Stability of Open Multi-Agent Systems and Applications to Dynamic Consensus

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***Abstract*—In this technical note we consider a class of multi-agent network systems that we refer to as Open Multi-Agent Systems (OMAS): in these multi-agent systems, an indefinite number of agents may join or leave the network at any time. Focusing on discrete-time evolutions of scalar agents, we provide a novel theoretical framework to study the dynamical properties of OMAS. Specifically, we propose a suitable notion of stability and derive sufficient conditions for it. Our analysis regards the arrival/departure of agents as a disturbance; consistently, our stability conditions require the effect of arrivals/departures to be bounded (in a precise sense) and the OMAS to be contractive in the absence of arrivals/departures. In order to provide an example of application for this theory, we re-formulate the well-known Proportional Dynamic Consensus for Open Multi-Agent Systems and we study the stability properties of the resulting Open Proportional Dynamic Consensus algorithm.**

I. Introduction

A multi-agent system is a dynamical model for the behavior of a possibly large group of agents, e.g., robots, devices, sensors, oscillators etc., whose pattern of interactions due to sensing, communication or physical coupling is modeled by a graph that represents the network structure of the system. Most literature on multi-agent systems considers networks of fixed size, i.e., number of agents, and then considers several kinds of scenarios such as time-varying network topologies. In this paper we explicitly consider a more radical scenario of *open* multi-agent systems where *the set of agents is time-varying*, i.e., agents may join or leave the network at any time. This situation is common to numerous applications, including the Internet of Things, smart power grids [1], [2]; social networks [3], vehicle platooning and robotic teams [4].

Despite their ubiquity, open multi-agent systems have received surprisingly little attention in either control or in contiguous fields. Notions of open systems can be found in the computer science literature [5], [6], for instance when referring to software agents and the problem of evaluating reputation in open environments, but not as dynamical systems. Sometimes, dynamically evolving populations have also been considered in game theory [7], [8]. Instead, despite the abundance of works in multi-agent systems from the systems and control community, openness is rarely explicitly included in a rigorous analysis, but rather explored by simulations as in [9]. In multi-robot systems, where adaptivity to addition/removal of robots is crucial, some architectures accommodate for dynamics teams but offer no performance guarantees [4]. Indeed, openness implies some conceptual difficulties in adapting control-theoretic notions such as state or stability. For this reason, some authors have recently proposed to circumvent some of the mathematical hurdles by embedding the time-varying agent set in a time-invariant superset [10]. In a different perspective, others have aimed to describe the open multi-agent system through significant statistical properties: insightful results have been presented in [11], [12], where the authors study the problem of average-consensus by gossiping, and in [13], where the authors study a max-consensus problem. Recently, some researchers have been considering continuous approximations of large graphs [14], [15], [16], which may accommodate for open agent sets.

In comparison with this literature, the contribution of this paper is twofold, as it covers both theoretical results and concrete examples. As a theoretical contribution, we introduce an abstract framework for discrete-time open multi-agent systems with an unbounded number of agents: this framework is based upon proper definitions of state evolution, equilibria, and stability, and allows to establish useful stability criteria for a class of "contractive" open multi-agent systems. Instrumental to this development is extending the notion of (Euclidean) distance to apply to vectors that belong to different spaces and therefore have different length: this goal is achieved by our definition of *open distance* function.

In order to provide a concrete example that can be studied by our analysis tools, we extend the distributed control protocol of Proportional Dynamic Consensus to the open scenario, thereby defining the Open Proportional Dynamic Consensus algorithm. In the classical *dynamic consensus problem*, each of the nodes receives an input signal and is tasked to track the average of all inputs over the network. Our interest in dynamic consensus originates from its fundamental role in distributed control in general and specifically in the domain of smart grids. In the latter application, the object of the distributed estimation can be the time-varying average power consumption by the network. Thus, by considering the planned power consumption of each device as an external reference signal for each agent, a dynamic consensus algorithm can be used to estimate the time-varying average value of this potentially large set of reference signals. Since devices login and logout from the network without notice, the set of reference signals is, in general, time-varying.

The dynamic consensus problem has received significant attention, as demonstrated by the tutorial [17]. Since the early work in [18], a fundamental idea to render consensus protocols "dynamic" has been adding the derivative of each agents' own reference signal to a consensus filter that would thus track the time-varying average of the references. Several algorithms that exploit this mechanism have been proposed [9], [19], [20]: their main advantages are convergence speed and accuracy (which can be perfect for constant reference signals), while their main drawback is their lack of robustness with respect to errors in their initialization and, consequently, with respect to changes in the network composition. If the number of agents changes, these algorithms accumulate estimation errors that can severely deteriorate the estimation performance. Some algorithms, for instance those in [21] and [22], [23], [24], [25], have instead shown superior robustness properties that can be useful to allow for the addition or removal of agents, even though their analysis has been so far limited to networks of fixed size.

