Technium Vol. 2, Issue 5 pp.11-42 (2020)
ISSN: 2668-778X
www.techniumscience.com

The explicit closure for the ultrarelativistic limit of Extended Thermodynamics of Rarefied Polyatomic Gas

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Abstract. In a recent paper the ultra-relativistic limit of a recent theory proposed by Pennisi and Ruggeri for polyatomic relativistic gas has been considered. This was important to check the general article. In particular, the explicitly expression of the characteristic velocities of the hyperbolic system were found for every value of the parameter a measuring "how much" the gas is polyatomic. This result was achieved in terms of the components of the main field as independent variables and without writing it in terms of the physical variables. Here the closure of the field equations is considered in terms of these physical variables and their ultrarelatistic limit is obtained.

Keywords. Extended thermodynamics, Relativistic fluids.

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1 Introduction

In the article [1], Pennisi and Ruggeri presented a casual relativistic theory for polyatomic rarefied gas. They proposed the following field equations

$$\partial_\alpha V^\alpha = 0 \quad , \quad \partial_\alpha T^{\alpha\beta} = 0 \quad , \quad \partial_\alpha A^{\alpha<\beta\gamma>} = I^{<\beta\gamma>} \, , \tag{1}$$

for the determination of the independent variables

$$\begin{aligned} V^\alpha(x^\mu) \quad &- \quad \text{particle, particle flux vector} \, , \\ T^{\alpha\beta}(x^\mu) \quad &- \quad \text{energy momentum tensor} \, . \end{aligned} \tag{2}$$

It is assumed that $T^{\alpha\beta}$, $A^{\alpha\beta\gamma}$ and $I^{\beta\gamma}$ are completely symmetric tensors and $< \cdots >$ denotes the traceless part of a tensor.
In the subsequent article [2] they have shown how the closure can be found in terms of a 4-potential h'^α and reads

$$V^\alpha = \frac{\partial \, h'^\alpha}{\partial \, \lambda} \, , \quad T^{\alpha\beta} = \frac{\partial \, h'^\alpha}{\partial \, \lambda_\beta} \, , \quad , \quad A^{\alpha\beta\gamma} = \frac{\partial \, h'^\alpha}{\partial \, \Sigma_{\beta\gamma}} \, . \tag{3}$$

After that, they have found the closure in terms of 3 scalar functions $\tilde{h}_0$, $\tilde{h}_2$, $\tilde{h}_5$; by comparing eqs. (9) and (11) of [2], we see that their expressions are

$$\begin{aligned} \tilde{h}_0 &= \frac{1}{(mc^2)^{a+1}\Gamma(a+1)} \int_0^{+\infty} J^*_{2,1} \mathcal{I}^a \, d\,\mathcal{I} \, , \\ \tilde{h}_2 &= \frac{1}{3} \frac{1}{(mc^2)^{a+1}\Gamma(a+1)} \int_0^{+\infty} J^*_{4,1} \left(1 + \frac{2\mathcal{I}}{mc^2}\right) \mathcal{I}^a \, d\,\mathcal{I} \, , \\ \tilde{h}_5 &= \frac{1}{(mc^2)^{a+1}\Gamma(a+1)} \int_0^{+\infty} J^*_{6,1} \left(1 + \frac{2\mathcal{I}}{mc^2}\right)^2 \mathcal{I}^a \, d\,\mathcal{I} \, , \\ &\text{where} \quad J_{m,n}(\gamma) = \int_0^{+\infty} \sinh^m s \cosh^n s \, d\, s \, , \quad \text{i.e., the Bessel function} \quad , \\ &J^*_{m,n} \quad \text{is} \quad J_{m,n}(\gamma) \quad \text{with} \quad \gamma \quad \text{replaced by} \quad \gamma^* = \gamma \left(1 + \frac{\mathcal{I}}{mc^2}\right) \, . \end{aligned} \tag{4}$$

Moreover, m is the particle mass, c the light speed, $a = -1 + f^i/2$ with f^i the internal degrees of freedom $f^i \geq 0$ due to the internal motion (rotation and vibration; for monatomic gases $f^i = 0$ and $a = -1$.), $\mathcal{I}$ is the internal energy and $\gamma = \frac{mc^2}{k_B T}$ with k_B the Boltzmann constant and T the absolute temperature.

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The fields $(V^\alpha, T^{\alpha\beta})$ are expressed in terms of the usual physical variables through the decomposition:

$$V^\alpha = nmU^\alpha, \qquad T^{\alpha\beta} = t^{<\alpha\beta>_3} + (p+\pi)h^{\alpha\beta} + \frac{2}{c^2}U^{(\alpha}q^{\beta)} + \frac{e}{c^2}U^\alpha U^\beta, \tag{5}$$

where U^α is the four-velocity ($U^\alpha U_\alpha = c^2$ because the metric tensor is chosen as $g^{\alpha\beta} = diag(1\,,\,-1\,,\,-1\,,\,-1)$), n is the number density, p is the pressure, $h^{\alpha\beta}$ is the projector tensor: $h^{\alpha\beta} = -g^{\alpha\beta} + \frac{1}{c^2}U^\alpha U^\beta$, e is the energy, π is the dynamical pressure, the symbol $< \cdots >_3$ denotes the 3-dimensional traceless part of a tensor, i.e., $t^{<\alpha\beta>_3} = T^{\mu\nu}\left(h^\alpha_\mu h^\beta_\nu - \frac{1}{3}h^{\alpha\beta}h_{\mu\nu}\right)$ is the viscous deviatoric stress, and $q^\alpha = -h^\alpha_\mu U_\nu T^{\mu\nu}$is the heat flux. In this case we can take n, T, U^α, $t^{<\alpha\beta>_3}$, π, q^α as independent variables.
The consequent expression of p, e and $A^{\alpha\beta\gamma}$ are:

$$p = \frac{m\,n\,c^2}{\gamma} \quad , \quad e = -\,\frac{\gamma}{\tilde{h}_0}\,\frac{\partial \tilde{h}_0}{\partial\gamma}\,\frac{m\,n\,c^2}{\gamma}\,, \tag{6}$$

$$A^{\alpha\beta\gamma} = -\,\frac{2\,\tilde{h}_2 + \gamma\,\frac{\partial \tilde{h}_2}{\partial\gamma}}{\tilde{h}_0}\,m\,n\,U^\alpha U^\beta U^\gamma + 3\,\frac{\tilde{h}_2}{\tilde{h}_0}\,m\,n\,c^2\,h^{(\alpha\beta}U^{\gamma)} - \tag{7}$$

$$\frac{3}{c^2}\,\frac{\tilde{N}_1}{D_1}\,\pi\,U^\alpha U^\beta U^\gamma - 3\,\frac{\tilde{N}_{11}}{D_1}\,\pi\,h^{(\alpha\beta}U^{\gamma)} +$$

$$+\,\frac{3}{c^2}\,\frac{\tilde{N}_3}{D_2}\,q^{(\alpha}U^\beta U^{\gamma)} + \frac{3}{5}\,\frac{\tilde{N}_{31}}{D_2}\,q^{(\alpha}h^{\beta\gamma)} + 3\tilde{C}_5\,t^{(<\alpha\beta>_3}U^{\gamma)}\,,$$

with

$$D_1 = \begin{vmatrix} -\tilde{h}_0 & \frac{\partial \tilde{h}_0}{\partial\gamma} & \tilde{h}_2 + \gamma\frac{\partial \tilde{h}_2}{\partial\gamma} \\ \frac{\partial \tilde{h}_0}{\partial\gamma} & -\frac{\partial^2 \tilde{h}_0}{\partial\gamma^2} & -\frac{\partial}{\partial\gamma}\left(\tilde{h}_2 + \gamma\frac{\partial \tilde{h}_2}{\partial\gamma}\right) \\ -\,\frac{\tilde{h}_0}{\gamma} & -\frac{1}{\gamma^2}\left(\tilde{h}_0 - \gamma\,\frac{\partial \tilde{h}_0}{\partial\gamma}\right) & \frac{\partial \tilde{h}_2}{\partial\gamma} - \frac{5}{3}\,\frac{\tilde{h}_2}{\gamma} \end{vmatrix}\,,$$

$$\tilde{N}_1 = \begin{vmatrix} -\tilde{h}_0 & \frac{\partial \tilde{h}_0}{\partial \gamma} & \tilde{h}_2 + \gamma \frac{\partial \tilde{h}_2}{\partial \gamma} \\ \frac{\partial \tilde{h}_0}{\partial \gamma} & -\frac{\partial^2 \tilde{h}_0}{\partial \gamma^2} & -\frac{\partial}{\partial \gamma}\left(\tilde{h}_2 + \gamma \frac{\partial \tilde{h}_2}{\partial \gamma}\right) \\ -\frac{1}{3}\left(2\,\tilde{h}_2 + \gamma \frac{\partial \tilde{h}_2}{\partial \gamma}\right) & \frac{\partial \tilde{h}_2}{\partial \gamma} + \frac{1}{3}\gamma \frac{\partial^2 \tilde{h}_2}{\partial \gamma^2} & \frac{1}{45}\left(6\,\tilde{h}_5 + 6\,\gamma \frac{\partial \tilde{h}_5}{\partial \gamma} + \gamma^2 \frac{\partial^2 \tilde{h}_5}{\partial \gamma^2}\right) \end{vmatrix},$$

$$\tilde{N}_{11} = \begin{vmatrix} -\tilde{h}_0 & \frac{\partial \tilde{h}_0}{\partial \gamma} & \tilde{h}_2 + \gamma \frac{\partial \tilde{h}_2}{\partial \gamma} \\ \frac{\partial \tilde{h}_0}{\partial \gamma} & -\frac{\partial^2 \tilde{h}_0}{\partial \gamma^2} & -\frac{\partial}{\partial \gamma}\left(\tilde{h}_2 + \gamma \frac{\partial \tilde{h}_2}{\partial \gamma}\right) \\ \tilde{h}_2 & -\frac{\partial \tilde{h}_2}{\partial \gamma} & -\frac{1}{45}\left(\tilde{h}_5 + 3\,\gamma \frac{\partial \tilde{h}_5}{\partial \gamma}\right) \end{vmatrix},$$

$$D_2 = \begin{vmatrix} \frac{\tilde{h}_0}{\gamma} & 2\,\tilde{h}_2 \\ \frac{\partial\left(\frac{\tilde{h}_0}{\gamma}\right)}{\partial \gamma} & 2\,\frac{\partial \tilde{h}_2}{\partial \gamma} \end{vmatrix},$$

$$\tilde{N}_3 = \begin{vmatrix} \frac{\tilde{h}_0}{\gamma} & 2\,\tilde{h}_2 \\ \frac{\partial \tilde{h}_2}{\partial \gamma} & \frac{2}{15}\left(2\,\tilde{h}_5 + \gamma \frac{\partial \tilde{h}_5}{\partial \gamma}\right) \end{vmatrix}, \quad \tilde{N}_{31} = \begin{vmatrix} \frac{\tilde{h}_0}{\gamma} & 2\,\tilde{h}_2 \\ -5\,\frac{\tilde{h}_2}{\gamma} & -\frac{2}{3}\,\tilde{h}_5 \end{vmatrix}, \quad \tilde{C}_5 = \frac{1}{15}\,\frac{\tilde{h}_5}{\tilde{h}_2}\,\gamma\,.$$

