbb{O}, \mathbb{I}),$$

with operations defined as follows, for any $P_1, P_2 \in \Pi(\mathcal{H})$:

- $P_1 \wedge P_2 = P_{C_1 \cap C_2}$;
- $P_1 \vee P_2 = P_{\langle C_1 \cup C_2 \rangle}$;
- $P^{\perp} = P_{C^{\perp}}$;
- $\mathbb{I}, \mathbb{O}$ are the identity resp. the null operators.

where P_C is meant to be the projection on the subspace C, $\langle C_1 \cup C_2 \rangle$ is the subspace generated by C_1 and C_2 and $C^{\perp}$ is the subspace orthogonal to C, i.e. $C^{\perp} = \{\phi \in \mathcal{H} : \langle \phi | \psi \rangle = 0, \text{ for any } \psi \in C\}$. $\Pi(\mathcal{H})$ turns out to be an (even modular, in the finite-dimensional case) atomic and complete orthomodular lattice, i.e. an ortholattice satisfying, for any $P, Q \in \Pi(\mathcal{H})$, the orthomodular law

$$P \leq Q \Rightarrow Q = (P \vee (Q \wedge P^{\perp})).$$

We are now ready to provide an abstract definition of the above structure.

Definition 2.1. An orthomodular lattice $\mathbf{A}$ is an ortholattice $(A, \wedge, \vee, ', 0, 1)$ satisfying the orthomodular law, i.e. for any $x, y \in A$

$$x \leq y \quad \text{implies} \quad y = x \vee (y \wedge x') \tag{OL}$$

Observe that OL can be equivalently replaced by the identity

$$(x \wedge y) \vee ((x \wedge y)' \wedge x) \approx x.$$

Therefore, since they constitute an equational class, by Theorem 1.33, orthomodular lattice (OMLs) form a variety. We denote by $\mathbb{OML}$ the variety of orthomodular lattices and by $\mathbb{OL}$ the variety of ortholattices. OMLs can be algebraically characterized in several ways.

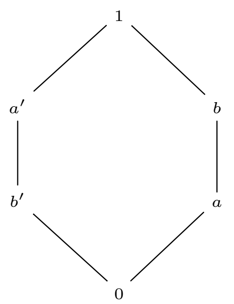
Theorem 2.2. *Let* $\mathbf{A} \in \mathbb{OL}$*. The following conditions are equivalent:*

1. $\mathbf{A}$ *is orthomodular;*
2. *for all* $a, b \in L$, $a \wedge ((b \wedge a) \vee a') \leq b$*;*
3. *for all* $a, b \in L$, $a \vee b = ((a \vee b) \wedge b') \vee b$*;*
4. *for all* $a, b \in L$*, if* $a \leq b$ *and* $a' \wedge b = 0$*, then* $a = b$*;*
5. *for all* $a, b \in L$*, if* $a \leq b$*, then there exists* $c \in L$ *such that* $a \leq c'$ *and* $b = a \vee c$*.*

Moreover, as for modular and distributive lattices (see Theorem 1.17), they are capable of a complete lattice-theoretical characterization.

Theorem 2.3. *Let* $\mathbf{A}$ *be an ortholattice. Then* $\mathbf{A}$ *is orthomodular if and only if it does not contain a subalgebra isomorphic to* $\mathbf{B}_6$*.*

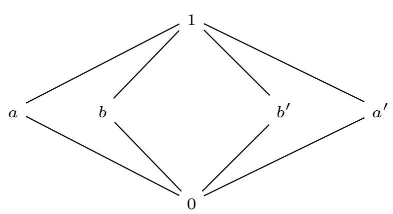
(2.1)



$\mathbf{B}_6$

It is worth noticing that Theorem 2.3 depends on the equivalence between items (1) and (4) in Theorem 2.2. This remark will be useful in subsequent sections.
Evidently, every distributive ortholattice, i.e. any Boolean algebra (see Definition 1.21), is orthomodular. The simplest example of a non-distributive orthomodular (modular) lattice is $\mathbf{MO}_2$ in figure (2.2).

(2.2)



Looking closely at the above figure one might have realized that $\mathbf{MO}_2$ can be seen as the "gluing" of two four-elements Boolean algebras, namely $\{a, a', 0, 1\}$ and $\{b, b', 0, 1\}$ (see Example 1.6) along their common Boolean subalgebra on $\{0, 1\}$. Surprisingly enough, it turns out that any orthomodular lattice can be seen as the "pasting" of Boolean algebras. More precisely, they are the pasting of their maximal Boolean subalgebras, as it will be clear in the next subsection.

Remark 2.4. Any orthomodular lattice is the *union* of its maximal Boolean subalgebras called *blocks*. In fact, if $\mathbf{A}$ is an OML and $x \in A$, the set $\{0, x, x', 1\}$ is a Boolean subalgebra. Since any chain of Boolean subalgebras is, of course, a Boolean subalgebra, by Zorn's Lemma, $x \in \mathbf{B}$, for some block $\mathbf{B} \leq \mathbf{A}$.

2.1.2 Orthomodular posets

Any statistical physical theory involves notions that are often taken as primitives: states, observables and probabilities. A *state* w can be seen as an abstract object which sums up observer's informations concerning a set of preparations of a given physical entity. An *observable* A is a physical quantity that can be measured on a given state w. Finally, a *probability measure* is a function assigning to any triple (w, A, Δ), where Δ is a Borel set, a value in the interval $[0, 1]$. We denote by $\mathcal{S}$, $\mathcal{O}$, $\mathcal{B}(\mathbb{R})$, the set of states, the set of observables, and the Borel σ-algebra over the real line, respectively.
Intuitively, a probability measure p assigns to any $A \in \mathcal{O}$ and $\Delta \in \mathcal{B}(\mathbb{R})$ a real number representing the probability that the measure of A on a state w lies in Δ.
To any statistical physical theory $\mathbf{T}$ we can associate a suitable class of triples $(\mathcal{S}, \mathcal{O}, p)$ called state-observable-probability systems.
A special class of observables is represented by the so-called *events*. Any event A is characterized by the condition

$$\text{for any } w \in \mathcal{S}, p(w, A, \{0, 1\}) = 1. \tag{*}$$

Roughly speaking, any event is an observable which can be assume, given a state w, only two values: 0 and 1. The importance of such observables rests on the fact that, under Mackey's axiomatization of orthodox QT, any statement like "the value of the physical quantity A lies in Δ" can be adequately represented by a $0 - 1$ valued observable. In fact, standing to Mackey's axioms, for any Borel function (see e.g. [33]) $f : \mathbb{R} \to \mathbb{R}$ and $A \in \mathcal{O}$, there exists $B \in \mathcal{O}$ such that for any $w \in \mathcal{S}$, $\Delta \in \mathcal{B}(\mathbb{R})$, one has

$$p(w, B, \Delta) = p(w, A, f^{-1}(\Delta)).$$

B can be seen to be uniquely determined, so we put $B = f(A)$. Moreover, since any characteristic function χ_Δ is a Borel function, for any $A \in \mathcal{O}, \Delta \in \mathcal{B}(\mathbb{R})$, there exists, by (*), an observable $\chi_\Delta(A)$ which is an easy exercise verifying to be an event. Moreover, by the above equality, it follows that

$$p(w, \chi_\Delta(A), \{1\}) = p(w, A, \Delta).$$

A moment's reflection shows that $\chi_\Delta(A)$ corresponds to the event "the measure of the physical quantity A lies in Δ". In fact, the above equality tells us nothing but the probability for a measure on A to be lying in Δ is equal to the probability that the statement "the measure of A lies in Δ" is true.
Of course, any event E can be represented in this way. We have indeed $E = \chi_{\Delta_{\{1\}}}(E)$ (it follows by [33, Lemma 2.2.1], cf. [33, Lemma 2.2.3]). Therefore, the class of events, that we denote by $\mathcal{E}_v$, is the mathematical representative of the whole set of statement concerning measures on physical quantities. Now, if we consider $m_w : \mathcal{E}_v \to [0,1]$ such that $m_w(x) = p(w, x, \{1\})$, then putting, for any $A, B \in \mathcal{E}_v$,

$$E \leq F \quad \text{whenever for any } w \in \mathcal{S}, m_w(E) \leq m_w(F),$$

$\mathbb{E}_v = (\mathcal{E}_v, \leq)$ turns out to be a partial order. The intuitive meaning of $\leq$ is the following: an event F is greater than E if, for any physical state w, the probability that F occurs in w is greater than the probability that E occurs in w.
Moreover, if we consider $\mathbb{O} = \chi_\emptyset(E)$ and $\mathbb{I} = \chi_\mathbb{R}(E)$, for some event E, then, since $m_w(\chi_\emptyset(E)) = 0$ and $m_w(\chi_\mathbb{R}(E)) = 1$, for any $w \in \mathcal{S}$, we have that $\mathbb{E}_v = (\mathcal{E}_v, \leq, \mathbb{O}, \mathbb{I})$ is a bounded poset. Finally, if for any $E = \chi_\Delta(B) \in \mathcal{E}_v$, we consider $\chi_{\Delta^c}(B)$ (where $\Delta^c = \mathbb{R} - \Delta$), namely the event representing the statement "the value of B does not lie in Δ", then it can be easily verified (see [33, Lemma]) that $m_w(\chi_{\Delta^c}(B)) = 1 - m_w(\chi_\Delta(B))$. Thus, the mapping $' : E = \chi_\Delta(B) \to \chi_{\Delta^c}(B)$ turns out to be an antitone involution. Therefore, $\mathbb{E}_v = (\mathcal{E}_v, \leq, ', \mathbb{O}, \mathbb{I})$ is an involution poset. Moreover, it can be seen that $\mathbb{E}_v$ is also an orthoposet. Finally, one shows that any countable set $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{E}_v$ of pairwise orthogonal elements has a l.u.b. in $\mathcal{E}_v$ under $\leq$ and for any $E, F \in \mathcal{E}_v$

$$E \leq F \quad \text{implies} \quad F = E \vee (E' \wedge F),$$

i.e. $\mathbb{E}_v$ satisfies the orthomodular law (see [33, Theorem 2.2.9]). It is worth briefly discussing what does the join of a family of pairwise orthogonal events mean in $\mathcal{E}_v$. As a consequence of one of the axioms of Mackey system (cf. [33, p.44]), for any set $\{E_n\}_{n\in\mathbb{N}}$ of pairwise orthogonal elements, there exists an $A \in \mathcal{O}$ and a Borel set Δ such that, for any w,

$$p(w, A, \Delta) = \sum_{n=0}^{\infty} p(w, E_n, \{1\}).$$

Clearly, the event "the measure of A lies in Δ" can be represented as the event $\chi_\Delta(A)$. A little thought shows that such event represents the statement "at least one among the *pairwise disjoint* events $E_0, E_1, ...$ occurs". $\chi_\Delta(A)$ is the join of the E_n's.
The structure $\mathbb{E}_v = (\mathcal{E}_v, \leq, ', \mathbb{O}, \mathbb{I})$ is called *orthomodular poset* which in the case discussed above is also σ-orthocomplete, namely the join of countable sets of pairwise orthogonal elements always exists. We are now ready to give an abstract definition of the structures we are dealing with in this subsection.

Definition 2.5. An orthomodular poset $\mathbf{A}$ is an orthoposet $(A, \leq, ', 0, 1)$ such that

(i) $x \leq y'$ implies $x \vee y$ exists;

(ii) $x \leq y'$ implies $y = x \vee (y \wedge x')$.

Let us observe that, from the fact that $a \leq b = b''$, we obtain that $a \vee b'$ exists, and so $(a \vee b')' = a' \wedge b$ exists. Since $a \leq a \vee b'$, then also $a \vee (a \vee b')' = a \vee (a' \wedge b)$ exists. This shows that equation in *(ii)* is correctly formulated. Moreover, it is easily seen that, since $'$ is an antitone involution, then (ii) is self-dual, i.e. one has also

$$x \leq y \quad \text{implies} \quad x = y \wedge (x \vee x').$$

From now on, given an involution poset $\mathbf{A}$ and $x, y \in A$, we say that x and y are *orthogonal* (written $x \perp y$) provided that $x \leq y'$.

Lemma 2.6. *Let* $\mathbf{A} = (A, \leq, ', 0, 1)$ *be an orthomodular poset. Then for any* $x, y \in A$

(i) $x \leq y$ *implies there exists* $z \in A$ *such that* $x \perp z$ *and* $y = x \vee z$*;*

(ii) $x \leq y$ *and* $x' \wedge y = 0$ *implies* $x = y$.

It is reasonable asking if, like orthomodular lattices, orthomodular posets can be seen as "pastings" of Boolean algebras as well. Moreover, we can see that

Remark 2.7. Reasoning as in Remark 2.4, it can be shown that orthomodular posets are unions of their maximal Boolean subalgebras, namely their blocks.

In order to provide an answer to the above question we introduce the notion of *pasting* of Boolean algebras.

Definition 2.8. A pasted family of Boolean algebras is a family $\mathcal{F}$ of Boolean algebras such that, for each $\mathbf{A}, \mathbf{B} \in \mathcal{F}$, $A \neq B$:

(i) $A \not\subseteq B$;

(ii) $A \cap B$ is a Boolean subalgebra of both $\mathbf{A}$ and $\mathbf{B}$ on which the operations of $\mathbf{A}$ and $\mathbf{B}$ coincide;

(iii) For any $x \in A \cap B$, there exists $\mathbf{C} \in \mathcal{F}$ such that $[0, x]_{\mathbf{A}} \cup [0, x']_{\mathbf{B}} \subseteq \mathbf{C}$.

Definition 2.9. Let $\mathcal{F}$ be a pasted family of Boolean algebras and consider $A = \bigcup \mathcal{F}$. On A we define the relation $\leq^A$ such that, for any $x, y \in A$, $x \leq^A y$ if there exists $\mathbf{B} \in \mathcal{F}$ such that $x \leq^{\mathbf{B}} y$. Moreover, for any $x \in A$, put $x'^A = x'^{\mathbf{B}}$, for some $\mathbf{B} \in \mathcal{F}$. Note that, if $x \in \mathbf{B} \cap \mathbf{C}$, then by (ii) $x'^{\mathbf{B}} = x'^{\mathbf{C}}$. Thus, $'$ is well defined. We call the triple $\mathbf{A}_{\mathcal{F}} = (A, \leq, ', 0, 1)$ the *pasting* of $\mathcal{F}$.

Lemma 2.10. *Let $\mathcal{F}$ be a pasted family of Boolean algebras. Then $\mathbf{A}_{\mathcal{F}} = (A, \leq, ', 0, 1)$ is an orthoposet.*

Roughly speaking, one can see that a pasting of a non empty pasted family of Boolean algebras $\mathcal{F}$ is an orthoposet obtained by "gluing" each other members of $\mathcal{F}$ along a common Boolean subalgebra.
One might ask when does the pasting of a pasted family of Boolean algebras $\mathcal{F}$ yields an orthomodular poset. Surprisingly enough, we can characterize those pastings which are indeed orthomodular posets.

Definition 2.11. Let $\mathcal{F}$ be a family of Boolean algebras. An *n-cycle* in $\mathcal{F}$ is a sequence of pairs $((\mathbf{A}_0, a_0), \ldots, (\mathbf{A}_n, a_n))$ of (non necessarily distinct) algebras in $\mathcal{F}$, and of (non necessarily distinct) elements $a_i \in A_i \cap A_{i+1}$, $a_i \neq 0$, so that $a_{i-1} \perp a_i$, and $[0, a_i]_{A_i} = [0, a_i]_{A_{i+1}}$ (indices modulo n).

Theorem 2.12. *[36] Let $\mathbf{L}$ be the pasting of a family $\mathcal{F}$ of Boolean algebras. $\mathbf{L}$ is an ortomodular poset if and only if for every 3-cycle $((\mathbf{A}_0, a_0), (\mathbf{A}_1, a_1)(\mathbf{A}_2, a_2))$ there is a Boolean algebra $\mathbf{B}$ such that $a_0, a_1, a_2 \in B$.*

Actually, we can prove something more. One can show that *any* orthomodular poset is a pasting of Boolean algebras

Theorem 2.13 (Theorem 3 , [113]). *Any orthomodular poset is the pasting of its blocks.*

It is worth noticing that, in its first formulation due to Dichtl ([36]), the definition of pasting was quite different. In fact, it employed the notion of *astroid.*

Definition 2.14. An astroid $(\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3)$ for m is a 4-cycle $((C_1, m_i))_{i=0}^{3}$ such that $m_0 = m = m_2$ and $m_1 = m' = m_3$ hold.

Dichtl's original definition was the following:

Definition 2.15. A pasted family of Boolean algebras is a family $\mathcal{F}$ of Boolean algebras such that, for each $\mathbf{A}, \mathbf{B} \in \mathcal{F}$, $A \neq B$:

(a) $(i) - (ii)$ of Definition 2.8 hold;

(b) if $m \in A \cap B$, then there exists an astroid $(\mathbf{C}_0, \mathbf{A}, \mathbf{C}_2, \mathbf{B})$ for m.

It can be proven that Definition 2.15 and Definition 2.8 are indeed equivalent. Since, at the best of our knowledge, the literature on the topic does not contain any explicit proof of this fact, we provide one below.

Proposition 2.16. *Let $\mathcal{F}$ be a family of Boolean algebras satisfying $(i) - (ii)$ of Definition 2.8. Then the following are equivalent:*

1. *$\mathcal{F}$ satisfies (iii) of Definition 2.8;*
2. *$\mathcal{F}$ satisfies (b) of Definition 2.15.*

Proof. (2) implies (1). Suppose that $\mathbf{C}_0, \mathbf{C}_2 \in \mathcal{F}$ and $m \in \mathbf{C}_0 \cap \mathbf{C}_2$. Then there exists an astroid $(\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3)$ for m. Note that $[0, m]_{C_0} = [0, m]_{C_0}$ and $[0, m']_{C_2} = [0, m']_{C_1}$. Therefore, $\mathbf{C}_1$ is the desired $\mathbf{C}$ such that $[0, m]_{C_0} \cup [0, m']_{C_2} \subseteq C$.
(1) implies (2). Let $m \in \mathbf{C}_0 \cap \mathbf{C}_2$. By (1) there exist $\mathbf{C}_1 \in \mathcal{F}$ such that $[0, m]_{C_0} \cup [0, m']_{C_2} \subseteq C_1$, and $\mathbf{C}_3$ such that $[0, m]_{C_2} \cup [0, m']_{C_0} \subseteq C_3$. Now, let us suppose by way of contradiction that there exists $z \in [0, m]_{C_1} - [0, m]_{C_0}$. Since $m \in C_1 \cap C_0$, we have that there exists $\mathbf{D} \in \mathcal{F}$ such that $[0, m]_{C_0} \cup [0, m']_{C_0} \subseteq [0, m]_{C_1} \cup [0, m']_{C_0} \subseteq D$. Since by (ii) $\mathbf{D} \cap \mathbf{C}_0$ is a subalgebra of $\mathbf{D}$ and $\mathbf{C}_0$ on which operations coincide, one has, for any $c \in C_0$: $c = c \wedge 1 = c \wedge (m \vee m') = (c \wedge m) \vee (c \wedge m') \in D$. Thus, it follows that $C_0 \subsetneq D$, against the assumption (i), a contradiction. Therefore we conclude $[0, m]_{C_0} = [0, m]_{C_1}$. Similarly we prove that $[0, m']_{C_1} = [0, m']_{C_2}$, $[0, m']_{C_0} = [0, m]_{C_3}$ and $[0, m]_{C_2} = [0, m]_{C_3}$. Hence, our conclusion follows. □

In light of the above proposition, exploiting properties of Boolean algebras we have the following

Remark 2.17. Let $\mathcal{F}$ be a pasted family of Boolean algebras. Then for any pair of blocks $\mathbf{A}, \mathbf{B} \in \mathcal{F}$ such that $m \in \mathbf{A} \cap \mathbf{B}$ there exists $\mathbf{C} \in \mathcal{F}$ such that $\mathbf{C} \cong [0, m]_{\mathbf{A}} \times [0, m']_{\mathbf{B}}$. Note that in any Boolean algebras $\mathbf{A}$, if $m \in A$ then the interval $[0, m]_{\mathbf{A}}$ can be regarded itself as a Boolean algebra.

Since any orthomodular lattice is also an orthomodular poset, Theorem 2.13 allows us to understand the "structure" of orthomodular lattices by shedding some light on the relationships between the blocks they are built of.

2.1.3 Effect algebras and Orthoalgebras

As we have seen in Subsection 2.1.2, given a state-observable-probability system $(\mathcal{S}, \mathcal{O}, p)$, the set of quantum events $\mathcal{E}_v$ naturally forms an orthomodular poset under the order induced by p. However, in Hilbert spaces framework there are operators which can be thought as events although they are not projectors. In fact, for a (finite or infinite-dimensional) Hilbert space $\mathcal{H}$, if we consider the set $\mathcal{E}(\mathcal{H})$ of linear bounded self-adjoint operators "satisfying" the Born rule, i.e. such that, for any $A \in \mathcal{E}(\mathcal{H})$, for any $\rho \in D(\mathcal{H})$ (the set of density operators on $\mathcal{H}$),

$$Tr(\rho A) \in [0, 1]$$

(where Tr is the *trace operator*) then it can be considered as a set of events as well. We call these operators *effects*. $\mathcal{E}(\mathcal{H})$ is of significance in representing unsharp measurements or observations on a physical system, and effect-valued measures play an important role in stochastic quantum mechanics (see [39] for a general account).
Note that such approach extends those that have been already investigated in the previous subsections. In fact, $\mathcal{E}(\mathcal{H})$ extends $\mathbf{\Pi}(\mathcal{H})$ properly, since now we consider operators like e.g. $(1/2)\mathbb{I}$ also. In light of the above considerations, the question if such ("fuzzy") events are capable of being treated algebraically naturally emerges.
It is well known that $\mathcal{E}(\mathcal{H})$ can be partially ordered by putting

$$E \leq_c F \quad \text{iff} \quad \forall \rho \in D(\mathcal{H}),\ Tr(\rho E) \leq Tr(\rho F),$$

where, for any $E \in \mathcal{E}(\mathcal{H})$ and $\rho \in D(\mathcal{H})$, $Tr(\rho E)$ can be read as "the probability that the event E occurs in state ρ". We call the above order the *canonical order* (CO). Moreover, note that, under $\leq_c$, the null operator $\mathbb{O}$ and the identity operator $\mathbb{I}$ behave as the bottom and the top element, respectively.
Now, let we define a partial sum $\oplus$ on $\mathcal{E}(\mathcal{H})$ such that

$$\text{for any } E, F \in \mathcal{E}(\mathcal{H}), (E \oplus F) \text{ exists iff } E + F \in \mathcal{E}(\mathcal{H}),$$

(where $+$ is the matrix sum) and an unary operation $^{\perp}$ such

$$E^{\perp} = \mathbb{I} - E.$$

It is easily seen that $^{\perp}$ is an *orthocomplementation* with respect to $\oplus$, i.e. for any $E \in \mathcal{E}(\mathcal{H})$ one has $E \oplus E^{\perp} = \mathbb{I}$. Moreover, $^{\perp}$ is an antitone involution on the poset induced by the canonical order.

$\oplus$ is said to be an *orthosum* over $\mathcal{E}(\mathcal{H})$, since it can be proven that

$$(E \oplus F) \text{ exists iff } E \leq_c F^{\perp},$$

i.e. $\oplus$ is defined for pairs of orthogonal elements.
Finally, we see that $\leq_c$ can be characterized algebraically. In fact, one has

$$E \leq_c F \quad \text{iff} \qquad \text{there exists } G \in \mathcal{E}(\mathcal{H}) \text{ such that } E \oplus G = F.$$

The (partial) algebra $\mathcal{E} = (\mathcal{E}(\mathcal{H}), \oplus, ^{\perp}, \mathbb{O}, \mathbb{I})$ is called a *Hilbert effect algebra*.

Effect algebras were discussed in 1994 by Foulis and Bennett in [47], and independently introduced, under the name of weak orthoalgebras, by Giuntini and Greuling in 1989 [57]. Since then, they constitute a fruitful direction of research which has brought to several achievements, from the axiomatization of unsharp logics of quantum mechanics [33] to deep algebraic investigations. For an extensive account on these obtainments the reader is referred to Dvurecěnskij and Pulmannová's monograph [39].

Definition 2.18. A structure $\mathbf{A} = (A, \oplus, 0, 1)$ is called an *effect algebra* if $0, 1$ are two distinguished elements and $\oplus$ is a partially defined binary operation on A that satisfied the following:

1. $a \oplus b = b \oplus a$, if $a \oplus b$ is defined;
2. $a \oplus (b \oplus c) = (a \oplus b) \oplus c$, if the expressions on either side is defined;
3. for any $a \in A$, there is a unique $b \in A$ such that $a \oplus b = 1$;
4. if $1 \oplus a$ is defined, then $a = 0$.

We will call an *orthoalgebra* any effect algebra with no isotropic element, i.e. with no element $x \neq 0$ such that $x \leq x'$.

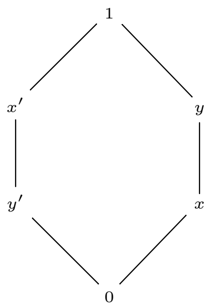
By item (3) of the above definition, any element a of an effect algebras $\mathbf{A}$ has exactly one orthocomplement. Thus we can define a total operation $' : a \mapsto a'$ which turns out to be an antitone involution (see [47, Theorem 2.4.]). Moreover, an interesting consequence of the unicity of orthocomplements is the following:

Proposition 2.19. *For any effect algebra* $\mathbf{A}$ *the following condition is satisfied, for any* $x, y \in A$*:*

$$x \leq y \textit{ and } x' \wedge y = 0 \textit{ imply } x = y. \tag{2.3}$$

Proof. Note that if condition 2.3 is not satisfied, then the underlying lattice of $\mathbf{A}$ contains a sublattice isomorphic to $\mathbf{B}_6$ (see Theorem 2.3).

(2.4)



Since $x \lneq y$, then $x \oplus y'$ is defined. Moreover, since $'$ is an antitone involution we have that $1 = x \vee y' \leq x \oplus y'$. Hence y' is an orthocomplement for x. Therefore $x' = y'$ and $x = y$, a contradiction. $\square$

Although the definition of effect algebras looks elementary, these structures are endowed with several rather remarkable features.

Lemma 2.20. *Let* $\mathbf{A}$ *be an effect algebra. Then it satisfies, for any* $a, b \in A$

$$x \leq y \quad \textit{implies } x \oplus (x \oplus y')' = y \tag{EOL}$$

Proof. See [47, Theorem 2.4]. $\square$

We call the condition (EOL) the *effect orthomodular law.*

Theorem 2.21. *Let* $\mathbf{A}$ *be an effect algebra and* $a, b, c \in A$ *with* $a, b \perp c$*. Then*

(i) $a \oplus c \leq b \oplus c$ *implies* $a \leq b$*;*

(ii) $a \oplus c = b \oplus c$ *implies* $a = b$*,*

i.e. the cancellation laws hold.

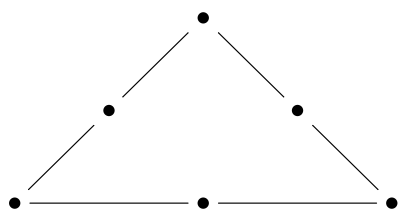
It is easy to notice several affinities between effect algebras and partially ordered Abelian groups (see e.g. [63]). In fact, we have the following:

Theorem 2.22. *Let* $\mathbf{G} = (G, +, -, 0)$ *be a partially ordered Abelian group* $0 \neq u \in G^+$*, where* G^+ *stands for the positive cone of* $\mathbf{G}$*. Defining, for any* $x, y \in G$*,* $x \oplus y = x + y$*, if* $x + y \in [0, u]$*, and* $x' = u - x$*, then the structure* $([0, u], \oplus, ', 0, u)$ *is an effect algebra.*

We call effect algebras which are intervals in partially ordered Abelian groups *interval* effect algebras.

Among effect algebras, orthoalgebras play an important role in the foundation of quantum mechanics. Indeed, many effect systems in a Hilbert space can be regarded as orthoalgebras, if adequately equipped with a suitable notion of orthogonality (see e.g. [57]).
It is easily seen that, since they cannot contain isotropic elements, orthoalgebras satisfy the identities $x \wedge x' \approx 0$ and $x \vee x' \approx 1$, i.e. their induced poset is an orthoposet. Moreover, it can be shown that they are a *proper* generalization of orthomodular posets. In fact, let $\mathbf{A} = (A, \leq, ', 0, 1)$ be an OMP. Let we define, for any $x, y \in A$, $x \oplus y = x \vee y$, if $x \leq y'$ and undefined, otherwise. Then it can be seen that $(A, \oplus, 0, 1)$ is an orthoalgebra. However, there are orthoalgebras which are not orthomodular posets, i.e. they cannot be obtained by some OMP by means of the aforementioned procedure.

Example 2.1. *There exist orthoalgebras which are proper, i.e. they cannot be obtained by an OMP as above. In fact, the* Wright Triangle*, whose Greechie diagram ([47]) is depicted in display* (6.17)*, is the smallest orthoalgebra which is not an OMP.*



(2.5)

We call orthoalgebras which are not OMP's *proper*. Intuitively, an orthoalgebra $\mathbf{A}$ is said to be *proper* if there exists a pair of elements $x, y \in A$ such that $x \perp y$ (thus $x \wedge y = 0$) but $x \vee y$ does not exist. Generalizing the orthoalgebraic case we define proper effect algebras as follows.

Definition 2.23. An effect algebra $\mathbf{A} = (A, \oplus, 0, 1)$ is said to be proper if there exists a pair of elements $x, y \in A$ such that $x \perp y$, $x \wedge y$ exists but $x \vee y$ is not defined.

Moreover, we have a complete characterization of OMP's in the framework of effect algebras. In fact, it turns out that an effect algebra is an OMP if and only if it satisfies the coherence law.

Proposition 2.24 (Theorem 5.3, [47])**.** *An effect algebra* $\mathbf{A}$ *is an OMP if and only if it satisfies the coherence law, i.e. for any* $x, y, z \in A$

$$\text{\textit{If } } x \oplus y, x \oplus z, y \oplus z \text{ \textit{are defined, then} } (x \oplus y) \oplus z \text{ \textit{exists.}} \tag{CL}$$

Moreover, we are able to characterize those effect algebras which are indeed orthoalgebras.

Theorem 2.25 (Theorem 5.1, [47]). *For an effect algebra* $\mathbf{A}$*, the following conditions are mutually equivalent:*

(i) $\mathbf{A}$ *is an orthoalgebra;*

(ii) If $x, y \in A$ *with* $x \perp y$*, then* $x \oplus y$ *is a* minimal upper bound *for* x *and* y *in* A*;*

(iii) $x \wedge x' = 0$*, for any* $x \in A$*;*

(iv) The mapping $' : x \mapsto x'$ *is an orthocomplementation with respect to* $\vee$*.*

Since orthoalgebras generalize orthomodular posets, one might ask if they can be seen as pastings of Boolean algebras as well. Firstly, we note that:

Remark 2.26. Any orthoalgebra is the union of its maximal Boolean subalgebras, i.e. its blocks.

Moreover, we have the following

Lemma 2.27 (Proposition 7.3, [97]). *The pasting of a pasted family of Boolean algebras* $\mathcal{F}$ *(cf. Definition 2.9) is an orthoalgebra.*

The orthoalgebra depicted in Example 2.1 is a prototypical example of pasting in the sense of Definition 2.9. In fact it is the "gluing" of three 8-elements Boolean algebras along three 4-elements Boolean subalgebras.
One might ask if a converse of Lemma holds true as well. Unfortunately, the answer is negative since the intersection of two blocks of an orthoalgebra need not be a Boolean subalgebra (see [99]). Therefore, we need to introduce a more general notion of pasting.

Definition 2.28. Le $\mathcal{F}$ be a family of Boolean algebras such that, for all $\mathbf{A}, \mathbf{B}$ in $\mathcal{F}$,

(PF1) $A \cap B$ is the universe a suborthoalgebra of both $\mathbf{A}, \mathbf{B}$ in which the operations coincide;

(PF2) $\forall x \in A \cap B \, \exists \mathbf{C} \in \mathcal{F} \, ([0, x]_A \cup [0, x']_B \subseteq C)$.

We call $\mathcal{F}$ an *o-pasted family* of Boolean algebras

In order to introduce a suitable notion of o-pasting of Boolean algebras we need the following

Theorem 2.29 ([99]). *Let $\mathcal{F}$ be a family of orthoalgebras that satisfies the following conditions for all distinct $\mathbf{A}, \mathbf{B}$ in $\mathcal{F}$:*

(P1) $1^{\mathbf{A}} = 1^{\mathbf{B}}$;

(P2) if $x \oplus^{\mathbf{A}} y = z$ and B contains at least two of x, y, z, then $x, y, z \in B$ and $x \oplus^{\mathbf{B}} y = z$;

(P3) if $(x \oplus^{\mathbf{A}} y) \oplus^{\mathbf{B}} z$ is defined, then there is a $\mathbf{C}$ in $\mathcal{F}$ such that $(x \oplus^{\mathbf{C}} y) \oplus^{\mathbf{C}} z = (x \oplus^{\mathbf{A}} y) \oplus^{\mathbf{B}} z$.

Let us set $P = \bigcup_{\mathbf{C} \in \mathcal{F}} C$, and define a partial operation $\oplus$ on P as follows: $x \oplus y = z$ if and only if there is a $\mathbf{C} \in \mathcal{F}$ such that $x \oplus^{\mathbf{C}} y = z$. Moreover, define $1 = 1^{\mathbf{C}}$, $0 = 0^{\mathbf{C}}$. Then, $\mathbf{P} = (P, \oplus, 0, 1)$ is an orthoalgebra.

Definition 2.30. The orthoalgebra $\mathbf{P} = (P, \oplus, 0, 1)$ of Theorem 2.29 is called the *o-pasting* of $\mathcal{F}$.

We can finally prove the following

Theorem 2.31 ([99]). *Let $\mathcal{F}$ be a family of Boolean algebras such that, for all $\mathbf{A}, \mathbf{B}$ in $\mathcal{F}$,*

(PF1) $A \cap B$ is the universe of a suborthoalgebra of both $\mathbf{A}, \mathbf{B}$ in which the operations coincide;

(PF2) $\forall x \in A \cap B \, \exists \mathbf{C} \in \mathcal{F} \, ([0, x]_A \cup [0, x']_B \subseteq C)$.

Then, the assumption of Theorem 2.29 are satisfied and the o-pasting $\mathbf{P}$ of $\mathcal{F}$ is an orthoalgebra. Furthermore, each orthoalgebra can be obtained in this way from the family of its blocks.

The above theorem completely characterizes orthoalgebras in terms of pastings of their maximal Boolean subalgebras.

2.1.4 Paraorthomodular lattices

As it has been pointed out above, effect algebras induce posets which are not, in general, lattices, since the canonical order induced by the Born's rule (see [33, Cap. 4]) on the set $\mathcal{E}(\mathcal{H})$ of effects on a Hilbert space $\mathcal{H}$ does not ensure that meets and joins of elements always exist. Therefore, in [59, 60] paraorthomodular lattices, i.e. regular bounded lattices with an antitone involution satisfying the paraorthomodular law have been introduced.

Definition 2.32. An involution lattice $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ is said to be paraorthomodular if the following conditions are satisfied, for any $x, y \in A$:

P1 $x \wedge x' \leq y \vee y'$ *(regularity)*;

P2 $x \leq y$ and $x' \wedge y = 0$ implies $x = y$ *(paraorthomodularity)*.

The importance of these structures rests on the fact that they represent a natural generalization of the lattice ordering on closed subspaces of a Hilbert space to the whole class of effects by means of the so-called *spectral* ordering.
A (bounded) spectral family on a separable Hilbert space $\mathcal{H}$ with set $\Pi(\mathcal{H})$ of projection operators is a map $M : \mathbb{R} \to \Pi(\mathcal{H})$ such that:

a. For any $\lambda, \mu \in \mathbb{R}$, if $\lambda \leq \mu$, then $M(\lambda) \leq M(\mu)$ (monotonicity);

b. For any $\lambda \in \mathbb{R}$, $M(\lambda) = \bigwedge_{\mu > \lambda} M(\mu)$ (right continuity);

c. There exist λ, μ such that, for any $\eta \in \mathbb{R}$, one has

$$M(\eta) = \begin{cases} \mathbb{O}, & \text{if } \eta < \lambda \\ \mathbb{I}, & \text{if } \eta \geq \mu. \end{cases}$$

Any self adjoint linear operator A of $\mathcal{H}$ can be uniquely "decomposed" as $A = \int_{-\infty}^{\infty} x dM(x)$, where the integral is meant in the sense of Riemann-Stieltjes norm-converging sums (see [119, cap. 1]). Now, we can introduce the spectral ordering on the set of effects $\mathcal{E}(\mathcal{H})$ of $\mathcal{H}$ defining, for any $E, F \in \mathcal{E}(\mathcal{H})$,

$$E \leq_s F \;\text{ iff }\; M^F(x) \leq M^E(x), \text{ for any } x \in \mathbb{R}. \tag{SO}$$

$\leq_s$ turns out to differ from (CO). In fact, the condition

$$E \leq_s F \text{ iff } F - E \geq_s \mathbb{O}$$

need not be satisfied (see [103]). Moreover, the order induced by (SO) *naturally* turns $\mathcal{E}(\mathcal{H})$ into a (conditionally) complete bounded lattice. Finally, setting $'$ as $E' = \mathbb{I} - E$, for any $E \in (\mathcal{H})$, $\mathcal{E}(\mathcal{H})$ becomes a paraorthomodular lattice. Therefore, under the spectral ordering, the whole set of effects is amenable of lattice theoretical analysis.
We remark that several varieties of involution lattices of prominent importance for mathematical logic are indeed paraorthomodular.

Proposition 2.33. *Any modular involution lattice satisfying regularity is paraorthomodular.*

Proof. It follows by noticing that any modular lattice cannot contain $\mathbf{B}_6$ as its sublattice (cf. Theorem 2.3). □

Distributive and regular involution lattices are called *Kleene* lattices while their non-necessarily-distributive generalizations are often denoted in the literature as *pseudo-Kleene* lattices. They will play an important role expecially in Chapter 5.
It is worth highlighting that paraorthomodular lattices are not, in general, ortholattices, since the identity $x \wedge x' \approx 0$ is no longer assumed (cf. e.g. Example 5.1). Therefore, in this much more general context, the paraorthomodularity condition is no longer equivalent to the orthomodular law since the latter implies that $x \wedge x' \approx 0$ holds. In fact, by $x \leq 1$, one has that $1 = (x \vee (x' \wedge 1)) = (x \vee x')$, i.e. $x \wedge x' = 0$.

2.1.5 Lattice effect algebras and Lattice Pseudoeffect algebras

Obviously, proper effect algebras (see Definition 2.23) are neither lattice-ordered nor completable via Dedekind-MacNeille completion [110]. In fact, they need not be (join-, meet-) densely embeddable into an effect algebra whose underlying poset is a complete lattice (cf. Subsection 1.1.1).
Due to their widespread application in logic and the foundation of quantum mechanics (see e.g. [51, 50]), and their multiple connections with algebraic structures of prominent relevance in abstract algebraic studies, lattice effect algebras represent a fruitful and increasing field of inquiry.

Definition 2.34. An effect algebra $\mathbf{A}$ is a *lattice* effect algebra if the partial order induced by $\oplus$ is a lattice.

Since lattice operations are always defined, they will be included in the type. Thus, any lattice effect algebra $\mathbf{A}$ will have the form $\mathbf{A} = (A, \wedge, \vee, \oplus, 0, 1)$.
Clearly, any orthomodular lattice can be turned into a lattice effect algebra and, by Theorem 2.25 (ii), it is easily seen that any lattice ordered orthoalgebra is an orthomodular lattice.
Interestingly enough, lattice effect algebra are capable of being described by means of their block structure. Infact, a celebrated result by Z. Riečanová ([112]) states that any lattice effect algebra is the union of its blocks, i.e. maximal subalgebras which are MV algebras (see Section 2.2). Some techniques for constructing lattice effect algebras starting from MV algebras and lattice effect algebras can be found in [128]. Nevertheless, at the best of our knowledge, the above achievements have been only partially extended to the general case of effect algebras. Indeed, the problem of finding a complete characterization of effect algebras in terms of blocks is still open.

Pseudoeffect algebras were introduced in 2001 by A. Dvurečenskij and T. Vetterlein ([40]) in order to provide a non-commutative generalization of effect algebras.

Definition 2.35. A structure $\mathbf{A} = (A, \oplus, 0, 1)$, with $\oplus$ a partial binary operation, and 0, 1 constant functions, is a *pseudoeffect algebra* if, for all $a, b, c \in A$, the following conditions are satisfied:

(E1) $a \oplus b$ and $(a \oplus b) \oplus c$ are defined iff so are $b \oplus c$ and $a \oplus (b \oplus c)$, and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;

(E2) there is exactly one $d \in A$, and exactly one $e \in A$ such that $a \oplus d = e \oplus a = 1$;

(E3) if $a \oplus b$ exists, then there are elements $c, d \in A$ such that $c \oplus a = a \oplus b = b \oplus d$;

(E4) if $1 \oplus a$ or $a \oplus 1$ exists, then $a = 0$.