The recent conference publication [26] contains a preliminary account of our work and variations of some of the results presented here. In comparison to [26], the present note explicitly introduces the notion of contractive OMAS (which was implicitly used in [26]) and defines a normalized notion of distance, where the distance between two states is divided by the square root of the number of agents. This

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[‡] This work was supported by the Italian Ministry of Research and Education (MIUR) under call SIR 2014 with the grant "CoNetDomeSys" code RBSI14OF6H, by Regione Autonoma della Sardegna with the 2016/2017 "Visiting Professor Program", and by the French Science Foundation ANR via project HANDY, number ANR-18-CE40-0010.

apparently simple idea allows for a fair comparison between states of different cardinality and for deriving stability results that do not depend on the network size.

Structure of the paper: In Section II we introduce the framework of open multi-agent systems (OMAS) from a theoretical perspective and present an adaptation of two known distributed control protocols to this new framework. In Section III we provide theoretical tools for the stability analysis of discrete-time OMAS and apply the results to dynamic consensus in Section IV. In Section V we corroborate our results with numerical examples and, finally, in Section VI we discuss some concluding remarks.

II. Open dynamical systems

For all time $k \in \mathbb{Z}_{\geq 0}$, let $G_k = (V_k, E_k)$ be a time-varying directed graph with time-varying set of agents (also called nodes) $V_k \subset \mathbb{Z}$ and time-varying set of edges $E_k \subseteq (V_k \times V_k)$. Set V_k contains the labels corresponding to the agents that are active at time k. The cardinality of set V_k, that is, the number of agents that belong to the network at time k, is denoted as $n_k = |V_k|$. To avoid trivialities, we assume that $n_k > 0$ for all k. Furthermore, we do not consider a maximum number of agents known a priori, i.e., as time passes the number of agents can grow unbounded. Two agents v and w are said to be neighbors at time k if they share an edge at time k, i.e., $(v, w) \in E_k$. Let N_k^v be the set of neighbors of node v at time k, i.e., $N_k^v = \{w \in V_k : (v, w) \in E_k\}$. Let $\Delta_k^v = |N_k^v|$ denote the number of neighbors of agent v at time k (that is, its degree).

For each time k and each agent $v \in V_k$, we associate a scalar *state* variable $x_k^v \in \mathbb{R}$ and an *input* variable $u_k^v \in \mathbb{R}$. Note that these variables are defined only at time instants such that $v \in V_k$. More generally, in this paper we shall call *open sequence* any sequence $\{y_k : k \in \mathbb{Z}_{\geq 0}\}$ where $y_k \in \mathbb{R}^{V_k}$. When appropriate we omit to specify $k \in \mathbb{Z}_{\geq 0}$ and denote the open sequence simply by $\{y_k\}$.

With these ingredients we can define laws that describe how the open sequence $\{x_k : k \in \mathbb{Z}_{\geq 0}\}$ evolves. However, we will not be able in general to write x_{k+1} as a function solely of x_k: therefore, the evolution of x_k does not constitute a "closed" dynamical system. Instead, we shall take as given the open sequences $\{V_k : k \in \mathbb{Z}_{\geq 0}\}$ and $\{E_k : k \in \mathbb{Z}_{\geq 0}\}$, as well as the open sequence of inputs $\{u_k : k \in \mathbb{Z}_{\geq 0}\}$. Provided the consistency conditions that $E_k \subseteq (V_k \times V_k)$ and $u_k \in \mathbb{R}^{V_k}$ for all k, we shall define the evolution of the open sequence $\{x_k\}$ by a law

$$x_{k+1} = f(x_k, V_k, V_{k+1}, E_k, E_{k+1}, u_k, u_{k+1}). \tag{1}$$

Such an update rule should distinguish three kinds of nodes v, respectively belonging to the sets:

- $R_k = V_k \cap V_{k+1}$, i.e., *remaining* nodes that belong to both V_k and V_{k+1};
- $D_k = V_k \setminus V_{k+1}$, i.e., *departing* nodes that belong to V_k but not to V_{k+1};
- $A_k = V_{k+1} \setminus V_k$, i.e., *arriving* nodes that belong to V_{k+1} but not to V_k.

Since x_k must take values in $\mathbb{R}^{V_k}$ for all k, the components corresponding to D_k are simply left out from x_{k+1}. Instead, components in A_k need to be "initialized" according to some rule. Finally, for all $v \in R_k$ there shall be a causal evolution law in the form

$$x_{k+1}^v = \bar{f}^v(x_k, V_k, E_k, u_k). \tag{2}$$

We observe that if $V_k = V_{k+1}$, that is, the set of agents does not change, then we can write in vector form

$$x_{k+1} = \bar{f}(x_k, V_k, E_k, u_k).$$

For concreteness, we now describe an example of such a map, which we call Open Proportional Dynamic Consensus (OPDC).

