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A NOTE ON A CLASS OF 4TH ORDER HYPERBOLIC PROBLEMS WITH WEAK AND STRONG DAMPING AND SUPERLINEAR SOURCE TERM

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To our friend Patrizia on the occasion of her sixty-fifth birthday.

Abstract. In this paper we study a initial-boundary value problem for 4th order hyperbolic equations with weak and strong damping terms and superlinear source term. For blow-up solutions a lower bound of the blow-up time is derived. Then we extend the results to a class of equations where a positive power of gradient term is introduced.

1. **Introduction.** Hyperbolic problems of 4th order provide models for various phenomena in Mathematical Physics as the motion of elasto-plastic bars. For more details, see for instance [2], [6].
A relevant feature of these models lies in finite time blow-up of solutions in higher dimensional settings.
An important field of investigation is to derive upper and lower bounds of blow-up time T^*, in particular lower bounds T, since they ensure a time interval $[0, T]$, $T < T^*$, where the solutions remain bounded.
In this paper we are concerned with blow-up solutions of the following class of 4th order hyperbolic problems

$$u_{tt} + k_1\Delta\Delta u - k_2\Delta u - k_3\Delta u_t + au_t|u_t|^{m-2} = b|u|^{p-2}u, \ in \ \Omega \times (t > 0), \tag{1}$$

$$u = 0, \quad \Delta u = 0 \qquad on \ \partial\Omega \times (t > 0), \tag{2}$$

$$u(x, 0) = u_0(x), \ \ u_t(x, 0) = u_1(x), \quad on \ \Omega, \tag{3}$$

where Ω is a bounded domain in $R^N, N > 2,\ p > m \geq 2,\ k_i\ (i = 1, 2, 3),\ a, b,$ are positive constants, u_0, u_1 suitable functions in Ω. T^* is the blow-up time (or lifespan) of the solution u defined as

$$T^* = \sup\{T > 0 : \ u \text{ exists in } \Omega \times [0, T]\}.$$

To 4th order hyperbolic problems different boundary conditions may be associated: not only (2) (Navier boundary conditions) but also the Dirichlet boundary conditions:

$$u = 0, \quad \frac{\partial u}{\partial n} = 0, \qquad on \ \partial\Omega \times (t > 0), \tag{4}$$

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with n the outward normal direction to the boundary. Some results hold for both Navier and Dirichlet boundary conditions (see [15], and for parabolic problems [16]). For the Petrovsky equation (obtained from (1) with $k_1 = 1, k_2 = k_3 = 0$):

$$u_{tt} + \Delta\Delta u + a u_t |u_t|^{m-2} = b|u|^{p-2}u, \ in\ \Omega \times (t > 0), \tag{5}$$

under Dirichlet boundary conditions (4) and initial conditions (3), in [11] Messaoudi proves that the solution is global if $m > p \geq 2$, while it blows up in finite time if $p > m \geq 2$ and the energy

$$E(t) := \frac{1}{2}\int_\Omega \left\{u_t^2 + (\Delta u)^2\right\}dx \ - \frac{b}{p}\int_\Omega |u|^p dx$$

is initially negative.

The result of Messaoudi is improved by Wu and Tsai in [19], where they show that the solution of (5) is global under some conditions without the relation between m and p, and the solution blows up if $p > m$ and the initial energy is nonnegative.

For the case $m = 2$, Wu ([18]) considers the equation

$$u_{tt} + \Delta\Delta u - \Delta u - \omega\Delta u_t + \alpha(t)u_t = |u|^{p-2}u, \ in\ \Omega \times (t \geq 0), \tag{6}$$

with $\omega > 0$ and $\alpha(t)$ a non increasing, positive and differentiable function, under (2) and (3), deriving both upper and lower bounds for blow-up time.

For the equation with the strong damping term

$$u_{tt} + \Delta\Delta u - \Delta u_t + a u_t |u_t|^{m-2} = b|u|^{p-2}u, \ in\ \Omega \times (t > 0), \tag{7}$$

under (4) and (3) in [7] the authors prove that the solution of (7) is global without a connection between p and m, while the local solution blows up in finite time if $p > m$ and $E(0)$ is less the potential well depth (see also [5]). In [7], [9] , [11] and [18] the existence of the solution is also investigated.