In view of condition (E2), with a slight notational abuse, we may define two complementation operations $^{\sim}, ^{-}$ on a pseudoeffect algebra $\mathbf{A}$ by requiring that, for any $a \in A$,

$$a \oplus a^{\sim} = 1 = a^{-} \oplus a. \tag{EC}$$

Therefore, we will freely consider a pseudoeffect algebra as a structure in the language $(\oplus, ^{\sim}, ^{-}, 0, 1)$, where $\oplus$ is a partial binary operation, $^{\sim}, ^{-}$ are unary operations, and $0, 1$ are constants such that conditions (E1),(EC),(E3) and (E4) are satisfied. Moreover, for notational clarity, since $\oplus$ is associative, we will omit unnecessary parentheses, whenever possible. Finally, it is perhaps worth recalling that, if the partial operation $\oplus$ is commutative, then the structure in question is indeed an effect algebra [40, Proposition 1.3].
As for effect algebras, it is possible to introduce a partial order on a pseudoeffect algebra $\mathbf{A}$ by setting, for any $x, y \in A$:

$$x \leq y \text{ if there exists } z \in A \text{ such that } x \oplus z = y \tag{2.6}$$

Moreover, by $(E3)$, the order is readily seen to be two-sided, i.e. for any pseudoeffect algebra $\mathbf{A}$ if $x \leq y$ in $\mathbf{A}$ then there exists z such that $z \oplus x = y$. It is not difficult to see that the induced poset is upper (lower) bounded by 1 (0) and the unary operations $^{\sim}$ and $^{-}$ are antitone although not necessarily involutive, i.e. one has $x \leq y$ implies $y^{\sim} \leq x^{\sim}$ and $y^{-} \leq x^{-}$ (see Chapter 4) but $x^{\sim\sim} \approx x$ $(x^{--} \approx x)$. Finally, given a pseudoeffect algebra $\mathbf{A}$, if the order induced by $\oplus$ is a lattice then $\mathbf{A}$ is said to be a *lattice pseudoeffect algebra*.
One of the interesting aspects of pseudoeffect algebras is that they can be put into relationship with well studied structures like partially ordered groups (see e.g. [41] for details).

Example 2.2 (Definition 2.1, [40]). *Consider a (not necessarily Abelian) partially ordered group* $\mathbf{G} = (G, \leq, \cdot, ^{-1}, e)$. *We call an element* $u \in G^+$ *(cf. Theorem 2.22) a* strong *unit if, for any* $g \in G$, *there exists* $n \in \mathbb{N}$ *such that* $u^{-n} \leq g \leq u^n$, *where* $x^n = (...(((x \cdot x) \cdot x)...)$ $(x^{-n} = (...(((x^{-1} \cdot x^{-1}) \cdot x^{-1})...))$ *n-times. For any element* $x \in [e, u]_{G^+}$ *consider the element* $x^- = u \cdot x^{-1}$ *and* $x^\sim = x^{-1} \cdot u$. *Then, putting for any* $x, y \in [e, u]_{G^+}$ $x \oplus y = x \cdot y$, *whenever* $x \cdot y \in [e, u]$, *the structure* $([e, u]_{G^+}, \cdot, ^\sim, ^-, e, u)$ *is provably a pseudoeffect algebra called* interval *pseudoeffect algebra.*

Moreover, it has been proven that pseudoeffect algebras satisfying RDP_1, namely a weaker form of the Riesz decomposition property are categorically equivalent to partially ordered groups with strong unit satisfying RDP_1. We refer the reader to [41] for details.

2.2 MV algebras, pseudo-MV algebras, Basic algebras

In this section we recall basic notions related to MV algebras and their generalizations.

2.2.1 MV algebras and pseudo-MV algebras

MV algebras were introduced by C.C. Chang in 1958 as an algebraic counterpart of the infinite-valued Łukasiewicz logic. In what follows we provide the definition and a very few facts concerning MV algebras. The interested reader is referred to [13] for overviews and deep investigations on the subject.
An MV-algebra $\mathbf{A}$ is an Abelian monoid $(A, \oplus, 0)$ equipped with an operation $\neg$ such that $\neg\neg 0 \approx x$, $x \oplus \neg 0 = \neg 0$ and the so-called Łukasiewicz identity holds:

$$\neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x \tag{LI}$$

Moreover, it can be proven that $\neg(\neg x \oplus y) \oplus y = x \vee y$.
The standard example of an MV-algebra is given by the real unit interval $[0, 1]$ equipped with the operations $\neg x = 1 - x$ and $x \oplus y = min(1, x + y)$. It is easily seen that the structure so defined is, under the natural order on the real interval $[0, 1]$, a totally ordered involution lattice with an order-preserving binary operation. We call the interval $[0, 1]$ equipped with operations as above the *standard MV algebra*. Although simple, this structure enjoys quite strong and interesting properties. In fact, Chang's completeness theorem ([13, Theorem 2.5.3]) states that the above algebra generates the whole variety of MV algebras. This means that any equation is valid in $[0, 1]$ if and only if it is valid in any MV algebra.

Definition 2.36. An MV algebra is an algebra $\mathbf{A} = (a, \oplus, \neg, 0)$ of type $\langle(2, 1, 0)$ satisfying the following equations:

(MV1) $(x \oplus y) \oplus z \approx x \oplus (y \oplus z)$;

(MV1) $x \oplus y \approx y \oplus x$;

(MV1) $x \oplus 0 \approx x$;

(MV1) $\neg\neg x \approx x$;

(MV1) $x \oplus 1 \approx 1$, where $1 = \neg 0$;

(MV1) $\neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x$.

Given an MV algebra $\mathbf{A}$, if we put $x \leq y$ if and only if $\neg x \oplus y = 1$, then it turns out that $\leq$ is a partial (lattice-) order. Moreover, $\neg$ is an antitone involution with respect to $\leq$. One might have noticed that the class of MV algebras has several affinities with lattice effect algebras. In fact, any MV algebra is a lattice effect algebra by letting $\oplus$ be defined for orthogonal elements only.
A famous theorem by C.C. Chang completely characterizes MV algebras as subdirect products of MV chains (i,e. totally ordered MV algebras).

Theorem 2.37. *Any MV algebra* $\mathbf{A}$ *is the subdirect product of MV chains.*

Clearly, since any totally ordered lattice is distributive and, of course, distributivity is preserved under direct products, subalgebras and onto homomorphisms, the lattice subreduct of any MV algebra is distributive. Finally, we recall the following famous theorem by D. Mundici

Theorem 2.38 ([13])**.** *Any MV algebra* $\mathbf{A}$ *is isomorphic to the interval* $[0, u]$ *of some lattice ordered group with strong unit* u*.*

As pseudoeffect algebras generalize effect algebras, pseudo-MV algebras represent a non-commutative generalization of MV algebras. Pseudo-MV algebras were introduced by Georgescu and Iorgulescu in [55], and independently considered, under a different name, by Rachůnek in [106]. For them, Dvurecĕnskij proved in [37] that any pseudo-MV algebra is always an interval in a unital (not necessarily Abelian) lattice ordered group (G, u).

Definition 2.39. A *pseudo-MV algebra* is an algebra $\mathbf{A} = (A, \oplus, \odot, ^{-}, ^{\sim}, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ satisfying the following equations:

(M1) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;

(M2) $x \oplus 0 = 0 \oplus x = x$;

(M3) $x \oplus 1 = 1 \oplus x = 1$;

(M4) $1^{\sim} = 1^{-} = 0$;

(M5) $(x^{-} \oplus y^{-})^{\sim} = (x^{\sim} \oplus y^{\sim})^{-}$;

(M6) $x \oplus (x^{\sim} \odot y) = y \oplus (y^{\sim} \odot x) = (x \odot y^{-}) \oplus y = (y \odot x^{-}) \oplus x$;

(M7) $x \odot (x^{-} \oplus y) = (x \oplus y^{\sim}) \odot y$;

(M8) $x^{-\sim} = x^{\sim -} = x$.

It turns out that (cf. [55, Proposition 1.13])

$$x \odot (x^{-} \oplus y) = y \odot (y^{-} \oplus x) = (y \oplus x^{\sim}) \odot x = (x \oplus y^{\sim}) \odot y.$$

Let us note that in any pseudo-MV algebra $\oplus$ and $\odot$ are inter-derivable.

$$x \odot y = (y^{-} \oplus x^{-})^{\sim} = (y^{\sim} \oplus x^{\sim})^{-}. \tag{2.7}$$

Moreover, in every pseudo-MV algebra a partial order is term-definable (see [55, Proposition 1.10]) by setting

$$x \leq y \text{ iff } x^{-} \oplus y = 1. \tag{2.8}$$

It is well known that this partial order is indeed a lattice ordering, whose infima and suprema are defined as follows:

(i) $x \vee y = x \oplus (x^{\sim} \odot y) = y \oplus (y^{\sim} \odot x) = (x \odot y^{-}) \oplus y = (y \odot x^{-}) \oplus x$;

(ii) $x \wedge y = x \odot (x^{-} \oplus y) = y \odot (y^{-} \oplus x) = (x \oplus y^{\sim}) \odot y = (y \oplus x^{\sim}) \odot x$.

Since pseudo-MV algebras will be the subject of subsequent chapters, we postpone an analysis of arithmetical properties of these structures when it will be needed.

2.2.2 Basic Algebras

Basic algebras were introduced in [21] by I. Chajda, R. Halaš and J. Kühr with the aim of providing a common generalization of orthomodular lattices and MV algebras. These structures grant a basic tool for discerning quantum structures and algebras arising in the framework of many-valued logics by simple additional conditions. Moreover, it represent an environment in which the above mentioned structures can be put into relationship from an arithmetical as well as from a structural perspective.

Definition 2.40. A basic algebra is an algebra $\mathbf{A} = (A, \oplus, \neg, 0)$ satisfying the following identities:

B1 $x \oplus 0 = x$;

B2 $\neg\neg x = x$;

B3 $\neg(\neg x \oplus y) \oplus y = \neg(y \oplus \neg x) \oplus x$;

B4 $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$ *(where* $\neg 0 = 1$*).*

In any basic algebras the order defined by $x \leq y$ if and only if $\neg x \oplus y = 1$ is a lattice order, whose corresponding join and meet are defined as

$$x \vee y = \neg(\neg x \oplus y) \oplus y \quad \text{ and } \quad x \wedge y = \neg(\neg x \vee \neg y).$$

It can be proven that the lattice induced by a basic algebra $\mathbf{A}$ is upper (lower) bounded by 1. Furthermore, the mapping $^a : x \mapsto \neg x \oplus a$ is an antitone involution on the section $[a, 1]$.

Definition 2.41. A *bounded lattice with sectional antitone involution* $\mathbf{A}$ is a system $(A, \wedge, \vee, \{^a\}_{a \in A}, 0, 1)$ such that:

- $(A, \wedge, \vee, 0, 1)$ is a bounded lattice
- For any $a \in A$ the operation ^{a} is an antitone involution on the interval $[a, 1]$, i.e. for any $x \in A$ it holds that $(x^a)^a \approx x$ and $x \leq y$ implies $y^a \leq x^a$.

Theorem 2.42 (Theorem 2.5, [21])**.** *Let* $\mathbf{A}$ *be a basic algebra. Then*

$$\mathcal{L}(\mathbf{A}) = (A, \vee, \wedge, \{^a\}_{a \in A}, 0, 1)$$

is a bounded lattice with sectional antitone involutions.

Theorem 2.43 (Theorem 2.6, [21])**.** *Let* $\mathbf{L} = (A, \wedge, \vee, \{^a\}_{a \in A}, 0, 1)$ *be a bounded lattice with sectional antitone involutions. Then the algebra* $\mathcal{A}(L) = (L, \oplus, \neg, 0)$*,where* $x \oplus y := (x^0 \vee y)^y$ *and* $\neg x = x^0$*, is a basic algebra.*

A little thought shows that every MV-algebra is a basic algebra, indeed, a commutative basic algebra. For the converse, it has been shown that any commutative basic algebra whose underlying lattice is complete is a complete MV algebra, but this does not hold in general. Indeed there exist examples of infinite commutative basic algebras which are not MV algebras (see [8] for details)

Theorem 2.44 (Theorem 5, [21]). *A basic algebra is an MV algebra if and only if it is associative, i.e. it satisfies*

$$x \oplus (y \oplus z) \approx (x \oplus y) \oplus z$$

It is worth observing that, also in the general framework of basic algebras, we can still define a notion of *block*.

Definition 2.45 (Definition 3.15, [21]). Given a basic algebra $\mathbf{A} = (A, \oplus, \neg, 0)$, we say that a non-empty subset $B \subseteq A$ is a block if B is a maximal set with the property that $x \oplus y = y \oplus x$ for all $x, y \in B$; in other words, a block is a maximal set whose elements *pairwise commute.*

By Zorn's lemma each element $a \in A$ is contained in a block of A, hence every basic algebra is the set-theoretical union of its blocks.
In light of the above considerations, one might wonder if, as for the orthomodular case, any block of a basic algebra $\mathbf{A}$ is a subalgebra. Surprisingly enough, we have the following.

Theorem 2.46 (Theorem 7.9, [21]). *For every basic algebra* $\mathbf{A} = (A, \oplus, \neg, 0)$*, the following are equivalent:*

1. *Every block of* A *is a subalgebra that is an MV algebra;*
2. $\mathbf{A}$ *is a lattice effect algebra.*

Thus, among basic algebras, lattice effect algebras can be characterized as "unions" of MV algebras which are indeed maximal subalgebras of mutually commuting elements. Moreover, it can be proven something more:

Theorem 2.47 (Corollary 7.10, [21]). *If* $\mathbf{A} = (A, \oplus, \neg, 0)$ *is a finite basic algebra, then the following are equivalent:*

1. *Every block of* $\mathbf{A}$ *is a subalgebra of* $\mathbf{A}$*;*
2. $\mathbf{A}$ *is a lattice effect algebra.*

This result might suggest that the distinguishing trait of quantum structures strongly relies on the notion of pasting. However, to the best of our knowledge, the problem as to wether Theorem 2.46 can be generalized to basic algebras of arbitrary cardinality has not been solved yet.

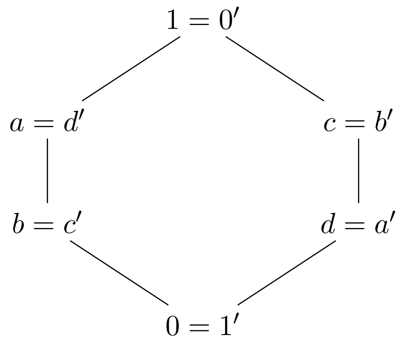
Clearly, orthomodular lattices, MV algebras and lattice effect algebras satisfy the paraorthomodularity condition (see Definition 2.32). However, this no longer holds when we consider the *whole* variety of Basic algebras.

Example 2.3. *Let us consider the Basic algebra* $\mathbf{A}$ *depicted below.*

$\oplus$	0	1	a	b	c	d
0	0	1	a	b	c	d
1	1	1	1	1	1	1
a	a	1	a	b	1	c
b	b	1	a	b	1	1
c	c	1	1	1	c	d
d	d	1	b	1	c	d

(2.9)

Clearly, $\mathbf{A}$ *does not satisfy the paraorthomodularity condition, since its underlying lattice is an instance of* $\mathbf{B}_6$*.*



(2.10)

As it has been pointed out in the previous subsections, the paraorthomodularity condition is a cross-cutting concept in quantum structures and in the wider context of "events algebras", i.e. the algebraic structures aimed at capturing the algebraic properties of events. Hence, it is worth asking if there are varieties of *proper* basic algebras (i.e. basic algebras which are neither MV algebras nor lattice effect algebras) fulfilling the paraorthomodularity condition. If such algebras exist, then we could conclude that paraorthomodularity is quite independent from particular properties of the aforementioned structures.

Of course, any basic algebra having an underlying distributive lattice, e.g. weakly monotone basic algebras ([10, Lemma 4.2]), are "paraorthomodular". However, there are basic algebras which are neither, in general, weakly monotone, nor lattice effect algebras, whose underlying involution lattice still have the paraorthomodularity property.

In fact, we close this subsection by showing that any basic algebra **A** satisfying the simple condition (A) below has the paraorthomodularity property. Moreover, as for weakly monotone basic algebras (see [10, Theorem 4.6]) the set $\mathcal{S}(\mathbf{A}) = \{a \in A : x \wedge \neg x = 0\}$ of "sharp" elements of **A** forms a Boolean sub(involution)lattice of the underlying involution lattice naturally induced by **A**. Therefore, the only OMLs satisfying (A) are Boolean algebras. Since, OMLs can be thought, in basic algebras framework, as a subvariety of lattice effect algebras (see [21]), we conclude that lattice effect algebras do not satisfy (A) as well. Finally, Example 2.4 shows that there are non-distributive basic algebras which still satisfy the aforementioned condition. Thus, basic algebras introduced in Proposition 2.48 represent a proper generalization of distributive basic algebras.

Proposition 2.48. *Let* **A** *be a basic algebra. If* **A** *satisfies the condition*

$$(x \vee \neg x) \wedge y \approx (x \wedge y) \vee (\neg x \wedge y), \tag{A}$$

then:

1. **A** *satisfies the paraorthomodularity condition;*
2. $\mathcal{S}(\mathbf{A})$ *forms a sub(involution)lattice of the involutive lattice reduct of* **A***;*
3. $\mathcal{S}(\mathbf{A})$ *is a Boolean algebra.*

Proof. As regards (1), just note that if $x \leq y$ and $x' \wedge y = 0$, then $y = 1 \wedge y = (x \vee x') \wedge y = (x \wedge y) \vee (x' \wedge y) = (x \wedge y) = x$. For (2), if $x \in \mathcal{S}(\mathbf{A})$, then $\neg x \in \mathcal{S}(\mathbf{A})$ as well. Now, suppose that $x \wedge \neg x = 0$ and $y \wedge \neg y = 0$. One has $x = (y \vee \neg y) \wedge x = (x \wedge y) \vee (x \wedge \neg y)$. Similarly, it follows that $y = (xy) \vee (\neg x \wedge y)$. Thus $y \vee \neg y = \neg y \vee (xy) \vee (\neg x \wedge y)$ and $x \vee \neg x = \neg x \vee (x \wedge y) \vee (x \wedge \neg y)$. Thus, $1 = \neg y \vee (xy) \vee (\neg x \wedge y) \vee \neg x \vee (x \wedge y) \vee (x \wedge \neg y) = (x \wedge y) \vee \neg(x \wedge y)$. We conclude that $\mathcal{S}(\mathbf{A})$ is closed under $\wedge$ and, since $\neg$ is an antitone involution, our claim is proved. Finally, we observe that $\mathcal{S}(\mathbf{A})$ contains 0 and 1 and, of course it is a paraorthomodular ortholattice, by $(1) - (2)$. Hence, $\mathcal{S}(\mathbf{A})$ is an orthomodular lattice, by Theorem 2.3. By general considerations on orthomodular lattices ([3]), since for any $x, y \in \mathcal{S}(\mathbf{A})$, one has $x = x \wedge (y \vee \neg y) = (x \wedge y) \vee (x \wedge \neg y)$, we conclude that $\mathcal{S}(\mathbf{A})$ is a Boolean sublattice of **A**. $\square$

Example 2.4. *Consider the the basic algebra* $\mathbf{A}$ *depicted below.*

$\oplus$	0	1	a	b	c
0	0	1	a	b	c
1	1	1	1	1	1
a	a	1	1	b	c
b	b	1	a	1	c
c	c	1	a	b	1

(2.11)

Note that $\mathbf{A}$ *satisfies* (A) *but it is neither distributive nor regular.*

$$
\begin{tikzcd}[arrows=-]
& 1 = 0' \arrow[dl] \arrow[d] \arrow[dr] & \\
a = a' \arrow[dr] & b = b' \arrow[d] & c = c' \arrow[dl] \\
& 0 = 1' &
\end{tikzcd}
\tag{2.12}
$$

Chapter 3

On the structure theory of Basic Algebras as near semirings

As it has been observed in Chapter 2, quantum structures as well as MV algebras have several distinguishing traits e.g. they have different types, operations which can be total or partial, induced orders which can be lattices or just posets...*etc*. Nevertheless, they enjoy several common features both from an order theoretical and an algebraic perspective.

In order to capture similarities among classes of algebras, it is a common practice in mathematics trying to find a sufficiently abstract framework in which seemingly different algebraic structures can be put under a common umbrella. As we have seen in Subsection 2.2.2, basic algebras provide a sufficiently powerful machinery in which most of the structures we are dealing with in this thesis can be embedded into. In fact, orthomodular lattices, lattice effect algebras, and MV algebras, can all be regarded as proper subvarieties of basic algebras.

A classical field of inquiry in abstract algebra is structure theory, i.e. the systematic study of kernels of homomorphisms, e.g. filters and ideals, and more generally congruences in classes of algebras. The importance of such a direction of research is given by the multiple relationships that the features of these objects might entertain with the general algebraic properties of structures they arise from. In particular, some varieties of algebras have ideals and filters which are into one-to-one correspondence with congruences. Therefore, an in-depth analysis of the former carries precious information about the latter.

Semirings were introduced by Vandiver [125] in 1934. In more recent times they have been deeply studied, especially in relation with applications. For example semirings have been used to model formal languages and automata theory (see [43]), and semirings over

real numbers, the so-called ((max, +)-semirings) have been fruitfully employed in idempotent analysis. A nice monograph on the subject is [62].
Recently, connections between semirings and infinite-valued logics algebraic semantics have been studied by Belluce, Di Nola and Ferraioli who have introduced MV semirings which turn out to be equivalent, as a category, to MV algebras [2]. The development of this theory has yielded several achievements and applications e.g. to automata theory ([56]). Moreover, semirings have received attention from logicians community due to their connections with residuation theory and, in particular, with residuated lattices, i.e. the algebraic counterpart of *substructural logics*. In fact, any residuated lattice can be turned into a semiring. Therefore, studies on semiring-like structures have been expedient for investigating the connections between structures arising in several, often different fields of logic and algebra (see e.g. [82]).

A semiring-like approach has been carried on by S. Bonzio, I. Chajda and A. Ledda in [6] for basic algebras. In fact, exploiting the notion of near semiring introduced by Länger and Chajda in [29, 30], they have shown that basic algebras are categorically equivalent, more precisely, term equivalent to the so called *Łukasiewicz near semirings*. These results suggest that, besides their theoretical meaningfulness, basic algebras might led to fruitful applications.
Clearly, any sub-variety of basic algebras can be framed within the context of Łukasiewicz near semirings. Thus, orthomodular lattices and their extensions as well as MV algebras can all be interpreted as semiring-like structures (with involution) satisfying further axioms. A natural question arises: is it possible to develop a general structure theory for near semirings in order to capture the "unifying" traits of quantum and many-valued logics algebras? Fortunately, the answer turns out to be yes. In fact, since Łukasiewicz near semirings are congruence regular, i.e. every congruence is completely determined by its 0-coset (its kernel, see Subsection 1.2.4), their structure is amenable of a smooth description. Therefore, a general theory can be developed by asking if kernels might be characterized set-theoretically, namely if a notion of *ideal* that properly matches with congruences can be introduced.
In light of the above considerations, in this chapter we define a notion of ideal for Łukasiewicz near semirings. It will turn out that these objects can be neatly characterized by means of sets closed under certain term operations. In particular, in case an element e is *central* (i.e. it generates an ideal corresponding to a factor congruence, cf. below and Chapter 1) then the ideal it generates is amenable of a neat order-theoretical characterization: it corresponds to the interval $[0, e]$. Although the notion of centrality is easily captured in the variety of involutive idempotent integral near semiring and Łukasiewicz near semiring, this concept yields rather strong properties. Indeed, by

virtue of this characterization, in the last section of this chapter, we propose a rather comprehensive algebraic version of the Cantor-Bernstein theorem for *left-residuable* involutive idempotent integral near semirings. It seems to us that this theorem implies a fairly general fact: even if an algebra is not a lattice (involutive idempotent integral near semirings, in general, need not be lattices, but semilattices) its inner structure is captured by means of the intervals $[0, e]$ (which are indeed ideals in Łukasiewicz near semirings!), with e a central element. This theorem subsumes analogous results for the structures we are dealing with in this thesis.

3.1 Basic algebras as semiring-like structures

In this section we define Łukasiewicz near semirings by recalling basic results from [6].

Definition 3.1. A *near semiring* is an algebra $\mathbf{A} = (A, +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ such that

(i) $(A, +, 0)$ is a commutative monoid;

(ii) $(A, \cdot, 1)$ is a groupoid satisfying $x \cdot 1 = x = 1 \cdot x$ (a unital groupoid);

(iii) $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$;

(iv) $x \cdot 0 = 0 \cdot x = 0$.

The algebra $\mathbf{A}$ is said to be *idempotent* if it satisfies

(v) $x + x = x$;

and *integral* if it satisfies

(vi) $x + 1 = 1$.

For brevity sake, since no danger of confusion will be possible, with a slight abuse of language, by *near semiring* (*semiring*) we will mean *idempotent integral near semiring* (*semiring*). An extensive guide to the bibliography on semirings is in Głazek's monograph [64].

In case $\mathbf{A}$ is idempotent, it can be seen that the reduct $(R, +)$ is a join semilattice, whose partial ordering $\leq^{\mathbf{A}}$ will be called the *induced order* of $\mathbf{R}$. A moment's reflection shows that 0 is its least element. From now on, for notational clarity, whenever no danger of confusion is impending we will omit unnecessary superscripts.

Definition 3.2. An *involutive near semiring* (briefly, ι-near semiring) is an algebra $\mathbf{A} = (A, +, \cdot, ^{\alpha}, 0, 1)$ of type $\langle 2, 2, 1, 0, 0\rangle$ such that

(i) the reduct $(A, +, \cdot, 0, 1)$ is an involutive idempotent integral near semiring;

(ii) $(A, \cdot, 1)$ is a groupoid satisfying $x \cdot 1 \approx x \approx 1 \cdot x$ (a unital groupoid);

(iii) $x^{\alpha\alpha} \approx x$;

(iv) if $x \leq y$, then $y^{\alpha} \leq x^{\alpha}$.

A ι-near semiring semiring is a *Łukasiewicz near semiring* if it satisfies the following further condition:

(vii) $(x \cdot y^{\alpha})^{\alpha} \cdot y^{\alpha} \approx (y \cdot x^{\alpha})^{\alpha} \cdot x^{\alpha}$.

Let us remark that in any ι-near semiring one has that $0^{\alpha} = 1$. Furthermore, it is easily seen that, since $x \leq x + y$ (by (i)), it follows that $(x + y)^{\alpha} \leq x^{\alpha}$ (by (vi)). Hence, we have that

$$(x + y)^{\alpha} + x^{\alpha} = x^{\alpha}. \tag{viii}$$

For notational clarity, whenever it's possible, we will omit the symbol "$\cdot$" and use juxtaposition: by xy we mean $x \cdot y$.

Theorem 3.3 (Theorem 5, [6]). *If $\mathbf{A} = (A, \oplus, \neg, 0)$ is a basic algebra, then the structure $\mathcal{R}(\mathbf{A}) = (B, +, \cdot, ^{\alpha}, 0, 1)$, where $x+y$, $x\cdot y$ and x^{α} are defined by $\neg(\neg x \oplus y) \oplus y$, $\neg(\neg x \oplus \neg y)$, $\neg x$, and $1 = \neg 0$, respectively, is a Łukasiewicz near semiring.*

Hence, any basic algebra can be converted into a Łukasiewicz near semiring. However, we can also prove that any Łukasiewicz near semiring can be turned into a basic algebra showing that these structures are indeed term equivalent.

Theorem 3.4 (Theorem 4, [6]). *If $\mathbf{A}$ is a Łukasiewicz near semiring, then the structure $\mathcal{B}(\mathbf{A}) = (A, \oplus, ^{\alpha}, 0)$, where $x \oplus y$ is defined by $((x^{\alpha} \cdot y) \cdot y^{\alpha})^{\alpha}$ is a basic algebra.*

Moreover, it can be proven (see [6, Theorem 6]) that the mapping $\mathcal{R}$ assigning to any basic algebra a Łukasiewicz near semiring and $\mathcal{B}$ of the above theorem are mutually inverses, i.e. if $\mathbf{A}$ is a Łukasiewicz near semiring resp. $\mathbf{B}$ is a basic algebra, one has $\mathcal{R}(\mathcal{B}(\mathbf{A})) = \mathbf{A}$ and $\mathcal{B}(\mathcal{R}(\mathbf{B})) = \mathbf{B}$, respectively. Therefore, basic algebras are into one-to-one correspondence with Łukasiewicz near semirings.
The following result is Lemma 3 in [6]. It will be useful for the development of our arguments.

Proposition 3.5. *In any Łukasiewicz near semiring the following identities hold:*

(a) $xx^\alpha \approx x^\alpha x \approx 0$;

(b) $x + y \approx ((x \cdot y^\alpha)^\alpha \cdot y^\alpha)^\alpha$.

A *Łukasiewicz semiring* $\mathbf{A}$ is a Łukasiewicz near semiring such that the reduct $(A, \cdot, 1)$ is a monoid.

Proposition 3.6. *Let* $\mathbf{A}$ *be a Łukasiewicz near semiring whose multiplication is associative. Then multiplication is also commutative, and therefore* $\mathbf{A}$ *is a commutative Łukasiewicz semiring.*

Thus we can conclude that in any Łukasiewicz semiring the groupoidal operation $\cdot$ is *commutative* and *right distributive*: the equation $z \cdot (x + y) = (z \cdot x) + (z \cdot y)$ is satisfied. In other words, the reduct $(A, +, \cdot, 0, 1)$ is a semiring.
As basic algebras correspond to Łukasiewicz near semirings, MV algebras match with Łukasiewicz semirings. In fact, since MV algebras are nothing but commutative and associative basic algebras, their near semiring "companions" enjoy the same properties.

3.2 Ideals in Łukasiewicz near semirings

Łukasiewicz near semirings are *arithmetical*, i.e. congruence distributive and permutable. [6, Theorem 8]. Moreover, they are also congruence regular (see [6, Theorem 7], Subsection 1.2.4). *A fortiori*, every congruence θ is fully specified by its kernel $[0]_\theta$. Therefore, it seems quite reasonable to wonder whether this class could be amenable of a smooth set-theoretical characterization. With this aim in mind we introduce the following definition:

Definition 3.7. Let $\mathbf{A}$ be a Łukasiewicz near semiring. A set $I \subseteq A$ is called an *ideal* if $0 \in I$ and the following conditions hold:

(I1) if $ab^\alpha \in I$ and $b \in I$, then $a \in I$;

(I2) if $a^\alpha b$, $b^\alpha a \in I$, then $(ac)^\alpha \cdot (bc)$, $(ca)^\alpha \cdot (cb) \in I$, for any $c \in A$.

Let us observe that, setting $c = b^\alpha$ in condition (I2) we immediately obtain

(I3) if $a^\alpha b$, $b^\alpha a \in I$, then $ab^\alpha \in I$.

We will denote by $\mathrm{Con}(\mathbf{A})$ and $\mathrm{Id}(\mathbf{A})$ the sets of congruences and ideals of $\mathbf{A}$, respectively.

Let us observe that, for any congruence θ on a Łukasiewicz near semiring $\mathbf{A}$, and any $a \in A$, $[a]_\theta$ is *convex*. In fact, if $c \in [a]_\theta$ and $a \leq b \leq c$, then $b = a + b\theta c + b = c$. The following lemma characterizes, for every congruence, the relative kernel. It can be seen that, for any Łukasiewicz near semiring $\mathbf{A}$, the following facts hold true.

Lemma 3.8. *If $\theta \in \mathrm{Con}(\mathbf{A})$, then $a\theta b$ if and only if $a^\alpha b$, $b^\alpha a \in [0]_\theta$.*

Proof. If $a\theta b$, then $a^\alpha b\theta b^\alpha b = 0$, and dually for $b^\alpha a$. Conversely, if $a^\alpha b$, $b^\alpha a \in [0]_\theta$, then if $(a^\alpha b)^\alpha \theta 0^\alpha = 1$, and so $(a^\alpha b)^\alpha b\theta 1b = b$, and dually $(b^\alpha a)^\alpha a\theta a$. But then, $b\theta(a^\alpha b)^\alpha b = (b^\alpha a)^\alpha a\theta a$. $\square$

It turns out that, for any congruence θ, the coset $[0]_\theta$ is an ideal.

Theorem 3.9. *If $\theta \in \mathrm{Con}(\mathbf{A})$, then $[0]_\theta \in \mathrm{Id}(\mathbf{A})$.*

Proof. It is clear that $0 \in [0]_\theta$. For (I1), if $ab^\alpha \in [0]_\theta$ and $[b]_\theta = [0]_\theta$, then $[0]_\theta = [ab^\alpha]_\theta = [a]_\theta[b^\alpha]_\theta = [a]_\theta[b]_\theta^\alpha = [a]_\theta[0]_\theta^\alpha = [a]_\theta[1]_\theta = [a]_\theta$. Finally, for condition (I2), if $a^\alpha b$, $b^\alpha a \in [0]_\theta$, again by Lemma 3.8, $a\theta b$. Hence, $ac\theta bc$ and $ca\theta cb$. Therefore, by Lemma 3.8, $(ac)^\alpha(bc)\theta 0$ and $(ca)^\alpha(cb)\theta 0$. $\square$

Conversely,

Theorem 3.10. *If $I \in \mathrm{Id}(\mathbf{A})$, then the relation $\theta(I)$, defined for all a, $b \in A$ by*

$$a\theta(I)b \Leftrightarrow a^\alpha b,\, b^\alpha a \in I, \tag{3.1}$$

is a congruence on $\mathbf{A}$, and $[0]_{\theta(I)} = I$.

Proof. Reflexivity and symmetry are straightforward. As regards transitivity, suppose that $a^\alpha b$, $b^\alpha a$, $b^\alpha c, c^\alpha b \in I$, then, by condition (I2), $(c^\alpha a)^\alpha(c^\alpha b)$, $(c^\alpha b)^\alpha(c^\alpha a) \in I$. So, by condition (I3), $(c^\alpha a)(c^\alpha b)^\alpha \in I$. Because $c^\alpha b \in I$, from condition (I1), $c^\alpha a \in I$. By assumption, and condition (I3) $a^{\alpha\alpha}b^\alpha$, $b^{\alpha\alpha}a^\alpha \in I$. From condition (I2) we obtain that $(a^\alpha c)^\alpha(b^\alpha c) \in I$ and $(b^\alpha c)^\alpha(a^\alpha c) \in I$. By (I3), $(a^\alpha c)(b^\alpha c)^\alpha \in I$. Now, $b^\alpha c \in I$, and so by (I1) $a^\alpha c \in I$. As regards the operations, it is straightforward from (I2) and (I3), respectively, that $\cdot$ and $^\alpha$ are preserved. From this fact we have that, if $a\theta(I)b$, then $a + c = ((a \cdot c^\alpha)^\alpha \cdot c^\alpha)^\alpha \theta(I)((b \cdot c^\alpha)^\alpha \cdot c^\alpha)^\alpha = b + c$, by Proposition 3.8. Finally, if $a \in I$, then $1a = 0^\alpha a \in I$ and $a^\alpha 0 = 0 \in I$, and so $a \in [0]_{\theta(I)}$. Conversely, if $a \in [0]_{\theta(I)}$, then $a^\alpha 0$, $0^\alpha a = 1a = a \in I$, which proves that $I = [0]_{\theta(I)}$. $\square$

As mentioned above, Łukasiewicz near semiring $\mathbf{A}$ whose multiplication operation is associative can be converted into an MV algebra and *viceversa*. Hence, it seems to us that it could be of some interest wondering how the notion of ideal, in the general setting of Łukasiewicz near semirings, would specify to the case of Łukasiewicz semirings. Actually, for a Łukasiewicz semiring $\mathbf{A}$, we have that:

Corollary 3.11. *A set $I \subseteq A$ such that $0 \in I$ is the kernel of some congruence θ if and only if it satisfies conditions (I1) and (I2) of Definition 3.7. Moreover, $I = [0]_{\theta(I)}$, where $\theta(I)$ is as in condition* (3.1) *in Theorem 3.10.*

Furthermore, one can easily prove that ideals in Łukasiewicz semirings can be defined in the same way they are defined in the case of commutative semirings. In fact, the next proposition shows that for Łukasiewicz semirings one has a finite basis of ideal terms.

Proposition 3.12. *Let $\mathbf{A} = (A, +, \cdot, ^\alpha, 0, 1)$ be a Łukasiewicz semiring. Then $I \subseteq A$ is an ideal if and only if the following conditions hold:*

(i) $0 \in I$;

(ii) $a, b \in I$ *implies* $a + b \in I$;

(iii) $a \in I$ *implies* $a \cdot c = c \cdot a \in I$, *for any* $c \in A$.

Proof. Let I be and ideal in $\mathbf{A}$. We only need to prove that conditions (ii) and (iii) are satisfied. Let $a \in I$. One has that $0 = c(aa^\alpha) = (ca)a^\alpha = (ac)a^\alpha \in I$. Hence, by (I1), it follows that $ac \in I$. So condition (iii) holds. Now, assume that $a, b \in I$. By condition (iii), we obtain that $(a + b)a^\alpha = 0 + ba^\alpha \in I$, so (I1) yields $(a + b) \in I$. Thus, (ii) is proved. Conversely, if (i)-(iii) hold, it is easily seen that (I1) and (I2) are satisfied. In fact, suppose that $ab^\alpha, b \in I$. By (iii) $ab \in I$, hence by (ii) $ab^\alpha + ab = a(b^\alpha + b) = a \cdot 1 \in I$. Finally, assuming that $a^\alpha b, b^\alpha a \in I$, one has by condition (iii) and Definition 3.2(viii) that $(ac)^\alpha(bc) = ((ac)^\alpha c)b = ((c^\alpha a^\alpha)^\alpha a^\alpha)b = (c^\alpha a^\alpha)^\alpha(a^\alpha b) \in I$. ☐

Let $\mathbf{A}$ be a Łukasiewicz near semiring. Hence, a straightforward verification proves that the structure $\langle \mathrm{Id}(\mathbf{A}), \wedge, \vee, \{0\}, A\rangle$ is a complete lattice under the set-theoretic ordering with operations $I \wedge J = I \cap J$ and $I \vee J = \langle I \cup J\rangle$ (i.e, the least ideal containing both I and J). In what follows, we will call this structure the *ideal lattice* of $\mathbf{A}$. Moreover, the one-to-one correspondence between $\mathrm{Id}(\mathbf{A})$ and $\mathrm{Con}(\mathbf{A})$ stated by Theorems 3.9 and 3.10 is, in fact, an isomorphism.