We want now to obtain the ultrarelatistic limit of the above coefficients and, consequently, the ultrarelativistic expressions of the balance equations. This particular is missing in [2], even if there is something in the artcicle [3] but limited to the case of only Euler Equations. To this end, some properties are necessary and we have reported and proved them in [4] in order not to excessively lengthen the present article. However, we observe that the ultrarelativistic limit is not a simple limit for γ going to zero, otherwise the independent variable γ disappears when it tends to zero and we have only 13 independent variables instead of 14. Instead of this, we do the following considerations: If a given function $F(\gamma)$ can be written as $F(\gamma) = F_1(\gamma) + F_2(\gamma)$ with $\lim_{\gamma \to 0} \frac{F_2(\gamma)}{F_1(\gamma)} = 0$, we say that $F_1(\gamma)$ is the leading term and substitute $F(\gamma)$ with $F_1(\gamma)$. In other words, we neglect terms which, in the limit for γ going to zero, are of less order with respect to the leading term. This will have the consequence that the balance equations

(depending on the parameter a) will have some points of discontinuity with respect to this parameter. The starting equations have the continuity property with respect to a, but passing through some values of a the leading term changes. This is not a problem because, when you make an application to a particular polyatomic gas, you know at what value of a it corresponds and then you use the equations pertinent to this value.
By using the results in [4], w can now find the leading terms of our closure and we do this in the next section, distinguihing some subcases according to different values of a. The results are given by eqs. (16) for $a < 0$, $a \neq -\frac{1}{2}$, (18) for $a = -\frac{1}{2}$, (20) for $a = 0$, (24) for $0 < a < 2$, (37) for $a = 2$, (49) for $2 < a < 3$, (61) for $a = 3$, (71) for $3 < a < 4$, (82) for $a = 4$, (92) for $a > 4$,

2 The closure of the field equations in the ultrarelativistic limit

Firstly, let us introduce the following notation

$$D_1 = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix} \quad , \quad \tilde{N}_1 = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ n_{31} & n_{32} & n_{33} \end{vmatrix} \tag{8}$$

$$\tilde{N}_{11} = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix} \quad , \quad D_2 = \begin{vmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{vmatrix} \quad ,$$

$$\tilde{N}_3 = \begin{vmatrix} e_{11} & e_{12} \\ f_{21} & f_{22} \end{vmatrix} \quad , \quad \tilde{N}_{31} = \begin{vmatrix} e_{11} & e_{12} \\ g_{21} & g_{22} \end{vmatrix} \quad , \tag{9}$$

with obvious meaning of the symbols. After that, let us distinguish some subcases.

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www.techniumscience.com

2.1 The case $a < 0$, $a \neq -\frac{1}{2}$

From eqs. (23), (31) and $(12)_1$ of [4] we obtain

$$
\begin{aligned}
&\tilde{h}_0 = \gamma^{-3}\,\Gamma(2-a)\; - \;\frac{1}{2}\,\gamma^{-1}\,\Gamma(-a)\,,\\
&\tilde{h}_2 = \frac{1}{3}\,\gamma^{-5}\,\Gamma(3-a)\,(a+5)\; - \;\frac{1}{2}\,\gamma^{-3}\,\Gamma(1-a)\,(a+3)\,,\\
&\tilde{h}_5 = \gamma^{-7}\,\Gamma(4-a)\,(a+4)(a+11)\,.
\end{aligned}
\tag{10}
$$

We recall that equilibrium is defined as the state where $\pi = 0$, $q^{\alpha} = 0$. $t^{<\alpha\beta>_3} = 0$. So, at equilibrium, eqs. (6) and (7) become

$$
\begin{aligned}
&p = \frac{m\,n\,c^2}{\gamma} \quad , \quad e = 3\,\frac{m\,n\,c^2}{\gamma}\,,\\
&A_E^{\alpha\beta\gamma} = \gamma^{-2}\,(2-a)(a+5)\,m\,n\,\left[U^{\alpha}U^{\beta}U^{\gamma} + c^2\,h^{(\alpha\beta}U^{\gamma)}\right]\,.
\end{aligned}
\tag{11}
$$

To find the leading terms of the non-equilibrium closure, we use these expressions and the follow scheme

$$
\begin{array}{cccc}
 & \gamma^3 & \gamma^4 & \gamma^5 \\
D_1\gamma^{12} = &
\left|\begin{array}{ccc}
d_{11} & d_{12} & d_{13} \\
d_{21} & d_{22} & d_{23} \\
d_{31} - \frac{1}{3}d_{21} & d_{32} - \frac{1}{3}d_{22} & d_{33} - \frac{1}{3}d_{23}
\end{array}\right|
\begin{array}{c} 1 \\ \gamma \\ \gamma^{-1} \end{array} & , &
\end{array}
\tag{12}
$$

where the notation indicates that we have multiplied the first column by γ^3, the second one by γ^4, the third ones by γ^5; after that, we have multiplied the first line by 1, the second line by γ and the third line by γ^{-1}; so we find

$$
\lim_{\gamma\to 0} D_1\gamma^{12} = -\,\frac{8}{9}\,(2a+1)(2a+5)\;\left[\Gamma(2-a)\right]^2\,\Gamma(-a)\,.
\tag{13}
$$

With a similar procedure we find

$$\tilde{N}_1\gamma^{15} = \begin{array}{c} \begin{matrix} \gamma^3 & \gamma^4 & \gamma^5 \end{matrix} \\ \left|\begin{matrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ n_{31} & n_{32} & n_{33} \end{matrix}\right| \begin{matrix} 1 \\ \gamma \\ \gamma^2 \end{matrix} \end{array},$$

$$\tilde{N}_{11}\gamma^{15} = \begin{array}{c} \begin{matrix} \gamma^3 & \gamma^4 & \gamma^5 \end{matrix} \\ \left|\begin{matrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ m_{31} & m_{32} & m_{33} \end{matrix}\right| \begin{matrix} 1 \\ \gamma \\ \gamma^2 \end{matrix} \end{array},$$

from which we obtain

$$\lim_{\gamma\to 0} \tilde{N}_1\gamma^{15} = \lim_{\gamma\to 0} \tilde{N}_{11}\gamma^{15} = \frac{4}{9}\,[\Gamma(2-a)]^2\,\Gamma(3-a)\;2\,(2a^3+10a^2+19a+23)\,, \tag{14}$$

and, consequently,

$$\lim_{\gamma\to 0} \frac{\tilde{N}_1}{D_1}\gamma^3 = \lim_{\gamma\to 0} \frac{\tilde{N}_{11}}{D_1}\gamma^3 = a\,(2-a)\,(1-a)\,\frac{2a^3+10a^2+19a+23}{(2a+1)(2a+5)}\,. \tag{15}$$

Going on with the same procedure, we find

$$D_2\gamma^{10} = \begin{array}{c} \begin{matrix} \gamma^4 & \gamma^5 \end{matrix} \\ \left|\begin{matrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{matrix}\right| \begin{matrix} 1 \\ \gamma \end{matrix} \end{array}, \quad \tilde{N}_3\gamma^{11} = \begin{array}{c} \begin{matrix} \gamma^4 & \gamma^5 \end{matrix} \\ \left|\begin{matrix} e_{11} & e_{12} \\ f_{21} & f_{22} \end{matrix}\right| \begin{matrix} 1 \\ \gamma^2 \end{matrix} \end{array}, \quad \tilde{N}_{31}\gamma^{11} = \begin{array}{c} \begin{matrix} \gamma^4 & \gamma^5 \end{matrix} \\ \left|\begin{matrix} e_{11} & e_{12} \\ g_{21} & g_{22} \end{matrix}\right| \begin{matrix} 1 \\ \gamma^2 \end{matrix} \end{array},$$

From these results we obtain

$$\lim_{\gamma\to 0} D_2\,\gamma^{10} = -\,\frac{2}{3}\,\Gamma(2-a)\,\Gamma(3-a)\,(a+5)\,,$$

$$\lim_{\gamma\to 0} \tilde{N}_3\gamma^{11} = \lim_{\gamma\to 0} \tilde{N}_{31}\gamma^{11} = -\,\frac{4}{9}\,\Gamma(2-a)\,\Gamma(3-a)\,(a^3+2a^2+14a+73)\,,$$

and, consequently,

$$\lim_{\gamma\to 0}\frac{\tilde{N}_3}{D_2}\gamma = \lim_{\gamma\to 0}\frac{\tilde{N}_{31}}{D_2}\gamma = \frac{2}{3}\,\frac{a^3+2a^2+14a+73}{a+5}\,.$$

Finally,

$$\lim_{\gamma\to 0}\tilde{C}_5\,\gamma = (3-a)\frac{(a+4)(a+11)}{5\,(a+5)}\,.$$

We note that, in the monoatomic limit $a=-1$, the present results are the same which have been found in [?]. We note also that, if we consider only the first leading terms of (10), we obtain

$$\lim_{\gamma\to 0} D_1\gamma^{14} = 0$$

which is not significative; so also the subsequent orders in the expansion around $\gamma = 0$ were necessary. Instead of this, for the other functions only the first leading terms of the expansion around $\gamma = 0$ played an active role.
So for this case we have obtained the closure

$$A^{\alpha\beta\gamma} = (2-a)\,(a+5)\,\frac{m\,n}{\gamma^2}\,\left(U^\alpha U^\beta U^\gamma + c^2\,h^{(\alpha\beta}U^{\gamma)}\right) - \tag{16}$$

$$3\,a\,(2-a)\,(1-a)\,\frac{2a^3+10a^2+19a+23}{(2a+1)(2a+5)}\,\pi\,\gamma^{-3}\left(\frac{1}{c^2}\,U^\alpha U^\beta U^\gamma + \,h^{(\alpha\beta}U^{\gamma)}\right) +$$

$$+\;2\,\frac{a^3+2a^2+14a+73}{a+5}\,\left(\frac{1}{c^2}q^{(\alpha}U^\beta U^{\gamma)} + \,\frac{1}{5}\,q^{(\alpha}h^{\beta\gamma)}\right)\,\gamma^{-1} +$$

$$+\,3(3-a)\frac{(a+4)(a+11)}{5\,(a+5)}\,t^{(<\alpha\beta>_3}U^{\gamma)}\,\gamma^{-1}\,.$$