Dynamics 1 (Open Proportional Dynamic Consensus (OPCD)) *Let $\varepsilon > 0$ and $\alpha \in (0, 0.5)$. At each time $k \in \mathbb{Z}_{\geq 0}$, each agent $v \in V_k$ measures a reference signal u_k^v and updates its state x_k^v as follows:*

$$x_{k+1}^v = \begin{cases} x_k^v - \alpha(x_k^v - u_k^v) - \varepsilon \sum_{w \in N_k^v} (x_k^v - x_k^w) & \text{if } v \in R_k \quad \text{(3a)} \\ u_{k+1}^v & \text{if } v \in A_k \quad \text{(3b)} \end{cases}$$

We observe that if $V_{k+1} = V_k$, i.e., the set of agents does not change, the OPDC reduces to what is called Proportional Dynamic Consensus. Namely, it reduces to the update (3a), which can be written in vector form as

$$\begin{aligned} x_{k+1} &= x_k - \alpha(x_k - u_k) - \varepsilon L_k x_k \\ &= \big((1-\alpha)I - \varepsilon L_k\big)x_k + \alpha u_k \\ &= P_k x_k + \alpha u_k \end{aligned} \tag{4}$$

where matrix $P_k = (1-\alpha)I - \varepsilon L_k$.

III. Stability of open multi-agent systems

In our general study of the stability of OMAS, we introduce our instruments in three steps: (i) we define suitable (sequences of) points that play the role of equilibria and define a notion of stability that is suitable for them; (ii) we extend the notion of distance to operate on vectors of unequal length; (iii) we give sufficient conditions for stability.

A. Points of interest and their stability

We now define the concept of *trajectory of points of interest*, which will take the role of the concept of equilibrium in the considered scenario of open multi-agent system.

Definition 3.1 (Trajectory of Points of Interest (TPI)) *Consider an open multi-agent system* (1). *Assume that for every $k \geq 0$, the equation*

$$y = \bar{f}(y, V_k, E_k, u_k)$$

has a unique solution y and denote that solution as x_k^e. Then, the open sequence $\{x_k^e : k \in \mathbb{Z}_{\geq 0}\}$ is called trajectory of points of interest *of the open multi-agent system.*

The existence of a TPI is guaranteed for some classes of OMAS.

Definition 3.2 (Contractive OMAS) *Consider the open multi-agent system in* (1)*. The OMAS is said to be* contractive *if there exists $\gamma \in [0, 1)$ such that for all $x, y \in \mathbb{R}^{V_k}$ and for all $k \geq 0$*

$$||\bar{f}(x, V_k, E_k, u_k) - \bar{f}(y, V_k, , E_k, u_k)|| \leq \gamma ||x - y||. \tag{5}$$

By Banach Fixed Point Theorem, every contractive OMAS has a TPI. As an example, consider system (3). Under Assumption 4.1, the OPDC is a contractive OMAS and

$$x_k^e = (I - P_k)^{-1} \alpha u_k = \Big(I + \frac{\varepsilon}{\alpha} L_k\Big)^{-1} u_k. \tag{6}$$

The next definition introduces a notion akin to a weak form of Lyapunov stability for open multi-agent systems.

Definition 3.3 (Open Stability of a Trajectory of Points of Interest) *Let x_k be the evolution of an open multi-agent system. A trajectory of points of interest x_k^e is said to be* open stable *if there exists a* stability radius *$R \geq 0$ with the following property: for every $\varepsilon > R$, there exists $\delta > 0$ such that if $\frac{1}{\sqrt{n_0}}||x_0 - x_0^e|| < \delta$, then $\frac{1}{\sqrt{n_k}}||x_k - x_k^e|| < \varepsilon$ for every $k \geq 0$.*

In this definition, distances are normalized by the number of agents. This normalization, which is trivial when the set of agents is invariant, is crucial here because it allows for a fair comparison of distances evaluated in spaces of different dimension. Therefore, it allows for making the stability radius independent of the number of agents. Without the normalization, for instance, a system where the number of agents increases with time (and thus the state norm increases as well) does not have a bounded stability radius, despite the fact that the distance between each new agent and its corresponding component in the trajectory of points of interest is bounded.

B. Open distance

We now define a so-called "open" distance which is used to evaluate the distance between two points with labeled components that belong to Euclidean spaces of different dimensions.

Definition 3.4 (Open distance) *Let V_1 and V_2 be two finite sets of node indices. Let $d : \mathbb{R}^{V_1} \times \mathbb{R}^{V_2} \to \mathbb{R}_{\geq 0}$ be defined as*

$$d(x, y) = \sqrt{\sum_{v \in V_1 \cap V_2} (x^v - y^v)^2 + \sum_{v \in V_1 \setminus V_2} (x^v)^2 + \sum_{v \in V_2 \setminus V_1} (y^v)^2}$$

for any $x \in \mathbb{R}^{V_1}$ and $y \in \mathbb{R}^{V_2}$.

In the particular case in which the two points have components with the same labels, i.e., the two OMAS have the same agents, then the open distance reduces to the Euclidean distance. Variants of Definition 3.4 can be given based on norms different from the 2-norm. The open distance in Definition 3.4 satisfies several natural properties, which we summarize in the next statement.