In [15] Philippin and Vernier-Piro derive a lower bound of the solution of the Petrovsky equation (5) under (2) or (4) boundary conditions and (3), when Ω is a bounded domain in $R^N, N = 2, 3$. The bound is explicit due to the application of a Sobolev type inequality (valid only for $N = 2, 3$), where the constant is explicitly computable (see [14] for the proof of the inequality).

Different classes of higher order hyperbolic equations with blow-up solutions have been examined: in particular we cite the study of systems of Petrovsky equations in [13] where the authors obtain lower bounds to blow-up time under some considerations on initial data.

Li, Sun and Liu in [8] for a particular choice of the source terms prove the global existence of solutions, establish the uniform decay rates, and derive the conditions on the weak damping terms to obtain the blow-up of the solutions and lifespan estimates. In [10] Marras and Vernier-Piro investigate blow-up solutions of a nonlinear hyperbolic system of 4th order with time dependent coefficients under Dirichlet or Navier boundary conditions. Moreover, we cite the paper [3] of Autori, Colasuonno, Pucci, where the blow-up at infinity of global solutions of strongly damped polyharmonic Kirchhoff systems is investigated (see also [4] for the aymptotic stability).

The aim of this paper is to obtain a lower bound for the blow-up time for solution to problem (1)-(2)-(3), extending the results in [15] to the more general equation (1) and to an equation containing also a positive power of the gradient term (cf. (14)). Compared with the results in [7] and [18], we remark that the equations under investigation are more general and the bounds obtained are more explicit (in the sense that in the cited papers the bounds have an integral form) and computable except for the constant B_q in the Sobolev -Poincaré inequality (see Lemma 18 below).

We state now our main results, concerning the behaviour in time of blow-up solutions of problem (1)-(2)-(3). In the sequel we assume that $p > m \geq 2$ and the conditions on initial data present in Theorem 2.5 in §2 are satisfied.

We introduce the function

$$\Phi(t) := \int_\Omega |u_t|^2 dx + k_1 \int_\Omega |\Delta u|^2 dx + k_2 \int_\Omega |\nabla u|^2 dx, \quad in\ \Omega \times (t > 0), \tag{8}$$

with

$$\Phi_0 := \Phi(0) = \int_\Omega |u_1|^2 dx + k_1 \int_\Omega |\Delta u_0|^2 dx + k_2 \int_\Omega |\nabla u_0|^2 dx, \quad in\ \Omega. \tag{9}$$

Definition.
The solution of problem (1)-(2)-(3) blows up at time T^* if

$$\lim_{T \to T^*} \Phi(t) = +\infty. \tag{10}$$

We stress that the results of the paper hold for $N > 2$.
For the superlinear source term $u|u|^{p-2}$ in (1), we assume

$$\begin{cases} \frac{2(N+1)}{N} < p < +\infty, & N = 3, 4, \\ \frac{2(N+1)}{N} < p < \frac{2(N-1)}{N-4}, & N \geq 5, \end{cases} \tag{11}$$

in order to apply a Sobolev -Poincaré inequality in Lemma 2.2 when proving the Theorems 1.1 and 1.2 (see in particular (25)).

Our first result is contained in the following Theorem.

Theorem 1.1. *Let $u(x,t)$ be a blow-up solution of (1)-(2)-(3) and let $\Phi(t)$ and Φ_0 be defined in (8)-(9). Suppose that p satisfies (11) and $k_i, i = 1, 2, 3$ and a, b are positive constants, then a lower bound T of the blow-up time T^* is given by*

$$T = \frac{\Phi_0^{2-p}}{(p-2)K} \tag{12}$$

with

$$K = \frac{C^2 b^2}{k_3} B_p^{2(p-1)}, \tag{13}$$

C and B_p the positive constants in the Gagliardo-Nirenberg and Sobolev-Poincaré inequalities, respectively.

Remark 1.
Note that in the bound (12)-(13), for $p = 2$, $C = \frac{1}{\sqrt{\pi N(N-2)}} \left(\frac{\Gamma(N)}{\Gamma(N/2)} \right)^{\frac{1}{N}}$ (see [17]).

Moreover we introduce in (1) a gradient term of power type to obtain the new problem

$$\begin{aligned} u_{tt} + k_1 \Delta\Delta u - k_2 \Delta u - k_3 \Delta u_t + a u_t |u_t|^{m-2} \\ = b|u|^{p-2}u + c|\nabla u|^s, \ in\ \Omega \times (t > 0), \end{aligned} \tag{14}$$

where $\Omega \subset R^N, N > 2,\ s \leq \frac{N}{N-2}$, under initial and boundary conditions (2)-(3).