2.2 The case $a = -\frac{1}{2}$

All the considerations of the previous case still hold, but there is now the negative aspect that $\lim_{\gamma\to 0} D_1\gamma^{12} = 0$. So in the present case we have only to analyze the behaviour of D_1. We note that, in the previous subsection, only 2 subsequent terms in the expansion around

$\gamma = 0$ played a role. Now we need also the third term in each of these expressions. So eqs. $(10)_{1,2}$ have to be generalized in

$$\tilde{h}_0 = \gamma^{-3}\,\Gamma\left(\frac{5}{2}\right) - \frac{1}{2}\,\gamma^{-1}\,\Gamma\left(\frac{1}{2}\right) + R_{-\frac{7}{2}}\,\gamma^{-\frac{1}{2}}\,,$$
$$\tilde{h}_2 = \frac{3}{2}\,\gamma^{-5}\,\Gamma\left(\frac{7}{2}\right) - \frac{5}{4}\,\gamma^{-3}\,\Gamma\left(\frac{3}{2}\right) + \frac{2}{7}\,R_{-\frac{5}{2}}\,\gamma^{-\frac{3}{2}}\,, \tag{17}$$

(as it can be found from eqs. (26) and (34) of [4]) while eq. (10)−3 remains unchanged. In these expressions the numbers R_k appears and they are defined by

$$R_k = \lim_{\gamma\to 0} \bar{R}_k \quad \text{with} \quad \bar{R}_k = \int_1^{+\infty} e^{-\gamma\,y}\sqrt{y^2-1}\,y^k\,d\,y\,.$$

Their expression is reported in Appendix A of [4].
After that, We see that in our case (12) becomes

$$D_1\,\gamma^{12} =$$
$$= \begin{vmatrix} -\Gamma\left(\frac{5}{2}\right)+\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\gamma^2 - R_{\frac{-7}{2}}\gamma^{\frac{5}{2}} & -3\Gamma\left(\frac{5}{2}\right)+\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\gamma^2 - \frac{1}{2}R_{\frac{-7}{2}}\gamma^{\frac{5}{2}} & -6\Gamma\left(\frac{7}{2}\right)+\frac{5}{2}\Gamma\left(\frac{3}{2}\right)\gamma^2 - \frac{1}{7}R_{\frac{-5}{2}}\gamma^{\frac{7}{2}} \\ -3\Gamma\left(\frac{5}{2}\right)+\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\gamma^2 - \frac{1}{2}R_{\frac{-7}{2}}\gamma^{\frac{5}{2}} & -12\Gamma\left(\frac{5}{2}\right)+\Gamma\left(\frac{1}{2}\right)\gamma^2 - \frac{3}{4}R_{\frac{-7}{2}}\gamma^{\frac{5}{2}} & -30\Gamma\left(\frac{7}{2}\right)+\frac{15}{2}\Gamma\left(\frac{3}{2}\right)\gamma^2 - \frac{3}{14}R_{\frac{-5}{2}}\gamma^{\frac{7}{2}} \\ \frac{1}{3}\Gamma\left(\frac{1}{2}\right) - \frac{5}{6}R_{\frac{-7}{2}}\gamma^{\frac{1}{2}} & \frac{2}{3}\Gamma\left(\frac{1}{2}\right) - \frac{5}{4}R_{\frac{-7}{2}}\gamma^{\frac{1}{2}} & \frac{10}{3}\Gamma\left(\frac{3}{2}\right) - \frac{5}{6}R_{\frac{-5}{2}}\gamma^{\frac{3}{2}} \end{vmatrix}.$$

Now we add to the third line the first one multiplied by $\frac{8}{9}$ and the second one multiplied by $-\frac{4}{27}$; in this way the terms independent on γ in the new third line disappear and we can take out of this new third line a factor $\gamma^{\frac{1}{2}}$. So we obtain

$$D_1\,\gamma^{\frac{23}{2}} =$$
$$= \begin{vmatrix} -\Gamma\left(\frac{5}{2}\right)+\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\gamma^2 - R_{\frac{-7}{2}}\gamma^{\frac{5}{2}} & -3\Gamma\left(\frac{5}{2}\right)+\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\gamma^2 - \frac{1}{2}R_{\frac{-7}{2}}\gamma^{\frac{5}{2}} & -6\Gamma\left(\frac{7}{2}\right)+\frac{5}{2}\Gamma\left(\frac{3}{2}\right)\gamma^2 - \frac{1}{7}R_{\frac{-5}{2}}\gamma^{\frac{7}{2}} \\ -3\Gamma\left(\frac{5}{2}\right)+\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\gamma^2 - \frac{1}{2}R_{\frac{-7}{2}}\gamma^{\frac{5}{2}} & -12\Gamma\left(\frac{5}{2}\right)+\Gamma\left(\frac{1}{2}\right)\gamma^2 - \frac{3}{4}R_{\frac{-7}{2}}\gamma^{\frac{5}{2}} & -30\Gamma\left(\frac{7}{2}\right)+\frac{15}{2}\Gamma\left(\frac{3}{2}\right)\gamma^2 - \frac{3}{14}R_{\frac{-5}{2}}\gamma^{\frac{7}{2}} \\ \frac{10}{27}\Gamma\left(\frac{1}{2}\right)\gamma^{\frac{3}{2}} - \frac{22}{27}R_{\frac{-7}{2}}\gamma^2 - \frac{5}{6}R_{\frac{-7}{2}} & \frac{8}{27}\Gamma\left(\frac{1}{2}\right)\gamma^{\frac{3}{2}} - \frac{1}{3}R_{\frac{-7}{2}}\gamma^2 - \frac{5}{4}R_{\frac{-7}{2}} & \frac{10}{9}\Gamma\left(\frac{3}{2}\right)\gamma^{\frac{3}{2}} - \frac{2}{21}R_{\frac{-5}{2}}\gamma^3 - \frac{5}{6}R_{\frac{-5}{2}}\gamma \end{vmatrix}.$$

Now we add to the third column the first one multiplied by 15 and the second one multiplied by -10; in this way the terms independent on γ in the new third column disappear and we can take out of this new third line a factor γ. So we obtain

$$D_1\,\gamma^{\frac{21}{2}} =$$
$$\begin{vmatrix} -\Gamma\left(\frac{5}{2}\right)+\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\gamma^2 - R_{\frac{-7}{2}}\gamma^{\frac{5}{2}} & -3\Gamma\left(\frac{5}{2}\right)+\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\gamma^2-\frac{1}{2}R_{\frac{-7}{2}}\gamma^{\frac{5}{2}} & \frac{15}{4}\Gamma\left(\frac{1}{2}\right)\gamma-\frac{1}{7}\gamma^{\frac{3}{2}}\left(70R_{\frac{-7}{2}}+R_{\frac{-5}{2}}\gamma\right) \\ -3\Gamma\left(\frac{5}{2}\right)+\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\gamma^2-\frac{1}{2}R_{\frac{-7}{2}}\gamma^{\frac{5}{2}} & -12\Gamma\left(\frac{5}{2}\right)+\Gamma\left(\frac{1}{2}\right)\gamma^2-\frac{3}{4}R_{\frac{-7}{2}}\gamma^{\frac{5}{2}} & \frac{5}{4}\Gamma\left(\frac{1}{2}\right)\gamma-\frac{3}{14}R_{\frac{-5}{2}}\gamma^{\frac{5}{2}} \\ \frac{10}{27}\Gamma\left(\frac{1}{2}\right)\gamma^{\frac{3}{2}}-\frac{22}{27}R_{\frac{-7}{2}}\gamma^2-\frac{5}{6}R_{\frac{-7}{2}} & \frac{8}{27}\Gamma\left(\frac{1}{2}\right)\gamma^{\frac{3}{2}}-\frac{1}{3}R_{\frac{-7}{2}}\gamma^2-\frac{5}{4}R_{\frac{-7}{2}} & \frac{85}{27}\Gamma\left(\frac{1}{2}\right)\gamma^{\frac{1}{2}}-\frac{80}{9}R_{\frac{-7}{2}}\gamma- \\ & & \frac{1}{42}R_{\frac{-5}{2}}(35+4\gamma^2) \end{vmatrix} .$$

After that, we see that $\lim_{\gamma\to 0} D_1\,\gamma^{\frac{21}{2}} = -\frac{5}{2}\left[\Gamma\left(\frac{5}{2}\right)\right]^2 R_{\frac{-5}{2}}$,

and $\lim_{\gamma\to 0}\frac{\tilde{N}_1}{D_1}\gamma^{\frac{9}{2}} = \lim_{\gamma\to 0}\frac{\tilde{N}_{11}}{D_1}\gamma^{\frac{9}{2}} = -\frac{28}{5}\Gamma\left(\frac{7}{2}\right)\frac{1}{R_{\frac{-5}{2}}}$.