Proposition 3.5 (Properties of open distance functions) *Function $d(x, y)$ in Definition 3.4 is such that for any vectors x, y, and z of possibly different dimensions:*

1) $d(x, y) \geq 0$*;*
2) $d(x, y) = d(y, x)$*;*
3) *If $x = y$, then $d(x, y) = 0$;*
4) $d(x, z) \leq d(x, y) + d(y, z)$

Proof: Properties 1), 2), and 3) being evident, we now prove property 4), i.e., the triangle inequality. Consider sets V_x, V_y, V_z and define the union set $R = V_x \bigcup V_y \bigcup V_z$ and new vectors $\bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^R$ where their generic component is defined as $\bar{x}^v = x^v$ if $i \in V_x$ and $\bar{x}^v = 0$ otherwise. Since $\bar{x}$, $\bar{y}$, $\bar{z}$ belong to the same space $\mathbb{R}^R$, it follows that the triangle inequality

$$d(\bar{x}, \bar{y}) \leq d(\bar{x}, \bar{z}) + d(\bar{z}, \bar{y})$$

holds true since the open distance reduces to the ordinary Euclidean one. The result follows because one can readily verify that $d(\bar{x}, \bar{y}) = d(x, y)$. □

Note that the converse of the third implication (identity of indiscernibles) does not hold. Indeed, consider $x \in \mathbb{R}^{\{1,2\}}$ to be $x = [1, 0]$ and $y \in \mathbb{R}^{\{1\}}$ to be $[1]$. Then, $d(x, y) = 0$ despite the two vectors being different.

Having this open distance available, we can naturally use it on open sequences to give the following definition.

Definition 3.6 (Open sequence of bounded variation) *An open sequence $\{y_k\}$ of points $y_k \in \mathbb{R}^{V_k}$ is said to have bounded variation if there exists a constant $B \geq 0$ such that $d(y_{k+1}, y_k) \leq \sqrt{|V_{k+1}|}B$ for all $k \in \mathbb{Z}_{\geq 0}$.*

Note that this definition in fact normalizes the open distance by the number of components of the vectors, consistently with Definition 3.3. An important special case of sequence is the trajectory of points of interest of an open multi-agent system: we will say that *a TPI $\{x_k^e\}$ has bounded variation* if there exists B such that $d(x_{k+1}^e, x_k^e) \leq \sqrt{|V_{k+1}|}B$ for all $k \in \mathbb{Z}_{\geq 0}$.

C. Stability: sufficient conditions

In order to provide a sufficient condition to ensure stability in the above sense, we will need to combine assumptions on both the associated TPI and on its arrival process, that is, on the process by which agents join the OMAS during time. The latter assumption will take the following form.

Definition 3.7 (Bounded arrival process) *An arrival process is said to be* bounded *if there exists $H \geq 0$ such that each agent joins the OMAS with a state value such that*

$$\sqrt{\sum_{v \in A_k} \left(x_{k+1}^v - x_{k+1}^{e,v}\right)^2} \leq \sqrt{|V_{k+1}|}H \qquad \forall k \in \mathbb{Z}_{\geq 0}$$

where $x_k^{e,v}$ denotes the v-th component of x_k^e.

We are now ready to state our main stability result.

Theorem 3.8 (Stability of Open Multi-Agent Systems) *Consider an open multi-agent system as in* (1) *with state trajectory $\{x_k\}$. Assume that*

1) *the OMAS is contractive with parameter $\gamma \in [0, 1)$;*
2) *its TPI $\{x_k^e\}$ has bounded variation with constant B;*
3) *the arrival process is bounded with constant H;*
4) $|V_{k+1}| \geq \beta^2 |V_k|$ *for all k with $\beta > \gamma$.*

Then, the trajectory of points of interest is open stable *(Definition 3.3) with stability radius*

$$R = \frac{B + H}{1 - \frac{\gamma}{\beta}}$$

Proof: At each iteration k the agents first update their state, then some new agents may join and some may leave. By considering the open distance function, it holds

$$\begin{aligned} d(x_{k+1}, x_{k+1}^e) =& \sqrt{\sum_{v \in V_{k+1} \cap V_k} (x_{k+1}^v - x_{k+1}^{e,v})^2 + \sum_{v \in V_{k+1} \setminus V_k} (x_{k+1}^v - x_{k+1}^{e,v})^2} \\ \leq& \sqrt{\sum_{v \in V_{k+1} \cap V_k} (x_{k+1}^v - x_{k+1}^{e,v})^2} + \sqrt{\sum_{v \in V_{k+1} \setminus V_k} (x_{k+1}^v - x_{k+1}^{e,v})^2} \\ \leq& \sqrt{\sum_{v \in V_{k+1} \cap V_k} (x_{k+1}^v - x_k^{e,v})^2} + \sqrt{\sum_{v \in V_{k+1} \cap V_k} (x_{k+1}^{e,v} - x_k^{e,v})^2} \\ &+ \sqrt{\sum_{v \in V_{k+1} \setminus V_k} (x_{k+1}^v - x_{k+1}^{e,v})^2}. \end{aligned} \tag{7}$$