Theorem 1.2. *Let $u(x,t)$ be a blow-up solution of (14)-(2)-(3) and let $\Phi(t)$ be defined in (8). Suppose that p satisfies (11), $k_i, i = 1, 2, 3$ and a, b, c are positive constants and $1 \leq s \leq \frac{N}{N-2}$, then $\Phi(t)$ remains bounded in the interval $[0, \mathcal{T}_i)$ with*

$$\begin{cases} \mathcal{T}_1 := \dfrac{1}{c(p-2)} log\Big(1 + \dfrac{c\eta}{\zeta_1}\Phi_0^{2-p}\Big), & if\ s = p-1, \\ \mathcal{T}_2 := \dfrac{1}{c(s-1)} log\Big(1 + \dfrac{c\eta}{\zeta_2}\Phi_0^{1-s}\Big), & if\ s > p-1, \\ \mathcal{T}_3 := \dfrac{1}{c(p-2)} log\Big(1 + \dfrac{c\eta}{\zeta_3}\Phi_0^{2-p}\Big), & if\ s < p-1, \end{cases} \tag{15}$$

with

$$\begin{cases} \zeta_1 := \frac{c\gamma}{k_1} + K, \\ \zeta_2 := \frac{c\gamma}{k_1} + K\Phi_0^{p-1-s}, \\ \zeta_3 := \frac{c\gamma}{k_1}\Phi_0^{s-p+1} + K, \end{cases} \tag{16}$$

$\eta = \eta(p, c)$ a positive constant, K defined in (13) and γ the constant in the embedding $W^{2,2} \subseteq W^{1,2s}$. $\mathcal{T}_i$ are lower bounds of $\mathcal{T}_i^*$.

The scheme of this paper is the following: in §2 we explain the notations used throughout the paper and collect some known results to be used in the proofs of our results. Subsequently, in §3 we prove Theorem 1.1. Finally, in §4 we exhibit an extension of our first result to an equation containing a positive power of the gradient term.

2. **Preliminaries.** In this section we introduce some notations and collect some known results that we will use in the sequel. Let us introduce the space

$$H = \{u \in H^2(\Omega) \cap H_0^1(\Omega) : u = \Delta u = 0 \ \ on\ \partial\Omega\}$$

and

$$||u||_H^2 = ||\Delta u||_2^2 + ||\nabla u||_2^2.$$

Lemma 2.1. *(Gagliardo-Nirenberg-Sobolev inequality).*

Let $1 \le p < N$*, then* $\forall u \in W^{1,p}(\Omega)$ *there exists a positive constant* C *such that*

$$||u||_{p^*} \le C||\nabla u||_p, \quad p^* = \frac{pN}{N-p} > p.$$

Lemma 2.2. [1] *(Sobolev -Poincaré inequality)*

Let $u \in H$*. If* q *satisfies*

$$\begin{cases} 2 < q < +\infty, & 1 \le N \le 4, \\ 2 < q < \frac{2N}{N-4}, & N \ge 5, \end{cases} \tag{17}$$

then

$$||u||_q \le B_q||u||_H, \tag{18}$$

with $B_q = \sup_{u\in H\setminus\{0\}} \frac{||u||_q}{||u||_H}$.

Now we define an energy function associated to (1)-(2)-(3):

$$E(t) := \frac{1}{2}\int_\Omega u_t^2 dx + \frac{k_1}{2}\int_\Omega (\Delta u)^2 dx + \frac{k_2}{2}\int_\Omega |\nabla u|^2 dx \tag{19}$$
$$-\frac{b}{p}\int_\Omega |u|^p dx.$$

Lemma 2.3. $E(t)$ *is a non increasing function for all* $t \ge 0$.