So for this case we have obtained the closure

$$A^{\alpha\beta\gamma} = \frac{45}{4}\frac{m\,n}{\gamma^2}\left(U^\alpha U^\beta U^\gamma + c^2\,h^{(\alpha\beta}U^{\gamma)}\right) + \tag{18}$$
$$+\frac{84}{5}\Gamma\left(\frac{7}{2}\right)\frac{1}{R_{\frac{-5}{2}}}\pi\,\gamma^{-\frac{9}{2}}\left(\frac{1}{c^2}U^\alpha U^\beta U^\gamma + h^{(\alpha\beta}U^{\gamma)}\right) +$$
$$+\frac{59}{2}\left(\frac{1}{c^2}q^{(\alpha}U^\beta U^{\gamma)}+\frac{1}{5}q^{(\alpha}h^{\beta\gamma)}\right)\gamma^{-1} + \frac{343}{20}t^{(<\alpha\beta>_3}U^{\gamma)}\gamma^{-1} .$$

2.3 The case $a = 0$

From eqs. (24) of [4] we obtain

$$\tilde{h}_0 = \gamma^{-3} + \frac{1}{2}\gamma^{-1}\ln\gamma\,, \tag{19}$$

while the expressions of $\tilde{h}_2$ and $\tilde{h}_5$ remain the same of $(10)_{2,3}$. After that, we see that the expressions (11) at equilibrium still hold, while for the non equilibrium closure, with the previous scheme we now calculate

$$\begin{matrix} & \gamma^3 & \gamma^4 & \gamma^5 & \\ D_1 \dfrac{\gamma^{12}}{\ln\,\gamma} = & \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31}-\frac{1}{3}d_{21} & d_{32}-\frac{1}{3}d_{22} & d_{33}-\frac{1}{3}d_{23} \end{vmatrix} & & & \begin{matrix} 1 \\ \gamma \\ \frac{\gamma^{-1}}{\ln\,\gamma} \end{matrix} \; ; \end{matrix}$$

$$\text{after that, we find} \quad \lim_{\gamma\to 0} D_1 \frac{\gamma^{12}}{\ln \gamma} = \frac{40}{9}\,.$$

Regarding $\tilde{N}_1$, $\tilde{N}_{11}$, D_2, $\tilde{N}_3$, $\tilde{N}_{31}$, $\tilde{C}_5$, we recall that in the case $a < o\,,\; a \neq 0$ only the first leding term in each of eqs. (10) played an active role; since they are the same in this case $a = 0$, we find the same previous results, but calculated in $a = 0$. So we find

$$\lim_{\gamma\to 0} \frac{\tilde{N}_1}{D_1}\gamma^3(-\ln\gamma) = \lim_{\gamma\to 0} \frac{\tilde{N}_{11}}{D_1}\gamma^3(-\ln\gamma) = -\frac{46}{5}\,.$$

For D_2, $\tilde{N}_3$, $\tilde{N}_{31}$, $\tilde{C}_5$, the results are the same (but calculated in $a = 0$).
So for this case we have obtained the closure

$$A^{\alpha\beta\gamma} = 10\,\frac{m\,n}{\gamma^2}\,\left(U^\alpha U^\beta U^\gamma + c^2\,h^{(\alpha\beta}U^{\gamma)}\right) + \tag{20}$$

$$+\,\frac{138}{5}\,\pi\,\frac{1}{-\,\gamma^3\,\ln\,\gamma}\,\left(\frac{1}{c^2}\,U^\alpha U^\beta U^\gamma + \,h^{(\alpha\beta}U^{\gamma)}\right) +$$

$$+\,\frac{146}{5}\,\left(\frac{1}{c^2}q^{(\alpha}U^\beta U^{\gamma)} + \,\frac{1}{5}\,q^{(\alpha}h^{\beta\gamma)}\right)\,\gamma^{-1} + \frac{396}{25}\,t^{(<\alpha\beta>_3}U^{\gamma)}\,\gamma^{-1}\,.$$

2.4 The case $0 < a < 1$

From eq. (25) of [4] we obtain

$$\tilde{h}_0 = \gamma^{-3}\,\Gamma(2-a)\, + \,\frac{a+1}{a-2}\,R_{-a-2}\,\gamma^{-a-1}\,, \tag{21}$$

while the expressions of $\tilde{h}_2$ and $\tilde{h}_5$ remain the same of $(10)_{2,3}$. After that, we see that the expressions (11) at equilibrium still hold, while for the non equilibrium closure, with the previous scheme we now calculate

$$D_1\gamma^{12+a} = \begin{array}{c} \begin{array}{ccc} \gamma^3 & \gamma^4 & \gamma^5 \end{array} \\ \left| \begin{array}{ccc} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} - \frac{1}{3}d_{21} & d_{32} - \frac{1}{3}d_{22} & d_{33} - \frac{1}{3}d_{23} \end{array} \right| \begin{array}{c} 1 \\ \gamma \\ \gamma^{a-1} \end{array} \end{array} ,$$

and find

$$\lim_{\gamma\to 0} D_1\,\gamma^{12+a} = -\frac{4}{9}\,\Gamma(3-a)\,\Gamma(2-a)\,R_{-a-2}\,(a+1)(2a+1)(a+5)\,. \tag{22}$$

Regarding $\tilde{N}_1$, $\tilde{N}_{11}$, D_2, $\tilde{N}_3$, $\tilde{N}_{31}$, $\tilde{C}_5$, we recall that in the case $a < o$, $a \neq 0$ only the first leading term in each of eqs. (10) played an active role; since they are the same in this case, we find the same previous results. The only new result is that

$$\lim_{\gamma\to 0}\frac{\tilde{N}_1}{D_1}\gamma^{3-a} = \lim_{\gamma\to 0}\frac{\tilde{N}_{11}}{D_1}\gamma^{3-a} = -\,2\,\frac{2a^3+10a^2+19a+23}{(a+1)(a+5)(2a+1)}\quad\frac{\Gamma(2-a)}{R_{-a-2}}\,. \tag{23}$$

So for this case we have obtained the closure

$$\begin{aligned} A^{\alpha\beta\gamma} = {} & (2-a)\,(a+5)\,\frac{m\,n}{\gamma^2}\,\left(U^\alpha U^\beta U^\gamma + c^2\,h^{(\alpha\beta}U^{\gamma)}\right) + \\ & + 6\,\frac{2a^3+10a^2+19a+23}{(a+1)(a+5)(2a+1)}\quad\frac{\Gamma(2-a)}{R_{-a-2}}\,\pi\,\gamma^{a-3}\left(\frac{1}{c^2}\,U^\alpha U^\beta U^\gamma +\,h^{(\alpha\beta}U^{\gamma)}\right) + \\ & +\;2\,\frac{a^3+2a^2+14a+73}{a+5}\,\left(\frac{1}{c^2}q^{(\alpha}U^\beta U^{\gamma)} +\,\frac{1}{5}\,q^{(\alpha}h^{\beta\gamma)}\right)\,\gamma^{-1} + \\ & + 3(3-a)\frac{(a+4)(a+11)}{5\,(a+5)}\,t^{(<\alpha\beta>_3}U^{\gamma)}\,\gamma^{-1}\,. \end{aligned} \tag{24}$$

2.5 The case $a = 1$

From eqs. (25), (32) and $(12)_1$ of [4] we obtain for $\tilde{h}_0$ the same expression of (21) and for $\tilde{h}_5$ the same expression of (10) but calculated in $a = 1$, while the expression of $\tilde{h}_2$ is

$$\tilde{h}_2 = 2\,\gamma^{-5} \,+\, 2\,\gamma^{-3}\,\ln\gamma\,. \tag{25}$$

After that, we see that the expressions (11) at equilibrium still hold, while for the non equilibrium closure, we proceed with the same steps as before in the case $0 < a < 1$; we find the same previous results, but calculated in $a = 1$. Consequently, the closure for the case $a = 1$ is the same of the case $0 < a < 1$, but calculated in $a = 1$.

2.6 The case $1 < a < 2$

From eqs. (25), (33) and (36) of [4] we obtain for $\tilde{h}_0$ the same expression of (21) and for $\tilde{h}_5$ the same expression of (10) but calculated in $a = 1$, while the expression of $\tilde{h}_2$ is

$$\tilde{h}_2 = \frac{1}{3}\,\gamma^{-5}\,(a+5)\,\Gamma(3-a) \,+\, 2\,\frac{a+1}{a-3}\,R_{-a-1}\,\gamma^{-2-a}\,. \tag{26}$$

After that, we see that the expressions (11) at equilibrium still hold, while for the non equilibrium closure, we proceed with the same steps as before in the case $0 < a < 1$; we find the same previous results, but computed for $1 < a < 2$. Consequently, the closure for the case is the same of the case $0 < a < 1$, but holding for $1 < a < 2$.

2.7 The case $a = 2$

From eqs. $(10)_2$, $(12)_1$, and (33) of [4] we obtain for $\tilde{h}_2$ the same expression of (26) and for $\tilde{h}_5$ the same expression of (10) but calculated for $a = 2$, while the expression of $\tilde{h}_0$ is

$$\tilde{h}_0 = -\,\gamma^{-3}\,\ln\,\gamma\,. \tag{27}$$

After that, we see that the expressions $(11)_{1,2}$ at equilibrium still hold, while $(11)_3$ has to be substituted by

$$A_E^{\alpha\beta\gamma} = \frac{7}{-\ln\,\gamma}\,\frac{m\,n}{\gamma^2}\,\left[U^\alpha U^\beta U^\gamma + c^2\,h^{(\alpha\beta}U^{\gamma)}\right]\,.$$

Technium Vol. 2, Issue 5 pp.11-42 (2020)
ISSN: 2668-778X
www.techniumscience.com

To find the non equilibrium closure, we note that

$$D_1 \frac{\gamma^{14}}{\ln \gamma} = \begin{vmatrix} \frac{\gamma^3}{\ln \gamma} & \frac{\gamma^4}{\ln \gamma} & \gamma^5 & \\ d_{11} & d_{12} & d_{13} & 1 \\ d_{21} & d_{22} & d_{23} & \gamma \\ d_{31} - \frac{1}{3} d_{21} & d_{32} - \frac{1}{3} d_{22} & d_{33} - \frac{1}{3} d_{23} & \gamma \ln \gamma \end{vmatrix} . \tag{28}$$

$$\tilde{N}_1 \frac{\gamma^{15}}{(\ln \gamma)^2} = \begin{vmatrix} \frac{\gamma^3}{\ln \gamma} & \frac{\gamma^4}{\ln \gamma} & \gamma^5 & \\ d_{11} & d_{12} & d_{13} & 1 \\ d_{21} & d_{22} & d_{23} & \gamma \\ n_{31} & n_{32} & n_{33} & \gamma^2 \end{vmatrix} . \tag{29}$$

$$\tilde{N}_{11} \frac{\gamma^{15}}{(\ln \gamma)^2} = \begin{vmatrix} \frac{\gamma^3}{\ln \gamma} & \frac{\gamma^4}{\ln \gamma} & \gamma^5 & \\ d_{11} & d_{12} & d_{13} & 1 \\ d_{21} & d_{22} & d_{23} & \gamma \\ m_{31} & m_{32} & m_{33} & \gamma^2 \end{vmatrix} . \tag{30}$$

After that, we use the above properties and find

$$\lim_{\gamma\to 0} D_1 \frac{\gamma^{14}}{\ln \gamma} = \frac{140}{9} , \quad \lim_{\gamma\to 0} \tilde{N}_1 \frac{\gamma^{15}}{(\ln \gamma)^2} = \lim_{\gamma\to 0} \tilde{N}_{11} \frac{\gamma^{15}}{(\ln \gamma)^2} = 104 .$$

$$\text{and, consequently,} \quad \lim_{\gamma\to 0} \frac{\tilde{N}_1}{D_1} \frac{\gamma}{\ln \gamma} = \lim_{\gamma\to 0} \frac{\tilde{N}_{11}}{D_1} \frac{\gamma}{\ln \gamma} = \frac{234}{35} . \tag{31}$$