Since the OMAS is contractive, we observe that

$$\begin{aligned} \sqrt{\sum_{v \in V_{k+1} \cap V_k} (x_{k+1}^v - x_k^{e,v})^2} \leq& \gamma \sqrt{\sum_{v \in V_{k+1} \cap V_k} (x_k^v - x_k^{e,v})^2} \\ \leq& \gamma d(x_k, x_k^e). \end{aligned} \tag{8}$$

Now, note that the trajectory of points of interest is of bounded variation, implying

$$\sqrt{\sum_{v \in V_{k+1} \cap V_k} (x_{k+1}^{e,v} - x_k^{e,v})^2} \leq d(x_{k+1}^e, x_k^e) \leq \sqrt{|V_{k+1}|}B, \tag{9}$$

and that the arrival process is bounded as per Definition 3.7, implying

$$\sqrt{\sum_{v\in V_{k+1}\setminus V_k} (x^v_{k+1} - x^{e,v}_{k+1})} \leq \sqrt{|V_{k+1}|}H. \qquad (10)$$

Thus, by upper bounding the righthand side of (7) by (8)-(9)-(10), we can write

$$d(x_{k+1}, x^e_{k+1}) \leq \gamma d(x_k, x^e_k) + \sqrt{|V_{k+1}|}B + \sqrt{|V_{k+1}|}H. \qquad (11)$$

Let us now divide both sides of (11) by the square root of the cardinality of $|V_{k+1}|$

$$\frac{d(x_{k+1}, x^e_{k+1})}{\sqrt{|V_{k+1}|}} \leq \gamma \frac{d(x_k, x^e_k)}{\sqrt{|V_{k+1}|}} + B + H.$$

By assumption, $|V_{k+1}| \geq \beta^2 |V_k|$ where $\beta > \gamma$, thus we can write

$$\frac{d(x_{k+1}, x^e_{k+1})}{\sqrt{|V_{k+1}|}} \leq \frac{\gamma}{\beta} \frac{d(x_k, x^e_k)}{\sqrt{|V_k|}} + B + H.$$

By this inequality, the TPI is open stable with stability radius R. □

IV. Application: Open Proportional Dynamic Consensus

We now apply the result of Theorem 3.8 to study the convergence properties of the Open Proportional Dynamic Consensus protocol. Convergence will require that the arrival/departure process guarantees some good behavior of the sequence of graphs.

Assumption 4.1 (Graphs for OPDC) *Consider the open dynamics* (3) *and assume that for every* $k \in \mathbb{Z}_{\geq 0}$*:*

1) *graph* G_k *is undirected, that is,* $(u, v) \in E_k$ *if and only if* $(v, u) \in E_k$*;*
2) $\max_{v\in V_k} \Delta^v_k \leq \frac{1}{2\varepsilon}$ *for all* $v \in V_k$*;*
3) $\beta^2 \leq \frac{|V_{k+1}|}{|V_k|}$ *for some positive scalar* β*.*

This assumption simply requires that the graph be undirected, the degrees of the agents be not too large, and the number of agents in the OMAS do not decrease too rapidly. This assumption does not require any global property of the network such as connectivity. Even though Theorem 3.8 does not assume kind any graph structure, undirected graphs are a convenient assumption for the OPDC example. Indeed, the OPDC dynamics turns out to be a contraction on undirected graphs.

Theorem 4.2 (Stability of Open Proportional Dynamic Consensus) *Consider the Open Proportional Dynamic Consensus algorithm (OPDC) under Assumption 4.1 and assume that* $\beta > 1 - \alpha$*.*

Let λ_k *be the algebraic connectivity of the Laplacian matrix* L_k *corresponding to graph* G_k*, and let* $\underline{\lambda} \geq 0$ *be such that* [1] $\underline{\lambda} \leq \lambda_k$*.*

Let $\bar{u}_k = \frac{\mathbf{1}^T u_k}{n}\mathbf{1}$*,* $\hat{u}_k = u_k - \bar{u}_k$*. If the sequence of reference signals* $\{u_k\}$ *satisfies*

$$\|\hat{u}_k\|_\infty \leq \Pi, \quad \Pi \geq 0 \qquad (12)$$

and

$$d(\bar{u}_{k+1}, \bar{u}_k) \leq \sqrt{|V_{k+1}|}U, \quad U \geq 0, \qquad (13)$$

then the OPDC is open stable with stability radius

$$R = \frac{\left(1 + \frac{2}{1+\frac{\varepsilon}{\alpha}\underline{\lambda}} + \frac{1}{\beta}\frac{1}{1+\frac{\varepsilon}{\alpha}\underline{\lambda}}\right)\Pi + U}{1 - \frac{1-\alpha}{\beta}}$$

Proof: The proof is divided into four steps which lead to the application of Theorem 3.8.