Proof. We derive $E(t)$ to obtain

$$E'(t) = \int_\Omega u_t u_{tt} dx + k_1\int_\Omega u_t \Delta\Delta u dx - k_2\int_\Omega u_t\Delta u dx - b\int_\Omega u|u|^{p-2}u_t dx$$

Now we use (1), multiplied by u_t, and integrate to obtain

$$\int_\Omega u_t(u_{tt} + k_1\Delta\Delta u - k_2\Delta u)dx$$
$$= k_3\int_\Omega u_t\Delta u_t dx - a\int_\Omega |u_t|^m dx + b\int_\Omega u|u|^{p-2}u_t dx.$$

Then we get

$$E'(t) = k_3\int_\Omega u_t\Delta u_t dx - a\int_\Omega |u_t|^m dx.$$

Moreover, by divergence theorem

$$k_3\int_\Omega u_t\Delta u_t dx = -k_3\int_\Omega |\nabla u_t|^2 dx,$$

so that

$$E'(t) = -\big(a||u_t||_m^m + k_3||\nabla u_t||_2^2\big) \le 0.$$

As a consequence $E(t) \le E(0)$.

Now for completeness, we state without the proof, the local existence and the blow-up theorems to the solutions of our problem. First we present the result in [18] for the equation (6) ($m = 2$) under (2)-(3).

Theorem 2.4. [18] *(Local existence)*
Suppose that

$$\begin{cases} 2 < p < +\infty, \quad N \leq 4, \\ 2 < p < \frac{2(N-2)}{N-4}, \quad N \geq 5. \end{cases} \tag{20}$$

Let $u_0 \in H$ and $u_1 \in L^2$. Then there exists a unique weak solution u of the problem (6)-(2)-(3) such that $u \in L^\infty([0,T]; H(\Omega))$ and $u_t \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$, for $T > 0$ small enough.

For the problem (1)-(2)-(3) $(m > 2)$, the existence theorem can be established by combining the arguments of [5], [7], [11], [20], under the condition $2 < m < +\infty,\ N \leq 4,\ \ 2 < m < \frac{2N}{N-4}, \quad N \geq 5$.

Next we introduce $I(u) = k_1 \int_\Omega (\Delta u)^2 dx + k_2 \int_\Omega |\nabla u|^2 dx - b \int_\Omega |u|^p dx$ and d the potential well depth ([12]) to state the following

Theorem 2.5. *(Blow-up)*
Assume $p > m \geq 2$ and (20). If $u_0 \in \{u \in H(\Omega), I(u) < 0\}$ and $E(0) < d$, then the local solution u of the problem (1)-(2)-(3) blows up in finite time.

For the function $\Phi(t)$, we prove the following

Lemma 2.6. *Let $u(x,t)$ be the solution of (1)-(2)-(3). Let $\Phi(t)$ and Φ_0 be defined in (8)-(9) and satisfying (10). Then there exists a time $\bar{t} \in [0, T^*)$ such that*

$$\Phi^{\mathrm{q}}(t) \leq \Phi^{\mathrm{p}}(t)\Phi_0^{\mathrm{q}-\mathrm{p}}, \quad \forall t \in [\bar{t}, T^*), \tag{21}$$

for any $1 < \mathrm{q} < \mathrm{p}$.

Proof.
By definition of blow-up solution, if $\Phi(t)$ is non decreasing for all $t \in [0, T^*)$, then $\Phi(t) \geq \Phi_0, \forall t \in [0, T^*)$. On the contrary, there exists a time $\bar{t} \in (0, T^*)$ such that $\Phi(\bar{t}) = \Phi_0$ and $\Phi(t) \geq \Phi_0$, for $t \in [\bar{t}, T^*)$. Then (21) holds.

3. Proof of Theorem 1.1.

In this section we derive a differential inequality for $\Phi(t)$, defined in (8), to obtain a lower bound T of the blow-up time T^*, so that we may fix a safe time interval $[0,t], t < T$, where the solution $u(x,t)$ is bounded.
Differentiating Φ, we have for every blow-up solution u

$$\begin{aligned} \Phi'(t) &:= 2\int_\Omega \{u_t u_{tt} + k_1 \Delta u \Delta u_t + k_2 \nabla u \nabla u_t\} dx \\ &= 2\int_\Omega \{u_t\ u_{tt} + k_1 u_t \Delta\Delta u - k_2 u_t \Delta u)\} dx, \end{aligned} \tag{22}$$

where the second Green identity was used. Next, by using (1) we get

$$\Phi'(t) = 2\big(k_3 \int_\Omega\ u_t \Delta u_t dx - a\int_\Omega |u_t|^m dx + b\int_\Omega u_t\ u\ |u|^{p-2} dx\big) = I_1 + I_2 + I_3$$