Technium Vol. 2, Issue 5 pp.11-42 (2020)
ISSN: 2668-778X
www.techniumscience.com

Let us consider now the following group of functions and observe that

$$D_2 \frac{\gamma^{10}}{\ln\gamma} = \begin{vmatrix} e_{11} & e_{12} & 1 \\ e_{21} & e_{22} & \gamma \\ \frac{\gamma^4}{\ln\gamma} & \gamma^5 & \end{vmatrix}, \tag{32}$$

$$\tilde{N}_3 \frac{\gamma^{11}}{\ln\gamma} = \begin{vmatrix} e_{11} & e_{12} & 1 \\ f_{21} & f_{22} & \gamma^2 \\ \frac{\gamma^4}{\ln\gamma} & \gamma^5 & \end{vmatrix}, \tag{33}$$

$$\tilde{N}_{31} \frac{\gamma^{11}}{\ln\gamma} = \begin{vmatrix} e_{11} & e_{12} & 1 \\ g_{21} & g_{22} & \gamma^2 \\ \frac{\gamma^4}{\ln\gamma} & \gamma^5 & \end{vmatrix}, \tag{34}$$

From these results we obtain

$$\lim_{\gamma\to 0} D_2 \frac{\gamma^{10}}{\ln\gamma} = \frac{14}{3} \quad , \quad \lim_{\gamma\to 0} \tilde{N}_3 \frac{\gamma^{11}}{\ln\gamma} = \lim_{\gamma\to 0} \tilde{N}_{31} \frac{\gamma^{11}}{\ln\gamma} = 52 \,,$$

$$\text{and, consequently,} \qquad \lim_{\gamma\to 0} \frac{\tilde{N}_3}{D_3}\gamma = \lim_{\gamma\to 0} \frac{\tilde{N}_{31}}{D_3}\gamma = \frac{78}{7} \,. \tag{35}$$

The calculations for C_5 are the same of the previous cases, but calculated in $a = 2$, so that we have

$$\lim_{\gamma\to 0} C_5\,\gamma = \frac{78}{35} \,. \tag{36}$$

So for this case we have obtained the closure

$$A^{\alpha\beta\gamma} = \frac{7}{-\ln\gamma}\,\frac{m\,n}{\gamma^2}\,\left(U^\alpha U^\beta U^\gamma + c^2\,h^{(\alpha\beta}U^{\gamma)}\right) - \tag{37}$$

$$\frac{702}{35}\,\frac{\ln\gamma}{\gamma}\,\pi\left(\frac{1}{c^2}\,U^\alpha U^\beta U^\gamma + h^{(\alpha\beta}U^{\gamma)}\right) +$$

$$+\frac{234}{7}\left(\frac{1}{c^2}\,q^{(\alpha}U^\beta U^{\gamma)} + \frac{1}{5}\,q^{(\alpha}h^{\beta\gamma)}\right)\gamma^{-1} + \frac{234}{35}\,t^{(<\alpha\beta>_3}U^{\gamma)}\,\gamma^{-1}\,.$$

2.8 The case $2 < a < 3$

From eqs. $(10)_3$, $(12)_1$, and (33) of [4] we obtain for $\tilde{h}_2$ the same expression of (26) and for $\tilde{h}_5$ the same expression of (10) (but now only th first leading term plays an active role) and for $\tilde{h}_5$ the same expression of (10), while the expression of $\tilde{h}_0$ is

$$\tilde{h}_0 = \gamma^{-a-1}\,R_{-a}\,. \tag{38}$$

After that, we see that the expressions $(11)_1$ at equilibrium still hold, while $(11)_{2,3}$ have to be substituted by

$$e = (a+1)\,\frac{m\,n\,c^2}{\gamma}\,,\qquad A_E^{\alpha\beta\gamma} = \frac{\Gamma(3-a)}{R_{-a}}\,(a+5)\,m\,n\,\left[U^\alpha U^\beta U^\gamma + c^2\,h^{(\alpha\beta}U^{\gamma)}\right]\,.$$

To find the non equilibrium closure, we note that

$$\begin{matrix} \gamma^{a+1} & \gamma^{a+2} & \gamma^5 & \\ \end{matrix}$$

$$D_1\,\gamma^{2a+10} = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix}\begin{matrix} 1 \\ \gamma \\ \gamma \end{matrix}\quad . \tag{39}$$

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ISSN: 2668-778X
www.techniumscience.com

$$\begin{array}{ccc} \gamma^{a+1} & \gamma^{a+2} & \gamma^{5} \end{array}$$

$$\tilde{N}_1\,\gamma^{2a+11} = \left|\begin{array}{ccc|c} d_{11} & d_{12} & d_{13} & 1 \\ d_{21} & d_{22} & d_{23} & \gamma \\ n_{31} & n_{32} & n_{33} & \gamma^2 \end{array}\right| . \tag{40}$$

$$\begin{array}{ccc} \gamma^{a+1} & \gamma^{a+2} & \gamma^{5} \end{array}$$

$$\tilde{N}_{11}\,\gamma^{2a+11} = \left|\begin{array}{ccc|c} d_{11} & d_{12} & d_{13} & 1 \\ d_{21} & d_{22} & d_{23} & \gamma \\ m_{31} & m_{32} & m_{33} & \gamma^2 \end{array}\right| . \tag{41}$$

After that, we find

$$\begin{gathered} \lim_{\gamma\to 0} D_1\,\gamma^{2a+10} = -\,\frac{20}{9}\,(a-2)\,(a+5)\,\Gamma(3-a)\,(R_{-a})^2\,, \\ \lim_{\gamma\to 0} \tilde{N}_1\gamma^{11+2a} = \lim_{\gamma\to 0} \tilde{N}_{11}\gamma^{11+2a} = \frac{4}{9}\,(a+4)(a+11)(a+1)\,\Gamma(4-a)\,\left(R_{-a}\right)^2\,. \end{gathered} \tag{42}$$

and, consequently,

$$\lim_{\gamma\to 0} \frac{\tilde{N}_1}{D_1}\,\gamma = \lim_{\gamma\to 0} \frac{\tilde{N}_{11}}{D_1}\,\gamma = -\,\frac{1}{5}\,\frac{(a+1)(a+4)(a+11)(3-a)}{(a-2)(a+5)}\,. \tag{43}$$

Regarding the following group of functions we observe that

$$\begin{array}{cc} \gamma^{a+2} & \gamma^{5} \end{array}$$

$$D_2\,\gamma^{a+8} = \left|\begin{array}{cc|c} e_{11} & e_{12} & 1 \\ e_{21} & e_{22} & \gamma \end{array}\right| , \tag{44}$$

$$\begin{array}{rcc|c} & \gamma^{a+2} & \gamma^{5} & \\ \tilde{N}_3\,\gamma^{a+9} = & e_{11} & e_{12} & 1 \\ & f_{21} & f_{22} & \gamma^2 \end{array}\ , \tag{45}$$

$$\begin{array}{rcc|c} & \gamma^{a+2} & \gamma^{5} & \\ \tilde{N}_{31}\,\gamma^{a+9} = & e_{11} & e_{12} & 1 \\ & g_{21} & g_{22} & \gamma^2 \end{array}\ , \tag{46}$$

From these results we obtain

$$\lim_{\gamma\to 0} D_2\,\gamma^{a+8} = -\,\frac{2}{3}\,(a+5)\,\Gamma(4-a)\,R_{-a}\,,$$
$$\lim_{\gamma\to 0} \tilde{N}_3\gamma^{a+9} = \lim_{\gamma\to 0} \tilde{N}_{31}\gamma^{a+9} = -\,\frac{2}{3}\,(a+4)\,(a+11)\,\Gamma(4-a)\,R_{-a}\,,$$

$$\text{and, consequently,}\qquad \lim_{\gamma\to 0}\frac{N_3}{D_3}\gamma = \lim_{\gamma\to 0}\frac{N_{31}}{D_3}\gamma = \frac{(a+4)(a+11)}{a+5}\,. \tag{47}$$

The calculations for C_5 are the same of the previous cases, so that we have

$$\lim_{\gamma\to 0} C_5\,\gamma = (3-a)\,\frac{(a+4)(a+11)}{5\,(a+5)}\,. \tag{48}$$

So for this case we have obtained the closure

$$A^{\alpha\beta\gamma} = \frac{\Gamma(3-a)}{R_{-a}}\,(a+5)\,\gamma^{a-4}\,n\,m\,\left(U^\alpha U^\beta U^\gamma + c^2\,h^{(\alpha\beta}U^{\gamma)}\right) + \tag{49}$$

$$+\,\frac{3}{5}\,\frac{(a+1)(a+4)(a+11)(3-a)}{(a-2)(a+5)}\,\pi\,\left(\frac{1}{c^2}\,U^\alpha U^\beta U^\gamma + h^{(\alpha\beta}U^{\gamma)}\right)\gamma^{-1}+$$

$$+\,3\,\frac{(a+4)(a+11)}{a+5}\,\left(\frac{1}{c^2}\,q^{(\alpha}U^\beta U^{\gamma)} + \frac{1}{5}\,q^{(\alpha}h^{\beta\gamma)}\right)\gamma^{-1}+$$

$$+\,3\,(3-a)\,\frac{(a+4)(a+11)}{5\,(a+5)}\,t^{(<\alpha\beta>_3}U^{\gamma)}\,\gamma^{-1}\,.$$

2.9 The case $a=3$

From eqs. $(11)_2$, $(12)_1$, and $(29)_1$ of [4] we obtain

$$\tilde{h}_0 = \gamma^{-4}\,R_{-3} + \gamma^{-3}\,\ln\gamma - \gamma^{-3}\,, \quad \tilde{h}_2 = -\frac{8}{3}\,\gamma^{-5}\,\ln\gamma\,, \quad \tilde{h}_5 = 98\,\gamma^{-7}\,. \tag{50}$$

After that, we see that the expressions $(11)_1$ at equilibrium still hold, while $(11)_{2,3}$ have to be substituted by

$$e = 4\,\frac{m\,n\,c^2}{\gamma}\,, \quad A_E^{\alpha\beta\gamma} = -\frac{8}{\gamma}\,\ln\gamma\,\frac{m\,n}{R_{-3}}\,\left[U^\alpha U^\beta U^\gamma + c^2\,h^{(\alpha\beta}U^{\gamma)}\right]\,.$$