Theorem 3.13. *The ideal lattice of $\mathbf{A}$ is isomorphic to $\mathrm{Con}(\mathbf{A})$. Hence, $\mathrm{Id}(\mathbf{A})$ is an algebraic and distributive lattice.*

Proof. Let $f : \mathrm{Id}(\mathbf{A}) \to \mathrm{Con}(\mathbf{A})$ be the mapping defined by $f(I) = \theta(I)$. By Theorems 3.9 and 3.10, f is a bijection, and its inverse $g : \mathrm{Con}(\mathbf{A}) \to \mathrm{Id}(\mathbf{A})$ is $g(\theta) = [0]_\theta$. Now, it should only be proved that f is an homomorphism. Clearly, $f(I \cap J) = \theta(I) \cap \theta(J) = f(I) \wedge f(J)$. Now we show that $f(I \vee J) = f(\langle I \cup J \rangle) = \theta(\langle I \cup J \rangle) = \theta(I) \vee \theta(J)$. By Lemma 3.8, we have that $(a, b) \in \theta(I) \vee \theta(J)$ if and only if $a^\alpha b, b^\alpha a \in [0]_{\theta(I) \vee \theta(J)}$. Note that, by congruence permutability (cf. page 56), $a^\alpha b \in [0]_{\theta(I) \vee \theta(J)}$ if and only if there is a c such that

$$a^\alpha b \theta(I) c \theta(J) 0.$$

Therefore, again by Lemma 3.8 and Theorem 3.10:

$$(a^\alpha b)^\alpha c, c^\alpha (a^\alpha b) \in I \text{ and } c \in J.$$

Therefore, by (I3), $(a^\alpha b) c^\alpha \in I$. Then, by condition (I1), we have that $a^\alpha b \in \langle I \cup J \rangle$, and by symmetry $b^\alpha a \in \langle I \cup J \rangle$. For the other inclusion, note that $I, J \subseteq [0]_{\theta(I) \vee \theta(J)}$. Hence $\langle I \cup J \rangle \subseteq [0]_{\theta(I) \vee \theta(J)}$. Therefore, by Theorem 3.10, $\theta(\langle I \cup J \rangle) \subseteq \theta(I) \vee \theta(J)$. Then, it turns out that $f(I) \vee f(J) = \theta(I) \vee \theta(J) = \theta(\langle I \cup J \rangle)$. Hence f is an isomorphism. Finally, since $\mathrm{Con}(\mathbf{A})$ is both distributive (see [6]) and, of course, algebraic, $\mathrm{Id}(\mathbf{A})$ is a distributive and algebraic lattice. □

It might be useful to emphasize that the result above is, in fact, an explicit proof of a general result due to H.P. Gumm and A. Ursini. In fact, [73, Corollary 1.9] proves that a variety $\mathcal{V}$, equipped with a constant 0, is *ideal determined* (namely, for any $\mathbf{A} \in \mathcal{V}$ there is a one to one correspondence between $\mathrm{Con}(\mathbf{A})$ and $\mathrm{Id}(\mathbf{A})$) if and only if $\mathcal{V}$ is 0-regular and there exists a binary term $s(x, y)$ such that

$$\mathcal{V} \models s(x, x) = 0 \text{ and } \mathcal{V} \models s(0, x) = x.$$

Thus, since Łukasiewicz near semirings are congruence regular, putting $s(x, y) = x^\alpha y$, they are an ideal determined variety. Futhermore, it can be easily seen that the previous result provides a rather concise description of the ideals of the form $\langle I \cup J \rangle$ with $I, J \in \mathrm{Id}(\mathbf{A})$. Let

$$[I]_{\theta(J)} = \{a \in A | (a, i) \in \theta(J) \text{ for some } i \in I\},$$

for any $I, J \in \mathrm{Id}(\mathbf{A})$. In any Łukasiewicz near semiring $\mathbf{A}$, we can prove:

Proposition 3.14. *For any* $I, J \in \mathrm{Id}(\mathbf{A})$*:*

$$\langle I \cup J \rangle = [I]_{\theta(J)}.$$

Proof. If $a \in \langle I \cup J \rangle$, then $a \in [0]_{\theta(I)\vee\theta(J)}$ and there exists $k < \omega$ such that

$$a\theta(I)c_1\theta(J)c_2\theta(I)...c_k\theta(J)0.$$

Since Łukasiewicz near semirings are congruence permutable, one has that there exists $c \in A$ such that $a\theta(J)c\theta(I)0$. Thus, $\langle I \cup J \rangle \subseteq [I]_{\theta(J)}$. Conversely, if $a \in [I]_{\theta(J)}$, then $a\theta(J)i\theta(I)0$ for some $i \in I$. Hence, $a \in [0]_{\theta(J)\vee\theta(I)} = \langle I \cup J \rangle$. □

As we have mentioned, $\mathrm{Id}(\mathbf{A})$ is algebraic with $\{0\}$ and A its least and the greatest element, respectively. Moreover, since $\mathrm{Id}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{A})$ and $\mathrm{Con}(\mathbf{A})$ is distributive (see above), we can conclude that *it has the infinite join distributive property*, by Proposition 1.18. It means that, for any ideal $J \in \mathrm{Id}(\mathbf{A})$, and an arbitrary family of ideals $\{I_\gamma\}_{\gamma\in\Gamma}$, it holds that

$$J \cap \bigvee\{I_\gamma | \gamma \in \Gamma\} = \bigvee\{J \cap I_\gamma | \gamma \in \Gamma\}. \tag{3.2}$$

From this fact, we can deduce that:

Theorem 3.15. *The ideal lattice* $\mathrm{Id}(\mathbf{A})$ *of any Łukasiewicz near semiring* $\mathbf{A}$ *is pseudo-complemented.*

Proof. Let $J \in \mathrm{Id}(\mathbf{A})$ and consider the set

$$S^J = \{I \in \mathrm{Id}(\mathbf{A}) | J \cap I = \{0\}\}.$$

Clearly, $S^J \neq \emptyset$, since it contains $\{0\}$. By equation (3.2), we have that:

$$J \cap \bigvee S^J = J \cap \bigvee\{I \in \mathrm{Id}(\mathbf{A}) | J \cap I = \{0\}\} = \bigvee\{J \cap I | J \cap I = \{0\}\} = \bigvee\{0\} = \{0\}.$$

In other words, $\bigvee S^J$ is the greatest ideal I in $\mathrm{Id}(\mathbf{A})$ such that $J \cap I = \{0\}$, which means that it is the pseudocomplement of J. □

In what follows, if $I \in \mathrm{Id}(\mathbf{A})$, we denote the pseudocomplement of I by I^*.
Let $\mathbf{A}$ be a Łukasiewicz near semiring. For any $a \in A$, we indicate by $I(a)$ the *principal ideal* generated by a, i.e. the least ideal of $\mathrm{Id}(\mathbf{A})$ that contains a.

Our next task will be to provide a full description of the principal ideals of $\mathrm{Id}(\mathbf{A})$, for any Łukasiewicz near semiring $\mathbf{A}$. As it has been pointed out above, the variety of Łukasiewicz near semirings is congruence-permutable. The witnessing Mal'cev term is the following

$$p(x, y, z) = ((x \cdot y^\alpha)^\alpha \cdot z^\alpha) + ((z \cdot y^\alpha)^\alpha \cdot x^\alpha))^\alpha. \tag{3.3}$$

By Proposition 1.34, every reflexive binary relation on A having the *compatibility property* with respect to operations of $\mathbf{A}$ is a congruence on $\mathbf{A}$. In particular, for any pair $(a, b) \in A^2$, the least reflexive relation having the compatibility property, say $R(a, b)$, is the principal congruence $\theta(a, b)$, generated by a, b in $\mathbf{A}$.

By Theorem 3.9, it follows that the ideal which is the 0-coset of $\theta(a, 0)$ is the least ideal containing a, namely $I(a)$. Recall that by a unary polynomial $p(x)$ we mean a unary term-function $t^{\mathbf{A}}(x, e_1, e_2, ..., e_n)$ where t is a $n+1$-ary term and $e_1, ..., e_n \in A$. Now, as shown in [15], one has that $(c, d) \in R(a, b)$ if and only if there exists a unary polynomial $p(x)$ on A such that $c = p(a)$ and $d = p(b)$. Hence, $b \in I(a)$ if and only if $(b, 0) \in \theta(a, 0)$. Upon denoting by $Pol_1(\mathbf{A})$ the set of all unary polynomials of $\mathbf{A}$, it follows directly that:

Theorem 3.16. *For any $a \in A$,*

$$I(a) = \{p(a) | p \in Pol_1(\mathbf{A}) \text{ with } p(0) = 0\}.$$

It is easily noticed that when dealing with a Łukasiewicz semiring $\mathbf{A}$, since $+$ is idempotent and due to the associativity and commutativity of $\cdot$ (cf. [6, Theorem 2]), polynomials in one variable on $\mathbf{A}$ must be necessarily of the form

$$p(x) = xb + c, \text{ for } b, c \in A. \tag{3.4}$$

Now, according to the reasoning above, it is also required that $p(0) = 0$. Therefore, c in condition (3.4) must be 0, and then we directly infer that the description of principal ideals in a Łukasiewicz semiring $\mathbf{A}$ can be simplified as follows:

Corollary 3.17. *For any $a \in A$,*

$$I(a) = \{a \cdot c | c \in A\}.$$

3.3 Central elements and decompositions

The aim of this section is discussing the notion of centrality in the variety of ι-near semirings. This discussion will be relevant for the structure theory of Łukasiewicz near semirings, since it provides a rather neat description of principal ideals generated by central elements, as well as for the application that the description of central elements has in the proof of a Cantor-Bernstein type theorem that we will propose in section 6.1.

This section is based on the ideas developed in [114] and [90] on the general theory of *Church algebras*.

The notion of Church algebra is based on the simple observation that many well-known algebras, including Heyting algebras, rings with unit and combinatory algebras, possess a ternary term operation q and term definable nullary operations 0, 1, satisfying the equations:

$$q(1, x, y) \approx x \text{ and } q(0, x, y) \approx y.$$

The term operation q simulates the behaviour of the if-then-else connective and, surprisingly enough, these rather simple conditions determine quite strong algebraic properties.

An algebra $\mathbf{A}$ of type ν is a *Church algebra* if there are term definable constants $0^{\mathbf{A}}, 1^{\mathbf{A}} \in A$ and a term operation $q^{\mathbf{A}}$ such that, for all $a, b \in A$,

$$q^{\mathbf{A}}\left(1^{\mathbf{A}}, a, b\right) = a \text{ and } q^{\mathbf{A}}\left(0^{\mathbf{A}}, a, b\right) = b.$$

A variety $\mathcal{V}$ of type ν is a Church variety if every member of $\mathcal{V}$ is a Church algebra with respect to the same term $q(x, y, z)$ and the same constants $0, 1$.

Following the seminal work of D. Vaggione [124], we say that an element e of a Church algebra $\mathbf{A}$ is *central* if the congruences $\theta(e, 0), \theta(e, 1)$ form a pair of factor congruences on $\mathbf{A}$. A central element is said to be *nontrivial* if it differs from 0 and 1. We denote the set of central elements (the *centre*) of $\mathbf{A}$ by $\mathrm{Ce}(A)$.

Setting

$$x \wedge y = q(x, y, 0), \quad x \vee y = q(x, 1, y) \quad x^* = q(x, 0, 1)$$

we recall a general result for Church algebras:

Theorem 3.18. [114] *Let* $\mathbf{A}$ *be a Church algebra. Then*

$$\mathrm{Ce}(\mathbf{A}) = \langle \mathrm{Ce}(A), \wedge, \vee, ^*, 0, 1 \rangle$$

is a Boolean algebra which is isomorphic to the Boolean algebra of factor congruences of $\mathbf{A}$.

If $\mathbf{A}$ is a Church algebra of type ν and $e \in A$ is a central element, then we define $\mathbf{A}_e = (A_e, g_e)_{g \in \nu}$ to be the ν-algebra defined as follows:

$$A_e = \{e \wedge b : b \in A\}; \quad g_e(e \wedge \overline{b}) = e \wedge g(e \wedge \overline{b}), \tag{3.5}$$

where $\overline{b}$ denotes the n-tuple $b_1, ..., b_n$ and $e \wedge \overline{b}$ is an abbreviation for $e \wedge b_1, ..., e \wedge b_n$.

In any Church algebra, central elements are amenable of a neat description as follows:

Theorem 3.19. *If* $\mathbf{A}$ *is a Church algebra of type* ν *and* $e \in A$*, the following conditions are equivalent:*

(1) e *is central;*

(2) for all $a, b, \in A$ *and for all* $\overline{a}, \overline{b} \in A^n$*:*

- ***a)*** $q(e, a, a) = a$,
- ***b)*** $q(e, q(e, a, b), c) = q(e, a, c) = q(e, a, q(e, b, c))$,
- ***c)*** $q(e, f(\overline{a}), f(\overline{b})) = f(q(e, a_1, b_1), ..., q(e, a_n, b_n))$*, for every* $f \in \nu$,
- ***d)*** $q(e, 1, 0) = e$.

By [90, Theorem 4], we obtain the following theorem:

Theorem 3.20. *Let* $\mathbf{A}$ *be a Church algebra of type* ν *and* e *be a central element. Then:*

1. *For every* n*-ary* $g \in \nu$ *and every sequence of elements* $\overline{b} \in A^n$, $e \wedge g(\overline{b}) = e \wedge g(e \wedge \overline{b})$*, so that the function* $h_e : A \to A_e$*, defined by* $h_e(b) = e \wedge b$*, is a homomorphism from* $\mathbf{A}$ *onto* $\mathbf{A}_e$.
2. $\mathbf{A}_e$ *is isomorphic to* $\mathbf{A}/\theta(e, 1)$*. It follows that* $\mathbf{A} \cong \mathbf{A}_e \times \mathbf{A}_{e^*}$ *for every central element* e*, as in the Boolean case, under the mapping* $f(a) \mapsto (h_e(a), h_{e^*}(a))$.

This facts will be expedient in the context of ι-near semirings. Indeed, they are a Church variety [114, Definition 3.1].

Lemma 3.21. *The class of* ι*-near semirings is a Church variety, with witness term*

$$q(x, y, z) = (x \cdot y) + (x^\alpha \cdot z).$$

Proof. Direct computation. ☐

A straightforward interpretation of items a)–d) of Theorem 3.19 in our framework immediately provides that, given a ι-near semiring $\mathbf{A}$, the operations $\wedge, \vee, ^*$ in the Boolean algebra Ce($\mathbf{A}$) coincide with $\cdot, +, ^\alpha$, respectively (cf. [6, Proposition 3]).

Lemma 3.22. *If* e *is central in a* ι*-near semiring* $\mathbf{A}$*, and* $a, b \in A$*, then,*

1. $e \cdot e = e$ *(idempotency);*
2. $e \cdot a = a \cdot e$ *(commutativity);*

3. $(e \cdot a) \cdot b = a \cdot (e \cdot b)$ *(associativity).*

Proof. In this proof we will freely use Theorem 3.19 and Lemma 3.21.
(1) $e = q(e,1,0) = q(e, 1 \cdot 1, 0 \cdot 0) = q(e,1,0) \cdot q(e,1,0) = e \cdot e$.
(2) $e \cdot a = q(e,1,0) \cdot q(e,a,a) = q(e, 1 \cdot a, 0 \cdot a) = q(e, a \cdot 1, a \cdot 0) = q(e,a,a) \cdot q(e,1,0) = a \cdot e$.
(3) $(e \cdot a) \cdot b = q(e,a,0) \cdot q(e,b,b) = q(e, a \cdot b, 0) = e \cdot (a \cdot b) = q(e,a,a) \cdot q(e,b,0) = a \cdot (e \cdot b)$. □

In Łukasiewicz near semirings conditions (a)-(d) in Theorem 3.19 translate as follows: a) is trivially satisfied, by Lemma 3.22. For condition b),

$$(e \cdot c) + (e^{\alpha} \cdot ((e \cdot b) + (e^{\alpha} \cdot a))) = (e \cdot c) + (e^{\alpha} \cdot a);$$

and

$$(e \cdot c) + (e^{\alpha} \cdot a) = (e \cdot ((e \cdot c) + (e^{\alpha} \cdot b))) + (e^{\alpha} \cdot a).$$

As regards condition c), if f is the constant 0 or 1, then clearly $(e \cdot 1) + (e^{\alpha} \cdot 1) = 1$, and $(e \cdot 0) + (e^{\alpha} \cdot 0) = 0$. If f is $+$,

$$(e \cdot (b_1 + b_2)) \cdot (e^{\alpha} \cdot (a_1 + a_2)) = ((e \cdot b_1) + (e^{\alpha} \cdot a_1)) + ((e \cdot b_2) + (e^{\alpha} \cdot a_2)).$$

In case f is $\cdot$,

$$(e \cdot (b_1 \cdot b_2)) + (e^{\alpha} \cdot (a_1 \cdot a_2)) = ((e \cdot b_1) + (e^{\alpha} \cdot a_1)) \cdot ((e \cdot b_2) + (e^{\alpha} \cdot a_2)).$$

In case f is ${}^{\alpha}$,

$$(e \cdot a^{\alpha}) + (e^{\alpha} \cdot b^{\alpha}) = ((e \cdot a) + (e^{\alpha} \cdot b))^{\alpha}.$$

Finally, condition d) is obviously satisfied by any element in a Łukasiewicz near semirings.

As we have already seen in section 4.1, Theorem 3.16 provides a full description of principal ideals generated by elements of a Łukasiewicz near semiring. Moreover, generalizing the Boolean case, central elements produce a direct decomposition of these algebras. Due to this fact, in what follows we will see that the ideals generated by central elements can be described easily.

Definition 3.23. Let $\mathbf{A} = (A, +, \cdot, {}^{\alpha}, 0, 1)$ be a Łukasiewicz near semiring and $I, J \in \mathrm{Id}(\mathbf{A})$. I, J form a pair of *factor ideals* if and only if

$$I \cap J = \{0\} \quad \text{and} \quad I \vee J = A.$$

By the fact that $\mathrm{Id}(\mathbf{A})$ and $\mathrm{Con}(\mathbf{A})$ are the universes of two isomorphic algebraic distributive lattices, it is direct to verify that I, J form a pair of factor ideals if and only if $\theta(I)$ and $\theta(J)$ form a pair of factor congruences.
Upon recalling that, for $a \in A$, the interval $[0, a]$ corresponds to the set $\{x | x \leq a\}$, from the last notion introduced, the following theorem is obtained.

Theorem 3.24. *Let e be a central element of a Łukasiewicz near semiring $\mathbf{A} = (A, +, \cdot, ^\alpha, 0, 1)$. Then $I(e) = [0, e]$.*

Proof. By Theorem 3.20, e is central if and only if, for any $a \in A$, the mapping $f : A \to [0, e] \times [0, e^\alpha]$, defined by $f(a) \mapsto (h_e(a), h_{e^\alpha}(a))$, is a direct decomposition of $\mathbf{A}$. Let θ_1 and θ_2 be the factor congruences associated to $\ker(\pi_2 \circ f)$ and $\ker(\pi_1 \circ f)$, respectively, where π_i $(i \in \{1, 2\})$ is the natural projection map. We denote by I_i $(i = 1, 2)$ these kernels. Then, $e \in I_1$ and $e^\alpha \in I_2$. Thus, $I(e) \subseteq I_1 = [0, e]$ and $I(e^\alpha) \subseteq I_2 = [0, e^\alpha]$. Hence, $I(e) \cap I(e^\alpha) = \{0\}$. It is clear that, for a central element e, one has that $1 = e + e^\alpha \in I(e) \vee I(e^\alpha)$. So $I(e) \vee I(e^\alpha) = A$. Hence, $I(e)$ and $I(e^\alpha)$ form a pair of factor ideals with $I(e) \subseteq I_1$, $I(e^\alpha) \subseteq I_2$. Since I_1 and I_2 are factor ideals, we have that

$$I(e) = I_1 = [0, e].$$

□

A few basic results about the pseudocomplements are subsumed in the following lemma.

Lemma 3.25. *Let $\mathbf{A} = (A, +, \cdot, ^\alpha, 0, 1)$ be a Łukasiewicz near semiring, $I, J \in \mathrm{Id}(\mathbf{A})$ and $a \in A$. Then:*

1. *$I \subseteq I^{**}$;*
2. *If $I \subseteq J$, then $J^* \subseteq I^*$;*
3. *$I^* = I^{***}$;*
4. *(I, I^*) is a pair of factor ideals if and only if $I \vee I^* = A$;*
5. *$(I(a), I(a^\alpha))$ is a pair of factor ideals if and only if*

$$I(a^\alpha) = I(a)^* \text{ and } I(a) \vee I(a^\alpha) = A.$$

Proof. (1), (2), (3) and (4) are straightforward. For (5), if $(I(a), I(a^\alpha))$ is a pair of factor ideals, obviously $I(a) \vee I(a^\alpha) = A$. Now, clearly $I(a^\alpha) \subseteq I(a)^*$. Furthermore, if $b \in I(a)^*$, then $b = 0$ or $b \notin I(a)$ and $b \in I(a^\alpha)$ for $I(a) \vee I(a^\alpha) = A$. In any case one has $b \in I(a^\alpha)$. The converse follows immediately. □

Let us now briefly elaborate on the previous results. Consider a Łukasiewicz near semiring $\mathbf{A} = (A, +, \cdot, ^{\alpha}, 0, 1)$, and let $\mathrm{Skel}(\mathrm{Id}(\mathbf{A}))$ be the *skeleton* of $\mathrm{Id}(\mathbf{A})$, namely

$$\mathrm{Skel}(\mathrm{Id}(\mathbf{A})) = \{I^* | I \in \mathrm{Id}(\mathbf{A})\}.$$

By a theorem due to V. Glivenko, later proved in its full generality by O. Frink (see e.g. [65]), since (by Theorems 3.13 and 3.15) $\mathrm{Id}(\mathbf{A})$ is an algebraic pseudocomplemented lattice, it turns out that $\mathrm{Skel}(\mathrm{Id}(\mathbf{A}))$ is a Boolean lattice bounded by the trivial ideals $\{0\}$ and A. With a slight abuse of language, we may identify the skeleton with the Boolean algebra $\mathrm{Skel}(\mathrm{Id}(\mathbf{A})) = \langle \mathrm{Skel}(\mathrm{Id}(\mathbf{A})), \wedge, \vee, ^*, \{0\}, A\rangle$ where, for any $I, J \in \mathrm{Skel}(\mathrm{Id}(\mathbf{A}))$, $\wedge$ is $\cap$, $\vee$ is defined by $I \vee J = (I^* \wedge J^*)^*$. Trivially, for any $I \in \mathrm{Skel}(\mathrm{Id}(\mathbf{A}))$, I and I^* form a pair of complementary factor ideals.

Now, by Theorem 3.10, if $I, I^* \in \mathrm{Skel}(\mathrm{Id}(\mathbf{A}))$, then $I = [0]_{\theta(I)}$ and $I^* = [0]_{\theta(I^*)}$. By $I \vee I^* = A$ one obviously has that $\theta(I) \vee \theta(I^*) = \nabla$ and $I \wedge I^* = \{0\}$ implies that $\theta(I) \wedge \theta(I^*) = \Delta$.

Conversely, if (θ, θ') is a pair of complementary factor congruences, then their 0-cosets, say I and J, respectively, form a pair of complementary factor ideals. Indeed, it is easily seen that $I = J^*$ and $J = I^*$.

In fact, by Lemma 3.25, one has that $I \subseteq I^{**}$ and $J \subseteq J^{**}$. Hence, $I^{**} \vee J^{**} = A$. Moreover, since I and J form a pair of complementary factor ideals, one has $I \subseteq J^*$ and $J \subseteq I^*$. Thus, by (2) of Lemma 3.25, $I^{**} \subseteq J^*$ implies that if $x \in I^{**}$ and $x \neq 0$, then $x \in J^*$ and $x \notin J^{**}$. Hence, one has that $I^{**} \wedge J^{**} = \{0\}$ and, by the unicity of complements in $\mathrm{Skel}(\mathrm{Id}(\mathbf{A}))$, we can conclude that $I^{**} = J^*$ and $J^{**} = I^*$. In fact, suppose *ex absurdo* that $x \notin J$ and $x \in J^{**}$. Hence, $x \notin J^* = I^{**}$ and $x \notin I$. So $I \subseteq J$ and since $I \cap J = \{0\}$ this is a contradiction. This implies that $J = J^{**} = I^*$. Similarly, $J^* = I$. Then, we can conclude that there is a one-to-one correspondence between pairs of complementary factor ideals in $\mathrm{Skel}(\mathrm{Id}(\mathbf{A}))$ and pairs of complementary factor congruences in $\mathrm{Con}(\mathbf{A})$.

Since $\mathbf{A}$ is congruence-distributive (see [6]) one has that the sublattice of $\mathrm{Con}(\mathbf{A})$ that contains all pairs of complementary factor congruences on $\mathbf{A}$, that we denote by $\mathrm{Con}(\mathbf{A})_{\mathrm{F}}$, is Boolean. Exploiting the same mapping f of Theorem 3.13, one can easily observe that $\mathrm{Skel}(\mathrm{Id}(\mathbf{A})) \cong \mathrm{Con}(\mathbf{A})_{\mathrm{F}}$.

Finally, by Theorem 3.7 in [114] one has that $\mathrm{Con}(\mathbf{A})_{\mathrm{F}} \cong \mathrm{Ce}(\mathbf{A})$, where $\mathrm{Ce}(\mathbf{A})$ is the Boolean lattice of central elements of $\mathbf{A}$. In particular, it shows that the map $e \mapsto \theta(e, 0)$ is a bijective correspondence between $\mathrm{Ce}(\mathbf{A})$ and $\mathrm{Con}(\mathbf{A})_{\mathrm{F}}$. Moreover, for any $e, d \in \mathrm{Ce}(\mathbf{A})$, the elements e^{α}, $e \wedge d$, $e \vee d$ are central and naturally associated with the factor congruences $\theta(e,1) = \theta(e^{\alpha},0)$, $\theta(e,0) \cap \theta(d,0)$ and $\theta(e,0) \vee \theta(d,0)$, respectively. Hence, for any pair of complementary factor congruences (θ,θ') one has that $(\theta,\theta') = (\theta(e,0),\theta(e^{\alpha},0))$, for some $e \in \mathrm{Ce}(\mathbf{A})$.

Summarizing the observations above, we have that $\mathrm{Skel}(\mathrm{Id}(\mathbf{A}))$ coincides with the Boolean lattice of pairs of complementary factor ideals (I, I^*) in $\mathrm{Id}(\mathbf{A})^2$, which are nothing but 0-cosets of pairs of complementary factor congruences of the form $(\theta(e,0),\theta(e^{\alpha},0))$, for an element e in $\mathrm{Ce}(\mathbf{A})$. Thus, it directly follows that

$$\mathrm{Skel}(\mathrm{Id}(\mathbf{A})) = \{I(e) | e \in \mathrm{Ce}(\mathbf{A})\}$$

and by Theorem 3.24, $I^* = [0,e]$, for some $e \in \mathrm{Ce}(\mathbf{A})$.

3.4 A Cantor-Bernstein-type Theorem for ι-near semirings

We close this chapter with an application of the theory of central elements to ι-near semirings. Namely, we propose a version of the Cantor-Bernstein Theorem for join σ-complete *left-residuable* ι-near semirings with σ-complete algebras of central elements. More specifically, for a ι-near semiring $\mathbf{A}$, if $\{a_i\}_{i\in I}$, such that $|I| \leq \sigma$, then $\bigvee_{i\in I} a_i$ exists, and $\mathrm{Ce}(\mathbf{A})$ is a σ-complete Boolean algebra. This result was first shown in [117] (see also [120]) for Boolean algebras and subsequently extended to MV-algebras (with Boolean elements), orthomodular lattices, and other classes of algebras enjoying suitable properties, such as having an underlying lattice structure (see [34], [53]). Since Łukasiewicz near semirings generalize the notion of MV-algebra, it is natural to wonder whether a version of the Cantor-Bernstein Theorem could be widened for weaker structures, like ι-near semirings.

Definition 3.26. Let $\mathbf{A} = (A, +, \cdot, ^{\alpha}, 0, 1)$ be an ι-near semiring. We say that $\mathbf{A}$ is left-residuable if for any $x \in A$ there exists a mapping $f_x : A \to A$ such that, for any $y, z \in A$,

$$y \cdot x \leq z \quad \text{iff} \quad y \leq f_x(z).$$

Upon recalling that central elements commute with any other element (cf. Lemma 3.22), in order to prove the main result of this section, we start with the following

Lemma 3.27. *Let $\mathbf{A}$ be a left-residuable ι-near semiring, and $e \in \mathrm{Ce}(\mathbf{A})$, and $a, b \in A$.*

1. *if $a \leq e$, then $ae = a$;*
2. *$eb = e \wedge b$;*

3. *if* $\{a_i\}_{i\in N} \subseteq A$, *then*

$$e \wedge (\Sigma_{n\in N} a_n) = \Sigma_{n\in N}(e \wedge a_n).$$

Proof. (1) If $a \leq e$, then $ae^{\alpha} \leq ee^{\alpha} = 0$, because e is central. Therefore, $ea = ea + 0 = ea + e^{\alpha}a$, because e^{α} commutes and so $ae = ea = (e + e^{\alpha})a = 1a = a$, because $e + e^{\alpha} = e \vee e^{\alpha} = 1$.

(2) First, observe that $be \leq b, e$, by [6, Lemma 1]. If $a \leq e, b$, then $a + e = e$ and $b + a = b$. Note that $eb + a = eb + ea$, by the previous item, and $eb + ea = e(b + a) = eb$, because e is central and therefore commutes. Consequently, $a \leq eb = e \wedge b$.

(3) One has $\Sigma_{n\in M} a_n \wedge e = (\Sigma_{n\in M} a_n \cdot e)$, by item 2. Now, clearly $a_n \cdot e \leq (\Sigma_{n\in N} a_n) \cdot e$. Moreover, if $a_n \cdot e \leq c$, $n \in N$, one has $a_n \leq f_e(c)$; so, $\Sigma_{n\in N} a_n \leq f_e(c)$ and $(\Sigma_{n\in N} a_n) \cdot e \leq c$. We conclude that $(\Sigma_{n\in N} a_n) \cdot e = \Sigma_{n\in M} a_n \wedge e = \Sigma_{n\in N}(a_n \wedge e)$. □

When there is no confusion possible, we will use $\cdot$ and $\wedge$ ($+$ and $\vee$) as synonyms, respectively. The following lemma completes the results of the previous lemma.

Lemma 3.28. *Let* $\mathbf{A}, \mathbf{B}$ *be a left-residuable join* σ*-complete near semirings and* $\gamma : \mathbf{A} \to \mathbf{B}$ *an isomorphism. Then,*

1. *if* $a \in \mathrm{Ce}(\mathbf{A})$, *then* $\gamma(a) \in \mathrm{Ce}(\mathbf{B})$;
2. *if* $a \in \mathrm{Ce}(\mathbf{A})$, *then* $\gamma \restriction [0, a]$ *is isomorphic to* $[0, \gamma(a)]$;
3. *if* $\{a_i\}_{i\in N} \subseteq \mathrm{Ce}(\mathbf{A})$, $\bigvee_{n\in N} a_n = 1$, *and for* $i \neq j$ $a_i \wedge a_j = 0$, *then* $\mathbf{A}$ *is isomorphic to* $\Pi_{n\in N}[0, a_n]$.

Proof. (1) and (2) are straightforward. (3) let $\{a_i\}_{i\in N}$ be a family of central elements with the required properties. Let us call β the map from $\mathbf{A}$ to $\Pi_{n\in N}[0, a_n]$ defined, for $a \in A$, by $a \mapsto (a \wedge a_i : i \in N)$. Clearly, if $i \neq j$, then, in case $b \leq a_i, a_j$, we have that $b \leq a_i \wedge a_j = 0$. Thus, $[0, a_i] \cap [0, a_j] = \{0\}$, which implies injectivity. Clearly, $\beta(1) = \beta(\bigvee_{n\in N} a_n) = (a_i : i \in N)$. Let $(b_i : i \in N)$. Then, $\beta(\bigvee_{i\in N} b_i) = ((\bigvee_{i\in N} b_i) \wedge a_i : i \in N) = (\bigvee_{i\in N} b_i \wedge a_i : i \in N) = (b_i : i \in N)$. The fact that β preserves the operations directly follows from general results on central elements in a Church algebra [114]. □

We now have all the elements required for proving our main theorem. Recall that, given a near semiring $\mathbf{A}$ and $a \in \mathrm{Ce}(\mathbf{A})$, the interval $[0, a]$ is an algebra whose operations are the same as in $\mathbf{A}$ although adequately "constrained" to the considered subset of A (see Theorem 3.20).

Theorem 3.29. *Let* $\mathbf{A} = (A, +, \cdot, ^{\alpha}, 0, 1)$ *and* $\mathbf{B} = (B, +, \cdot, ^{\alpha}, 0, 1)$ *be left-residuable join* σ*-complete* ι*-near semirings, such that* $\mathrm{Ce}(\mathbf{A})$ *and* $\mathrm{Ce}(\mathbf{B})$ *are* σ*-complete Boolean algebras. If* $\mathbf{A} \cong [0, b]$ *and* $\mathbf{B} \cong [0, a]$ *with* $b \in \mathrm{Ce}(\mathbf{B})$ *and* $a \in \mathrm{Ce}(\mathbf{A})$*, then* $\mathbf{A} \cong \mathbf{B}$.

Proof. Let $\gamma : \mathbf{A} \to [0, b]$ and $\beta : \mathbf{B} \to [0, a]$ be isomorphisms with $a \in \mathrm{Ce}(\mathbf{A})$ and $b \in \mathrm{Ce}(\mathbf{B})$. Without loss of generality, we can safely assume that $0 < a, b < 1$. We recursively define, as in the proof of [34, Theorem 4.1], the following pair of infinite sequences:

$$\begin{aligned} &v_0 = 1 && u_0 = 1 \\ &v_{n+1} = \beta(u_n) && u_{n+1} = \gamma(v_n). \end{aligned}$$

Since $1 \in \mathrm{Ce}(\mathbf{A}) \cap \mathrm{Ce}(\mathbf{B})$ and γ, β are isomorphisms, one has, by Lemma 3.28(1), that $u_n \in \mathrm{Ce}(\mathbf{B})$ and $v_n \in \mathrm{Ce}(\mathbf{A})$ for any $n \in N$. Indeed, by induction on n, we obtain that $v_n = \beta(u_{n-1}) = \beta(\gamma(v_{n-2}))$. Since $\beta \circ \gamma$ is still an isomorphism, a straightforward application of the induction hypothesis yields $v_n \in \mathrm{Ce}(\mathbf{A})$. Similarly, $u_n \in \mathrm{Ce}(\mathbf{B})$. Furthermore, it can be seen that

$$v_0 > v_1 > ... > ... \quad \text{and} \quad u_0 > u_1 > ... > ... \,.$$

In fact, by induction on n, one has that $v_0 = 1 + v_1 = 1$ (since any ι-near semiring is integral). Hence, $v_0 \geq v_1$. Now, suppose that $v_k + v_{k+1} = v_k$ for any $k < n$. It can be seen that $v_n + v_{n+1} = \beta(u_{n-1}) + \beta(u_n) = (\beta \circ \gamma)(v_{n-2} + v_{n-1}) = (\beta \circ \gamma)(v_{n-2}) = v_n$. Similarly, $u_n + u_{n+1} = u_n$, for any $n \in N$. Clearly, $v_n \gneq v_{n+1}$ ($u_n \gneq u_{n+1}$) follows from the injectivity of β (γ).

Indeed, since $\mathrm{Ce}(\mathbf{A})$ and $\mathrm{Ce}(\mathbf{B})$ are σ-complete Boolean algebras (see Theorem 3.18), we can define the following

$$v_\infty = \bigwedge_{n \in N} v_n \qquad \text{and} \qquad u_\infty = \bigwedge_{n \in N} u_n.$$

Recall that, by Lemma 3.27(2), since all v_n, $n \in N$, are central, we obtain that $\bigwedge_{n \in N} v_n = \prod_{n \in N} v_n$. Similarly for u_n, $n \in N$.

Moreover, a simple computation proves that $\gamma(v_\infty) = \gamma(\bigwedge_{n \in N} v_n) = \bigwedge_{n \in N} \gamma(v_n) = \bigwedge_{n \in N} u_{n+1} = u_\infty$ as well as $\beta(u_\infty) = v_\infty$. We define the following

$$e_n = v_n \wedge v_{n+1}^{\alpha} \quad \text{and} \quad d_n = u_n \wedge u_{n+1}^{\alpha}.$$

Let us note that $\gamma(e_n) = \gamma(v_n \wedge v_{n+1}^{\alpha}) = \gamma(v_n) \wedge \gamma(v_{n+1})^{\alpha} = u_{n+1} \wedge u_{n+2}^{\alpha} = d_{n+1}$. Similarly, $\beta(d_n) = e_{n+1}$. Now, it is easily seen that $e_{n-1} = v_n^{\alpha}$ and $d_{n-1} = u_n^{\alpha}$ for any

$n \in N^+$. Indeed, since the latter case can be handled similarly, we prove the former. We have that $e_0 = v_0 \wedge v_1^\alpha = 1 \cdot v_1^\alpha = v_1^\alpha$. Suppose that $e_{k-1} = v_k^\alpha$ for any $k < n$. We obtain that $e_{n-1} = v_{n-1} \wedge v_n^\alpha = (v_{n-1}^\alpha)^\alpha \wedge v_n^\alpha = (v_{n-2} \cdot v_{n-1}^\alpha)^\alpha \cdot v_n^\alpha = (v_{n-2}^\alpha + v_{n-1}) \cdot v_n^\alpha$, by centrality and De Morgan laws, and then $v_{n-2}^\alpha \cdot v_n^\alpha + v_{n-1} \cdot v_n^\alpha = v_n^\alpha + v_{n-1} \cdot v_n^\alpha = (v_{n-1} + 1) \cdot v_n^\alpha = v_n^\alpha$. Hence:

$$\bigvee_{n \in N^+} e_{n-1} = \bigvee_{n \in N^+} v_n^\alpha = (\bigwedge_{n \in N^+} v_n)^\alpha = v_\infty^\alpha$$

as well as

$$\bigvee_{n \in N^+} d_{n-1} = u_\infty^\alpha.$$

Thus, we have that $v_\infty \vee (\bigvee_{n \in N} e_n) = 1$ and $u_\infty \vee (\bigvee_{n \in N} d_n) = 1$. Furthermore, let us note that $e_m \wedge e_n = 0$ and $d_m \wedge d_n = 0$ for any $n \neq m$. In fact, suppose without loss of generality that $m > n$. It can be verified that $e_m \wedge e_n = v_m \wedge v_{m+1}^\alpha \wedge v_n \wedge v_{n+1}^\alpha = (v_m \wedge v_{n+1}) \wedge v_{m+1}^\alpha \wedge v_n \wedge v_{n+1}^\alpha = 0$. The latter case can be handled similarly. Moreover, a little thought shows that $v_\infty \wedge e_n = 0$ as well as $u_\infty \wedge d_n = 0$, for any $n \in N$.

Finally, a direct application of Lemma 3.28 yields

$$\mathbf{A} \cong [0, v_\infty] \times [0, e_0] \times [0, e_1] \times \cdots \times \ldots$$

and

$$\mathbf{B} \cong [0, u_\infty] \times [0, d_0] \times [0, d_1] \times \cdots \times \ldots \ .$$

Recall that $\gamma(v_\infty) = u_\infty$ and $\gamma(e_n) = d_{n+1}$ as well as $\beta(d_n) = e_{n+1}$, for any $n \in N$. Hence, by Lemma 3.28, we obtain that $[0, e_\infty] \cong [0, \gamma(e_\infty)] = [0, d_\infty]$, $[0, e_n] \cong [0, \gamma(e_n)] = [0, d_{n+1}]$ and $[0, d_n] \cong [0, \beta(d_n)] = [0, e_{n+1}]$. Thus, in general, we have that $\mathbf{A} \cong \mathbf{B}$. □

By virtue of Definition 3.2, Theorem 3.24 and the definition of $\mathbf{A}_e$, with $e \in \mathrm{Ce}(\mathbf{A})$ (see 3.5), if $\mathbf{A}$ and $\mathbf{B}$ are Łukasiewicz near semirings, then they can be regarded as *trivial* principal ideals $[0, 1]$ (generated by $1 \in \mathrm{Ce}(\mathbf{A}) \cap \mathrm{Ce}(\mathbf{B})$) of $\mathrm{Id}(\mathbf{A})$ and $\mathrm{Id}(\mathbf{B})$, respectively, Hence, we conclude the following:

Corollary 3.30. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be join* σ*-complete Łukasiewicz near semirings such that* $\mathrm{Ce}(\mathbf{A})$ *and* $\mathrm{Ce}(\mathbf{B})$ *are* σ*-complete Boolean algebras. Then,* $\mathbf{A} \cong \mathbf{B}$ *if and only if there are central elements* $a \in \mathrm{Ce}(\mathbf{A})$ *and* $b \in \mathrm{Ce}(\mathbf{B})$ *such that* $\mathbf{A} \cong I(b)$ *and* $\mathbf{B} \cong I(a)$.

Proof. Suppose that there are central elements $a \in \mathrm{Ce}(\mathbf{A})$ and $b \in \mathrm{Ce}(\mathbf{B})$ such that $\mathbf{B} \cong I(a)$ and $\mathbf{A} \cong I(b)$, respectively. By Theorem 3.24, it follows that $I(a) = [0, a]$ as well as $I(b) = [0, b]$. Hence, Theorem 3.29 ensures that $\mathbf{A} \cong \mathbf{B}$. Conversely, if $\mathbf{A} \cong \mathbf{B}$, then, upon noticing that 1 is the greatest element in any Łukasiewicz near semiring, and it is also central, we have that $\mathbf{A} \cong [0, 1^{\mathbf{B}}] = \mathbf{B}$ and viceversa. $\square$

Theorem 3.29 states something that, in our opinion, is interesting: the "structure" of a Łukasiewicz near semiring $\mathbf{A}$ is already contained *in nuce* in each of its intervals $[0, e]$, where $e \in \mathrm{Ce}(\mathbf{A})$. Moreover, it is worth observing that an abstract proof of this result can be given by exploiting results in [52] where categorical sufficient and necessary conditions for the Cantor-Bernstein Theorem to be proven for a variety of algebras are given. However, we have included a completely algebraic proof of this fact in order to highlight the algebraic properties of Łukasiewicz near semirings it depends on.

In this chapter we have shown that a general structure theory for quantum algebras and MV algebras can be given within the framework of Łukasiewicz near semirings. Therefore, one could ask if the aforementioned formal machinery can be applied to non-commutative generalizations of quantum algebras and many-valued logics algebras. Providing an answer to the above question will be indeed the aim of the next chapter.

Chapter 4

Non-commutative lattice quantum structures as near semirings

In the previous chapter we discussed the structure theory for quantum and MV algebras in the common framework of Łukasiewicz near semirings. In what follows, we extend the semiring approach to non-commutative generalizations of lattice effect algebras and MV algebras, namely lattice pseudoeffect algebras and pseudo-MV algebras. To this aim, we define *near pseudoeffect semirings* and generalized Łukasiewicz semirings.
Such perspective has two fundamental advantages. On one hand it allows the representation of pseudo effect algebras as total algebras. On the other, it grants a very simple axiomatization of pseudo-MV algebras which yields, in turn, a rather straight as well as clean explanation of their relationship with pseudoeffect algebras. In fact, we will provide an alternative proof of [41, Theorem 8.7 and Proposition 8.15(γ) and (δ)] showing that any generalized Łukasiewicz semiring is a near pseudoeffect semiring (Theorem 4.20). Finally, by simple considerations, we will conclude that the class of gl-semirings is a proper subclass of near-p semirings (see Theorem 4.22).

4.1 Representing pseudoeffect algebras as near semirings

Recall by Chapter 2 that a pseudoeffect algebra $\mathbf{A}$ is a structure $\mathbf{A} = (A, \oplus, 0, 1)$ satisfying axioms $(E1) - (E4)$ of Definition 2.35. In what follows, since for any pseudoeffect algebra $\mathbf{A}$ and $a \in A$, a is uniquely right(left)-complemented, we will denote its right (left) complement by $a^{\sim}$ (a^{-}).

Let us recall some basic facts on pseudoeffect algebras that will be expedient for the development of our arguments.

Proposition 4.1 (Lemma 1.4, [40])**.** *Let* **A** *be a pseudoeffect algebra. Then, for* $a, b, c \in A$ *the following conditions are satisfied:*

(i) $a \oplus 0 = 0 \oplus a = a$;

(ii) if $a \oplus b = 0$, *then* $a = b = 0$;

(iii) $0^{\sim} = 0^{-} = 1$ *and* $1^{\sim} = 1^{-} = 0$;

(iv) $a^{-\sim} = a = a^{\sim -}$;

(v) if $a \oplus b = a \oplus c$, *then* $b = c$, *and if* $b \oplus a = c \oplus a$, *then* $b = c$;

(vi) $a \oplus b = c$ *iff* $a = (b \oplus c^{\sim})^{-}$ *iff* $b = (c^{-} \oplus a)^{\sim}$.