[1]Constant $\underline{\lambda}$ is a uniform lower bound on the algebraic connectivities at all times. Such a constant always exists (since we allow it to be zero): when it is positive, it implies that all graphs are connected and that connectivity is uniformly good.

Step 1: As we have already observed right before (6), system (3) under Assumption 4.1 is a contractive OMAS with $\gamma = 1 - \alpha$.

Step 2: We claim that if the sequence of reference signals u_k satisfies (12) and (13), then the TPI is of bounded variation with constant

$$B = \frac{1}{1 + \frac{\varepsilon}{\alpha}\underline{\lambda}}\left(1 + \frac{1}{\beta}\right)\Pi + U.$$

We start the proof of this claim by exploiting the triangle inequality property of the open distance function

$$d(x^e_{k+1}, x^e_k) \leq d(x^e_{k+1}, \bar{u}_{k+1}) + d(x^e_k, \bar{u}_k) + d(\bar{u}_{k+1}, \bar{u}_k). \qquad (14)$$

The points of interest are

$$x^e_k = (I - P_k)^{-1}\alpha u_k = (\alpha I + \varepsilon L_k)^{-1}\alpha(\bar{u}_k + \hat{u}_k).$$

Since $(\alpha I + \varepsilon L_k)^{-1}\bar{u}_k = \bar{u}_k$ for any L_k we can write

$$x^e_k - \bar{u}_k = \alpha(\alpha I + \varepsilon L_k)^{-1}\hat{u}_k.$$

Now, since the eigenvector corresponding to the largest eigenvalue of $(\alpha I + \varepsilon L_k)^{-1}$ is $\mathbf{1}$ and $\mathbf{1}^T\hat{u}_k = 0$, it holds

$$\|x^e_k - \bar{u}_k\|_2 = \|\alpha(\alpha I + \varepsilon L_k)^{-1}\hat{u}_k\|_2 \leq \frac{1}{1 + \frac{\varepsilon\lambda_k}{\alpha}}\|\hat{u}_k\|_2, \qquad (15)$$

where $\left(1 + \frac{\varepsilon\lambda_k}{\alpha}\right)^{-1}$ is the second largest eigenvalue of $\alpha\left(\alpha I + \varepsilon L_k\right)^{-1}$. Then, the distance between the point of interest and the reference signal at time k satisfies

$$\begin{aligned} d(x^e_k, \bar{u}_k) = \|x^e_k - \bar{u}_k\|_2 &\leq \frac{\alpha}{\alpha + \varepsilon\lambda_k}\|\hat{u}_k\|_2 \\ &\leq \frac{1}{1 + \frac{\varepsilon}{\alpha}\lambda_k}\sqrt{|V_k|}\|\hat{u}_k\|_\infty. \end{aligned}$$

By noting that $\frac{|V_{k+1}|}{\beta^2} \geq |V_k|$, $\|\hat{u}_k\|_\infty \leq \Pi$ and $d(\bar{u}_{k+1}, \bar{u}_k) \leq \sqrt{|V_{k+1}|}U$ for some $U \geq 0$, it follows from (14) that

$$\begin{aligned} d(x^e_{k+1}, x^e_k) &\leq \sqrt{|V_{k+1}|}\left(\frac{1}{1 + \frac{\varepsilon}{\alpha}\underline{\lambda}}\left(1 + \frac{1}{\beta}\right)\Pi + U\right) \\ &= \sqrt{|V_{k+1}|}B. \end{aligned}$$

Step 3: The arrival process of the OPCD is bounded according to Definition 3.7 with

$$H = \left(1 + \frac{1}{1 + \frac{\varepsilon\underline{\lambda}}{\alpha}}\right)\Pi.$$

In the OPDC algorithm new agents join with a state value equal to their reference signal. Since from (15) at time $k+1$,

$$\|x^e_{k+1} - \bar{u}_{k+1}\|_2 \leq \frac{\alpha}{\alpha + \varepsilon\lambda_k}\|\hat{u}_{k+1}\|_2,$$

and $|u^v_{k+1} - \bar{u}_{k+1}| \leq \Pi$ by assumption, by recalling that $A_{k+1} = V_{k+1}\setminus V_k$, it holds

$$\begin{aligned} \sqrt{\sum_{v\in A_{k+1}} \left(x^v_{k+1} - x^{e,v}_{k+1}\right)^2} &\leq \sqrt{|V_{k+1}\setminus V_k|}(1 + \frac{\alpha}{\alpha + \varepsilon\lambda_{k+1}})\Pi \\ &\leq \sqrt{|V_{k+1}|}(1 + \frac{1}{1 + \frac{\varepsilon\underline{\lambda}}{\alpha}})\Pi \end{aligned}$$

Thus, the arrival process of the OPCD algorithm is bounded according to Definition 3.3 with $H = \left(1 + \frac{1}{1+\frac{\varepsilon\underline{\lambda}}{\alpha}}\right)\Pi$.