To find the non equilibrium closure, we note that

$$\begin{matrix} & \gamma^4 & \gamma^5 & \frac{\gamma^5}{\ln\gamma} & \\ D_1\,\frac{\gamma^{16}}{\ln\gamma} = & \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix} & & & \begin{matrix} 1 \\ \gamma \\ \gamma \end{matrix} \end{matrix}\,. \tag{51}$$

$$\tilde{N}_1\,\gamma^{17} = \begin{array}{c} \begin{array}{ccc} \gamma^4 & \gamma^5 & \gamma^5 \end{array} \\ \left| \begin{array}{ccc} d_{11} & d_{12} & d_{13} + \frac{8}{3}\,\frac{\ln\gamma}{R_{-3}}\,d_{12} \\ d_{21} & d_{22} & d_{23} + \frac{8}{3}\,\frac{\ln\gamma}{R_{-3}}\,d_{22} \\ n_{31} & n_{32} & n_{33} + \frac{8}{3}\,\frac{\ln\gamma}{R_{-3}}\,n_{32} \end{array} \right| \begin{array}{c} 1 \\ \gamma \\ \gamma^2 \end{array} \end{array} . \tag{52}$$

$$\tilde{N}_{11}\,\gamma^{17} = \begin{array}{c} \begin{array}{ccc} \gamma^4 & \gamma^5 & \gamma^5 \end{array} \\ \left| \begin{array}{ccc} d_{11} & d_{12} & d_{13} + \frac{8}{3}\,\frac{\ln\gamma}{R_{-3}}\,d_{12} \\ d_{21} & d_{22} & d_{23} + \frac{8}{3}\,\frac{\ln\gamma}{R_{-3}}\,d_{22} \\ m_{31} & m_{32} & m_{33} + \frac{8}{3}\,\frac{\ln\gamma}{R_{-3}}\,m_{32} \end{array} \right| \begin{array}{c} 1 \\ \gamma \\ \gamma^2 \end{array} \end{array} . \tag{53}$$

After that, we use the above properties and find

$$\begin{aligned} &\lim_{\gamma\to 0} D_1\,\frac{\gamma^{16}}{\ln\,\gamma} = \frac{160}{9}\,(R_{-3})^2\,, \\ &\lim_{\gamma\to 0} \tilde{N}_1\gamma^{17} = \lim_{\gamma\to 0} \tilde{N}_{11}\gamma^{17} = 32\cdot\frac{49}{9}\;(R_{-3})^2\;. \end{aligned} \tag{54}$$

$$\text{and, consequently,}\qquad \lim_{\gamma\to 0}\frac{\tilde{N}_1}{D_1}\,\gamma\,\ln\gamma = \lim_{\gamma\to 0}\frac{\tilde{N}_{11}}{D_1}\,\gamma\,\ln\gamma = \frac{49}{5}\,. \tag{55}$$

Regarding the following group of functions we observe that

$$D_2\,\gamma^{11} = \begin{array}{c} \begin{array}{cc} \gamma^5 & \gamma^5 \end{array} \\ \left| \begin{array}{cc} e_{11} & e_{12} + \frac{16}{3}\,\frac{\ln\gamma}{R_{-3}}\,e_{11} \\ e_{21} & e_{22} + \frac{16}{3}\,\frac{\ln\gamma}{R_{-3}}\,e_{21} \end{array} \right| \begin{array}{c} 1 \\ \gamma \end{array} \end{array} , \tag{56}$$

$$\tilde{N}_3\,\gamma^{12} = \begin{vmatrix} \overset{\gamma^5}{e_{11}} & \overset{\gamma^5}{e_{12} + \frac{16}{3}\frac{\ln\gamma}{R_{-3}}\,e_{11}} \\ f_{21} & f_{22} + \frac{16}{3}\frac{\ln\gamma}{R_{-3}}\,f_{21} \end{vmatrix} \begin{matrix} 1 \\ \gamma^2 \end{matrix} \,, \tag{57}$$

$$\tilde{N}_{31}\,\gamma^{12} = \begin{vmatrix} \overset{\gamma^5}{e_{11}} & \overset{\gamma^5}{e_{12} + \frac{16}{3}\frac{\ln\gamma}{R_{-3}}\,e_{11}} \\ g_{21} & g_{22} + \frac{16}{3}\frac{\ln\gamma}{R_{-3}}\,g_{21} \end{vmatrix} \begin{matrix} 1 \\ \gamma^2 \end{matrix} \,, \tag{58}$$

From these results we obtain

$$\lim_{\gamma\to 0} D_2\,\gamma^{11} = -\,\frac{16}{3}\,R_{-3}\,,$$
$$\lim_{\gamma\to 0} \tilde{N}_3\gamma^{12} = \lim_{\gamma\to 0} \tilde{N}_{31}\gamma^{12} = -\,49\cdot\frac{4}{3}\,R_{-3}\,.$$

From the above results we deduce that.

$$\lim_{\gamma\to 0}\frac{\tilde{N}_3}{D_2}\gamma = \lim_{\gamma\to 0}\frac{\tilde{N}_{31}}{D_2}\gamma = \frac{49}{4}\,. \tag{59}$$

$$\text{Regarding } C_5 \text{ we have that} \quad \lim_{\gamma\to 0}\,\gamma\ln\gamma\,C_5 = -\,\frac{49}{20}\,. \tag{60}$$

So for this case we have obtained the closure

$$A^{\alpha\beta\gamma} = -\,\frac{8}{\gamma}\,\ln\,\gamma\,\frac{m\,n}{R_{-3}}\,\left(U^\alpha U^\beta U^\gamma + c^2\,h^{(\alpha\beta}U^{\gamma)}\right)\,- \tag{61}$$
$$\frac{147}{5}\,\pi\,\left(\frac{1}{c^2}\,U^\alpha U^\beta U^\gamma +\,h^{(\alpha\beta}U^{\gamma)}\right)\frac{1}{\gamma\,\ln\,\gamma}+$$
$$+\,\frac{147}{4}\,\left(\frac{1}{c^2}\,q^{(\alpha}U^\beta U^{\gamma)} + \frac{1}{5}\,q^{(\alpha}h^{\beta\gamma)}\right)\,\gamma^{-1}\,-\,\frac{147}{20}\,t^{(<\alpha\beta>_3}U^{\gamma)}\frac{1}{\gamma\,\ln\,\gamma}\,.$$

2.10 The case $3 < a < 4$

From eqs. (30), (35) and $(12)_1$ of [4] we obtain

$$\begin{aligned} \tilde{h}_0 &= \gamma^{-a-1} R_{-a} - \gamma^{-a} R_{1-a}\,, \\ \tilde{h}_2 &= 2\,\frac{a+1}{a-3}\,R_{-1-a}\,\gamma^{-a-2} - \frac{1}{3}\,\Gamma(4-a)\,\frac{a+5}{a-3}\,\gamma^{-5}\,, \\ \tilde{h}_5 &= \gamma^{-7}\,\Gamma(4-a)\,(a+4)(a+11)\,. \end{aligned} \tag{62}$$

After that, we see that the expressions $(11)_1$ at equilibrium still holds, while $(11)_{2,3}$ have to be substituted by

$$\begin{aligned} e &= (a+1)\,\frac{m\,n\,c^2}{\gamma}\,, \\ A_E^{\alpha\beta\gamma} &= 2\,\frac{a(a+1)}{a-3}\,\frac{R_{-1-a}}{R_{-a}}\,\frac{m\,n}{\gamma}\,U^\alpha U^\beta U^\gamma + 6\,\frac{a+1}{a-3}\,\frac{R_{-1-a}}{R_{-a}}\,\frac{m\,n}{\gamma}\,c^2\,h^{(\alpha\beta}U^{\gamma)}\,. \end{aligned}$$

To find the non equilibrium closure, we note that

$$D_1\,\gamma^{3a+7} = \begin{matrix} \gamma^{a+1} & \gamma^{a+2} & \gamma^{a+2} & \\ \left|\begin{matrix} d_{11} \\ d_{21} \\ d_{31} \end{matrix}\right. & \begin{matrix} d_{12} \\ d_{22} \\ d_{32} \end{matrix} & \begin{matrix} d_{13} \\ d_{23} \\ d_{33} \end{matrix} & \left|\begin{matrix} 1 \\ \gamma \\ \gamma \end{matrix}\right. \end{matrix}\,. \tag{63}$$

$$\tilde{N}_1\,\gamma^{2a+11} = \begin{matrix} \gamma^{a+1} & \gamma^{a+2} & \gamma^{a+2} & \\ \left|\begin{matrix} d_{11} \\ d_{21} \\ n_{31} \end{matrix}\right. & \begin{matrix} d_{12} \\ d_{22} \\ n_{32} \end{matrix} & \begin{matrix} d_{13} \\ d_{23} \\ n_{33} \end{matrix} & \left|\begin{matrix} 1 \\ \gamma \\ \gamma^{5-a} \end{matrix}\right. \end{matrix}\,. \tag{64}$$

$$\begin{array}{lccc|c} & \gamma^{a+1} & \gamma^{a+2} & \gamma^{a+2} & \\ \tilde{N}_{11}\,\gamma^{2a+11} = & d_{11} & d_{12} & d_{13} & 1 \\ & d_{21} & d_{22} & d_{23} & \gamma \\ & m_{31} & m_{32} & m_{33} & \gamma^{5-a} \end{array} . \tag{65}$$

After that, we find

$$\lim_{\gamma\to 0} D_1\,\gamma^{3a+7} = -\,\frac{10}{3}\,(R_{-a})^2\,R_{-a-1}\,\frac{(a+1)^2}{a-3}\,,$$
$$\lim_{\gamma\to 0} \tilde{N}_1\,\gamma^{2a+11} = \lim_{\gamma\to 0} \tilde{N}_{11}\,\gamma^{2a+11} = \frac{4}{9}\,(a+1)(a+4)(a+11)\,\Gamma(4-a)\,(R_{-a})^2\,.$$

and, consequently,

$$\lim_{\gamma\to 0} \frac{\tilde{N}_1}{D_1}\,\gamma^{4-a} = \lim_{\gamma\to 0} \frac{\tilde{N}_{11}}{D_1}\,\gamma^{4-a} = -\,\frac{2}{15}\,\frac{(a-3)(a+4)(a+11)}{a+1}\,\frac{\Gamma(4-a)}{R_{-1-a}}\,.$$