It is worth noticing that in this framework unary complementations, albeit non involutive, are still antitone.

Lemma 4.2 (Lemma 1.6 [40])**.** *Let* **A** *be a pseudoeffect algebra. Then, for* $a, b, c, d \in A$:

(i) $a \leq b$ *iff* $b^{-} \leq a^{-}$ *iff* $b^{\sim} \leq a^{\sim}$;

(ii) if $a \oplus b$ *exists, and* $c \leq a$, $d \leq b$, *then* $c \oplus d$ *exists;*

(iii) $a \oplus b$ *exists iff* $a \leq b^{-}$ *iff* $b \leq a^{\sim}$;

(iv) if $b \oplus c$ *exists, then* $a \leq b$ *iff* $a \oplus c$ *exists and* $a \oplus c \leq b \oplus c$;

(v) if $c \oplus b$ *exists, then* $a \leq b$ *iff* $c \oplus a$ *exists and* $c \oplus a \leq c \oplus b$.

The next proposition explains the relationship between $\oplus$ and the induced lattice order.

Proposition 4.3 (Lemma 1.7 [40])**.** *Let* **A** *be a lattice pseudoeffect algebra.*

(i) If $c \oplus (a \vee b)$ *exists, then* $c \oplus a$ *and* $c \oplus b$ *exist, and*

$$(c \oplus a) \vee (c \oplus b) = c \oplus (a \vee b).$$

If $(a \vee b) \oplus c$ *exists, then* $a \oplus c$ *and* $b \oplus c$ *exist, and*

$$(a \oplus c) \vee (b \oplus c) = (a \vee b) \oplus c.$$

(ii) If $c \oplus a$ and $c \oplus b$ exist, then $c \oplus (a \wedge b)$ exists and

$$(c \oplus a) \wedge (c \oplus b) = c \oplus (a \wedge b).$$

If $a \oplus c$ and $b \oplus c$ exist, then $(a \wedge b) \oplus c$ exists, and

$$(a \oplus c) \wedge (b \oplus c) = (a \wedge b) \oplus c.$$

We now prove an easy lemma providing an useful characterization of the operation $\oplus$ in every pseudoeffect algebra.

Lemma 4.4. *Every pseudoeffect algebra satisfies the equations*

$$(a \oplus (a \oplus b)^{\sim})^{-} \oplus a = a \oplus b \text{ and } b \oplus ((a \oplus b)^{-} \oplus b)^{\sim} = a \oplus b \tag{CD}$$

Proof. By Proposition 4.1(vi), we can see that $(a \oplus (a \oplus b)^{\sim})^{-} \oplus a = a \oplus b$ is equivalent to $(a \oplus (a \oplus b)^{\sim})^{-} = (a \oplus (a \oplus b)^{\sim})^{-}$, which trivially holds true. Analogously for the second equation. □

Let us remark that equations (CD) can equivalently substitute condition (E3) in Definition 2.35, simply take $(a \oplus (a \oplus b)^{\sim})^{-}$, $((a \oplus b)^{-} \oplus b)^{\sim}$ as c, d, respectively, in (E3).

If **A** is a lattice pseudoeffect algebra, then it can be seen that full De Morgan laws hold. Indeed, for any a, $b \in A$:

$$a^{\sim} \vee b^{\sim} = (a \wedge b)^{\sim} \text{ and } a^{-} \vee b^{-} = (a \wedge b)^{-}. \tag{DM}$$

In fact, a routine verification shows that, if $a^{\sim}$, $b^{\sim} \leq c$, then $c^{-} \leq a^{\sim -} = a$, $c^{-} \leq b^{\sim -} = b$. Therefore, $c^{-} \leq a \wedge b$, and so $(a \wedge b)^{\sim} \leq c^{-\sim} = c$, i.e. $(a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim}$. Analogously, we have that $(a \vee b)^{\sim} = a^{\sim} \wedge b^{\sim}$ and $(a \vee b)^{-} = a^{-} \wedge b^{-}$.

Lemma 4.5. *Let* **A** *be a lattice pseudoeffect algebra and set*

$$x \cdot y = ((x^{\sim} \wedge y) \oplus y^{\sim})^{-}.$$

Then,

(i) "$\cdot$" is defined everywhere in **A**;

(ii) $x \cdot y = 0$ iff $x \oplus y$ exists;

(iii) $x \cdot y = 0$ iff $y \leq x^{\sim}$ iff $x \leq y^{-}$.

Proof. (i) $a \cdot b$ is defined if $(a^{\sim} \wedge b) \oplus b^{\sim}$ is defined. By Lemma 4.2(iii), this happens whenever $a^{\sim} \wedge b \leq b^{\sim -} = b$, which is always the case.

(ii) If $a \cdot b = 0$, then $(a^{\sim} \wedge b) \oplus b^{\sim} = 1$. Therefore, by Proposition 4.1(vi),

$$b^{\sim} = (1^{-} \oplus (a^{\sim} \wedge b))^{\sim} = (0 \oplus (a^{\sim} \wedge b))^{\sim} = (a^{\sim} \wedge b)^{\sim}.$$

We obtain that $b = b^{\sim -} = (a^{\sim} \wedge b)^{\sim -} = a^{\sim} \wedge b$, i.e. $b \leq a^{\sim}$. This, by Lemma 4.2(iii), implies that $a \oplus b$ is defined. If $a \oplus b$ is defined, then $b \leq a^{\sim}$. Thus,

$$a \cdot b = ((a^{\sim} \wedge b) \oplus b^{\sim})^{-} = (b \oplus b^{\sim})^{-} = 1^{-} = 0.$$

(iii) Straightforward. □

From Lemma 4.5, the following corollary easily follows.

Corollary 4.6. *If* $\mathbf{A}$ *is a lattice pseudoeffect algebra, and* $a \leq b^{-}$ *in* $\mathbf{A}$*, then* $a \oplus b = (a^{-} \cdot b^{-})^{\sim}$.

Proof. Let $a \leq b^{-}$ in $\mathbf{A}$. Then, by Lemma 4.2(iii), $a \oplus b$ exists. So, by Lemma 4.5, we have that

$$\begin{aligned}
(a^{-} \cdot b^{-})^{\sim} &= ((a^{-\sim} \wedge b^{-}) \oplus b^{-\sim})^{-\sim} \\
&= ((a \wedge b^{-}) \oplus b^{-\sim})^{-\sim} \\
&= (a \oplus b^{-\sim})^{-\sim} \\
&= (a \oplus b)^{-\sim} \\
&= a \oplus b
\end{aligned}$$

and so our claim follows. □

We are now ready to state, and prove, our first result:

Theorem 4.7. *Let* $\mathbf{A}$ *be a lattice pseudoeffect algebra. Upon setting*

$$x \cdot y = ((x^{\sim} \wedge y) \oplus y^{\sim})^{-},$$

the structure

$$\mathcal{P}(\mathbf{A}) = (A, \vee, \cdot, 0, 1)$$

is a near semiring.

Proof. Evidently, $(A, \vee, 0)$ is an idempotent commutative monoid and $a \vee 1 = 1$. As regards condition (ii),

$$\begin{aligned} a \cdot 1 =& ((a^{\sim} \wedge 1) \oplus 1^{\sim})^{-} \\ =& ((a^{\sim} \wedge 1) \oplus 0)^{-} \\ =& (a^{\sim} \oplus 0)^{-} \\ =& a^{\sim -} \\ =& a. \end{aligned}$$

The fact that $1 \cdot a = a$ holds can be proven similarly. For equation (iii), $(a \vee b) \cdot c = (((a \vee b)^{\sim} \wedge c) \oplus c^{\sim})^{-} = ((a^{\sim} \wedge b^{\sim} \wedge c) \oplus c^{\sim})^{-}$, which exists by Lemma 4.5(i). And then,

$$\begin{aligned} ((a^{\sim} \wedge b^{\sim} \wedge c) \oplus c^{\sim})^{-} =& (((a^{\sim} \wedge c) \wedge (b^{\sim} \wedge c)) \oplus c^{\sim})^{-} \\ =& (((a^{\sim} \wedge c) \oplus c^{\sim}) \wedge ((b^{\sim} \wedge c) \oplus c^{\sim}))^{-}, \end{aligned}$$

because both $(a^{\sim} \wedge c) \oplus c^{\sim}$ and $(b^{\sim} \wedge c) \oplus c^{\sim}$ are defined by Lemma 4.2(iii), and then [40, Lemma 1.7(i)] applies. Thus,

$$\begin{aligned} (((a^{\sim} \wedge c) \oplus c^{\sim}) \wedge ((b^{\sim} \wedge c) \oplus c^{\sim}))^{-} =& ((a^{\sim} \wedge c) \oplus c^{\sim})^{-} \vee ((b^{\sim} \wedge c) \oplus c^{\sim})^{-} \\ =& (a \cdot c) \vee (b \cdot c). \end{aligned}$$

Finally, for equation (iv), it can be seen that

$$a \cdot 0 = ((a^{\sim} \wedge 0) \oplus 0^{\sim})^{-} = (0 \oplus 0^{\sim})^{-} = (0 \oplus 1)^{-} = 1^{-} = 0,$$

and also

$$0 \cdot a = ((0^{\sim} \wedge a) \oplus a^{\sim})^{-} = ((1 \wedge a) \oplus a^{\sim})^{-} = (a \oplus a^{\sim})^{-} = 1^{-} = 0.$$

□

Let us observe that, in general, not every near semiring can be regarded as a lattice pseudoeffect algebra. In order to see this, we need some additional requirements.

Definition 4.8. A *near pseudoeffect semiring* (*near-p semiring*) is an algebra $\mathbf{R} = (R, +, \cdot, f, g, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ such that the reduct $(R, +, \cdot, 0, 1)$ is an idempotent near semiring, and f, g are operations on R such that the following conditions are fulfilled:

(P1) $f(0) = 1$ and $f(1) = 0$;

(P2) $y \leq x$ iff $g(x) \leq g(y)$ iff $f(x) \cdot y = 0$ iff $f(x) \leq f(y)$ iff $y \cdot g(x) = 0$;

(P3) $f(g(x)) = g(f(x)) = x$;

(P4) $g(x) \cdot g(y) = 0$ and $g(x \cdot y) \cdot g(z) = 0$ iff $g(y) \cdot g(z) = 0$ and $g(x) \cdot g(y \cdot z) = 0$;

(P5) if $g(x) \cdot g(y) = 0$ and $g(x \cdot y) \cdot g(z) = 0$, then $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;

(P6) if $x \leq f(y)$, then $f(x) \cdot f(y) = f(f(x) \cdot g(f(x) \cdot f(y))) \cdot f(x) = f(y) \cdot g(f(f(x) \cdot f(y)) \cdot f(y))$;

(P7) if $x \leq y$, then there is a z such that $f(z) \cdot f(x) = f(y)$, and $z \leq f(x)$;

where $\leq$ is the induced order.

Let us remark that, in general, in a near-p semiring the product "$\cdot$" need not be associative. However, it can be seen that any near-p semiring is lattice ordered. In fact, an easy check shows that a meet operation $\wedge$ can be defined, using De Morgan laws, by $x \wedge y = g(f(x) \vee f(y))$, where the join is defined as $+$ (cf. page 50). Actually, if $z \leq x, y$, then, by antitonicity, $f(x), f(y) \leq f(z)$. Therefore, $f(x) \vee f(y) \leq f(z)$, and so $g(f(x) \vee f(y)) \geq g(f(z)) = z$. Again, a brief reflection shows that the lattice reduct is bounded below and above by 0 and 1, respectively. Moreover, $g(f(1)) = 1 = g(0)$ and $g(f(0)) = 0 = g(1)$.

We are now going to provide a characterization of pseudoeffect algebras in terms of near-p semirings.

Theorem 4.9. *Let* $\mathbf{A}$ *be a lattice pseudoeffect algebra. Upon setting* $x \cdot y = ((x^{\sim} \wedge y) \oplus y^{\sim})^{-}$*, the structure*

$$\mathcal{P}(\mathbf{A}) = (A, \vee, \cdot, ^{-}, ^{\sim}, 0, 1)$$

is a near-p semiring.

Proof. From Theorem 4.7, we have that the reduct $(A, \vee, \cdot, 0, 1)$ is an idempotent near semiring. Upon setting $f(x) = x^{-}$ and $g(x) = x^{\sim}$, we now show that conditions (P1)–(P7) hold true as well. (P1) follows directly from Proposition 4.1(iii). (P2) follows from Lemma 4.2(i), Proposition 4.1(iv) and Lemma 4.5(iii). That $x^{\sim -} = x^{-\sim} = x$ follows directly from Proposition 4.1(iv). As regards (P4), if $x^{\sim} \cdot y^{\sim} = 0$, then, by Lemma 4.5(ii), $x^{\sim} \oplus y^{\sim}$ exists and $x \cdot y = ((x^{\sim} \wedge y) \oplus y^{\sim})^{-} = (x^{\sim} \oplus y^{\sim})^{-}$, because $x^{\sim} \leq y^{\sim -} = y$ (see Lemma 4.5(iii)). From the fact that $(x \cdot y)^{\sim} \cdot z^{\sim} = 0$, it follows that $(x \cdot y)^{\sim} \oplus z^{\sim}$ exists, by Lemma 4.5(ii), and it is equal to $(x^{\sim} \oplus y^{\sim}) \oplus z^{\sim}$. Then, by (E2), $y^{\sim} \oplus z^{\sim}$

and $x^{\sim} \oplus (y^{\sim} \oplus z^{\sim})$ both exist. Again from Lemma 4.5, we derive that $y^{\sim} \cdot z^{\sim} = 0$ and $x^{\sim} \cdot (y^{\sim} \oplus z^{\sim}) = 0$. Then, by Corollary 4.6,

$$0 = x^{\sim} \cdot (y^{\sim} \oplus z^{\sim}) = x^{\sim} \cdot (y^{\sim -} \cdot z^{\sim -})^{\sim} = x^{\sim} \cdot (y \cdot z)^{\sim} = g(x) \cdot g(y \cdot z).$$

The converse holds symmetrically.
As regards (P5), if $x^{\sim} \cdot y^{\sim} = 0$ and $(x \cdot y)^{\sim} \cdot z^{\sim} = 0$, then, building on the proof for (P4), one has that

$$\begin{aligned}(x \cdot y) \cdot z =& ((x^{\sim} \oplus y^{\sim})^{-\sim} \oplus z^{\sim})^{-} \\ =& ((x^{\sim} \oplus y^{\sim}) \oplus z^{\sim})^{-} \\ =& (x^{\sim} \oplus (y^{\sim} \oplus z^{\sim}))^{-} \\ =& x \cdot (y \cdot z).\end{aligned}$$

For (P6), if $x \leq y^{-}$, then

$$\begin{aligned}(x^{-}(x^{-} \cdot y^{-})^{\sim})^{-} \cdot x^{-} =& (x^{-} \cdot (x \oplus y))^{-} \cdot x^{-} \\ =& ((x^{-} \cdot (x \oplus y)) \oplus x)^{-} && \text{(Corollary 4.6)} \\ =& ((x \oplus (x \oplus y)^{\sim})^{-} \oplus x)^{-} \\ =& (x \oplus y)^{-} && \text{(condition (CD))} \\ =& x^{-} \cdot y^{-}\end{aligned}$$

by Proposition 4.1(iv). Note that $(x^{-} \cdot (x \oplus y)) \oplus x$ is defined, since

$$\begin{aligned}(x^{-} \cdot (x \oplus y)) =& ((x^{-\sim} \wedge (x \oplus y)) \oplus (x \oplus y)^{\sim})^{-} \\ =& ((x \wedge (x \oplus y)) \oplus (x \oplus y)^{\sim})^{-} \\ =& (x \oplus (x \oplus y)^{\sim})^{-} \leq x^{-},\end{aligned}$$

by Proposition 4.1(iii), the definition of $\leq$, the fact that $x \oplus (x \oplus y)^{\sim}$ is exists by Lemma 4.2(iii) and the antitonicity of $^{-}$.
Similarly, again by (CD) and Corollary 4.6, we have that

$$\begin{aligned}y^{-} \cdot ((x^{-} \cdot y^{-})^{-} y^{-})^{\sim} =& y^{-}((x \oplus y)^{--} \cdot y^{-})^{\sim} \\ =& y^{-} \cdot ((x \oplus y)^{-} \oplus y) \\ =& (y^{-} \cdot ((x \oplus y)^{-} \oplus y)^{\sim -})^{\sim -} \\ =& (y \oplus ((x \oplus y)^{-} \oplus y)^{\sim})^{-} \\ =& (x \oplus y)^{-} = x^{-} \cdot y^{-}\end{aligned}$$

(P7) If $x \leq y$, then there is a z such that $z \oplus x = y$. Therefore, by Lemma 4.5(ii) and (iii), $z \leq x^-$. Hence, $z^- \cdot x^- = ((z^{-\sim} \wedge x^-) \oplus x^{-\sim})^- = ((z \wedge x^-) \oplus x)^- = (z \oplus x)^- = y^-$. $\square$

As a converse of the previous result, the next theorem shows that any near-p semiring can be turned into a lattice pseudoeffect algebra.

Theorem 4.10. *Let* $\mathbf{R} = (R, +, \cdot, f, g, 0, 1)$ *be a near-p semiring, and* $\leq$ *its induced lattice order. Then the structure*

$$\mathcal{E}(\mathbf{R}) = (R, \oplus, g, f, 0, 1)$$

is a lattice pseudoeffect algebra whose order coincides with $\leq$*, where*

$$x \oplus y = g(f(x) \cdot f(y)) \text{ is defined in case } x \leq f(y).$$

Proof. (E1) Suppose $x \oplus y$ and $(x \oplus y) \oplus z$ are defined. Then, $x \leq f(y)$, and $x \oplus y \leq f(z)$. So, by (P2) and (P3), $g(f(x)) \cdot g(f(y)) = 0$ and $g(f(x) \cdot f(y)) \cdot g(f(z)) = 0$. Therefore, by (P4), $g(f(y)) \cdot g(f(z)) = 0$ and $g(f(x)) \cdot g(f(y) \cdot f(z)) = 0$. This means that $y \leq f(z)$ and $x \leq f(y) \cdot f(z) = f(g(f(y) \cdot f(z)))$. Hence, $y \oplus z$ and $x \oplus (y \oplus z)$ are defined and $(x \oplus y) \oplus z = x \oplus (y \oplus z)$, by (P4) and (P5). The converse implication is straightforward.

We now prove (E2). Immediately, it can be seen that $g(x) \leq g(x)$ implies that $x \cdot g(x) = 1$, and $x \leq x$ implies that $f(x) \cdot x = 0$. Then, $g(f(x) \cdot f(g(x))) = x \oplus g(x) = 1$. On the other hand, since $f(f(x)) \cdot f(x) = 0$, then $g(f(f(x)) \cdot f(x)) = f(x) \oplus x = 1$. As regards uniqueness, suppose $x \oplus k = 1$. Hence, $x \leq f(k)$ and $g(f(x) \cdot f(k)) = 1$ and $f(x) \cdot f(k) = 0$. So, $f(k) \leq x$ and $f(k) = x$, namely $k = g(x)$. Finally, assuming that $k \oplus x = 1$, $k = f(x)$ follows by a similar argument.

For (E3), suppose that $x \oplus y$ is defined. Then, $x \leq f(y)$. Thus, applying (P6),

$$\begin{aligned} x \oplus y =& g(f(x) \cdot f(y)) \\ =& g(f(f(x) \cdot g(f(x) \cdot f(y))) \cdot f(x)) \\ =& (f(x) \cdot g(f(x) \cdot f(y))) \oplus x. \end{aligned}$$

Note that $(f(x) \cdot g(f(x) \cdot f(y))) \oplus x$ is defined. Indeed, by (P6) and $x \leq 1$ implies $x \cdot z \leq z$, one has that $x \leq f(y)$ implies $f(x) \cdot f(y) = f(f(x) \cdot g(f(x) \cdot f(y))) \cdot f(x) \leq f(x)$. Now, the fact that $f(x) \cdot f(y) \leq f(x)$ implies that $x \leq g(f(x) \cdot f(y)) = f(g(g(f(x) \cdot f(y))))$. Hence, $x \leq f(g(g(f(x) \cdot f(y))))$, making use of (P6) as above. Then, $x \leq f(g(g(f(x) \cdot f(y))))$ implies that $f(x) \cdot g(f(x) \cdot f(y)) = f(x) \cdot f(g(g(f(x) \cdot f(y)))) \leq f(x)$. Thus, from the previous reasoning, whenever $x \leq f(y)$ we have proven that $f(x) \cdot g(f(x) \cdot f(y)) \leq f(x)$, namely $(f(x) \cdot g(f(x) \cdot f(y))) \oplus x$ exists. Furthermore, one has that

$$\begin{aligned} x \oplus y =& g(f(x) \cdot f(y)) \\ =& g(f(y) \cdot g(f(f(x) \cdot f(y)) \cdot f(y))) \\ =& g(f(y) \cdot f(g(g(f(f(x) \cdot f(y)) \cdot f(y))))) \\ =& y \oplus g(g(f(f(x) \cdot f(y)) \cdot f(y))) \end{aligned}$$

and $y \oplus g(g(f(f(x) \cdot f(y)) \cdot f(y)))$ is defined because $f(f(x) \cdot f(y)) \leq 1$ implies that $f(f(x) \cdot f(y)) \cdot f(y) \leq 1 \cdot f(y)$, and then $y = g(f(y)) \leq g(f(f(x) \cdot f(y)) \cdot f(y)) = f(g(g(f(f(x) \cdot f(y)) \cdot f(y))))$.

As regards (E4), suppose that $1 \oplus x$ is defined. Then, $1 \leq f(x)$. Consequently, $x \leq 0$, i.e. $x = 0$. Dually, if $x \oplus 1$ is defined, then $x \leq f(1) = 0$, and again $x = 0$.

We now show that $x \leq^{\mathbf{R}} y$ if and only if $x \leq^{\mathcal{E}(\mathbf{R})} y$. If $x \leq^{\mathbf{R}} y$, then, by (P7), there is a $z \in R$ such that $f(z) \cdot f(x) = f(y)$ and $z \leq f(x)$. So, $g(f(z) \cdot f(x)) = z \oplus^{\mathcal{E}(\mathbf{R})} x = g(f(y)) = y$, i.e. $x \leq^{\mathcal{E}(\mathbf{R})} y$. Conversely, suppose that $x \leq^{\mathcal{E}(\mathbf{R})} y$. Then, since in pseudoeffect algebras the order induced by the operation $\oplus$ is two-sided (cf. page **??**), there exists a $z \in R$ such that $z \oplus^{\mathcal{E}(\mathbf{R})} x = y$. So, $g(f(z) \cdot f(x)) = y$. The fact that $f(z) \leq^{\mathbf{R}} 1$ implies that $f(z) \cdot f(x) \leq^{\mathbf{R}} 1 \cdot f(x)$, then $f(z) \cdot f(x) = f(y) \leq^{\mathbf{R}} f(x)$. Therefore, by antitonicity, $g(f(x)) = x \leq^{\mathbf{R}} y = g(f(y))$. In conclusion, the orders $x \leq^{\mathbf{R}} y$ and $x \leq^{\mathcal{E}(\mathbf{R})} y$ coincide, and then, because $\mathbf{R}$ is lattice ordered, $\mathcal{E}(\mathbf{R})$ is a lattice pseudoeffect algebra. □

Finally, we can state, as a corollary, that the correspondences above are indeed mutually inverse.

Corollary 4.11. *If* $\mathbf{A}$, $\mathbf{R}$ *are a lattice pseudoeffect algebra and a near-p semiring, respectively, then*

1. $\mathbf{A} = \mathcal{E}(\mathcal{P}(\mathbf{A}))$*;*

2. $\mathbf{R} = \mathcal{P}(\mathcal{E}(\mathbf{R}))$.

Proof. Clearly, unary operations and constants are all preserved by both compositions $\mathcal{P} \circ \mathcal{E}$ and $\mathcal{E} \circ \mathcal{P}$. Moreover the lattice orders $\leq^{\mathcal{E}(\mathcal{P}(\mathbf{A}))}$ and $\leq^{\mathcal{P}(\mathcal{E}(\mathbf{R}))}$ coincide, by virtue of the last part of the proof of Theorem 4.10.

(1) Let $\mathbf{A}$ be a lattice pseudoeffect algebra. If $x \oplus^{\mathbf{A}} y$ is defined, then $x \leq^{\mathbf{A}} y^-$, that is $x \leq f(y)$ in $\mathcal{P}(\mathbf{A})$. Therefore, $x \oplus^{\mathcal{E}(\mathcal{P}(\mathbf{A}))} y$ is defined and

$$\begin{aligned} x \oplus^{\mathcal{E}(\mathcal{P}(\mathbf{A}))} y =& g(f(x) \cdot^{\mathcal{P}(\mathbf{A})} f(y)) \\ =& g(f((g(f(x)) \wedge f(y)) \oplus^{\mathbf{A}} g(f(y)))) \\ =& (x \wedge f(y)) \oplus^{\mathbf{A}} y \\ =& (x \oplus^{\mathbf{A}} y) \wedge (f(y) \oplus^{\mathbf{A}} y) \\ =& (x \oplus^{\mathbf{A}} y) \wedge 1 \\ =& x \oplus^{\mathbf{A}} y. \end{aligned}$$

Note that this result rests on Proposition 4.3(ii).
(2) If $\mathbf{R}$ is a near-p semiring, then

$$\begin{aligned} x \cdot^{\mathcal{P}(\mathcal{E}(\mathbf{R}))} y =& ((x^{\sim} \wedge y) \oplus^{\mathcal{E}(\mathbf{R})} y^{\sim})^- \\ =& ((x^{\sim} \wedge y)^- \cdot^{\mathbf{R}} y^{\sim -})^{\sim -} \\ =& (x^{\sim} \wedge y)^- \cdot^{\mathbf{R}} y \\ =& (x^{\sim -} \vee y^-) \cdot^{\mathbf{R}} y \\ =& (x \vee y^-) \cdot^{\mathbf{R}} y \\ =& (x \cdot^{\mathbf{R}} y) \vee (y^- \cdot^{\mathbf{R}} y) \\ =& (x \cdot^{\mathbf{R}} y) \vee 0 \\ =& x \cdot^{\mathbf{R}} y. \end{aligned}$$

□

4.2 Semiring representation of pseudo-MV algebras

The notion of generalized Łukasiewicz semiring was introduced by A. Kadji, C. Lele and J.B. Nganou in [84] as a natural semiring counterpart of pseudo-MV algebras. In fact, they observed that any pseudo-MV algebra $\mathbf{A}$ has an underlying semiring reduct which entertains several relationships with the fundamental operations on $\mathbf{A}$.

Proposition 4.12 (Proposition 2.1, [84]). *Let $A = (A, \oplus, \odot, ^-, ^\sim, 0, 1)$ be a pseudo MV-algebra and $\mathcal{R}(\mathbf{A}) = (A, +, \cdot, 0, 1)$. Then $\mathcal{R}(\mathbf{A})$ is an additively idempotent semiring satisfying:*

(i) $x \cdot y = 0$ *iff* $y \leq x^-$ *iff* $x \leq y^\sim$;

(ii) $x + y = ((x^\sim \cdot y)^\sim \cdot x^\sim)^- = (x^\sim \cdot (y \cdot x^-)^\sim)^-$;

(iii) $(y^{\sim} \cdot x^{\sim})^{-} = (y^{-} \cdot x^{-})^{\sim}$;

where $x + y = x \vee y, x \cdot y = x \oplus y$, *and* $x \leq y$ *iff* $x^{-} \oplus y = 1$.

However, although their construction is interesting, we found that their results can be refined by streamlining the number of axioms (cf. [84], p. 3) to a simple and rather natural condition. Then, in what follows, we outline an alternative one-axiom characterization of gl-semirings showing how the main arithmetical properties of these structures readily derive from our simple axiomatization, and then we propose a proof, in semirings' framework, of Dvurečenskij and Vetterlein ([41]) results stating that pseudo MV- algebras can be regarded as a proper subvariety of pseudoeffect algebras.
Note that Proposition 4.12 strongly relies on the distributivity of $\odot$ over $\vee$.

Proposition 4.13 (Proposition 1.16, [55]). *Let* $\mathbf{A}$ *be a pseudo-MV algebra. Then, for any* $\{x\} \cup \{y_i\}_{i \in I} \subseteq A$,

$$x \odot (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \odot y_i) \text{ and } (\bigvee_{i \in I} y_i) \odot x = \bigvee_{i \in I} (y_i \odot x),$$

whenever $\bigvee_{i \in I} y_i$ *exists.*

We are now ready to re-state the notion of generalized Łukasiewicz semiring.

Definition 4.14. A *generalized Łukasiewicz semiring* (*gl-semiring*) is an algebra $\mathbf{R} = (R, +, \cdot, ^{-}, ^{\sim}, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ such that the reduct $(R, +, \cdot, 0, 1)$ is a semiring, $^{-}$ and $^{\sim}$ are unary operations satisfying $x + y = y$ implies $x^{-} + y^{-} = x^{-}$ and $x^{\sim} + y^{\sim} = x^{\sim}$, and the following equations are satisfied:

$$x + y = ((x^{\sim} \cdot y)^{\sim} \cdot x^{\sim})^{-} = (x^{\sim} \cdot (y \cdot x^{-})^{\sim})^{-}. \tag{4.1}$$

This rather concise definition yields quite strong properties. First, it is possible to verify that the structure $(R, \leq)$, where

$$x \leq y \text{ iff } x + y = y, \tag{4.2}$$

is a bouded poset. In fact, it is clearly transitive and antisymmetric and item (9) of the next lemma shows that it is also reflexive. Since 0 is the neutral element with respect to the addition and by integrality, proved in (8), any gl-semiring induces a poset with a least and a greatest element. Hence, $^{-}$ and $^{\sim}$ can be seen as order-reversing unary operations.
Some of the items in the next lemma can be found also in [84, Lemma 2.2]. However, for reader's convenience, we provide new proofs for any of them.

Lemma 4.15. *Every gl-semiring satisfies the following conditions:*

1. $0 = (0^{\sim} \cdot 0^{\sim})^{-}$;
2. $1 = (0^{\sim\sim} \cdot 0^{\sim})^{-}$;
3. $0^{\sim} = 1$;
4. $0 = 1^{-}$;
5. $0^{\sim -} = 1^{-}$;
6. $1^{-\sim} = 1$;
7. $1 = 1^{\sim -}$;
8. $x + 1 = 1$;
9. $x = x + x$;
10. $x = x^{\sim -}$;
11. $1^{\sim} = 0$;
12. $0^{-} = 1$;
13. $0^{\sim\sim} = 0$;
14. $0^{-\sim} = 0$;
15. $x^{\sim} \cdot x = 0$;
16. $(x^{\sim} \cdot y)^{\sim} \cdot x^{\sim} =$
 $(x + y)^{\sim} = x^{\sim} \cdot (y \cdot x^{-})^{\sim}$;
17. $(x \cdot x^{-})^{\sim} \cdot x^{-} = x^{-}$;
18. $x \cdot x^{-} = 0$;
19. $x = x^{-\sim}$.

Proof. (1) $0 = 0 + 0 = ((0^{\sim} \cdot 0)^{\sim} \cdot 0^{\sim})^{-} = (0^{\sim} \cdot 0^{\sim})^{-}$.
(2) $1 = 0 + 1 = ((0^{\sim} \cdot 1)^{\sim} \cdot 0^{\sim})^{-} = (0^{\sim\sim} \cdot 0^{\sim})^{-}$.
(3) Using item (2),

$$0^{\sim} = 0^{\sim} + 0 = (0^{\sim\sim} \cdot (0 \cdot 0^{\sim -})^{\sim})^{-} = (0^{\sim\sim} \cdot 0^{\sim})^{-} = 1.$$

(4) $0 = (0^{\sim} \cdot 0^{\sim})^{-} = (1 \cdot 1)^{-} = 1^{-}$, by items (1) and (3).
(5) Items (3) and (4) imply that $0^{\sim -} = 1^{-} = 0$.
(6) Items (4) and (3) imply that $1^{-\sim} = 0^{\sim} = 1$.
(7) Using item (3),

$$1 = 1 + 0 = (1^{\sim} \cdot (0 \cdot 1^{-})^{\sim})^{-} = (1^{\sim} \cdot 0^{\sim})^{-} = (1^{\sim} \cdot 1)^{-} = 1^{\sim -}.$$

(8) $x + 1 = 1 + x = (1^{\sim} \cdot (x \cdot 1^{-})^{\sim})^{-} = (1^{\sim} \cdot 0^{\sim})^{-} = (1^{\sim} \cdot 1)^{-} = 1^{\sim -} = 1$, by items (4), (3) and (7).
(9) by item (8) one has that $1 + 1 = 1$, hence by distributivity it follows that $(1+1) \cdot x = 1 \cdot x$, namely $x + x = x$.
(10) See the proof of item (7).
(11) Note that $1^{\sim} = 1^{\sim} + 1^{\sim} = (1^{\sim\sim} \cdot (1^{\sim} \cdot 1^{\sim -})^{\sim})^{-} = (1^{\sim\sim} \cdot 1^{\sim\sim})^{-}$. Now, using item (10), $(x^{\sim} \cdot x^{-\sim})^{-} = (x^{\sim} \cdot (1 \cdot x^{-})^{\sim})^{-} = x + 1 = 1$. Putting $x = (1^{\sim\sim} \cdot 1^{\sim\sim})$, one has that $1 = ((1^{\sim\sim} \cdot 1^{\sim\sim})^{\sim} \cdot (1^{\sim\sim} \cdot 1^{\sim\sim})^{-\sim})^{-}$; since $1^{\sim} = (1^{\sim\sim} \cdot 1^{\sim\sim})^{-}$, substituting we obtain that $1 = ((1^{\sim\sim} \cdot 1^{\sim\sim})^{\sim} \cdot 1^{\sim\sim})^{-} = 1^{\sim} + 1^{\sim\sim} = 1^{\sim\sim}$, because $1^{\sim} \leq 1$ and ${}^{\sim}$ is antitone. Thus, $1^{\sim} = 1^{\sim\sim -} = 1^{-} = 0$.
(12) follows by items (7) and (11).

(13) $0^{\sim\sim} = 1^{\sim} = 0$, by items (3) and (11).
(14) From items (12) and (11), one has $0^{-\sim} = 1^{\sim} = 0$.
(15) As in (11), we have $1 = (x^{\sim} \cdot x^{-\sim})^{-}$. Replace x by $x^{\sim} \cdot (x \cdot x^{-})^{\sim}$. Then,

$$1 = ((x^{\sim} \cdot (x \cdot x^{-})^{\sim})^{\sim} \cdot (x^{\sim} \cdot (x \cdot x^{-})^{\sim})^{-\sim})^{-} = ((x^{\sim} \cdot (x \cdot x^{-})^{\sim})^{\sim} \cdot x^{\sim})^{-} = x + (x \cdot x^{-})^{\sim},$$

since $(x^{\sim} \cdot (x \cdot x^{-})^{\sim})^{-} = x + x = x$, by item (9). Replacing x by $x^{\sim}$, by item (10), we have that $1 = x^{\sim} + (x^{\sim} \cdot x^{\sim-})^{\sim} = x^{\sim} + (x^{\sim} \cdot x)^{\sim} = (x^{\sim} \cdot x)^{\sim}$, by $x^{\sim} \cdot x \leq x$ and $^{\sim}$ is order reversing. So, $x^{\sim} \cdot x = 0$.
(16) First, let us observe that item (15) implies that $x^{-\sim} + x = x + x^{-\sim} = (x^{\sim} \cdot (x^{-\sim} \cdot x^{-})^{\sim})^{-} = x^{\sim-} = x$. So,

$$(x+y)^{\sim} + ((y^{\sim} \cdot x)^{\sim} \cdot y^{\sim}) = ((y^{\sim} \cdot x)^{\sim} \cdot y^{\sim})^{-\sim} + ((y^{\sim} \cdot x)^{\sim} \cdot y^{\sim}) = (y^{\sim} \cdot x)^{\sim} \cdot y^{\sim}.$$

Hence, $(x+y)^{\sim} \leq (y^{\sim} \cdot x)^{\sim} \cdot y^{\sim}$. Conversely, by item (3), (10), (15) and distributivity, it follows that

$$\begin{aligned}
(x+y)^{\sim} + ((y^{\sim} \cdot x)^{\sim} \cdot y^{\sim}) =& ((x+y)^{\sim\sim} \cdot (((y^{\sim} \cdot x)^{\sim} \cdot y^{\sim}) \cdot (x+y)^{\sim-})^{\sim})^{-} \\
=& ((x+y)^{\sim\sim} \cdot (((y^{\sim} \cdot x)^{\sim} \cdot y^{\sim}) \cdot (x+y))^{\sim})^{-} \\
=& ((x+y)^{\sim\sim} \cdot ((((y^{\sim} \cdot x)^{\sim} \cdot y^{\sim}) \cdot x) + (((y^{\sim} \cdot x)^{\sim} \cdot y^{\sim}) \cdot y))^{\sim})^{-} \\
=& ((x+y)^{\sim\sim} \cdot (((y^{\sim} \cdot x)^{\sim} \cdot (y^{\sim} \cdot x)) + ((y^{\sim} \cdot x)^{\sim} \cdot (y^{\sim} \cdot y)))^{\sim})^{-} \\
=& ((x+y)^{\sim\sim} \cdot 1)^{-} \\
=& (x+y)^{\sim}.
\end{aligned}$$

The second equality follows similarly. In fact, from the previous reasoning, we have $(x+y)^{\sim} \leq x^{\sim} \cdot (y \cdot x^{-})^{\sim}$. Now, note that (by (15), (3) and (10)) $x + (y \cdot x^{-})^{\sim} \cdot y = (x^{\sim} \cdot ((y \cdot x^{-})^{\sim} \cdot (y \cdot x^{-}))^{\sim})^{-} = (x^{\sim} \cdot 0^{\sim})^{-} = x^{\sim-} = x$. Hence, by (15) and the distributivity of $\cdot$ over $+$, it follows that $x^{\sim} \cdot (y \cdot x^{-})^{\sim} \cdot y = 0 + x^{\sim} \cdot (y \cdot x^{-})^{\sim} \cdot y = x^{\sim} \cdot x + x^{\sim} \cdot (y \cdot x^{-})^{\sim} = x^{\sim} \cdot (x + (y \cdot x^{-})^{\sim} \cdot y) = x^{\sim} \cdot x = 0$. Furthermore, since any element is under the unit and $\cdot$ preserves the order in both arguments, we have that $(y \cdot x^{-})^{\sim} \cdot x \leq x$ implies that $x^{\sim} \cdot (y \cdot x^{-})^{\sim} \cdot x \leq x^{\sim} \cdot x = 0$, namely $x^{\sim} \cdot (y \cdot x^{-})^{\sim} \cdot x = 0$. Thus, it is easily seen that

$$\begin{aligned}
(x+y)^{\sim} + (x^{\sim} \cdot (y \cdot x^{-})^{\sim}) =& ((x+y)^{\sim\sim} \cdot ((x^{\sim} \cdot (y \cdot x^{-})^{\sim}) \cdot (x+y)^{\sim-})^{\sim})^{-} \\
=& ((x+y)^{\sim\sim} \cdot ((x^{\sim} \cdot (y \cdot x^{-})^{\sim}) \cdot (x+y))^{\sim})^{-} \\
=& ((x+y)^{\sim\sim} \cdot ((x^{\sim} \cdot (y \cdot x^{-})^{\sim} \cdot x) + (x^{\sim} \cdot (y \cdot x^{-})^{\sim} \cdot y))^{\sim})^{-} \\
=& ((x+y)^{\sim\sim} \cdot 0^{\sim})^{-} \\
=& ((x+y)^{\sim\sim} \cdot 1)^{-} \\
=& (x+y)^{\sim}.
\end{aligned}$$

The identity clearly follows from the definition of the induced order. (17) Note that, by (9) and (16), we have $x^{\sim} = (x+x)^{\sim} = x^{\sim} \cdot (x \cdot x^{-})^{\sim}$. Now, by (16), (9), (15) and (12), one has $x + (x \cdot x^{-})^{\sim} = ((x^{\sim} \cdot (x \cdot x^{-})^{\sim})^{\sim} \cdot x^{\sim})^{-} = (x^{\sim\sim} \cdot x^{\sim})^{-} = 1$. Thus, again by (16), we compute

$$\begin{aligned} x+((x\cdot x^-)^\sim\cdot x^-)^\sim &= ((x^\sim\cdot((x\cdot x^-)^\sim\cdot x^-)^\sim)^\sim\cdot x^\sim)^- \\ &= (((x+(x\cdot x^-)^\sim)^\sim)^\sim\cdot x^\sim)^- \\ &= (1^{\sim\sim}\cdot x^\sim)^- \\ &= x, \end{aligned}$$

namely $((x\cdot x^-)^\sim\cdot x^-)^\sim \leq x$ and by antitonicity of $^-$, $x^- \leq ((x\cdot x^-)^\sim\cdot x^-)^{\sim -} = (x\cdot x^-)^\sim\cdot x^-$. Furthermore, since $(x\cdot x^-)^\sim \leq 1$ (by item (8)), it follows that $(x\cdot x^-)^\sim\cdot x^- \leq x^-$, namely $x^- = (x\cdot x^-)^\sim\cdot x^-$.