In order to estimate I_1, we note that

$$\int_\Omega u_t \Delta u_t dx = -\int_\Omega |\nabla u_t|^2 dx.$$

The negative term I_2 can be neglected. For I_3 , from the Hölder inequality we have

$$\Big|\int_\Omega u_t\ u\ |u|^{p-2} dx\Big| \leq \int_\Omega |u_t|\ |u|^{p-1} dx \leq \Big(\int_\Omega |u|^{\frac{2(p-1)N}{N+2}} dx\Big)^{\frac{N+2}{2N}} \Big(\int_\Omega |u_t|^{\frac{2N}{N-2}} dx\Big)^{\frac{N-2}{2N}}.$$

By Lemma (2.1) with $p = 2$ and $p^* = \frac{2N}{N-2} > 2$, applied to the last integral, it follows

$$I_3 \leq 2bC\Big(\int_\Omega |u|^{\frac{2(p-1)N}{N+2}} dx\Big)^{\frac{N+2}{2N}} \Big(\int_\Omega |\nabla u_t|^2 dx\Big)^{1/2}.$$

By using

$$\mathcal{A}^\theta\ \mathcal{B}^{1-\theta} \leq \theta\mathcal{A} + (1-\theta)\mathcal{B}, \tag{23}$$

with $\mathcal{A}, \mathcal{B} > 0,\ 0 < \theta < 1$,

$$I_3 \leq \Big(\frac{C^2 b^2}{k_3}\Big)\Big(\int_\Omega |u|^{\frac{2(p-1)N}{N+2}} dx\Big)^{\frac{N+2}{N}} + 2k_3 \int_\Omega |\nabla u_t|^2 dx.$$

At the end

$$\Phi'(t) \leq \Big(\frac{C^2 b^2}{k_3}\Big)\Big(\int_\Omega |u|^{\frac{2(p-1)N}{N+2}} dx\Big)^{\frac{N+2}{N}}. \tag{24}$$

Now, with q in Lemma (2.2) replaced by $\frac{2(p-1)N}{N+2}$, we have

$$\Big(\int_\Omega |u|^{\frac{2(p-1)N}{N+2}} dx\Big)^{\frac{N+2}{N}} \leq B_p^{2(p-1)} \Phi^{p-1}, \tag{25}$$

where, without loss of generality, we suppose $k_1,\ k_2$ greater than 1. Finally, by inserting (25) in (24) with $K = \Big(\frac{C^2 b^2}{k_3}\Big) B_p^{2(p-1)}$, the following inequality holds

$$\Phi'(t) \leq K\ \Phi^{p-1}(t), \quad t \in [0, T^*), \tag{26}$$

that may be rewritten as

$$\Big(\Phi^{2-p}(t)\Big)' \geq -(p-2)K.$$

If we integrate in the time interval $[0, t), t < T^*$, we get

$$\Phi^{2-p}(t) \geq \Phi_0^{2-p} - (p-2)Kt. \tag{27}$$

From the definition (10), and letting $t \to T^*$, we obtain

$$T^* \geq \frac{\Phi_0^{2-p}}{K(p-2)} := T.$$

From (27) it follows that $\Phi(t)$ remains bounded in any time interval $[0, t],\ t < T$, with T defined in (12). T is a lower bound for T^*.
It is important to observe that the boundedness of $\Phi(t)$ in $[0, T)$ implies the boundedness of the solution $u(x, t)$ in L^2-norm.
From the hinged plate eigenvalue problem

$$\begin{cases} \Delta\Delta v - \Lambda v = 0, & in \quad \Omega, \\ v = 0, \quad \Delta v = 0, & on \quad \partial\Omega, \end{cases} \tag{28}$$

it is known that

$$||v||_2^2 \leq \frac{1}{\Lambda_1} ||\Delta v||_2^2$$

with Λ_1 the first eigenvalue of the problem (28). The boundedness follows from the definition of $\Phi(t)$ (8) and

$$||u||_2^2 \leq \frac{1}{\Lambda_1 k_1}\ \Phi(t).$$

.

4. **Extensions.** In this section we consider a fourth-order hyperbolic equation with a gradient term $|\nabla u|^s$. We prove that Theorem 1.1 may be extended to blow-up solutions of the equation (14) under Navier boundary condition (2) and (3). By using the same auxiliary function $\Phi(t)$, defined in (8), we obtain a lower bound of the blow-up time.