Regarding the subsequent group of functions we observe that

$$\begin{array}{lcc|c} & \gamma^{a+2} & \gamma^{a+2} & \\ D_2\,\gamma^{a+8} = & e_{11} & e_{12} & 1 \\ & e_{21} + (a+2)\,\gamma^{-1}\,e_{11} & e_{22} + (a+2)\,\gamma^{-1}\,e_{12} & \gamma^{4-a} \end{array} , \tag{66}$$

$$\begin{array}{lcc|c} & \gamma^{a+2} & \gamma^{a+2} & \\ \tilde{N}_3\,\gamma^{a+9} = & e_{11} & e_{12} & 1 \\ & f_{21} & f_{22} & \gamma^{5-a} \end{array} , \tag{67}$$

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$$\tilde{N}_{31}\ \gamma^{a+9} = \begin{vmatrix} \gamma^{a+2} & \gamma^{a+2} \\ e_{11} & e_{12} \\ g_{21} & g_{22} \end{vmatrix} \begin{matrix} 1 \\ \gamma^{5-a} \end{matrix}\ , \tag{68}$$

From these results we obtain

$$\lim_{\gamma\to 0} D_2\, \gamma^{a+8} = -\frac{2}{3}\,(a+5)\,\Gamma(4-a)\,R_{-a}\,,$$
$$\lim_{\gamma\to 0} \tilde{N}_3\gamma^{a+9} = \lim_{\gamma\to 0} \tilde{N}_{31}\gamma^{a+9} = -\frac{2}{3}\,(a+4)(a+11)\,\Gamma(4-a)\,R_{-a}\,,$$

$$\text{and, consequently,}\quad \lim_{\gamma\to 0}\frac{\tilde{N}_3}{D_2}\gamma = \lim_{\gamma\to 0}\frac{\tilde{N}_{31}}{D_2}\gamma = \frac{(a+4)(a+11)}{a+5}\,. \tag{69}$$

$$\text{Finally,}\quad \lim_{\gamma\to 0}\tilde{C}_5\,\gamma^{4-a} = \frac{1}{30}\,\frac{\Gamma(4-a)}{R_{-1-a}}\,\frac{(a-3)(a+4)(a+11)}{a+1}\,. \tag{70}$$

So for this case we have obtained the closure

$$\begin{aligned} A^{\alpha\beta\gamma} = {} & 2\,\frac{a(a+1)}{a-3}\,\frac{R_{-1-a}}{R_{-a}}\,\frac{m\,n}{\gamma}\,U^{\alpha}U^{\beta}U^{\gamma} + 6\,\frac{a+1}{a-3}\,\frac{R_{-1-a}}{R_{-a}}\,\frac{m\,n}{\gamma}\,c^2\,h^{(\alpha\beta}U^{\gamma)} + \\ & + \frac{2}{5}\,\frac{(a-3)(a+4)(a+11)}{a+1}\,\frac{\Gamma(4-a)}{R_{-1-a}}\,\pi\,\left(\frac{1}{c^2}\,U^{\alpha}U^{\beta}U^{\gamma} + h^{(\alpha\beta}U^{\gamma)}\right)\,\gamma^{a-4} + \\ & + 3\,\frac{(a+4)(a+11)}{a+5}\,\left(\frac{1}{c^2}\,q^{(\alpha}U^{\beta}U^{\gamma)} + \frac{1}{5}\,q^{(\alpha}h^{\beta\gamma)}\right)\,\gamma^{-1} + \\ & + \frac{1}{10}\,\frac{\Gamma(4-a)}{R_{-1-a}}\,\frac{(a-3)(a+4)(a+11)}{a+1}\,t^{(<\alpha\beta>_3}U^{\gamma)}\,\gamma^{a-4}\,. \end{aligned} \tag{71}$$

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2.11 The case $a = 4$

From eqs. (30), $(35)_2$ and $(12)_1$ of [4] we obtain

$$\begin{aligned} \tilde{h}_0 &= \gamma^{-5} R_{-4} - \gamma^{-4} R_{-3} , \\ \tilde{h}_2 &= 10 R_{-5} \gamma^{-6} + 3 \gamma^{-5} \ln \gamma - 3 \gamma^{-5} , \\ \tilde{h}_5 &= - 120 \gamma^{-7} \ln \gamma . \end{aligned} \tag{72}$$

After that, we see that the expressions $(11)_1$ at equilibrium still holds, while $(11)_{2,3}$ have to be substituted by

$$\begin{aligned} & e = 5 \frac{m n c^2}{\gamma} , \\ & A_E^{\alpha\beta\gamma} = 40 \frac{R_{-5}}{R_{-4}} \frac{m n}{\gamma} U^\alpha U^\beta U^\gamma + 30 \frac{R_{-5}}{R_{-4}} \frac{m n}{\gamma} c^2 h^{(\alpha\beta} U^{\gamma)} . \end{aligned}$$

To find the non equilibrium closure, we note that

$$D_1 \gamma^{19} = \begin{array}{ccc} \gamma^5 & \gamma^6 & \gamma^6 \\ \left| \begin{array}{ccc} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{array} \right| \end{array} \begin{array}{l} 1 \\ \gamma \\ \gamma \end{array} . \tag{73}$$

$$\tilde{N}_1 \frac{\gamma^{19}}{\ln \gamma} = \begin{array}{ccc} \gamma^5 & \gamma^6 & \gamma^6 \\ \left| \begin{array}{ccc} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ n_{31} & n_{32} & n_{33} \end{array} \right| \end{array} \begin{array}{l} 1 \\ \gamma \\ \frac{\gamma}{\ln \gamma} \end{array} . \tag{74}$$

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$$
\tilde{N}_{11}\,\frac{\gamma^{19}}{\ln\gamma} = \begin{array}{|ccc|c}
\gamma^{5} & \gamma^{6} & \gamma^{6} & \\
d_{11} & d_{12} & d_{13} & 1 \\
d_{21} & d_{22} & d_{23} & \gamma \\
n_{31} & n_{32} & n_{33} & \frac{\gamma}{\ln\gamma}
\end{array} \,. \tag{75}
$$

After that, we find

$$
\lim_{\gamma\to 0} D_1\,\gamma^{19} = -\,\frac{250}{3}\,(R_{-4})^2\,R_{-5}\,,
$$
$$
\lim_{\gamma\to 0} \tilde{N}_1\,\frac{\gamma^{19}}{\ln\,\gamma} = \lim_{\gamma\to 0} \tilde{N}_{11}\,\frac{\gamma^{19}}{\ln\,\gamma} = -\,\frac{800}{3}\,(R_{-4})^2\,.
$$

$$
\text{and, consequently,}\quad \lim_{\gamma\to 0}\frac{\tilde{N}_1}{D_1}\,\frac{1}{\ln\,\gamma} = \lim_{\gamma\to 0}\frac{\tilde{N}_{11}}{D_1}\,\frac{1}{\ln\,\gamma} = \frac{16}{5}\,\frac{1}{R_{-5}}\,.
$$

Regarding the subsequent group of functions we observe that

$$
D_2\,\frac{\gamma^{12}}{\ln\,\gamma} = \begin{array}{|cc|c}
\gamma^{6} & \gamma^{6} & \\
e_{11} & e_{12} & 1 \\
e_{21} + 6\,\gamma^{-1}\,e_{11} & e_{22} + 6\,\gamma^{-1}\,e_{12} & \frac{1}{\ln\,\gamma}
\end{array} \,, \tag{76}
$$

$$
\tilde{N}_3\,\frac{\gamma^{13}}{\ln\,\gamma} = \begin{array}{|cc|c}
\gamma^{6} & \gamma^{6} & \\
e_{11} & e_{12} & 1 \\
f_{21} & f_{22} & \frac{\gamma}{\ln\,\gamma}
\end{array} \,, \tag{77}
$$

$$\begin{array}{ccc} & \gamma^6 & \gamma^6 \end{array}$$

$$\tilde{N}_{31}\,\frac{\gamma^{13}}{\ln\,\gamma} = \begin{vmatrix} e_{11} & e_{12} & 1 \\ g_{21} & g_{22} & \frac{\gamma}{\ln\,\gamma} \end{vmatrix} \,, \tag{78}$$

From these results we obtain

$$\lim_{\gamma\to 0} D_2\,\frac{\gamma^{12}}{\ln\,\gamma} = 6\,R_{-4}\,, \quad \lim_{\gamma\to 0}\tilde{N}_3\,\frac{\gamma^{13}}{\ln\,\gamma} = \lim_{\gamma\to 0}\tilde{N}_{31}\,\frac{\gamma^{13}}{\ln\,\gamma} = 80\,R_{-4}\,, \tag{79}$$

$$\text{and, consequently,} \quad \lim_{\gamma\to 0}\frac{\tilde{N}_3}{D_2}\,\gamma = \lim_{\gamma\to 0}\frac{\tilde{N}_{31}}{D_2}\,\gamma = \frac{40}{3}\,. \tag{80}$$

$$\text{Finally,} \quad \lim_{\gamma\to 0}\tilde{C}_5\,\frac{1}{\ln\,\gamma} = -\,\frac{4}{5}\,\frac{1}{R_{-5}}\,. \tag{81}$$

So for this case we have obtained the closure

$$A^{\alpha\beta\gamma} = 40\,\frac{R_{-5}}{R_{-4}}\,\frac{m\,n}{\gamma}\,U^\alpha U^\beta U^\gamma + 30\,\frac{R_{-5}}{R_{-4}}\,\frac{m\,n}{\gamma}\;c^2\,h^{(\alpha\beta}U^{\gamma)} - \tag{82}$$

$$\frac{48}{5}\,\frac{1}{R_{-5}}\,\pi\left(\frac{1}{c^2}\,U^\alpha U^\beta U^\gamma + \,h^{(\alpha\beta}U^{\gamma)}\right)\,\cdot\,\ln\,\gamma +$$

$$+\,40\left(\frac{1}{c^2}\,q^{(\alpha}U^\beta U^{\gamma)} + \frac{1}{5}\,q^{(\alpha}h^{\beta\gamma)}\right)\,\gamma^{-1} \;-\; \frac{12}{5}\,\frac{1}{R_{-5}}\,t^{(<\alpha\beta>_3}U^{\gamma)}\,\cdot\,\ln\,\gamma\,.$$