(18) By items (10), (17), (15) and (12), one has

$$\begin{aligned} ((x^{-\sim\sim}\cdot(x\cdot x^-)^\sim)^\sim\cdot x^{-\sim\sim})^- &= x^{-\sim}+(x\cdot x^-)^\sim \\ &= (x^{-\sim\sim}\cdot((x\cdot x^-)^\sim\cdot x^{-\sim-})^\sim)^- \\ &= (x^{-\sim\sim}\cdot((x\cdot x^-)^\sim\cdot x^-)^\sim)^- \\ &= (x^{-\sim\sim}\cdot x^{-\sim})^- = 1. \end{aligned}$$

Setting $a = x^-$, $b = x$, we obtain that $((a^{\sim\sim}\cdot(b\cdot a)^\sim)^\sim\cdot a^{\sim\sim})^- = 1$. Let us note that $(a^{\sim\sim}\cdot(b\cdot a)^\sim)^\sim\cdot a^{\sim\sim} = (a^\sim+(b\cdot a)^\sim)^\sim = (b\cdot a)^{\sim\sim}$, by (16) and antitonicity. Therefore, $1 = (b\cdot a)^{\sim\sim-} = (b\cdot a)^\sim$, and then $(b\cdot a)^{\sim-} = b\cdot a = 1^- = 0$ (items (10) and (12)), namely $x\cdot x^- = 0$.

(19) By the previous item, (10) and (3) we have that $x^{-\sim}+x = (x^{-\sim\sim}\cdot(x\cdot x^-)^\sim)^- = x^{-\sim\sim-} = x^{-\sim}$. Therefore, by the definition of the induced order, $x \leq x^{-\sim}$. Finally, it follows that $x+x^{-\sim} = (x^\sim\cdot(x^{-\sim}\cdot x^-)^\sim)^- = x$ (by item (15), (3) and (10)). □

The items of the next lemma are taken in [84] as axioms.

Lemma 4.16. *Let* $\mathbf{R}$ *be a gl-semiring. Then, the following conditions are satisfied:*

1. $x\cdot y = 0$ *iff* $y \leq x^-$ *iff* $x \leq y^\sim$;
2. $(x^\sim\cdot y^\sim)^- = (x^-\cdot y^-)^\sim$.

Proof. (1) Clearly, by Definition 4.14 and Lemma 4.15, $y \leq x^-$ iff $x \leq y^\sim$. On the one hand, suppose $x \leq y^\sim$. Note that:

$$\begin{aligned} 1 &= x+1 && \text{(Lemma 4.15(8))} \\ &= ((x^\sim\cdot 1)^\sim\cdot x^\sim)^- && \text{(Def. 4.14)} \\ &= (x^{\sim\sim}\cdot x^\sim)^-. \end{aligned}$$

Upon setting $x = y^-$, by Lemma 4.15(19), we have that $1 = (y^{-\sim\sim} \cdot y^{-\sim})^- = (y^\sim \cdot y)^-$. And so, by Lemma 4.15(11),(19), $0 = 1^\sim = (y^\sim \cdot y)^{-\sim} = y^\sim \cdot y \geq x \cdot y$.
On the other hand, if $x \cdot y = 0$, then

$$\begin{array}{rcll} y + x^- = & & x^- + y & \text{(Def. 4.14)} \\ & = & ((x^{-\sim} \cdot y)^\sim \cdot x^{-\sim})^- & \text{(Def. 4.14)} \\ & = & ((x \cdot y)^\sim \cdot x)^- & \text{(Lemma 4.15(19))} \\ & = & (0^\sim \cdot x)^- & \\ & = & (1 \cdot x)^- & \text{(Lemma 4.15(3))} \\ & = & x^-. & \end{array}$$

Therefore, $y \leq x^-$.
(2) Let us remark that the statement "$z \leq (x^- \cdot y^-)^\sim$ implies $z \leq (x^\sim \cdot y^\sim)^-$" is equivalent, by the previous item, to "$z \cdot (x^- \cdot y^-) = 0$ implies $(x^\sim \cdot y^\sim) \cdot z = 0$". In turn, by associativity of $\cdot$, item (1) and Lemma 4.15, this is equivalent to "$z \cdot x^- \leq y$ implies $y^\sim \cdot z \leq x$". Thus, we have that $z \cdot x^- \leq y$ implies $y^\sim \cdot z \cdot x^- \leq y^\sim \cdot y$, which implies that $(y^\sim \cdot z) \cdot x^- = 0$, and so, by the previous item and Lemma 4.15, $y^\sim \cdot z \leq x^{-\sim} = x$. The converse inequality is proved dually. □

By virtue of Lemma 4.15, it follows that the structure $(R, \leq, 0, 1)$, where $\leq$ is defined by condition (4.2), is a bounded lattice, with $x \vee^{\mathbf{R}} y = x + y$, $x \wedge^{\mathbf{R}} y = (x^- + y^-)^\sim = (x^\sim + y^\sim)^-$ (cf. [84, Lemma 2.2(v)]).

Now, let $\mathbf{R}$ be a gl-semiring . Upon setting, for all $x, y \in R$,

$$x \oplus y = (y^- \cdot x^-)^\sim = (y^\sim \cdot x^\sim)^- \text{ and } x \odot y = x \cdot y,$$

it is possible to prove that

Theorem 4.17. *The structure $\mathcal{A}(\mathbf{R}) = (R, \oplus, \odot, ^-, ^\sim, 0, 1)$ is a pseudo-MV algebra.*

Proof. See [84, Proposition 2.3]. □

Let $\mathcal{R}(\mathbf{A})$ be the gl-semiring associated to the pseudo MV-algebra $\mathbf{A}$ (cf. [84, Proposition 2.5]). The previous theorem yields the following

Corollary 4.18. *If $\mathbf{A}$, $\mathbf{R}$ are a pseudo-MV algebra and a gl-semiring, respectively, then*

1. $\mathbf{A} = \mathcal{A}(\mathcal{R}(\mathbf{A}))$;

2. $\mathbf{R} = \mathcal{R}(\mathcal{A}(\mathbf{R}))$.

Proof. See [84, Proposition 2.5] $\square$

Finally, we close this chapter recalling that, by Dvurečenskij and Vetterlein [41], any pseudo-MV algebra can be regarded as a lattice pseudoeffect algebra. In fact, in what follows, we propose an alternative proof of [41, Theorem 8.3 and Theorem 8.7].
With this aim in mind, we start with the following technical lemma.

Lemma 4.19. *Let* $\mathbf{R} = (R, +, \cdot, ^-, ^\sim, 0, 1)$ *be a gl-semiring. Then, the following equations hold:*

(i) $(x^\sim \cdot y)^- \cdot x^- = x^- \cdot (y \cdot x^-)^-$;

(ii) $(x^{\sim\sim} \cdot y)^- \cdot x = (y^- \cdot x^-)^\sim \cdot y^-$;

(iii) $x \cdot (y \cdot x)^- = (y^- \cdot x^-)^\sim \cdot y^-$.

Proof. (i) Just note that $((x^\sim \cdot y)^- \cdot x^-)^\sim = ((x^\sim \cdot y)^\sim \cdot x^\sim)^- = x + y = (x^\sim \cdot (y \cdot x^-)^\sim)^- = (x^- \cdot (y \cdot x^-)^-)^\sim$ by Lemma 4.16(ii). Then, the identity follows by Lemma 4.15(10). As regards (ii), by commutativity and the definition of $+$, repeatedly applying Lemma 4.16(ii), we compute $((x^{\sim\sim} \cdot y)^- \cdot x)^\sim = ((x^{\sim\sim} \cdot y)^- \cdot x^{\sim -})^\sim = ((x^{\sim\sim} \cdot y)^\sim \cdot x^{\sim\sim})^- = x^\sim + y = ((y^\sim \cdot x^\sim)^\sim \cdot y^\sim)^- = ((y^\sim \cdot x^\sim)^- \cdot y^-)^\sim = ((y^- \cdot x^-)^\sim \cdot y^-)^\sim$. The conclusion follows by Lemma 4.15(10). Finally, (iii) derives from item (i) replacing x by $x^\sim$ and then applying (ii). $\square$

Theorem 4.20. *Any gl-semiring is a near-p semiring.*

Proof. Let $\mathbf{R} = (R, +, \cdot, ^-, ^\sim, 0, 1)$ be a gl-semiring. Then, putting $g(x) = x^-$ and $f(x) = x^\sim$, we show that $(R, +, \cdot, f, g, 0, 1)$ is a near-p semiring. Since any integral idempotent semiring is also an integral, idempotent, bounded, near-semiring, we just need to show that (P1)-P(7) of Definition 4.8 hold. (P1)-(P3) are straightforward and (P5) holds trivially. As regards (P4), assume that $g(x) \cdot g(y) = 0$ and $g(x \cdot y) \cdot g(z) = 0$. Hence, $x^- \cdot y^- = 0$ and $(x \cdot y)^- \cdot z^- = 0$. By Lemma 4.16(1), it follows that $x^- \leq y$ and $(x \cdot y)^- \leq z$. Thus $z^\sim \leq x \cdot y \leq y$ and $y^- \leq z$. So, another application of Lemma 4.16(1) yields $0 = y^- \cdot z^- = g(y) \cdot g(z)$. Furthermore, note that $(y \cdot z)^\sim \cdot y = ((y^{-\sim} \cdot z)^\sim \cdot y^{-\sim})^{-\sim} = (y^- + z)^\sim = z^\sim$ and $z^\sim \leq (x \cdot y)$ implies $z^\sim \cdot (x \cdot y)^- = 0$. By Lemma 4.19(iii), one has $x^- = x^{--\sim} = (x^{--} + y^-)^\sim = ((x^{--\sim} \cdot y^-)^\sim \cdot x^{--\sim})^{-\sim} = (x^- \cdot y^-)^\sim \cdot x^- = y \cdot (x \cdot y)^-$. Therefore, we compute $(y \cdot z)^\sim \cdot x^- = (y \cdot z)^\sim \cdot (y \cdot (x \cdot y)^-) = z^\sim \cdot (x \cdot y)^- = 0$, namely (by Lemma 4.16(1)) $x^- \leq (y \cdot z)$ and $x^- \cdot (y \cdot z)^- = 0$.

For the converse, suppose that $g(y) \cdot g(z) = 0$ and $g(x) \cdot g(y \cdot z) = 0$. Hence, we have $x^- \cdot (y \cdot z)^- = 0$ and $y^- \cdot z^- = 0$, namely $x^- \leq y \cdot z \leq y$ and $y^- \leq z$. We readily obtain that $x^- \cdot y^- = 0$. Let us note that $y \cdot (x \cdot y)^- = x^-$, $(y \cdot z)^\sim \cdot x^- = 0$ and $(y \cdot z)^\sim \cdot y = z^\sim$ (by Lemma 4.16 and Lemma 4.19(iii)). Therefore, $0 = (y \cdot z)^\sim \cdot x^- = (y \cdot z)^\sim \cdot (y \cdot (x \cdot y)^-) = ((y \cdot z)^\sim \cdot y) \cdot (x \cdot y)^- = z^\sim \cdot (x \cdot y)^-$. Thus, by Lemma 4.16(1), from the fact that $z^\sim \cdot (x \cdot y)^- = 0$ we have that $(x \cdot y)^- \leq z$, namely $(x \cdot y)^- z^- = 0$.

As regards (P6), we must show that if $x \leq y^\sim$, then $x^\sim \cdot y^\sim = (x^\sim \cdot (x^\sim \cdot y^\sim)^-)^\sim \cdot x^\sim = y^\sim \cdot ((x^\sim \cdot y^\sim)^\sim \cdot y^\sim)^-$. Now, the first equality immediately follows by the definition of the sum: $((x^\sim \cdot (x^\sim \cdot y^\sim)^-)^\sim \cdot x^\sim)^- = x + (x^\sim \cdot y^\sim)^- = (x^\sim \cdot y^\sim)^-$. The second equality follows similarly by noticing that $(y^\sim \cdot ((x^\sim \cdot y^\sim)^\sim \cdot y^\sim)^-)^- = y + (x^\sim \cdot y^\sim)^-$ (by a simple application of Lemma 4.16(2)). Finally, (P7) is straightforward: just note that if $x \leq y$, then $x + y = y$ and $(x^\sim \cdot y)^\sim \cdot x^\sim = y^\sim$ and clearly $(x^\sim \cdot y) \leq x^\sim$. □

Corollary 4.21. *Let* $\mathbf{A}$ *be a pseudo MV-algebra. Then* $\mathcal{E}(\mathcal{R}(\mathbf{A}))$ *is a lattice pseudoeffect algebra.*

Proof. By Corollary 4.18, Theorem 4.20 and Theorem 4.10. □

Conversely, it is not difficult to provide condition(s) under which a near-p semiring can be turned into a gl-semiring.

Theorem 4.22. *A near-p semiring* $\mathbf{A}$ *is a gl-semiring if and only if for any* $x, y \in A$, *there exists a unique* z *such that*

$$f(y) \cdot f(z) = f(x \vee y) \quad \textit{and} \quad f(x \wedge y) \cdot f(z) = f(x).$$

In fact, let $\mathbf{A}$ be a near-p semiring. By [41, Theorem 8.7 and Proposition 8.15(γ) and (δ)], a lattice pseudoeffect algebra $\mathbf{B}$ is a pseudo MV-algebra if and only if its elements are pairwise *compatible* (cf. [126, Theorem 4.8]), namely for any $x, y \in B$ one has

$$(x \vee y)/y = x/(x \wedge y),$$

where $(x \vee y)/y$ and $x/(x \wedge y)$ are (unique, cf. [40, Lemma 1.4(v)]) elements in B such that $((x \vee y)/y) \oplus y = x \vee y$ and $(x/(x \wedge y)) \oplus (x \wedge y) = x$ (cf. [41, Definition 8.5]). Hence, by Theorem 4.10, since any pseudo-MV algebra is a gl-semiring, under suitable translations into our framework, we readily have that $\mathcal{R}(\mathcal{E}(\mathbf{A}))$ is a gl-semiring if and only if for any $x, y \in A$, there exists a unique z satisfying the conditions of the above theorem, namely x and y are compatible in $\mathcal{E}(\mathbf{A})$.
Although its proof is straightforward, the previous remark emphasizes an interesting

relationship between the structures we have dealt with in this chapter.

Finally, as it might have been noticed, near-p semirings do not form a variety since they are not defined by means of equations. Therefore, in our opinion, solving the following problem might yield a relevant achievement, since it would make pseudoeffect algebras amenable of universal algebraic investigations:

Problem 1. Do near-p semirings admit a finite basis of axioms?

If yes,

Problem 2. Which properties does the variety of near-p semirings enjoy?

Chapter 5

Paraorthomodular lattices, residuation and near semirings

In Chapter 3 it has been shown that basic algebras, which represent a "unifying" framework in which lattice effect algebras as well as orthomodular lattices can be interpreted into, are amenable of a smooth structural analysis if converted into Łukasiewicz near semirings. Subsequently, in Chapter 4, we have seen that lattice pseudoeffect algebras as well as pseudo-MV algebras can be described in a semiring-like fashion by means of near-p semirings. As it has been pointed out in Chapter 2, the aforementioned algebraic structures satisfy the paraorthomodular condition. Therefore, a natural question arises: may paraorthomodular lattices be represented as near-semirings?
Actually, if we would have been able to interpret paraorthomodular lattices into basic algebras framework, i.e. as bounded lattices with sectional antitone involutions, then we would have been done. Unfortunately, the next example shows this is not the case.

Example 5.1. *Consider the distributive paraorthomodular lattice depicted in Fig. 5.1. Let us suppose, by way of contradiction, it can be equipped with sectional antitone involutions and let us consider the interval* $[b, 1]$*. If* $x^b = 1$*, for* $x \in \{a, a', b'\}$*, then* $x = b$*, which is impossible. Similarly, we prove that* $x^b \neq b$*. Now, if* $b'^b = b'$*, then* $a \leq b'^b$ *implies* $b' = b'^{bb} \leq a^b$ *and* $a^b = b'$*, i.e.* $a = b'^b = b'$*, a contradiction. Thus one has* $b'^b \in \{a, a'\}$*. If* $b'^b = a$*, then* $a^b = b'$*. Since,* $a' \leq a^b$*, we conclude that* $a \leq a'^b$*. If* $a'^b = a$*, then* $a' = a^b = b'$*. Thus* $a'^b = b'$ *and* $a' = b'^b = a$*, again a contradiction. Similarly, we conclude that* $b'^b \neq a'$*. Therefore, the above paraorthomodular lattice cannot be converted into a basic algebra.*

Thus, in order to turn paraorthomodular lattices into near-semirings, we must go another way.

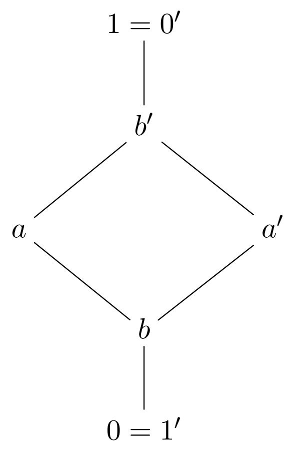


FIGURE 5.1

In this chapter, we show that paraorthomodular lattices of a certain sort can be converted into near-semirings by "going through" the theory of left-residuated ℓ-groupoids. In other words, we will show that, under certain conditions, a parorthomodular lattice $\mathbf{A}$ can be equipped with binary operations $*, \to$ such that $(x * a, a \to x)$ forms a residuated pair, for any $a \in A$.
The above approach has two advantages: on one hand, it will allow us to frame paraorthomodular lattices, basic algebras, and lattice pseudoeffect algebras in a same theoretical "environment". On the other, it will provide conditions granting a paraorthomodular lattice to be equipped with a (partially) residuated material implication operation. Therefore, we will conclude that, under certain conditions, paraorthomodular lattices can be thought as logics in their own right.

5.1 Material implication in Kleene, modular and paraorthomodular lattices

Since 1930s, the class of ortholattices (see e.g. [5]) and, subsequently, orthomodular lattices and posets were introduced as "quantum logics". Therefore, a natural question raised: "Is the quantum logic really a logic?". The debate as to whether OMLs can be regarded as logics in their own right commenced since the seminal paper [81] by Jauch and Piron appeared in 1970. According to their ideas further developed by R. Greechie and S. Gudder in [68, 69], the lattice of closed subspaces of a Hilbert space does not admit an algebraic counterpart of the *modus ponens* inference scheme, namely it cannot be equipped with a conditional operation by means of which such a deductive scheme can be incorporated.

Neverthless, G.M. Hardegree stated that any orthomodular lattice **A** allows the definition of a privileged term-operation ([76]), the so-called Sasaki hook $\to_s$, by putting, for any $a, b \in A$:

$$a \to_s b = a' \vee (a \wedge b). \qquad \text{(Sasaki hook)}$$

Surprisingly enough, it reflects all the essential traits of the Boolean horseshoe. In fact, it satisfies the following minimal implicative conditions [76, pp. 165-166]:

N1 $x \wedge (x \to_s y) \leq y$;

N2 $x' \wedge (y \to x) \leq y'$;

N3 $x \wedge y' \leq (x \to_s y)'$.

Moreover, if we define, for any $a, b \in A$

$$a \odot b = b \wedge (b' \vee a), \qquad \text{(Sasaki projection)}$$

then we have (see [25, 27])

$$x \odot y \to z \quad \text{iff} \quad x \leq y \to_s z.$$

In view of the above discussion, due to the prominent importance of paraorthomodular lattices for the logico-algebraic approach to QT, it is worth asking if they can be regarded as logics in their own right. In other words, one might wonder if paraorthomodular lattices can be equipped with a residuated material implication operation playing the same role of the Sasaki hook in orthomodular lattices.
In the framework of structures arising in alternative approaches to quantum theory, several achievements have been obtained in [24] for weakly orthomodular and dually weakly orthomodular lattices. Interesting results can be found e.g. in [22, 9, 50, 7] for effect algebras and lattice effect algebras. In this chapter, we will see that modular paraorthomodular lattices (cf. Proposition 2.33) satisfying a strenghtened form of regularity (see Definition 2.32) can be organized into left-residuated structures.

Let us introduce the notions that will be expedient for our purpose.

Definition 5.1. An algebra $\mathbf{A} = (A, \wedge, \vee, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a *left-residuated ℓ-groupoid* if:

(i) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;

(ii) $x \odot 1 = x = 1 \odot x$, for any $x \in A$;

(iii) $x \odot y \leq z$ if and only if $x \leq y \to z$ (left-residuation), for any $x, y, z \in A$.

If $(A, \odot)$ is also a semigroup, then $\mathbf{A}$ is an integral bounded left-residuated lattice. Finally, it is easily seen that, if $\odot$ is also commutative, then $\mathbf{A}$ is an integral bounded commutative residuated lattice (cf. [54]).

Given a modular involution lattice $\mathbf{A}$, let us define the operations the following operations, for any $x, y \in A$:

$$x \odot y = \begin{cases} 0, & \text{if } x \leq y' \\ y \wedge (x \vee y'), & \text{otherwise.} \end{cases} \tag{5.1}$$

$$x \to y = \begin{cases} 1, & \text{if } x \leq y \\ x' \vee (x \wedge y), & \text{otherwise.} \end{cases} \tag{5.2}$$

The next lemma proves that $\to$ and $\odot$ are indeed interdefinable.

Lemma 5.2. *Let $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ be a bounded lattice with antitone involution. Define $x \odot y$ and $x \to y$ as in (5.1) resp. (5.2). Then, $x \to y = (y' \odot x)'$.*

Proof. If $x \not\leq y$, i.e. $y' \not\leq x'$, then $(y' \odot x)' = (x \wedge (x' \vee y'))' = x' \vee (x \wedge y) = x \to y$. If $x \leq y$, then $y' \leq x'$ and $(y' \odot x)' = 0' = 1 = x \to y$. □

Proposition 5.3. *Let $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ be a bounded modular lattice with antitone involution. Then, defining $\to$ and $\odot$ as in (5.2) resp. (5.1), one has for any $x, y \in A$:*

(1) $x \leq y$ if and only if $x \to y = 1$.

(2) $x \leq y'$ if and only if $x \odot y = 0$.

Proof. As regards (1), suppose by way of contradiction that $x \to y = 1$ but $x \not\leq y$. Then $x' \vee (y \wedge x) = 1$. Since $x' \leq y' \vee x'$, by Proposition **??** and (**??**) we have $x' = x' \vee y'$, that is $y' \leq x'$, i.e. $x \leq y$, a contradiction.

(2). If $x \odot y = 0$, then $(x \odot y)' = 1$, i.e. by Lemma 5.2 $y \to x' = 1$ and (by (1)) $y \leq x'$. We conclude that $x \leq y'$, since $'$ is an antitone involution. □

It was shown in [26] that every bounded complemented modular lattice $\mathbf{A}$ can be organized into a left-residuated ℓ-groupoid $(A, \wedge, \vee, \odot, \to, 0, 1)$ by putting $x \odot y = (x \vee y') \wedge y$ and $x \to y = x' \vee (x \wedge y)$ without any regard if the complementation $'$ is or is not an antitone involution. On the contrary, we can show that if $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ is a modular lattice with an antitone involution which is not a complementation, then the previous prescription for $\odot$ and $\to$ does not convert $\mathbf{A}$ into a left-residuated ℓ-groupoid. In fact, consider the modular lattice M_3 with an antitone involution defined as visualized in Fig. 5.2.

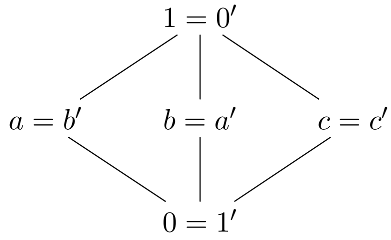


FIGURE 5.2

Taking $x \odot y = (x \vee y') \wedge y$ and $x \to y = x' \vee (x \wedge y)$, one has that $a \odot c = (a \vee c') \wedge c = c$ but $a \not\leq c \to c = c \vee c' = c$. However, if we ask that the modular lattice in question satisfies one more condition which is, as it will be clear later, a strengthened form of regularity, we are able to define the operations $\odot$ and $\to$ in such a way that the resulting algebra is a left-residuated ℓ-groupoid. In particular, we will be concerned with modular involution lattices satisfying the following condition:

$$x \not\leq y' \quad \text{implies} \quad (x \wedge y) \vee (x \wedge x') = (x \wedge y) \vee (y \wedge y'). \tag{*}$$

Observe that, given a modular involution lattice $\mathbf{A}$ satisfying (*), then the converse of this condition holds provided that, given $x, y \in A$, $x \vee x' \neq y \vee y'$.

Lemma 5.4. *Let* $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ *be a modular involution lattice and* $x, y \in A$*. If* $x \vee x' \neq y \vee y'$*, then* $(x \wedge y) \vee (x \wedge x') = (x \wedge y) \vee (y \wedge y')$ *implies* $x \not\leq y'$*.*

Proof. Suppose that $(x \wedge y) \vee (x \wedge x') = (x \wedge y) \vee (y \wedge y')$ and $x \leq y'$. We compute

$$\begin{aligned}(x \wedge y) \vee (x \wedge x') =& ((x \wedge x') \vee y) \wedge x \\ =& ((x \vee y) \wedge x') \wedge x = x \wedge x'.\end{aligned}$$

Similarly, we can show that $(x \wedge y) \vee (y \wedge y') = y \wedge y'$. Hence $x \vee x' = y \vee y'$, a contradiction. □

The next lemma will be expedient for proving that modular involution lattices satisfying (*) can be endowed with a left-residuated product.

Lemma 5.5. *Let* $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ *be a bounded modular lattice with antitone involution satisfying* (*)*. Then,* $x \not\leq y$ *and* $(x \vee y) \wedge (x \vee x') \neq (x \vee y) \wedge (y \vee y')$ *imply* $y \leq x$*.*

Proof. Suppose that $x \not\leq y$, $y \not\leq x$, and $(x \vee y) \wedge (x \vee x') \neq (x \vee y) \wedge (y \vee y')$. If $y \leq x'$, then $x \vee y \neq x \vee y$, a contradiction. Thus $y \not\leq x'$. If $x' \not\leq y$, then, by (*), $(x' \wedge y') \vee (x \wedge x') = (x' \wedge y') \vee (y \wedge y')$, i.e. $(x \vee y) \wedge (x \vee x') = (x \vee y) \wedge (y \vee y')$. Contradiction. It follows that $x' \leq y$. By $y \not\leq x$, one has that $(y \wedge x') \vee (x \wedge x') = (y \wedge x') \vee (y \wedge y')$. Hence, $x' \geq (y \wedge y')$ and $x \wedge x' \geq (y \wedge y')$. Moreover, by $x' \leq y$, one has $x \wedge x' \leq y$ and $x \not\leq y$ (by (*)) implies that $(x \wedge y') \vee (x \wedge x') = (x \wedge y') \vee (y \wedge y')$, i.e. $y' \vee (x \wedge x') = y'$. Thus, $x \wedge x' \leq y'$ and $x \wedge x' \leq y \wedge y'$. $x \wedge x' \geq y \wedge y'$ and $y \wedge y' \geq x \wedge x'$ jointly imply $x \wedge x' = y \wedge y'$. Contradiction. Hence, $y \leq x$. $\square$

Corollary 5.6. *Let* $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ *be a bounded modular lattice with antitone involution satisfying* (*). *Then,* $x \not\leq y$ *and* $(x \wedge y) \vee (y \wedge y') \neq (x \wedge y) \vee (x \wedge x')$ *imply* $y \leq x$.

Proof. By our hypotheses it follows that $y' \not\leq x'$ and $(x' \vee y') \wedge (y \vee y') \neq (x' \vee y') \wedge (x \vee x')$. By Lemma 5.5, we have $x' \leq y'$, namely $y \leq x$. $\square$

Theorem 5.7. *Let* $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ *be a bounded modular lattice with antitone involution satisfying* (*). *Put* $x \odot y$ *and* $x \to y$ *as in (5.1) resp. (5.2). Then,* $R(\mathbf{A}) = (A, \wedge, \vee, \odot, \to, 0, 1)$ *is a left-residuated* ℓ*-groupoid.*

Proof. By the assumption, $(A, \wedge, \vee, 0, 1)$ is a bounded lattice. Now, if $x \neq 0$, then $x \not\leq 1'$ and $x \odot 1 = 1 \wedge (x \vee 1') = x = x \wedge (1 \vee x')$, since $1 \not\leq x'$. If $x = 0$, then $0 \odot 1 = 0 = 1 \odot 0$, by definition.

Let us prove left-adjointness. Suppose that $x \leq y \to z$. If $x \leq y'$, then clearly $x \odot y = 0 \leq z$. If $x \not\leq y'$ and $y \not\leq z$ then $x \odot y = y \wedge (x \vee y') \leq y \wedge ((y \to z) \vee y') = y \wedge ((y' \vee (y \wedge z)) \vee y') = (y \wedge z) \vee (y \wedge y')$, by modularity. If $y \not\leq z'$,then $(y \wedge z) \vee (y \wedge y') = (y \wedge z) \vee (z \wedge z') \leq z$, by (*) and we conclude $x \odot y \leq z$. Now, let us suppose that $y \leq z'$. We show that $(y \wedge z) \vee (y \wedge y') = (y \wedge z) \vee (z \wedge z')$ holds as well. In fact, assume by way of contradiction that $(y \wedge z) \vee (y \wedge y') \neq (y \wedge z) \vee (z \wedge z')$. By $y \not\leq z$ and Corollary 5.6, one has $z \leq y$. However, $x \leq y \to z$ implies $x \leq y' \vee (y \wedge z) = y' \vee z = y'$, against our assumption $x \not\leq y'$. We conclude that $(y \wedge z) \vee (y \wedge y') = (y \wedge z) \vee (z \wedge z')$ and, reasoning as above, we have that $x \odot y \leq z$. Finally, if $y \leq z$, then $x \odot y \leq y \wedge ((y \to z) \vee y') = y \wedge (1 \vee y') = y \leq z$. Thus, in each case, $x \leq y \to z$ implies $x \odot y \leq z$.

Now, assume that $x \odot y \leq z$, $x \not\leq y'$ and $y \not\leq z$. If $(x \vee y') \wedge (y \vee y') = (x \vee y') \wedge (x \vee x')$

we compute

$$
\begin{aligned}
y \to z =& y' \vee (y \wedge z) \\
\geq& y' \vee (y \wedge (x \odot y)) \\
=& y' \vee (y \wedge (y \wedge (x \vee y'))) \\
\geq& (x \vee y') \wedge (y \vee y') \\
=& (x \vee y') \wedge (x \vee x') \geq x,
\end{aligned}
$$

by modularity. Otherwise, suppose that $(x \vee y') \wedge (y \vee y') \neq (x \vee y') \wedge (x \vee x')$. Thus, one has $(x \vee x') \neq (y \vee y')$. By Lemma 5.5, it follows that $y' \leq x$.
If $x \not\leq y$, then if $(x \vee y) \wedge (x \vee x') = (x \vee y) \wedge (y \vee y')$, by $x' \leq y$ and $y' \leq x$, it follows that $(x \vee x') = (y \vee y')$, a contradiction. Moreover, if $(x \vee y) \wedge (x \vee x') \neq (x \vee y) \wedge (y \vee y')$ then, by Lemma 5.5, $y \leq x$. Hence, $x \odot y = y \wedge (x \vee y') = y \leq z$, again a contradiction. Therefore, we conclude that $x \not\leq y$ is impossible.
If $x \leq y$, then $x \odot y = y \wedge (x \vee y') = x \wedge y = x \leq z$. Hence, $x \leq y' \vee (y \wedge z) = y \to z$. Thus, if $x \not\leq y'$ and $y \not\leq z$, in each possible case one has that $x \odot y \leq z$ implies $x \leq y \to z$.

If $x \leq y'$, i.e. $x \odot y = 0$, and $y \not\leq z$, then $y \to z = y' \vee (z \wedge y) \geq y' \geq x$. Finally, if $y \leq z$, then $y \to z = 1$ and obviously $x \leq y \to z$. Hence, in each case, it follows that $x \odot y \leq z$ if and only if $x \leq y \to z$. □

Note that the converse of Theorem 5.7 does not hold in general. In fact, consider the modular (non-distributive) involution lattice $\mathbf{A} = (\{x, x', z, 0, 1\}, \wedge, \vee, ', 0, 1)$ displayed in Fig. 5.3.

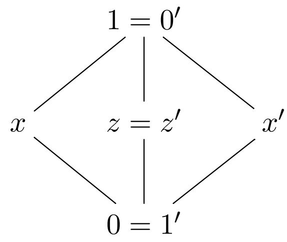


FIGURE 5.3

$\odot$	0	x	z	x'	1
0	0	0	0	0	0
x	0	x	z	0	x
z	0	x	0	x'	z
x'	0	0	z	x'	x'
1	0	x	z	x'	1

$\to$	0	x	z	x'	1
0	1	1	1	1	1
x	x'	1	x'	x'	1
z	z	z	1	z	1
x'	x	x	x	1	1
1	0	x	z	x'	1

An easy check shows that the above involution lattice does not satisfy $(^*)$. Nevertheless, it can be organized into a left-residuated ℓ-groupoid.

Example 5.2. *Consider the Kleene lattice depicted in Figure 5.1. A routine check shows that it is a bounded modular lattice satisfying $(^*)$. We can define $\odot$ and $\to$ as in (5.1) resp. (5.2) given by the following tables of operations:*

$\odot$	0	a	b	a'	b'	1
0	0	0	0	0	0	0
a	0	0	0	0	a	a
b	0	0	0	0	0	b
a'	0	0	0	a'	a'	a'
b'	0	a	0	a'	b'	b'
1	0	a	b	a'	b'	1

$\to$	0	a	b	a'	b'	1
0	1	1	1	1	1	1
a	a'	1	a'	a'	1	1
b	b'	1	1	1	1	1
a'	a	a	a	1	1	1
b'	b	a	b	a'	1	1
1	0	a	b	a'	b'	1

It is worth noticing that, as a left-residuated ℓ-groupoid, it is commutative and associative. Hence, it is a bounded integral commutative residuated lattice which is distributive.

It is reasonable to ask if condition $(^*)$ of Theorem 5.7 can be dropped. Unfortunately, the answer is negative. In fact, the next example shows the existence of a modular involution lattice which is even a pseudo-Kleene lattice but it does not satisfy $(^*)$. It will be clear that $\odot$ and $\to$ cannot be defined as above.

Example 5.3. *Consider the bounded involution lattice $\mathbf{A} = (\{a, b, c, a', b', c', 0, 1\}, \wedge, \vee, ', 0, 1)$ depicted in Figure 5.4. A straightforward verification shows that it is modular and regular.*

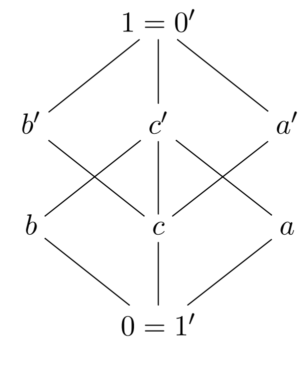


FIGURE 5.4

$\mathbf{A}$ *does not satisfy $(^*)$, since $c' \not\leq b'$ but $(c' \wedge c) \vee (c' \wedge b) = c \vee b = c' \neq b = b \vee 0 = (b' \wedge b) \vee (b \wedge c')$. Note that $\mathbf{A}$ cannot be turned into a left residuated groupoid as in Theorem 5.7 or 5.17, since e.g. $b' \odot c' = c' \wedge (b' \vee c) = c$ but $b' \not\leq c = c \vee (c \wedge c') = c' \to c$.*

5.2 Properties of modular lattices with antitone involution

In this section we investigate basic properties of bounded modular lattices with antitone involution satisfying (*). It will turn out that the above condition can be expressed in a simpler form. We start with the following

Lemma 5.8. *Let* $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ *be a modular involution lattice satisfying* (*)*. Then, for any* $x, y \in A$*,* $x \wedge x' \leq y$ *or* $y \wedge y' \leq x$*.*

Proof. If (*) hold, then $\mathbf{A}$ can be turned into a left-residuated ℓ-groupoid by Theorem 5.7. If $x \leq y$, then obviously $x \wedge x' \leq y$. Otherwise, suppose that $x \not\leq y$. Hence, $x \to y = x' \vee (x \wedge y)$. If $x \wedge y \not\leq x'$, then by $(x \to y) \odot x \leq y$, one has that $x \wedge ((x' \vee (x \wedge y)) \vee x') = x \wedge (x' \vee (x \wedge y)) = (x \wedge y) \vee (x \wedge x') \leq y$. Thus, $(x \wedge x') \leq y$. If $x \wedge y \leq x'$, then $x \leq x' \vee y'$. By $y' \not\leq x'$, if $(x' \vee y') \wedge (x \vee x') = (x' \vee y') \wedge (y \vee y')$, then $(x \vee x') = (x' \vee y') \wedge (y \vee y')$ and $(x \vee x') \leq (y \vee y')$. A moment reflection shows that $y \wedge y' \leq x$. If $(x' \vee y') \wedge (x \vee x') \neq (x' \vee y') \wedge (y \vee y')$, then, by Lemma 5.5, one has $x' \leq y'$ and $y \leq x$. Hence, $y \wedge y' \leq x$. □

Now, we are going to formulate simpler conditions equivalent to (*).

Theorem 5.9. *Let* $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ *be a modular involution lattice. The following are equivalent:*

(1) $\mathbf{A}$ *satisfies* (*)*;*

(2) For any $x, y \in A$ *it holds that* $x \not\leq y'$ *implies* $x \wedge x' \leq y$*;*

(3) For any $x, y \in A$*, if* $x || y$*, then* $x \wedge x' = y \wedge y'$*.*

Proof. (2) ⇒ (1). Suppose that $x \not\leq y'$. Then $x \wedge x' \leq y$ (by (2)) and $x \wedge x' \leq x \wedge y$. Similarly, by $y \not\leq x'$ it follows that $y \wedge y' \leq x \wedge y$ and $(x \wedge y) \vee (x \wedge x') = x \wedge y = (x \wedge y) \vee (y \wedge y')$.

(1) ⇒ (2). Suppose that (*) holds and, by way of contradiction, that (2) is not satisfied. Then, there exist $x, y \in A$ such that $x \not\leq y'$ but $x \wedge x' \not\leq y$. By Lemma 5.8, $y \wedge y' \leq x$. By (*), one has that $(x \wedge y) \vee (x \wedge x') = (x \wedge y) \vee (y \wedge y')$, i.e. $(x \wedge y) = (x \wedge y) \vee (x \wedge x')$ and $(x \wedge x') \leq y$, a contradiction.

(2) ⇒ (3). By $x \not\leq y$ it follows that $x \wedge x' \leq y'$. Moreover, by $y \not\leq x$, i.e. $x' \not\leq y'$, one has that $x \wedge x' \leq y$. So, we obtain that $x \wedge x' \leq y \wedge y'$. By a symmetrical argument, one concludes that $y \wedge y' \leq x \wedge x'$. Hence, $y \wedge y' = x \wedge x'$.

(3) $\Rightarrow$ (2). Assume that $x \not\leq y'$ but $x \wedge x' \not\leq y$. If $y' \leq x$, then $x \wedge x' \leq y$, a contradiction. Hence we have that $y' \not\leq x$. By hypothesis it follows that $x \wedge x' = y \wedge y'$, again a contradiction. Thus, one has that $x \wedge x' \leq y$. ☐

It is easily seen that any bounded lattice satisfying (*) is regular but not the other way around (see Fig. 5.3). Therefore, (*) can be regarded as a strengthened form of regularity. It is worth noticing that, in any modular involution lattice $\mathbf{A}$ satisfying (*), $\{x \wedge x' : x \in A\} \cup \{x \vee x' : x \in A\}$ forms a linearly ordered sub-algebra of $\mathbf{A}$.

The next proposition shows that if a bounded modular involution lattice satisfying (*) has a fixed point with respect to $'$, then it must be a chain.

Proposition 5.10. *Let* $\mathbf{A} = (A, \wedge, \vee,', 0, 1)$ *be a bounded modular lattice satisfying* (*). *If there exists* $a \in A$ *such that* $a = a'$ *(*a *is a fixed point), then* $\mathbf{A}$ *is linearly ordered.*

Proof. Let $x, y \in A$ be such that $x \not\leq y$. By (*) one has $x \wedge x' \leq y'$. Let we consider the following cases:

1. $a \leq x$. Therefore, $x' \leq a = a' \leq x$ (by antitonicity) and $x' \leq y'$, i.e. $y \leq x$;
2. $a \not\leq x$. We have $a \leq x'$ (by (*)). Since $x \not\leq y$, one has $y' \not\leq a$, i.e. $a \not\leq y$ and $a \leq y'$, again by (*). Therefore $y \leq y'$. By $y' \not\leq x'$, (*) ensures that $y = y \wedge y' \leq x$.

☐

Proposition 5.11. *Let* $\mathbf{A}$ *be a bounded modular lattice with antitone involution satisfying* (*). *Then, for any* $x, y \in A$, $x \wedge x' \leq y \wedge y'$ *or* $y \wedge y' \leq x \wedge x'$.

Proof. If $x \wedge x'$ and $y \wedge y'$ are incomparable, then by Theorem 5.9 (3) we have that $(x\wedge x') = (x\wedge x')\wedge(x\vee x') = (x\wedge x')\wedge(x\wedge x')' = (y\wedge y')\wedge(y\wedge y')' = (y\wedge y')\wedge(y\vee y') = (y\wedge y')$, a contradiction. ☐

In what follows we will refer to any of the equivalent conditions stated in Theorem 5.9 by (*) as well.