Step 4: By Theorem 3.8, the TPI of the OPCD algorithm is open stable with stability radius $R = \frac{B+H}{1-\frac{\gamma}{\beta}}$. □

When $\underline{\lambda} = 0$, that is, the arrival process does not guarantee a uniform connectivity, then the stability radius in Theorem 4.2 takes the simpler form

$$R_0 = \frac{(3\beta + 1)\Pi + \beta U}{\alpha + \beta - 1}.$$

This bound also holds true in the extreme case where the network is always completely disconnected. Clearly, for positive $\underline{\lambda}$ we have that $R < R_0$. This simple remark is consistent with the intuition that better connectivity is beneficial to OPDC, just like it is beneficial in the non-open case.

Both expressions of R and R_0 make evident that if the inputs have large variations, that is if U is large, then the stability radius can become arbitrarily large. This behavior is not an artifact of our analysis but is a drawback that is inherited from the original Proportional Dynamic Consensus algorithm [17].

Finally, we find useful to further discuss some consequences of the demands of Assumption 4.1:

1) *Undirected networks.* Our general Open Multi-Agent framework is defined for the more general case of directed graphs. However, the application to Dynamic Consensus is done for undirected graphs, because this assumption guarantees the OPDC to be a contractive OMAS.
2) *Bounded degrees.* Limiting the degrees of the agents may be detrimental to the connectivity and therefore increase the stability radius.
3) *Bounded rate of departure.* Due to this assumption, if the rate of departure is large, then β cannot be chosen too large, which makes the radius larger.

All these conditions are stated for all times: this uniformity makes the result of Theorem 4.2 rather conservative, as will be apparent in the simulations that we propose in the next section.

V. Numerical examples

In this section we show a numerical example of the OPCD algorithm. Our simulations are performed as follows. We considered as tuning parameters $\varepsilon = 0.01$, $\alpha = 0.1$. The simulated scenario consists of a network of 200 agents at the initial time, with initial values chosen uniformly at random in the interval $[-5000, 5000]$. The initial graph is generated as an Erdős-Rényi graph with edge probability $p = 0.05$. At each iteration, with probability 0.1 one random agent leaves and with probability 0.7 one new agent arrives: the arriving agent creates random edges with probability 0.05 with all other agents. Input reference signals are constant and randomly sampled in the interval $[0, 1]$ when agents join the network.

After describing our simulation setup, we present one typical realization. To begin with, in Figure 1 we show the evolution of the number of agents. By our choice of the arrival and departure probabilities, the set of network agents is constantly renewed and the plot shows its clear trend to increase size (since the arrival probability is larger than the departure probability). Figure 1 also shows the normalized open distance between the current point of interest and the average of the input reference signals given to the agents. The value of the latter quantity depends on the OPCD parameters, in particular it could be reduced by decreasing the parameter α.

We then proceed to showcase the stability properties of the OPDC. To this purpose, Figure 2 shows the evolution of the normalized open distance $|V_k|^{-1/2} d(x_k, x_k^e)$, that is, the distance of the state x_k of the network from the current point of interest x_k^e. This distance remains bounded after decreasing during a transient phase that leads to a state of approximate consensus. This behavior is consistent with the stability analysis given in Theorem 4.2. The transient lasts approximately 75 time steps and the network size grows during this time by about 25% of the initial size.

Even though our analysis makes deterministic assumptions and therefore does not allow a priori conclusions on this randomized evolution, we can a posteriori verify that the simulated arrival/departure process has satisfied the assumptions of Theorem 4.2 with minimum

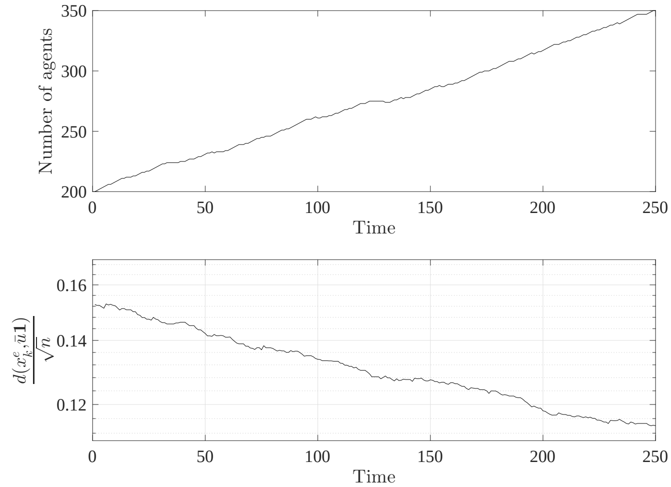

Fig. 1. Evolution of the number of agents $|V_k|$ and of the normalized open distance between average reference input and point of interest $|V_k|^{-1/2} d(x_k^e, \bar{u}\mathbf{1})$.