2.12 The case $a > 4$

From eqs. (30), $(35)_3$ and $(12)_3$ of [4] we obtain the same expression (62) for $\tilde{h}_0$, while $\tilde{h}_2$, $\tilde{h}_5$ have to be substituted by

$$\tilde{h}_2 = 2\,\frac{a+1}{a}\,R_{1-a}\,\gamma^{-a-2} \;-\; \frac{2a+1}{a-4}\,R_{-a}\,\gamma^{-a-1}\,, \quad \tilde{h}_5 = 60\,\frac{a+2}{a-4}\,R_{-a}\,\gamma^{-a-3} \tag{83}$$

where we substituted R_{-1-a} and R_{2-a} with $\frac{a-3}{a}\,R_{1-a}$ and $\frac{1-a}{4-a}\,R_{-a}$ respectively; this can be done because from (37) of [2] it follows the property $R_k = \frac{k-1}{k+2}\,R_{k-2}$ which hods for $k < -2$. After that, we see that the expressions $(11)_1$ at equilibrium still holds, while $(11)_{2,3}$ have to be substituted by

$$e = (a+1)\,\frac{m\,n\,c^2}{\gamma}\,,$$
$$A_E^{\alpha\beta\gamma} = 2(a+1)\,\frac{R_{1-a}}{R_{-a}}\,\frac{m\,n}{\gamma}\,U^\alpha U^\beta U^\gamma + 6\,\frac{a+1}{a}\,\frac{R_{1-a}}{R_{-a}}\,\frac{m\,n}{\gamma}\;c^2\,h^{(\alpha\beta}U^{\gamma)}\,.$$

To find the non equilibrium closure, we note that

$$\begin{array}{ccc} \gamma^{a+1} & \gamma^{a+2} & \gamma^{a+2} \end{array}$$
$$D_1\,\gamma^{3a+7} = \left|\begin{array}{ccc} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{array}\right| \begin{array}{c} 1 \\ \gamma \\ \gamma \end{array}\,. \tag{84}$$

$$\begin{array}{ccc} \gamma^{a+1} & \gamma^{a+2} & \gamma^{a+2} \end{array}$$
$$\tilde{N}_1\,\gamma^{3a+7} = \left|\begin{array}{ccc} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ n_{31} & n_{32} & n_{33} \end{array}\right| \begin{array}{c} 1 \\ \gamma \\ \gamma \end{array}\,. \tag{85}$$

$$\begin{array}{ccc} \gamma^{a+1} & \gamma^{a+2} & \gamma^{a+2} \end{array}$$
$$\tilde{N}_{11}\,\gamma^{3a+7} = \left|\begin{array}{ccc} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ m_{31} & m_{32} & m_{33} \end{array}\right| \begin{array}{c} 1 \\ \gamma \\ \gamma \end{array}\,. \tag{86}$$

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ISSN: 2668-778X
www.techniumscience.com

After that, we find

$$\lim_{\gamma\to 0} D_1\,\gamma^{3a+7} = -\,\frac{10}{3}\,\frac{(a+1)^2}{a-3}\,(R_{-a})^2\,R_{-1-a}\,,$$

$$\lim_{\gamma\to 0} \tilde{N}_1\gamma^{3a+7} = \frac{4}{3}\,a\,(a+1)^2\,(a+2)\,(R_{-a})^3\,\left[\frac{1}{a-4}\;-\;(a+1)\left(\frac{R_{1-a}}{R_{-a}}\,\frac{1}{a}\right)^2\right]\,,$$

$$\lim_{\gamma\to 0} \tilde{N}_{11}\gamma^{3a+7} = \frac{4}{3}\,(a+1)\,(a+2)\,(R_{-a})^3\,\left[\frac{3a+8}{a-4}\;-\;3\left(\frac{R_{1-a}}{R_{-a}}\,\frac{a+1}{a}\right)^2\right]\,,$$

and, consequently,

$$\lim_{\gamma\to 0}\frac{\tilde{N}_1}{D_1} = \frac{2}{5}\,a^2\,(a+2)\,\left[(a+1)\left(\frac{R_{1-a}}{R_{-a}}\,\frac{1}{a}\right)^2\;-\;\frac{1}{a-4}\right]\,\frac{R_{-a}}{R_{1-a}}\,,$$

$$\lim_{\gamma\to 0}\frac{\tilde{N}_{11}}{D_1} = \frac{2}{5}\,\frac{a(a+2)}{a+1}\,\left[3\left(\frac{R_{1-a}}{R_{-a}}\,\frac{a+1}{a}\right)^2\;-\;\frac{3a+8}{a-4}\right]\,\frac{R_{-a}}{R_{1-a}}\,.$$

We note that only in the present case D_1, $\tilde{N}_1$, $\tilde{N}_{11}$ are of the same order and, moreover, $\lim_{\gamma\to 0}\frac{\tilde{N}_1}{D_1} \neq \lim_{\gamma\to 0}\frac{\tilde{N}_{11}}{D_1}$.
Regarding the subsequent group of functions we observe that

$$D_2\,\gamma^{2a+4} = \begin{vmatrix} \gamma^{a+2} & \gamma^{a+2} & \\ e_{11} & e_{12} & 1 \\ e_{21} + (a+2)\,\gamma^{-1}\,e_{11} & e_{22} + (a+2)\,\gamma^{-1}\,e_{12} & 1 \end{vmatrix}\,, \tag{87}$$

$$\tilde{N}_3\,\gamma^{2a+5} = \begin{vmatrix} \gamma^{a+2} & \gamma^{a+2} & \\ e_{11} & e_{12} & 1 \\ f_{21} & f_{22} & \gamma \end{vmatrix}\,, \tag{88}$$

$$\tilde{N}_{31}\,\gamma^{2a+5} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc|c} \gamma^{a+2} & \gamma^{a+2} & \\ \hline e_{11} & e_{12} & 1 \\ g_{21} & g_{22} & \gamma \end{array}\,, \tag{89}$$

From these results we obtain

$$\begin{aligned}
\lim_{\gamma\to 0} D_2\,\gamma^{2a+4} &= 4\,\frac{a+1}{a}\,(R_{1-a})^2 \;-\; 2\,\frac{2a+1}{a-4}\,(R_{-a})^2\,, \\
\lim_{\gamma\to 0} \tilde{N}_3\gamma^{2a+5} &= 8\,(a+1)(a+2)\,\left[\frac{a+1}{a^2}\,(R_{1-a})^2 \;-\; \frac{1}{a-4}\,(R_{-a})^2\right]\,, \\
\lim_{\gamma\to 0} \tilde{N}_{31}\gamma^{2a+5} &= 40\,\left(\frac{a+1}{a}\right)^2\,(R_{1-a})^2 \;-\; 40\,\frac{a+2}{a-4}\,(R_{-a})^2\,,
\end{aligned}$$

and, consequently,

$$\begin{aligned}
\lim_{\gamma\to 0}\frac{\tilde{N}_3}{D_2}\,\gamma &= 4\quad\frac{(a+1)\,(a+2)}{a}\quad\frac{(a+1)(a-4)(R_{1-a})^2 \;-\; a^2\,(R_{-a})^2}{2\,(a+1)(a-4)\,(R_{1-a})^2 \;-\; a\,(2a+1)\,(R_{-a})^2}\,, \\
\lim_{\gamma\to 0}\frac{\tilde{N}_{31}}{D_2}\,\gamma &= \frac{20}{a}\quad\frac{(a-4)(a+1)^2\,(R_{1-a})^2 \;-\; a^2\,(a+2)\,(R_{-a})^2}{2\,(a+1)(a-4)\,(R_{1-a})^2 \;-\; a\,(2a+1)\,(R_{-a})^2}\,.
\end{aligned} \tag{90}$$

We note that only in the present case we have obtained $\lim_{\gamma\to 0}\frac{\tilde{N}_{31}}{D_2}\,\gamma \neq \lim_{\gamma\to 0}\frac{\tilde{N}_3}{D_2}\,\gamma$.
Finally,

$$\lim_{\gamma\to 0}\tilde{C}_5 \;=\; 2\,\frac{a+2}{a+1}\,\frac{a}{a-4}\,\frac{R_{-a}}{R_{1-a}}\,. \tag{91}$$

So for this case we have obtained the closure

$$A^{\alpha\beta\gamma} = 2(a+1)\,\frac{R_{1-a}}{R_{-a}}\,\frac{n\,m}{\gamma}\,U^\alpha U^\beta U^\gamma + 6\,\frac{(a+1)}{a}\,\frac{R_{1-a}}{R_{-a}}\,\frac{n\,m}{\gamma}\,c^2\,h^{(\alpha\beta}U^{\gamma)} - \tag{92}$$

$$\frac{6}{5}\,a^2\,(a+2)\left[(a+1)\left(\frac{R_{1-a}}{R_{-a}}\,\frac{1}{a}\right)^2 - \frac{1}{a-4}\right]\frac{R_{-a}}{R_{1-a}}\,\pi\,\frac{1}{c^2}\,U^\alpha U^\beta U^\gamma -$$

$$\frac{6}{5}\,\frac{a(a+2)}{a+1}\left[3\left(\frac{R_{1-a}}{R_{-a}}\,\frac{a+1}{a}\right)^2 - \frac{3a+8}{a-4}\right]\frac{R_{-a}}{R_{1-a}}\,\pi\,h^{(\alpha\beta}U^{\gamma)} +$$

$$+\,\frac{12}{c^2}\quad\frac{(a+1)\,(a+2)}{a}\quad\frac{(a+1)(a-4)(R_{1-a})^2\,-\,a^2\,(R_{-a})^2}{2\,(a+1)(a-4)\,(R_{1-a})^2\,-\,a\,(2a+1)\,(R_{-a})^2}\,q^{(\alpha}U^\beta U^{\gamma)} +$$

$$+\,\frac{12}{a}\quad\frac{(a-4)(a+1)^2\,(R_{1-a})^2\,-\,a^2\,(a+2)\,(R_{-a})^2}{2\,(a+1)(a-4)\,(R_{1-a})^2\,-\,a\,(2a+1)\,(R_{-a})^2}\,q^{(\alpha}h^{\beta\gamma)} +$$

$$+\,6\,\frac{a+2}{a+1}\,\frac{a}{a-4}\,\frac{R_{-a}}{R_{1-a}}\,t^{(<\alpha\beta>_3}U^{\gamma)}\,.$$

3 Conclusions

We have found here a set of balance equations which approximate in the ultrarlativistic limit the original one and they are suitable for describing a polyatomic gas. They are divided into some subcases depending on how polyatomic the gas is or, to be more precise, on the basis of the parameter $a = -1 + f^i/2$ with f^i the internal degrees of freedom $f^i \geq 0$ due to the internal motions. As these degrees of freedom should be expressed by an integer, $2a$ is also an integer. However, as a mathematical abstraction, one can always think of a continuous trend of a. This model allows this. Obviously, it is physically valid as long as the absolute temperature is high enough to make γ tend to zero, but not so much as to break the molecules and thus deprive the gas of its polyatomic nature. Therefore the model is more physically significant, when a is not too big.

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