Remark 5.12. The class of bounded (even distributive) modular lattices with an antitone involution satisfying (*) is not an equational class. In fact, let we consider the three-elements Kleene lattice $\mathbf{D}_3$ in Fig. 5.5. Clearly, $\mathbf{D}_3$ satisfies (*). However, in $\mathbf{D}_3^3$ one has $(x, x, x) \not\leq (0, 1, 1)$ but $(x, x, x) \wedge (x, x, x)' = (x, x, x) \not\leq (1, 0, 0) = (0, 1, 1)'$. Thus, bounded modular lattices with antitone involution satisying (*) do *not* form a variety. However, it is worth noticing that, since (*) can be expressed by a positive universal formula, it is preserved by subalgebras and quotients.

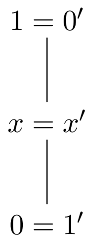


FIGURE 5.5

Note that the bounded modular lattice with an antitone involution described in Example 5.2 satisfies the following further condition, for any $x, y \in A$ such that $x \neq 1$:

$$x \not\leq y \quad \text{implies} \quad (x \vee y) \wedge (x \vee x') = (x \vee y) \wedge (y \vee y'). \tag{**}$$

The next proposition represents an analogue of Theorem 5.9 for (**).

Proposition 5.13. *Let* $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ *be a modular involution lattice satisfying* (*). *The following are equivalent:*

(1) $\mathbf{A}$ *satisfies* (**)*;*

(2) For any $x, y \in A$*, if* $x \not\leq y$ *and* $x \neq 1$*, then* $x \leq y \vee y'$*.*

Proof. (2) $\Rightarrow$ (1). Let us suppose that $x \neq 1$ and $x \not\leq y$. By (*) one has that $x \wedge x' \leq y'$, i.e. $y \leq x \vee x'$ and $x \vee y \leq x \vee x'$. Moreover, by (2), one has that $x \leq y \vee y'$ and $x \vee y \leq y \vee y'$. Hence, $y \vee y' = x \vee x'$ and $(x \vee y) \wedge (x \vee x') = (x \vee y) \wedge (y \vee y')$.

(1) $\Rightarrow$ (2). Suppose that there exist $x, y \in A$ such that $x \neq 1$, $x \not\leq y$ and $x \not\leq y \vee y'$. By (*) one has that $x \wedge x' \leq y'$ and $y \leq x \vee x'$. Hence $x \vee y \leq x \vee x'$. Moreover, by (**) one has that $(x \vee y) \wedge (x \vee x') = (x \vee y) = (x \vee y) \wedge (y \vee y')$, i.e. $x \leq y \vee y'$, a contradiction. □

Actually, we can show that bounded lattices with antitone involution satisfying (*) and (**) have a very specific lattice structure.

Proposition 5.14. *Let* $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ *be a modular involution lattice satisfying* (*). *Then,* $\mathbf{A}$ *satisfies* (**) *if and only if, for any* $x, y \in A - \{0, 1\}$*,* $x \vee x' = y \vee y'$*. Consequently,* (**) *holds in* $\mathbf{A}$ *if and only if* $\mathbf{A}$ *is either an ortholattice or there exists an atom* $a \in A$ *such that for any* $x \in A - \{0, 1\}$*,* $x \wedge x' = a$*.*

Proof. Suppose that (**) holds and let x, y be elements in$A - \{0, 1\}$. If $x \leq y$, then obviously $x \leq y \vee y'$ and $x \not\leq y$ implies (by (**)) $x \leq y \vee y'$ as well. By repeating the same argument for x', one has that $x \vee x' \leq y \vee y'$. Symmetrically, it follows that

$y \vee y' \leq x \vee x'$. Hence, $y \vee y' = x \vee x'$. If there exists $x \in A - \{0,1\}$ such that $x \wedge x' = 0$, then $\mathbf{A}$ is an ortholattice. Otherwise, suppose that for any $x \in A - \{0,1\}$, $x \wedge x' \neq 0$. Put $x \wedge x' = a$. Since, for any $x \neq 0$ one has that $0 \neq a \leq x$, it follows that a is an atom. The converse of the statement is trivial. □

One might ask if it is possible to prove an analogue of Theorem 5.7 assuming (**) instead of (*). Indeed, the next lemma shows this is not the case.

Lemma 5.15. *Let* $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ *be a modular involution lattice satisfying* (**). *Then,* $R(\mathbf{A}) = (A, \wedge, \vee, \odot, \to, 0, 1)$, *where* $\odot$ *and* $\to$ *are defined as in (5.1) resp. (5.2), is a left-residuated* ℓ*-groupoid if and only if* (*) *holds.*

Proof. ($\Leftarrow$) follows directly from Theorem 5.7. As regards ($\Rightarrow$), let us assume that (*) does not hold. Hence, there exist $x, y \in A$ such that $x \not\leq y'$ but $(x \wedge y) \vee (x \wedge x') \neq (x \wedge y) \vee (y \wedge y')$. If $x \leq y$, then $x \neq x \vee (y \wedge y')$, i.e. $y \wedge y' \not\leq x$. Moreover, by modularity, $x \odot y = y \wedge (x \vee y') = x \vee (y \wedge y')$. Note that $x \leq y' \vee x = y' \vee (x \wedge y) = y \to x$, since $y \leq x$ would imply $x = y$. Hence, $x \odot y = x \vee (y \wedge y') \leq x$, by left residuation. So, $y \wedge y' \leq x$, a contradiction. We conclude that $x \not\leq y$ and $y' \not\leq x'$. Clearly, $y' \neq 1$ and, by (**), we have that $(x' \vee y') \wedge (x \vee x') = (x' \vee y') \wedge (y \vee y')$. Since $'$ is an antitone involution, we conclude that $(x \wedge y) \vee (x \wedge x') = (x \wedge y) \vee (y \wedge y')$, a contradiction. □

The next example reveals the existence of bounded modular lattices with antitone involution satisfying (*) but not (**) which can be organized into a left-residuated ℓ-groupoid.

Example 5.4. *Consider the involution lattice*

$$\mathbf{A} = (\{0, x, y, x', y', 1\}, \wedge, \vee, ', 0, 1)$$

depicted in Fig. 5.6. If we define $\odot$ *and* $\to$ *as in Theorem 5.7, then we have the following tables:*

$\odot$	0	x	y	x'	y'	1
0	0	0	0	0	0	0
x	0	x	y	y'	0	x
y	0	y	y	0	0	y
y'	0	y'	0	0	0	y'
x'	0	0	0	0	0	x'
1	0	x	y	y'	x'	1

$\to$	0	x	y	x'	y'	1
0	1	1	1	1	1	1
x	x'	1	y	y'	x'	1
y	y'	1	1	y'	y'	1
y'	y	y	1	1	y	1
x'	x	1	1	1	1	1
1	0	x	y	y'	x'	1

Note that $\mathbf{A}$ *can be regarded as a left-residuated* $\ell -$*groupoid. Moreover, a little thought shows that* $\mathbf{A}$ *satisfies* (*) *but* (**) *does not hold. In fact, one has that* $x \not\leq y$ *but* $(x \vee y) \wedge (x \vee x') = x \neq y = (x \vee y) \wedge (y \vee y')$.

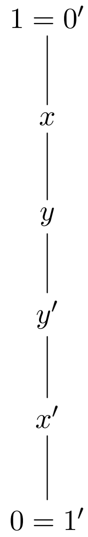


FIGURE 5.6

5.3 Distributive lattices with antitone involution

It is worth asking if, given a distributive involution lattice $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$, putting $x \odot y = x \wedge y$ and $x \to y = x' \vee y$, it is possible to convert $\mathbf{A}$ into a bounded integral commutative residuated lattice as for Boolean algebras. Unfortunately, this is not the case. In fact, it is well known that in distributive involution lattices (see e.g. [45]) the pair $(\wedge, \to)$ is not necessarily residuated, as shown in the next example. It is worth asking if, given a bounded distributive lattice with antitone involution $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$, putting $x \odot y = x \wedge y$ and $x \to y = x' \vee y$, it is possible to convert $\mathbf{A}$ into a bounded integral commutative residuated lattice as for Boolean algebras. Unfortunately, this is not the case. In fact, it is well known that in a Kleene lattice $\mathbf{A}$ the pair $(x \wedge y, y \to x)$, where $x \to y = x' \vee y$, for any $x, y \in A$, need not be residuated (see e.g. [45]), as shown in the next example.

Example 5.5. *Consider the Kleene lattice* $\mathbf{A}$ *depicted in Fig. 5.7. If we define* $x \to y = x' \vee y$ *and* $x \odot y = x \wedge y$*, for any* $x, y \in A$*, one has* $b \wedge c = a \leq c$ *but* $b \not\leq c \to c = c' \vee c = c$.

However, the next result shows that, for distributive lattices with an antitone involution satisying (*), the definition of $\odot$ and $\to$ as in (5.1) resp. (5.2) can be semplified.

Lemma 5.16. *Let* $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ *be a distributive involution lattice satisfying* (*)*. Then, the following hold:*

(1) $x \not\leq y'$ *implies* $y \wedge (x \vee y') = x \wedge y$*;*

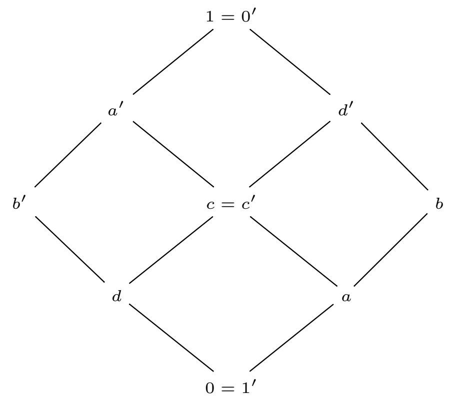


FIGURE 5.7

(2) $x \not\leq y$ implies $x' \vee (y \wedge x) = x' \vee y$.

Proof. As regards (1), assume $x \not\leq y'$. Hence, $y \wedge (x \vee y') = (y \wedge x) \vee (y \wedge y') = (y \wedge x) \vee (x \wedge x') = y \wedge x$, since (*) implies that $x \wedge x' \leq y$.

(2). Assume that $x \not\leq y$. Thus, we have $x' \vee (y \wedge x) = (x' \vee y) \wedge (x \vee x') = x' \vee y$, since (*) implies that $x \wedge x' \leq y'$, i.e. $y \leq x \vee x'$. □

We can now show that, for a distributive involution lattice **A**, (*) is not only a sufficient but also a necessary condition for **A** to be turned into a bounded integral commutative residuated lattice. This yields a complete characterization of those Kleene lattices which can be equipped with a material Boolean-like implication.

Theorem 5.17. *Let $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ be a bounded distributive lattice with antitone involution satisfying (*). Define $x \odot y = x \wedge y$ if $x \not\leq y'$, and $x \odot y = 0$ otherwise. Furthermore, put $x \to y = x' \vee y$, if $x \not\leq y$ and $x \to y = 1$, otherwise. Then, $R(\mathbf{A}) = (A, \wedge, \vee, \odot, \to, 0, 1)$ is a bounded integral commutative residuated lattice if and only if (*) holds.*

Proof. ($\Leftarrow$). In view of Lemma 5.16, since any distributive lattice is modular and the definitions of $\odot$ and $\to$ coincide with those in the statement of Theorem 5.7, we conclude that $R(\mathbf{A}) = (A, \wedge, \vee, \odot, \to, 0, 1)$ is a left-residuated ℓ-groupoid. Moreover, $\odot$ is commutative. Hence, $R(\mathbf{A})$ is indeed residuated. We only need to prove the associativity of $\odot$. Let us distinguish several cases:

(a) If $x \odot y = x \wedge y$, $(x \wedge y) \odot z = (x \wedge y) \wedge z$, $y \odot z = y \wedge z$, and $x \odot (y \wedge z) = x \wedge (y \wedge z)$, then clearly $(x \odot y) \odot z = x \odot (y \odot z)$.

(b) $x \odot y = 0$, i.e. $x \leq y'$. Then, $(x \odot y) \odot z = 0$. If $y \leq z'$, then $(x \odot y) \odot z = 0 = x \odot (y \odot z)$. Otherwise, let us note that $x \leq y' \vee z' = (y \wedge z)'$ and $x \odot (y \odot z) = 0$.

(c) If $y \odot z = 0$, i.e. $y \leq z'$, then $x \odot (y \odot z) = 0$ and by $x \wedge y \leq z'$, one obtains in each case the desired result.

Therefore, we can assume that $x \odot y \neq 0$ and $y \odot z \neq 0$, i.e. $x \nleq y'$ and $y \nleq z'$, respectively.

(d) If $x \wedge y \leq z'$, then, if $(x \vee y') \wedge (y \vee y') = (x \vee y') \wedge (x \vee x')$, we have

$$\begin{aligned} z' \vee y' \geq & (x \wedge y) \vee y' \\ = & (x \vee y') \wedge (y \vee y') \\ = & (x \vee y') \wedge (x \vee x') \\ = & x \vee (y' \wedge x') \geq x, \end{aligned}$$

by distributivity, so associativity easily follows. Otherwise, suppose $(x \vee y') \wedge (y \vee y') \neq (x \vee y') \wedge (x \vee x')$. By Lemma 5.5, one has $y' \leq x$. Now, if $x \leq y$, then $x = x \wedge y \leq z' \leq z' \vee y'$. Hence, $x \odot (y \odot z) = 0 = (x \odot y) \odot z$. Suppose $x \nleq y$. If $y \nleq x$, then by (*) and Theorem 5.9 we have that $x \wedge x' = y \wedge y'$ and one obtains $(x \vee y') \wedge (y \vee y') = (x \vee y') \wedge (x \vee x')$, a contradiction. Hence, we must have that $y \leq x$. So, by $y = x \wedge y \leq z'$, it follows that $x \odot (y \odot z) = x \odot 0 = 0 = (x \odot y) \odot z$.

(e) Finally, if $x \leq y' \vee z'$, then $x \wedge y \leq y \wedge (y' \vee z') = (y \wedge y') \vee (y \wedge z') = (z \wedge z') \vee (y \wedge z') \leq z'$, by (*), and again $(x \odot y) \odot z = (x \odot y) \odot z$.

($\Rightarrow$). Assume that $R(\mathbf{A}) = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a bounded commutative residuated lattice with operations defined as above, and suppose by way of contradiction that $x \nleq y'$ but $x \wedge x' \nleq y$. One has that $(x \rightarrow (y' \odot x))' \leq y$. If $y' \leq x'$ then $x \leq y$ and $x \wedge x' \leq y$, a contradiction. Hence we have that $x \rightarrow (y' \odot x) = x \rightarrow (y' \wedge x)$. Since $x \leq y' \wedge x$ contradicts our assumptions, we have that $(x \rightarrow (y' \wedge x))' = (x' \vee (y' \wedge x))' = (x \wedge x') \vee (x \wedge y) \leq y$, by distributivity and the De Morgan law. Thus, $x \wedge x' \leq y$, again a contradiction. □

5.4 Modular involution lattices as ι-near semirings

In light of the above arguments, we can now prove that any modular involution lattice satisfying (*) can be turned into a ι-near semiring (see Definition 3.2).

Proposition 5.18. *Let* $\mathbf{A}$ *be a left-residuated* ℓ*-groupoid. Then, if* $\{y\} \cup X \subseteq A$ *and* $\bigvee X$ *exists, one has*

$$(\bigvee X) \odot y = \bigvee_{x \in X} (x \odot y).$$

Proof. Let x, y, z be elements in $\mathbf{A}$. Assume that $x \leq y$. Observe that $y \odot z \leq y \odot z$ implies $x \leq y \leq z \to (y \odot z)$. Thus we have $x \odot z \leq y \odot z$. We conclude that $\odot$ is left-monotonic. Now, clearly $x \odot y \leq (\bigvee X) \odot y$. Suppose that $x \odot y \leq z$, for any $x \in X$. thus $\bigvee X \leq y \to z$ and our statement follows by left-adjointness. □

It can be observed that, if $\mathbf{A}$ is a modular involution lattice satisfying (*), then $x \to 0 = x'$ in $\mathcal{R}(\mathbf{A})$.

Proposition 5.19. *Let* $\mathbf{A}$ *be a modular involution lattice satisfying* (*) *and* $\mathcal{R}(\mathbf{A})$ *its associated left-residuated* ℓ*-groupoid. Then, putting for any* $x \in A$, $x^\alpha = x \to 0$, $^\alpha$ *is an antitone involution.*

Finally, we have the following

Theorem 5.20. *Let* $\mathbf{A}$ *be a modular involution lattice satisfying* (*)*. Then* $\mathcal{R}(\mathbf{A})$ *can be turned into a* ι*-near semiring* $\mathcal{S}(\mathbf{A})$*. Furthermore, if* $\mathbf{A}$ *is also distributive, then* $\mathcal{S}(\mathbf{A})$ *is a semiring.*

Proof. Put $x \vee y = x + y$, $x \cdot y = x \odot y$ and $x^\alpha = x \to z$. Then $(A, +, 0)$ is obviously a integral idempotent commutative monoid, $x \cdot 1 = 1 \cdot x = x$ holds by the definition of left-residuated ℓ-groupoid, $(x + z) \cdot y = c \cdot y + z \cdot y$ is ensured by Proposition 5.18, and $^\alpha$ is an antitone involution by Proposition 5.19. Finally, observe that $x \odot 0 = 0$, by $x \leq 0' = 1$ and $0 \odot x = 0$ for $0 \leq x'$. As regards the "furthermore" part, note that if $\mathbf{A}$ is distributive, then $\mathcal{R}(\mathbf{A})$ is a bounded integral commutative residuated lattice. Thus, right distributivity is ensured. □

Finally, we apply Theorem 5.20 in order to prove a Cantor-Bernstein type theorem for modular involution lattices satisfying (*).
Let us observe that modular involution lattices form a Church variety (see Section 4.2) as witnessed by the term

$$(x \wedge y) \vee (x' \wedge z).$$

Therefore, if $\mathbf{A}$ is a modular involution lattice satisfying (*), then $\mathcal{R}(\mathbf{A})$ is a Church algebra as well with the same witnessing term (just exchange x' with $x \to 0$). Clearly, $\mathrm{Ce}(\mathbf{A}) = \mathrm{Ce}(\mathcal{R}(\mathbf{A})) = \mathrm{Ce}(\mathcal{S}(\mathbf{A}))$. Hence, if $\mathrm{Ce}(\mathbf{A})$, as a *Boolean algebra*, is σ-complete, then so do $\mathrm{Ce}(\mathcal{R}(\mathbf{A}))$ and $\mathrm{Ce}(\mathcal{S}(\mathbf{A}))$. Moreover, in light of the above considerations, any modular involution lattice satisfying $(^*)$ is clearly left-residuable. Therefore we have the following

Theorem 5.21. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be join* σ*-complete modular involution lattices satisfying* $(^*)$*. Moreover, let us suppose that* $\mathrm{Ce}(\mathbf{A})$ *and* $\mathrm{Ce}(\mathbf{B})$ *are* σ*-complete Boolean algebras. If* $\mathbf{A} \cong [0, b]$ *and* $\mathbf{B} \cong [0, a]$ *with* $b \in \mathrm{Ce}(\mathbf{B})$ *and* $a \in \mathrm{Ce}(\mathbf{A})$*, then* $\mathbf{A} \cong \mathbf{B}$*.*

Proof. If $\mathbf{A} \cong [0, b]$ under and isomorphism ϕ, and $\mathbf{B} \cong [0, a]$ under ψ, then $\mathcal{R}(\mathbf{A}) \cong [0, b]_{\mathcal{R}(\mathbf{B})}$ and $\mathcal{R}(\mathbf{B}) \cong [0, b]_{\mathcal{R}(\mathbf{A})}$, respectively. In fact, if $a \odot^{\mathcal{R}(\mathbf{A})} b = 0$, then $a \leq b'$. Therefore, since ψ is an involution lattice isomorphism, it follows that $\phi(a) \leq \phi(b)'$ and $\phi(a) \odot^{\mathcal{R}(\mathbf{B})} \phi(b) = 0^{\mathcal{R}(\mathbf{B})} = \phi(a \odot^{\mathcal{R}(\mathbf{A})} b)$. Otherwise, suppose that $a \nleq b'$ then $\phi(a) \nleq \phi(b)'$ and $\phi(a \odot^{\mathcal{R}(\mathbf{A})} b) = \phi(b \wedge (a \vee b')) = \phi(b) \wedge (\phi(a) \vee \phi(b)') = \phi(a) \odot^{\mathcal{R}(\mathbf{B})} \phi(b)$. That $\phi(a \to^{\mathcal{R}(\mathbf{A})} b) = \phi(a) \to^{\mathcal{R}(\mathbf{B})} \phi(b)$ can be proven similarly. So we conclude that $\mathcal{R}(\mathbf{A}) \cong [0, b]_{\mathcal{R}(\mathbf{B})}$. Similarly, we prove that $\mathcal{R}(\mathbf{B}) \cong [0, b]_{\mathcal{R}(\mathbf{A})}$ under ψ. Now, it is easily seen that $S(\mathbf{A}) \cong [0, b]_{\mathcal{S}(\mathbf{B})}$ and $S(\mathbf{B}) \cong [0, a]_{\mathcal{R}(\mathbf{A})}$. Since $\mathcal{S}(\mathbf{A})$ and $\mathcal{S}(\mathbf{B})$ are left-residuable, Theorem 3.29 applies and we conclude that $S(\mathbf{A}) \cong S(\mathbf{B})$. Let $\delta : S(\mathbf{A}) \to S(\mathbf{B})$ be the isomorphism. Note that $\delta(a \vee b) = \delta(a + b) = \delta(a) \vee \delta(b)$ and $\delta(a^{\alpha}) = \delta(a') = \delta(a)^{\alpha} = \delta(a)'$. Thus, since $'$ is an antitone involution, we have also $\delta(a \wedge b) = \delta(a) \wedge \delta(b)$. Therefore δ is an involution lattice isomorphism and we are done. $\square$

Chapter 6

A general framework for orthomodular structures

In the previous chapters, we abstracted from quantum structures and (pseudo-)MV algebras, in order to find a sufficiently general environment in which common properties of their structure can be easily captured. To that aim, we introduced Łukasiewicz near semirings as a semiring-like counterpart of basic algebras.
In what follows, our approach will be different. In fact, we set a general framework in which orthomodular quantum structures (i.e. orthomodular posets, orthomodular lattices and orthoalgebras, see e.g. [97]) can be studied from an order theoretical point of view with the aim of understanding not only their common features but also their distinguishing traits. In fact, we generalize the notion of orthomodularity for posets to the concept of the generalized orthomodularity property (GO-property) by considering the LU-operators (see Chapter 1). This seemingly mild generalization of orthomodular posets yields rather strong applications to orthomodular structures and effect algebras. As a consequence, this approach will yield a completely order-theoretical characterization of the coherence law for several classes of orthoalgebras.

It is well known that orthomodular structures are particular pastings of their Boolean blocks. However, any pasting of Boolean algebras need not be either an orthomodular lattice, or a orthomodular poset. Sufficient and necessary conditions, that specify the block structure and pasting sets of orthomodular posets, were discussed by Greechie and Rogalewicz [87], as well as, in the orthoalgebraic framework, by Navara [99] (see Chapter 2).
In this chapter, we will weaken the notion of orthomodularity for posets by relaxing the condition ensuring that joins of orthogonal pairs exist (see e.g. [32] and section 6.1),

and then analyzing the order theoretical properties it determines. It will turn out that orthomodular posets can be neatly characterized by means of the properties of orthoposets. In fact, it will be proven that any GO-poset **A** is orthomodular provided that for any $a, b \in A$, if $a \leq b$ then there exists $c \perp a$ such that $a \vee c = b$.
Furthermore, we will develop a general theory of configurations for GO-posets, and determine those that characterize GO-posets and orthomodular posets with respect to orthoposets and GO-posets, respectively. Surprisingly enough, these results will shine a new light on Greechie's celebrated Theorems. In fact, we will prove that, for atomic amalgams of Boolean algebras, the GO-property is equivalent to the absence of loops of order three. Moreover, we will also show that one of our forbidden configurations (see display (6.6)) is the smallest orthoposet which does not contain the benzene ring (cf. below) and whose Dedekind-MacNeille completion is not orthomodular.
Finally, we will widen our perspective to orthoalgebraic pastings of (not necessarily finite) Boolean algebras. In this framework, we will characterize the order-theoretical meaning of the coherence law for classes of orthoalgebras: tame and Riesz orthoalgebras. In fact, it will be shown that tame and Riesz orthoalgebras are generalized orthomodular if and only if they are not proper, i.e. they are indeed orthomodular posets.

6.1 The generalized orthomodularity property

Since 1990s, poset versions of well known algebraic structures were introduced with the aim of putting into relationship their order-theoretical and algebraic properties, e.g. the so-called *Boolean orthoposets*. Let us recall that an orthoposet **A** is *Boolean* in the sense of Tkadlec and Klukowski [121, 89] if, for any $a, b \in A$, $a \wedge^{\mathbf{A}} b = 0$ implies that $a \leq b'$. An example of a Boolean poset which is not a lattice is depicted in figure (6.2). These structures are well known for admitting a Dedekind-MacNeille completion (see Chapter 1) which is a Boolean algebra. Moreover, let us also observe that any Boolean orthoposet is indeed distributive in the above sense, as the next lemma shows.

Lemma 6.1. *An orthoposet* **A** *is distributive if and only if for* $a, b \in A$*, whenever* $a \wedge b = 0$*, then* $a \perp b$*.*

Proof. ($\Rightarrow$) If $\mathbf{A}$ is distributive, and $a \wedge b = 0$, then

$$\begin{aligned} U(a') &= U(a', 0) \\ &= U(a', L(a, b)) \\ &= U(L(U(a', a), U(a', b))) \\ &= U(L(U(a', b))) \\ &= U(a', b) \end{aligned}$$

So, if $a' \leq c$, then $b \leq c$. Therefore, $a' \geq b$. ($\Leftarrow$) In [121, Theorem 4.2], it has been shown that any Boolean orthoposet (in Tkadlec' sense) admits a Dedekind-MacNeille completion, which is a complete Boolean algebra. Therefore, the image of the orthoposet $\mathbf{A}$ in its completion is distributive, i.e. it satisfies $U(a, L(b, c)) = U(L(U(a, b), U(a, c)))$, and then $\mathbf{A}$ will be distributive too. $\square$

As orthomodular lattices generalize Boolean algebras, we will weaken the notion of Boolean orthoposet by introducing the concept of GO-poset.

Definition 6.2. A bounded poset $\mathbf{A} = (A, \leq, ', 0, 1)$ equipped with an antitone involution is said to have the *generalized orthomodularity property* (*GO-property*) if it satisfies the following condition for any $a, b \in A$:

$$a \leq b \text{ implies } U(b) = U(a, L(b, a')). \tag{6.1}$$

From now on, posets fulfilling the GO-property will be called GO-posets. Furthermore, condition (6.1) will be referred to as the GO-condition.

Many well known structures arising from sharp quantum theory naturally induce GO-posets. In fact, it can be easily seen that orthomodular posets, and therefore intervals in *weak generalized orthomodular posets* ([95],[104, Definition 1.5.12],[105]) enjoy the GO-property. A straightforward verification shows that:

Remark 6.3. Orthomodular posets and orthomodular lattices enjoy the GO-property.

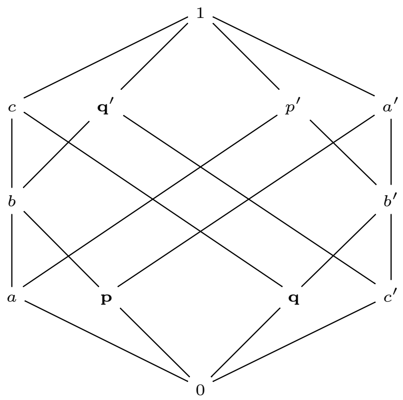
Indeed, we have that $L(y, x') = L(y \wedge x')$, and $U(x \wedge (y \vee x')) = U(x, (y, x'))$, thus Lemma 1.7 yields (6.1) immediately.

However, on the contrary, there are generalized orthomodular posets which are not orthomodular posets, as the following example shows.

Example 6.1. *Consider the poset* $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ *depicted in figure* (6.2). *A routine check shows that* $\mathbf{A}$ *is a GO-poset. However, it is not orthomodular. In fact,*

$p \leq q'$, but $p \vee q$ does not exists in A. Nonetheless, we still have that $U(p, L(q', p')) = U(p, \{0, a, c'\}) = \{q', 1\} = U(q')$, as required by condition (6.1).

(6.2)



The next lemma characterizes those GO-poset that are in point of fact orthomodular.

Lemma 6.4. *Let $\mathbf{A} = (A, \leq, ', 0, 1)$ be a GO-poset. Then $\mathbf{A}$ is an orthomodular poset if and only if for any $a, b \in A$, $a \leq b$ implies that there exists $c \in A$ such that $a \perp c$ and $U(b) = U(a, c)$.*

Proof. The left-to-right direction follows by noticing that $a \leq b$ implies $U(b) = U(a \vee (b \wedge a')) = U(a, b \wedge a')$. Conversely, suppose that $a \leq b'$. By hypothesis, there exists $c \in A$ such that $a \leq c'$ and $U(b') = U(a, c)$. Therefore, by the GO-property it follows that $L(a', b') = L(a') \cap L(b') = L(a') \cap L(U(b')) = L(a') \cap L(U(a, c)) = L(a', U(a, c)) = L(c)$, since $c \leq a'$. Therefore $a' \wedge b'$ exists and so does $a \vee b$. □

The following result shows that a generalized formulation by means of upper and lower sets of the De Morgan laws holds in any poset equipped with an antitone involution.

Lemma 6.5. *Let $\mathbf{A} = (A, \leq, ')$ be a poset equipped with an antitone involution, and $B \subseteq A$. Then,*

$$U(B)' = L(B') \text{ and } L(B)' = U(B').$$

Proof. Let $a \in U(B)'$. Then, $a' \in U(B)$. Hence, for all $b \in B$, $b \leq a'' = a$. So, $a \leq b'$, for any $b' \in B'$. Thus, $a \in L(B')$. Conversely, if $a \in L(B')$, then for all $b' \in B'$, $a \leq b'$. So, for all $b \in B$, $b \leq a'' = a$. Hence, $a \in U(B)'$. The second equality is proven similarly. □

Remark 6.6. Let us observe that any GO-poset is indeed an orthoposet. In fact, suppose that $L(a, a') \neq \{0\}$. Then there is a non-zero $b \in L(a, a')$. So, $b' \in U(a', a)$, and therefore $b \leq b'$. Applying the GO property, $b \leq 1$ implies that $\{1\} = U(1) = U(b, L(b', 1)) = U(b, b') = U(b')$. Consequently, $b' = 1$, and then $b = 0$.

As for orthomodularity, it can be seen that the GO-condition admits a dual definition.

Theorem 6.7. *An orthoposet* $\mathbf{A} = (A, \leq, ', 0, 1)$ *has the GO-property if and only if it satisfies the following, for all* $a, b \in A$*:*

$$\text{if } a \leq b \text{ then } L(a) = L(b, U(a, b')). \tag{6.3}$$

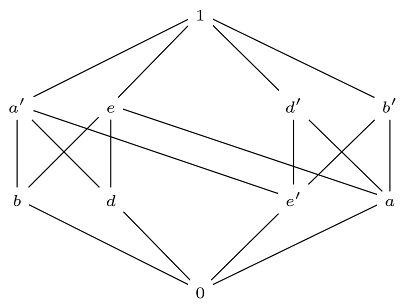
Proof. Suppose $a \leq b$. Then, $b' \leq a'$. Therefore, $U(b)' = L(b') = L(a', U(b', a)) = L(a', L(b, a')') = U(a, L(b, a'))'$. Whence, $U(b)'' = U(b) = U(a, L(b, a')) = U(a, L(b, a'))''$. The converse direction can be shown dually. □

The next lemma shows that distributive and modular orthoposets follow in the framework of GO-posets.

Lemma 6.8. *Every modular orthoposet enjoys the GO-property.*

Proof. Straightforward. □

This observation naturally leads to wonder whether every orthoposet is a GO-poset. The next example answers this question in the negative, showing that the generalized orthomodularity is a non-trivial property of orthoposets.



(6.4)

Example 6.2. *It is immediate to verify that the orthoposet in figure* (6.4) *is an orthoposet, however it does not fulfill the GO-condition. Indeed,* $b \leq a'$*, but* $U(b, L(a', b')) = U(b, e) = \{d', a', 1\} \neq \{a', 1\} = U(a')$*. Moreover, let us remark that this orthoposet fulfills the condition*

$$\text{if } x \leq y \text{ and } U(x, y') = \{1\}, \text{ then } x = y. \tag{6.5}$$

This condition is a poset-version of condition $(P1)$ *of Definition* (2.32)*. If the structure is a lattice, then it is, of course, an orthomodular lattice and the paraorthomodularity condition is equivalent to orthomodularity (cf. Theorem 2.2). However, as this example shows, this is not the case for the orthoposet in figure* (6.4)*.*

6.2 Forbidden configurations of GO-posets

As modular and distributive ordered sets generalize modular and distributive lattices, GO-posets extend the notion of orthomodular lattice. In [32], J. Rachůnek and one of the present author presented a generalized notion of the well known Dedekind-Birkhoff's type "forbidden configurations" for modular and distributive posets. Over the last fifteen years, their results have stirred increasing attention from several scholars. In particular, in [88] the concept of strong elements in posets was introduced and a characterization of strong posets in terms of forbidden configurations was proposed, showing that many classical results in lattice theory can be extended to posets. Those results were further developed in [83], where several known forbidden configurations for lattices and are generalized to posets, and in [118], in which 0-distributive lattices are taken into account.

Inspired by this research stream, in this chapter we present results on forbidden configurations in terms of strong subposets and LU-subposets. The arguments in this section generalize to the framework of GO-posets (that need not be lattice-ordered or orthomodular, in general) the well known fact that an ortholattice is orthomodular if and only if it does ot contain the so-called benzene ring as its sublattice.

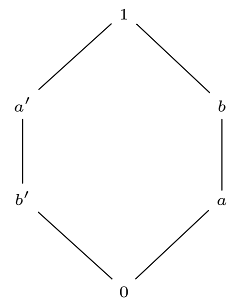
First, let us discuss a rather technical lemma that will be expedient for the development of our discourse.

Lemma 6.9. *Let* $\mathbf{A} = (A, \leq, ', 0, 1)$ *be an orthoposet.*

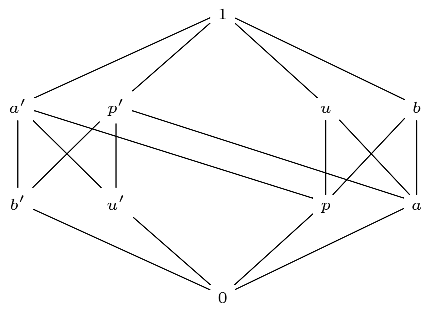
1. *if* $\mathbf{B}_6$ *(figure* (6.6)*) is a subposet of* $\mathbf{A}$*, and* $L(a', b) = \{0\}$ *in* A*, then* $\mathbf{A}$ *does not have the generalized orthomodular property;*
2. *if* $\mathbf{B}_{10}$ *(figure* (6.6)*) is a subposet of* $\mathbf{A}$*, and* $L(a', b) \subseteq L(u)$ *in* A*, then* $\mathbf{A}$ *does not have the generalized orthomodular property;*
3. *if* $\mathbf{B}_{10^*}$ *(figure* (6.7)*) is a subposet of* $\mathbf{A}$*, and* $U(a, b') \subseteq U(u)$ *in* A*, then* $\mathbf{A}$ *does not have the generalized orthomodular property.*

Proof. (1) We have that $a \leq b$, but $U(a, L(b, a')) = U(a) \neq U(b)$. (2) It can be seen that $a \leq b$, and, since $L(a', b) \subseteq L(u)$, then $U(u) \subseteq U(L(a', b))$. So, $U(u) = U(a, u) \subseteq U(a, L(a', b))$. Therefore $u \in U(a, L(a', b))$, but $u \notin U(b)$. (3) Similarly, $a \leq b$, but $U(a, b') \subseteq U(u)$ implies $L(u) \subseteq L(U(a, b'))$, which implies $L(u, b) = L(u) \subseteq L(U(a', b), b)$. Theorefore, $u \in L(U(a', b), b)$, but $u \notin L(a)$. □

(6.6)

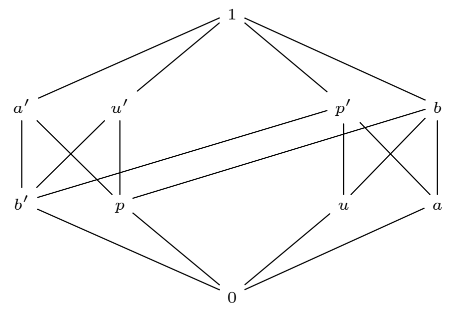


$\mathbf{B}_6$



$\mathbf{B}_{10}$

(6.7)



$\mathbf{B}_{10^*}$

Let us observe that $\mathbf{B}_{10}$ and $\mathbf{B}_{10^*}$ are order ortho-isomorphic. However, whenever it will be expedient for readability, we will consider them separately.

Definition 6.10. Let $\mathbf{A} = (A, \leq, ', 0, 1)$ be an orthoposet. A set $M \subseteq A$ is said to be a *strong subposet* if, for any $a, b \in M$, $U_{\mathbf{A}}(L_{\mathbf{M}}(a, b)) = U_{\mathbf{A}}(L_{\mathbf{A}}(a, b))$ and $L_{\mathbf{A}}(U_{\mathbf{M}}(a, b)) = L_{\mathbf{A}}(U_{\mathbf{A}}(a, b))$.

Combining Lemma 6.9 together with the notion of strong subposet, we can obtain an alternative description of forbidden configurations for posets having the GO-property.

Lemma 6.11. *Let $\mathbf{A} = (A, \leq, ', 0, 1)$ be an orthoposet. If $\mathbf{A}$ contains a strong subposet isomorphic either to $\mathbf{B}_6$, or to $\mathbf{B}_{10}$, or to $\mathbf{B}_{10^*}$, then $\mathbf{A}$ does not have the GO-property.*

Proof. If $\mathbf{A}$ contains a strong subposet isomorphic to $\mathbf{B}_6$, then

$$U_{\mathbf{A}}(L_{\mathbf{A}}(a', b)) = U_{\mathbf{A}}(L_{\mathbf{B}_6}(a', b)) = U_{\mathbf{A}}(0) = A.$$

So $\{0\} = L_{\mathbf{A}}(a', b)$, and then $\mathbf{A}$ does not fulfills the GO-condition by Lemma 6.9-(1). If $\mathbf{A}$ contains a strong subposet isomorphic to $\mathbf{B}_{10}$, then $U_{\mathbf{A}}(L_{\mathbf{A}}(a', b)) = U_{\mathbf{A}}(L_{\mathbf{B}_{10}}(a', b)) =$

$U_{\mathbf{A}}(p,0)$, which implies that $L_{\mathbf{A}}(a',b) = L_{\mathbf{A}}(p) \subset L_{\mathbf{A}}(u)$, and then $\mathbf{A}$ does not have the GO-property by Lemma 6.9-(2). If $\mathbf{A}$ contains a strong subposet isomorphic to $\mathbf{B}_{10^*}$, then $L_{\mathbf{A}}(U_{\mathbf{A}}(a',b)) = L_{\mathbf{A}}(U_{\mathbf{B}_{10^*}}(a',b)) = L_{\mathbf{A}}(p') = A$, which implies that $U_{\mathbf{A}}(a',b) = \{1\} \subset U_{\mathbf{A}}(u)$, and then $\mathbf{A}$ is not a GO-poset by Lemma 6.9-(3). $\square$

Taking up an idea from [32], let us now reintroduce, in the present context, the notion of LU-subset of an orthoposet, which weakens the concept of strong subposet (cf. Lemma 6.13).

Definition 6.12. Let $\mathbf{A} = (A, \leq, ', 0, 1)$ be an orthoposet. A set $M \subseteq A$ is said to be an *LU-subset* if the following conditions are satisfied, for $a, b \in M$:

1. $L_{\mathbf{M}}(a,b) = \{0\}$ if and only if $L_{\mathbf{A}}(a,b) = \{0\}$;
2. if $U_{\mathbf{M}}(a,b) = \{1\}$ if and only if $U_{\mathbf{A}}(a,b) = \{1\}$.

It is not difficult to observe that the previous definition is partially redundant, for the only if part of conditions (1) and (2) in Definition 6.12 are fulfilled by any poset. Let us now show an easy technical lemma that will prove useful for the development of our arguments.