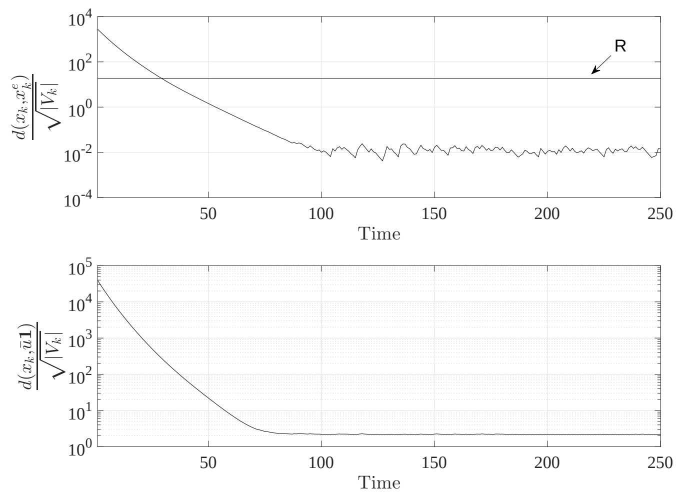

Fig. 2. Evolution of the normalized open distances between network state and point of interest $|V_k|^{-1/2} d(x_k, x_k^e)$ and between network state and average reference input $|V_k|^{-1/2} d(x_k, \bar{u}\mathbf{1})$).

algebraic connectivity $\underline{\lambda} = 0.9037$, $|V_{k+1}| \geq \beta^2 |V_k|$ with $\beta = 0.9975$, largest degree equal to 20, $\Pi = 0.5139$, and $U = 0.0001785$. Therefore, the result implies a stability radius equal to $R = 17.375$, which appears to be a conservative estimate according to Figure 2.

Figure 2 also shows the evolution of the normalized open distance $|V_k|^{-1/2} d(x_k, \bar{u}\mathbf{1})$, which represents the distance between the network state and the average of the input reference signals. Estimating the latter quantity is the objective of the OPCD protocol. This estimation error can be seen to converge to a bounded value despite the open nature of the multi-agent system.

For a useful comparison, in Figure 3 we show a simulation of the PDC algorithm with a *fixed set of agents* ($n = 200$) and constant reference signals. It can be seen that the network state converges to its equilibrium point (up to machine precision), in contrast with the finite error in Figure 2. At the same time, the network state converges to a steady-state which has a bounded error with respect to the average reference signals: in comparison with Figure 2, the Open PDC reaches a similar steady-state error (albeit at slower pace) as its classical PDC counterpart.

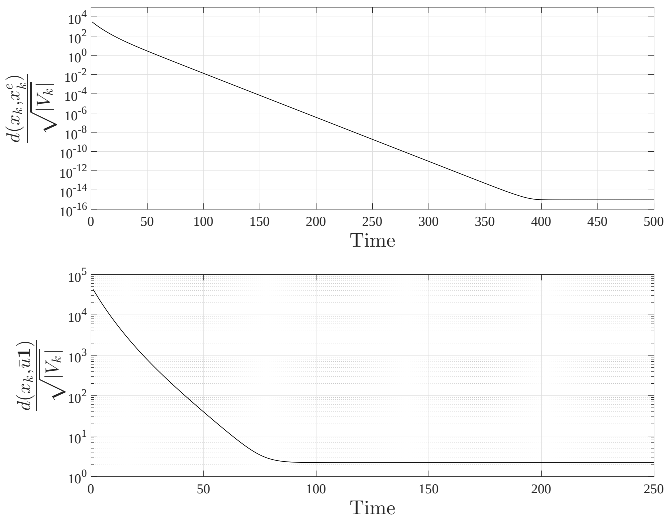

Fig. 3. Evolution of the normalized open distance between network state and average reference input $|V_k|^{-1/2} d(x_k, \bar{u}\mathbf{1})$ and between network state and point of interest $|V_k|^{-1/2} d(x_k, x_k^e)$, in the case of time-invariant number of agents $n = 200$.

VI. Conclusion

In this paper we proposed a theoretical framework for stability analysis of discrete-time open multi-agent systems. Standard system-theoretic tools do not apply directly to OMAS, because of the evolution of their state space. For this reason, we had to propose several new definitions, including suitable definitions of state evolution and of stability. The proposed definition of stability has two features: (1) it normalizes the distance from the origin by the number of agents; and (2) it disregards what happens within a certain distance from the origin (we refer to this distance as to the stability radius). In order to study the evolution and the stability of OMAS, it is necessary to compare states that belong to different spaces. To this purpose, we defined the open distance function and used it to establish criteria for stability in the proposed open scenario. In particular, we showed that multi-agent systems whose dynamics (up to arrivals and departures of agents) can be defined by contraction maps are stable according to our definition and their stability radius depends upon the properties of the arrival and departure mechanisms in the network. Furthermore, we applied our results to an adaptation to OMAS of the proportional dynamic consensus protocol. Future work should pursue two complementary direction: building up a more general and comprehensive theory, thereby also including the possibility to have endogenous sequences of graphs, and investigating other classes of open-multi agent systems in order to propose novel open distributed coordination algorithms.

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