Proof of Theorem 1.2.

Following the proof of Theorem 1.1 in §3

$$\Phi'(t) := 2\int_\Omega u_t\ \{u_{tt} + k_1 \Delta\Delta u - k_2 \Delta u\} dx \tag{29}$$

$$= 2(k_3 \int_\Omega\ u_t \Delta u_t dx - a\int_\Omega |u_t|^m dx + b\int_\Omega u_t\ u\ |u|^{p-2} dx) + c\int_\Omega u_t |\nabla u|^s dx$$

The new term that we have to estimate is

$$\int_\Omega u_t |\nabla u|^s dx$$

By Hölder inequality and (23)

$$\int_\Omega u_t|\nabla u|^s dx \leq \Big(\int_\Omega u_t^2 dx\Big)^{\frac{1}{2}}\Big(\int_\Omega |\nabla u|^{2s} dx\Big)^{\frac{1}{2}}$$

$$\leq \frac{1}{2}\int_\Omega u_t^2 dx + \frac{1}{2}\int_\Omega |\nabla u|^{2s} dx.$$

By the Sobolev embedding theorem

$$W^{2,2}(\Omega) \subseteq W^{1,2s}(\Omega)$$

with $s \leq \frac{N}{N-2}$, applied to a function $w \in W_0^{2,2}(\Omega)$ to obtain the inequality

$$||\nabla u||_{2s}^{2s} \leq \gamma ||\Delta u||_2^{2s} \tag{30}$$

Then for all $t \in [0, T^*)$

$$\Phi'(t) \leq c\int_\Omega u_t^2 dx + c\gamma\Big(\int_\Omega (\Delta u)^2 dx\Big)^s \tag{31}$$

$$+\frac{C^2b^2}{k_3}B_p^{2(p-1)}\Big(\int_\Omega \big(k_1(\Delta u)^2 + a|\nabla u|^2\big)dx\Big)^{p-1}.$$

By the definition of Φ and K, we can derive the following differential inequality

$$\Phi'(t) \leq c\Phi + \frac{c\gamma}{k_1}\Phi^s(t) + K\ \Phi^{p-1}(t), \quad t \in [0, T^*), \tag{32}$$

Our aim is to semplify the inequality (32) in a way that it is possible to integrate and then to derive a lower bound to T^*. By using (2.6) and comparing s with $p-1$ we rewrite inequality (32) in the following three cases:

$$\begin{cases} \Phi'(t) \leq c\Phi(t) + (\frac{c\gamma}{k_1} + K)\ \Phi^{p-1}(t), & if\ s = p-1, \\ \Phi'(t) \leq c\Phi(t) + (\frac{c\gamma}{k_1} + K\Phi_0^{p-1-s})\ \Phi^s(t), & if\ s > p-1, \\ \Phi'(t) \leq c\Phi(t) + (\frac{c\gamma}{k_1}\Phi_0^{s-p+1} + K)\ \Phi^{p-1}(t), & if\ s < p-1. \end{cases} \tag{33}$$

Next step is to integrate each inequality in (33) in $[\bar{t}, t]$. For instance, the first inequality may be rewritten as

$$\Big(\frac{e^{c(p-2)t}\Phi^{2-p}}{2-p}\Big)' \leq \zeta_1 e^{c(p-2)t},$$

with $\zeta_1 := (\frac{c\gamma}{k_1} + K)$. It follows , with $\eta = e^{c(p-2)\bar{t}}$

$$\frac{e^{c(p-2)t}\Phi^{2-p}}{2-p} - \frac{\eta\Phi_0^{2-p}}{2-p} \leq \zeta_1\int_{\bar{t}}^t e^{c(p-2)\tau}d\tau \tag{34}$$

$$\leq \zeta_1\int_0^t e^{c(p-2)\tau}d\tau = \frac{\zeta_1}{c(p-2)}(e^{c(p-2)t} - 1).$$

We can easily repeat the computations in (34) for the second and third inequality in (33). Then if $t \to T^*$, we deduce that $\Phi(t)$ remains bounded in the intervals $[0, t_i], i = 1, 2, 3$ with $t_i < \mathcal{T}_i$ defined in (15). As mentioned above, $\mathcal{T}_i$ are lower bounds of the blow-up time in each case. The proof of Theorem 1.2 is finished.

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