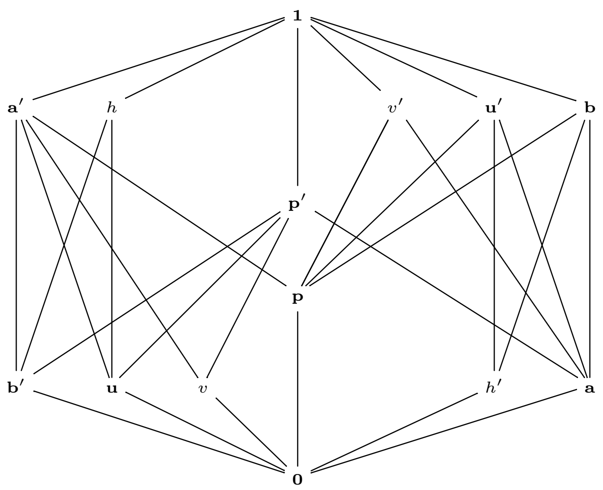
Lemma 6.13. *Let $\mathbf{A} = (A, \leq, ', 0, 1)$ be an orthoposet, and $M \subseteq A$ a strong subposet. Then, M is an LU-subposet.*

Proof. Let $L_{\mathbf{M}}(a,b) = \{0\}$. Then, $L_{\mathbf{A}}(a,b) = L_{\mathbf{A}}(U_{\mathbf{A}}(L_{\mathbf{A}}(a,b))) = L_{\mathbf{A}}(U_{\mathbf{A}}(L_{\mathbf{M}}(a,b))) = L_{\mathbf{A}}(U_{\mathbf{A}}(0)) = L_{\mathbf{A}}(A) = \{0\}$. The second condition is proven similarly. $\square$

Let us remark that from [32, page 411], if M is a strong subset of a poset $\mathbf{A}$, and M is lattice ordered with respect to the same order, then M is a sublattice of $\mathbf{A}$. Moreover, if $M \subseteq A$ is a sublattice of $\mathbf{A}$, it can be readily seen that M is also a strong subset of $\mathbf{A}$. Indeed, if M is a sublattice, it can be seen that $U_{\mathbf{A}}(L_{\mathbf{M}}(a,b)) = U_{\mathbf{A}}(L_{\mathbf{M}}(a \wedge^{\mathbf{M}} b)) = U_{\mathbf{A}}(a \wedge^{\mathbf{A}} b) = U_{\mathbf{A}}(L_{\mathbf{A}}(a \wedge^{\mathbf{A}} b)) = U_{\mathbf{A}}(L_{\mathbf{A}}(a,b))$. A similar argument shows the remaining condition. It is also not complicate to verify by direct inspection that if M is an LU-subset of $\mathbf{A}$ isomorphic to $\mathbf{B}_6$, then it is also a strong subset of $\mathbf{A}$. However, it may happen that an orthoposet $\mathbf{A}$ admits subposets isomorphic to $\mathbf{B}_{10}$ that are not strong, as the orthoposet $\mathbf{B}_{14}$ (figure ($\mathbf{B}_{14}$)). In fact, the subposet M whose universe is $\{1, a, b, u, p, p', a', b', u', 0\}$ is isomorphic to $\mathbf{B}_{10}$, but $h \in L_{\mathbf{A}}(U_{\mathbf{A}}(a', p'))$ and

$h \notin L_{\mathbf{A}}(U_{\mathbf{M}}(a', p'))$. Similar counterexamples can be found for $\mathbf{B}_{10^*}$.

($\mathbf{B}_{14}$)



We are now ready to present the main result of this section, which is a kind of converse of Lemma 6.11 in terms of LU-subposets.

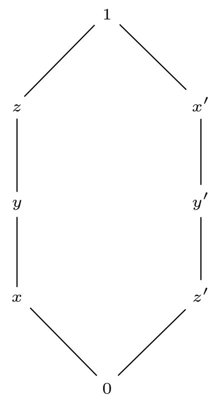
Theorem 6.14. *Let* $\mathbf{A} = (A, \leq,', 0, 1)$ *be an orthoposet. If* $\mathbf{A}$ *is not a GO-poset, then* $\mathbf{A}$ *contains an LU-subposet isomorphic either to* $\mathbf{B}_6$*, or to* $\mathbf{B}_{10}$*, or to* $\mathbf{B}_{10^*}$*.*

Proof. Let $x < y$, and $x, y \notin \{0, 1\}$. It can be seen that $L(x', y) \subseteq L(y)$ implies that $U(L(y)) = U(y) \subseteq U(L(x', y))$, and then $U(x, y) = U(y) \subseteq U(x, L(x', y))$. Let us assume that $U(y) \subset U(x, L(x', y))$. Preliminary, we observe that $x' \parallel y$. In fact, if $y \leq x'$, then $x < x'$, thus $x' = 1$ and $x = 0$, and if $x' \leq y$, then $y = 1$, a contradiction. We now proceed through a case-splitting argument.

Case (1). $U(y) = \{y, 1\}$ and $L(x', y) = \{0\}$. Let us note that $L(y', x) = \{0\}$, since $L(y') = \{y', 0\}$, and $y' \parallel x$, and dually $U(y, x') = \{1\}$. Then the set $M = \{1, x, y, x', y', 0\}$ is an LU-subset isomorphic to $\mathbf{B}_6$.

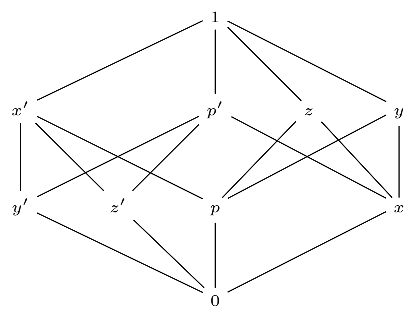
Case (2). $U(y) \neq \{y, 1\}$ and $L(x', y) = \{0\}$. Then, let $y, 1 \neq z \in U(y)$. Observe that $L(y, z') = \{0\}$, and also $L(z', x) = \{0\}$, because $x < y < z$. Therefore, we are as in figure (Case (2)).

(Case (2))



So, the set $M = \{1, x, y, x', y', 0\}$ is an LU-subposet isomorphic to $\mathbf{B}_6$. **Case (3).** $U(y) = \{y, 1\}$ and $L(x', y) \neq \{0\}$. Then, let $p \in L(x', y)$. By hypothesis, there is a $z \in U(x, L(x', y))$ such that $z \notin U(y)$. Let us observe that $z \neq x$; otherwise, since $p \leq x'$, $x \leq p'$, and then, if $x = z$, $p \leq z \leq p'$. So, $p \leq p'$, implies that $p' = 1$. Therefore, because p was arbitrary, $L(x', y) = \{0\}$, a contradiction. A few subcases are possible: **Case (3)(i).** $z \parallel y$. We are in the situation depicted in figure (Case (3)(i)).

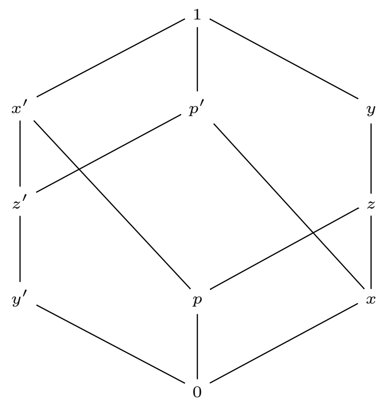
(Case (3)(i))



In fact, we notice that $U(x', p') = \{1\}$. Indeed, if $k \geq x', p'$, then, since $x \leq p'$, $1 = k \geq x, x'$. Also, if $k \geq z, p'$, then, since $z \geq p$, we have that $1 = k \geq p, p'$, i.e. $U(z, p') = \{1\}$, and $U(x', z) = \{1\}$, since $z > x$. Consequently, $L(p, z') = \{0\} = L(x, z')$. Because $U(y) = \{y, 1\}$, $U(z, y) = \{1\}$. So, the set $M = \{1, x', p', z, y, x, p, z', y', 0\}$ is an LU-subset isomorphic to $\mathbf{B}_{10}$. **Case (3)(ii).** $z < y$. Under our hypotheses, and the fact that $U(z', y) = \{1\}$, since $z' > y'$, we are in the situation depicted in figure

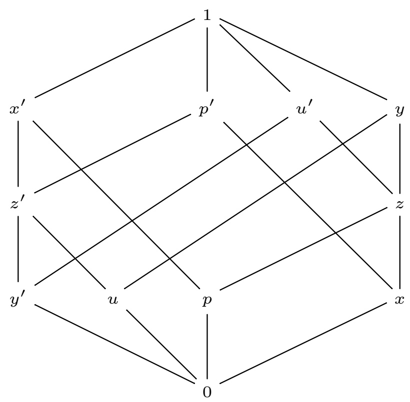
(Case (3)(ii)).

(Case (3)(ii))



Two further subcases are possible. **Case (3)(ii)(a)** $L(y, z') = \{0\}$. Then, the subposet $M = \{1, z', y, y', z, 0\}$ is an LU-subset isomorphic to $\mathbf{B}_6$. **Case (3)(ii)(b)** $L(y, z') \neq \{0\}$. Thus, there is a $u \in L(z', y)$, and so we are in the situation depicted in figure (Case (3)(ii)(b)).

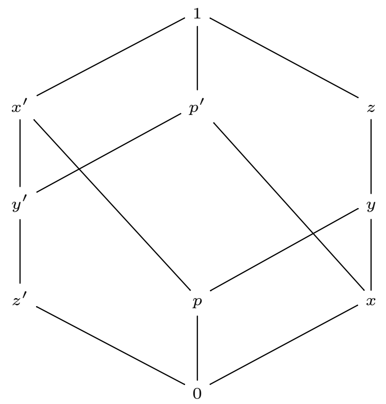
(Case (3)(ii)(b))



In fact, $L(y', u) = \{0\}$, since $u < y$, $L(u, p) = \{0\}$, since $u < p'$. Consequently, also $U(y, u') = \{1\} = U(z', u') = U(u', p')$. So, the LU-subposet $M = \{1, x', p', u', y, y', u, p, x, 0\}$ is isomorphic to $\mathbf{B}_{10^*}$.

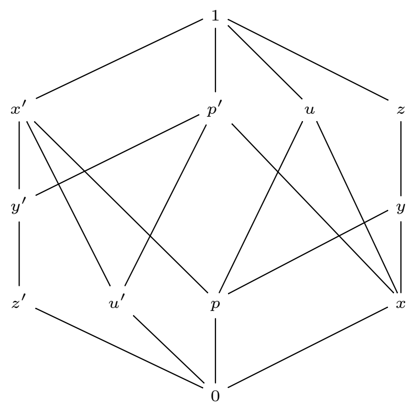
Case (4) $U(y) \neq \{y, 1\}$ and $L(x', y) \neq \{0\}$. Hence, there is a $z \in U(y)$, $y \neq z \neq 1$, and a $p \in L(x', y)$, $p \neq 0$. Then, we are in the situation depicted in figure (Case (4)).

(Case (4))



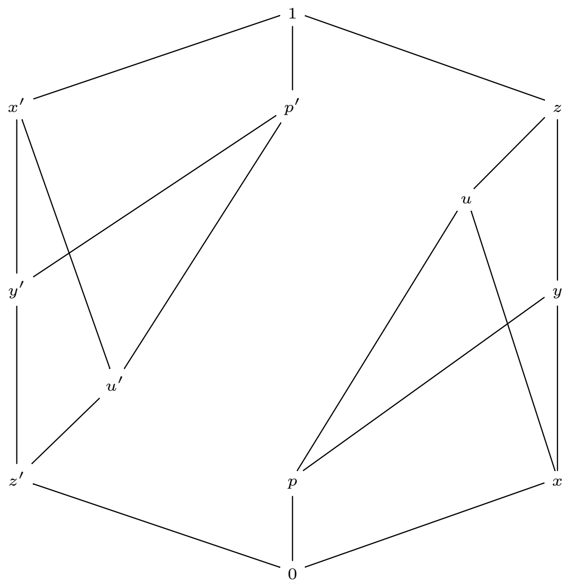
Let us notice that $U(p', z) = \{1\}$, since $p' > z'$, $U(p', x') = \{1\}$, since $p' > x$, and clearly $U(y', z) = \{1\} = U(x', z)$. As a consequence, $L(p, z') = L(p, x) = L(p, y') = L(y, z') = L(x, z') = \{0\}$. By hypothesis, there is a $u \in U(x, L(y, x'))$, but $u \notin U(y)$. Several cases may arise in base of the position of u with respect to y. **Case (4)(i)** $u \parallel y$. Then, we are in the situation depicted in figure (Case (4)(i)).

(Case (4)(i))



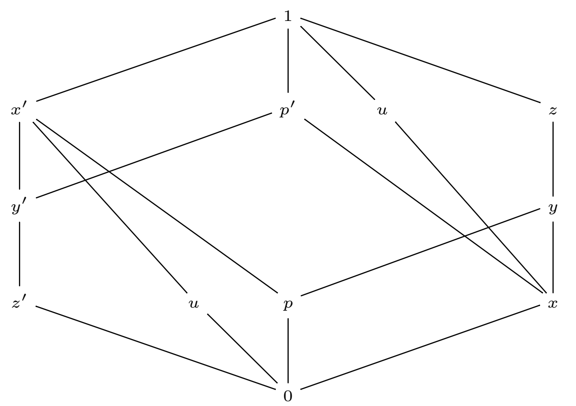
Let us notice that $L(u', p) = \{0\}$, since $p' > u'$, and $U(u, p') = \{1\}$. Moreover, $z \not\leq u$, because $z > y$ and $u \parallel y$. Few subcases are possible. **Case (4)(i)(a)** $u < z$. In this case, we are in the situation depicted in figure (Case (4)(i)(a)).

(Case (4)(i)(a))

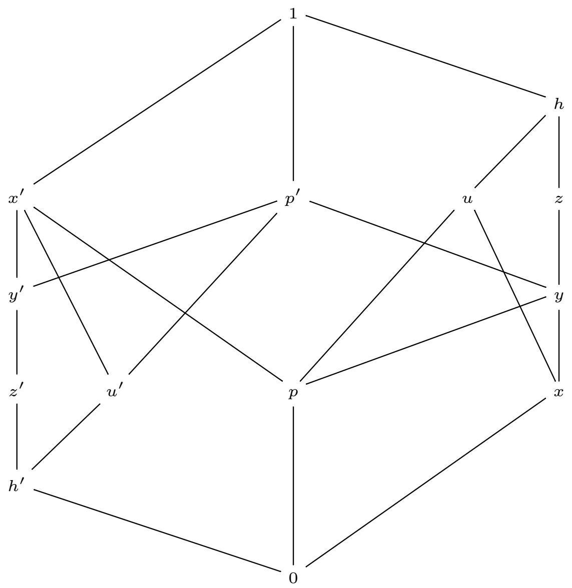


Let us notice that $U(x', z) = \{1\}$, since $z > x$, $U(y', z) = \{1\}$, since $y' > z'$, $U(p', x') = \{1\}$, since $p' > x$, $U(p', z) = \{1\}$, since $p' > z'$, $L(x, y') = \{0\}$, since $y' < x'$, and $L(x, z') = \{0\}$, since $z' < x'$. Two degrees of freedom are now possible. **Case (4)(i)(a)($\aleph_0$)** $L(y', z) = \{0\}$. Then, the set $M = \{1, y', z, z', y, 0\}$ is an LU-subposet isomorphic to $\mathbf{B}_6$. **Case (4)(i)(a)($\aleph_1$)** $h \in L(y', z) \neq \{0\}$. Reasoning as in case (3)(ii)(b), we see that the set $M = \{1, y, z, p, p', h, h', y', z', 0\}$ is an LU-subset isomorphic to $\mathbf{B}_{10^*}$. **Case (4)(i)(b)** $u \parallel z$. Then, we are in the situation depicted in figure (Case (4)(i)(b)).

(Case (4)(i)(b))

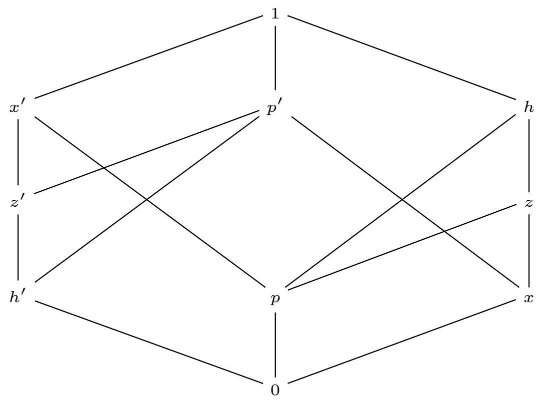


In case $U(u, z') = \{1\}$, then the set $M = \{1, x, z, u, p, x', z', u', p', 0\}$ is an LU-subposet isomorphic to $\mathbf{B}_{10}$. On the contrary, if $1 \neq h \in U(u, z')$, then we have the following configuration (figure (6.8)):



(6.8)

Let us consider the configuration in figure (6.9):

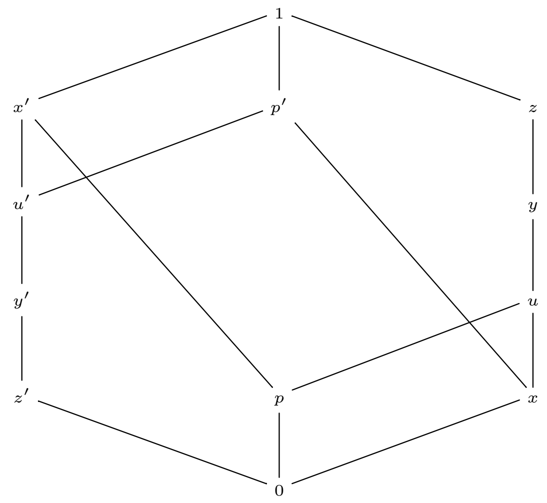


(6.9)

It can be seen that $U(p', h) = \{1\}$, since $p < h$, $U(p', z) = \{1\}$, since $p' > z'$, $U(p', x') = \{1\}$, since $p' > x'$; dually $L(p, h') = L(p, z') = L(p, z) = \{0\}$. We may have two subcases in base of the elements of $L(z', h)$. **Case (4)(i)(b)($\aleph_0$).** $L(z', h) = \{0\}$. The subposet $M = \{1, z', h, h', z, 0\}$ is an LU-subset isomorphic to $\mathbf{B}_6$. **Case (4)(i)(b)($\aleph_1$).** $0 \neq l \in L(z', h)$. Then, the set $M = \{1, x, h, l, p, x', h', l'p', 0\}$ is an LU-subset isomorphic to $\mathbf{B}_{10}$. **Case (4)(ii)** $u < y$. Then, we have the following configuration (cf. figure

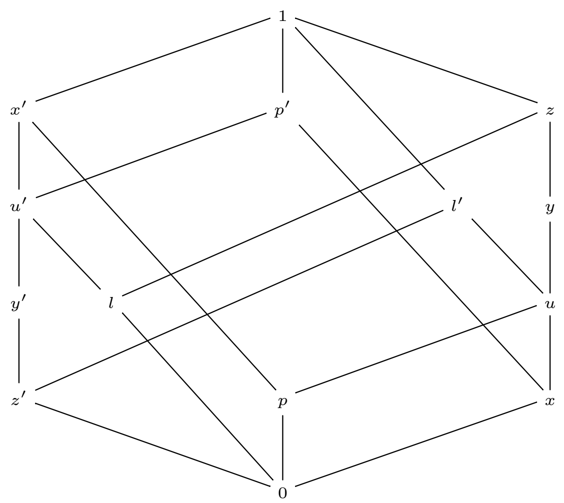
(Case (4)(ii))):

(Case (4)(ii))



Consider the structure whose universe is the following ten-element set $M = \{1, x', p', z, u', u, z', p, x, 0\}$. We have a few cases according with the position of $L(u', z)$. **Case (4)(ii)(a)** $L(u', z) = \{0\}$. Then, $M = \{1, z, u, z', u', 0\}$ is an LU-subset isomorphic to $\mathbf{B}_6$. **Case (4)(ii)(b)** $0 \neq l \in L(u', z)$. Then, we have the following configuration (figure (Case (4)(ii)(b))):

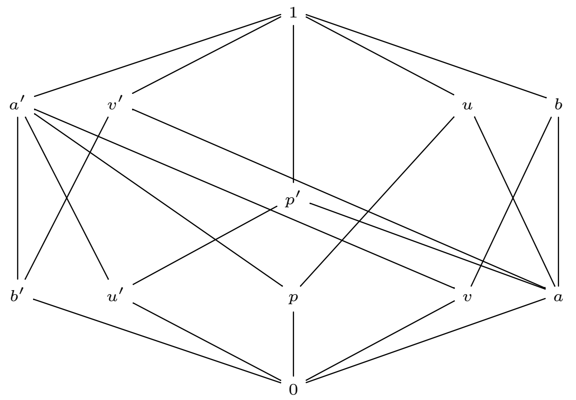
(Case (4)(ii)(b))



Consider the set $M = \{1, x, z, p, l, x', z', p', l', 0\}$. An easy check shows that M is an LU-subposet isomorphic to $\mathbf{B}_{10^*}$. □

Let us note that, even if by Lemma 6.11 an orthoposet containing an LU-subset isomorphic to $\mathbf{B}_6$ is not a GO-poset, there are orthoposets containing an LU-subset isomorphic to $\mathbf{B}_{10}$ that are GO-posets (figure ($\mathbf{B}_{12}$)). In fact, $\mathbf{B}_{12}$ contains the subposet $\mathbf{M}$ whose universe is $\{1, a', u, b, p', b', u', p, a, 0\}$ which is isomorphic to $\mathbf{B}_{10}$, but $\mathbf{B}_{12}$ has the GO-property.

$(\mathbf{B}_{12})$

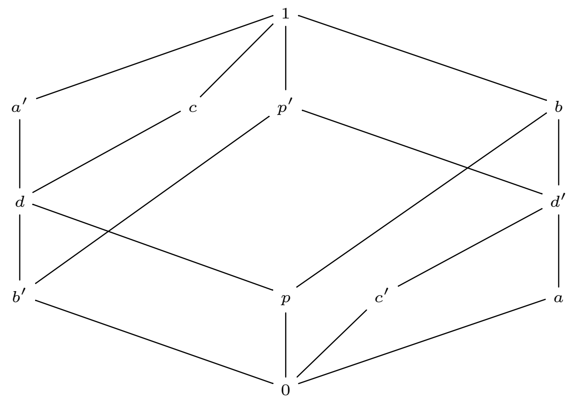


We close this section by showing that $\mathbf{B}_{10}$ (see display (6.6)) has the rather interesting property of being the smallest orthoposet which does not contain $\mathbf{B}_6$ and whose MacNeille completion is not orthomodular.

Theorem 6.15. $\mathbf{B}_{10}$ *($\mathbf{B}_{10^*}$) is the smallest orthoposet such that:* (i) *it does not contain* $\mathbf{B}_6$ *as a sub-lattice;* (ii) *its MacNeille completion is not orthomodular.*

Proof. As regards (ii), let us note that the Dedekind-MacNeille completion of $\mathbf{B}_{10}$ has the form depicted below

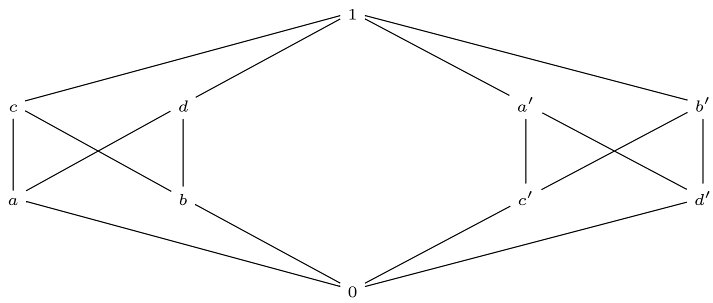
$(\mathrm{DM}(\mathbf{B}_{10}))$



It is easily seen that the set $\{0, 1, c, d, c', d'\}$ forms a sub-lattice isomorphic to $\mathbf{B}_6$. Therefore, $\mathrm{DM}(()\mathbf{B}_{10})$ is an ortholattice which is not orthomodular.

(i). Note that if $\mathbf{A} = (A, \leq, ', 0, 1)$ is a finite ortholattice, then $\mathrm{DM}(()\mathbf{A}) \cong \mathbf{A}$. Therefore if $\mathbf{A}$ does not admit an orthomodular Dedekind-MacNeille completion, then it must contain itself a sub-lattice isomorphic to $\mathbf{B}_6$. Hence, $\mathbf{A}$ cannot be a lattice, so it must contain at least a pair of uncomparable elements $a, b \in A$ having two distinct uncomparable upper bounds c, d. Thus, for cardinality reasons, $|A| \geq 10$ and we are in the situation depicted below:

(6.10)



Let us observe that the set $\{0, 1, d, b, d', b'\}$ forms a sub-lattice isomorphic to $\mathbf{B}_6$. Moreover, if $c' \leq d$ and $a \leq b'$, an easy exercise shows that for any $x, y \in A$, one has that $x \wedge y = 0$ implies $x \leq y'$. Therefore, $\mathbf{A}$ is a Boolean orthoposet and, by [121, Theorem 4.2], it follows that $DM(\mathbf{A})$ is a Boolean algebra, contradicting our assumptions. So the only possible cases are:

a. $c' \leq d$ and $a \not\leq b'$. Thus, it is easily seen that $\mathbf{A} \cong \mathbf{B}_{10}$;

b. $a \leq b'$ and $c' \not\leq d$. So, we have $\mathbf{A} \cong \mathbf{B}_{10^*}$.

Summarizing the above considerations one has that an orthoposet $\mathbf{A}$ which does not contain a sub-lattice isomorphic to $\mathbf{B}_6$ must have at least 10 elements. Moreover, if it does not admit an orthomodular Dedekind-MacNeille completion, then it must be isomorphic to $\mathbf{B}_{10}$ or $\mathbf{B}_{10^*}$. Hence our statement is proved. $\square$

6.3 Amalgams of Boolean algebras: applications to Greechie's Theorems

In the context of orthomodular posets, the first results using pasting techniques are to be credited to R. Greechie, and date back to the second half of the sixties [66, 67]. Indeed, he put the pasting technique to good use and came up with orthomodular posets possessing no measures ("stateless logics"). Taking up these ideas, other scholars fruitfully ventured upon the construction of other "unconventional" orthomodular posets [36, 115]. In particular, F. Schultz' state space characterization [116], together with the enlargement construction independent on centres, and state spaces by M. Navara and his coauthors [100] brought significant contributions to the foundations of quantum mechanics. In addition, in the field of orthomodular lattices viewed as universal algebras, pasting techniques had led to significant advancements and generalizations [94, 101] (see also [99]).

In this section we take advantage of the results of section 6.2 in order to obtain a novel characterization of atomic amalgams of Boolean algebras (cf. [3, Chapter 4.4]) which represent a subclass of pastings already encountered in Chapter ?? (see Subsection 2.1.2, Definition 2.9). In particular, a development of our arguments will yield Greechie's celebrated Theorems as corollaries [66, 67], shedding, perhaps, some light from a new perspective upon well developed algebraic tools.

For readability convenience, we now recall the main notions that will be used in this section (for a specific account we refer the reader to [3, p. 142], and for a wide overview to [?]). For any $i \in I \neq \emptyset$, with I a finite set, let $\mathbf{B}_i$ be a finite Boolean algebra. Suppose that the following conditions are satisfied:

(a1) any $\mathbf{B}_i$ has cardinality at least 8;

(a2) for any $i \neq j$ of I, $\mathbf{B}_i \cap \mathbf{B}_j$ is either $\{0, 1\}$ or $\{0, a, a', 1\}$, for some atom a, that belongs to both $\mathbf{B}_i$ and $\mathbf{B}_j$. In case $B_i \cap B_j = \{0, a, a', 1\}$, then the complement of a coincides in both $\mathbf{B}_i$ and $\mathbf{B}_j$ and will be denoted by a'.

Then, the system $\mathbf{B} = (\mathbf{B}_i : i \in I)$, with universe $\bigcup_i \{\mathbf{B}_i\}_{i \in I}$, is said to be an *atomic amalgam* of the family $\{\mathbf{B}_i\}_{i \in I}$. The elements of the set $\{\mathbf{B}_i\}_{i \in I}$ are called the *initial blocks of the amalgam*. In what follows, the ordering in a block $\mathbf{B}_i$ will be denoted by $\leq_{\mathbf{B}_i}$, and the order in the amalgam will be the union of the orders in the block (see [3, p.144]), more formally:

$$x \leq y \text{ if and only if } x \leq_{\mathbf{B}_i} y, \text{ for some } \mathbf{B}_i. \tag{6.11}$$

We now recollect some well known results that will be of some importance for our arguments.

Theorem 6.16. *[3, Theorem 4.1] An atomic amalgam of Boolean algebras is an orthocomplemented poset.*

Proof. It follows directly by Lemma 2.10. □

Theorem 6.17. *[3, Theorem 4.4] Let* $\mathbf{B} = (\mathbf{B}_i : i \in I)$ *be an atomic amalgam of Boolean algebras.*

(i) If $i \neq j$, $i, j \in I$, *then* $B_i \cup B_j$ *is the universe of an orthomodular lattice which is isomorphic to the amalgam pasting together the lattices* $(B_i, \wedge, \vee)$ *and* $(B_j, \wedge, \vee)$ *along the sublattice* $(B_i \cap B_j, \wedge, \vee)$.

(ii) The poset $(B, \leq)$ *is order-isomorphic to the amalgam pasting together the posets* $\mathbf{S}_1 = (B_k, \leq)$, $k \in I$, $\mathbf{S}_2 = (\bigcup_{i \in I \setminus \{k\}} B_i, \leq)$ *along the subposet* $(B_k \cap \bigcup_{i \in I \setminus \{k\}} B_i, \leq)$.

In order to keep the notation as compact as possible, we will write "$x \lessdot y$" to express that x is *covered* by y, i.e. $x < y$ and there is no z such that $x < z < y$.

Lemma 6.18. *[3, Lemma 4.5] If* $a \lessdot b$ *holds in an atomic amalgam, then* $a \lessdot b$ *holds in every block* B_i *that contains both* a *and* b*. Conversely, if* $a \lessdot b$ *holds in some initial block* B_i*, then* $a \lessdot b$ *holds in the atomic amalgam.*

Finally, let us introduce the last preliminary concept that will be required in this section: the notion of atomic loop (cf. [3, p.150]).

Let n be an integer greater or equal to 3. We will say that the initial blocks $B_1 = B_{i_1}, B_2 = B_{i_2}, \ldots, B_n = B_{i_n}$, $i_1, i_2, \ldots, i_n \in I$, form an *atomic loop of order* n if and only if the following conditions are satisfied:

(L1) for every $i = 1, 2, \ldots, n-1$, $B_i \cap B_{i+1} = \{0, 1, a_i, a_i'\}$, where a_i is an atom both in B_i and B_{i+1}, moreover $B_n \cap B_1 = \{0, 1, a_n, a_n'\}$, where a_n is an atom both in B_n and B_1.

(L2) $B_i \cap B_j = \{0, 1\}$, for all indices $i \neq j$ not mentioned in condition (L1).

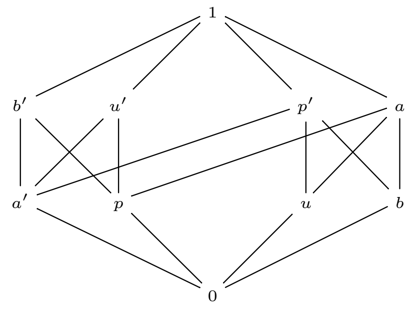
(L3) For any $1 \leq i < j < k \leq n$, $B_i \cap B_j \cap B_k = \{0, 1\}$.

Remark 6.19. [3, Remark 4.6] (A) by (L3), $\{0, 1\} = B_i \cap B_{i+1} \cap B_{i+2} = (B_i \cap B_{i+1}) \cap (B_{i+1} \cap B_{i+2}) = \{0, 1, a_i, a_i'\} \cap \{0, 1, a_{i+1}, a_{i+1}'\}$. This means that a_i and a_{i+1} are always distinct atoms. A completely analogous reasoning shows that any two atoms of $a_1, \ldots, a_n$ are distinct. This remark reveals the meaning of (L3).

We are now ready to state and prove the first result of this section. Interestingly enough, this result ties the existence of an atomic loop of a certain order with the theory of GO-posets. In particular, it shows that, in an atomic amalgam of Boolean algebras $\mathbf{B}$, the presence of an atomic loop of order 3 is bounded to the presence in $\mathbf{B}$ of a subposet isomorphic to $\mathbf{B}_{10^*}$, one of the forbidden configurations of the theory of GO-posets that have been discussed in section 6.2.

Lemma 6.20. *An atomic amalgam* $\mathbf{B}$ *of Boolean algebras* $(\mathbf{B}_i : i \in I)$ *contains an atomic loop of order* 3 *if and only if it contains a subposet isomorphic to* $\mathbf{B}_{10^*}$ *(see* *figure* (6.12)*)*

(6.12)



such that $U(a', b) \subseteq U(u)$*.*

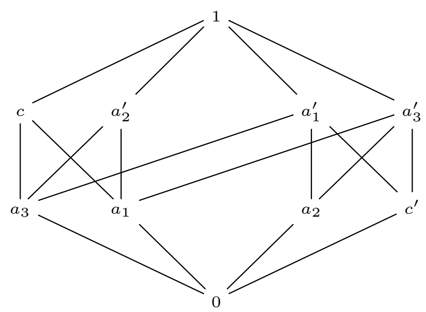
Proof. ($\Rightarrow$) Suppose that $\mathbf{B}$ contains an atomic loop of order 3. By its very definition $B_1 \cap B_2 = \{0, 1, a_1, a_1'\}$, $B_2 \cap B_3 = \{0, 1, a_2, a_2'\}$, $B_3 \cap B_1 = \{0, 1, a_3, a_3'\}$, $B_1 \cap B_2 \cap B_3 = \{0, 1\}$, and, for $1 \leq i \leq 3$, a_i is an atom in its respective blocks. Let us now consider the set $\{0, a_1, a_2, a_3, a_1', a_2', a_3', c, c', 1\}$, where $c = a_1 \vee^{\mathbf{B}_1} a_3$, which is defined because $a_1, a_3 \in B_1$. Note that $a_1, a_3 < a_2'$, $a_1 < a_3'$, and $a_3 < a_1'$. Indeed, since a_1 is an atom, if $a_1 \not< a_2'$, then $a_1 \vee^{\mathbf{B}_2} a_2' = 1$, because a_2' is a coatom. Therefore, since a_1 is an atom, a_1 is the complement of a_2', and since $\mathbf{B}_2$ is Boolean, this means that $a_1 = a_2$. Likewise, the other inequalities follow from analogous reasons. Now, let us observe that $c \neq a_2'$, otherwise $a_2' \in B_1 \cap B_2 \cap B_3$, and so $a_2' = 1$, which is absurd. If $c \leq_{\mathbf{B}_i} a_2'$, for $1 \leq i \leq 3$, two cases are possible.

(1) $i = 1$ and $c \leq_{\mathbf{B}_1} a_2'$. Therefore, $a_2' \in B_1$, and then $a_2' = 1$, impossible.

(2) $i \neq 1$. Let us observe that $0 \lessdot a_1 \lessdot c$. In fact, suppose that there is a $b \in B_1$ such that $a_1 \lessdot b \leq c = a_1 \vee^{\mathbf{B}_1} a_3$. Then, $b = b \wedge^{\mathbf{B}_1} (a_1 \vee^{\mathbf{B}_1} a_3) = (b \wedge^{\mathbf{B}_1} a_1) \vee^{\mathbf{B}_1} (b \vee^{\mathbf{B}_1} a_3) = a_1 \vee^{\mathbf{B}_1} (b \wedge^{\mathbf{B}_1} a_3) = a_1 \vee^{\mathbf{B}_1} a_3$, since otherwise $b = a_1 \vee^{\mathbf{B}_1} (b \vee^{\mathbf{B}_1} a_3) = a_1 \vee^{\mathbf{B}_1} 0 = a_1$, which is impossible. Moreover, $c < a_2' \lessdot 1$, because if $c = a_2'$, we would have that $c \in B_1 \cap B_2 \cap B_3$, and so $c = 1$, absurd. Therefore, we have obtained that $0 \lessdot a_1 \lessdot c < a_2' \lessdot 1$ in $\mathbf{B}$, by Lemma 6.18. And again by Lemma 6.18, $c \in B_1 \cap B_i$ for $i = 2$ or $i = 3$. But, c is neither an atom, nor a coatom, nor 0, nor 1. A contradiction. Thus, $c \parallel a_2'$.

We claim that M is the poset in figure (6.13)

(6.13)



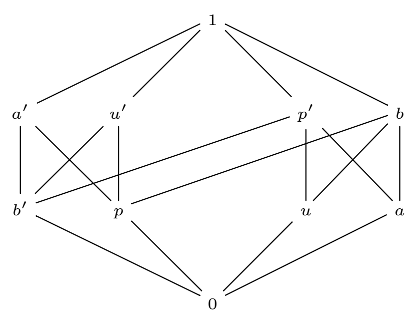
Now, suppose that there is a $p \in L(c, a_3')$. Then, by construction, $p \leq_{\mathbf{B}_i} c$ and $p \leq_{\mathbf{B}_j} a_2'$, for some i, j. But, independently from the fact that either $i = j$ or $i \neq j$, $p \in B_i \cap B_j \cap B_1 \cap B_3 = B_i \cap B_j \cap \{0, 1, a_3, a_3'\}$, and $a_3 \neq p \neq a_3'$, on pain of contradiction. Therefore, $a_1 = p$, and so $L(c, a_3') = \{0, a_1\}$. Consequently, $L(c, a_3')' = U(c', a_3) = \{1, a_1'\} \subseteq U(a_2)$.

($\Leftarrow$) For the converse direction, let us suppose that there is a subposet of $\mathbf{B}$ isomorphic to $\mathbf{B}_{10^*}$ so that $U(a', b) \subseteq U(u)$, as in figure (6.12). Without any loss of generality, we may assume that $\{0, 1, a, b, p, , a', b', p'\}$ is a subset of B_1. Since Lemma 6.9(3), and the fact that $U(a', b) \subseteq U(u)$, we have that $\mathbf{B}$ is not generalized orthomodular. Then, $u \notin B_1$, since $\mathbf{B}_1$ would not be Boolean. So, $p \leq_{\mathbf{B}_{i_1}} u'$ and $a' \leq_{\mathbf{B}_{i_2}} u'$, with $i_1 \neq i_2$, otherwise $M \subseteq B_1 \cup B_{i_1}$, which would not be an orthomodular lattice, contradicting Theorem 6.17. Therefore, by condition (6.11), there are three distinct Boolean algebras $\mathbf{B}_1, \mathbf{B}_{i_1}, \mathbf{B}_{i_2}$ such that $\mathbf{B}_1 \cap \mathbf{B}_{i_1} = \{0, 1, p, p'\}$, $\mathbf{B}_{i_1} \cap \mathbf{B}_{i_2} = \{0, 1, u, u'\}$, $\mathbf{B}_1 \cap \mathbf{B}_{i_2} = \{0, 1, a, a'\}$, and $\mathbf{B}_1 \cap \mathbf{B}_{i_1} \cap \mathbf{B}_{i_2} = \{0, 1\}$. This establishes our claim. $\square$

Taking advantage of Lemma 6.20, the next theorem characterizes completely those atomic amalgam that are not orthomodular posets. This results is essential for our arguments, in facts it reveal from a wider perspective the concealed facts upon which Greechie's results rely. Indeed, putting to good use our results on the forbidden configurations in section 6.2, Theorem 6.21 will offer a general tool from which Greechie's First and Second Theorem derive.

Theorem 6.21. *An atomic amalgam* $\mathbf{B}$ *of Boolean algebras* $(\mathbf{B}_i : i \in I)$ *is not an orhomodular poset if and only if it contains a subposet isomorphic to* $\mathbf{B}_{10^*}$ *(see figure* (6.14)*)*

(6.14)



such that $U(a, b') \subseteq U(u)$.

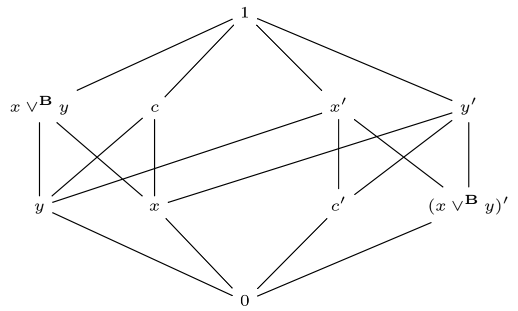
Proof. ($\Leftarrow$) Follows from Lemma 6.9(3).

($\Rightarrow$) Let us assume that $\mathbf{B}$ is not an orhomodular poset. The following cases may arise.

(1) $x \perp y$, i.e. $x \leq y'$, but $x \vee^{\mathbf{B}} y$ does not exist. Ergo, there is a $c \in B$ such that $c \parallel x \vee^{\mathbf{B}} y$, but $x, y \leq c$. Since the definition of order in $\mathbf{B}$ (condition 6.11), without any loss of

generality, we may assume that $x, y \in B_1$. By Lemma 6.18, $x \leq_{\mathbf{B}_{i_1}} c$, and $y \leq_{\mathbf{B}_{i_2}} c$, for some i_1, i_2. Then, in case $i_1 = i_2$, we have the situation depicted in figure (6.15).

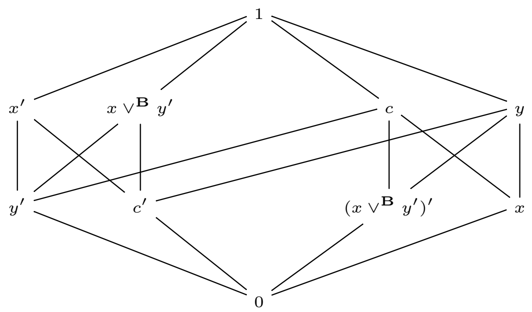
(6.15)



However, by Theorem 6.17, $B_1 \cup B_{i_1}$ is the universe of an orhomodular lattice, a contradiction. Therefore, $i_1 \neq i_2$. So, $\mathbf{B}_1 \cap \mathbf{B}_{i_1} = \{0, 1, y, y'\}$, $\mathbf{B}_{i_1} \cap \mathbf{B}_{i_2} = \{0, 1, c, c'\}$, and $\mathbf{B}_1 \cap \mathbf{B}_{i_2} = \{0, 1, x, x'\}$. Thus, since $\mathbf{B}$ contains an atomic loop of order 3, by Lemma 6.20, $\mathbf{B}$ contains a subposet M isomorphic to $\mathbf{B}_{10^*}$ so that $U(a, b') \subseteq U(u)$.

(2) There are $x, y \in B$ such that $x \leq y$, but $(x \vee^{\mathbf{B}} y')' \vee^{\mathbf{B}} x \neq y$. Without any loss of generality, we may assume that $x, y \in B_1$, and that $(x \vee^{\mathbf{B}} y')' \vee^{\mathbf{B}} x$ exists, since otherwise we are as discussed in case (1). Because $\mathbf{B}_1$ is Boolean, and so a fortiori an orthomodular lattice, $(x \vee^{\mathbf{B}} y')' \vee^{\mathbf{B}} x = c \neq y = (x \vee^{\mathbf{B}_1} y')' \vee^{\mathbf{B}_1} x$. Then, we are in the situation depicted in figure (6.16).

(6.16)



Repeating exactly the same reasoning applied in case (1), we obtain a subposet $\mathbf{M}$ isomorphic to $\mathbf{B}_{10^*}$ that satisfies the required condition.

□

Taking advantage of the previous results, we can now state, as corollaries, Greechie's celebrated First and Second Theorem.

Corollary 6.22. *[3, Greechie's First Theorem 4.9] An atomic amalgam* $\mathbf{B}$ *of Boolean algebras* $(\mathbf{B}_i : i \in I)$ *is an orthomodular poset if and only if* $\mathbf{B}$ *does not contain an atomic loop of order* 3.

Proof. Follows from Theorem 6.21. □

Corollary 6.23. *[3, Greechie's Second Theorem 4.10] An atomic amalgam* $\mathbf{B}$ *of Boolean algebras* $(\mathbf{B}_i : i \in I)$ *is an orthomodular lattice if and only if* $\mathbf{B}$ *does not contain an atomic loop of order* 3 *or* 4.

Proof. The result substantially rests on Corollary 6.22 (see [3]). □

Finally, a direct application of the results in this section yields the following:

Corollary 6.24. *An atomic amalgam of Boolean algebras is a GO-poset if and only if it is an orhomodular poset.*

Proof. Straightforward. □

Perhaps, a few words on the significance of Corollary 6.24 may be worth saying. In fact, this results expresses the fact that, in the context of atomic amalgam of Boolean algebras, the notions of orthomodular poset and GO-poset coincide. In other words, there is no chance to go beyond the concept of orthomodular poset, starting from an atomic amalgam of Boolean algebras. In other words, it is impossible to construct an atomic amalgam of Boolean algebras, that satisfies the generalized orthomodularity property, which is not an orthomodular poset at the same time.

6.4 Effect algebras and the GO-property

Making use of the techniques developed in section 6.2, we will prove that the GO-condition, for orthoalgebras of a certain sort, i.e. effect algebras with no isotropic elements, corresponds exactly to orthomodularity. Hence, the only orthoalgebras satisfying an even weaker orthomodularity condition, in its order theoretical sense, are orthomodular posets. It will appear clear soon that the GO-property represents the order-theoretical counterpart of the coherence law for effect algebras. To the best of our knowledge, these results are new and subsume under a unifying framework many well known facts sparsely scattered in the literature [102, 110].

Let us briefly recall the basic machinery that will be employed in the next considerations. We recall that any Boolean algebra is, obviously, an orthoalgebra if we set as orthosum $\oplus$ the join of orthogonal elements in the underlying lattice. As customary, we will denote the *orhocomplement* (cf. Definition 2.18) of an element x by x'. In any orthoalgebra a partial order is naturally defined stipulating that $x \leq y$ if and only if there is a z so that $x \oplus z = y$. A *suborthoalgebra* $\mathbf{M}$ of an orthoalgebra $\mathbf{L}$ (see [109]) is a structure such that

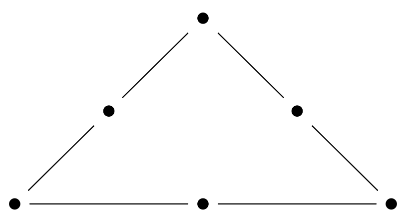
- $M \subseteq L$;
- $0^{\mathbf{L}} = 0^{\mathbf{M}}$, $1^{\mathbf{L}} = 1^{\mathbf{M}}$;
- $\forall x, y, z \in L$: $(x \oplus y = z \,\&\, |\{x, y, z\} \cap M| \geq 2) \Rightarrow \{x, y, z\} \subseteq M$;
- $\forall x, y, z \in M$: $x \oplus^{\mathbf{M}} y = x \oplus^{\mathbf{L}} y$.

In case $\mathbf{M}$ is a suborthoalgebra which is Boolean, then we will refer to $\mathbf{M}$ as a *Boolean subalgebra* of $\mathbf{L}$. If $\mathbf{M}$ is maximal with respect to the property of being Boolean, then we say that $\mathbf{M}$ is a *block*, and we denote by $\mathcal{B}(\mathbf{L})$ the collection of the blocks of $\mathbf{L}$.

In order to maintain our arguments as smooth as possible, we now introduce the following concept:

Definition 6.25. An orthoalgebra $\mathbf{P}$ will be called *tame* if, for distinct blocks $\mathbf{B}_i, \mathbf{B}_j$, and $x, y \in B_i \cup B_j$, if $x \leq^{\mathbf{B}_i \cup \mathbf{B}_j} y'$, then $x \oplus^{\mathbf{P}} y = x \oplus^{\mathbf{B}_i \cup \mathbf{B}_j} y \in \mathbf{B}_i \cup \mathbf{B}_j$.

Let us remark that there are proper, in the sense that are not orthomodular posets, orthoalgebras that are tame, e.g. the *Wright Triangle* (See Chapter ??), whose Greeche diagram is depicted in display (6.17):



(6.17)

The next technical lemma will be useful in proving Theorem 6.27, the main result of this section.

Lemma 6.26. *Let* $\mathbf{P}$ *be a tame orthoalgebra, and* $\mathbf{B}_i, \mathbf{B}_j$ *be distinct blocks of* $\mathbf{P}$*. Then,*

1. $\mathbf{C} = (B_i \cup B_j, \oplus^{\mathbf{C}}, '^{\mathbf{C}}, 0^{\mathbf{C}}, 1^{\mathbf{C}})$ *is an orthoalgebra;*
2. $\mathbf{C}$ *is an orthomodular poset.*

Proof. Set $'^{\mathbf{C}}, 0^{\mathbf{C}}, 1^{\mathbf{C}}$ as $'^{\mathbf{P}}, 0^{\mathbf{P}}, 1^{\mathbf{P}}$, respectively. When possible, we will omit unnecessary superscripts. Moreover, let us define:

$$x \oplus^{\mathbf{C}} y = \begin{cases} x \oplus^{\mathbf{P}} y, \text{ if } x \leq^{\mathbf{C}} y'; \\ \text{not defined, otherwise.} \end{cases}$$

(1) Let us check that $\mathbf{C}$ is an orthoalgebra. Concerning commutativity, if $x \oplus^{\mathbf{C}} y$ is defined, then $x \leq^{\mathbf{C}} y'$, i.e. there is a $u \in B_i \cap B_j$ so that $x \leq^{\mathbf{B}_i} u \leq^{\mathbf{B}_j} y'$. Therefore, $y \leq^{\mathbf{B}_j} u' \leq^{\mathbf{B}_i} x'$. Then, $y \oplus^{\mathbf{C}} x$ is defined. Hence, $x \oplus^{\mathbf{C}} y = x \oplus^{\mathbf{P}} y = y \oplus^{\mathbf{P}} x = y \oplus^{\mathbf{C}} x$. Suppose that $(x \oplus^{\mathbf{C}} y) \oplus^{\mathbf{C}} z$ is defined. Then, $x \leq^{\mathbf{C}} y'$ and $x \oplus^{\mathbf{C}} y \leq^{\mathbf{C}} z'$. If $\{x, y, z\} \subseteq \mathbf{B}_k, i \in \{i, j\}$, then the claim is straightforward. In case $\{x, y\} \subseteq B_i$ and $z \in B_j \setminus B_i$, then $x \oplus^{\mathbf{C}} y = x \oplus^{\mathbf{P}} y$. Also, we observe that $y \leq^{\mathbf{B}_i} x \oplus^{\mathbf{C}} y \leq^{\mathbf{C}} z'$. So, there is a $u \in B_i \cap B_j$ such that $y \leq^{\mathbf{B}_i} x \oplus^{\mathbf{C}} y \leq^{\mathbf{B}_i} u \leq^{\mathbf{B}_j} z'$. By transitivity, $y \leq^{\mathbf{C}} z'$, and then $y \oplus^{\mathbf{C}} z = y \oplus^{\mathbf{P}} z$ is defined. If $y \oplus^{\mathbf{C}} z \in B_i$, then by properties of orthoalgebras, since also $y \in B_i$, $z \in B_i$, a contradiction. Therefore, $y \oplus^{\mathbf{C}} z \in B_j$, and so $y \in B_j$. Consequently, $y \in B_i \cap B_j$. Let us also observe that, for some $u \in B_i \cap B_j$, $x \leq^{\mathbf{B}_i} x \oplus^{\mathbf{C}} y \leq^{\mathbf{B}_i} u \leq^{\mathbf{B}_j} z'$. Then $x \oplus^{\mathbf{C}} z$ is defined and it belongs to $\mathbf{B}_j$. And so $x \in B_i \cap B_j$. Hence, because $x \oplus^{\mathbf{P}} (y \oplus^{\mathbf{P}} z)$ is defined, $x \leq^{\mathbf{P}} (y \oplus^{\mathbf{P}} z)'$. So, $x \leq^{\mathbf{C}} (y \oplus^{\mathbf{C}} z)' = (y \oplus^{\mathbf{P}} z)'$. We obtain that

$$
\begin{aligned}
(x \oplus^{\mathbf{C}} y) \oplus^{\mathbf{C}} z = &\quad (x \oplus^{\mathbf{P}} y) \oplus^{\mathbf{P}} z \\
= &\quad x \oplus^{\mathbf{P}} (y \oplus^{\mathbf{P}} z) \\
= &\quad x \oplus^{\mathbf{C}} (y \oplus^{\mathbf{C}} z)
\end{aligned}
$$

This establishes our first claim since the other possible case is treated similarly and the other axioms that define orthoalgebras are trivially satisfied.

(2) Let $p, q, r \in C = B_i \cup B_j$ such that $p \oplus^{\mathbf{C}} q, p \oplus^{\mathbf{C}} r, q \oplus^{\mathbf{C}} r$ are defined. We will show that also $(p \oplus^{\mathbf{C}} q) \oplus^{\mathbf{C}} r$ is defined. Therefore, by the coherence law (see [**?**, Theorem 5.3]) $\mathbf{C}$ this would imply that is an orthomodular poset. If $\{p, q, r, \} \subseteq B_k$ with $k \in \{i, j\}$, then $p \oplus^{\mathbf{C}} q = p \oplus^{\mathbf{P}} q = p \vee^{\mathbf{B}_k} q \leq^{\mathbf{B}_k} r'$. Therefore, $p \oplus^{\mathbf{C}} q \leq^{\mathbf{B}_k} r'$ and $(p \oplus^{\mathbf{C}} q) \oplus^{\mathbf{C}} r$ is defined. Now, without any loss of generality, let us suppose that $\{p, q\} \subseteq B_i$ and $r \in B_j \smallsetminus B_i$. Clearly, both $\mathbf{B}_i$ and $\mathbf{B}_j$ are Boolean subalgebras of $\mathbf{P}$. Now, if $p \oplus^{\mathbf{P}} r \in B_i$, then $r \in B_i$, a contradiction. Thus we have $p \oplus^{\mathbf{P}} r \in B_j$ and $p \in B_j$. The same argument applies to q. In conclusion, we have obtained that $\{p, q, r, \} \subseteq B_j$, and so our claim is proved. $\square$

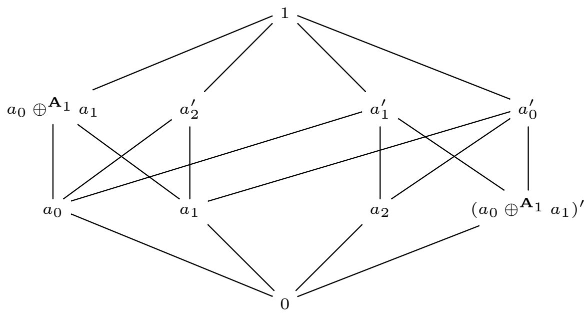
The next theorem completely characterize those tame orthoalgebras that are not orthomodular posets in terms of forbidden configurations induced by their blocks.

Theorem 6.27. *A tame orthoalgebra* $\mathbf{P} = (P, \oplus, ', 0, 1)$ *is a GO-poset if and only if it is an orthomodular poset.*

Proof. For the non-trivial direction, let $\mathbf{P}$ be a tame orthoalgebra which is not an orthomodular poset. Thus, there exist $a_0, a_1, a_2 \in P$ such that $a_0 \oplus^{\mathbf{P}} a_1, a_0 \oplus^{\mathbf{P}} a_2, a_1 \oplus^{\mathbf{P}} a_2$ are defined but $(a_0 \oplus^{\mathbf{P}} a_1) \oplus^{\mathbf{P}} a_2$ is not, in other words the coherence law fails: $(a_0 \oplus^{\mathbf{P}} a_1) \not\leq a_2'$. In case $a_2' \leq a_0 \oplus^{\mathbf{P}} a_1$, then by the minimality of $a_0 \oplus^{\mathbf{P}} a_1$, and the fact that

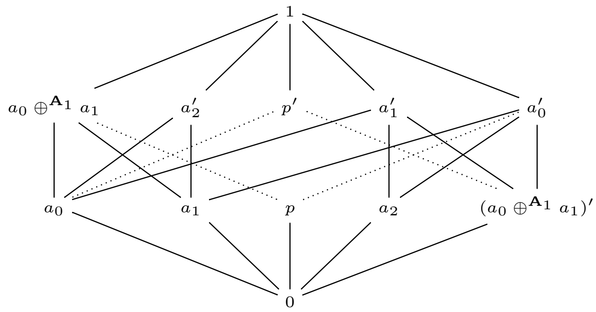
$a_0 \perp a_2$, $a_1 \perp a_2$, we have $a_0 \oplus^{\mathbf{P}} a_1 = a_2'$ and $(a_0 \oplus^{\mathbf{P}} a_1) \oplus^{\mathbf{P}} a_2 = a_2' \oplus^{\mathbf{P}} a_2$ is defined. Contradiction. Thus $a_0 \oplus^{\mathbf{P}} a_1 \parallel a_2'$. Moreover, since a_0 and a_1 are orthogonal, they are contained in a common block $\mathbf{A}_1$ such that $a_0 \oplus^{\mathbf{P}} a_1 = a_0 \oplus^{\mathbf{A}_1} a_1$. So we are in the situation depicted in figure (6.18).

(6.18)



We want to show that $L(a_0 \oplus^{\mathbf{A}_1} a_1, a_0') \subseteq L(a_2')$. Suppose that $p \in L(a_0 \oplus^{\mathbf{A}_1} a_1, a_0')$ (see figure (6.19)).

(6.19)



Two cases may arise.

(1) $p \in A_1$. Let us note that $a_0, a_0 \oplus^{\mathbf{A}_1} a_1, a_1 \in A_1$, and $\mathbf{A}_1$ is Boolean. We have also that $a_0 \leq_{\mathbf{A}_1} p'$ and $(a_0 \oplus^{\mathbf{A}_1} a_1)' \leq_{\mathbf{A}_1} p'$. So, $a_1' = a_0 \oplus^{\mathbf{A}_1} (a_0 \oplus^{\mathbf{A}_1} a_1)' = a_0 \vee^{\mathbf{A}_1} (a_0 \oplus^{\mathbf{A}_1} a_1)' \leq p'$, and so $p \leq a_1 \leq a_2'$.

(2) $p \notin A_1$. So, for distinct $\mathbf{H}, \mathbf{F} \in \mathcal{F}$, $a_0' \geq_{\mathbf{H}} p$ and $a_0 \oplus^{\mathbf{A}_1} a_1 \geq_{\mathbf{F}} p$. By Lemma 6.26, $H \cup F$ is the universe of an orthomodular poset. So, $p \oplus^{\mathbf{H}} a_0 = p \vee^{H \cup F} a_0 \leq a_0 \oplus^{\mathbf{A}_1} a_1$ in $\mathbf{P}$. Therefore, $a_0, p \leq p \vee^{H \cup F} a_0 \leq a_0 \oplus^{\mathbf{A}_1} a_1$. Applying [97, Proposition 6.15], there exists a $\mathbf{C} \in \mathcal{F}$ such that $a_0, p, p \vee^{H \cup F} a_0, a_0 \oplus^{\mathbf{A}_1} a_1 \in C$. By (PF1) and the notion of suborthoalgebra, $a_0, a_0 \oplus^{\mathbf{H}} a_1 = a_0 \oplus^{\mathbf{A}_1} a_1 \in C \cap H$ implies that $a_1 \in C \cap H$, and, a fortiori, $a_1 \in C$. Then $a_0, a_1, a_0 \oplus^{\mathbf{A}_1} a_1, p \in C$. We observe that $a_1' = a_0 \oplus^{\mathbf{C}} (a_0 \oplus^{\mathbf{C}} a_1)' = a_0 \vee^{\mathbf{C}} (a_0 \oplus^{\mathbf{C}} a_1)' \leq p'$, because $p \leq a_0', (a_0 \oplus^{\mathbf{C}} a_1)$. Then, $p \leq a_1$, and consequently $p \leq a_2'$.

Then, in both cases Lemma 6.9 applies, and our claim follows. □

Combining Theorem 6.27 together with Theorem 2.12, we obtain that, if $\mathbf{P}$ is the pasting of a family $\mathcal{F}$ of Boolean algebras and it is tame, then

Corollary 6.28. $\mathbf{P}$ *is a GO-poset if and only if for every* 3*-cycle* $((\mathbf{A}_0, a_0), (\mathbf{A}_1, a_1)(\mathbf{A}_2, a_2))$ *there is a Boolean algebra* $\mathbf{B}$ *such that* $a_0, a_1, a_2 \in B$.

Let us remark that a result analogous to Theorem 6.27 can be proved for the class of *Riesz orthoalgebras* [35], i.e. those orthoalgebras that satisfy the *Riesz decomposition property*: if $x \leq y_1 \oplus \ldots \oplus y_n$ then $x = x_1 \oplus \ldots \oplus x_n$, where $x_i \leq y_i$, for $1 \leq i \leq n$.

Theorem 6.29. *A Riesz orthoalgebra* $\mathbf{P} = (P, \oplus, ', 0, 1)$ *is a GO-poset if and only if it is an orthomodular poset.*

Proof. The proof is the same of Theorem 6.27, except for case (2), which can be proven as follows. Suppose that there is a $p \leq a_0 \oplus^{\mathbf{A}_1} a_1, a_0'$. Then, by the Riesz decomposition property, there are b_0, b_1 such that $p = b_0 \oplus^{\mathbf{A}_1} b_1$ and $b_0 \leq a_0$, $b_1 \leq a_1$. Note that $b_0 \in L(a_0, a_0') = \{0\}$, and so $b_0 = 0$. Therefore, $p = b_1 \leq a_1 \leq a_2'$. □

As a side remark, we observe that there are tame orthoalgebras that are not Riesz, for example the Wright Triangle (see e.g. [33, Example 5.1.11]) in display (6.17).

Let us remark that from Theorem 6.27 (Theorem 6.29) it follows directly that a tame (Riesz) orthoalgebra $\mathbf{A}$ is not an orthomodular poset if and only if it there are elements $a, b \in A$ such that $a \oplus b$ is defined and $a \wedge b = 0$, but $a \vee b$ is not defined. By its proof, our result presents from a novel perspective Riečănová's Theorem 2.2 in [111].

Proposition 6.30. *Every atomic amalgam of Boolean algebras is tame.*

Proof. Let $\mathbf{B}_i, \mathbf{B}_j$ be distinct blocks of an orthoalgebra $\mathbf{A}$, which is an atomic amalgam of Boolean algebras. Then, either $B_i \cap B_j = \{0, x, x', 1\}$, with x an atom, or $B_i \cap B_j = \{0, 1\}$. If $x \leq y'$ in $B_i \cup B_j$, then if $x, y \in B_i$ we have that $x \oplus^{\mathbf{A}} y = x \vee^{\mathbf{B}_i} y$. Otherwise, if there is a $z \in B_i \cap B_j$ such that $x \leq^{\mathbf{B}_i} z \leq^{\mathbf{B}_j} y'$, then $z = x = y$. □

6.5 Open problems and future research

Finally, we close this chapter with five open problems.

Problem 3. The following questions seem to be relevant for the theory of GO-posets:

1. Is it possible to define an appropriate notion of commutator in the context of GO-posets?
2. If yes, is it the case that every GO-poset is the union of maximal suborthoposets of pairwise commuting elements?
3. Is any maximal suborthoposet of pairwise commuting elements Boolean?
4. Is it possible to introduce a notion of "pasting" that generalizes the notion of pasting of Boolean algebras [97]?

Problem 4. Are all orthoalgebras tame?

If all orthoalgebras are tame, then Theorem 6.27 would hold true for all orthoalgebras. Since every GO-poset is an orthoposet (Remark 6.6), so any effect algebra enjoying the GO-property is an orthoalgebra, we would have, as an easy consequence of Theorem 6.27, Corollary 6.31, which would completely characterize all effect algebras that are orthomodular as posets in terms of the GO-property.

Corollary 6.31. *An effect algebra has the GO-property if and only if it is an orthomodular poset.*

Also, from Corollary 6.31 would follow directly that, in the context of effect algebras, the GO-condition is just equivalent to the coherence law [47, Theorem 5.3].

If not all orthoalgebras are tame, then:

Problem 5. Exhibit an example of an orthoalgebra which is generalized orthomodular, but it is not an orthomodular poset.

Also,

Problem 6. Is it possible to express the generalized orthomodularity property within the language of orthoalgebras?

Bibliography

[1] Amemiya I., Araki H., "A remark on Piron?s paper", *Publ. Res. Inst. Math. Sci.*, Sect. A 12, 1966, pp. 423-427.

[2] L.P. Belluce, A. Di Nola, and A.R Ferraioli. MV-semirings and their sheaf representations. *Order*, 30(1):165–179, 2013.

[3] L. Beran. *Orthomodular Lattices: An Algebraic Approach.* Mathematics and its Applications. Springer Netherlands, 2011.

[4] Birkhoff G., *Lattice Theory*, Amer. Math. Soc. Colloq. Pub., AMS, Providence, R.I., 1979.

[5] Birkhoff G., Von Neumann J., "The logic of quantum mechanics", *Annals of Mathematics*, 37 (4), 1936, pp. 823 - 843.

[6] S. Bonzio, I. Chajda, and A. Ledda. Representing quantum structures as near semirings. *Logic Journal of the IGPL*, 24(5):719–742, 2016.

[7] Borzooei R.A., Dvurečenskij A., Sharafi A.H., "Material implications in lattice effect algebras", *Information Sciences*, 433-434, 2018, pp. 233 - 240.

[8] Botur M. "An example of a commutative basic algebra which is not an MV-algebra", *Mathematica Slovaca*, 60(2), 2010, pp.171-178

[9] Botur M., Chajda I., Halaš R., "Are basic algebras residuated structures?" *Soft Computing*, 14, 2010, pp. 251 - 255.

[10] Botur M. Kühr J., "On (finite) distributive lattices with antitone involutions", *Soft Computing*,18, 2014,1033?1040.

[11] Burris S., Sankappanavar H. P. *A Course in Universal Algebra*, Springer, Verlag, 1981.

[12] Bruns G., Harding J., "Algebraic Aspects of Orthomodular Lattices", In: Coecke B., Moore D., Wilce A. (eds) Current Research in Operational Quantum Logic. Fundamental Theories of Physics, vol 111. Springer, Dordrecht, 2000.

[13] Cignoli R.L., d'Ottaviano I. M., Mundici D. , *Algebraic Foundations of Many-Valued Reasoning*, Dordrecht; Boston : Kluwer Academic Publishers, 1999.

[14] I. Chajda. Basic algebras and their applications. An overview. In J.Czermak et al., editor, *Proc. 81st Workshop on General Algebra, Salzburg, Austria, 2011*, pages 1–10. Johannes Heyn, Klagenfurt, 2012.

[15] I. Chajda. *Algebraic theory of tolerance relations.* Univ. Palack., Olomouc. 1991.

[16] Chajda I., "A note on pseudo-Kleene algebras", *Acta Univ. Palach. Olom.*, Mathematica, 55, 2016, pp. 39-45.

[17] Chajda I., Fazio D., Ledda A., "On the structure theory of L??ukasiewicz near semirings", *Logic Journal of the IGPL*, 2018.

[18] Chajda I., Fazio D., Ledda A., "A semiring-like representation of pseudoeffect algebras", *Soft Computing*, 2018.

[19] Chajda I., Fazio D., "On residuation in paraorthomodular lattices", (submitted).

[20] Chajda I., Fazio D., Ledda A., "The generalized orthomodularity property: configurations and pastings", (submitted).

[21] Chajda I., Halaš R., Kühr J., "Many-valued quantum algebras" *Algebra Universalis*, Volume 60, Issue 1, 2009, pp 63?90.

[22] Chajda I., Halaš R., Kühr J., "Every effect algebra can be made into a total algebra", *Algebra Universalis*, 61, 2009, pp. 139-150.

[23] Chajda I., Kolařík M., "Variety of orthomodular posets", *Miskolc Mathematical Notes*, 15, 2, 2014, pp. 361-371.

[24] Chajda I., Länger H., "Weakly Orthomodular and Dually Weakly Orthomodular Lattices", *Order*, 35, 2018, pp 541-555.

[25] Chajda I., Länger H., "Residuation in orthomodular lattices", *Topological Algebra and its Applications*, 5, 2017, pp. 1-5.

[26] Chajda I., Länger H., "Residuation in modular lattices and posets", *Asian-European J. Mathem.*, to appear.

[27] Chajda I., Länger H., "Orthomodular lattices can be converted into left-residuated l-groupoids", *Miskolc Mathematical Notes*, 18, 2017, pp. 685-689.

[28] Chajda I., Länger H., "Coupled Right Orthosemirings Induced by Orthomodular Lattices", *Order*, 34, 1, 2017, pp. 1-7.

[29] I. Chajda and H. Länger. Commutative basic algebras and coupled near semirings. *Soft Computing*, 19(5):1129–1134, 2015.

[30] I. Chajda and H. Länger. A representation of basic algebras by coupled right near semirings. *Acta Sci. Math. (Szegel)*, 81(34):361–374, 2015.

[31] I. Chajda and H. Länger. A representation of lattice effect algebras by means of right near semirings with involution. *International Journal of Theoretical Physics*, (56): 3719-3726, 2017

[32] Chajda I., Rachůnek J., “Forbidden configurations for distributive and modular ordered sets”, *Order*, 5, 1989, pp. 407–423.

[33] M. L. Dalla Chiara, R. Giuntini , and R. Greechie *Reasoning in Quantum Theory–Sharp and Unsharp Quantum Logic.* Kluwer Dordrecht, 2004.

[34] A. De Simone, D. Mundici, and M. Navara. A Cantor-Bernstein theorem for σ-complete MV-algebras. *Czechoslovak Mathematical Journal, Vol. 53 (2003), No. 2, 437-447.*

[35] Di Nola A., Dvurečenskij A., Hyčko M., Manara C., “Entropy on effect algebras with the Riesz decomposition property: Basic properties”, *Kibernetika*, 41, 2, 2005, pp. 143-160.

[36] Dichtl M.,“Astroids and pastings”, *Algebra Universalis*, 18, 1984, pp. 380-385.

[37] A. Dvurečenskij. Pseudo MV algebras are intervals in ℓ-groups. *Journal of The Australian Mathematical Society*, (72):427–445, 2001.

[38] A. Dvurečenskij. Lexicographic pseudo MV-algebras. *Journal of Applied Logic*, (13):825–841, 2015.

[39] A. Dvurečenskij and S. Pulmannová. *New Trends in Quantum Structures.* Mathematics and Its Applications. Kluwer Dordrecht, 2000.

[40] A. Dvurečenskij and T. Vetterlein. Pseudoeffect algebras I, basic properties. *International Journal of Theoretical Physics*, (40):685–701, 2001.

[41] A. Dvurečenskij and T. Vetterlein. Pseudoeffect algebras II, group representations. *International Journal of Theoretical Physics*, (40):703–726, 2001.

[42] A. Dvurečenskij and T. Vetterlein. Non-commutative algebras and quantum structures. *International Journal of Theoretical Physics*, 43(7/8):15–63, 2004.

[43] Eilenberg S., *Automata, Languages, and Machines*, Academic Press, 1974.

[44] Finch P. D., “On orthomodular posets”, *Bulletin of the Australian Mathematical Society*, 2, 1970, pp. 57-62.

[45] Font J.M., “Belnap’s Four-Valued Logic and De Morgan Lattices”, *Logic journal of IGPL*, 5, 1997, pp. 1-29.

[46] Foulis D. J., “A note on orthomodular lattices”, *Portugaliae Math.*, 21, 1962, pp. 65-72.

[47] D. J. Foulis and M. K. Bennett. Effect algebras and unsharp quantum logics. *Foundations of Physics*, (24):719–742, 1994.

[48] Foulis D.J., Greechie R.J., Rüttiman G.T., “Filters and Supports in Orthoalgebras”, *International Journal of Theoretical Physics*, 31, 5, 1992, pp. 789-807.

[49] Foulis D.J., Gudder S. “Observables, Calibration, and Effect Algebras”, *Foundation of Physics*, Vol. 31, No. 11, 2001.

[50] Foulis D.J., Pulmannov S., “Logical connectives on lattice effect algebras”, *Studia Logica*, 100 (6), 2012, pp. 1291-1315.

[51] D.J. Foulis, S. Pulmannová, and E. Vincenková. Lattice pseudoeffect algebras as double residuated structures. *Soft Computing*, (12):2479–2488, 2011.

[52] Freytes H., “The Cantor?Bernstein?Schro´oder theorem via universal algebra”, *Algebra Universalis*, 80 (17), 2019.

[53] Freytes. H. An algebraic version of the Cantor-Bernstein-Schröder theorem. *Czechoslovak Mathematical Journal, Vol. 54 (2004), Issue 3, 609-621.*

[54] Galatos N., Jipsen P., Kowalski T., Ono H., *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, Elsevier Science, 2007.

[55] G. Georgescu and A. Iorgulescu. Pseudo MV-algebras. *Multiple-Valued Logic*, (6):95–135, 2001.

[56] Gerla B., “Automata over MV-algebras [many-valued logic]”, *Proceedings of The International Symposium on Multiple-Valued Logic*, 2004

[57] Giuntini R., Greuling H. Toward a formal language for unsharp properties. *Foundations of Physics*, pages 931–945, 1989.

[58] Giuntini R., “Quantum MV algebras”, *Studia Logica*, 56,1996, pp. 393 - 417.

[59] Giuntini R., Ledda A., Paoli F., “A new view of effects in a Hilbert space”, *Studia Logica*, 104, 2016, pp. 1145-1177.

[60] Giuntini R., Ledda A., Paoli F., "On Some Properties of PBZ^*-lattices", *International Journal of Theoretical Physics*, 56, 2017, pp. 3895-3911.

[61] Givant S., Halmos P. *Introduction to Boolean Algebras*. Springer, Verlag, 2008.

[62] Golan J. S.,*The theory of semirings with applications in mathematics and theoretical computer science'*, Longman Scientific and Technical, 1992.

[63] Glass A. M. W. Partially ordered groups. World Scientific, Series in Algebra, Volume 7, 1999.

[64] K. Głazek. *A guide to the literature on semirings and their applications in mathematics and information sciences.* Springer, Verlag, 2002.

[65] Grätzer G. *Lattice Theory: Foundation*. Springer Basel, 1st edition. edition, 2011.

[66] Greechie R. J., "On the structure of orthomodular lattices satisfying the chain condition", *Journal of Combinatorial Theory*, 4, 1968, pp. 210-218.

[67] Greechie R. J., "Orthomodular lattices admitting no states", *Journal of Combinatorial Theory*, 10, 1971, pp. 119-132.

[68] Greechie R., Gudder S., "Is a quantum logic a logic?", *Helv. Phys. Acta*, 44, 1971, pp. 238 - 240.

[69] Greechie R., Gudder S., Reidel D., "Quantum logics" in Hooker C. (Ed.) *Contemporary Research in the Foundations and Philosophy of Quantum Theory*, Dordrect, 1973, pp. 143 - 173.

[70] Gudder S., "Inner product spaces", *Amer. Math. Monthly*, 81, 1974, pp. 29-36.

[71] Gudder S., "Correction to Inner product spaces", *Amer. Math. Monthly*, 82, 1975, pp. 251-252.

[72] Gudder S., "Second correction to Inner product spaces", *Amer. Math. Monthly*, 82, 1975, p. 818.

[73] Gumm H. P. and Ursini A. Ideals in universal algebras. *Algebra Universalis*, 19:45–54, 1984.

[74] Habil E., "Orthosummable orthoalgebras", *International Journal of Theoretical Physics*, 33, 10, 1994, pp. 1957-1984.

[75] Hamhalter J., Navara M., Pták P., "States on orthoalgebras", *International Journal of Theoretical Physics*, 34, 8, 1995, pp. 1439-1465.

[76] Hardegree G.M.,“ Material implication in orthomodular (and boolean) lattices”, *Notre Dame J. Form. Logic*, 22 (2), 1981, pp. 163 - 182.

[77] Harding J., “Decompositions in quantum logic”, *Transactions of the American Mathematical Society*, Volume 348, Number 5, 1996

[78] Holland S. H., “The current interest in orthomodular lattices”, in Hooker, C. H. ed. *The Logico-Algebraic Approach to Quantum Mechanics*, Springer Netherlands, 1979, pp. 437-496.

[79] Hooker, C. A. (Ed.), *The Logico-Algebraic Approach to Quantum Mechanics*. Vol. 1-2, Dordrecht: D. Reidel Pub. Co, 1975.

[80] Husimi K., “Studies on the foundations of quantum mechanics, I”, *Proceedings of the Physics-Mathematics Society of Japan*, 19, 1937, pp. 766-789.

[81] Jauch J., Piron C., “What is quantum logic?”, in Freund P., Goebel C., Nabir Y. (Eds.) *Quanta*, University of Chicago Press, 1970, pp. 842?848.

[82] Jipsen P., “From Semirings to Residuated Kleene Lattices”, *Studia Logica*, Volume 76, Issue 2, 2004, pp 291?303.

[83] Joshi V., “On forbidden configuration of 0-distributive lattices”, *Mathematica Bohemica*, 134, 1, 2009, pp. 59-65.

[84] A. Kadji, C. Lele, J.B. Nganou. A non-commutative generalization of Łukasiewicz rings. *Journal of Applied Logic*, (16):1–13, 2016.

[85] Kalman J. A., “Lattices with antitone involution”, *Trans. Amer. Math. Soc.*, 87, 1958, 485-491.

[86] Kalmbach G., “Orthomodular logic”, *Z. Math. Logic Grundlagen J. Math.*, 20, 1, 1974, pp. 395-406.

[87] G. Kalmbach. *Orthomodular lattices*, volume 18 of *London Mathematical Society Monographs*. Academic Press, 1983.

[88] Kharat V. S., Waphare B. N., Thakare N. K., “On forbidden configurations for strong posets”, *Algebra Universalis*, 51, 2004, pp. 111-124.

[89] Klukowski J., “On Boolean orthomodular posets”, *Demonstratio Math.*, 8, 1975, pp. 5-14.

[90] Ledda A., Paoli F., and Salibra A.. On semi-Boolean-like algebras. *Acta Univ. Palack. Olom.*, 52:101–120, 2013.

[91] Mackey G., *The Mathematical Foundations of Quantum Mechanics*, W. A. Bejamin Inc., New York, 1963.

[92] MacLaren M. D., "Atomic orthocomplemented lattices", *Pacific Journal of Mathematics*, 14, 1964, pp. 597-612.

[93] Matoušek M, Pták P.,"Orthocomplemented Posets with a Symmetric Difference", *Order*, 26, 1, 2009, pp. 1-21.

[94] Mayet R., "Varieties of orthomodular lattices related to states", *Algebra Universalis*, 20, 1985, pp. 368-396.

[95] Mayet-Ippolito A., "Generalized orthomodular posets", *Demonstratio Math.*, 24, 1991, pp. 263-274.

[96] D. Mundici. Interpretation of AFC*-algebras in Lukasiewicz sentential calculus. *Journal of Functional Analysis*, (65):15–63, 1986.

[97] Navara M., "State spaces of orthomodular structures", *Rend. Istit. Mat. Univ. Trieste*, 31, 1, 2000, pp. 143-201.

[98] Navara M., "Constructions of quantum structures", In: Gabbay, D., Lehmann, D., Engesser, K. (eds.) *Handbook of Quantum Logic*, Vol. 1, Elsevier, 2007, pp. 335-366.

[99] Navara M., "Orthoalgebras as Pastings of Boolean Algebras", *International Journal of Theoretical Physics*, 56, 2017, pp. 4126-4132.

[100] Navara M., Pták P., Rogalewicz V., "Enlargements of quantum logics", *Pacific Journal of Mathematics*, 135, 1988, pp. 361-369.

[101] Navara M., Rogalewicz V., "Construction of orthomodular lattices with given state spaces", *Demonstratio Mathematica*, 21, 1988, pp. 481-493.

[102] Navara M., Rogalewicz V.,"The Pasting Constructions for Orthomodular Posets", *Mathematische Nachrichten*, 154, 1991, pp. 157-168.

[103] Olson M. P., "The self-adjoint operators of a von Neumann algebra form a conditionally complete lattice", *Proceedings of the American Mathematical Society*, 28, 1971. 537?544.

[104] Pták P., Pulmannová S., *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht 1991.

[105] Pulmannová S., Riečanová Z., Vinceková E.,"Representation of Concrete Logics and Concrete Generalized Orthomodular Posets", *Reports on Mathematical Physics*, 73, 2, 2014, pp. 225-239

[106] J. Rachůnek. A non-commutative generalization of MV-algebras. *Czechoslovak Mathematical Journal*, (52):255–273, 2001.

[107] Randall C., Foulis D., "New Definitions and Theorems", *University of Massachusetts Mimeographed Notes*, Amherst, Massachusetts, Autumn 1979.

[108] Randall C., Foulis D., "What are quantum logics and what ought they to be?", in Beltrametti E., van Fraassen B. (Eds.) *Current Issues in Quantum Logic*, Ettore Majorana International Science Series 8, Plenum Press, New York, 1981, pp. 35-52.

[109] Riečanová Z., "Subalgebras, intervals and central elements of generalized effect algebras", *International Journal of Theoretical Physics*, 38, 1999, pp. 3209-3220.

[110] Riečanová Z., "MacNeille Completions of D-Posets and Effect Algebras", *International Journal of Theoretical Physics*, 39, 2000, pp. 859-869.

[111] Riečanová Z., "Sub-effect algebras and Boolean sub-effect algebras", *Soft Computing*, 5, 2001, p. 400-403.

[112] Riečanová Z., "Generalization of Blocks for D-Lattices and Lattice-Ordered Effect Algebras, *International Journal of Theoretical Physics*, Volume 39, Issue 2, 2000, pp 231?237.

[113] Rogalewicz V. "Any orthomodular poset is a pasting of Boolean algebras", *Commentationes Mathematicae Universitatis Carolinae*. 29 (3), 1988.

[114] A. Salibra, A. Ledda, F. Paoli, and T. Kowalski. Boolean-like algebras. *Algebra Universalis*, 69(2):113–138, 2013.

[115] Schindler C., *Decomposition of Measures on Orthologics*, PhD. Dissertation, University of Bern, 1986.

[116] Schultz F. W., "A characterization of state spaces of orthomodular lattices", *Journal of Combinatorial Theory*, 17, 1974, pp. 317-328.

[117] R. Sikorski. A generalization of a theorem of Banach and Cantor-Bernstein. *Colloq. Math., 1 (1948), 140-144 and 242.*

[118] Shewale R. S., Kharat V., "Forbidden configurations for distributive, modular and semidistributive posets", *Discrete Mathematics*, 339, 2016, pp. 3005-3016.

[119] Stroock D. W. *A Concise Introduction to the Theory of Integration*, 3rd ed., Birkhäuser, Basel, 1998.

[120] Tarski A. *Cardinal Algebras*. Oxford University Press, New York, 1949.

[121] Tkadlec J., "Properties of Boolean Orthoposets", *International Journal of Theoretical Physics*, 32, 10, 1993, pp. 1993-1997.

[122] Tkadlec J., "Partially additive measures and set representations of orthoposets", *Journal of Pure and Applied Algebra*, 86, 1993, pp. 79-94.

[123] Schroeder B., *Ordered Sets*, II Edition, Springer, Verlag, 2016.

[124] D. Vaggione. Varieties in which the pierce stalks are directly indecomposable. *Journal of Algebra*, 184:424–434, 1996.

[125] Vandiver H.S., "Note on a simple type of algebra in which cancellation law of addition does not hold" *Bull. Amer. Math. Soc.*, 1934, 40, pp. 914-920.

[126] P. Vitolo. Compatibility and central elements in pseudo-effect algebras. *Kybernetica*, 46(6):996–1008, 2010.

[127] H. Werner. A Mal'cev condition for admissible relations. *Algebra Universalis, 3 (1973), 263.*

[128] Xie Y, Li Y., Yang A., "The pasting constructions of lattice ordered effect algebras", *Information Science*, Volume 180, Issue 12, 2010, pp. 2476-2486.

[129] Young N. *An introduction to Hilbert space.* Cambridge University Press, 1988.